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CONFORMAL GRAVITY AND TIME

by

Jeffrey Shafiq Hazboun

A dissertation submitted in partial fulfillment
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Physics

Approved:

Dr. James T. Wheeler Major Professor	Dr. Charles Torre Committee Member
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UTAH STATE UNIVERSITY
Logan, Utah

2014

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ABSTRACT

Conformal Gravity and Time

by

Jeffrey Shafiq Hazboun, Doctor of Philosophy

Utah State University, 2014

Major Professor: Dr. James T. Wheeler
Department: Physics

Cartan geometry provides a rich formalism from which to look at various geometrically motivated extensions to general relativity. In this manuscript, we start by motivating reasons to extend the theory of general relativity. We then introduce the reader to our technique, called the quotient manifold method, for extending the geometry of spacetime. We will specifically look at the class of theories formed from the various quotients of the conformal group.

Starting with the conformal symmetries of Euclidean space, we construct a manifold where time manifests as a part of the geometry. Though there is no matter present in the geometry studied here, geometric terms analogous to dark energy and dark matter appear when we write down the Einstein tensor. Specifically, the quotient of the conformal group of Euclidean four-space by its Weyl subgroup results in a geometry possessing many of the properties of relativistic phase space, including both a natural symplectic form and nondegenerate Killing metric. We show the general solution possesses orthogonal Lagrangian submanifolds, with the induced metric and the spin connection on the submanifolds necessarily Lorentzian, despite the Euclidean starting point. By examining the structure equations of the biconformal space in an orthonormal frame adapted to its phase space properties, we also find two new tensor fields exist in this geometry, not present in Riemannian geometry. The first is a combination of the Weyl vector with the scale factor on the metric, and determines the time-like directions on the submanifolds. The second comes from the components of the spin connection, symmetric with

respect to the new metric. Though this field comes from the spin connection, it transforms homogeneously. Finally, we show in the absence of Cartan curvature or sources, the configuration space has geometric terms equivalent to a perfect fluid and a cosmological constant.

We complete the analysis of this homogeneous space by transforming the known, general solution of the Maurer-Cartan equations into the orthogonal, Lagrangian basis. This results in a signature-changing metric, just as in the work of Spencer and Wheeler, however without any conditions on the curvature of the momentum sector. The Riemannian curvatures of the two submanifolds are directly related. We investigate the case where the curvature on the momentum submanifold vanishes, while the curvature of the configuration submanifold gives an effective energy-momentum tensor corresponding to a perfect fluid.

In the second part of this manuscript, we look at the most general curved biconformal geometry dictated by the Wehner-Wheeler action. We use the assemblage of structure equations, Bianchi identities, and field equations to show how the geometry of the manifolds self-organizes into trivial Weyl geometries, which can then be gauged to Riemannian geometries. The Bianchi identities reveal the strong relationships between the various curvatures, torsions, and cotorsions. The discussion of the curved case culminates in a number of simplifying restrictions that show general relativity as the base of the more general theory.

(179 pages)

PUBLIC ABSTRACT

Conformal Gravity and Time

Within the last year, two acclaimed physics experiments have probed further into the extremes of our physical understanding. The Large Hadron Collider, the largest experiment ever constructed, has detected a Higgs boson, which establishes a mass scale for the fundamental particles. The Planck mission satellite has made the most accurate measurements of the cosmic microwave background radiation, which is the oldest data about the early universe we are currently able to measure directly. The mission corroborated the proportions of dark matter and dark energy are all very close to expected values. While these experiments have helped solidify the current working model of physics (general relativity plus the standard model of particle physics), large questions remain about the origins of the main constituents of the universe. Galactic and cosmological scale observations indicate something is missing from the standard model of our universe. The current ΛCDM model of cosmology is named after dark energy (Λ) and cold dark matter, place holders in a model where we know the constituents' phenomenology, but not their origin. The need for an extension of current physical models is obvious.

Most research in gravity has focused on understanding the geometry of spacetime. We demonstrate how the geometry of spacetime may emerge by starting with a space where time does not exist. Time can emerge as part of a physical theory, instead of assuming its existence from the beginning. Specifically, we look at the symmetry of the equations that define the gravitational interaction and extend those existing symmetries, i.e. giving a theory with more symmetries than standard general relativity. We investigate the consequences of making a theory of gravity that is fully scale symmetric. When we change the units (i.e. meters, feet, pounds, seconds) of our physical measurements locally, we expect the laws of physics to undergo no change. Biconformal space is constructed by requiring this broader class of symmetries. Here, we show how time comes necessarily from the construction of biconformal space. The gravitational theory derived from this construction is more complex than general relativity; however, general relativity arises as a special case of biconformal gravity, a feature any candidate alternative

theory of gravity must possess. We illustrate biconformal gravity is a viable successor to general relativity and discuss this in the context of dark matter and dark energy candidates.

Jeffrey S. Hazboun

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Jeffrey S. Hazboun

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CHAPTER 1

INTRODUCTION

1.1. Our current understanding

Within the last year, two acclaimed physics experiments have probed further into the extremes of our physical understanding than has been possible in the past. The Planck satellite has made the most accurate measurements of the anisotropies in the cosmic microwave background to date obtaining the oldest data about the early universe we are currently able to directly measure. The mission found the least exotic theories for the mechanisms of inflation, and the proportions of dark matter and dark energy are all very close to accepted values from other astrophysical measurements [1]. The Large Hadron Collider, the largest experiment yet constructed by humans, has found the Higgs boson in the mass range where exotic particle physics is unnecessary. While these experiments have helped solidify the current working model of physics (general relativity + the standard model of particle physics), large questions remain about the origins of the main constituents of the universe and how the dichotomous parts of physics interact at a basal level.

Work in gravity has been an effort to understand the geometry of spacetime; however, in this manuscript we will demonstrate how the geometry of spacetime emerges by considering the symmetries of a Euclidean signature space. In fact, we will show the natural geometry is that of a symplectic manifold with spacetime as one of the Lagrangian submanifolds: biconformal space in an orthogonal, Lagrangian basis. Our current work relies heavily on differential geometry in the Cartan formalism and is based on the pioneering work into gravitational gauge theory of Kibble [2], Ne'eman and Regge [3,4], Ivanov and Niederle [5,6], and others. An extensive history of the field is included in Section 2.2 of Chapter 2.

We extend the work of Wheeler, Wehner, and Spencer to combine the main results concerning biconformal space, that biconformal space reproduces the physics of general relativity, with the natural emergence of time as a special direction within the submanifolds of a phase space. The orthonormal version of the time basis of [7] is used to investigate the Lorentzian properties of the submanifolds, specifically the spin connection. We strengthen the result of [7]

by showing that, starting in the Euclidean case, no assumptions need to be made about the space to derive a Lorentzian (signature-changing) metric. We will also see the appearance of two new tensors and a self organization of the submanifold geometry into that of a Riemannian one from a Weyl geometry.

It is important to note that general relativity is extremely successful as a predictive theory, describing the gravitational interaction for almost a century. The accuracy of both strong and weak field tests of GR continues to grow [8]. The last, unseen, prediction of general relativity, gravitational waves, will likely be directly detected by the end of the decade. Their dissipative effects have already been seen in the Hulse-Taylor binary pulsar system in a way that agrees with general relativity to better than a half a percent [8].

So the question stands, "Why do we need an extension to such a successful physical theory?" The answer lies partly in the galactic and cosmological scale observations mounting up that seem to point out something is missing from the standard model of our universe. The current ΛCDM model of cosmology is named after dark energy and dark matter, place holders in a model where we know the constituent's phenomenology, but not their origin. If this outstanding 95% of the universe's content is not reason enough, there is the century old quest for a theory of quantum gravity.

1.1.1. Dark matter

The need for a large amount of unseen gravitationally interacting matter in the universe originated in the calculations of Fritz Zwicky who first realized the Coma galaxy cluster seemed to have a large amount of matter missing [9]. These observations gained more modern traction after Rubin, Thonnard, and Ford [10] used the Doppler shift of edge-on galaxies to show their rotation curves seemed to necessitate the existence of large amounts of unseen matter. The amount of dark matter has now been corroborated by measurements of the gravitational lensing of galaxies [11] where the mass of the intervening galaxies can be calculated from the lensing of more distant sources. Galaxy formation simulations also corroborate the proportion of dark matter needed first for star formation, and then the large–scale structure of galaxies [12]. Simulations with only slight deviations from those proportions can drastically change the outcome of these

simulations. Since the estimates of the missing matter comprises $\sim$25% of all the energy density in the universe, there is reason to take these observations seriously.

There have been quite a few hypotheses put forward to explain the origin of this mass. Most current searches for dark matter center around various particles that fall into the weakly interacting massive particle (WIMP) category. All of these particles are extensions to the currently accepted theories of physics. The axion [13, 14] can either be seen as an extension of the standard model of particle physics, or as arising generically from string theory. There are also a number of supersymmetric particles, which have been put forward as dark matter candidates. Searches for these particles rely on them having a small, but nonzero interaction cross section with baryonic matter [15].

Many theories have been put forward to explain dark matter and, another possibility is the extra gravitational degrees of freedom arising in modified theories of gravity play the role of dark matter [15]. While in some cases the degrees of freedom can be interpreted as new matter, there are other theories, like modified Newtonian dynamics (MOND) that try to explain the observations, normally attributed to dark matter, as a modification of gravitational dynamics (the potential is not Newtonian $\propto \frac{1}{r^2}$) on the galactic size scale [16]. The phenomenological predictions of MOND (and its relativistic relative Tensor Vector Scalar theory [17]) stem from a change to the Newtonian gravitational potential at the galactic (and larger) scale. MOND has seen strong opposition since observations of colliding galaxy clusters, most famously the Bullet cluster, have allowed the mapping of dark matter within the colliding clusters. MOND and dark matter give distinct predictions about where the gravitational lensing will be centered in the collision of two galaxy clusters. In MOND, the lensing is expected to be centered at the center of mass of the luminous matter (since there is no dark matter, only a different potential). If dark matter exists, it is expected it would continue to pass through the luminous matter, which is slowed down due to a larger cross section of interaction. The lensing would then be centered around the dark matter that has continued to move due to inertia. The latter seems to be the case in the Bullet cluster [18]. Nonetheless, while MOND/TVS may be invalidated, it seems an extension to our current theories, whether in the particle physics sector or the gravitational

sector, is necessary to explain what constitutes dark matter.

1.1.2. Dark energy

The observation of the acceleration of the expansion of the universe [19,20], seen through a small number of high redshift Type Ia supernovae data points, and for which the Nobel Prize was recently awarded, is another arena of cosmology that has befuddled theorists. The acceleration is well–modeled by resurrecting the cosmological constant, the constant term consistent with a fully diffeomorphism-invariant theory of gravity [21], included in the Einstein field equation. The vacuum energy of spacetime is often proposed as a source of the negative pressure needed to create such an acceleration. However, when calculated through quantum field theoretic means this energy density is around 100 orders of magnitude larger than the observed value [22]. The source of dark energy is an especially interesting problem because it represents a majority of the energy density composition of the universe. The most recent data from the Planck mission [1] substantiated that dark energy makes up 68.3% of the critical density of the universe.

There are three commonly stated reasons the cosmological constant is not considered the end of the story [15]. The first is the value for the cosmological constant is unexpectedly small with regard to any physical scale (especially the predicted vacuum energy), except the current Hubble horizon scale. Another reason is the energy density of dark energy is surprisingly close to the current matter-energy density. This means the time at which humans are able to start measuring the acceleration of the expansion of the universe happens to be the exact epoch when these densities are comparable. Many physicists see this as a fine-tuning problem [15, 22]. Lastly, the existence of coherent acoustical oscillations (baryon acoustic oscillations) in the CMB have made inflation (exponential acceleration in the early universe) an integral part of the cosmological model. Since the accelerated expansion of inflation stopped, this gives reason to believe the current acceleration is temporary and not due to the cosmological constant [15].

A measurement solely of the expansion rate of the universe does not allow observers to differentiate between the possible mechanisms of the acceleration. It is unknown whether the acceleration is due to a heretofore unknown dynamical fluid or field (dark energy) or an extension of the theory of general relativity. A number of modifications to general relativity have been

proposed that explain the expansion without adding in a new form of matter, such as new massive gravity [23], but often add other complications to the theory [24]. The fields necessary for this acceleration have been phenomenologically modeled [25]; however, the source of these fields is not yet known. How these fields will emerge from an extension of the standard model of particle physics, or general relativity or a unified field theory such as string theory is difficult to predict; however, it is certain such an extension is needed. That it will be centered on the gravitational sector is certainly possible.

1.1.3. Quantum gravity

One of the last motivations we mention for extending general relativity is quantization. Quantization of the gravitational interaction is a long–open field of current theoretical effort. There are varied opinions as to how close we are to realizing a quantum theory of gravity, but we are certainly lacking any experimental verifications of any quantum gravity candidate [26, 27]. The Large Hadron Collider in CERN is projected to eventually run at $14\ TeV$ [28], while the Planck scale (the energy scale at which we expect to see quantum gravity effects) is $1.22 \times 10^{16}\ TeV$, a factor of approximately 10^{15} larger. The scales at which the effects of quantum gravity are predicted are at small enough lengths (large enough energies) that we are far from being able to directly measure them through the normal route of particle accelerators. There are efforts to look for quantum gravity in the signatures of astrophysical events [26], but these are still nascent. The most straightforward route to quantization, as a quantum field theory of the spin-two graviton with the Einstein-Hilbert action is perturbatively nonrenormalizable [29]. This can most easily be seen from the superficial degree of divergence, where the relative mass term has a dimension of –1 in four dimensions [29].

The construction of a fully diffeomorphism invariant theory of spacetime is one of the main contributions of relativity. Unfortunately, the most commonly used quantization schemes necessitate separating the direction of time from space. The problem of time is a term used in slightly different ways within the quantum gravity community. According to [29] the incompatibility lies in the fact that quantum field theory treats the background as external to the physics, while general relativity treats them as dynamical. Thiemann [30] more specifically points to the fact

the Hamiltonian, in a diffeomorphism invariant theory, vanishes on the constraint surface. There is, therefore, no evolution in the normal sense we think of it in a quantum system. A partial solution to the problem of time in canonical quantum gravity is the use of dust fields to characterize the passage of time [31, 32]. It is an issue at the heart of the problem between gravity and quantum theory, and it is listed as a motivation for looking at the problem of quantum gravity [29]. In this manuscript we will show another resolution to the problem of time. In fact, we will see time emerge as part of the symmetries of a Euclidean space.

String theory is naturally seen as a theory of quantum gravity since a closed, quantized string has a massless spin–two mode [33], describing the graviton at the first–order interaction. It does not suffer in the same way from the problem of time since the gravitational interaction happens on a nondynamical $10\,(11)$-dimensional Minkowski background. However, string theory struggles to make predictions on the physical scales of current astrophysical observations.

Extending general relativity

Since there are so many unanswered questions surrounding gravitational phenomena, it seems reasonable to consider extensions to general relativity in order to gain understanding about them. There are myriad avenues available to extend general relativity. There is a long history [34] of extending or changing altogether the paradigms, which are the basis of general relativity.[1] Here, we will quickly summarize a number of schemes for constructing an alternative theory of gravity. One can imagine these theories as effective theories of some larger unified field theory, or solely as an extension of GR. Here we focus on the latter, but point out biconformal space can also be seen as a low–energy limit of some string models. In this manuscript we will show, while the starting point may seem like an alternative theory, we reproduce general relativity with our approach.

The most straightforward way to alter the physics of a field theory is to change the action principle for that theory. There are changes to the theory that can be viewed as changing the fundamental way in which one views the world, or the characteristics of its constituents. However, most of these can be best understood in how they affect the Lagrangian of the theory.

[1] One must remember that at the time, solving the questions about the advance of the perihelion of Mercury was a triumph of modified gravity theory. Granted the modification was a complete paradigm shift.

Perhaps the most straightforward way to change the Einstein-Hilbert action is by adding terms not linear in the curvature. These type of theories, including Gauss-Bonnet gravity and $f(R)$-theories have been extensively studied in the literature. Many of them manifest as lower energy limits of string theory. Various other structures can be added to the action, for instance Lovelock gravity, or tensor-vector-scalar gravity. It can even be shown one can recover much of general relativity by dropping the Riemann curvature and instead considering torsion to be the leading curvature to consider in teleparallel theory.

In this manuscript we are most interested in those extensions of general relativity that change the symmetry of the action. It is out of the scope of this introduction to cover the various alternate theories, but in Table 1.1 we list a number of extensions that change the symmetry of the action. We also note the change in symmetry with respect to Poincaré symmetry and references.

Table 1.1. Symmetry Extensions of General Relativity

Theory/ Model	Symmetry change from Poincaré	Reference
Supergravity	Supersymmetric Poincaré invariance	[35]
String Theory	Supersymmetric and up to $E(8) \times E(8)$	[33, 36, 37]
Hořava-Lifshitz	Galilean Invariance	[38]
MacDowell-Mansouri	de Sitter or anti-de Sitter invariance	[39, 40]
Weyl Gravity	4-dim Weyl invariance	[41–44]
Dynamical Cartan	Various extensions broken by dynamical vector	[45]
Observer Space	de Sitter Symmetry	[46]
Shape Dynamics	3-dim Weyl invariance	[47, 48]
Einstein-Aether	Vector field which can break 4-diffeos	[49]
Biconformal Space	n-dimensional Weyl invariance	[50, 51]

Biconformal space is an extension of the symmetries of general relativity. In Chapter 4 we will show one can regain scale invariant general relativity from an action that has Weyl symmetry.

1.2. Conformal symmetry

1.2.1. Extending the symmetry of general relativity

The development of the Standard Model of particle physics gives an interesting lesson about developing working physical models. The $SU(3) \times SU(2) \times U(1)$ symmetry of the Standard Model is not a symmetry we actually see in nature (i.e. particle accelerators) today. It is a symmetry spontaneously broken via the Higgs mechanism, which gives mass to the fields within the theory. The full $SU(3) \times SU(2) \times U(1)$ symmetry would only be seen in a high–energy limit. However, a full understanding of how the various fermions and vector bosons of the model interact necessitates an understanding of the underlying, broken symmetry. It is by starting with a fully symmetric theory and then breaking that symmetry we get to the widely successful Standard Model.

The idea of extending the symmetries of physics goes back much further than particle physics. One can interpret Newton's law of inertia (first due to Galileo), at the time, as an extension of the symmetry principles within physics. It was one of Galileo's most powerful insights to see that without friction, inertia would keep objects at rest or in motion [52]. In modern language, one would say that the equations of mechanics are invariant under Galilean transformations. Again, the common theme here is the theoretical framework possesses a symmetry, which must be broken (in this case by friction) to find experimental resolution. While the laws of classical physics are invariant under Galilean transformations, this is not always evident from everyday experience. Friction allows for inertia to be worked against, allowing Newton's first law of mechanics to be realized. In other words, Newton's insight was to see the symmetries of the world are broken by dissipative forces, which can then be included into the theory.

Special relativity is another theory where we have been able to change the symmetries of nature in order to broaden our physical model of the world. With hindsight, it can be shown all one needs in order to understand special relativity is to change the symmetries of nature from those of the Galilean group (spatial rotations, classical boosts and space + time translations) in three-dim to those of the Poincaré group (spacetime "rotations" and translations) in four-dim [53].

In fact, string theory is so successful at including all physical interactions in part because it includes huge symmetry groups. Of course, this is a double-edged sword, and the symmetry of string theory makes it difficult to make specific predictions about the world.

As illustrated here, the technique of extending the symmetry of a physical theory is a common method utilized by theorists to gain understanding of the world. In this manuscript, we investigate the outcome of considering an extension of the Poincaré symmetry group of general relativity, to that of the full conformal group.

1.2.2. Why conformal symmetry?

In this manuscript conformal symmetry will refer to the freedom to choose the units of a physical measurement. For a particle, the position $x^{\mu}(t)$ is the dynamical variable and therefore scales by a scale factor $x^{\mu} \rightarrow e^{\phi} x^{\mu}, \quad \phi \in \mathbb{R}$. In field theory, it is the fields, not the coordinates, that transform so we have $g_{\alpha\beta} \rightarrow e^{2\varphi} g_{\alpha\beta}$ as the principal transformation, where $g_{\alpha\beta}$ is a metric on spacetime. Of course, in physics the units on either side of an equation must match, so where there are other units besides length, those units must also change with a scale transformation. For instance, a mass transforms as $\frac{1}{Length}$ in geometric coordinates. There is often confusion in the literature about various versions of conformal or scaling symmetry. If one scales the coordinates of a theory, but not the masses, for instance, often the theory will seem inconsistent [54]. This arises from being inconsistent with the implementation of the scaling. In this manuscript it is acknowledged that all physical measurements are comparisons, and so what is important in physics is ratios of unit-ful observations.

Apart from the obvious observation that every physical measurement is only a comparison of ratios, there are a number of other motivations to specifically consider conformal symmetry in a theory of gravity. One compelling reason stems from looking at the symmetry of the combined action of the known physical interactions.

$$S_{\text{Universe(Currently)}} = \underbrace{\int S_{\text{Gravity}}}_{\text{Poincaré}} + \underbrace{\int S_{\text{Yang-Mills}}}_{\text{Conformal}}$$

The action of Yang-Mills theories (i.e. the Standard Model) are conformally invariant, only later

broken by the Higgs boson. This is fair reason to hypothesize that the whole action should be conformally invariant. In fact in [55], and earlier work cited therein this idea is taken a step further to show the scale factor of a conformal theory can be related to the Higgs field on the Yang-Mills sector.

In the renowned paper by Ehlers, Pirani, and Schild [56] they define axioms about the measurement of light rays and freely falling particles to bootstrap to the geometry of spacetime. They show by making the fully conformal geometry defined by light rays and the projective geometry defined by measurement of freely falling particles consistent with each other, the connection is, at best, a connection of a Weyl geometry. Figure 1.1 summarizes the methodology of the paper.

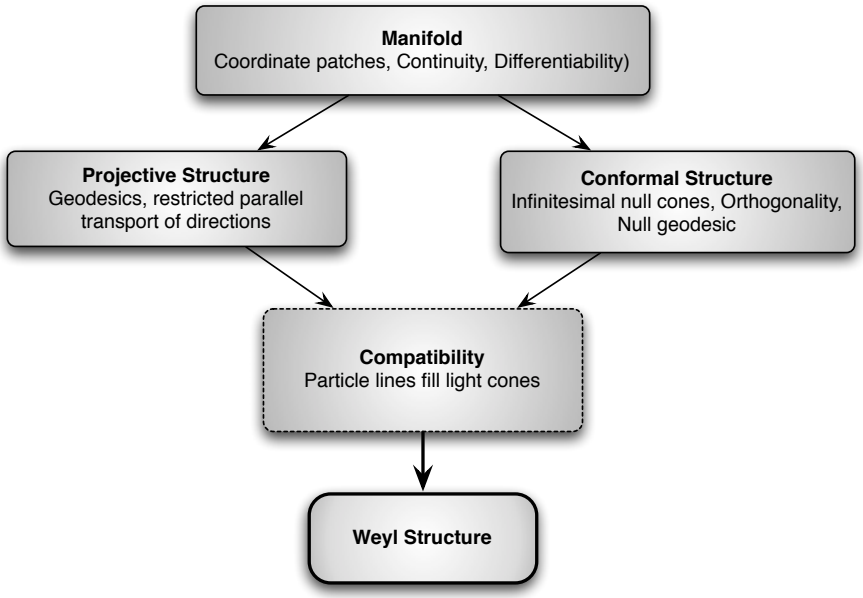


Figure 1.1. This diagram summarizes how a Weyl geometry can be built from the symmetries of light rays and freely falling particles. In [56] Ehlers, Pirani, and Schild start by considering what types of geometry can be defined by observations of light rays and freely falling dust. Requiring these geometries to be compatible, they show the connection is that of a Weyl geometry.

It should be noted the paper ends imposing what they refer to as Einstein simultaneity to restrict the Weyl geometry to a Riemannian and regain the background for general relativity. We have excised this ad hoc assumption from the diagram, as we hope to give the reader ample reason to take (trivial) Weyl geometry as the natural background for a gravitational theory.

CHAPTER 2

BICONFORMAL SPACE

2.1. Introduction

Through the course of this work, we will study the consequences of choosing conformal symmetry as the symmetry of our gravitational theory, first by considering strictly the geometry of a homogeneous space based on conformal symmetries, then by adding a gravitational action to that geometry. The tool we will use to understand this choice is Cartan geometry in the language of differential forms. A review of differential forms[1] is out of the scope of this manuscript, but the classic reference is Flanders [57].

In the remainder of this introduction, we give a brief historical overview of techniques leading up to, related to, or motivating our own, then describe the layout of our presentation.

We show, by basing a gravitational gauge theory on underlying symmetry, how the presence of a time-like direction can emerge from an initially Euclidean geometry. In addition, we show it is possible to produce a cosmological constant and cosmological dust as part of an initial geometry rather than as matter sources. Both of these changes occur as a result of increased symmetry. For the first, a new vector field, built as the difference of two gauge–dependent quantities, necessarily gives a time-like direction. The cosmological constant and dust arise in much the same way as the emergence of a cosmological constant in the MacDowell-Mansouri treatment of the de Sitter group [39], with the extra symmetry adding terms to the curvature.

By gauge theory, we typically understand a theory (i.e. the specification of an action functional), which is invariant under a local symmetry group – the gauge symmetry. Thus, there may be many gauge theories having the same gauge group. However, gauge theories having the same gauge group share a common structure: the underlying principal fiber bundle in which the base manifold is spacetime or some other world manifold and the fibers are copies of the gauge group. Such a principal fiber bundle is most simply constructed as the quotient of a larger group by the symmetry group. Constructed in this way, we have immediate access to

[1] For a more modern approach, in the form of class notes see
`http://www.physics.usu.edu/Wheeler/GaugeTheory/Lectures09SpringGaugeTheory.htm` .

relevant tensor fields: any group invariant tensors, the curvatures of the bundle, and the vectors of the group representation. Then any functional built invariantly from these tensors is a gauge theory. For example, in Sec. 2.3.3 below, we show how the quotient of the Poincaré group by its Lorentz subgroup may be generalized to a principal fiber bundle with Lorentz group fibers and a general base manifold having arbitrary Riemannian curvature. Identifying the curvature, solder form, Lorentz metric, and Levi-Civita tensor as tensors with respect to this local Lorentz symmetry, it is clear any functional built invariantly from them is a gauge theory. In addition, if we use a linear representation, $SO(3,1)$ or $SL\,(2,C)$, of the Lorentz group, then the action functional may include vectors or spinors from that representation and their covariant derivatives. For these reasons, we will define a gauging to be the fiber bundle of a specific quotient, along with the identification of its associated tensors. A gauge theory remains the specification of an action functional invariant on this bundle.

We develop a gauging based on the conformal group of a Euclidean space, and show its group properties necessarily lead to a symplectic manifold with Lagrangian submanifolds of Lorentzian signature. Though we deal almost exclusively with the homogeneous quotient space, we always have in mind the class of biconformal gauge theories presented in Sec. 2.5.2. This theory has been studied extensively [50]. In particular, we note from the field equations given in [50] that for specific relations of the action coefficients the homogeneous space is a vacuum solution. We find these vacuum solutions carry both a cosmological constant and a cosmological perfect fluid as geometric generalizations of the Einstein tensor. In curved models, this geometric background may explain or contribute to dark matter and dark energy. To emphasize the purely geometric character of the construction, we give a description of our use of the quotient manifold method for building gauge theories. Our use of the conformal group, together with our choice of local symmetry lead to several structures not present in other related gauge theories. Specifically, we show the generic presence of a symplectic form, there exists an induced metric from the nondegenerate Killing form, demonstrate (but do not use) Kähler structure, and find natural orthogonal, Lagrangian submanifolds. All of these properties arise directly from group theory.

2.2. Historical introduction

As mathematicians began studying the various incarnations of non-Euclidean geometry, Klein started his Erlangen Program in 1872 as a way to classify all forms of geometries that could be constructed using quotients of groups. These homogeneous spaces allowed for straightforward classification of the spaces dependent on their symmetry properties. Much of the machinery necessary to understand these spaces originated with Cartan, beginning with his doctoral dissertation [58]. The classification of these geometries according to symmetry foreshadowed gauge theory, the major tool that would be used by theoretical physicists as the twentieth century continued. We will go into extensive detail about how these methods are used in a modern context in section 2.3. Most of the development, in modern language, can be found in [59].

The use of symmetries to construct physical theories can be greatly credited to Weyl's attempts at constructing a unified theory of gravity and electromagnetism by adding dilatational symmetry to general relativity. These attempts failed until Weyl looked at a $U(1)$ symmetry of the action, thus constructing the first gauge theory of electromagnetism. These efforts were extended to non-Abelian groups by Yang and Mills [60], including all $SU(n)$ and described by the Yang-Mills action. The success of these theories as quantum precursors inspired relativists to try and construct general relativity as a gauge theory. Utiyama [61] looked at GR based on the the Lorentz group, followed by Kibble [2] who first gauged the Poincaré group to form general relativity.

Standard approaches to gauge theory begin with a matter action, globally invariant under some symmetry group $\mathcal{H}$. This action generally fails to be locally symmetric due to the derivatives of the fields, but can be made locally invariant by introducing an $\mathcal{H}$-covariant derivative. The connection fields used for this derivative are called gauge fields. The final step is to make the gauge fields dynamical by constructing their field strengths, which may be thought of as curvatures of the connection, and including them in a modified action.

In the 1970s, the success of the standard model and the growth of supersymmetric gravity theories inspired physicists to extend the symmetry used to construct a gravitational theory. MacDowell and Mansouri [39] obtained general relativity by gauging the de Sitter or anti-de

Sitter groups, and using a Wigner-Inönu contraction to recover Poincaré symmetry. As a precursor to supersymmetrizing Weyl gravity, two groups [62–65] looked at a gravitational theory based on the conformal group, using the Weyl curvature-squared action. These approaches are top down, in the sense they are often based on constructing an action with specified local symmetry, then investigating any new structures and the new field equations. However, as this work expanded, physicists started using the techniques of Cartan and Klein to organize and develop the structures systematically.

In [3,4] Ne'eman and Regge develop what they refer to as the quotient manifold technique to construct a gauge theory of gravity based on the Poincaré group. Theirs is the first construction of a gravitational gauge theory that uses Klein (homogeneous) spaces as generalized versions of tangent spaces, applying methods developed by Cartan [66] to characterize a more general geometry. In their 1982 papers [5, 6], Ivanov and Niederle exhaustively considered quotients of the Poincaré, de Sitter, anti-de Sitter and Lorentzian conformal groups ($ISO\left(3,1\right)$, $SO\left(4,1\right)$, $SO\left(3,2\right)$, and $SO\left(4,2\right)$ respectively) by various subgroups containing the Lorentz group.

There are a number of more recent implementations of Cartan geometry in the modern literature. One good introduction is Wise's use of Cartan methods to look at the MacDowell-Mansouri action [40]. The waywiser approach of visualizing these geometries is advocated strongly, and gives a clear geometric way of undertsanding Cartan geometry. The use of Cartan techniques in [67] to look at the Chern-Simons action in $2+1$ dimensions provides a nice example of the versatility of the method. This action can be viewed as having either Minkowski, de Sitter or anti-de Sitter symmetry, and Cartan methods allow a straightforward characterization of the theory given the various symmetries. The analysis is extended to look first at the conformal representation of these groups on the Euclidean surfaces of the theory (two-dimensional spatial slices). The authors then look specifically at shape dynamics, which is found equivalent to the case when the Chern-Simons action has de Sitter symmetry. Tractor calculus is another example using a quotient of the conformal group, in which the associated tensor bundles are based on a linear, $(n+2)$-dim representation of the group. This is a distinct gauging from the one we study here, but one studied in [68].

Our research focuses primarily on gaugings of the conformal group. Initially motivated by a desire to understand the physical role of local scale invariance, the growing prospects of twistor string formulations of gravity [69] elevate the importance of understanding its low-energy limit, which is expected to be a conformal gauge theory of gravity. Interestingly, there are two distinct ways to formulate gravitational theories based on the conformal group, first identified in [5, 6] and developed in [50], [51], and [68]. Both of these lead directly to scale-invariant general relativity. This is surprising since the best known conformal gravity theory is the fourth-order theory developed by Weyl [41, 42, 70–72] and Bach [43]. Wheeler recently showed when a Palatini variation is applied to Weyl gravity, it becomes second-order, scale-invariant general relativity.

The second gauging of the conformal group identified in these works is the biconformal gauging. Leading to scale-invariant general relativity formulated on a $2n$-dimensional symplectic manifold, the approach took a novel twist for homogeneous spaces in [7]. There it is shown that, because the biconformal gauging leads to a zero-signature manifold of doubled dimension, we can start with the conformal symmetry of a non-Lorentzian space while still arriving at spacetime gravity. We describe the resulting signature theorem in detail below, and considerably strengthen its conclusions. In addition to necessarily developing a direction of time from a Euclidean-signature starting point, we show these models give a group-theoretically driven candidate for dark matter.

2.3. Quotient manifold method

We are interested in geometries – ultimately spacetime geometries – which have continuous local symmetries. The structure of such systems is that of a principal fiber bundle with Lie group fibers. The quotient method starts with a Lie group, $\mathcal{G}$, with the desired local symmetry as a proper Lie subgroup, $\mathcal{H}$. To develop the local properties any representation will give equivalent results, so without loss of generality we assume a linear representation, $\mathcal{V}^{n+2}$, i.e. a vector space, $\mathcal{V}^{n+2}$, on which $\mathcal{G}$ acts. Typically this will be either a signature (p, q) (pseudo-)Euclidean space or the corresponding spinor space. This vector space is useful for describing the larger symmetry group, but is only a starting point and will not appear in the theory.

The quotient method, laid out below, is identical in many respects to the approaches of [40, 67]. The nice geometric interpretation of using a Klein space in place of a tangent space to both characterize a curved manifold and take advantage of its metric structure are also among the motivations for using the quotient method. In what follows, not all the manifolds we look at will be interpreted as spacetime; so, we choose not to use the interpretation of a Klein space moving around on spacetime in a larger ambient space. Rather, we directly generalize the homogeneous space to add curvatures. The homogeneous space becomes a local model for a more general curved space, similar to the way that $\mathbb{R}^n$ provides a local model for an n-dim Riemannian manifold.

We include a concise introduction here, but the reader can find a more detailed exposition in [59]. Our intention is to make it clear that our ultimate conclusions have rigorous roots in group theory, rather than to present a comprehensive mathematical description.

2.3.1. Construction of a principal $\mathcal{H}$-bundle $\mathfrak{B}(\mathcal{G}, \pi, \mathcal{H}, M_0)$ with connection

Consider a Lie group, $\mathcal{G}$, and a nonnormal Lie subgroup, $\mathcal{H}$, on which $\mathcal{G}$ acts effectively and transitively on $\mathcal{H}$. The quotient of these is a homogeneous manifold, M_0. The points of M_0 are the left cosets, $g\mathcal{H} = \{g' \mid g' = gh \text{ for some } h \in \mathcal{H}\}$, so, there is a natural one to one mapping $g\mathcal{H} \leftrightarrow \mathcal{H}$. The cosets are disjoint from one another and together cover $\mathcal{G}$. There is a projection, $\pi : \mathcal{G} \to M_0$, defined by $\pi(g) = g\mathcal{H} \in M_0$. There is also a right action of $\mathcal{G}$, $g\mathcal{H}\mathcal{G}$, given, for all elements of $\mathcal{G}$, by right multiplication.

Therefore, $\mathcal{G}$ is a principal $\mathcal{H}$-bundle, $\mathfrak{B}(\mathcal{G}, \pi, \mathcal{H}, M_0)$, where the fibers are the left cosets. This is the mathematical object required to carry a gauge theory of the symmetry group $\mathcal{H}$. Let the dimension of $\mathcal{G}$ be m, the dimension of $\mathcal{H}$ be k. Then the dimension of the quotient manifold is $n = m - k$ and we write $M_0^{(n)}$. Choosing a gauge amounts to picking a cross section of this bundle, i.e. one point from each of these copies of $\mathcal{H}$. Local symmetry amounts to dynamical laws, which are independent of the choice of cross section.

Lie groups have a natural Cartan connection given by the one-forms, $\boldsymbol{\xi}^A$, dual to the group generators, G_A. Rewriting the Lie algebra in terms of these dual forms leads immediately to the

Maurer-Cartan structure equations,

$$\mathbf{d}\boldsymbol{\xi}^A = -\frac{1}{2}c^A_{BC}\boldsymbol{\xi}^B \wedge \boldsymbol{\xi}^C, \tag{2.1}$$

where c^A_{BC} are the group structure constants, and $\wedge$ is the wedge product. The integrability condition for this equation follows from the Poincaré lemma, $\mathbf{d}^2 = 0$, and turns out to be precisely the Jacobi identity. Therefore, the Maurer-Cartan equations together with their integrability conditions are completely equivalent to the Lie algebra of $\mathcal{G}$.

Let $\boldsymbol{\xi}^a$ (where $a = 1, \ldots, k$) be the subset of one-forms dual to the generators of the subgroup, $\mathcal{H}$. Let the remaining independent forms be labeled $\boldsymbol{\chi}^\alpha$. Then the $\boldsymbol{\xi}^a$ give a connection on the fibers while the $\boldsymbol{\chi}^\alpha$ span the cotangent spaces to $M_0^{(n)}$. We denote the manifold with connection by $\mathcal{M}_0^{(n)} = \left(M_0^{(n)}, \boldsymbol{\xi}^A\right)$.

2.3.2. Cartan generalization

For a gravity theory, we require in general a curved geometry, $\mathcal{M}^{(n)}$. To achieve this, the general method allows us to generalize both the connection and the manifold. Since the principal fiber bundle from the quotient is a local direct product, this is not changed if we allow a generalization of the manifold, $M_0^{(n)} \to M^{(n)}$. We will not consider such topological issues here. Generalizing the connection is more subtle. If we change $\boldsymbol{\xi}^A = (\boldsymbol{\xi}^a, \boldsymbol{\chi}^\alpha)$ to a new connection $\boldsymbol{\xi}^A \to \boldsymbol{\omega}^A, \boldsymbol{\xi}^a \to \boldsymbol{\omega}^a, \boldsymbol{\chi}^\alpha \to \boldsymbol{\omega}^\alpha$ arbitrarily, the Maurer-Cartan equation is altered to $\mathbf{d}\boldsymbol{\omega}^A = -\frac{1}{2}c^A_{BC}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C + \boldsymbol{\Omega}^A$, where $\boldsymbol{\Omega}^A$ is a two-form determined by the choice of the new connection. We need restrictions on $\boldsymbol{\Omega}^A$ so it represents curvature of the geometry, $\mathcal{M}^{(n)} = \left(M^{(n)}, \boldsymbol{\omega}^A\right)$, and not of the full bundle, $\mathfrak{B}$. We restrict $\boldsymbol{\Omega}^A$ by requiring it to be independent of lifting, i.e. horizontality of the curvature.

To define horizontality, recall the integral of the connection associated with $\mathcal{G}$ around a closed curve in the bundle is given by the integral of $\boldsymbol{\Omega}^A$ over any surface bounded by the curve. We require this integral to be independent of lifting, i.e. horizontal. This means the two-form bases for the curvatures $\boldsymbol{\Omega}^A$ cannot include any of the one-forms, $\boldsymbol{\omega}^a$, that span the fiber group,

$\mathcal{H}$. With the horizontality condition, the curvatures take the simpler form

$$\mathbf{\Omega}^A = \frac{1}{2}\Omega^A_{\mu\nu}\boldsymbol{\omega}^\mu \wedge \boldsymbol{\omega}^\nu.$$

More general curvatures than this will destroy the homogeneity of the fibers, so we would no longer have a *principal* $\mathcal{H}$-bundle.

In addition to horizontality, we require integrability. Again using the Poincaré lemma, $\mathbf{d}^2\boldsymbol{\omega}^A \equiv 0$, we always find a term $\frac{1}{2}c^A_{B[C}c^B_{DE]}\boldsymbol{\omega}^C \wedge \boldsymbol{\omega}^D \wedge \boldsymbol{\omega}^E$, which vanishes by the Jacobi identity, $c^A_{B[C}c^B_{DE]} \equiv 0$, while the remaining terms give the general form of the Bianchi identities,

$$\mathbf{d}\mathbf{\Omega}^A + c^A_{BC}\boldsymbol{\omega}^B \wedge \mathbf{\Omega}^C = 0.$$

2.3.3. Example: pseudo-Riemannian manifolds

To see how this works in a familiar example, consider the construction of the pseudo-Riemannian spacetimes used in general relativity, for which we take the quotient of the 10-dim Poincaré group by its six-dim Lorentz subgroup. The result is a principal Lorentz bundle over $\mathbb{R}^4$. Writing the one-forms dual to the Lorentz $(M^a_{\ b})$ and translation (P_a) generators as $\boldsymbol{\xi}^a_{\ b}$ and $\boldsymbol{\omega}^a$, respectively, the 10 Maurer-Cartan equations are

$$\begin{aligned}\mathbf{d}\boldsymbol{\xi}^a_{\ b} &= \boldsymbol{\xi}^c_{\ b} \wedge \boldsymbol{\xi}^a_{\ c},\\ \mathbf{d}\boldsymbol{\omega}^a &= \boldsymbol{\omega}^b \wedge \boldsymbol{\xi}^a_{\ b}.\end{aligned}$$

Notice the first describes a pure gauge spin connection, $\mathbf{d}\boldsymbol{\xi}^a_{\ b} = -\bar{\Lambda}^c_{\ b}\mathbf{d}\Lambda^a_{\ c}$ where $\Lambda^a_{\ c}$ is a local Lorentz transformation. Therefore, there exists a local Lorentz gauge such that $\boldsymbol{\xi}^a_{\ b} = 0$. The second equation then shows the existence of an exact orthonormal frame, which tells us the space is Minkowski.

Now generalize the geometry, $\left(M^4_0, \boldsymbol{\xi}^A\right) \to \left(M^4, \boldsymbol{\omega}^A\right)$, where $M^4_0 = \mathbb{R}^4$ and we denote the new connection forms by $\boldsymbol{\omega}^A = \left(\boldsymbol{\omega}^a_{\ b}, \mathbf{e}^b\right)$. In the structure equations, this leads to the presence

of 10 curvature two-forms,

$$\begin{aligned} \mathbf{d}\boldsymbol{\omega}^a_{\ b} &= \boldsymbol{\omega}^c_{\ b} \wedge \boldsymbol{\omega}^a_{\ c} + \mathbf{R}^a_{\ b}, \\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a_{\ b} + \mathbf{T}^a. \end{aligned}$$

Since the $\boldsymbol{\omega}^a_{\ b}$ span the Lorentz subgroup, horizontality is accomplished by restricting the curvatures to

$$\begin{aligned} \mathbf{R}^a_{\ b} &= \frac{1}{2} R^a_{\ bcd} \mathbf{e}^c \wedge \mathbf{e}^d, \\ \mathbf{T}^a_{\ b} &= \frac{1}{2} T^a_{\ bc} \mathbf{e}^b \wedge \mathbf{e}^c. \end{aligned}$$

That is, there are no terms such as, $\frac{1}{2} R^{a\ c}_{\ b\ de} \boldsymbol{\omega}^d_{\ c} \wedge \mathbf{e}^e$ or $\frac{1}{2} T^{a\ c\ e}_{\ b\ d} \boldsymbol{\omega}^b_{\ c} \wedge \boldsymbol{\omega}^d_{\ e}$, for example. Finally, integrability is guaranteed by the pair of Bianchi identities,

$$\begin{aligned} \mathbf{d}\mathbf{R}^a_{\ b} + \mathbf{R}^c_{\ b} \wedge \boldsymbol{\omega}^a_{\ c} - \mathbf{R}^a_{\ c} \wedge \boldsymbol{\omega}^c_{\ b} &= 0, \\ \mathbf{d}\mathbf{T}^a + \mathbf{T}^b \wedge \boldsymbol{\omega}^a_{\ b} + \mathbf{e}^b \wedge \mathbf{R}^a_{\ b} &= 0. \end{aligned}$$

By looking at the transformation of $\mathbf{R}^a_{\ b}$ and $\mathbf{T}^a$ under local Lorentz transformations, we find despite originating as components of a single Poincaré-valued curvature, they are independent Lorentz tensors. The translations of the Poincaré symmetry were broken when we curved the base manifold (see [2–4], but note Kibble effectively uses a 14-dimensional bundle, whereas ours and related approaches require only 10-dim). We recognize $\mathbf{R}^a_{\ b}$ and $\mathbf{T}^a$ as the Riemann curvature and the torsion two-forms, respectively. Since the torsion is an independent tensor under the fiber group, it is consistent to consider the subclass of Riemannian geometries, $\mathbf{T}^a = 0$. Alternatively, vanishing torsion follows from the tetradic Palatini action, $S = \int \mathbf{R}^{ab} \mathbf{e}^c \mathbf{e}^d \epsilon_{abcd}$.

With vanishing torsion, the quotient method has resulted in the usual solder form, $\mathbf{e}^a$, and related metric-compatible spin connection, $\boldsymbol{\omega}^a_{\ b}$,

$$\mathbf{d}\mathbf{e}^a - \mathbf{e}^b \wedge \boldsymbol{\omega}^a_{\ b} = 0,$$

the expression for the Riemannian curvature in terms of these,

$$\mathbf{R}^a_{\ b} = \mathbf{d}\boldsymbol{\omega}^a_{\ b} - \boldsymbol{\omega}^c_{\ b} \wedge \boldsymbol{\omega}^a_{\ c},$$

and the first and second Bianchi identities,

$$\begin{aligned} \mathbf{e}^b \wedge \mathbf{R}^a_{\ b} &= 0, \\ \mathbf{D}\mathbf{R}^a_{\ b} &= 0. \end{aligned}$$

This is a complete description of the class of Riemannian geometries.

In order to return to the usual language of general relativity we note we can endow the manifold with an orthonormal Lorentzian inner product $\left\langle \mathbf{e}^a, \mathbf{e}^b \right\rangle \equiv \eta^{ab}$, which gives a metric of the form $\eta_{ab} = diag\,(-1, 1, 1, 1)$. We can pass between the orthonormal frame fields and the coordinate frame by using the coefficients of the solder form, $\mathbf{e}^a = e_\mu^{\ a} \mathbf{d}x^\mu$. The coordinate metric is then defined by the relationship $g_{\mu\nu} \equiv e_\mu^{\ a} e_\nu^{\ b} \eta_{ab}$.

This is a nice point in the discussion to point out the difference between the quotient method of [3, 4] that we use in this manuscript, and the method originally used to gauge the Poincaré group used by Kibble [2]. Kibble's original treatment effectively uses a 14-dimensional bundle, see Figure 2.1, whereas ours and related approaches require only 10-dim. Kibble uses the group manifold of the Poincaré group together with a four-dimensional manifold to describe general relativity. He identifies the degrees of freedom of the frame fields, $\mathbf{e}^a$ as the cotangent space to the manifold, soldering the forms to the manifold.

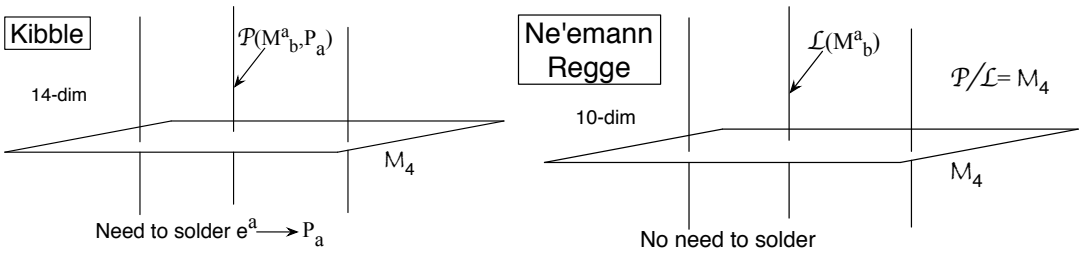


Figure 2.1. These diagrams summarize the relationship between the two techniques to gauge the Poincaré group. The figure on the left represents Kibble's original construction [2], where he has soldered the degrees of freedom of the frame fields to the manifold. The figure on the right shows the quotient of the Poincaré by the Lorentz group and gives a cleaner description.

2.3.4. Example: $SO(3)/SO(2)$

The following description of the Hopf fibration, becoming a well–regarded one in the gravitational Cartan geometry community, is originally due to Wise [40] and has recently been extended in [67] and [73].

In the Poincaré example we demonstrated how the full connection of $ISO(3,1)$ can be separated into an $SO(3,1)$ connection and a frame field. This homogeneous space is then used to characterize a curved manifold of the same dimension. In order to understand the geometric meaning of such a split we use a simpler example; the quotient of $SO(3)/SO(2)$. This quotient can be viewed as a sphere, S^2. This sphere can then be used to characterize a two-manifold, $\mathcal{M}$, by rolling the sphere around on it. We can visualize the situation as a rodent ball (here a gerbil) where the gerbil is standing over the point of contact, see Figure 2.2.

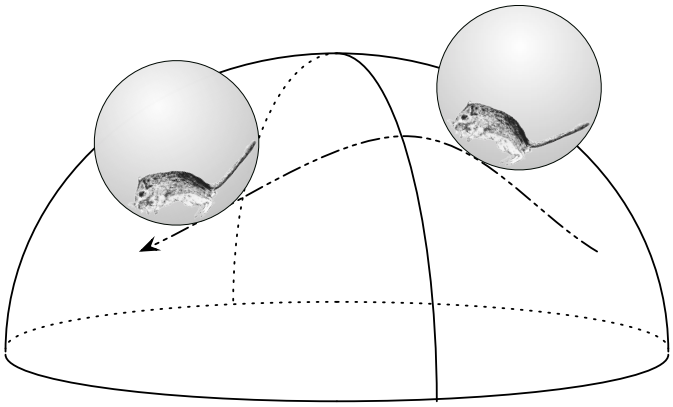


Figure 2.2. The gerbil in the ball is free to map out the geometry of the curved surface, $\mathcal{M}$, by rolling without slipping along its surface.

The information in the connection can be probed by allowing the gerbil to roll the ball over $\mathcal{M}$. The three different directions the gerbil can move the ball correspond to the three degrees of freedom in $SO(3)$, see Figure 2.3.

One of these, $\mathcal{H} = SO(2)$, stabilizes the point of contact on the manifold. This rotation, by construction, does not change the point of contact with $\mathcal{M}$ and represents vertical motion in the fiber bundle. The remaining degrees of freedom correspond to the two independent directions the gerbil can move the ball. These horizontal directions give a natural metric structure on $\mathcal{M}$. A general Cartan geometry can then be thought of as a generalized ball, $\mathcal{M}_0 = \mathcal{G}/\mathcal{H}$, that can

be rolled on an arbitrary manifold, $\mathcal{M}$. The subgroup $\mathcal{H}$, called the isotropy subgroup, then stabilizes the point of contact of $\mathcal{M}_0$ on $\mathcal{M}$.

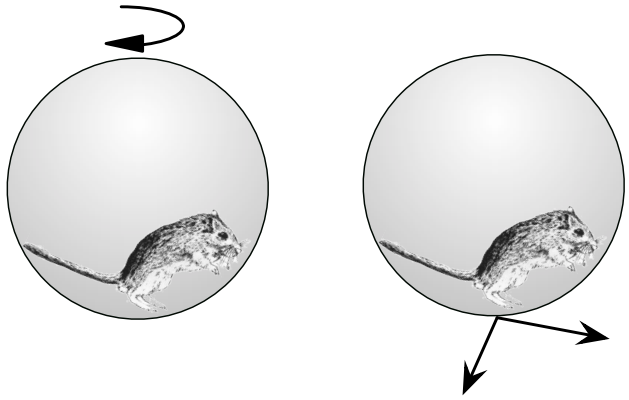


Figure 2.3. The degree of freedom represented by the gerbil ball on the left is the $SO(2)$ rotation about the contact point. The degrees of freedom on the right are the remaining degrees of freedom that tell how the gerbil is moving around on the curved surface.

2.4. Quotients of the conformal group

2.4.1. General properties of the conformal group

Physically, we are interested in measurements of relative magnitudes, so the relevant group is the conformal group, $\mathcal{C}$, of compactified $\mathbb{R}^n$ together with a metric. The one-point compactification at infinity allows a global definition of inversion, with translations of the point at infinity defining the special conformal transformation. Then $\mathcal{C}$ has a real linear representation in $n+2$ dimensions, $\mathcal{V}^{n+2}$; alternatively, we could choose the complex representation $\mathbb{C}^{2^{[(n+2)/2]}}$ for $Spin\,(p+1,q+1)$. The isotropy subgroup of $\mathcal{V}^{n+2}$ is the rotations, $SO\,(p,q)$, together with dilatations. We call this subgroup the homogeneous Weyl group, $\mathcal{W}$, and require our fibers to contain it. There are then only three allowed subgroups: $\mathcal{W}$ itself; the inhomogeneous Weyl group, $\mathcal{IW}$, found by appending the translations; and $\mathcal{W}$ together with special conformal transformations, isomorphic to $\mathcal{IW}$. The quotient of the conformal group by either inhomogeneous Weyl group, called the auxiliary gauging, leads most naturally to Weyl gravity (for a review, see [68]). We concern ourselves with the only other meaningful conformal quotient, the biconformal gauging: the principal $\mathcal{W}$-bundle formed by the quotient of the conformal group by its Weyl subgroup. To help clarify the method and our model, it is useful to consider both these

gaugings.

All parts of this construction work for any (p,q) with $n=p+q$. The conformal group is then $SO\left(p+1,q+1\right)$ (or $Spin\left(p+1,q+1\right)$ for the twistor representation). The Maurer-Cartan structure equations are immediate. In addition to the $\frac{n(n-1)}{2}$ generators M^{α}_{β} of $SO\left(n\right)$ and n translational generators P_{α}, there are n generators of translations of a point at infinity (special conformal transformations) K^{α}, and a single dilatational generator, D. Dual to these, we have the connection forms $\xi^{\alpha}_{\beta},\boldsymbol{\chi}^{\alpha},\boldsymbol{\pi}_{\alpha},\boldsymbol{\delta}$, respectively. Substituting the structure constants into the Maurer-Cartan dual form of the Lie algebra, eq.(2.1) gives

$$\begin{aligned}
\mathbf{d}\boldsymbol{\xi}^{\alpha}_{\beta} &= \boldsymbol{\xi}^{\mu}_{\beta}\wedge\boldsymbol{\xi}^{\alpha}_{\mu}+2\Delta^{\alpha\mu}_{\nu\beta}\boldsymbol{\pi}_{\mu}\wedge\boldsymbol{\chi}^{\nu}, && (2.2)\\
\mathbf{d}\boldsymbol{\chi}^{\alpha} &= \boldsymbol{\chi}^{\beta}\wedge\boldsymbol{\xi}^{\alpha}_{\beta}+\boldsymbol{\delta}\wedge\boldsymbol{\chi}^{\alpha}, && (2.3)\\
\mathbf{d}\boldsymbol{\pi}_{\alpha} &= \boldsymbol{\xi}^{\beta}_{\alpha}\wedge\boldsymbol{\pi}_{\beta}-\boldsymbol{\delta}\wedge\boldsymbol{\pi}_{\alpha}, && (2.4)\\
\mathbf{d}\boldsymbol{\delta} &= \boldsymbol{\chi}^{\alpha}\wedge\boldsymbol{\pi}_{\alpha}, && (2.5)
\end{aligned}$$

where $\Delta^{\alpha\mu}_{\nu\beta}\equiv\frac{1}{2}\left(\delta^{\alpha}_{\nu}\delta^{\mu}_{\beta}-\delta^{\alpha\mu}\delta_{\nu\beta}\right)$ antisymmetrizes with respect to the original (p,q) metric, $\delta_{\mu\nu}=diag\left(1,\ldots,1,-1,\ldots,-1\right)$. These equations, which are the same regardless of the gauging chosen, describe the Cartan connection on the conformal group manifold. Before proceeding to the quotients, we note the conformal group has a nondegenerate Killing form,

$$K_{AB}\equiv tr\left(G_{A}G_{B}\right)=c^{C}_{AD}c^{D}_{BC}=\begin{pmatrix}\Delta^{ac}_{db} & & \\ & 0 & \delta^{a}_{b} \\ & \delta^{a}_{b} & 0 \\ & & & 1\end{pmatrix}.$$

This provides a metric on the conformal Lie algebra. As we show below, when restricted to $\mathcal{M}_{0}$, it may or may not remain nondegenerate, depending on the quotient.

Finally, we note the conformal group is invariant under inversion. Within the Lie algebra, this manifests itself as the interchange between the translations and special conformal transformations, $P_{\alpha}\leftrightarrow\delta_{\alpha\beta}K^{\beta}$, along with the interchange of conformal weights, $D\rightarrow-D$. The

corresponding transformation of the connection forms is easily seen to leave equations (2.2)–(2.5) invariant. In the biconformal gauging below, we show this symmetry leads to a Kähler structure.

2.4.2. Specialized notation

Much of the notation used in this manuscript is standard for those working in Cartan formalism. Bold-faced symbols represent one-forms. In any equation where there is more than one differential form, it should be assumed they are multiplied with a wedge product. In other words, $\mathbf{e}^a \wedge \mathbf{e}^b$ and $\mathbf{e}^a\mathbf{e}^b$ are equivalent. The conformal weights of connections and various tensors are encoded in their index position. For instance, $\mathbf{e}^a \to e^{\phi}\mathbf{e}^a$ scales with a conformal weight of $+1$, $\mathbf{f}_a \to e^{-\phi}\mathbf{f}_a$ scales with a conformal weight of -1, while $\mathbf{\Omega}^a_{\ b} \to \mathbf{\Omega}^a_{\ b}$ has a conformal weight of zero, since the number of contravariant and covariant indices are equal. We denote differential forms associated with, what we eventually identify as the configuration submanifold with an (x) and those associated with the momentum submanifold with a (y), when it is not obvious from the index position or the presence of basis forms. This predicates our eventual choice of Darboux coordinates, x^α and y_α for the configuration and momentum subspaces respectively, such that $\mathbf{e}^a = e_\mu^{\ a}\mathbf{d}x^\mu$ and $\mathbf{f}_a = f_a^{\ \mu}\mathbf{d}y_\mu$.

In what follows, the terms Euclidean and Lorentzian will be used to distinguish between the signatures of metric manifolds, while the term Riemannian will refer to a geometry with no torsion, cotorsion (Cartan curvature of the cosolder form) or dilatational curvature, analogous to the geometry on which general relativity is based. One of our major conclusions is, though we start with the conformal symmetries of a Euclidean space in a fully general Cartan formalism, we show the orthogonal, Lagrangian submanifolds have a Lorentzian metric with the structure of a Riemannian geometry.

2.4.3. Curved generalizations

In this subsection and Section 2.5, we will complete the development of the curved auxiliary and biconformal geometries and show how one can easily construct actions with the curvatures. In this subsection, we construct the two possible fiber bundles, $\mathcal{C}/\mathcal{S}$ where $\mathcal{W} \subseteq \mathcal{S}$. For each, we carry out the generalization of the manifold and connection. The results in this subsection depend only on whether the local symmetry is $\mathcal{S} = \mathcal{I}\mathcal{W}$ or $\mathcal{S} = \mathcal{W}$. In Section 3.1 and Section

3.2 we will return to the uncurved case to present a number of new calculations characterizing the homogenous space formed from the biconformal gauging.

The first subsection below describes the auxiliary gauging, given by the quotient of the conformal group by the inhomogeneous Weyl group, $\mathcal{IW}$. Since $\mathcal{IW}$ is a parabolic subgroup of the conformal group, the resulting quotient can be considered a tractor space, for which there are numerous results [74]. Tractor calculus is a version of the auxiliary gauging, where the original conformal group is tensored with $\mathbb{R}^{(p+1,q+1)}$. This allows for a linear representation of the conformal group with $(n+2)$-dimensional tensorial (physical) entities called tractors. This linear representation, first introduced by Dirac [75], makes a number of calculations much easier and also allows for straightforward building of tensors of any rank. The main physical differences stem from the use of Dirac's action, usually encoded as the scale tractor squared in the $n+2$-dimensional linear representation, instead of the Weyl action we introduce in Section 2.5.

In subsection 2.4.3, we quotient by the homogeneous Weyl group, giving the biconformal gauging. This is not a parabolic quotient and therefore represents a less conventional option, which turns out to have a number of rich structures not present in the auxiliary gauging. The biconformal gauging will occupy our attention for the bulk of our subsequent discussion.

The auxiliary gauging: $\mathcal{S} = \mathcal{IW}$

Given the quotient $\mathcal{C}/\mathcal{IW}$, the one-forms $\left(\boldsymbol{\xi}^{\alpha}_{\ \beta}, \boldsymbol{\pi}_{\mu}, \boldsymbol{\delta}\right)$ span the $\mathcal{IW}$-fibers, with $\boldsymbol{\beta}^{\alpha}$ spanning the cotangent space of the remaining n independent directions. This means $\mathcal{M}_0^{(n)}$ has the same dimension, n, as the original space. Generalizing the connection, we replace $\left(\boldsymbol{\xi}^{\alpha}_{\ \beta}, \boldsymbol{\chi}^{\alpha}, \boldsymbol{\pi}_{\alpha}, \boldsymbol{\delta}\right) \rightarrow \left(\boldsymbol{\omega}^{\alpha}_{\ \beta}, \mathbf{e}^{\alpha}, \boldsymbol{\omega}_{\alpha}, \boldsymbol{\omega}\right)$ and the Cartan equations now give the conformal curvatures in terms of the new connection forms,

$$\mathbf{d}\boldsymbol{\omega}^{\alpha}_{\ \beta} = \boldsymbol{\omega}^{\mu}_{\ \beta} \wedge \boldsymbol{\omega}^{\alpha}_{\ \mu} + 2\Delta^{\alpha\mu}_{\nu\beta}\boldsymbol{\omega}_{\mu} \wedge \boldsymbol{\omega}^{\nu} + \boldsymbol{\Omega}^{\alpha}_{\ \beta}, \tag{2.6}$$

$$\mathbf{d}\mathbf{e}^{\alpha} = \mathbf{e}^{\beta} \wedge \boldsymbol{\omega}^{\alpha}_{\ \beta} + \boldsymbol{\omega} \wedge \mathbf{e}^{\alpha} + \mathbf{T}^{\alpha}, \tag{2.7}$$

$$\mathbf{d}\boldsymbol{\omega}_{\alpha} = \boldsymbol{\omega}^{\beta}_{\ \alpha} \wedge \boldsymbol{\omega}_{\beta} - \boldsymbol{\omega} \wedge \boldsymbol{\omega}_{\alpha} + \mathbf{S}_{\alpha}, \tag{2.8}$$

$$\mathbf{d}\boldsymbol{\omega} = \boldsymbol{\omega}^{\alpha} \wedge \boldsymbol{\omega}_{\alpha} + \boldsymbol{\Omega}. \tag{2.9}$$

Up to local gauge transformations, the curvatures depend only on the n nonvertical forms, $\mathbf{e}^{\alpha}$, so the curvatures are similar to what we find in an n-dim Riemannian geometry. For example, the $SO(p,q)$ piece of the curvature takes the form $\mathbf{\Omega}^{\alpha}_{\ \beta} = \frac{1}{2}\Omega^{\alpha}_{\ \beta\mu\nu}\mathbf{e}^{\alpha} \wedge \mathbf{e}^{\beta}$. The coefficients have the same number of degrees of freedom as the Riemannian curvature of an n-dim Weyl geometry.

Finally, each of the curvatures has a corresponding Bianchi identity, to guarantee integrability of the modified structure equations,

$$
\begin{aligned}
0 &= \mathbf{D\Omega}^{\alpha}_{\ \beta} + 2\Delta^{\alpha\mu}_{\nu\beta}\left(\mathbf{\Omega}_{\mu} \wedge \boldsymbol{\omega}^{\nu} - \boldsymbol{\omega}_{\mu} \wedge \mathbf{\Omega}^{\nu}\right), && (2.10)\\
0 &= \mathbf{DT}^{\alpha} - \mathbf{e}^{\beta} \wedge \mathbf{\Omega}^{\alpha}_{\ \beta} + \mathbf{\Omega} \wedge \mathbf{e}^{\alpha}, && (2.11)\\
0 &= \mathbf{\Omega}^{\alpha}_{\ \beta} \wedge \boldsymbol{\omega}_{\beta} - \boldsymbol{\omega}^{\beta}_{\ \alpha} \wedge \mathbf{S}_{\beta} + \mathbf{S}_{\alpha} \wedge \boldsymbol{\omega} - \boldsymbol{\omega}_{\alpha} \wedge \mathbf{\Omega} + \mathbf{dS}_{\alpha}, && (2.12)\\
0 &= \mathbf{D\Omega} + \mathbf{T}^{\alpha} \wedge \boldsymbol{\omega}_{\alpha} - \boldsymbol{\omega}^{\alpha} \wedge \mathbf{S}_{\alpha}, && (2.13)
\end{aligned}
$$

where D is the Weyl covariant derivative,

$$
\begin{aligned}
\mathbf{D\Omega}^{\alpha}_{\ \beta} &= \mathbf{d\Omega}^{\alpha}_{\ \beta} + \mathbf{\Omega}^{\mu}_{\ \beta} \wedge \boldsymbol{\omega}^{\alpha}_{\ \mu} - \mathbf{\Omega}^{\alpha}_{\ \mu} \wedge \boldsymbol{\omega}^{\mu}_{\ \beta},\\
\mathbf{DT}^{\alpha} &= \mathbf{dT}^{\alpha} + \mathbf{T}^{\beta} \wedge \boldsymbol{\omega}^{\alpha}_{\ \beta} - \boldsymbol{\omega} \wedge \mathbf{T}^{\alpha},\\
\mathbf{DS}_{\alpha} &= \mathbf{dS}_{\alpha} - \boldsymbol{\omega}^{\beta}_{\ \alpha} \wedge \mathbf{S}_{\beta} + \mathbf{S}_{\alpha} \wedge \boldsymbol{\omega},\\
\mathbf{D\Omega} &= \mathbf{d\Omega}.
\end{aligned}
$$

Equations (2.6-2.9) give the curvature two-forms in terms of the connection forms. We have, therefore, constructed an n-dim geometry based on the conformal group with local $\mathcal{IW}$ symmetry.

We note no additional special properties of these geometries from the group structure. In particular, the restriction (in square brackets, [ ], below) of the Killing metric, K_{AB}, to $\mathcal{M}^{(n)}$

vanishes identically,

$$\left.\begin{pmatrix} \Delta^{ac}_{db} & & & \\ & [0] & \delta^a_b & \\ & \delta^a_b & 0 & \\ & & & 1 \end{pmatrix}\right|_{\mathcal{M}^{(n)}} = \begin{pmatrix} 0 \end{pmatrix}_{n\times n},$$

so there is no induced metric on the spacetime manifold. We may add the usual metric by hand, of course, but our goal here is to find those properties, which are intrinsic to the underlying group structures.

The biconformal gauging

We next consider the biconformal gauging, first considered by Ivanov and Niederle [6], given by the quotient of the conformal group by its Weyl subgroup. The resulting geometry has been shown to contain the structures of general relativity [50, 51].

Given the quotient $\mathcal{C}/\mathcal{W}$, the one-forms $\left(\boldsymbol{\xi}^{\alpha}_{\ \beta}, \boldsymbol{\delta}\right)$ span the $\mathcal{W}$-fibers, with $(\boldsymbol{\chi}^{\alpha}, \boldsymbol{\pi}_{\alpha})$ spanning the remaining $2n$ independent directions. This means $\mathcal{M}^{(2n)}_0$ has twice the dimension of the original space. Generalizing, we replace $\left(\boldsymbol{\xi}^{\alpha}_{\ \beta}, \boldsymbol{\chi}^{\alpha}, \boldsymbol{\pi}_{\alpha}, \boldsymbol{\delta}\right) \rightarrow \left(\boldsymbol{\omega}^{\alpha}_{\ \beta}, \boldsymbol{\omega}^{\alpha}, \boldsymbol{\omega}_{\alpha}, \boldsymbol{D}\right)$ and the modified structure equations appear identical to Equations (2.6-2.9). However, the curvatures now depend on the $2n$ nonvertical forms, $(\boldsymbol{\omega}^{\alpha}, \boldsymbol{\omega}_{\alpha})$, so there are far more components than for an n-dim Riemannian geometry. For example,

$$\boldsymbol{\Omega}^{\alpha}_{\ \beta} = \frac{1}{2}\Omega^{\alpha}_{\ \beta\mu\nu}\boldsymbol{\omega}^{\mu}\wedge\boldsymbol{\omega}^{\nu} + \Omega^{\alpha\ \mu}_{\ \beta\ \nu}\boldsymbol{\omega}_{\mu}\wedge\boldsymbol{\omega}^{\nu} + \frac{1}{2}\Omega^{\alpha\ \mu\nu}_{\ \beta}\boldsymbol{\omega}_{\mu}\wedge\boldsymbol{\omega}_{\nu}.$$

The coefficients of the pure terms, $\Omega^{\alpha}_{\ \beta\mu\nu}$ and $\Omega^{\alpha\ \mu\nu}_{\ \beta}$ each have the same number of degrees of freedom as the Riemannian curvature of an n-dim Weyl geometry, while the cross-term coefficients $\Omega^{\alpha\ \mu}_{\ \beta\ \nu}$ have more, being asymmetric on the final two indices.

For our purpose, it is important to notice the spin connection, $\boldsymbol{\xi}^{\alpha}_{\ \beta}$, is antisymmetric with respect to the original (p, q) metric, $\delta_{\alpha\beta}$, in the sense that $\boldsymbol{\xi}^{\alpha}_{\ \beta} = -\delta^{\alpha\mu}\delta_{\beta\nu}\boldsymbol{\xi}^{\nu}_{\ \mu}$. It is crucial to note that $\boldsymbol{\omega}^{\alpha}_{\ \beta}$ retains this property, $\boldsymbol{\omega}^{\alpha}_{\ \beta} = -\delta^{\alpha\mu}\delta_{\beta\nu}\boldsymbol{\omega}^{\nu}_{\ \mu}$. This expresses metric compatibility with the $SO\,(p, q)$-covariant derivative, since it implies $\mathbf{D}\delta_{\alpha\beta} \equiv \mathbf{d}\delta_{\alpha\beta} - \delta_{\mu\beta}\boldsymbol{\omega}^{\mu}_{\ \alpha} - \delta_{\alpha\mu}\boldsymbol{\omega}^{\mu}_{\ \beta} = 0$.

Therefore, the curved generalization has a connection, which is compatible with a (p, q)-metric. This relationship is general. If $\kappa_{\alpha\beta}$ is any metric, its compatible spin connection will satisfy $\boldsymbol{\omega}^{\alpha}_{\ \beta} = -\kappa^{\alpha\mu}\kappa_{\beta\nu}\boldsymbol{\omega}^{\nu}_{\ \mu}$. Since we also have local scale symmetry, the full covariant derivative we use will also include a Weyl vector term. The Bianchi identities, written as three-form equations, also appear the same as Equations (2.10-2.13), but expand into more components.

In the conformal group, translations and special conformal transformations are related by inversion. Indeed, a special conformal tranformation is a translation centered at the point at infinity instead of the origin. Because the biconformal gauging maintains the symmetry between translations and special conformal transformations, it is useful to name the corresponding connection forms and curvatures to reflect this. Therefore, the biconformal basis will be described as the solder form and the cosolder form, and the corresponding curvatures as the torsion and cotorsion. Thus, when we speak of torsion-free biconformal space we do not imply the cotorsion (Cartan curvature of the cosolder form) vanishes. In phase space interpretations, the solder form is taken to span the cotangent spaces of the spacetime manifold, while the cosolder form is taken to span the cotangent spaces of the momentum space. The opposite convention is equally valid.

Unlike other quotient manifolds arising in conformal gaugings, the biconformal quotient manifold possesses natural invariant structures. The first is the restriction of the Killing metric, which is now nondegenerate,

$$\left.\begin{pmatrix} \Delta^{ac}_{db} & & \\ & \begin{bmatrix} 0 & \delta^a_b \\ \delta^a_b & 0 \end{bmatrix} & \\ & & 1 \end{pmatrix}\right|_{\mathcal{M}^{(2n)}} = \begin{pmatrix} 0 & \delta^a_b \\ \delta^a_b & 0 \end{pmatrix}_{2n\times 2n},$$

and this gives an inner product for the basis,

$$\begin{bmatrix} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{\omega}^{\beta} \rangle & \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{\omega}_{\beta} \rangle \\ \langle \boldsymbol{\omega}_{\alpha}, \boldsymbol{\omega}^{\beta} \rangle & \langle \boldsymbol{\omega}_{\alpha}, \boldsymbol{\omega}_{\beta} \rangle \end{bmatrix} \equiv \begin{bmatrix} 0 & \delta^{\alpha}_{\beta} \\ \delta^{\beta}_{\alpha} & 0 \end{bmatrix}. \tag{2.14}$$

This metric remains unchanged by the generalization to curved base manifolds.

The second natural invariant property is the generic presence of a symplectic form. The original fiber bundle always has this, because the structure equation (2.5) shows $\boldsymbol{\chi}^\alpha \wedge \boldsymbol{\pi}_\alpha$ is exact; hence, closed $\mathbf{d}^2\boldsymbol{\omega} = 0$, while it is clear the two-form product is nondegenerate because $(\boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha)$ together span $\mathcal{M}_0^{(2n)}$. Moreover, the symplectic form is in canonical form,

$$[\Omega]_{AB} \;=\; \begin{bmatrix} 0 & \delta^\beta_\alpha \\ -\delta^\alpha_\beta & 0 \end{bmatrix},$$

so $\boldsymbol{\chi}^\alpha$ and $\boldsymbol{\pi}_\alpha$ are canonically conjugate. The symplectic form persists for the two-form, $\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}_\alpha + \boldsymbol{\Omega}$, as long as it is nondegenerate, so curved biconformal spaces are generically symplectic.

Next, we consider the effect of inversion symmetry. As a $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ tensor, the basis interchange takes the form

$$I^A_B \boldsymbol{\chi}^B = \begin{pmatrix} & \delta^{\alpha\nu} \\ \delta_{\beta\mu} & \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi}^\mu \\ \boldsymbol{\pi}_\nu \end{pmatrix} = \begin{pmatrix} \delta^{\alpha\nu}\boldsymbol{\pi}_\nu \\ \delta_{\beta\mu}\boldsymbol{\chi}^\mu \end{pmatrix}.$$

In order to interchange conformal weights, I^A_B must anticommute with the conformal weight operator, which is given by

$$W^A_B \boldsymbol{\chi}^B = \begin{pmatrix} \delta^\alpha_\mu & \\ & -\delta^\nu_\beta \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi}^\mu \\ \boldsymbol{\pi}_\nu \end{pmatrix} = \begin{pmatrix} +\boldsymbol{\chi}^\alpha \\ -\boldsymbol{\pi}_\beta \end{pmatrix}.$$

This is the case: we easily check that $\{I, W\}^A_B = I^A_C W^C_B + W^A_C I^C_B = 0$. The commutator gives a new object,

$$J^A_B \;\equiv\; [I, W]^A_B = \begin{pmatrix} & -\delta^{\alpha\beta} \\ \delta_{\alpha\beta} & \end{pmatrix}.$$

Squaring, $J^A_C J^C_B = -\delta^A_B$, we see J^A_B provides an almost complex structure. That the almost complex structure is integrable follows immediately in this (global) basis by the obvious vanishing

of the Nijenhuis tensor, $N^A_{BC} = J^D_C \partial_D J^A_B - J^D_C \partial_D J^A_B - J^A_D \left(\partial_C J^D_B - \partial_B J^D_C\right) = 0$.

Next, using the symplectic form to define the compatible metric $g(u,v) \equiv \Omega(u, Jv)$, we find in this basis $g = \begin{pmatrix} \delta_{\alpha\beta} & \\ & \delta^{\alpha\beta} \end{pmatrix}$, and we check the remaining compatibility conditions of the triple (g, J, Ω),

$$\begin{aligned} \omega(u,v) &= g(Ju, v), \\ J(u) &= (\phi_g)^{-1}(\phi_\omega(u)), \end{aligned}$$

where ϕ_ω and ϕ_g are defined by

$$\begin{aligned} \phi_\omega(u) &= \omega(u, \cdot), \\ \phi_g(u) &= g(u, \cdot). \end{aligned}$$

These are easily checked to be satisfied, showing $\mathcal{M}_0^{(2n)}$ is a Kähler manifold. Notice, however, the metric of the Kähler manifold is not the restricted Killing metric, which we use in the following considerations.

Finally, a surprising result emerges if we require $\mathcal{M}_0^{(2n)}$ to match our usual expectations for a relativistic phase space. To make the connection to phase space clear, the precise requirements were studied in [7], where it was shown the flat biconformal gauging of $SO(p,q)$ in any dimension $n = p + q$ will have Lagrangian submanifolds that are orthogonal with respect to the 2n-dim biconformal (Killing) metric and have nondegenerate n-dim restrictions only if the original space is Euclidean or signature zero $\left(p \in \left\{0, \frac{n}{2}, n\right\}\right)$, and then the signature of the submanifolds is severely limited $(p \to p \pm 1)$, leading in the two Euclidean cases to Lorentzian configuration space, and hence, the origin of time. For the case of flat, eight-dim biconformal space [7] has the following theorem:

Flat eight-dim biconformal space is a metric phase space with Lagrangian submanifolds that are orthogonal with respect to the 2n-dim biconformal (Killing) metric and have nondegenerate n-dim metric restrictions if and only if the initial four-dim space we gauge is Euclidean or

signature zero. In either of these cases the resulting configuration submanifold is necessarily Lorentzian [7].

Thus, it is possible to impose the conditions necessary to make biconformal space a metric phase space only in a restricted subclass of cases, and the configuration space metric must be Lorentzian. In [7], it was found that with a suitable choice of gauge, the metric may be written in coordinates y_α as

$$h_{\alpha\beta} = \frac{1}{(y^2)^2}\left(2y_\alpha y_\beta - y^2\delta_{\alpha\beta}\right), \tag{2.15}$$

where the signature changing character of the metric is easily seen.

In [7], there is a restriction necessary to prove the signature theorem. In Section 4 of [7] the conformal flatness of the momentum submanifold is assumed. This is then proven to result in an expression for the metric similar to (2.15), but with a generic vector, u^a. In Section 5 of [7] this generic vector is then related to the coordinates of the momentum space, y_a. In Chapter 3 we will arrive at the signature changing form of the metric without assuming the conformal flatness of the momentum sector. In fact, we will show the momentum sector can be consistently chosen to be fully flat.

In the metric above, (2.15), $y_\alpha = W_\alpha$ is the Weyl connection of the space. (This is modified to a true vector in the new treatment.) This points to another unique characteristic of flat biconformal space. The structures of the conformal group, treated as described above, give rise to a natural direction of time, given by the gauge field of dilatations. The situation is reminiscent of previous studies. In 1979, Stelle and West introduced a special vector field to choose the local symmetry of the MacDowell-Mansouri theory. The vector breaks the de Sitter symmetry, eliminating the need for the Wigner-Inönu contraction. Recently, Westman and Zlosnik [45] have looked in depth at both the de Sitter and anti-de Sitter cases using a class of actions, which extend that of Stelle and West by including derivative terms for the vector field and, therefore, lead to dynamical symmetry breaking. In [76,77] and Einstein-Aether theory [49], there is also a special vector field introduced into the action by hand that makes the Lorentzian metric Euclidean. These approaches are distinct from that of the biconformal approach, where the vector necessary for specifying the timelike direction occurs naturally from the underlying

group structure. We will have more to say about this below, where we show explicitly the Euclidean gauge theory necessarily possesses a special one-form, $\mathbf{v} = \boldsymbol{\omega} - \frac{1}{2}\eta_{ab}\mathbf{d}\eta^{bc}$. This gives the time direction on a biconformal submanifold, which is necessarily Lorentzian. The full manifold retains its original symmetry.

2.5. A brief note on gravitation

Notice our development to this point was based solely on group quotients and generalization of the resulting principal fiber bundle. We have arrived at the form of the curvatures in terms of the Cartan connection, and Bianchi identities required for integrability, thereby describing certain classes of geometry. Within the biconformal quotient, the demand for orthogonal Lagrangian submanifolds with nondegenerate n-dim restrictions of the Killing metric leads to the selection of certain Lorentzian submanifolds. Our concern in Chapter 3 has to do with the geometric background rather than with gravitational theories on those backgrounds. For continuity, we briefly digress to specify the action functionals for gravity. The main results of Chapter 3 concern only the homogeneous space, $\mathcal{M}_0^{(2n)}$. We will return to the curved case, $\mathcal{M}^{(2n)}$, in Chapter 4.

We are guided in the choice of action functionals by the example of general relativity. Given the Riemannian geometries of Section 2.3, we may write the Einstein-Hilbert action and proceed. More systematically, however, we may write the most general, even-parity action linear in the curvature and torsion. This still turns out to be the tetradic Palatini action and, as noted above, one of the classical field equations under a full variation of the connection $\left(\delta\mathbf{e}^b, \delta\boldsymbol{\omega}^a{}_b\right)$, implies vanishing torsion. The latter, more robust approach is what we follow for conformal gravity theories.

It is generally of interest to build the simplest class of actions possible, and we use the following criteria:

1. The pure-gravity action should be built from the available curvature tensor(s) and other tensors which occur in the geometric construction.

2. The action should be of lowest possible order ≥ 1 in the curvatures.

3. The action should be of even parity.

These are of sufficient generality not to bias our choice. It may also be a reasonable assumption to set certain tensor fields, for example, the spacetime torsion to zero. This can significantly change the available tensors, allowing a wider range of action functionals.

Notice, if we perform an infinitesimal conformal transformation to the curvatures, $\left(\mathbf{\Omega}^{\alpha}_{\ \beta}, \mathbf{\Omega}^{\alpha}, \mathbf{\Omega}_{\beta}, \mathbf{\Omega}\right)$, they all mix with one another, since the conformal curvature is really a single Lie-algebra-valued two-form. However, the generalization to a curved manifold breaks the nonvertical symmetries, allowing these different components to become independent tensors under the remaining Weyl group. Thus, to find the available tensors, we apply an infinitesimal transformation of the fiber symmetry. Tensors are those objects which transform linearly and homogeneously under these transformations.

2.5.1. The auxiliary gauging and Weyl gravity

According to our rules above, an action for the auxiliary gauging is constructible from the available tensors, $\mathbf{e}^{c}, \mathbf{\Omega}^{A}_{\ B}$, together with the invariant metric and Levi-Civita tensors, $\eta_{ab}, \varepsilon_{abcd}$. In $2n$-dimensions, scale invariance requires n factors of the curvature, so it is the $(p, q) = (4, 2)$ case that is of interest here. Then the most general even parity, $\mathcal{IW}$-invariant possibility is uniquely determined (up to an overall multiple) to be

$$\begin{aligned} S^{\mathcal{IW}}_{auxiliary} &= \alpha \int \mathbf{\Omega}^{A}_{\ B} \wedge {}^{*}\mathbf{\Omega}^{B}_{\ A} \\ &= \alpha \int \left(\mathbf{\Omega}^{a}_{\ b} \wedge {}^{*}\mathbf{\Omega}^{b}_{\ a} + 4\mathbf{T}^{c} \wedge {}^{*}\mathbf{S}_{c} + 2\mathbf{\Omega} \wedge {}^{*}\mathbf{\Omega} \right), \end{aligned}$$

where $\mathbf{\Omega}^{A}_{\ B}$ is the full $SO(4, 2)$ curvature two-form. This leads to a Weyl-Cartan geometry (i.e. one having nontrivial dilatation and torsion). To achieve Weyl gravity on the $\mathcal{IW}$ bundle, we need to break the special conformal symmetry with our choice of the action (putting aside the question of whether this might be done dynamically). Since the curvature has already broken the translational symmetry, we expect both nondynamical torsion and nondynamical special conformal curvature. Dropping the center term in $S^{\mathcal{IW}}_{auxiliary}$, we have the $\mathcal{W}$-invariant

Weyl-Bach action [43],

$$S^{\mathcal{W}}_{auxiliary} \;=\; \int \left(\alpha \mathbf{\Omega}^a_{\ b} \wedge {}^*\mathbf{\Omega}^b_{\ a} + \beta \mathbf{\Omega} \wedge {}^*\mathbf{\Omega}\right). \tag{2.16}$$

Various special cases of this action have been studied. With the absence of translations and special conformal transformations, $\mathbf{\Omega}^a_{\ b}, \mathbf{T}^a, \mathbf{S}_a$, and $\mathbf{\Omega}$ all become independent tensors under the remaining $\mathcal{W}$ symmetry, making the choice of α and β arbitrary. Bach [43] examined the exceptional case $\beta = 2\alpha$, for which the dilatation $\mathbf{\Omega}$ is nontrivial. Assuming a suitable metric dependence of the connection components $\left(\boldsymbol{\omega}^\alpha_{\ \beta}, \mathbf{f}_\alpha, \boldsymbol{\omega}\right)$, metric variation leads to the fourth-order Bach equation. In efforts to study superconformal gravity, two collaborations, [62–65, 78] set $\beta = 0$, and showed the action reduces to the Weyl curvature squared. Both these sets of investigations assumed vanishing torsion. Recently (with $\beta \neq 2\alpha$) it has been shown when the full connection is varied independently and the torsion set to zero only in the resulting field equations, $S^{\mathcal{W}}_{auxiliary}$ leads to the locally dilatationally invariant generalization of the vacuum Einstein equation [68].

In dimensions higher than four, our criteria lead to still higher order actions. Alternatively, curvature-linear actions can be written in any dimension by introducing a suitable power of a scalar field [75, 79]. This latter reference [79] gives the $\phi^2 R$ action often used in tractor studies.

2.5.2. Gravity in the biconformal gauging

The biconformal gauging, based on $\mathcal{C}/\mathcal{W}$, also has tensorial basis forms $(\boldsymbol{\omega}^\alpha, \boldsymbol{\omega}_\alpha)$. Moreover, each of the component curvatures $\left(\mathbf{\Omega}^\alpha_{\ \beta}, \mathbf{\Omega}^\alpha, \mathbf{\Omega}_\beta, \mathbf{\Omega}\right)$ becomes an independent tensor under the Weyl group.

In the biconformal case, the volume form $e^{\rho\sigma\ldots\lambda}{}_{\alpha\beta\ldots\nu}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \wedge \ldots \wedge \boldsymbol{\omega}^\nu \wedge \boldsymbol{\omega}_\rho \wedge \boldsymbol{\omega}_\sigma \wedge \ldots \wedge \boldsymbol{\omega}_\lambda$ has zero conformal weight. Since both $\mathbf{\Omega}^\alpha_{\ \beta}$ and $\mathbf{\Omega}$ also have zero conformal weight, there exists a curvature-linear action [50] in any dimension. The most general case is

$$S = \int \left(\alpha \mathbf{\Omega}^\alpha_{\ \beta} + \beta \mathbf{\Omega} \delta^\alpha_\beta + \gamma \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}_\beta\right) \wedge e^{\beta\rho\ldots\sigma}{}_{\alpha\mu\ldots\nu}\boldsymbol{\omega}^\mu \wedge \ldots \wedge \boldsymbol{\omega}^\nu \wedge \boldsymbol{\omega}_\rho \wedge \ldots \wedge \boldsymbol{\omega}_\sigma.$$

Notice, we now have three important properties of biconformal gravity that arise because of the

doubled dimension: (1) the nondegenerate conformal Killing metric induces a non-degenerate metric on the manifold, (2) the dilatational structure equation generically gives a symplectic form, and (3) there exists a Weyl symmetric action functional linear in the curvature, valid in any dimension.

There are a number of known results following from the linear action. In [50] torsion-constrained solutions are found, which are consistent with scale-invariant general relativity. Subsequent work along the same lines shows the torsion-free solutions are determined by the spacetime solder form, and reduce to describe spaces conformal to Ricci-flat spacetimes on the corresponding spacetime submanifold.[2] A supersymmetric version is presented in [80], and studies of Hamiltonian dynamics [81, 82] and quantum dynamics [83] support the idea that the models describe some type of relativistic phase space determined by the configuration space solution.

2.6. Summary of chapters

The characteristics of a given Cartan geometry depend exclusively on the structure of the chosen homogenous space. This homogeneous space can then be used to characterize the gravitational theory. In Chapter 3, we investigate the properties of flat biconformal space (BCS), $\mathcal{M}_0 = Conformal\,(n)\,/Weyl\,(n)$. Specifically, we look at BCS in the orthonormal version of the basis found in [7]. The orthonormal basis clarifies a number of characteristics of the space, making it easier to see how, though based on conformal symmetry, the submanifold structure equations organize themselves to look just like the usual Riemannian geometry of GR, with a Lorentzian signature. Chapter 3 will investigate extensively the characteristics of the spin connection (identified with rotations and Lorentz boosts) in this new basis. A major point is that the spin connection, in the time basis, is not fully antisymmetric with respect to the new metric. There is a symmetric part, which transforms like a tensor. Another tensor, built from the Weyl connection and the metric scale factor, can also be defined. We will use the general flat solution of BCS to show the degrees of freedom of the symmetric part of the spin connection are all due to this last tensor.

[2] http://www.physics.usu.edu/Wheeler/GaugeTheory/VanishingT03May12.pdf

In Chapter 4, we look at curved biconformal space and the gravitational theory defined by the Wehner-Wheeler action. We lay out the theory in full generality with all curvatures, torsion, cotorsions and dilatational curvatures present. We show some of the Bianchi identities (integrability conditions) of the theory are more restrictive than in a Riemannian geometry. The consequences of combining the the full-field equations together with the Bianchi identities and structure equations is investigated. We conclude by making a number of simplifying assumptions and recover scale invariant general relativity on the cotangent bundle of spacetime. The chosen assumptions turn out to be overly restrictive.

In Chapter 5, we summarize the conclusions of the manuscript and discuss future directions of research.

CHAPTER 3

DARK MATTER FROM EUCLIDEAN CONFORMAL SYMMETRIES

3.1. Homogeneous biconformal space in a conformally orthonormal, symplectic basis

The central goal of this chapter is to examine properties of the homogeneous manifold, $\mathcal{M}_0^{(2n)}$, which become evident in a conformally orthonormal basis; that is, a basis which is orthonormal up to an overall conformal factor. Generically, the properties we discuss will be inherited by the related gravity theories as well.

As noted above, biconformal space is immediately seen to possess several structures not seen in other gravitational gauge theories: a nondegenerate restriction of the Killing metric,[1] a symplectic form, and Kähler structure. In addition, the signature theorem in [7] shows if the original space has signature $\pm n$ or zero, the imposition of involution conditions leads to orthogonal Lagrangian submanifolds that have nondegenerate n-dim restrictions of the Killing metric. Further, constraining the momentum space to be as flat as permitted requires the restricted metrics to be Lorentzian. We strengthen these results in this Section and the next. Concerning ourselves only with elements of the geometry of the Euclidean $(s = \pm n)$ cases (as opposed to the additional restrictions of the field equations, involution conditions, or other constraints), we show the presence of exactly such Lorentzian signature Lagrangian submanifolds without further assumptions.

We go on to study the transformation of the spin connection when we transform the basis of an eight-dim biconformal space to one adapted to the Lagrangian submanifolds. We show, in addition to the Lorentzian metric, a Lorentzian connection emerges on the configuration and momentum spaces and there are two new tensor fields. Finally, we examine the curvature of these Lorentzian connections and find both a cosmological constant and cosmological dust. While it is premature to make quantitative predictions, these new geometric features provide novel candidates for dark energy and dark matter.

[1] There are nondegenerate restrictions in anti-de Sitter and de Sitter gravitational gauge theories.

3.1.1. The biconformal quotient

We start with the biconformal gauging of Section 2.4, specialized to the case of compactified, Euclidean $\mathbb{R}^4$ in a conformally orthonormal, symplectic basis. The Maurer-Cartan structure equations are

$$\mathbf{d}\boldsymbol{\omega}^{\alpha}_{\ \beta} = \boldsymbol{\omega}^{\mu}_{\ \beta} \wedge \boldsymbol{\omega}^{\alpha}_{\ \mu} + 2\Delta^{\alpha\mu}_{\nu\beta}\boldsymbol{\omega}_{\mu} \wedge \boldsymbol{\omega}^{\nu}, \tag{3.1}$$

$$\mathbf{d}\boldsymbol{\omega}^{\alpha} = \boldsymbol{\omega}^{\beta} \wedge \boldsymbol{\omega}^{\alpha}_{\ \beta} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}^{\alpha}, \tag{3.2}$$

$$\mathbf{d}\boldsymbol{\omega}_{\alpha} = \boldsymbol{\omega}^{\beta}_{\ \alpha} \wedge \boldsymbol{\omega}_{\beta} + \boldsymbol{\omega}_{\alpha} \wedge \boldsymbol{\omega}, \tag{3.3}$$

$$\mathbf{d}\boldsymbol{\omega} = \boldsymbol{\omega}^{\alpha} \wedge \boldsymbol{\omega}_{\alpha}, \tag{3.4}$$

where the connection one-forms represent $SO\left(4\right)$ rotations, translations, special conformal transformations and dilatations, respectively. The projection operator $\Delta^{\alpha\mu}_{\gamma\beta} \equiv \frac{1}{2}\left(\delta^{\alpha}_{\gamma}\delta^{\mu}_{\beta} - \delta^{\alpha\mu}\delta_{\gamma\beta}\right)$ in equation (3.1) gives that part of any $\binom{1}{1}$-tensor antisymmetric with respect to the original Euclidean metric, $\delta_{\alpha\beta}$. As discussed in Section 2.4.3, this group has a nondegenerate, 15-dim Killing metric. We stress the structure equations and Killing metric – and hence, their restrictions to the quotient manifold – are intrinsic to the conformal symmetry.

The gauging begins with the quotient of this conformal group, $SO\left(5,1\right)$, by its Weyl subgroup, spanned by the connection forms, $\boldsymbol{\omega}^{\alpha}_{\ \beta}$ (here dual to $SO(4)$ generators) and $\boldsymbol{\omega}$. The cotangent space of the quotient manifold is then spanned by the solder form, $\boldsymbol{\omega}^{\alpha}$, and the cosolder form, $\boldsymbol{\omega}_{\alpha}$, and the full conformal group becomes a principal fiber bundle with local Weyl symmetry over this eight-dim quotient manifold. The independence of $\boldsymbol{\omega}^{\alpha}$ and $\boldsymbol{\omega}_{\alpha}$ in the biconformal gauging makes the two-form, $\boldsymbol{\omega}^{\alpha} \wedge \boldsymbol{\omega}_{\alpha}$, nondegenerate, and equation (3.4) immediately shows $\boldsymbol{\omega}^{\alpha} \wedge \boldsymbol{\omega}_{\alpha}$ is a symplectic form.

The involution evident in equation (3.2) shows the solder forms, $\boldsymbol{\omega}^{\alpha}$, span a submanifold, and from the simultaneous vanishing of the symplectic form, this submanifold is Lagrangian. Similarly, equation (3.3) shows the $\boldsymbol{\omega}_{\beta}$ span a Lagrangian submanifold. However, notice neither of these submanifolds, spanned by either $\boldsymbol{\omega}^{\alpha}$ or $\boldsymbol{\omega}_{\alpha}$ alone, has an induced metric, since by

equation (2.14), $\left\langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{\omega}^{\beta} \right\rangle = \langle \boldsymbol{\omega}_{\alpha}, \boldsymbol{\omega}_{\beta} \rangle = 0$. The orthonormal basis will make the Killing metric block diagonal, guaranteeing its restriction to the configuration and momentum submanifolds have well-defined, nondegenerate metrics.

It was shown in [7] it is consistent (for signatures $\pm n, 0$ only) to impose involution conditions and momentum flatness in this rotated basis in such a way that the new basis still gives Lagrangian submanifolds. Moreover, the restriction of the Killing metric to these new submanifolds is necessarily Lorentzian. In what follows, we do not need the assumptions of momentum flatness or involution, and work only with intrinsic properties of $\mathcal{M}_0^{(2n)}$. This section describes the new basis and resulting connection, while the next establishes that for initial Euclidean signature, the principal results of [7] follow necessarily. Our results show the time-like directions in these models arise from intrinsically conformal structures. We now change to a new canonical basis, adapted to the Lagrangian submanifolds.

3.1.2. The conformally-orthonormal Lagrangian basis

In [7] the $(\boldsymbol{\omega}^{\alpha}, \boldsymbol{\omega}_{\alpha})$ basis is rotated so the metric, h_{AB} becomes block diagonal

$$\begin{bmatrix} 0 & \delta^{\alpha}_{\beta} \\ \delta^{\alpha}_{\beta} & 0 \end{bmatrix} \Rightarrow [h_{AB}] \;=\; \begin{bmatrix} h_{ab} & 0 \\ 0 & -h^{ab} \end{bmatrix},$$

while the symplectic form remains canonical. This makes the Lagrangian submanifolds orthogonal with a nondegenerate restriction to the metric. Here we use the same basis change, but in addition define coefficients, $h_a^{\ \alpha}$, to relate the orthogonal metric to one conformally orthonormal on the submanifolds, $\eta_{ab} = h_a^{\ \alpha} h_{\alpha\beta} h_b^{\ \beta}$, where η_{ab} is conformal to $diag\,(\pm 1, \pm 1, \pm 1, \pm 1)$. From [7] we know h_{ab} is necessarily Lorentzian, $h_{ab} = \eta_{ab} = e^{2\phi} diag\,(-1, 1, 1, 1) = e^{2\phi}\eta^0_{ab}$, and we give a more general proof below. Notice the definition of η_{ab} includes an unknown conformal factor. The required change of basis is then

$$\mathbf{e}^{a} \;=\; h_{\alpha}^{\ a}\left(\boldsymbol{\omega}^{\alpha} + \frac{1}{2} h^{\alpha\beta}\boldsymbol{\omega}_{\beta}\right), \tag{3.5}$$

$$\mathbf{f}_{a} \;=\; h_{a}^{\ \alpha}\left(\frac{1}{2}\boldsymbol{\omega}_{\alpha} - h_{\alpha\beta}\boldsymbol{\omega}^{\beta}\right) \tag{3.6}$$

with inverse basis change

$$\boldsymbol{\omega}^{\alpha} = \frac{1}{2} h_a^{\ \alpha} \left(\mathbf{e}^a - \eta^{ab} \mathbf{f}_b \right), \tag{3.7}$$

$$\boldsymbol{\omega}_{\alpha} = h_{\alpha}^{\ a} \left(\mathbf{f}_a + \eta_{ab} \mathbf{e}^b \right). \tag{3.8}$$

Using (2.14), the Killing metric is easily checked to be

$$\begin{bmatrix} \langle \mathbf{e}^a, \mathbf{e}^b \rangle & \langle \mathbf{e}^a, \mathbf{f}_b \rangle \\ \langle \mathbf{f}_a, \mathbf{e}^b \rangle & \langle \mathbf{f}_a, \mathbf{f}_b \rangle \end{bmatrix} = \begin{bmatrix} h_{\alpha}^{\ a} h_{\beta}^{\ b} h^{(\alpha\beta)} & 0 \\ 0 & -h_a^{\ \alpha} h_b^{\ \beta} h_{(\alpha\beta)} \end{bmatrix}$$
$$= \begin{bmatrix} e^{-2\phi} \eta_0^{ab} & 0 \\ 0 & -e^{2\phi} \eta^0_{ab} \end{bmatrix},$$

where $h_{\alpha\beta} = h_{(\alpha\beta)}$, and $h^{\alpha\beta} h_{\beta\gamma} = \delta^{\alpha}_{\gamma}$.

By transforming the dilatation equation (3.4) to find $\mathbf{d}\boldsymbol{\omega} = \mathbf{e}^a \mathbf{f}_a$, we immediately see these submanifolds are Lagrangian. We refer to the $\mathbf{f}_a = 0$ and $\mathbf{e}^a = 0$ submanifolds as the configuration and momentum submanifolds, respectively.

3.1.3. Properties of the structure equations in the new basis

We now explore the properties of the biconformal system in this adapted basis. Rewriting the remaining structure equations (3.1 3.2, 3.3), in terms of $\mathbf{e}^a$ and $\mathbf{f}_a$, we show some striking cancelations that lead to the emergence of a connection compatible with the Lorentzian metric, and two new tensors.

We begin with the exterior derivative of equation (3.5), using structure equations (3.2) and equation (3.3), and then using the basis change equations (3.7, 3.8). Because equations (3.7, 3.8) involve the sum and difference of $\mathbf{e}^a$ and $\mathbf{f}_b$, separating by these new basis forms leads to a separation of symmetries. This leads to a cumbersome expansion, which reduces considerably and in significant ways, to

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \Theta_{cb}^{ad} \boldsymbol{\tau}^c_{\ d} - \eta^{bc} \mathbf{f}_c \wedge \Xi_{db}^{ae} \boldsymbol{\tau}^d_{\ e} + \frac{1}{2} \eta_{bc} \mathbf{d}\eta^{ab} \wedge \mathbf{e}^c + \frac{1}{2} \mathbf{d}\eta^{ab} \wedge \mathbf{f}_b + 2\eta^{ab} \mathbf{f}_b \wedge \boldsymbol{\omega}, \tag{3.9}$$

where we define projections $\Theta^{ac}_{db} \equiv \frac{1}{2}\left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{bd}\right)$ and $\Xi^{ad}_{cb} \equiv \frac{1}{2}\left(\delta^a_c\delta^d_b + \eta^{ad}\eta_{cb}\right)$ that separate symmetries with respect to the new metric η_{ab} rather than $\delta_{\alpha\beta}$. These give the antisymmetric and symmetric parts, respectively, of a $\binom{1}{1}$-tensor with respect to the new orthonormal metric, η_{ab}. Notice that these projections are independent of the conformal factor on η_{ab}.

The significance of the reduction lies in how the symmetries separate between the different subspaces. Just as the curvatures split into three parts, equation (3.9) and each of the remaining structure equations splits into three parts. Expanding these independent parts separately allows us to see the Riemannian structure of the configuration and momentum spaces. It is useful to first define

$$\boldsymbol{\tau}^a_{\ b} \equiv \boldsymbol{\alpha}^a_{\ b} + \boldsymbol{\beta}^a_{\ b}, \tag{3.10}$$

where $\boldsymbol{\alpha}^a_{\ b} \equiv \Theta^{ad}_{cb}\boldsymbol{\tau}^c_{\ d}$ and $\boldsymbol{\beta}^a_{\ b} \equiv \Xi^{ad}_{cb}\boldsymbol{\tau}^c_{\ d}$. Then, to facilitate the split into $\mathbf{e}^a \wedge \mathbf{e}^b$, $\mathbf{e}^a \wedge \mathbf{f}_b$, and $\mathbf{f}_a \wedge \mathbf{f}_b$ parts, we partition the spin connection and Weyl vector by submanifold, defining

$$\boldsymbol{\alpha}^a_{\ b} \equiv \boldsymbol{\sigma}^a_{\ b} + \boldsymbol{\gamma}^a_{\ b} = \sigma^a_{\ bc}\mathbf{e}^c + \gamma^{a\ c}_{\ b}\mathbf{f}_c, \tag{3.11}$$

$$\boldsymbol{\beta}^a_{\ b} \equiv \boldsymbol{\mu}^a_{\ b} + \boldsymbol{\rho}^a_{\ b} = \mu^a_{\ bc}\mathbf{e}^c + \rho^{a\ c}_{\ b}\mathbf{f}_c, \tag{3.12}$$

$$\boldsymbol{\omega} \equiv W_a\mathbf{e}^a + W^a\mathbf{f}_a. \tag{3.13}$$

We also split the exterior derivative, $\mathbf{d} = \mathbf{d}_{(x)} + \mathbf{d}_{(y)}$, where coordinates x^α and y_α are used on the $\mathbf{e}^a = e_\alpha\ ^a\mathbf{d}x^\alpha$ and $\mathbf{f}_a = f_a\ ^\alpha\mathbf{d}y_\alpha$ submanifolds, respectively. Using these, we expand each of the structure equations into three $\mathcal{W}$-invariant parts. The complete set (with curvatures included for completeness) is given in Appendix A.

The simplifying features and notable properties include:

1. The new connection: The first thing is that all occurrences of the spin connection, $\boldsymbol{\omega}^\alpha_{\ \beta}$, may be written in terms of the combination

$$\boldsymbol{\tau}^a_{\ b} \equiv h_\alpha^{\ a}\boldsymbol{\omega}^\alpha_{\ \beta}h_b^{\ \beta} - h_b^{\ \alpha}\mathbf{d}h_\alpha^{\ a}, \tag{3.14}$$

which, as we show below, transforms as a Lorentz spin connection. Although the basis change is not a gauge transformation, the change in the connection has a similar inhomogeneous form. Because h_α^a is a change of basis rather than local $SO(n)$ or local Lorentz, the inhomogeneous term has no particular symmetry property, so $\boldsymbol{\tau}^a_{\ b}$ will have both symmetric and antisymmetric parts.

2. Separation of symmetric and antisymmetric parts: Notice in equation (3.9) how the antisymmetric part of the new connection, $\boldsymbol{\alpha}^a_{\ b}$, is associated with $\mathbf{e}^b$, while the symmetric part, $\boldsymbol{\beta}^a_{\ b}$, pairs with $\mathbf{f}_c$. This surprising correspondence puts the symmetric part into the cross terms while leaving the connection of the configuration submanifold metric compatible, up to the conformal factor.

3. Cancellation of the submanifold Weyl vector: The Weyl vector terms cancel on the configuration submanifold, while the $\mathbf{f}_a$ terms add. The expansion of the $\mathbf{df}_a$ structure equation shows the Weyl vector also drops out of the momentum submanifold equations. Nonetheless, these submanifold equations are scale invariant because of the residual metric derivative. Recognizing the combination of $\mathbf{d}h$ terms that arises as $\mathbf{d}\eta^{ab}$, and recalling that $\eta_{ab} = e^{2\phi}\eta^0_{ab}$, we have $-\frac{1}{2}\mathbf{d}\eta^{ac}\eta_{cb} = \delta^a_b\mathbf{d}\phi$. When the metric is rescaled, this term changes with the same inhomogeneous term as the Weyl vector.

4. Covariant derivative and a second Weyl-type connection: It is natural to define the $\boldsymbol{\tau}^b_{\ c}$-covariant derivative of the metric. Since $\eta^{cb}\boldsymbol{\alpha}^a_{\ c} + \eta^{ac}\boldsymbol{\alpha}^b_{\ c} = 0$, it depends only on $\boldsymbol{\beta}^a_{\ c}$ and the Weyl vector,

$$\begin{aligned} \mathbf{D}\eta^{ab} &\equiv \mathbf{d}\eta^{ab} + \eta^{cb}\boldsymbol{\tau}^a_{\ c} + \eta^{ac}\boldsymbol{\tau}^b_{\ c} - 2\boldsymbol{\omega}\eta^{ae}, && (3.15)\\ &= \mathbf{d}\eta^{ab} + 2\eta^{cb}\boldsymbol{\beta}^a_{\ c} - 2\boldsymbol{\omega}\eta^{ab}. && (3.16) \end{aligned}$$

This derivative allows us to express the structure of the biconformal space in terms of the Lorentzian properties.

When all of the identifications and definitions are included, and carrying out similar calculations for the remaining structure equations, the full set becomes

$$\mathbf{d}\boldsymbol{\tau}^a{}_b = \boldsymbol{\tau}^c{}_b \wedge \boldsymbol{\tau}^a{}_c + \Delta^{ae}_{db}\eta_{ec}\mathbf{e}^c \wedge \mathbf{e}^d - \Delta^{ac}_{eb}\eta^{ed}\mathbf{f}_c \wedge \mathbf{f}_d + 2\Delta^{ae}_{fb}\Xi^{fc}_{de}\mathbf{f}_c \wedge \mathbf{e}^d, \tag{3.17}$$

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^c \wedge \boldsymbol{\alpha}^a{}_c + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac} \wedge \mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ab} \wedge \mathbf{f}_b, \tag{3.18}$$

$$\mathbf{d}\mathbf{f}_a = \boldsymbol{\alpha}^b{}_a \wedge \mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab} \wedge \mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ab} \wedge \mathbf{e}^b, \tag{3.19}$$

$$\mathbf{d}\boldsymbol{\omega} = \mathbf{e}^a \wedge \mathbf{f}_a, \tag{3.20}$$

with the complete $\mathcal{W}$-invariant separation in Appendix A.

3.1.4. Gauge transformations and new tensors

The biconformal bundle now allows local Lorentz transformations and local dilatations on $\mathcal{M}_0^{(2n)}$. Under local Lorentz transformations, $\Lambda^a{}_c$, the connection $\boldsymbol{\tau}^a{}_b$ changes with an inhomogeneous term of the form $\bar{\Lambda}^c{}_b\mathbf{d}\Lambda^a{}_c$. Since this term lies in the Lie algebra of the Lorentz group, it is antisymmetric with respect to η_{ab}, $\Theta^{ac}_{db}\left(\bar{\Lambda}^e{}_c\mathbf{d}\Lambda^d{}_e\right) = \bar{\Lambda}^e{}_b\mathbf{d}\Lambda^a{}_e$ and therefore, only changes the corresponding Θ^{ac}_{db}-antisymmetric part of the connection, with the symmetric part transforming homogeneously:

$$\begin{aligned}\tilde{\boldsymbol{\alpha}}^a{}_b &= \Lambda^a{}_c\boldsymbol{\alpha}^c{}_d\bar{\Lambda}^d{}_b - \bar{\Lambda}^c{}_b\mathbf{d}\Lambda^a{}_c,\\ \tilde{\boldsymbol{\beta}}^a{}_b &= \Lambda^a{}_c\boldsymbol{\beta}^c{}_d\bar{\Lambda}^d{}_b.\end{aligned}$$

Having no inhomogeneous term, $\boldsymbol{\beta}^a{}_b$ is a Lorentz tensor. In Appendix B, we go through the gauge transformations of all of the structure equations. This new tensor field, $\boldsymbol{\beta}^a{}_b$, necessarily includes degrees of freedom from the original connection that cannot be present in $\boldsymbol{\alpha}^a{}_b$, the total equaling the degrees of freedom present in $\boldsymbol{\tau}^a{}_b$. As there is no obvious constraint on the connection $\boldsymbol{\alpha}^a{}_b$, we expect $\boldsymbol{\beta}^a{}_b$ to be highly constrained. Clearly, $\boldsymbol{\alpha}^c{}_d$ transforms as a Lorentzian spin connection, and the addition of the tensor, $\boldsymbol{\beta}^a{}_b$, preserves this property, so $\boldsymbol{\tau}^a{}_b$ is a local Lorentz connection.

Transformation of the connection under dilatations reveals another new tensor. The Weyl

vector transforms inhomogeneously in the usual way, $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} + \mathbf{d}f$, but, as noted above, the expression $\frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}$ also transforms, $\frac{1}{2}\tilde{\eta}_{cb}\mathbf{d}\tilde{\eta}^{ac} = \delta^a_b\mathbf{d}\tilde{\phi} = \delta^a_b\left(\mathbf{d}\phi - \mathbf{d}f\right)$, so the combination $\mathbf{v} = \boldsymbol{\omega} + \mathbf{d}\phi$ is scale invariant, see Appendix B. Notice the presence of two distinct scalars here. Obviously, given $\frac{1}{2}\eta^{ac}\mathbf{d}\eta_{cb} = \delta^a_b\mathbf{d}\phi$, we can choose a gauge function, $f_1 = -\phi$, such that $\frac{1}{2}\eta^{ac}\mathbf{d}\eta_{cb} = 0$. We also have $\mathbf{d}\boldsymbol{\omega} = 0$ on the configuration submanifold, so $\boldsymbol{\omega} = \mathbf{d}f_2$, for some scalar f_2, and this might be gauged to zero instead. But while one or the other of $\boldsymbol{\omega}$ or $\mathbf{d}\phi$ can be gauged to zero, their sum is gauge invariant. As we show below, it is the resulting vector, $\mathbf{v}$, that determines the timelike directions.

Recall certain involution relationships must be satisfied to ensure spacetime and momentum space are each submanifolds. The involution conditions in homogeneous biconformal space are

$$0 = \boldsymbol{\mu}^a_{\ b} \wedge \mathbf{e}^b - \mathbf{v}_{(x)} \wedge \mathbf{e}^a, \tag{3.21}$$

$$0 = \boldsymbol{\rho}^b{}_a \wedge \mathbf{f}_b - \mathbf{u}_{(y)} \wedge \mathbf{f}_a, \tag{3.22}$$

where $\mathbf{v} \equiv \mathbf{v}_{(x)} + \mathbf{u}_{(y)} \equiv v_a\mathbf{e}^a + u^a\mathbf{f}_a$. These were imposed as constraints in [7], but are shown below to hold automatically in Euclidean cases.

3.2. Riemannian spacetime in Euclidean biconformal space

The principal result of [7] was to show the flat biconformal space, $\mathcal{M}_0^{(2n)}$, arising from any $SO\left(p,q\right)$ symmetric biconformal gauging can be identified with a metric phase space only when the initial n-space is of signature $\pm n$ or zero. To make the identification, involution of the Lagrangian submanifolds was imposed, and it was assumed the momentum space is conformally flat. With these assumptions the Lagrangian configuration and momentum submanifolds of the signature $\pm n$ cases are necessarily Lorentzian.

Here we substantially strengthen this result, by considering only the Euclidean case. We are able to show further assumptions are unnecessary. The gauging always leads to Lorentzian configuration and momentum submanifolds, the involution conditions are automatically satisfied by the structure equations, and both the configuration and momentum spaces are conformally flat. We make no assumptions beyond the choice of the quotient $\mathcal{C}/\mathcal{W}$ and the structures that

follow from these groups. Because this result shows the development of the Lorentzian metric on the Lagrangian submanifolds, we give details of the calculation.

3.2.1. Solution of the structure equations

A complete solution of the structure equations in the original basis, equations (3.1-3.4), is given in [51] and derived in [50], with a concise derivation presented in [81]. By choosing the gauge and coordinates (w^α, s_β) appropriately, where Greek indices now refer to coordinates and will do so for the remainder of this manuscript,[6] the solution may be given the form

$$\boldsymbol{\omega}^\alpha_{\ \beta} = 2\Delta^{\alpha\mu}_{\nu\beta} s_\mu \mathbf{d}w^\nu, \tag{3.23}$$

$$\boldsymbol{\omega}^\alpha = \mathbf{d}w^\alpha, \tag{3.24}$$

$$\boldsymbol{\omega}_\alpha = \mathbf{d}s_\alpha - \left(s_\alpha s_\beta - \frac{1}{2}s^2\delta_{\alpha\beta}\right)\mathbf{d}w^\beta, \tag{3.25}$$

$$\boldsymbol{\omega} = -s_\alpha \mathbf{d}w^\alpha, \tag{3.26}$$

as is easily checked by direct substitution. Our first goal is to express this solution in the adapted basis and find the resulting metric. See Appendix C for detailed calculations.

From the original form of the Killing metric, equation (2.14), we find

$$\begin{bmatrix} \langle \mathbf{d}w^\alpha, \mathbf{d}w^\beta \rangle & \langle \mathbf{d}w^\alpha, \mathbf{d}s_\beta \rangle \\ \langle \mathbf{d}s_\alpha, \mathbf{d}w^\beta \rangle & \langle \mathbf{d}s_\alpha, \mathbf{d}s_\beta \rangle \end{bmatrix} = \begin{bmatrix} 0 & \delta^\alpha_\beta \\ \delta^\beta_\alpha & -k_{\alpha\beta} \end{bmatrix},$$

where we define $k_{\alpha\beta} \equiv s^2\delta_{\alpha\beta} - 2s_\alpha s_\beta$. This shows $\mathbf{d}w^\alpha$ and $\mathbf{d}s_\alpha$ do not span orthogonal subspaces. We want to find the most general set of orthogonal Lagrangian submanifolds, and the restriction of the Killing metric to them.

Suppose we find linear combinations of the orginal basis $\boldsymbol{\kappa}^\beta, \boldsymbol{\lambda}_\alpha$ that make the metric block diagonal, with $\boldsymbol{\lambda}_\alpha = 0$ and $\boldsymbol{\kappa}^\beta = 0$ giving Lagrangian submanifolds. Then any further

[6] The connection forms could be written with distinct indices, for example as $\boldsymbol{\omega}^a = \delta^a_\alpha \mathbf{d}w^\alpha$, but this is unnecessarily cumbersome.

transformation,

$$\begin{aligned}\tilde{\boldsymbol{\kappa}}^{\alpha} &= A^{\alpha}_{\beta}\boldsymbol{\kappa}^{\beta},\\ \tilde{\boldsymbol{\lambda}}_{\alpha} &= B^{\beta}_{\alpha}\boldsymbol{\lambda}_{\beta},\end{aligned}$$

leaves these submanifolds unchanged and is therefore equivalent. Now suppose one of the linear combinations is

$$\begin{aligned}\tilde{\boldsymbol{\lambda}}_{\alpha} &= \alpha A^{\beta}_{\alpha}\mathbf{d}s_{\beta} + \beta\tilde{C}_{\alpha\mu}\mathbf{d}w^{\mu}\\ &= A^{\beta}_{\alpha}\left(\alpha\mathbf{d}s_{\beta} + \beta C_{\beta\mu}\mathbf{d}w^{\mu}\right),\end{aligned}$$

where $C = A^{-1}\tilde{C}$ and the constants are required to keep the transformation nondegenerate. Then $\boldsymbol{\lambda}_{\alpha} = \alpha\mathbf{d}s_{\alpha} + \beta C_{\alpha\beta}\mathbf{d}w^{\beta}$ spans the same subspace. A similar argument holds for $\tilde{\boldsymbol{\kappa}}^{\beta}$, so if we can find a basis at all, there is also one of the form

$$\begin{aligned}\boldsymbol{\lambda}_{\alpha} &= \alpha\mathbf{d}s_{\alpha} + \beta C_{\alpha\beta}\mathbf{d}w^{\beta},\\ \boldsymbol{\kappa}^{\alpha} &= \mu\mathbf{d}w^{\alpha} + \nu B^{\alpha\beta}\mathbf{d}s_{\beta}.\end{aligned}$$

Now check the symplectic condition,

$$\boldsymbol{\kappa}^{\alpha}\boldsymbol{\lambda}_{\alpha} = \left(\mu\beta C_{\alpha\mu}\right)\mathbf{d}w^{\alpha}\mathbf{d}w^{\mu} + \alpha\mu\left(\delta^{\beta}_{\mu} - \nu\beta C_{\alpha\mu}B^{\alpha\beta}\right)\mathbf{d}w^{\mu}\mathbf{d}s_{\beta} + \left(\nu\alpha B^{\alpha\beta}\right)\mathbf{d}s_{\beta}\mathbf{d}s_{\alpha}.$$

To have $\boldsymbol{\kappa}^{\alpha}\boldsymbol{\lambda}_{\alpha} = \mathbf{d}w^{\alpha}\mathbf{d}s_{\alpha}$, $B^{\alpha\beta}$ and $C_{\alpha\beta}$ must be symmetric and

$$B = B^{t} = \frac{\alpha\mu - 1}{\nu\beta}C^{-1} \equiv \alpha\beta\bar{C}.$$

Replacing $B_{\alpha\beta}$ in the basis, we look at orthogonality of the inner product, requiring

$$\begin{aligned}0 &= \langle\boldsymbol{\kappa}^{\alpha}, \boldsymbol{\lambda}_{\beta}\rangle\\ &= \left\langle \mu\mathbf{d}w^{\alpha} + \frac{\alpha\mu - 1}{\beta}\bar{C}^{\alpha\mu}\mathbf{d}s_{\mu}, \alpha\mathbf{d}s_{\beta} + \beta C_{\beta\nu}\mathbf{d}w^{\nu}\right\rangle\end{aligned}$$

$$= \quad (2\alpha\mu - 1)\,\delta^{\alpha}_{\beta} - \frac{1}{\beta}\alpha\,(\alpha\mu - 1)\,\bar{C}^{\alpha\mu}k_{\mu\beta},$$

with solution $C_{\alpha\beta} = \frac{\alpha(\alpha\mu-1)}{\beta(2\alpha\mu-1)}k_{\alpha\beta}$. Therefore, the basis

$$\begin{aligned}
\boldsymbol{\lambda}_{\alpha} &= \alpha\mathbf{d}s_{\alpha} + \frac{\alpha\,(\alpha\mu-1)}{(2\alpha\mu-1)}k_{\alpha\beta}\mathbf{d}w^{\beta}, \\
\boldsymbol{\kappa}^{\alpha} &= \mu\mathbf{d}w^{\alpha} + \frac{2\alpha\mu-1}{\alpha}k^{\alpha\beta}\mathbf{d}s_{\beta}
\end{aligned}$$

satisfies the required properties and is equivalent to any other basis which does.

The metric restrictions to the submanifolds are now immediate from the inner products:

$$\begin{aligned}
\left\langle \boldsymbol{\kappa}^{\alpha}, \boldsymbol{\kappa}^{\beta} \right\rangle &= \frac{2\alpha\mu-1}{\alpha^2}k^{\alpha\beta}, \\
\langle \boldsymbol{\lambda}_{\alpha}, \boldsymbol{\lambda}_{\beta} \rangle &= -\frac{\alpha^2}{2\alpha\mu-1}k_{\alpha\beta}.
\end{aligned}$$

This shows the metric on the Lagrangian submanifolds is proportional to $k_{\alpha\beta}$, and we normalize the proportionality to 1 by choosing $\mu = \frac{1+k\alpha^2}{2\alpha}$ and $\beta \equiv k\alpha$, where $k = \pm 1$. This puts the basis in the form

$$\begin{aligned}
\boldsymbol{\kappa}^{\alpha} &= \frac{k}{2\beta}\left(\left(k\beta^2+1\right)\mathbf{d}w^{\alpha} + 2k\beta^2 k^{\alpha\beta}\mathbf{d}s_{\beta}\right). \\
\boldsymbol{\lambda}_{\alpha} &= \frac{1}{2\beta}\left(2k\beta^2\mathbf{d}s_{\alpha} + \left(k\beta^2-1\right)k_{\alpha\beta}\mathbf{d}w^{\beta}\right).
\end{aligned}$$

Now that we have established the metric $k_{\alpha\beta} = s^2\left(\delta_{\alpha\beta} - \frac{2}{s^2}s_{\alpha}s_{\beta}\right)$, where $\delta_{\alpha\beta}$ is the Euclidean metric and $s^2 = \delta^{\alpha\beta}s_{\alpha}s_{\beta} > 0$, and have found one basis for the submanifolds, we may form an orthonormal basis for each, setting $\eta_{ab} = h_a^{\alpha}h_b^{\beta}k_{\alpha\beta}$.

$$\mathbf{e}^{a} = \frac{k}{2\beta}h_{\alpha}^{a}\left(\left(1+k\beta^2\right)\mathbf{d}w^{\alpha} + 2k\beta^2 k^{\alpha\beta}\mathbf{d}s_{\beta}\right), \tag{3.27}$$

$$\mathbf{f}_{a} = \frac{1}{2\beta}h_{a}^{\alpha}\left(2k\beta^2\mathbf{d}s_{\alpha} - \left(1-k\beta^2\right)k_{\alpha\beta}\mathbf{d}w^{\beta}\right). \tag{3.28}$$

We see from the form $k_{\alpha\beta} = s^2\left(\delta_{\alpha\beta} - \frac{2}{s^2}s_{\alpha}s_{\beta}\right)$ that at any point s^0_{α}, a rotation that takes $\frac{1}{\sqrt{s^w}}s^0_{\alpha}$ to a fixed direction $\hat{\mathbf{n}}$ will take $k_{\alpha\beta}$ to $s^2 diag\,(-1,1,\ldots,1)$ so the orthonormal metric

η_{ab} is Lorentzian. This is one of our central results. Since equations (3.23-3.26) provide an exact, general solution to the structure equations, the induced configuration and momentum spaces of Euclidean biconformal spaces are always Lorentzian, without restrictions. We now find the connections forms in the orthogonal basis and check the involution conditions required to guarantee the configuration and momentum subspaces are Lagrangian submanifolds.

3.2.2. The connection in the adapted solution basis

We have defined $\boldsymbol{\tau}_b^a$ in equation (3.14) with antisymmetric and symmetric parts $\boldsymbol{\alpha}_b^a$ and $\boldsymbol{\beta}_b^a$, subdivided between the $\mathbf{e}^a$ and $\mathbf{f}_a$ subspaces, equations (3.11, 3.12). All quantities may be written in terms of the new basis. We calculate these transformations in explicit detail in Appendix C. We will make use of $s_a \equiv h_a^{\ \alpha} s_\alpha$ and $\delta_{ab} \equiv h_a^{\ \alpha} h_b^{\ \beta} \delta_{\alpha\beta}$. In terms of these, the orthonormal metric is $\eta_{ab} = s^2 \left(\delta_{ab} - \frac{2}{s^2} s_a s_b\right)$, where $s^2 \equiv \delta^{ab} s_a s_b > 0$. Solving for δ_{ab}, we find $\delta_{ab} = \frac{1}{s^2}\eta_{ab} + \frac{2}{s^2} s_a s_b$. Similar relations hold for the inverses, η^{ab}, δ^{ab}, see Appendix C. In addition, we may invert the basis change to write the coordinate differentials,

$$\begin{aligned} \mathbf{d}w^\beta &= k\beta h_a^{\ \beta}\left(\mathbf{e}^a - k\eta^{ab}\mathbf{f}_b\right), \\ \mathbf{d}s_\alpha &= \frac{1}{2\beta} h_\alpha^{\ a}\left(\left(1-k\beta^2\right)\eta_{ab}\mathbf{e}^b + k\left(1+k\beta^2\right)\mathbf{f}_a\right). \end{aligned}$$

The known solution for the spin connection and Weyl form, equations (3.23,3.26) immediately become

$$\boldsymbol{\omega}_b^a = 2\Delta_{db}^{ac} s_c k\beta\left(\mathbf{e}^d - k\eta^{de}\mathbf{f}_e\right), \tag{3.29}$$

$$\boldsymbol{\omega} = -k\beta s_a \mathbf{e}^a + \beta\eta^{ab} s_a \mathbf{f}_b, \tag{3.30}$$

where we easily expand the projection Δ_{db}^{ac} in terms of the new metric. Substituting this expansion to find $\boldsymbol{\tau}_b^a$, results in

$$\boldsymbol{\tau}_b^a = \beta\left(2\Theta_{db}^{ac} s_c + 2\eta^{ae}\eta_{bd} s_e + 2\eta^{ae} s_e s_b s_d\right)\left(k\mathbf{e}^d - \eta^{dg}\mathbf{f}_g\right) - h_b^{\ \alpha}\mathbf{d}h_\alpha^{\ a}.$$

The antisymmetric part is then $\boldsymbol{\alpha}^a{}_b \equiv \Theta^{ad}_{cb}\boldsymbol{\tau}^c{}_d = -\Theta^{ad}_{cb}h_d{}^\alpha \mathbf{d}h_\alpha{}^c$ with the remaining terms canceling identically. Furthermore, as described above, $h_\alpha{}^c$ is a purely s_α-dependent rotation at each point. Therefore, the remaining $h_d{}^\alpha \mathbf{d}h_\alpha{}^c$ term will lie totally in the subspace spanned by $\mathbf{d}s_\alpha$, giving the parts of $\boldsymbol{\alpha}^a{}_b$ as

$$\boldsymbol{\sigma}^a{}_b = -\frac{1-k\beta^2}{2\beta}\Theta^{ad}_{cb}\left(h_b{}^\alpha \frac{\partial}{\partial s_\beta}h_\alpha{}^a\right)h_\beta{}^c\eta_{cd}\mathbf{e}^d, \tag{3.31}$$

$$\boldsymbol{\gamma}^a{}_b = -\frac{k+\beta^2}{2\beta}\Theta^{ad}_{cb}\left(h_b{}^\alpha \frac{\partial}{\partial s_\beta}h_\alpha{}^a\right)h_\beta{}^c\mathbf{f}_c. \tag{3.32}$$

Recall the value of k or β in these expressions is essentially a gauge choice and should be physically irrelevant. If we choose $\beta^2 = 1$, we get either $\boldsymbol{\sigma}^a{}_b = 0$ or $\boldsymbol{\gamma}^a{}_b = 0$, depending on the sign of k.

Continuing, we are particularly interested in the symmetric pieces of the connection since they constitute a new feature of the theory. Applying the symmetric projection to $\boldsymbol{\tau}^a{}_b$, we expand $\boldsymbol{\beta}^a{}_b \equiv \Xi^{ad}_{cb}\boldsymbol{\tau}^c{}_d$. Using $\Xi^{cd}_{ab}\left(h_d{}^\mu \mathbf{d}h_\mu{}^a\right) = \frac{1}{2}h_\alpha{}^c h_b{}^\beta k^{\alpha\mu}\mathbf{d}k_{\mu\beta}$ (see Appendix C) to express the derivative term in terms of v_a, we find the independent parts

$$\begin{aligned}
\boldsymbol{\mu}^a{}_b &= \left(-k\beta\delta^a_b s_c + \beta\gamma_+\left(\delta^a_b s_c + \delta^a_c s_b + \eta^{ad}\eta_{bc}s_d + 2\eta^{ad}s_b s_c s_d\right)\right)\mathbf{e}^c, \\
\boldsymbol{\rho}^a{}_b &= \left(\beta\delta^a_b\eta^{cd}s_d + k\beta\gamma_-\left(\delta^a_b\eta^{cd}s_d + \delta^c_b\eta^{ad}s_d + \eta^{ac}s_b + 2\eta^{ad}\eta^{ce}s_b s_d s_e\right)\right)\mathbf{f}_c,
\end{aligned}$$

where $\gamma_\pm \equiv \frac{1}{2\beta}\left(1 \pm k\beta^2\right)$. Written in this form, the tensor character of $\boldsymbol{\mu}^a{}_b$ and $\boldsymbol{\rho}^a{}_b$ is not evident, but since we have chosen η_{ab} orthonormal (referred to later as the orthonormal gauge), $\phi = 0$, and $\mathbf{v} = \boldsymbol{\omega} + \mathbf{d}\phi = \boldsymbol{\omega}$ we have $\mathbf{v}_{(e)} + \mathbf{u}_{(f)} = -k\beta s_a\mathbf{e}^a + \beta\eta^{ab}s_a\mathbf{f}_b$ so we may equally well write

$$\boldsymbol{\mu}^a{}_b = \left(\delta^a_b v_c - k\gamma_+\left(\delta^a_b v_c + \delta^a_c v_b + \eta^{ad}\eta_{bc}v_d + \frac{2}{\beta^2}\eta^{ad}v_b v_c v_d\right)\right)\mathbf{e}^c, \tag{3.33}$$

$$\boldsymbol{\rho}^a{}_b = \left(\delta^a_b u^c + k\gamma_-\left(\delta^a_b u^c + \delta^c_b u^a + \eta^{ac}\eta_{bd}u^d + \frac{2}{\beta^2}\eta_{bd}u^a u^c u^d\right)\right)\mathbf{f}_c, \tag{3.34}$$

which are manifestly tensorial.

The involution conditions, equations (3.21-3.22), are easily seen to be satisfied identically by equations (3.33, 3.34). Therefore, the $\mathbf{f}_a = 0$ and $\mathbf{e}^a = 0$ subspaces are Lagrangian submanifolds spanned respectively by $\mathbf{e}^a$ and $\mathbf{f}_a$. There exist coordinates such that these basis forms may be written

$$\mathbf{e}^a = e_\mu{}^a \mathbf{d}x^\mu, \tag{3.35}$$

$$\mathbf{f}_a = f_a{}^\mu \mathbf{d}y_\mu. \tag{3.36}$$

To find such submanifold coordinates, the useful thing to note is $\mathbf{d}\left(\frac{s_\alpha}{s^2}\right) = \delta_{\alpha\nu}k^{\mu\nu}\mathbf{d}s_\mu$, so the basis may be written as

$$\begin{aligned}
\mathbf{e}^a &= h_\alpha{}^a \mathbf{d}\left(k\gamma_+ w^\alpha + \beta\delta^{\alpha\beta}\left(\frac{s_\alpha}{s^2}\right)\right) \equiv h_\alpha{}^a \mathbf{d}x^\alpha, \\
\mathbf{f}_a &= \left(h_a{}^\alpha k_{\alpha\beta}\delta^{\beta\mu}\right)\mathbf{d}\left(k\beta\left(\frac{s_\mu}{s^2}\right) - \gamma_-\delta_{\mu\nu}w^\nu\right) \equiv f_a{}^\mu \mathbf{d}y_\mu,
\end{aligned}$$

with $x^\alpha = k\gamma_+ w^\alpha + \beta\delta^{\alpha\beta}\left(\frac{s_\alpha}{s^2}\right)$ and $y_\mu = k\beta\left(\frac{s_\mu}{s^2}\right) - \gamma_-\delta_{\mu\nu}w^\nu$. This confirms the involution.

3.3. Curvature of the submanifolds

The nature of the configuration or momentum submanifold may be determined by restricting the structure equations by $\mathbf{f}_a = 0$ or $\mathbf{e}^a = 0$, respectively. To aid in the interpretation of the resulting submanifold structure equations, we define the curvature of the antisymmetric connection $\boldsymbol{\alpha}^a_b$

$$\mathbf{R}^a_b \equiv \mathbf{d}\boldsymbol{\alpha}^a_b - \boldsymbol{\alpha}^c_b \wedge \boldsymbol{\alpha}^a_c \tag{3.37}$$

$$= \frac{1}{2}R^a_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d + R^a_{b\,d}{}^c\mathbf{f}_c \wedge \mathbf{e}^d + \frac{1}{2}R^a_b{}^{cd}\mathbf{f}_c \wedge \mathbf{f}_d. \tag{3.38}$$

While all components of the overall Cartan curvature, $\boldsymbol{\Omega}^A = (\boldsymbol{\Omega}^a_b, \mathbf{T}^a, \mathbf{S}_a, \boldsymbol{\Omega})$ are zero on $\mathcal{M}_0^{(2n)}$, the curvature, $\mathbf{R}^a_b$, and in particular the curvatures $\frac{1}{2}R^a_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d$ and $\frac{1}{2}R^a_b{}^{cd}\mathbf{f}_c \wedge \mathbf{f}_d$ on the submanifolds, may or may not be. Here, we examine this question using the structure equations to find the Riemannian curvature of the connections, $\boldsymbol{\sigma}^a_b$ and $\boldsymbol{\gamma}^a_b$, of the Lorentzian

submanifolds.

3.3.1. Momentum space curvature

To see the Lagrangian submanifold equations describe a Riemannian geometry, we set $\mathbf{e}^a = 0$ in the structure equations, (3.17-3.20), and choose the $\phi = 0$ (orthonormal) gauge (or see Appendix A, equations (A.13-A.16), with the Cartan curvatures set to zero). Then, taking the Θ^{ac}_{db} projection, we have

$$\begin{aligned} 0 &= \frac{1}{2} R^{a}_{\ b}{}^{cd} \mathbf{f}_c \wedge \mathbf{f}_d - \boldsymbol{\rho}^{c}_{\ b} \wedge \boldsymbol{\rho}^{a}_{\ c} + \Theta^{ac}_{db} \eta^{ac} \Delta^{eb}_{cf} \mathbf{f}_b \wedge \mathbf{f}_a, \\ 0 &= \mathbf{d}_{(y)} \mathbf{f}_b - \boldsymbol{\gamma}^{a}_{\ b} \wedge \mathbf{f}_a. \end{aligned} \tag{3.39}$$

These are the structure equations of a Riemannian geometry with additional geometric terms, $-\boldsymbol{\rho}^{c}_{\ b} \wedge \boldsymbol{\rho}^{a}_{\ c} + \Theta^{ac}_{db} \eta^{ac} \Delta^{eb}_{cf} \mathbf{f}_b \wedge \mathbf{f}_a$, reflecting the difference between Riemannian curvature and Cartan curvature. The symmetric projection is

$$\begin{aligned} \mathbf{D}^{(y)} \boldsymbol{\rho}^{a}_{\ b} &= -k \Xi^{ac}_{db} \Delta^{df}_{ec} \eta^{eg} \mathbf{f}_f \wedge \mathbf{f}_g, \\ \mathbf{d}_{(y)} \mathbf{u}_{(f)} &= 0, \end{aligned}$$

where $\mathbf{u}_{(f)}, \boldsymbol{\gamma}^{a}_{\ b}$ and $\boldsymbol{\rho}^{a}_{\ b}$ are given by equations (3.30,3.32,3.34), respectively. Rather than computing $R^{a}_{\ b}{}^{cd}$ directly from $\boldsymbol{\gamma}^{a}_{\ b}$, which requires a complicated expression for the local rotation, h^{a}_{α}, we find it using the rest of equation (3.39).

Letting $\beta = e^{\lambda}$ so that

$$k + \gamma_{-}^{2} = \begin{cases} \cosh^2 \lambda & k = 1 \\ \sinh^2 \lambda & k = -1 \end{cases},$$

the curvature is

$$\frac{1}{2} R^{a}_{\ b}{}^{cd} \mathbf{f}_c \mathbf{f}_d = \begin{cases} \cosh^2 \lambda \Theta^{ag}_{cb} \left(\eta^{cf} + 2 \eta^{cd} \eta^{fe} s_d s_e \right) \mathbf{f}_f \wedge \mathbf{f}_g & k = 1 \\ \sinh^2 \lambda \Theta^{ag}_{cb} \left(\eta^{cf} + 2 \eta^{cd} \eta^{fe} s_d s_e \right) \mathbf{f}_f \wedge \mathbf{f}_g & k = -1 \end{cases}.$$

Now consider the symmetric equations. Notice the Weyl vector has totally decoupled, with its equation showing that $\mathbf{u}_{(f)}$ is closed, a result that also follows from its definition. For the symmetric projection, we find $\Xi^{ac}_{db}\eta^{ac}\Delta^{eb}_{cf}\mathbf{f}_b\mathbf{f}_a \equiv 0$. Then, contraction of $D^a\rho^a_{\ b}{}^c$ with $\eta_{ad}\eta_{ce}u^au^e$, together with $\mathbf{d}_{(y)}\mathbf{u}_{(f)} = 0$ shows u^a is covariantly constant, $D^a_{(y)}u^b = 0$.

If we choose $k = -1$ and $\lambda = 0$, the Riemann curvature of the momentum space vanishes. This is a stronger result than in [7], where only the Weyl curvature could consistently be set to zero. In this case, the Lagrangian submanifold becomes a vector space and there is a natural interpretation as the cotangent space of the configuration space. However, the orthonormal metric in this case, $\langle \mathbf{f}_a, \mathbf{f}_b\rangle = \eta_{ab}$, has the opposite sign from the metric of the configuration space, $\left\langle \mathbf{e}^a, \mathbf{e}^b\right\rangle = -\eta^{ab}$. This reversal of sign of the metric, together with the the units, suggests the physical (momentum) tangent space coordinates are related to the geometrical ones by $p_\alpha \sim i\hbar y_\alpha$. This has been suggested previously [84] and explored in the context of quantization [83].

Leaving β and k unspecified, we see that in general, momentum space has nonvanishing Riemannian curvature of the connection $\boldsymbol{\gamma}^a_{\ b}$, a situation suggested long ago for quantum gravity [85,86]. We consider this further in Section 7.3. Whatever the values of β and k, the momentum space is conformally flat. We see this from the decomposition of Riemannian curvature into the Weyl curvature, $\mathbf{C}^a_{\ b}$, and Schouten tensor, $\boldsymbol{\mathcal{R}}_a$, given by

$$\mathbf{R}^a_{\ b} \;=\; \mathbf{C}^a_{\ b} - 2\Theta^{ae}_{db}\boldsymbol{\mathcal{R}}_e\mathbf{e}^d.$$

The Schouten tensor,$\boldsymbol{\mathcal{R}}_a \equiv \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)}R\eta_{ab}\right)\mathbf{e}^b$ is algebraically equivalent to the Ricci tensor, R_{ab}. It is easy to prove that when the curvature two-form can be expressed as a projection in the form $\mathbf{R}^a_{\ b} = -2\Theta^{ae}_{db}\mathbf{X}_e\mathbf{e}^d$, then $\mathbf{X}_a$ is the Schouten tensor, and the Weyl curvature vanishes. Vanishing Weyl curvature implies conformal flatness.

3.3.2. Spacetime curvature and geometric curvature

The curvature on the configuration space takes the same basic form. Still in the orthonormal gauge, and separating the symmetric and antisymmetric parts as before, we again find a

Riemannian geometry with additional geometric terms,

$$0 = \mathbf{R}^a_{\ b}(\boldsymbol{\sigma}) - \boldsymbol{\mu}^c_{\ b}\boldsymbol{\mu}^a_{\ c} - \Theta^{ac}_{db}\Delta^{de}_{fc}\eta_{eg}\mathbf{e}^g\mathbf{e}^f, \tag{3.40}$$

$$0 = \mathbf{d}_{(x)}\mathbf{e}^a - \mathbf{e}^b\boldsymbol{\sigma}^a_{\ b}, \tag{3.41}$$

together with

$$0 = \mathbf{D}_{(x)}\boldsymbol{\mu}^a_{\ b} - \Xi^{ac}_{db}\Delta^{de}_{fc}\eta_{eg}\mathbf{e}^g\mathbf{e}^f,$$

$$0 = \mathbf{d}_{(x)}\mathbf{v}.$$

Looking first at all the Θ^{ad}_{cb}-antisymmetric terms and substituting in (3.33) for $\boldsymbol{\mu}^a_{\ b}$, we find the Riemannian curvature is

$$\mathbf{R}^a_{\ b} = \left(\gamma_+^2 - k\right)\Theta^{ac}_{db}\left(\eta_{ce} + 2s_c s_e\right)\mathbf{e}^d\mathbf{e}^e,$$

so the Weyl curvature vanishes and the Schouten tensor is

$$\boldsymbol{\mathcal{R}}_a = \frac{1}{2}\left(\gamma_+^2 - k\right)\left(\eta_{ab} + 2s_a s_b\right)\mathbf{e}^b. \tag{3.42}$$

The vanishing Weyl curvature tensor shows the spacetime is conformally flat. This result is discussed in detail below.

The equation, $\mathbf{d}_{(x)}\mathbf{v} = 0$ shows $\mathbf{v}$ is hypersurface orthogonal. Expanding the remaining equation with $\mathbf{d}_{(x)}\mathbf{v} = 0$, $\mathbf{D}_{(x)}\eta_{ab} = 0$ and $\mathbf{D}_{(x)}\mathbf{e}^a = 0$, contractions involving η_{ab} and v_a quickly show

$$D^{(x)}_a v_b = 0.$$

This, combined with $\mathbf{D}^{(y)}u^a = 0$ and $u^a = -k\eta^{ab}v_b$ shows the full covariant derivative vanishes, $D_a v_b = 0$. The scale vector is, therefore, a covariantly constant, hypersurface orthogonal, unit timelike Killing vector of the spacetime submanifold.

3.3.3. Curvature invariant

Substituting $\beta = e^{\lambda}$ as before, the components of the momentum and configuration curvatures become

$$\eta_{df}\eta_{eg}R^{a\ fg}_{\ \ b} = \begin{cases} \cosh^2\lambda\left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c) & k=1 \\ \sinh^2\lambda\left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c) & k=-1 \end{cases},$$

and

$$R^a_{\ bde} = \begin{cases} \sinh^2\lambda\left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c) & k=1 \\ \cosh^2\lambda\left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c) & k=-1 \end{cases}.$$

Subtracting these,

$$\eta_{df}\eta_{eg}R^{a\ fg}_{\ \ b} - R^a_{\ bde} = k\left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c)$$

so the difference of the configuration and momentum curvatures is independent of the linear combination of basis forms used. This coupling between the momentum and configuration space curvatures adds a sort of complementarity that goes beyond the suggestion by Born [85, 86] that momentum space might also be curved. As we continuously vary β^2, the curvature moves between momentum and configuration space but this difference remains unchanged. We may even make one or the other Lagrangian submanifold flat.

For the Einstein tensors,

$$\eta_{ac}\eta_{bd}G^{cd}_{(y)} - G^{(x)}_{ab} = \frac{1}{2}k\left((n-3)\,\eta_{ab} + (n-2)\,s_a s_b\right).$$

3.3.4. Candidate dark matter

There is a surprising consequence of the tensor $\boldsymbol{\mu}^a_{\ b}$ in the Lorentz structure equation. The structure equations for the configuration Lagrangian submanifold above describe an ordinary curved Lorentzian spacetime with certain extra terms from the conformal geometry that exist even in the absence of matter. We gain some insight into the nature of these additional terms

from the metric and Einstein tensor. In coordinates, the metric takes the form

$$h_{\alpha\beta} = s^2 \left(\delta_{\alpha\beta} - \frac{2}{s^2} s_\alpha s_\beta \right),$$

which is straightforwardly boosted to $s^2 \eta^0_{\alpha\beta}$ at a point. Since the spacetime is conformally flat, gradients of the conformal factor must be in the time direction, s_α, so we may rescale the time, $dt' = \sqrt{s^2} dt$ to put the line element in the form

$$ds^2 = -dt'^2 + s^2\left(t'\right)\left(dx^2 + dy^2 + dz^2\right).$$

That is, the vacuum solution is a spatially flat FRW cosmology. Putting the results in terms of the Einstein tensor and a coordinate basis, we expect an equation of the form $\tilde{G}_{\alpha\beta} = \kappa T^{matter}_{\alpha\beta}$ where the Cartan Einstein tensor is modified to

$$\tilde{G}_{\alpha\beta} \equiv G_{\alpha\beta} - 3\left(n-2\right) s^2 s_\alpha s_\beta + \frac{3}{2}\left(n-2\right)\left(n-3\right) s^2 h_{\alpha\beta}, \tag{3.43}$$

where $G_{\alpha\beta}$ is the familiar Einstein tensor. The new geometric terms may be thought of as a combination of a cosmological constant and a cosmological perfect fluid. With this interpretation, we may write the new cosmological terms as

$$\kappa T^{cosm}_{\alpha\beta} = \left(\rho_0 + p_0\right) v_\alpha v_\beta + p_0 h_{\alpha\beta} - \Lambda h_{\alpha\beta},$$

where $\kappa T^{cosm}_{ab} \equiv 3\left(n-2\right) s^2 v_\alpha v_\beta - \frac{3}{2}\left(n-2\right)\left(n-3\right) s^2 h_{\alpha\beta}$. In $n =$ four-dimensions, $\frac{1}{2}\left(\rho_0 + p_0\right) = \Lambda - p_0$, with the equation of state and the overall scale undetermined. If we assume an equation of state $p_0 = w\rho_0$, this becomes

$$\frac{1}{2}\left(1 + 3w\right)\rho_0 = \Lambda.$$

This relation alone does not account for the values suggested by the current Planck data: about 0.68 for the cosmological constant, 0.268 for the density of dark matter, and vanishing pressure, $w = 0$. However, these values are based on standard cosmology, while we have not yet included

matter terms in equation (3.43). Moreover, the proportions of the three geometric terms in equation (3.43) may change when curvature is included. Such a change is suggested by the form of known solutions in the original basis, where $h_{\alpha\beta}$ is augmented by a Schouten term. If this modification also occurs in the adapted basis, the ratios above will be modified. We are currently examining such solutions.

3.4. Discussion

Using the quotient method of gauging, we constructed the class of biconformal geometries. The construction starts with the conformal group of an $SO(p,q)$-symmetric pseudo-metric space. The quotient by $\mathcal{W}(p,q) \equiv SO(p,q) \times$ dilatations gives the homogeneous manifold, $\mathcal{M}_0^{2n}$. We show this manifold is metric and symplectic (as well as Kähler with a different metric). Generalizing the manifold and connection while maintaining the local $\mathcal{W}$ invariance, we display the resulting biconformal spaces, $\mathcal{M}^{2n}$ [5, 6, 51].

This class of locally symmetric manifolds becomes a model for gravity when we recall the most general curvature-linear action [50].

It is shown in [7] that $\mathcal{M}_0^{(2n)}(p,q)$ in any dimension $n = p + q$ will have Lagrangian submanifolds that are orthogonal with respect to the 2n-dim biconformal (Killing) metric and have nondegenerate n-dim metric restrictions on those submanifolds only if the original space is Euclidean or signature zero $\left(p \in \left\{0, \frac{n}{2}, n\right\}\right)$, and then the signature of the submanifolds is severely limited $(p \to p \pm 1)$. This leads in the two Euclidean cases to Lorentzian configuration space, and hence the origin of time [7]. For the case of flat, eight-dim biconformal space, the Lagrangian submanifolds are necessarily Lorentzian.

Our investigation explores properties of the homogeneous manifold, $\mathcal{M}_0^{2n}(n,0)$. Starting with Euclidean symmetry, $SO(n)$, we clarify the emergence of Lorentzian signature Lagrangian submanifolds. We extend the results of [7], eliminating all but the group-theoretic assumptions. By writing the structure equations in an adapted basis, we reveal new features of these geometries. We summarize our new findings below.

A new connection

There is a natural $SO(n)$ Cartan connection on $\mathcal{M}_0^{2n}$. Rewriting the biconformal structure

equations in an orthogonal, canonically conjugate, conformally orthonormal basis automatically introduces a Lorentzian connection and decouples the Weyl vector from the submanifolds. The submanifold equations remain scale invariant because of the residual metric derivative, $\frac{1}{2}\mathbf{d}\eta^{ac}\eta_{cb} = \delta^a_b\mathbf{d}\phi$. When the metric is rescaled, this term changes with the negative of the inhomogeneous term acquired by the Weyl vector. This structure emerges directly from the transformation of the structure equations, as detailed in points 1 through 4 in Section 3.1.3.

Specifically, we showed all occurrences of the $SO\,(4)$ spin connection $\boldsymbol{\omega}^{\alpha}_{\ \beta}$ may be written in terms of the new connection, $\boldsymbol{\tau}^{a}_{\ b} \equiv h_{\alpha}^{\ a}\boldsymbol{\omega}^{\alpha}_{\ \beta}h_{b}^{\ \beta} - h_{b}^{\ \alpha}\mathbf{d}h_{\alpha}^{\ a}$, which has both symmetric and antisymmetric parts. These symmetric and antisymmetric parts separate automatically in the structure equations, with only the Lorentz part of the connection, $\boldsymbol{\alpha}^{a}_{\ b} = \Theta^{ac}_{db}\boldsymbol{\tau}^{d}_{\ c}$ describing the evolution of the configuration submanifold solder form. The spacetime and momentum space connections are metric compatible, up to a conformal factor.

Two new tensors

It is especially striking how the Weyl vector and the symmetric piece of the connection are pushed from the basis submanifolds into the mixed basis equations. These extra degrees of freedom are embodied in two new Lorentz tensors.

The factor $\delta^a_b\mathbf{d}\phi$, which replaces the Weyl vector in the submanifold basis equations, allows us to form a scale-invariant one-form, $\mathbf{v} = \boldsymbol{\omega} + \mathbf{d}\phi$, in the mixed basis equations. It is ultimately this vector that determines the time direction.

We showed the symmetric part of the spin connection, $\boldsymbol{\beta}^{a}_{\ b}$, despite being a piece of the connection, transforms as a tensor. The solution of the structure equations shows the two tensors, $\mathbf{v}$ and $\boldsymbol{\beta}^{a}_{\ b}$ are related, with $\boldsymbol{\beta}^{a}_{\ b}$ constructed cubically, purely from $\mathbf{v}$ and the metric. Although the presence of $\boldsymbol{\beta}^{a}_{\ b}$ changes the form of the momentum space curvature, we find the same signature changing metric as found in [7]. Rather than imposing vanishing momentum space curvature as in [7], we make use of a complete solution of the Maurer-Cartan equations to derive the metric. The integrability of the Lagrangian submanifolds, the Lorentzian metric and connection, and the possibility of a flat momentum space are all now seen as direct consequences of the structure equations, without assumptions.

Riemannian spacetime and momentum space

The configuration and momentum submanifolds have vanishing dilatational curvature, making them gauge equivalent to Riemannian geometries. Together with the signature change from the original Euclidean space to these Lorentzian manifolds, we arrive at a suitable arena for general relativity in which time is constructed covariantly from a scale-invariant Killing field. This field is provided automatically from the group structure.

Effective cosmological fluid and cosmological constant

Though we work in the homogeneous space, $\mathcal{M}_0^{2n}$, so that there are no Cartan curvatures, there is a net Riemannian curvature remaining on the spacetime submanifold. We show this to describe a conformally flat spacetime with the deviation from flatness provided by additional geometric terms of the form

$$\tilde{G}_{\alpha\beta} \equiv G_{\alpha\beta} - \rho_0 v_\alpha v_\beta + \Lambda h_{\alpha\beta} = 0; \tag{3.44}$$

that is, a background dust and a cosmological constant. The values $\rho_0 = 3\,(n-2)\,s^2$ and $\Lambda = \frac{3}{2}\,(n-2)\,(n-3)\,s^2$ give, in the absence of physical sources, the relation $(2+3w)\,\rho_0 = \Lambda$ for an equation of state $p_0 = w\rho_0$. An examination of more realistic cosmological models involving matter fields and curved biconformal spaces, $\mathcal{M}^{2n}$, is underway.

CHAPTER 4

BICONFORMAL GRAVITY WITH EMERGENT LORENTZIAN STRUCTURE

4.1. Introduction

In the previous chapters, we have shown flat biconformal space, the homogeneous space formed from the quotient of the conformal group by the rotation group cross dilatations, $SO(5,1)/SO(4) \times \mathbb{R}^+$, possesses two new tensors and a natural Lorentz spin connection. The thrust of the current chapter is to show how these natural structures meld with a gravitational action principle in this geometry. General relativity on curved biconformal space has been shown to arise from the Wehner–Wheeler action [50]. This action is conformally invariant, despite linearity in the curvatures. More recently, it has been shown the torsion–free case generically gives rise to general relativity. This work was done in a context where the Lorentzian signature of the submanifolds was imposed by starting with a Lorentzian signature before gauging. Here we examine solutions to biconformal gravity in the basis where the Lorentzian structure emerges directly from the gauge theoretic construction when we start from Euclidean space. We show that with a simple ansatz, we again regain general relativity, but on both submanifolds. This is due to a complete symmetry between the two Lagrangian submanifolds, interpreted as configuration and momentum space. We can break that symmetry to show biconformal gravity reproduces, in a special case, general relativity. The solution will be kept as general as possible through the early development of this manuscript. Not until Sections (4.4) and (4.5) will we make some simplifying assumptions. For a comprehensive review of the construction of homogeneous biconformal space see Chapter 3.

There exists a torsion-free solution to biconformal space [50] that reproduces all of general relativity; however, the current work focuses on showing general relativity emerge in the time basis of [7]. Unfortunately, when the basis is rotated, the torsion and cotorsions mix nontrivially, so we are unable to use the existing solution in the same way we used the existing solution of the homogeneous space in Chapter 3 to find the symmetric parts of the spin connection. Instead we use an ansatz, inspired by the homogeneous solution, that simplifies the system of equations extensively.

4.1.1. Notation

We follow the same notational conventions as in the previous chapters of this manuscript. Of particular importance in this chapter will be the positions of the last two indices on curvatures. The last two indices will always appear in their original positions; all metrics will be explicit. This then means $R^a_{\ bcd}$, $R^{a\ c}_{\ b\ d}$, and $R^{a\ cd}_{\ b}$ are distinct tensors, referring to the configuration, mixed-basis and momentum curvatures, respectively.

4.1.2. Organization of chapter

The organization of the chapter is as follows. In Section 4.2, we first review the structure equations of curved biconformal space in the orthonormal, Lagrangian basis. Then we survey the integrability conditions of these equations, which will turn out to strongly couple the curvatures. The last part of Section 4.2 looks at the curvature-linear action of [50] and the field equations therefrom. In Section 4.3, we show how the combination of the field equations and structure equations dictate submanifold structures very similar to two Riemannian geometries, but which, in general, have a number of new structures. Through this section, we still consider the fully general biconformal space. In Section 4.4, we investigate the consequences of choosing a simple linear ansatz for one of these new structures, the tensorial, symmetric part of the spin connection. In Section 4.5, we will show general relativity on a cotangent bundle is a special case of curved biconformal space, by choosing a number of the torsion and cotorsion tensors to vanish. In the last section we make some concluding remarks.

4.2. Curved biconformal space in the orthonormal canonical basis

In Chapter 3, we laid out the construction of biconformal space in the Lagrangian, orthonormal frame. The quotient manifold method gives us a manifold described by the Maurer-Cartan structure equations of the conformal group and possessing local Weyl symmetry ($SO(n)$ and scale). Those structure equations, now expanded in terms of the basis forms, can be generalized to include curvatures for each of the connection types. These break out into four types.

1. The curvature of the spin connection, which is referred to as the $SO(4)$ curvature.

2. The curvature of only the antisymmetric part of the spin connection, referred to as the Riemannian curvature.

3. The curvature of the Weyl connection, referred to here as the dilatational curvature.

4. The curvature of the $\mathbf{e}^a$ basis forms is referred to as the torsion.

5. The curvature of the $\mathbf{f}_a$ basis forms is referred to as the cotorsion.

Since the scale vector is tensorial, it is useful to substitute out the Weyl vector wherever possible. Therefore, we will work with the following version of the dilatational structure equation.

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega} &= \mathbf{d}\left(\boldsymbol{\omega}_{(x)} + \boldsymbol{\omega}_{(y)}\right) \\
&= \mathbf{d}\left(\mathbf{v} + \mathbf{d}_{(x)}\phi + \mathbf{u} + \mathbf{d}_{(y)}\phi\right) \\
\mathbf{dv} + \mathbf{du} &= \mathbf{e}^a\mathbf{f}_a + \boldsymbol{\Omega}, \qquad (4.1)
\end{aligned}$$

where ϕ is the scale factor on the metric, as in Section 3.1.4. This will prove especially useful in the calculations of the Bianchi identities we undertake in Section 4.2.1. Following the same conventions and notation of Chapter 3, the structure equations written out in the orthonormal basis, with the scale vector and symmetric spin connection written out explicitly are

$$\begin{aligned}
\mathbf{d}\boldsymbol{\tau}^a_{\ b} &= \boldsymbol{\tau}^c_{\ b}\boldsymbol{\tau}^a_{\ c} + \Delta^{ah}_{gb}\eta_{hj}\mathbf{e}^j\mathbf{e}^g - \Delta^{ah}_{gb}\eta^{gi}\mathbf{f}_h\mathbf{f}_i + 2\Delta^{ah}_{gb}\Xi^{gi}_{jh}\mathbf{f}_i\mathbf{e}^j + \boldsymbol{\Omega}^a_{\ b}, \\
\mathbf{de}^a &= \mathbf{e}^b\boldsymbol{\sigma}^a_{\ b} + \mathbf{d}_{(x)}\phi\mathbf{e}^a + \frac{1}{2}T^a_{bc}\mathbf{e}^b\mathbf{e}^c \\
&\quad -\boldsymbol{\gamma}^a_{\ c}\mathbf{e}^c + \mathbf{d}_{(y)}\phi\mathbf{e}^a + \eta^{ad}\boldsymbol{\mu}^b_{\ d}\mathbf{f}_b - \eta^{ab}\mathbf{v}\mathbf{f}_b + T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c \\
&\quad +\eta^{ad}\boldsymbol{\rho}^b_{\ d}\mathbf{f}_b - \eta^{ac}\mathbf{u}\mathbf{f}_c + \frac{1}{2}T^{abc}\mathbf{f}_b\mathbf{f}_c, \\
\mathbf{df}_a &= \boldsymbol{\gamma}^b_{\ a}\mathbf{f}_b - \mathbf{d}_{(y)}\phi\mathbf{f}_a + \frac{1}{2}S_a^{\ bc}\mathbf{f}_b\mathbf{f}_c \\
&\quad +\boldsymbol{\sigma}^b_{\ a}\mathbf{f}_b - \mathbf{d}_{(x)}\phi\mathbf{f}_a + \eta_{ae}\boldsymbol{\rho}^e_{\ b}\mathbf{e}^b - \eta_{ab}\mathbf{u}\mathbf{e}^b + S_a^{\ \ b}_{\ \ c}\mathbf{f}_b\mathbf{e}^c \\
&\quad -\mathbf{v}\mathbf{e}^a + \boldsymbol{\mu}^a_{\ b}\mathbf{e}^b + \frac{1}{2}\eta^{ab}S_{bcd}\mathbf{e}^c\mathbf{e}^d, \\
\mathbf{dv} + \mathbf{du} &= \mathbf{e}^a\mathbf{f}_a + \frac{1}{2}\Omega_{ab}\mathbf{e}^a\mathbf{e}^b + \Omega^a_{\ b}\mathbf{f}_a\mathbf{e}^b + \frac{1}{2}\Omega^{ab}\mathbf{f}_a\mathbf{f}_b.
\end{aligned}$$

Here, we have only expanded the Cartan curvatures[1] explictly. To see the equations expanded fully into the configuration ($\mathbf{e}^a\mathbf{e}^b$), momentum ($\mathbf{f}_a\mathbf{f}_b$), and mixed ($\mathbf{f}_a\mathbf{e}^b$) bases, see Appendix A. Since there is no index on the scale vector one-form the covariant derivative is equivalent to the exterior derivative. The relationship between the two is highlighted by equation 4.1. Notice the combinations $\delta^a_b\mathbf{v} - \boldsymbol{\mu}^a_{\ b}$ and $\delta^a_b\mathbf{u} - \boldsymbol{\rho}^a_{\ b}$ appear throughout the basis structure equations. These terms are equivalent to the full Weyl-covariant derivative of a conformally orthonormal metric (with both the symmetric and antisymmetric parts of the spin connection).

$$\begin{aligned}\mathbf{D}\eta^{ab} &\equiv \mathbf{d}\eta^{ab} + \eta^{cb}\boldsymbol{\tau}^a_{\ c} + \eta^{ac}\boldsymbol{\tau}^b_{\ c} - 2\boldsymbol{\omega}\eta^{ae} = \mathbf{d}\eta^{ab} + 2\eta^{cb}\boldsymbol{\beta}^a_{\ c} - 2\boldsymbol{\omega}\eta^{ab} \\ &= 2\left(\eta^{cb}\boldsymbol{\beta}^a_{\ c} - \mathbf{v}\eta^{ab}\right). \end{aligned} \tag{4.2}$$

Note, the covariant derivative that we will use in the remainder of this manuscript, appearing with an (x) or (y), is the scale covariant derivative defined by the basis ($\mathbf{de}^a$ and $\mathbf{df}_a$) equations

$$\begin{aligned}\mathbf{D}^{(x)}\mathbf{e}^a &= \mathbf{d}^{(x)}\mathbf{e}^a - \mathbf{e}^b\boldsymbol{\sigma}^a_{\ b} - \mathbf{d}_{(x)}\phi\mathbf{e}^a = \frac{1}{2}T^a_{\ bc}\mathbf{e}^b\mathbf{e}^c, \\ \mathbf{D}^{(y)}\mathbf{f}_a &= \mathbf{d}^{(y)}\mathbf{f}_a - \boldsymbol{\gamma}^b_{\ a}\mathbf{f}_b + \mathbf{d}_{(y)}\phi\mathbf{f}_a = \frac{1}{2}S_a^{\ bc}\mathbf{f}_b\mathbf{f}_c.\end{aligned}$$

4.2.1. Bianchi identities

The integrability conditions on the structure equations with curvature, referred to here as the Bianchi identities, will prove useful in the following calculations. These conditions are generated by enforcing the integrability of the structure equations using the Poincaré lemma, $\mathbf{d}^2 = 0$. In the generic Cartan formalism, these integrability conditions relate the Cartan curvatures and their covariant derivatives. In Section 4.5, we set a number of conditions on the Bianchi identities, while the field equations will give us other relations.

Submanifold basis Bianchi identities

In Chapter 3, the Bianchi identities on the submanifolds have been calculated in the case of no torsion (the flat case). Here they would simply generalize to include the submanifold torsion

[1] By Cartan curvature we mean all four types of curvature, $\boldsymbol{\Omega}^a_{\ b}$, $\mathbf{T}^a$, $\mathbf{S}_a$, and $\boldsymbol{\Omega}$.

and cotorsion

$$\begin{aligned}\frac{1}{2}R^{a}{}_{bcd}\mathbf{e}^{b}\mathbf{e}^{c}\mathbf{e}^{d} &= \mathbf{D}_{(x)}\left(\frac{1}{2}T^{a}{}_{bc}\mathbf{e}^{b}\mathbf{e}^{c}\right),\\ \frac{1}{2}R^{b\,cd}{}_{a}\mathbf{f}_{b}\mathbf{f}_{c}\mathbf{f}_{d} &= -\mathbf{D}_{(y)}\left(\frac{1}{2}S_{a}{}^{bc}\mathbf{f}_{b}\mathbf{f}_{c}\right).\end{aligned}$$

Without the submanifold torsion and cotorsion they lead to the familiar first Bianchi identity of Riemannian geometry that tells us the triply antisymmetrized Riemann tensor is zero, and the Ricci tensor is symmetric. Note, the curvature appearing in these equations is not the Cartan curvature of the spin connection, $\mathbf{\Omega}^{a}_{\ b}$, appearing in the spin connection structure equation. It turns out to be the case in all the Bianchi identities, except the integrability condition of the spin connection, not included here, that $\mathbf{R}^{a}_{\ b}$ appears naturally.

Involution Bianchi identities

The involution condition sets the momentum part of the exterior derivative of the solder form and configuration part of the exterior derivative of the cosolder form to zero, $\mathbf{de}^{a}|_{\mathbf{ff}} = 0$ and $\mathbf{df}_{a}|_{\mathbf{ee}} = 0$, see equations (A.7) and (A.15) in Appendix A. These equations relate the scale vector to the symmetric part of the spin connection. When we look at the integrability conditions

$$\begin{aligned}0 &= \mathbf{D}_{(x)}\left(\boldsymbol{\mu}^{a}_{\ b} - \delta^{a}_{b}\mathbf{v}\right)\mathbf{e}^{b} + \left(\delta^{a}_{e}\mathbf{v} - \boldsymbol{\mu}^{a}_{\ e}\right)\frac{1}{2}T^{e}_{\ bc}\mathbf{e}^{b}\mathbf{e}^{c} + \mathbf{D}_{(x)}\left(\frac{1}{2}\eta^{ab}S_{bcd}\mathbf{e}^{c}\mathbf{e}^{d}\right),\\ 0 &= \mathbf{D}_{(y)}\left(\boldsymbol{\rho}^{b}_{\ a} - \delta^{b}_{a}\mathbf{u}\right)\mathbf{f}_{b} + \left(\delta^{d}_{a}\mathbf{u} - \boldsymbol{\rho}^{d}_{\ a}\right)\frac{1}{2}S_{d}{}^{bc}\mathbf{f}_{b}\mathbf{f}_{c} + \mathbf{D}_{(y)}\left(\frac{1}{2}\eta_{ad}T^{dbc}\mathbf{f}_{b}\mathbf{f}_{c}\right),\end{aligned}$$

we see they involve the covariant derivative of the symmetric spin connection and the submanifold dilatational curvature, $\mathbf{D}_{(x)}\mathbf{v} = \mathbf{\Omega}_{(x)}$. These will later be combined with the field equations to show the symmetric spin connection sources the dilatational curvature.

Mixed basis Bianchi identities

The full expressions are

$$R^{a\ c}_{\ b\ d}\mathbf{f}_{c}\mathbf{e}^{d}\mathbf{e}^{b} = \eta^{ad}\mathbf{D}_{(x)}\left(\boldsymbol{\mu}^{b}_{\ d} - \delta^{b}_{d}\mathbf{v}\right)\mathbf{f}_{b}$$

$$
\begin{aligned}
& + \left(\delta^a_d \mathbf{v} - \boldsymbol{\mu}^a{}_d\right)\left(\boldsymbol{\rho}^d{}_b - \delta^d_b \mathbf{u}\right)\mathbf{e}^b + \left(\delta^e_d \mathbf{v} - \boldsymbol{\mu}^e{}_d\right)\eta^{ad} S_e{}^b{}_c \mathbf{f}_b \mathbf{e}^c \\
& + \mathbf{D}_{(x)}\left(T^{ab}{}_c \mathbf{f}_b \mathbf{e}^c\right) + \mathbf{D}_{(y)}\left(\frac{1}{2} T^a_{bc} \mathbf{e}^b \mathbf{e}^c\right), \\
-R^{b\,c}_{a\,d} \mathbf{f}_b \mathbf{f}_c \mathbf{e}^d &= \eta_{ae} \mathbf{D}_{(y)}\left(\boldsymbol{\rho}^e{}_b - \delta^e_b \mathbf{u}\right)\mathbf{e}^b \\
& + \left(\boldsymbol{\rho}^d{}_a - \delta^d_a \mathbf{u}\right)\left(\delta^b_d \mathbf{v} - \boldsymbol{\mu}^b{}_d\right)\mathbf{f}_b + \left(\eta_{ae}\mathbf{u} - \eta_{af}\boldsymbol{\rho}^f{}_e\right) T^{eb}{}_c \mathbf{f}_b \mathbf{e}^c \\
& + \mathbf{D}_{(x)}\left(\frac{1}{2} S_a{}^{bc} \mathbf{f}_b \mathbf{f}_c\right) + \mathbf{D}_{(y)}\left(S_a{}^b{}_c \mathbf{f}_b \mathbf{e}^c\right), \\
\frac{1}{2} R^{a\ de}_{\ c} \mathbf{f}_d \mathbf{f}_e \mathbf{e}^c &= \eta^{ad} \mathbf{D}_{(y)}\left(\boldsymbol{\mu}^b{}_d - \delta^b_d \mathbf{v}\right)\mathbf{f}_b + \eta^{ad} \mathbf{D}_{(\sigma)}\left(\boldsymbol{\rho}^b{}_d - \delta^b_d \mathbf{u}\right)\mathbf{f}_b \\
& - \left(\delta^a_c \mathbf{u} - \boldsymbol{\rho}^a{}_c\right)\left(\delta^c_b \mathbf{u} - \boldsymbol{\rho}^c{}_b\right)\mathbf{e}^b \\
& + \left(\eta^{ae}\mathbf{v} - \eta^{ad}\boldsymbol{\mu}^e{}_d\right)\frac{1}{2} S_e{}^{bc} \mathbf{f}_b \mathbf{f}_c + \left(\eta^{ae}\mathbf{u} - \eta^{ad}\boldsymbol{\rho}^e{}_d\right) S_e{}^b{}_c \mathbf{f}_b \mathbf{e}^c \\
& + \mathbf{D}_{(x)}\left(\frac{1}{2} T^{abc} \mathbf{f}_b \mathbf{f}_c\right) + \mathbf{D}_{(y)}\left(T^{ab}{}_c \mathbf{f}_b \mathbf{e}^c\right), \\
-\frac{1}{2} R^b_{acd} \mathbf{f}_b \mathbf{e}^c \mathbf{e}^d &= \eta_{ac} \mathbf{D}_{(y)}\left(\boldsymbol{\mu}^c{}_b - \delta^c_b \mathbf{v}\right)\mathbf{e}^b + \eta_{ae} \mathbf{D}_{(x)}\left(\boldsymbol{\rho}^e{}_b - \delta^e_b \mathbf{u}\right)\mathbf{e}^b \\
& - \left(\delta^c_a \mathbf{v} - \boldsymbol{\mu}^c{}_a\right)\left(\delta^b_c \mathbf{v} - \boldsymbol{\mu}^b{}_c\right)\mathbf{f}_b \\
& + \left(\eta_{af}\mathbf{u} - \eta_{ae}\boldsymbol{\rho}^e{}_f\right)\left(\frac{1}{2} T^f_{bc} \mathbf{e}^b \mathbf{e}^c\right) + \left(\eta_{ae}\mathbf{v} - \eta_{ac}\boldsymbol{\mu}^c{}_e\right) T^{eb}{}_c \mathbf{f}_b \mathbf{e}^c \\
& + \mathbf{D}_{(x)}\left(S_a{}^b{}_c \mathbf{f}_b \mathbf{e}^c\right) + \mathbf{D}_{(y)}\left(\frac{1}{2} \eta^{ab} S_{bcd} \mathbf{e}^c \mathbf{e}^d\right).
\end{aligned}
$$

In our example solutions to follow, the last two identities will prove very useful for simplification of the geometry.

The integrability conditions of the mixed basis structure equations lead to relationships between the mixed torsion and the momentum Riemannian curvature and the mixed cotorsion and the configuration Riemannian curvature. Since these appear as either $\mathbf{e}^a \mathbf{e}^b \mathbf{f}_c$ or $\mathbf{e}^a \mathbf{f}_b \mathbf{f}_c$ we are able to strip off the basis forms without having to triply antisymmetrize the curvature. This gives stronger relationships between the curvatures than a "normal" Bianchi identity would. For example,

$$
-\frac{1}{2} R^b_{acd} = \eta_{ae} D^b_{(y)}\left(\mu^e_{[dc]} - \delta^e_{[d} v_{c]}\right) + \cdots .
$$

Dilatation Bianchi identities

On the submanifolds the Bianchi identities for the dilatational curvature

$$\begin{aligned} 0 &= \mathbf{d}_{(x)}\left(\frac{1}{2}\Omega_{ab}\mathbf{e}^a\mathbf{e}^b\right), \\ 0 &= \mathbf{d}_{(y)}\left(\frac{1}{2}\Omega^{ab}\mathbf{f}_a\mathbf{f}_b\right), \end{aligned}$$

show the submanifold dilatational curvatures are closed. In the mixed basis the Bianchi identities

$$\begin{aligned} T^{ab}{}_c\mathbf{e}^c\mathbf{f}_a\mathbf{f}_b &= \mathbf{d}_{(y)}\left(\Omega^a{}_b\mathbf{f}_a\mathbf{e}^b\right) + \mathbf{d}_{(x)}\left(\frac{1}{2}\Omega^{ab}\mathbf{f}_a\mathbf{f}_b\right), \\ -S_a{}^b{}_c\mathbf{f}_b\mathbf{e}^a\mathbf{e}^c &= \mathbf{d}_{(x)}\left(\Omega^a{}_b\mathbf{f}_a\mathbf{e}^b\right) + \mathbf{d}_{(y)}\left(\frac{1}{2}\Omega_{ab}\mathbf{e}^a\mathbf{e}^b\right), \end{aligned}$$

show the x and y derivatives of the dilatational curvatures are related to the anti-symmetric part of the mixed cotorsion and mixed torsion.

Closure of dilatational curvature and symplectic form

Since the dilatational curvature is closed on the submanifolds the full dilatational structure equation generically defines a symplectic form over the full biconformal space. Since $\mathbf{d}^2\mathbf{v} = 0$, the RHS of the Weyl connection structure equation is closed, while $\mathbf{e}^a\mathbf{f}_a$ spans the cotangent space of biconformal space. We can interpret the RHS of $\mathbf{dv} = \mathbf{e}^a\mathbf{f}_a + \mathbf{\Omega} = \boldsymbol{\chi}$ as a symplectic form except for the special case when the mixed dilatational curvature is such that $\boldsymbol{\chi}$ is degenerate. When the dilatational curvature is closed, $\mathbf{d\Omega} = 0$, over the whole manifold, we are free to interpret the two-form $\mathbf{e}^a\mathbf{f}_a$ as the symplectic form. We will use the appearance of this condition later to find Darboux coordinates for the manifold.

4.2.2. Field equations

Until this point, we have been exploring only the geometry of curved biconformal space. The gravitational theory we consider in this chapter is based on the Wehner–Wheeler action in [50]. There, it was shown one can write an action that is linear in the curvatures in biconformal

space. The action, in terms of curvature two-forms, is

$$S = \int \left(\alpha \mathbf{\Omega}^a_b + \beta \delta^a_b \mathbf{\Omega} + \gamma \mathbf{e}^a \mathbf{f}_b\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f, \tag{4.3}$$

where $\mathbf{\Omega}^a_b$ is the curvature associated with $SO(n)$ transformations and $\mathbf{\Omega}$ is the dilatational curvature. The torsion, $\mathbf{T}^a$, and cotorsion, $\mathbf{S}_a$ cannot appear in a linear, conformally invariant action, but do appear in the field equations. The classical extrema of the action are then found by doing a Palatini (first order) variation of all the connection forms. Here we survey the various field equations we obtain from the Palatini variation. See Appendix F for explicit details of the variation of the Wehner-Wheeler action.

Dilatation and $SO(4)$ curvature field equations

There are four field equations relating traces of the Cartan curvature of the spin connection, $\mathbf{\Omega}^a_b$, the dilatational curvature, $\mathbf{\Omega}$, and the metric. These come from varying the basis forms $\mathbf{e}^a$ and $\mathbf{f}_a$.

$$\begin{aligned}
0 &= \alpha \Omega^a_{bac} + \beta \Omega_{bc} - \alpha \frac{1}{2}\left(\left(n-2\right)\eta_{bc} + \delta_{bc}\left(\delta^{ad}\eta_{da}\right)\right), && (4.4)\\
0 &= \alpha \Omega^{a\ b}_{b\ c} - \alpha \delta^a_c \Omega^{d\ b}_{b\ d} + \beta \Omega^a_{\ c} - \beta \delta^a_c \Omega^b_{\ b} + \delta^a_c \left(\alpha \frac{1}{2} n - \beta + \gamma n^2\right) \\
&\quad + \alpha \frac{1}{2} \delta^{ad} \eta_{dc} \left(\delta_{bh}\eta^{hb}\right) - \alpha \eta^{ah} \delta_{hc} \left(\delta^{dg}\eta_{gd}\right), && (4.5)\\
0 &= \alpha \Omega^{c\ a}_{b\ c} - \alpha \delta^a_b \Omega^{c\ d}_{d\ c} + \beta \Omega^a_{\ b} - \beta \delta^a_b \Omega^c_{\ c} + \delta^a_b \left(\alpha \frac{1}{2} n - \beta + \gamma n^2\right) \\
&\quad + \alpha \frac{1}{2} \eta^{ad} \delta_{db} \left(\delta^{cg}\eta_{gc}\right) - \alpha \delta^{ag} \eta_{gb} \left(\delta_{ch}\eta^{hc}\right), && (4.6)\\
0 &= \alpha \Omega^{a\ cb}_{\ c} + \beta \Omega^{ab} - \alpha \frac{1}{2}\left(\left(n-2\right)\eta^{ab} + \delta^{ab}\left(\delta_{dc}\eta^{dc}\right)\right). && (4.7)
\end{aligned}$$

There are four because $\delta \mathbf{e}^a \sim A^a_{\ b} \mathbf{e}^b + B^{ab} \mathbf{f}_b$ and $\delta \mathbf{f}_a \sim C_{ab} \mathbf{e}^b + D_a^{\ b} \mathbf{f}_b$. There is one each for the submanifolds and two for the mixed basis. It has been shown these field equations, in the original basis, lead to general relativity on the cotangent bundle of spacetime [50]. In that case, combining these submanifold field equations with the basis equation integrability conditions leads to the vanishing of the dilatational curvature generically and the Einstein field equation

on the submanifolds. We will show the same result through a different route, using a simple linear ansatz for the symmetric spin connection.

Torsion/ cotorsion field equations

There are two sets of field equations that relate the torsion, cotorsion, symmetric spin connection and the scale vector. One set comes from the variation of the Weyl connection

$$\begin{aligned} 0 &= \beta\left(T^{ba}{}_{b} - T^{ab}{}_{b} + S_{b}^{\ ab}\right), \\ 0 &= \beta\left(T^{b}_{ab} + S_{a}^{\ b}{}_{\ b} - S_{b}^{\ b}{}_{\ a}\right), \end{aligned}$$

and the other set comes from the variation of the spin connection,

$$\begin{aligned} 0 &= \Delta^{ap}_{qb}\left(T^{cb}{}_{a} - \delta^{c}_{a}T^{eb}{}_{e} - \delta^{c}_{a}S_{e}^{\ be}\right) \\ &\quad +\Delta^{ap}_{qb}\left(-\frac{1}{2}\partial_{a}\eta^{cb} + \frac{1}{2}\delta^{c}_{a}\partial_{f}\eta^{bf} - \eta^{cd}\mu^{b}_{\ da} + \delta^{c}_{a}\eta^{bd}\mu^{f}_{\ df} + W_{a}\eta^{cb} - \delta^{c}_{a}\eta^{bf}W_{f}\right) \\ &\quad +\Delta^{ap}_{qb}\left(-\frac{1}{2}\eta^{cd}\partial^{b}\eta_{ad} + \frac{1}{2}\delta^{c}_{a}\eta^{bd}\partial^{e}\eta_{ed} + \rho^{c\,b}_{\ a} - \delta^{c}_{a}\rho^{b\,e}_{\ e}\right), \\ 0 &= \Delta^{ap}_{qb}\left(S_{c\,a}^{\ b} - \delta^{b}_{c}S_{e\,a}^{\ e} + \delta^{b}_{c}T^{e}_{\ ae}\right) \\ &\quad +\Delta^{ap}_{qb}\left(-\frac{1}{2}\partial^{b}\eta_{ac} + \delta^{b}_{c}\frac{1}{2}\partial^{e}\eta_{ea} + \eta_{ec}\rho^{e\,b}_{\ a} - \delta^{b}_{c}\eta_{da}\rho^{d\,e}_{\ e} - \eta_{ac}W^{b} + \delta^{b}_{c}\eta_{ea}W^{e}\right) \\ &\quad +\Delta^{ap}_{qb}\left(\frac{1}{2}\eta_{cd}\partial_{a}\eta^{bd} - \frac{1}{2}\delta^{b}_{c}\eta_{ad}\partial_{f}\eta^{df} + \eta_{cd}\eta^{de}\mu^{b}_{\ ea} - \delta^{b}_{c}\eta_{ad}\eta^{de}\mu^{f}_{\ ef}\right). \end{aligned}$$

Note, the torsion and cotorsion from the involution conditions, T^{abc} and S_{abc}, do not appear in the field equations and are therefore, in general, undetermined by them. The symmetric part of the spin connection and the scale vector can, and do, appear in the field equations precisely because they transform tensorially. We immediately combine the trace equations with the trivalent equations to give the following set of field equations.

$$\begin{aligned} 0 &= \Delta^{ap}_{qb}\left(T^{cb}{}_{a} - \delta^{c}_{a}T^{be}{}_{e}\right) + \Delta^{ap}_{qb}\left(\rho^{c\,b}_{\ a} - \delta^{c}_{a}\rho^{b\,e}_{\ e} - \eta^{cd}\mu^{b}_{\ da} + \delta^{c}_{a}\eta^{bd}\mu^{f}_{\ df}\right) \\ &\quad +\Delta^{ap}_{qb}\left(\eta^{cb}v_{a} - \delta^{c}_{a}\eta^{bf}v_{f}\right), \\ 0 &= \Delta^{ap}_{qb}\left(S_{c\,a}^{\ b} - \delta^{b}_{c}S_{a\,e}^{\ e}\right) + \Delta^{ap}_{qb}\left(\mu^{b}_{\ ca} - \delta^{b}_{c}\mu^{f}_{\ af} + \eta_{ec}\rho^{e\,b}_{\ a} - \delta^{b}_{c}\eta_{da}\rho^{d\,e}_{\ e}\right) \end{aligned} \tag{4.8}$$

$$+\Delta^{ap}_{qb}\left(\delta^b_c\eta_{ea}u^e-\eta_{ac}u^b\right). \tag{4.9}$$

We have also replaced the partial derivatives of the metric and the Weyl vector with the scale vector as defined above and in Chapter 3. We preemptively write the equations with three terms.

These field equations will be shown to restrict the form of the metric in the example solutions worked out in this chapter. In that example, the curvature/dilatational curvature field equations (4.4 - 4.7) give us the vanishing of the dilatational curvature generically and the vacuum Einstein field equations, which are exactly the results we would hope for. The balance between the conditions set by these torsion field equations with the ansatz for the form of the symmetric spin connection gives us the remaining restrictions on the solution.

4.3. Combining the field equations and structure equations

Once we define the curvature of the anti-symmetric part of the spin connection, $\mathbf{R}^a_{\ b}$, as in equation (3.37), we can write the $SO(n)$ structure equation in three parts

$$\begin{aligned}
\frac{1}{2}\Omega^a_{\ bcd}\mathbf{e}^c\mathbf{e}^d &= \frac{1}{2}R^a_{\ bcd}\mathbf{e}^c\mathbf{e}^d+\mathbf{D}_{(x)}\boldsymbol{\mu}^a_{\ b}-\boldsymbol{\mu}^e_{\ b}\boldsymbol{\mu}^a_{\ e}-\Delta^{ah}_{db}\eta_{hc}\mathbf{e}^c\mathbf{e}^d,\\
\Omega^{a\ c}_{\ b\ d}\mathbf{f}_c\mathbf{e}^d &= R^{a\ c}_{\ b\ d}\mathbf{f}_c\mathbf{e}^d+\mathbf{D}_{(y)}\boldsymbol{\mu}^a_{\ b}+\mathbf{D}^{(x)}\boldsymbol{\rho}^a_{\ b}-\boldsymbol{\mu}^e_{\ b}\boldsymbol{\rho}^a_{\ e}-\boldsymbol{\rho}^e_{\ b}\boldsymbol{\mu}^a_{\ e}-2\Delta^{ah}_{gb}\Xi^{gi}_{jh}\mathbf{f}_i\mathbf{e}^j,\\
\frac{1}{2}\Omega^{a\ cd}_{\ b}\mathbf{f}_c\mathbf{f}_d &= \frac{1}{2}R^{a\ cd}_{\ b}\mathbf{f}_c\mathbf{f}_d+\mathbf{D}_{(y)}\boldsymbol{\rho}^a_{\ b}-\boldsymbol{\rho}^e_{\ b}\boldsymbol{\rho}^a_{\ e}+\Delta^{ac}_{eb}\eta^{ed}\mathbf{f}_c\mathbf{f}_d,
\end{aligned}$$

where the covariant derivatives can be written equivalently as Lorentz covariant or Weyl covariant derivatives, since the conformal weight of $\boldsymbol{\mu}^a_{\ b}$ and $\boldsymbol{\rho}^a_{\ b}$ are zero. When written in this form, it is easy to see there is a relationship between the natural, overall curvature of biconformal space, $\boldsymbol{\Omega}^a_{\ b}$ and the naturally defined Riemannian curvature on the submanifolds, $\mathbf{R}^a_{\ b}$.

4.3.1. Configuration submanifold

When we combine the submanifold field equations (4.4 - 4.7) with the spin connection structure equation, a number of simplifications occur. First, it replaces the trace of the $SO(n)$

curvature of the conformal group,

$$\Omega^{f}_{\ bfd} \;=\; R^{f}_{\ bfd} + D^{(x)}_{f}\mu^{f}_{\ bd} - D^{(x)}_{d}\mu^{f}_{\ bf} - \mu^{e}_{\ bf}\mu^{f}_{\ ed} + \mu^{e}_{\ bd}\mu^{f}_{\ ef} + \frac{1}{2}\left(n-2\right)\eta_{bd} + \frac{1}{2}\delta_{db}\left(\eta_{ef}\delta^{fe}\right),$$

with the Ricci curvature defined from the antisymmetric spin connection.

$$0 \;=\; R^{f}_{\ bfd} + \frac{\beta}{\alpha}\Omega_{bd} + D^{(x)}_{f}\mu^{f}_{\ bd} - D^{(x)}_{d}\mu^{f}_{\ bf} - \mu^{e}_{\ bf}\mu^{f}_{\ ed} + \mu^{e}_{\ bd}\mu^{f}_{\ ef}.$$

Notice the metric terms also cancel. We are left with a relationship between the Ricci curvature, the dilatational curvature and the symmetric spin connection. Since the Ricci curvature is symmetric by the Bianchi identity and the dilatational curvature is antisymmetric by definition the equation separates into two independent parts, which relate these curvatures to the symmetric spin connection

$$\begin{aligned} R^{f}_{\ bfd} &= D^{(x)}_{(d}\mu^{f}_{\ b)f} - D^{(x)}_{f}\mu^{f}_{\ (bd)} + \mu^{f}_{\ e(d}\mu^{e}_{\ b)f} - \mu^{e}_{\ (bd)}\mu^{f}_{\ ef}, \\ \frac{\beta}{\alpha}\Omega_{bd} &= D^{(x)}_{[d}\mu^{f}_{\ b]f} - D^{(x)}_{f}\mu^{f}_{\ [bd]} + \mu^{f}_{\ e[d}\mu^{e}_{\ b]f} - \mu^{e}_{\ [bd]}\mu^{f}_{\ ef}. \end{aligned}$$

As usual, the parentheses and square brackets on the indices mean the parts symmetric and antisymmetric, respectively, on those indices, $A_{(ab)} \equiv \frac{1}{2}\left(A_{ab} + A_{ba}\right)$ and $A_{[ab]} \equiv \frac{1}{2}\left(A_{ab} - A_{ba}\right)$.

4.3.2. Momentum submanifold

In the momentum sector we get an analogous expression by combining the field equations and the structure equations. Again it relates the curvature of the anti-symmetric spin connection, the dilatational curvature and the symmetric spin connection.

$$0 \;=\; R^{a\ fd}_{\ f} + \frac{\beta}{\alpha}\Omega^{ab} + D^{f}_{(y)}\rho^{a\ d}_{\ f} - D^{d}_{(y)}\rho^{a\ f}_{\ f} - \rho^{e\ f}_{\ f}\rho^{a\ d}_{\ e} + \rho^{e\ d}_{\ f}\rho^{a\ f}_{\ e}.$$

This also decomposes into a symmetric and antisymmetric part.

$$R^{a\ fd}_{\ f} \;=\; D^{(d}_{(y)}\rho^{a)\ f}_{\ f} - D^{f}_{(y)}\rho^{(a\ d)}_{\ f} + \rho^{e\ f}_{\ f}\rho^{(a\ d)}_{\ e} - \rho^{e\ (d}_{\ f}\rho^{a)\ f}_{\ e},$$

$$\frac{\beta}{\alpha}\Omega^{ab} \;=\; D^{[d}_{(y)}\rho^{a]\,f}_{\;\;f} - D^{f}_{(y)}\rho^{[a\,d]}_{\;\;f} + \rho^{e\,f}_{\;\;f}\,\rho^{[a\,d]}_{\;\;e} - \rho^{e\,[d}_{\;\;f}\rho^{a]\,f}_{\;\;e}.$$

4.3.3. Mixed-basis equations

The mixed-basis field equations can similarly be used to substitute out the Riemannian curvature in the mixed basis spin connection structure equations. We refrain from including them here until we set some simplifying conditions.

Note the combination of the field equations and structure equations has completely decoupled $\boldsymbol{\Omega}^a_{\;b}$ from the Riemannian curvature naturally defined on the submanifolds. This will prove important both from a physical standpoint and because this decoupling allows for the time metric to be derived from the form of $\boldsymbol{\Omega}^a_{\;b}$ even when $R^{a\;cd}_{\;\;b} = 0$.

4.4. Vector ansatz

Here we examine a solution of biconformal gravity with the aim of recovering general relativity. We motivate the use of a vector ansatz for the symmetric spin connection by looking at the involution conditions in the homogeneous space.

It can be shown a spin connection of the form

$$\boldsymbol{\mu}^a_{\;b} \;=\; A\delta^a_b v_c \mathbf{e}^c + (A-1)\left(\delta^a_c v_b + \eta_{bc}\eta^{ad}v_d\right)\mathbf{e}^c + B\eta^{ad}v_d v_b v_c \mathbf{e}^c, \tag{4.10}$$

$$\boldsymbol{\rho}^a_{\;b} \;=\; C\delta^a_b u^c \mathbf{f}_c + (C-1)\left(\eta^{ac}\eta_{bd}u^d + \delta^c_b u^a\right)\mathbf{f}_c + Du^a\eta_{bd}u^d u^c \mathbf{f}_c \tag{4.11}$$

has the same symmetries as $\boldsymbol{\beta}^a_{\;b}$ and satisfies the involution conditions of homogeneous biconformal space. The form of $\boldsymbol{\mu}^a_{\;b}$ and $\boldsymbol{\rho}^a_{\;b}$ in Chapter 3 are special cases of this general vector form where $A = -1$, $B = -4$, $C = 0$, and $D = -2$. By satisfying the involution conditions in the homogeneous case, an ansatz of this form sets the torsion and cotorsion that appear in the involution conditions of the curved case identically to zero. Here the form of the Bianchi identities illustrates the relationship between $\boldsymbol{\beta}^a_{\;b}$ and $\mathbf{v}$ can be used to greatly simplify the geometry. In this chapter, motivated by the Bianchi identities and the goal of regaining general relativity, we

will look in detail at the consequences of choosing the simplest linear vector ansatz,

$$\boldsymbol{\mu}^a_{\ b} = \delta^a_b v_c \mathbf{e}^c, \tag{4.12}$$

$$\boldsymbol{\rho}^a_{\ b} = \delta^a_b u^c \mathbf{f}_c, \tag{4.13}$$

which amounts to setting $A = B = 1$ and $C = D = 0$. From the point of view of finding a solution, this is a straightforward choice given the cascade of simplifications it gives in the Bianchi identities, structure equations and field equations. Another reason, even more geometrically motivated, is that equations (4.12-4.13) are equivalent to setting the τ-covariant derivative of the orthonormal metric to zero. Choosing the above ansatz is equivalent to ensuring the entire connection, $\boldsymbol{\tau}^a_{\ b}$, is metric compatible. This greatly simplifies the mixed torsion and cotorsion field equations, (4.8-4.9), canceling the symmetric spin connection terms with the scale vector terms,[2] and leaving

$$\begin{aligned} 0 &= \Delta^{ap}_{qb}\left(T^{cb}{}_{a} - \delta^c_a T^{be}{}_{e}\right), \\ 0 &= \Delta^{ap}_{qb}\left(S_c{}^b{}_a - \delta^b_c S_a{}^e{}_e\right). \end{aligned}$$

4.4.1. Spin connection structure equations

Possibly the most important effect of this ansatz is to greatly simplify the spin connection structure equation. This ansatz cancels the $\boldsymbol{\beta}^c_{\ b}\boldsymbol{\beta}^a_{\ c}$ terms. We can also rewrite the derivative term as the dilatational curvature so the three parts of the Lorentz structure equation now look like

$$\begin{aligned} \frac{1}{2}\Omega^a_{\ bcd}\mathbf{e}^c\mathbf{e}^d &= \frac{1}{2}R^a_{\ bcd}\mathbf{e}^c\mathbf{e}^d + \frac{1}{2}\delta^a_b\Omega_{cd}\mathbf{e}^c\mathbf{e}^d - \Delta^{ah}_{db}\eta_{hc}\mathbf{e}^c\mathbf{e}^d, \\ \Omega^{a\ c}_{\ b\ d}\mathbf{f}_c\mathbf{e}^d &= R^{a\ c}_{\ b\ d}\mathbf{f}_c\mathbf{e}^d + \delta^a_b\left(\Omega^c_{\ d}\mathbf{f}_c\mathbf{e}^d - \delta^c_d\mathbf{f}_c\mathbf{e}^d\right) - 2\Delta^{ah}_{gb}\Xi^{gi}_{jh}\mathbf{f}_i\mathbf{e}^j, \\ \frac{1}{2}\Omega^{a\ cd}_{\ b}\mathbf{f}_c\mathbf{f}_d &= \frac{1}{2}R^{a\ cd}_{\ b}\mathbf{f}_c\mathbf{f}_d + \frac{1}{2}\delta^a_b\Omega^{cd}\mathbf{f}_c\mathbf{f}_d + \Delta^{ac}_{eb}\eta^{ed}\mathbf{f}_c\mathbf{f}_d. \end{aligned}$$

[2] Note, if the original Euclidean metric was the effective metric of this space, then these equations would look very similar to the field equation one gets from Einstein-Cartan theory.

When the field equations are folded in with the traces of these equations, we are left with the following four relationships on the submanifolds.

$$\begin{aligned} R^{f}{}_{bfd} &= 0, \\ R^{a\,fd}_{f} &= 0, \\ \left(1+\frac{\beta}{\alpha}\right)\Omega_{bd} &= 0, \\ \left(1+\frac{\beta}{\alpha}\right)\Omega^{ab} &= 0. \end{aligned}$$

These relationships elucidate two very important results. First, the field equations are now the vacuum Einstein field equations on the submanifolds. The second important result is the dilatational curvature vanishes generically on the submanifolds.[3]

The mixed-basis terms of the field equations are now

$$0 = R^{a\,f}{}_{f\,d} - R^{e\,a}{}_{d\,e} + 2\delta^{ae}\eta_{ec}\left(\delta_{fh}\eta^{hf}\right) - 2\eta^{ae}\delta_{ec}\left(\delta^{fg}\eta_{gf}\right), \tag{4.14}$$

$$\begin{aligned} 0 = {} & R^{a\,f}{}_{f\,d} + R^{e\,a}{}_{d\,e} + 2\left(1+\frac{\beta}{\alpha}\right)\Omega^{a}_{d} \\ & -\delta^{a}_{d}\frac{2}{(n-1)}\left(\frac{1}{2}n^2 - \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}n^2\right) + \delta^{a}_{d}\frac{1}{(n-1)}\left(\delta_{ge}\eta^{ge}\right)\left(\delta^{fh}\eta_{fh}\right), \end{aligned} \tag{4.15}$$

where we have written the equations as the symmetrized and antisymmetrized pieces of the mixed curvature.

4.4.2. Bianchi identities

The Bianchi identities are substantially simplified by the simple vector ansatz. They now only relate the various Cartan curvatures and their covariant derivatives. The dilatational Bianchi identities now have the following form,

$$T^{ab}{}_{c}\mathbf{e}^{c}\mathbf{f}_{a}\mathbf{f}_{b} = \mathbf{d}_{(y)}\left(\Omega^{a}{}_{b}\mathbf{f}_{a}\mathbf{e}^{b}\right), \tag{4.16}$$

$$-S_{a}{}^{b}{}_{c}\mathbf{f}_{b}\mathbf{e}^{a}\mathbf{e}^{c} = \mathbf{d}_{(x)}\left(\Omega^{a}{}_{b}\mathbf{f}_{a}\mathbf{e}^{b}\right). \tag{4.17}$$

[3] We choose to work in the generic case and save the $\frac{\beta}{a} = -1$ case for later work.

The configuration Bianchi identities have the following form,

$$\begin{aligned}
\frac{1}{2}R^{a}{}_{bcd}\mathbf{e}^{b}\mathbf{e}^{c}\mathbf{e}^{d} &= \mathbf{D}_{(x)}\left(\frac{1}{2}T^{a}{}_{bc}\mathbf{e}^{b}\mathbf{e}^{c}\right), && (4.18)\\
R^{a\ c}_{\ b\ d}\mathbf{f}_{c}\mathbf{e}^{d}\mathbf{e}^{b} &= \mathbf{D}_{(x)}\left(T^{ab}{}_{c}\mathbf{f}_{b}\mathbf{e}^{c}\right)+\mathbf{D}_{(y)}\left(\frac{1}{2}T^{a}{}_{bc}\mathbf{e}^{b}\mathbf{e}^{c}\right), && (4.19)\\
\frac{1}{2}R^{a\ de}_{\ c}\mathbf{f}_{d}\mathbf{f}_{e}\mathbf{e}^{c} &= \mathbf{D}_{(y)}\left(T^{ab}{}_{c}\mathbf{f}_{b}\mathbf{e}^{c}\right). && (4.20)
\end{aligned}$$

And, the momentum Bianchi identities have the following form,

$$\begin{aligned}
\frac{1}{2}R^{b\ cd}_{\ a}\mathbf{f}_{b}\mathbf{f}_{c}\mathbf{f}_{d} &= -\mathbf{D}_{(y)}\left(\frac{1}{2}S_{a}{}^{bc}\mathbf{f}_{b}\mathbf{f}_{c}\right), && (4.21)\\
R^{b\ c}_{\ a\ d}\mathbf{f}_{c}\mathbf{e}^{d}\mathbf{f}_{b} &= -\mathbf{D}_{(x)}\left(\frac{1}{2}S_{a}{}^{bc}\mathbf{f}_{b}\mathbf{f}_{c}\right)-\mathbf{D}_{(y)}\left(S_{a}{}^{b}{}_{\ c}\mathbf{f}_{b}\mathbf{e}^{c}\right), && (4.22)\\
\frac{1}{2}R^{b}{}_{acd}\mathbf{f}_{b}\mathbf{e}^{c}\mathbf{e}^{d} &= -\mathbf{D}_{(x)}\left(S_{a}{}^{b}{}_{\ c}\mathbf{f}_{b}\mathbf{e}^{c}\right). && (4.23)
\end{aligned}$$

Note, this ansatz sets the involution torsion (T^{abc}) and cotorsion (S_{abc}) to zero, automatically satisfying the involution conditions. The Bianchi identities now reveal the Riemannian curvatures are related to the covariant derivative of the four remaining torsions and cotorsions. Since $\mathbf{e}^{a}$ and $\mathbf{f}_{a}$ are distinguishable, we may strip the odd form off of the nonstandard submanifold Bianchi identities (4.20) and (4.23). This shows the curvatures are highly restricted by the choices of torsion and cotorsion, since we are dealing with the full curvatures. The anti-symmetric part of the mixed torsion and cotorsion are in turn related to the covariant derivative of the mixed dilatational curvature, (4.16) and (4.17).

4.4.3. Mixed basis structure equations

This ansatz also simplifies the mixed basis equations greatly. When we make the substitution we are left with

$$\begin{aligned}
T^{ab}{}_{c}\mathbf{f}_{b}\mathbf{e}^{c} &= \mathbf{d}_{(y)}\mathbf{e}^{a}+\left(\gamma^{a}_{\ c}{}^{\ b}+\delta^{a}_{c}\partial^{b}\phi\right)\mathbf{f}_{b}\mathbf{e}^{c},\\
S_{a}{}^{b}{}_{\ c}\mathbf{f}_{b}\mathbf{e}^{c} &= \mathbf{d}_{(x)}\mathbf{f}_{a}+\left(\sigma^{b}_{\ ac}+\delta^{b}_{a}\partial_{c}\phi\right)\mathbf{f}_{b}\mathbf{e}^{c}.
\end{aligned}$$

We can relate these to the submanifold equations by using the definitions of the Christoffel symbols (which can be found in Appendix D).

$$\begin{aligned} e_c{}^\mu \partial_b e_\mu{}^a + \sigma^a_{cb} + \delta^b_a \partial_c \phi - \tilde{\Gamma}^a_{cb} &= 0, \\ f_\mu{}^c \partial^b f_a{}^\mu - \gamma^{c\,b}_{a} - \delta^c_a \partial^b \phi + \tilde{\Gamma}^{c\,b}_{a} &= 0. \end{aligned}$$

These show the mixed torsion/cotorsion are, in fact, the submanifold Christoffel symbols written in the orthonormal basis

$$\begin{aligned} T^{ab}{}_c &= \tilde{\Gamma}^{a\,b}_{c} + \left(e_c{}^\mu \partial^b e_\mu{}^a - f_c{}^\mu \partial^b f_\mu{}^a \right), \\ S_a{}^b{}_c &= \tilde{\Gamma}^b_{ac} + \left(e_\mu{}^b \partial_c e_a{}^\mu - f_\mu{}^b \partial_c f_a^\mu \right), \end{aligned} \tag{4.24}$$

where the terms in parenthesis encode information about the symplectic structure of the space and vanish when the coordinates are Darboux.

4.5. Vanishing torsion solution

The ansatz we have chosen above gives us a much simpler set of equations to work with, while at the same time elucidating the relationship between the torsions/cotorsions and the Riemannian curvatures. Here, we will take advantage of these relationships by looking at the solution when the most physically relevant torsion/cotorsions vanish. The submanifold torsion, T^a_{bc}, is the one that appears in Einstein-Cartan theory and has yet to be measured. We will immediately set it and the analogous cotorsion on the momentum submanifold, S_a^{bc}, to zero.

The mixed torsion and cotorsion are the only remaining components. Again, motivated by finding general relativity within biconformal gravity, we set the mixed torsion to zero. We immediately see (4.20) sets the momentum Riemannian curvature to zero. Having a momentum sector with no curvature allows us to regard it as a vector space. We can then interpret it as the cotangent bundle to spacetime. The only nonvanishing tensor from the basis structure equations is the mixed cotorsion, $S_a{}^b{}_c$. It is easy to see why we choose not to set the mixed basis cotorsion to zero by looking at (4.23). If $S_a{}^b{}_c = 0$, then there would be no configuration

Riemannian curvature. We investigate this below.

4.5.1. Time metric

In [7] and in Chapter 3, it was shown the homogeneous biconformal space the Killing metric on the configuration space is Lorentzian, despite the Euclidean starting place. Here the vanishing torsion, through the Bianchi identities, sets the Riemannian part of the curvature to zero on the momentum submanifold.

$$\frac{1}{2}\Omega_{\ b}^{a\ cd}\mathbf{f}_c\mathbf{f}_d \quad = \quad \Delta_{eb}^{ac}\eta^{ed}\mathbf{f}_c\mathbf{f}_d.$$

This allows us to use the result of [7] to show the presence of the time metric. There the Weyl curvature of $R_{\ b}^{a\ cd}$ was set to zero to make the momentum sector as flat as possible. Here $R_{\ b}^{a\ cd} = 0$ by the Bianchi identity and the vanishing torsion condition. Instead we set the traceless part of $\Omega_{\ b}^{a\ cd}$ to zero, i.e. its Weyl curvature. From here the calculation is analogous to the one in [7] and leads to the same time form of the metric

$$\eta_{ab} = \chi\left(\delta_{ab} - \frac{2}{z^2}z_a z_b\right). \tag{4.25}$$

Since we are in an orthonormal basis and the vector is normalized explicitly, the conformal factor must be unity in the orthonormal gauge. The field equation $\Omega_{\ c}^{a\ cb} = \frac{1}{2}\left[(n-2)\,\eta^{ab} + \delta^{ab}\left(\delta_{dc}\eta^{dc}\right)\right]$ keeps us from setting the trace to zero, since this would make the new metric proportional to the old metric, breaking involution. Unlike in Chapter 3, the vector in the metric is not necessarily the scale vector, $W_a - \partial_a\phi$.

Notice here the restriction of the momentum submanifold curvature $R_{\ b}^{a\ cd} = 0$ is not as problematic as it is in the case of the homogeneous space. Here the Riemannian curvature is zero on the momentum sector because we have set the torsion on the mixed sector to zero and they are related through a Bianchi identity, (4.20). In the homogeneous space the solution gives a more complicated form of the symmetric part of the spin connection. This leads to a form for the momentum curvature that is only zero in a special case, see Section 3.3.3. This leads to an interesting tension between the solution found in this chapter, using the simplest

ansatz possible, and the solution found using an already known solution to biconformal space. This tension points in the direction of further work using a more complicated form of the the symmetric spin connection to find other gravitational solutions.

4.5.2. Mixed basis curvatures

We also see the mixed Riemannian curvature antisymmetrized on its lower indices is zero $R^{a\ c}_{b\ d} - R^{a\ c}_{d\ b} = 0$, again from the Bianchi identities (4.19). Note this makes one of the traces vanish,[4] $R^{e\ c}_{b\ e} = 0$. In turn this simplifies (4.14) and (4.15) to give

$$\begin{aligned} 0 &= R^{a\ f}_{f\ d} + 2\delta^{ae}\eta_{ec}\left(\delta_{fh}\eta^{hf}\right) - 2\eta^{ae}\delta_{ec}\left(\delta^{fg}\eta_{gf}\right), \\ 0 &= R^{a\ f}_{f\ d} + 2\left(1+\frac{\beta}{\alpha}\right)\Omega^{a}_{d} \\ &\quad -\delta^{a}_{d}\left(\frac{2}{(n-1)}\left(\frac{1}{2}n^2 - \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}n^2\right) - \frac{1}{(n-1)}\left(\delta_{ge}\eta^{ge}\right)\left(\delta^{fh}\eta_{fh}\right)\right). \end{aligned}$$

We can actually show the entire mixed Riemannian curvature vanishes in two different ways. Taking the antisymmetric and symmetric projections of the equations above would reveal that $R^{a\ f}_{f\ d} = 0$. Alternatively, equation (4.20) shows us the Riemannian curvature of the momentum submanifold, $R^{a\ de}_{c}$, is zero if there is no mixed torsion. This leads to a cascade of simplifications. First, since $R^{a\ de}_{c} = 0$, we are free to gauge the momentum spin connection away, $\boldsymbol{\gamma}^{a}_{\ b} = 0$. The momentum submanifold basis is then exact, with respect to y-derivatives, $\mathbf{d}_{(y)}\mathbf{f}_a = 0$. The mixed Riemannian curvature is greatly simplified by gauging away $\boldsymbol{\gamma}^{a}_{\ b}$,

$$R^{a\ c}_{b\ d}\mathbf{f}_c\mathbf{e}^d = \mathbf{d}^{(y)}\boldsymbol{\sigma}^{a}_{\ b} + \mathbf{d}^{(x)}\boldsymbol{\gamma}^{a}_{\ b} - \boldsymbol{\sigma}^{c}_{\ b}\boldsymbol{\gamma}^{a}_{\ c} - \boldsymbol{\gamma}^{c}_{\ b}\boldsymbol{\sigma}^{a}_{\ c} = \mathbf{d}^{(y)}\boldsymbol{\sigma}^{a}_{\ b}.$$

Looking at the mixed basis equation, we see $\mathbf{d}_{(y)}\mathbf{e}^a = 0$, which means the tetrad coefficients, $e^{\ a}_{\mu} = e^{\ a}_{\mu}(x)$, are only functions of the configuration coordinates. Since the configuration submanifold is a Riemannian geometry, we can solve for the spin connection, $\boldsymbol{\sigma}^{a}_{\ b}$, in terms of the tetrad and its de'rivatives, $\boldsymbol{\sigma}^{a}_{\ b}(e,\ \partial e)$. Therefore, the spin connection on the configuration side is only a function of the x-coordinates as well. This makes the last remaining term in the

[4] $R^{e\ a}_{e\ b} = 0$ from the antisymmetry of $\boldsymbol{\alpha}^{a}_{\ b}$

definition of the mixed Riemannian curvature vanish and we have $R^{a\ c}_{\ b\ d} = 0$. We then have

$$\begin{aligned} 0 &= \delta^{ae}\eta_{ec}\left(\delta_{fh}\eta^{hf}\right) - \eta^{ae}\delta_{ec}\left(\delta^{fg}\eta_{gf}\right), \\ \Omega^a_{\ d} &= \frac{\alpha}{\alpha+\beta}\left(\frac{1}{(n-1)}\left(\frac{1}{2}n^2 - \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}n^2\right) - \frac{1}{2}\frac{1}{(n-1)}\left(\delta_{ge}\eta^{ge}\right)\left(\delta^{fh}\eta_{fh}\right)\right)\delta^a_d \\ &= \kappa\delta^a_d. \end{aligned}$$

We choose to work in the orthonormal gauge where the scale factor on the metric is one. This gives us a form of the metric where the trace is constant. Therefore, all the terms on the RHS of the above equation are constant, $\mathbf{d}\kappa = 0$. Since the mixed dilatational curvature is constant, we then see the Bianchi identity concerning its x-derivative (4.17) puts a condition on the anti-symmetric part of the mixed cotorsion, $S_{[a\ \ c]}^{\ \ b} = 0$. The only piece left of any of the torsions or cotorsions is now the symmetric part of the mixed cotorsion, $S_{(a\ \ c)}^{\ \ b}$.

4.5.3. Symplectic structure

We are now in a much simpler geometry where most of the torsions and cotorsions are zero, $T^{ab}_{\ \ c} = T^a_{bc} = T^{abc} = S^{bc}_a = S_{abc} = 0$, two of the dilatational curvatures are zero, $\Omega_{ab} = \Omega^{ab} = 0$, and two of the curvatures are zero, $R^{a\ cd}_{\ b} = R^{a\ c}_{\ b\ d} = 0$. The only curvatures that remain are $S_{(a\ \ c)}^{\ \ b}$, $\Omega^a_{\ d}$, and $R^a_{\ bcd}$. Since the mixed dilatational curvature is constant it is closed and we are able to interpret $\mathbf{e}^a\mathbf{f}_a$ as a symplectic form. The Darboux theorem then allows us to choose a set of coordinates such that $\mathbf{d}\boldsymbol{\omega} = \mathbf{d}x^\alpha\mathbf{d}y_\alpha$. This implies $e^{\ a}_{\mu\ } = f^{\ a}_{\mu\ }$. In this basis the already simplified mixed basis equations (4.24) show the mixed cotorsion is equal to the configuration Christoffel symbol, $\Gamma^a_{\ bc}$.

The Bianchi identity relating the configuration curvature to the mixed cotorsion (4.23) is then identically satisfied.

$$\begin{aligned} \frac{1}{2}R^b_{\ acd}\mathbf{f}_b\mathbf{e}^c\mathbf{e}^d &= -\mathbf{D}_{(x)}\left(S_a^{\ b}_{\ \ c}\mathbf{f}_b\mathbf{e}^c\right) = \mathbf{D}_{(x)}\left(\Gamma^b_{\ ac}\mathbf{f}_b\mathbf{e}^c\right) \\ &= \mathbf{D}_{(x)}\Gamma^b_{\ ac}\mathbf{f}_b\mathbf{e}^c + \Gamma^b_{\ ac}\mathbf{D}_{(x)}\mathbf{f}_b\mathbf{e}^c \\ &= \mathbf{D}_{(x)}\Gamma^b_{\ ac}\mathbf{f}_b\mathbf{e}^c + \Gamma^e_{\ ad}S_e^{\ b}_{\ \ c}\mathbf{f}_b\mathbf{e}^c\mathbf{e}^d \\ &= \mathbf{D}_{(x)}\Gamma^b_{\ ac}\mathbf{f}_b\mathbf{e}^c + \Gamma^e_{\ ad}\Gamma^b_{\ ec}\mathbf{f}_b\mathbf{e}^c\mathbf{e}^d. \end{aligned}$$

This works because the covariant derivative of the cotetrad is again the mixed cotorsion. This cancels the extra Γ-squared pieces from the covariant derivative.

4.5.4. Mixed cotorsion field equation

Finally, we look at the consequences of the remaining field equations. The field equation for the mixed torsion is identically satisfied; however, the field equation for the only remaining cotorsion gives conditions on the configuration Christoffel symbol, and hence, the metric. The mixed cotorsion field equation puts a condition on the derivatives of the metric since the mixed cotorsion is equal to the Christoffel symbol (written in the orthonormal frame). We therefore have the following condition

$$0 \;=\; \Delta^{ap}_{qb}\left(\Gamma^{b}_{\ ca} - \delta^{b}_{c}\Gamma^{e}_{\ ae}\right).$$

See Appendix E for a detailed discussion of the role of the Christoffel symbol in the extrinsic curvature on submanifolds. If we look at the time metric of the form $\eta_{ab} = \chi\left(\delta_{ab} - \frac{2}{z^2}z_a z_b\right)$, where z_a defines the time coordinate via $z^\alpha \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial t}$, then we can write the condition in the following form,

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\left(\partial_\nu g_{\beta\pi} - \partial_\beta g_{\nu\pi}\right) - \frac{1}{2}\left(g_{\pi\beta}s_\nu - g_{\nu\pi}s_\beta\right) \\ & -Z^2\left(Z^2\delta^0_\nu\delta^0_\beta g^{\sigma 0}g^{\alpha 0} - \frac{1}{2}\delta^0_\nu g^{\sigma 0}\delta^\alpha_\beta - \frac{1}{2}\delta^\sigma_\nu g^{\alpha 0}\delta^0_\beta\right)\left(\partial_\pi g_{\alpha\sigma} + \partial_\alpha g_{\sigma\pi} - \partial_\sigma g_{\alpha\pi}\right) \\ & +Z^2\left(2Z^2\delta^0_\nu\delta^0_\pi\delta^0_\beta g^{\alpha 0} - \delta^0_\nu\delta^0_\pi\delta^\alpha_\beta - \delta^0_\beta g_{\nu\pi}g^{\alpha 0}\right)s_\alpha, \end{aligned}$$

where $\Gamma^\rho_{\ \beta\rho} = \frac{1}{2}g^{\epsilon\rho}\partial_\beta g_{\rho\epsilon} = \partial_\beta \ln\sqrt{|g|} = \partial_\beta s \equiv s_\beta$, $Z^2 = z_a z_b \eta^{ab}$ and 0 index designates the time direction. Looking at the various components of this, we get the following conditions

$$\begin{aligned} 0 \;&=\; \partial_i g_{k0} - \partial_k g_{i0} - g_{0k}s_i + g_{i0}s_k, \\ 0 \;&=\; \partial_i g_{jk} - \partial_k g_{ij} - g_{jk}s_i + g_{ij}s_k. \end{aligned}$$

The first condition is the one necessary to write the metric as block diagonal $diag\left(g_{00},\, g_{ij}\right)$. The second condition shows the metric times a conformal factor, $e^{-s}g_{ij}$, comes from a potential.

When the metric is written as block diagonal, then $0 = s_\mu$. We then have

$$0 \;=\; \partial_i g_{jk} - \partial_k g_{ij} \Rightarrow g_{ij} = \partial_i \chi_j,$$

where χ_j is some vector potential. See Appendix G for a detailed list of the conditions from this field equation on the coordinate metric.

4.6. Discussion

In this chapter, we have looked at curved biconformal space and the gravitational theory defined by the curvature-linear action, 4.3. Biconformal gravity has been laid out in full generality with all possible Cartan curvatures present. In $2n$-dimensional biconformal space, the Bianchi identities turn out to be very restrictive of the n-dimensional curvatures. The consequences of combining the full field equations together with the Bianchi identities and structure equations is investigated. Section 4.3 shows the general approach to investigating this class of geometries. This section allows a straightforward starting point from which to work toward any solution to biconformal gravity.

The simple linear ansatz, $\boldsymbol{\mu}^a_{\ b} = \delta^a_b v_c \mathbf{e}^c$ and $\boldsymbol{\rho}^a_{\ b} = \delta^a_b u^c \mathbf{f}_c$, shows how we can recover a subset of scale invariant general relativity on the cotangent bundle of spacetime. We then look specifically at torsion-free solutions. We are then left with a geometry where

$$\begin{aligned} 0 &= T^{ab}_{\ \ c} = T^a_{bc} = T^{abc} = S_a^{\ bc} = S_{abc}, \\ 0 &= R^{a\ cd}_{\ b} = R^{a\ c}_{\ b\ d} = R^e_{\ bed}, \\ 0 &= \Omega_{ab} = \Omega^{ab}. \end{aligned}$$

Note, the configuration subspace is Ricci flat, but the whole Riemann tensor is not zero. The chosen assumptions turn out to be overly restrictive and give us general relativity restricted to those solutions where the coordinate metric can be written in the form $diag\,(g_{00},\ g_{ij})$ and where $g_{ij} = \partial_i \chi_j$. Since we know the torsion-free solution should give us all of general relativity [50], we assume the simple linear ansatz must be loosened in order to incorporate more generality.

Unlike in general relativity (where there is only T^a_{bc}), we are unable to set all the torsions and cotorsions to zero and still have a gravitational theory with any curvature. This means the torsions and cotorsions play a slightly different role in biconformal space. By looking at a more general theory, where we do not restrict them so stringently, we should be able to better understand the role they play.

Note, the case where $S_a{}^b{}_c = 0$ results in a flat (in the Riemannian sense) configuration space, $R^a_{bcd} = 0$, but not in a homogeneous space. The mixed dilatational curvature, $\Omega^a_d = \kappa\delta^a_d$, and the Cartan curvature associated with the $SO(n)$ rotations are nonzero.

This work has shown the time result of Chapter 3 is fully compatible with scale invariant general relativity. We hope the interesting results of that chapter, that the Einstein tensor contains purely geometrical source terms, can be reproduced within the framework of a gravitational theory we have established here.

CHAPTER 5
CONCLUSION

In this dissertation, we have laid out the current status of biconformal geometry, based on the Cartan geometry of the homogeneous manifold $\mathcal{M}_0 = Conformal\,(p,q)\,/Weyl\,(p,q)$ looking specifically at the interesting Euclidean case, $p = n, q = 0$. Though biconformal spaces are endowed with more symmetries (and hence more connections) than a Riemannian space, we see Riemannian geometry as the effective geometry on the Lagrangian submanifolds of any given biconformal space. This is in one of the major advancements in the work presented here. By casting the original basis in a conformally-orthonormal Lagrangian, one can easily prove the connection on the submanifolds is that of a Riemannian geometry. It can be written as the spin connection plus a trivial scale factor term, that is not the Weyl connection. Once these Riemannian structures are recognized, the next step is to look at how the whole of biconformal geometry relates to these Riemannian geometries. The astonishing result is instead of the biconformal submanifolds looking like Minkowski space, as expected, the resulting Ricci tensor has an effective stress energy term corresponding to that of a perfect fluid.

In Chapter 2, we have shown how biconformal space is constructed as a special case of Cartan geometry. There it was shown how biconformal space fits in to the spectrum of a number of other active lines of research. The biconformal quotient is unique in giving a homogeneous space endowed naturally with both a Killing metric and a symplectic form. These stem directly from the properties of the conformal group and the chosen quotient.

In Chapter 3, we investigated the properties of the homogeneous space referred to as flat biconformal space. We have outlined the methods of [7] to show these spaces have a characteristic heretofore unknown in other geometries, a geometric mechanism whereby the spacetime signature on the n-dimensional submanifolds is different from those of the original space considered.

Specifically, we looked at biconformal space in the orthonormal version of the basis found in [7]. The orthonormal basis clarifies a number of characteristics of the space, making it easier to see how, though based on conformal symmetry, the submanifold structure equations

organize themselves to look just like the usual Riemannian geometry of general relativity, with a Lorentzian signature. We have shown the spin connection in this new basis, though not fully antisymmetric with respect to the new metric, naturally separates in the structure equations. The antisymmetric part appears naturally in the submanifold basis equations where it helps define a trivial Weyl geometry, which we are free to gauge to a run-of-the-mill Riemannian geometry. The symmetric part of the spin connection, which transforms like a tensor, can be cast in terms of another tensor, the scale vector, that defines the direction of time in the solution. This vector appears as part of an effective source to the Einstein field equation. The form is characteristic of a perfect fluid. These results are intriguing, but need to be cast as part of a full gravitational theory in order to be correctly interpreted.

The existence of these Riemannian submanifolds leads the way for the latter research in this dissertation. The basis we use in Chapter 3 is used in Chapter 4 to outline an efficient way to approach investigations of a gravitational theory with biconformal geometry as the background.

We looked at curved biconformal space and the gravitational theory defined by the Wehner–Wheeler action. We laid out the theory in full generality with all curvatures, torsion, cotorsions and dilatational curvatures present. We showed some of the Bianchi identities (integrability conditions) of the theory are more restrictive of the Riemannian curvatures on the submanifolds than in a Riemannian geometry, determining them in terms of other fields. The consequences of combining the field equations with the Bianchi identities and structure equations is investigated. We concluded by making a number of simplifying assumptions and recovering scale invariant general relativity on the cotangent bundle of spacetime. The assumptions chosen turn out to be overly restrictive and give us general relativity restricted to metrics of the form $g_{\mu\nu} = (g_{00}, g_{ij})$.

Unlike in general relativity (where there is only T^a_{bc}), we are unable to set all the torsions and cotorsions to zero and still have a gravitational theory with curvature. This means the torsions and cotorsions play a slightly different role in biconformal space. By looking at a more general theory, where we do not restrict them so stringently, we should be able to better understand the role that they play.

In Chapter 4, we demonstrated general relativity can emerge as a subsector of biconformal

gravity. We know from the work of Wehner and Wheeler [50] that torsion-free biconformal gravity generically leads to all of general relativity. Their result was not derived in the time basis, but was done without restricting the form of the symmetric spin connection. The next obvious calculation is to use the symmetric spin connection calculated in Chapter 3 as the ansatz in the curved space. Presumably this will again lead to an effective energy-momentum tensor for a perfect fluid. In the curved case, we will then be able to interpret this as a source to the Einstein field equation.

The quantization of the gravitational interaction to produce a predictive theory is still an active area of research within the theoretical physics community. Weyl gravity has a storied history as a direction to help with the quantization of gravity. The theory has been shown to be more well-behaved, from a quantization perspective, than the Einstein-Hilbert action in that it is renormalizable [87] and its Poisson algebra closes [88]. Others have worked out the one-loop corrections to the theory [89]. However, the physicality of the theory has been questioned for a long time, mostly due to the fact the Bach equation is a fourth-order equation whose correlation functions, upon quantization, lead to ghosts (negative norm states) [90–92]. Recently the work of Wheeler [68] has shown, when the Weyl action is looked at as being fully conformally symmetric with all of the connections of the conformal group varied independently, using a Palatini variation, the theory is equivalent to general relativity.

This result opens the door to the question, of how we reconcile the following facts:

1. Weyl gravity is perturbatively renormalizable, but has problems owing to its the fourth-order field equations, i.e. probably has ghost fields.

2. General relativity is not perturbatively renormalizable, but is a well-behaved theory with second-order field equations, i.e. no ghosts.

Quantum mechanics seems to follow as a natural consequence of the characteristics of biconformal space [83, 84]. It has many structures that make it amenable to quantization. The canonical-orthonormal basis, by construction, has a natural phase space structure. There is also a Kähler structure to the manifold, which might allow for straightforward geometric quantization of the manifold. The natural notion of a time direction defined by the scale vector would

allow an ADM-type decomposition of the action in order to look at canonical quantization. Lastly, since time emerges as a part of the theory, it might be possible to try and quantize the theory before gauging, without time, and then see what this quantized theory looks like, thereby skirting the problem of time altogether.

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APPENDICES

Appendix A

Structure equations in the orthonormal, Lagrangian basis

Here, we write the structure equations, including Cartan curvature. We expand the configuration, mixed and momentum terms separately. Note, the $\mathbf{f}_a\mathbf{f}_b$ part of the $\mathbf{d}\mathbf{e}^a$ equation and the $\mathbf{e}^a\mathbf{e}^b$ part of the $\mathbf{d}\mathbf{f}_a$ equation are set to zero. These are the involution conditions, which guarantee the configuration and momentum subspaces are integrable submanifolds by the Frobenius theorem.

In the conformal-orthonormal basis, we have $g^{ab}\mathbf{d}g_{bc} = e^{-2\phi}\eta^{ab}\mathbf{d}\left(e^{2\phi}\eta_{bc}\right) = 2\delta^a_c\mathbf{d}\phi$. The structure equations in the conformal-orthonormal basis are

$$\begin{aligned}
\mathbf{d}\boldsymbol{\tau}^a_{\ b} &= \boldsymbol{\tau}^c_{\ b}\boldsymbol{\tau}^a_{\ c} + \Delta^{ae}_{db}\eta_{ec}\mathbf{e}^c\mathbf{e}^d - \Delta^{ac}_{eb}\eta^{ed}\mathbf{f}_c\mathbf{f}_d + 2\Delta^{ae}_{fb}\Xi^{fc}_{de}\mathbf{f}_c\mathbf{e}^d + \boldsymbol{\Omega}^a_{\ b}, && \text{(A.1)}\\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c\boldsymbol{\alpha}^a_{\ c} + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ab}\mathbf{f}_b + \mathbf{T}^a, && \text{(A.2)}\\
\mathbf{d}\mathbf{f}_a &= \boldsymbol{\alpha}^b_{\ a}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ab}\mathbf{e}^b + \mathbf{S}_a, && \text{(A.3)}\\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a\mathbf{f}_a + \boldsymbol{\Omega}. && \text{(A.4)}
\end{aligned}$$

We then define

$$\begin{aligned}
\mathbf{D}^{(x)}\boldsymbol{\mu}^a_{\ b} &\equiv \mathbf{d}^{(x)}\boldsymbol{\mu}^a_{\ b} - \boldsymbol{\mu}^c_{\ b}\boldsymbol{\sigma}^a_{\ c} - \boldsymbol{\sigma}^c_{\ b}\boldsymbol{\mu}^a_{\ c},\\
\mathbf{D}^{(x)}\boldsymbol{\rho}^a_{\ b} &\equiv \mathbf{d}^{(x)}\boldsymbol{\rho}^a_{\ b} - \boldsymbol{\rho}^c_{\ b}\boldsymbol{\sigma}^a_{\ c} - \boldsymbol{\sigma}^c_{\ b}\boldsymbol{\rho}^a_{\ c},\\
\mathbf{D}^{(y)}\boldsymbol{\mu}^a_{\ b} &\equiv \mathbf{d}^{(y)}\boldsymbol{\mu}^a_{\ b} - \boldsymbol{\mu}^c_{\ b}\boldsymbol{\gamma}^a_{\ c} - \boldsymbol{\gamma}^c_{\ b}\boldsymbol{\mu}^a_{\ c},\\
\mathbf{D}^{(y)}\boldsymbol{\rho}^a_{\ b} &\equiv \mathbf{d}^{(y)}\boldsymbol{\rho}^a_{\ b} - \boldsymbol{\rho}^c_{\ b}\boldsymbol{\gamma}^a_{\ c} - \boldsymbol{\gamma}^c_{\ b}\boldsymbol{\rho}^a_{\ c},
\end{aligned}$$

allowing the separation of the structure equations into independent parts.

A.1. Configuration space:

$$\begin{aligned}
\frac{1}{2}\Omega^a_{\ bcd}\mathbf{e}^c\mathbf{e}^d &= \mathbf{d}^{(x)}\boldsymbol{\sigma}^a_{\ b} - \boldsymbol{\sigma}^c_{\ b}\boldsymbol{\sigma}^a_{\ c} + \mathbf{D}^{(x)}\boldsymbol{\mu}^a_{\ b} - \boldsymbol{\mu}^c_{\ b}\boldsymbol{\mu}^a_{\ c} - k\Delta^{ac}_{eb}\eta_{cd}\mathbf{e}^d\mathbf{e}^e, && \text{(A.5)}\\
\frac{1}{2}T^a_{\ bc}\mathbf{e}^b\mathbf{e}^c &= \mathbf{d}_{(x)}\mathbf{e}^a - \mathbf{e}^b\boldsymbol{\sigma}^a_{\ b} + \frac{1}{2}\eta^{ac}\mathbf{d}^{(x)}\eta_{cb}\mathbf{e}^b, && \text{(A.6)}
\end{aligned}$$

$$\frac{1}{2}S_{abc}\mathbf{e}^b\mathbf{e}^c = k\eta_{ab}\mathbf{e}^c\left(\boldsymbol{\mu}^b{}_c - \delta^b_c W_d\mathbf{e}^d + \frac{1}{2}\eta_{ce}\mathbf{d}^{(x)}\eta^{be}\right), \tag{A.7}$$

$$\frac{1}{2}\Omega_{ab}\mathbf{e}^a\mathbf{e}^b = \mathbf{d}_{(x)}\left(W_a\mathbf{e}^a\right). \tag{A.8}$$

A.2. Cross-term:

$$\begin{aligned}
\Omega^a_{b\ d}{}^c\mathbf{f}_c\mathbf{e}^d &= \mathbf{d}^{(y)}\boldsymbol{\sigma}^a_{\ b} + \mathbf{d}^{(x)}\boldsymbol{\gamma}^a_{\ b} - \boldsymbol{\gamma}^c_{\ b}\boldsymbol{\sigma}^a_{\ c} - \boldsymbol{\sigma}^c_{\ b}\boldsymbol{\gamma}^a_{\ c} \\
&\quad + \mathbf{D}^{(x)}\boldsymbol{\rho}^a_{\ b} + \mathbf{D}^{(y)}\boldsymbol{\mu}^a_{\ b} - \boldsymbol{\rho}^c_{\ b}\boldsymbol{\mu}^a_{\ c} - \boldsymbol{\mu}^c_{\ b}\boldsymbol{\rho}^a_{\ c} \\
&\quad - 2\Delta^{ac}_{db}\Xi^{fd}_{ce}\mathbf{f}_f\mathbf{e}^e, \qquad \text{(A.9)} \\
T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c &= \mathbf{d}^{(y)}\mathbf{e}^a - \mathbf{e}^b\boldsymbol{\gamma}^a_{\ b} + \frac{1}{2}\eta^{ac}\mathbf{d}^{(y)}\eta_{cb}\mathbf{e}^b \\
&\quad - k\eta^{ac}\left(\boldsymbol{\mu}^b{}_c\mathbf{f}_b + W_d\mathbf{f}_c\mathbf{e}^d - \frac{1}{2}\eta^{bd}\mathbf{d}^{(x)}\eta_{cd}\mathbf{f}_b\right), \qquad \text{(A.10)} \\
S_a{}^b{}_c\mathbf{f}_b\mathbf{e}^c &= \mathbf{d}^{(x)}\mathbf{f}_a - \boldsymbol{\sigma}^b{}_a\mathbf{f}_b - \frac{1}{2}\eta^{cb}\mathbf{d}^{(x)}\eta_{ac}\mathbf{f}_b \\
&\quad + k\eta_{ab}\left(\mathbf{e}^c\boldsymbol{\rho}^b{}_c + W^c\mathbf{f}_c\mathbf{e}^b + \frac{1}{2}\eta^{bc}\mathbf{d}^{(y)}\eta_{cd}\mathbf{e}^d\right), \qquad \text{(A.11)} \\
\Omega^a_b\mathbf{f}_a\mathbf{e}^b &= \mathbf{d}_{(y)}\left(W_a\mathbf{e}^a\right) + \mathbf{d}_{(x)}\left(W^a\mathbf{f}_a\right) - \mathbf{e}^a\mathbf{f}_a. \qquad \text{(A.12)}
\end{aligned}$$

A.3. Momentum space:

$$\begin{aligned}
\frac{1}{2}\Omega^a_{b\ }{}^{cd}\mathbf{f}_c\mathbf{f}_d &= \mathbf{d}^{(y)}\boldsymbol{\gamma}^a_{\ b} - \boldsymbol{\gamma}^c_{\ b}\boldsymbol{\gamma}^a_{\ c} + \mathbf{D}\boldsymbol{\rho}^a_{\ b} - \boldsymbol{\rho}^c_{\ b}\boldsymbol{\rho}^a_{\ c} + k\Delta^{ac}_{eb}\eta^{ed}\mathbf{f}_c\mathbf{f}_d, \qquad \text{(A.13)} \\
\frac{1}{2}S_a{}^{bc}\mathbf{f}_b\mathbf{f}_c &= \mathbf{d}^{(y)}\mathbf{f}_a - \boldsymbol{\gamma}^b{}_a\mathbf{f}_b - \frac{1}{2}\eta^{cb}\mathbf{d}^{(y)}\eta_{ac}\mathbf{f}_b, \qquad \text{(A.14)} \\
\frac{1}{2}T^{abc}\mathbf{f}_b\mathbf{f}_c &= -k\eta^{ac}\left(\boldsymbol{\rho}^b{}_c\mathbf{f}_b - W^b\mathbf{f}_b\mathbf{f}_c - \frac{1}{2}\eta^{bd}\mathbf{d}^{(y)}\eta_{cd}\mathbf{f}_b\right), \qquad \text{(A.15)} \\
\frac{1}{2}\Omega^{bc}\mathbf{f}_b\mathbf{f}_c &= \mathbf{d}^{(y)}\left(W^a\mathbf{f}_a\right). \qquad \text{(A.16)}
\end{aligned}$$

Appendix B

Gauge transformations in the orthonormal, Lagrangian basis

B.1. Definitions

In the conformally orthonormal frame our structure equations are given by equations (A.1) through (A.4), where the $\mathbf{e}^c$ components of $\mathbf{D}\eta^{ab}$ become

$$\begin{aligned}
\mathbf{D}\eta^{ab} &= \mathbf{d}_{(x)}\eta^{ab} + \eta^{cb}\boldsymbol{\alpha}^a_{\ c} + \eta^{ac}\boldsymbol{\alpha}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}_{(x)}\eta^{ab} + \eta^{cb}\left(\boldsymbol{\sigma}^a_{\ c} + \boldsymbol{\mu}^a_{\ c}\right) + \eta^{ac}\left(\boldsymbol{\sigma}^b_{\ c} + \boldsymbol{\mu}^b_{\ c}\right) - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}_{(x)}\eta^{ab} + \left(\eta^{cb}\boldsymbol{\sigma}^a_{\ c} + \eta^{ac}\boldsymbol{\sigma}^b_{\ c}\right) + \eta^{cb}\boldsymbol{\mu}^a_{\ c} + \eta^{ac}\boldsymbol{\mu}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}_{(x)}\eta^{ab} + 2\eta^{ac}\boldsymbol{\mu}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab}.
\end{aligned}$$

We assume the involution equations are already satisfied. Similarly, for the third equation, we need only the $\mathbf{f}_a$ components,

$$\begin{aligned}
\mathbf{D}\eta_{ab} &= \mathbf{d}_{(y)}\eta_{ab} - \eta_{cb}\boldsymbol{\beta}^c_{\ a} - \eta_{ac}\boldsymbol{\beta}^c_{\ b} + 2W^c\mathbf{f}_c\eta^{ab} \\
&= \mathbf{d}_{(y)}\eta_{ab} - \eta_{cb}\left(\boldsymbol{\gamma}^c_{\ a} + \boldsymbol{\rho}^c_{\ a}\right) - \eta_{ac}\left(\boldsymbol{\gamma}^c_{\ b} + \boldsymbol{\rho}^c_{\ b}\right) + 2W^c\mathbf{f}_c\eta^{ab} \\
&= \mathbf{d}_{(y)}\eta_{ab} - \eta_{cb}\boldsymbol{\gamma}^c_{\ a} - \eta_{ac}\boldsymbol{\gamma}^c_{\ b} - \eta_{cb}\boldsymbol{\rho}^c_{\ a} - \eta_{ac}\boldsymbol{\rho}^c_{\ b} + 2W^c\mathbf{f}_c\eta^{ab} \\
&= \mathbf{d}_{(y)}\eta_{ab} - 2\eta_{ac}\boldsymbol{\rho}^c_{\ b} + 2W^c\mathbf{f}_c\eta_{ab} \\
&= \mathbf{D}_{(y,\gamma)}\eta_{ab} - 2\eta_{ac}\boldsymbol{\rho}^c_{\ b} + 2W^c\mathbf{f}_c\eta_{ab}.
\end{aligned}$$

B.2. Lorentz transformations

The basis equations

We compute the gauge transformation properties of the different pieces of the connection. Notice first we have the antisymmetry of the inhomogeneous part of the transformation,

$$\begin{aligned}
\eta_{ab} &= \eta_{ef}\Lambda^e_{\ a}\Lambda^f_{\ b} \\
\delta^c_b &= \eta^{ca}\eta_{ef}\Lambda^e_{\ a}\Lambda^f_{\ b}
\end{aligned}$$

$$
\begin{aligned}
\delta^c_b \bar{\Lambda}^b{}_d &= \eta^{ca}\eta_{ef}\Lambda^e{}_a\Lambda^f{}_b\bar{\Lambda}^b{}_d \\
\bar{\Lambda}^c{}_d &= \eta^{ca}\eta_{de}\Lambda^e{}_a \\
\eta_{ac}\eta^{de}\bar{\Lambda}^c{}_d &= \Lambda^e{}_a \\
\eta^{de}\bar{\Lambda}^c{}_d &= \eta^{cd}\Lambda^e{}_d.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\bar{\Lambda}^b{}_e\partial^d\Lambda^e{}_c &= \bar{\Lambda}^b{}_e\partial^d\left(\eta^{ef}\eta_{cg}\bar{\Lambda}^g{}_f\right) \\
&= \eta^{ef}\eta_{cg}\bar{\Lambda}^b{}_e\partial^d\bar{\Lambda}^g{}_f \\
&= \eta^{eb}\eta_{cg}\Lambda^f{}_e\partial^d\bar{\Lambda}^g{}_f \\
\bar{\Lambda}^b{}_e\partial^d\Lambda^e{}_c &= -\eta^{eb}\eta_{cg}\bar{\Lambda}^g{}_f\partial^d\Lambda^f{}_e \\
\delta^b_g\delta^e_c\bar{\Lambda}^g{}_f\partial^d\Lambda^f{}_e + \eta^{eb}\eta_{cg}\bar{\Lambda}^g{}_f\partial^d\Lambda^f{}_e &= 0 \\
\Xi^{eb}_{cg}\left(\bar{\Lambda}^g{}_f\partial^d\Lambda^f{}_e\right) &= 0.
\end{aligned}
$$

Now consider the basis equations (dropping the involution terms),

$$
\begin{aligned}
\mathbf{de}^a &= \mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d{}_c + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}_{(x)}\mathbf{e}^a + \mathbf{d}_{(y)}\mathbf{e}^a &= \mathbf{e}^b\left(\boldsymbol{\sigma}^a{}_b + \boldsymbol{\gamma}^a{}_b\right) + \frac{1}{2}\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac} + \mathbf{d}_{(y)}\eta^{ac}\right)\mathbf{e}^b \\
&\quad + \frac{1}{2}\left(\mathbf{d}_{(x)}\eta^{ab} + 2\eta^{ac}\boldsymbol{\mu}^b{}_c - 2W_c\mathbf{e}^c\eta^{ab}\right)\mathbf{f}_b \\
&\quad + T^{ab}{}_c\mathbf{f}_b\mathbf{e}^c + \frac{1}{2}T^a_{bc}\mathbf{e}^b\mathbf{e}^c, \\
\mathbf{df}_a &= \Theta^{bc}_{da}\boldsymbol{\tau}^d{}_c\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + \mathbf{S}_a \\
\mathbf{d}_{(x)}\mathbf{f}_a + \mathbf{d}_{(y)}\mathbf{f}_a &= \left(\boldsymbol{\sigma}^b{}_a + \boldsymbol{\gamma}^b{}_a\right)\mathbf{f}_b + \frac{1}{2}\eta^{bc}\left(\mathbf{d}_{(x)}\eta_{ab} + \mathbf{d}_{(y)}\eta_{ab}\right)\mathbf{f}_c \\
&\quad - \frac{1}{2}\left(\mathbf{d}_{(y)}\eta_{ab} - 2\eta^{ac}\boldsymbol{\rho}^b{}_c + 2W^c\mathbf{f}_c\eta^{ab}\right)\mathbf{e}^b \\
&\quad + S_a{}^b{}_c\mathbf{f}_b\mathbf{e}^c + \frac{1}{2}S_a{}^{bc}\mathbf{f}_b\mathbf{f}_c,
\end{aligned}
$$

so we have two pieces from each equation

$$\begin{aligned}
\mathbf{d}_{(x)}\mathbf{e}^a &= \mathbf{e}^b\boldsymbol{\sigma}^a{}_b + \frac{1}{2}\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac}\right)\mathbf{e}^b + \frac{1}{2}T^a{}_{bc}\mathbf{e}^b\mathbf{e}^c, \\
\mathbf{d}_{(y)}\mathbf{e}^a &= \mathbf{e}^b\boldsymbol{\gamma}^a{}_b + \eta^{ac}\boldsymbol{\mu}^b{}_c\mathbf{f}_b - W_c\mathbf{e}^c\eta^{ab}\mathbf{f}_b + T^{ab}{}_c\mathbf{f}_b\mathbf{e}^c \\
&\quad + \frac{1}{2}\eta_{cb}\mathbf{d}_{(y)}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{d}_{(x)}\eta^{ab}\mathbf{f}_b \\
&= \left(-\gamma^a{}_c{}^b - \eta^{ad}\mu^b{}_{dc} + W_c\eta^{ab} + T^{ab}{}_c\right)\mathbf{f}_b\mathbf{e}^c \\
&\quad + \frac{1}{2}\eta_{cb}\mathbf{d}_{(y)}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{d}_{(x)}\eta^{ab}\mathbf{f}_b, \\
\mathbf{d}_{(y)}\mathbf{f}_a &= \boldsymbol{\gamma}^b{}_a\mathbf{f}_b + \frac{1}{2}\eta^{bc}\left(\mathbf{d}_{(y)}\eta_{ab}\right)\mathbf{f}_c + \frac{1}{2}S_a{}^{bc}\mathbf{f}_b\mathbf{f}_c, \\
\mathbf{d}_{(x)}\mathbf{f}_a &= \boldsymbol{\sigma}^b{}_a\mathbf{f}_b + \frac{1}{2}\eta^{bc}\left(\mathbf{d}_{(x)}\eta_{ab}\right)\mathbf{f}_c - \frac{1}{2}\left(\mathbf{d}_{(y)}\eta_{ab} - 2\eta_{ac}\boldsymbol{\rho}^c{}_b + 2W^c\mathbf{f}_c\eta_{ab}\right)\mathbf{e}^b \\
&\quad + S_a{}^b{}_c\mathbf{f}_b\mathbf{e}^c.
\end{aligned}$$

Transforming the configuration basis equation

If we choose a different orthonormal basis, $\tilde{\mathbf{e}}^a = \Lambda^a{}_b\mathbf{e}^b$, $\tilde{\eta}_{ab} = \eta_{ab}$, and $\mathbf{f}_b = \bar{\Lambda}^c{}_b\mathbf{f}_c$, the first piece becomes

$$\begin{aligned}
\mathbf{d}_{(x)}\left(\Lambda^a{}_b\mathbf{e}^b\right) &= \left(\Lambda^b{}_c\mathbf{e}^c\right)\tilde{\boldsymbol{\sigma}}^a{}_b + \frac{1}{2}\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac}\right)\left(\Lambda^b{}_d\mathbf{e}^d\right) + \frac{1}{2}\tilde{T}^a{}_{bc}\Lambda^b{}_d\Lambda^c{}_e\mathbf{e}^d\mathbf{e}^e \\
\Lambda^a{}_b\mathbf{d}_{(x)}\mathbf{e}^b &= \Lambda^b{}_c\mathbf{e}^c\left(\tilde{\boldsymbol{\sigma}}^a{}_b + \bar{\Lambda}^d{}_b\mathbf{d}_{(x)}\Lambda^a{}_d\right) \\
&\quad + \frac{1}{2}\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac}\right)\Lambda^b{}_d\mathbf{e}^d + \frac{1}{2}\tilde{T}^a{}_{bc}\Lambda^b{}_d\Lambda^c{}_e\mathbf{e}^d\mathbf{e}^e \\
\Lambda^a{}_b\mathbf{e}^c\boldsymbol{\sigma}^b{}_c &= -\frac{1}{2}\Lambda^a{}_b\eta_{dc}\left(\mathbf{d}_{(x)}\eta^{bd}\right)\mathbf{e}^c - \frac{1}{2}\Lambda^a{}_bT^b{}_{cd}\mathbf{e}^c\mathbf{e}^d \\
&\quad + \Lambda^b{}_c\mathbf{e}^c\left(\tilde{\boldsymbol{\sigma}}^a{}_b + \bar{\Lambda}^d{}_b\mathbf{d}_{(x)}\Lambda^a{}_d\right) \\
&\quad + \frac{1}{2}\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac}\right)\Lambda^b{}_d\mathbf{e}^d + \frac{1}{2}\tilde{T}^a{}_{bc}\Lambda^b{}_d\Lambda^c{}_e\mathbf{e}^d\mathbf{e}^e.
\end{aligned}$$

So combining like terms,

$$\begin{aligned}
0 &= \Lambda^a{}_e\mathbf{e}^c\bar{\Lambda}^d{}_b\Lambda^b{}_c\boldsymbol{\sigma}^e{}_d - \Lambda^b{}_c\mathbf{e}^c\left(\tilde{\boldsymbol{\sigma}}^a{}_b + \bar{\Lambda}^d{}_b\mathbf{d}_{(x)}\Lambda^a{}_d\right) \\
&\quad + \frac{1}{2}\left(\Lambda^a{}_b\eta_{cd}\left(\mathbf{d}_{(x)}\eta^{bc}\right) - \eta_{cb}\Lambda^b{}_d\left(\mathbf{d}_{(x)}\eta^{ac}\right)\right)\mathbf{e}^d + \frac{1}{2}\left(\Lambda^a{}_bT^b{}_{de} - \tilde{T}^a{}_{bc}\Lambda^b{}_d\Lambda^c{}_e\right)\mathbf{e}^d\mathbf{e}^e \\
&= \Lambda^b{}_c\mathbf{e}^c\left(\Lambda^a{}_e\bar{\Lambda}^d{}_b\boldsymbol{\sigma}^e{}_d - \bar{\Lambda}^d{}_b\mathbf{d}_{(x)}\Lambda^a{}_d - \tilde{\boldsymbol{\sigma}}^a{}_b\right)
\end{aligned}$$

$$+\frac{1}{2}\left(\Lambda^a{}_b\eta_{cd}\left(\mathbf{d}_{(x)}\eta^{bc}\right)-\eta_{cb}\Lambda^b{}_d\left(\mathbf{d}_{(x)}\eta^{ac}\right)\right)\mathbf{e}^d+\frac{1}{2}\left(\Lambda^a{}_bT^b_{de}-\tilde{T}^a_{bc}\Lambda^b{}_d\Lambda^c{}_e\right)\mathbf{e}^d\mathbf{e}^e,$$

we may identify

$$\begin{aligned}
\tilde{\boldsymbol{\sigma}}^a{}_b &= \Lambda^a{}_e\bar{\Lambda}^d{}_b\boldsymbol{\sigma}^e{}_d-\bar{\Lambda}^d{}_b\mathbf{d}_{(x)}\Lambda^a{}_d,\\
\tilde{\sigma}^a_{bc} &= \Lambda^a{}_e\bar{\Lambda}^d{}_b\sigma^e_{df}\bar{\Lambda}^f{}_c-\bar{\Lambda}^d{}_b\bar{\Lambda}^f{}_c\partial_f\Lambda^a{}_d,\\
\tilde{T}^a_{bc} &= \Lambda^a{}_fT^f_{de}\bar{\Lambda}^d{}_b\bar{\Lambda}^e{}_c,\\
\Lambda^a{}_b\eta_{cd}\left(\mathbf{d}_{(x)}\eta^{bc}\right)-\eta_{cb}\Lambda^b{}_d\left(\mathbf{d}_{(x)}\eta^{ac}\right) &= \Lambda^a{}_be^{2\varphi}\eta_{0cd}\left(-2e^{-2\varphi}\mathbf{d}_{(x)}\varphi\eta_0^{bc}\right)\\
&\quad -e^{2\varphi}\eta_{0cb}\Lambda^b{}_d\left(-2e^{-2\varphi}\mathbf{d}_{(x)}\varphi\eta_0^{ac}\right)\\
&= -2\Lambda^a{}_b\eta_0^{bc}\eta_{0cd}\mathbf{d}_{(x)}\varphi+2\eta_0^{ac}\eta_{0cb}\Lambda^b{}_d\mathbf{d}_{(x)}\varphi\\
&= -2\Lambda^a{}_d\mathbf{d}_{(x)}\varphi+2\Lambda^a{}_d\mathbf{d}_{(x)}\varphi\\
&\equiv 0.
\end{aligned}$$

These are the expected transformation properties.

Transforming the momentum basis equation

Now we repeat the calculation for the momentum submanifold. Starting with,

$$\mathbf{d}_{(y)}\mathbf{f}_a=\boldsymbol{\gamma}^b{}_a\mathbf{f}_b+\left(\mathbf{d}_{(y)}\varphi\right)\mathbf{f}_a+\frac{1}{2}S_a{}^{bc}\mathbf{f}_b\mathbf{f}_c,$$

we gauge to

$$\begin{aligned}
\mathbf{d}_{(y)}\left(\bar{\Lambda}^c{}_a\mathbf{f}_c\right) &= \tilde{\gamma}^b_a\ {}^d\mathbf{f}_d\left(\bar{\Lambda}^c{}_b\mathbf{f}_c\right)+\frac{1}{2}\eta_0^{bc}e^{-2\varphi}\left(\mathbf{d}_{(y)}\eta^0_{ab}e^{2\varphi}\right)\left(\bar{\Lambda}^d{}_c\mathbf{f}_d\right)\\
&\quad +\frac{1}{2}\tilde{S}_a{}^{bc}\left(\bar{\Lambda}^d{}_b\mathbf{f}_d\right)\left(\bar{\Lambda}^e{}_c\mathbf{f}_e\right)\\
0 &= \tilde{\gamma}^b_a\ {}^d\bar{\Lambda}^c{}_b\mathbf{f}_d\mathbf{f}_c-\left(\mathbf{d}_{(y)}\bar{\Lambda}^c{}_a\right)\mathbf{f}_c+\mathbf{d}_{(y)}\varphi\left(\bar{\Lambda}^d{}_a\mathbf{f}_d\right)\\
&\quad +\frac{1}{2}\tilde{S}_a{}^{bc}\bar{\Lambda}^d{}_b\bar{\Lambda}^e{}_c\mathbf{f}_d\mathbf{f}_e-\bar{\Lambda}^c{}_a\mathbf{d}_{(y)}\mathbf{f}_c\\
&= \tilde{\gamma}^b_a\ {}^d\bar{\Lambda}^c{}_b\mathbf{f}_d\mathbf{f}_c-\mathbf{d}_{(y)}\bar{\Lambda}^c{}_a\mathbf{f}_c+\mathbf{d}_{(y)}\varphi\left(\bar{\Lambda}^d{}_a\mathbf{f}_d\right)\\
&\quad +\frac{1}{2}\tilde{S}_a{}^{bc}\bar{\Lambda}^d{}_b\bar{\Lambda}^e{}_c\mathbf{f}_d\mathbf{f}_e
\end{aligned}$$

$$-\bar{\Lambda}^c{}_a\left(\boldsymbol{\gamma}^b{}_c\mathbf{f}_b+\left(\mathbf{d}_{(y)}\varphi\right)\mathbf{f}_c+\frac{1}{2}\tilde{S}_c^{\ bd}\mathbf{f}_b\mathbf{f}_d\right).$$

Comparing like terms

$$\begin{aligned}0 &= \tilde{\gamma}^b_{\ a}{}^{\ d}\bar{\Lambda}^c{}_b\mathbf{f}_d\mathbf{f}_c-\mathbf{d}_{(y)}\bar{\Lambda}^c{}_a\mathbf{f}_c-\bar{\Lambda}^c{}_a\gamma^b_{\ c}{}^{\ d}\mathbf{f}_d\mathbf{f}_b\\ &\quad+\mathbf{d}_{(y)}\varphi\left(\bar{\Lambda}^d{}_a\mathbf{f}_d\right)-\bar{\Lambda}^c{}_a\left(\mathbf{d}_{(y)}\varphi\right)\mathbf{f}_c\\ &\quad+\frac{1}{2}S_a^{\ bc}\bar{\Lambda}^d{}_b\bar{\Lambda}^e{}_c\mathbf{f}_d\mathbf{f}_e-\bar{\Lambda}^c{}_a\frac{1}{2}S_c^{\ bd}\mathbf{f}_b\mathbf{f}_d,\end{aligned}$$

so the metric terms cancel. We then have

$$\begin{aligned}\tilde{\boldsymbol{\gamma}}^b_{\ a} &= \Lambda^b{}_c\bar{\Lambda}^d{}_a\boldsymbol{\gamma}^c{}_d-\bar{\Lambda}^c{}_a\mathbf{d}_{(y)}\Lambda^b{}_c\\ \tilde{\gamma}^b_{\ a}{}^{\ e} &= \Lambda^b{}_c\Lambda^e{}_f\bar{\Lambda}^d{}_a\gamma^c_{\ d}{}^{\ f}-\bar{\Lambda}^c{}_a\Lambda^e{}_f\partial^f\Lambda^b{}_c\\ \tilde{S}_a^{\ bc} &= \bar{\Lambda}^f{}_aS_f^{\ de}\Lambda^b{}_d\Lambda^c{}_e.\end{aligned}$$

This leaves the two cross-terms, where we have now determined the transformations of $\boldsymbol{\sigma}^a_{\ b}$ and $\boldsymbol{\gamma}^a_{\ b}$.

Transforming the first cross-term equation

The first cross-term,

$$\begin{aligned}\mathbf{d}_{(y)}\mathbf{e}^a &= \left(-\gamma^a_{\ c}{}^{\ b}-\eta^{ad}\mu^b_{\ dc}+W_c\eta^{ab}+T^{ab}{}_c\right)\mathbf{f}_b\mathbf{e}^c\\ &\quad+\frac{1}{2}\eta_{cb}\mathbf{d}_{(y)}\eta^{ac}\mathbf{e}^b+\frac{1}{2}\left(\mathbf{d}_{(x)}\eta^{ab}\right)\mathbf{f}_b\\ &= \left(-\gamma^a_{\ c}{}^{\ b}-\eta^{ad}\mu^b_{\ dc}+W_c\eta^{ab}+T^{ab}{}_c\right)\mathbf{f}_b\mathbf{e}^c\\ &\quad-\left(\mathbf{d}_{(y)}\varphi\right)\mathbf{e}^a-\eta^{ab}\mathbf{d}_{(x)}\varphi\mathbf{f}_b,\end{aligned}$$

becomes

$$\begin{aligned}\mathbf{d}_{(y)}\left(\Lambda^a{}_b\mathbf{e}^b\right) &= \left(-\tilde{\gamma}^a_{\ c}{}^{\ b}-\eta^{ad}\tilde{\mu}^b_{\ dc}+\tilde{W}_c\eta^{ab}+\tilde{T}^{ab}{}_c\right)\left(\bar{\Lambda}^e{}_b\mathbf{f}_e\right)\left(\Lambda^c{}_f\mathbf{e}^f\right)\\ &\quad-\left(\mathbf{d}_{(y)}\varphi\right)\Lambda^a{}_b\mathbf{e}^b-\mathbf{d}_{(x)}\varphi\left(\eta^{ab}\bar{\Lambda}^c{}_b\mathbf{f}_c\right)\end{aligned}$$

$$
\begin{aligned}
0 &= \left(-\tilde{\gamma}^{a}_{\ c}{}^{\ b} - \eta^{ad}\tilde{\mu}^{b}_{\ dc} + \tilde{W}_{c}\eta^{ab} + \tilde{T}^{ab}{}_{c}\right)\left(\bar{\Lambda}^{e}_{\ b}\mathbf{f}_{e}\right)\left(\Lambda^{c}_{\ f}\mathbf{e}^{f}\right) \\
&\quad - \left(\mathbf{d}_{(y)}\varphi\right)\Lambda^{a}_{\ b}\mathbf{e}^{b} - \mathbf{d}_{(x)}\varphi\left(\eta^{ab}\bar{\Lambda}^{c}_{\ b}\mathbf{f}_{c}\right) \\
&\quad - \mathbf{d}_{(y)}\Lambda^{a}_{\ b}\mathbf{e}^{b} - \Lambda^{a}_{\ b}\mathbf{d}_{(y)}\mathbf{e}^{b} \\
&= \left(-\tilde{\gamma}^{a}_{\ c}{}^{\ b}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} - \eta^{ad}\tilde{\mu}^{b}_{\ dc}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} + \tilde{W}_{c}\eta^{ab}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} + \tilde{T}^{ab}{}_{c}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f}\right)\mathbf{f}_{e}\mathbf{e}^{f} \\
&\quad - \mathbf{d}_{(y)}\Lambda^{a}_{\ b}\mathbf{e}^{b} + \left(\Lambda^{a}_{\ b}\gamma^{b}_{\ c}{}^{\ e} + \Lambda^{a}_{\ b}\eta^{bd}\mu^{e}_{\ dc} - \Lambda^{a}_{\ b}W_{c}\eta^{be} - \Lambda^{a}_{\ b}T^{be}{}_{c}\right)\mathbf{f}_{e}\mathbf{e}^{c} \\
&\quad - \left(\mathbf{d}_{(y)}\varphi\right)\Lambda^{a}_{\ b}\mathbf{e}^{b} - \mathbf{d}_{(x)}\varphi\left(\eta^{ab}\bar{\Lambda}^{c}_{\ b}\mathbf{f}_{c}\right) + \Lambda^{a}_{\ b}\left(\mathbf{d}_{(y)}\varphi\right)\mathbf{e}^{b} + \Lambda^{a}_{\ b}\eta^{bc}\mathbf{d}_{(x)}\varphi\mathbf{f}_{c}.
\end{aligned}
$$

Checking the metric terms,

$$
\begin{aligned}
-\left(\mathbf{d}_{(y)}\varphi\right)\Lambda^{a}_{\ b}\mathbf{e}^{b} + \Lambda^{a}_{\ b}\left(\mathbf{d}_{(y)}\varphi\right)\mathbf{e}^{b} &= 0 \\
-\mathbf{d}_{(x)}\varphi\left(\eta^{ab}\bar{\Lambda}^{c}_{\ b}\mathbf{f}_{c}\right) + \Lambda^{a}_{\ b}\eta^{bc}\mathbf{d}_{(x)}\varphi\mathbf{f}_{c} &= \Lambda^{a}_{\ b}\left(\eta^{bc} - \eta^{de}\bar{\Lambda}^{b}_{\ d}\bar{\Lambda}^{c}_{\ e}\right)\mathbf{d}_{(x)}\varphi\mathbf{f}_{c} \\
&= 0.
\end{aligned}
$$

Then, there remains

$$
\begin{aligned}
0 &= \left(-\tilde{\gamma}^{a}_{\ c}{}^{\ b}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} - \eta^{ad}\tilde{\mu}^{b}_{\ dc}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} + \tilde{W}_{c}\eta^{ab}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} + \tilde{T}^{ab}{}_{c}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f}\right)\mathbf{f}_{e}\mathbf{e}^{f} \\
&\quad - \mathbf{d}_{(y)}\Lambda^{a}_{\ b}\mathbf{e}^{b} + \left(\Lambda^{a}_{\ b}\gamma^{b}_{\ f}{}^{\ e} + \Lambda^{a}_{\ b}\eta^{bd}\mu^{e}_{\ df} - \Lambda^{a}_{\ b}W_{f}\eta^{be} - \Lambda^{a}_{\ b}T^{be}{}_{f}\right)\mathbf{f}_{e}\mathbf{e}^{f} \\
0 &= -\left(\Lambda^{a}_{\ g}\Lambda^{b}_{\ h}\bar{\Lambda}^{d}_{\ c}\gamma^{g}_{\ d}{}^{\ h} - \bar{\Lambda}^{g}_{\ c}\Lambda^{b}_{\ h}\partial^{h}\Lambda^{a}_{\ g}\right)\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} - \partial^{e}\Lambda^{a}_{\ f} + \Lambda^{a}_{\ b}\gamma^{b}_{\ f}{}^{\ e} \\
&\quad - \eta^{ad}\tilde{\mu}^{b}_{\ dc}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} + \Lambda^{a}_{\ b}\eta^{bd}\mu^{e}_{\ df} - \Lambda^{a}_{\ b}W_{f}\eta^{be} + \tilde{W}_{c}\eta^{ab}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} \\
&\quad + \tilde{T}^{ab}{}_{c}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} - \Lambda^{a}_{\ b}T^{be}{}_{f} \\
&= -\Lambda^{a}_{\ g}\gamma^{g}_{\ f}{}^{\ e} + \partial^{e}\Lambda^{a}_{\ f} - \partial^{e}\Lambda^{a}_{\ f} + \Lambda^{a}_{\ b}\gamma^{b}_{\ f}{}^{\ e} \\
&\quad - \left(\eta^{ad}\tilde{\mu}^{b}_{\ dc}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} - \Lambda^{a}_{\ b}\eta^{bd}\mu^{e}_{\ df}\right) + \Lambda^{a}_{\ b}\left(\tilde{W}_{c}\Lambda^{c}_{\ f} - W_{f}\right)\eta^{be} \\
&\quad + \left(\tilde{T}^{ab}{}_{c}\bar{\Lambda}^{e}_{\ b}\Lambda^{c}_{\ f} - \Lambda^{a}_{\ b}T^{be}{}_{f}\right)
\end{aligned}
$$

and we, therefore, have

$$
\tilde{\mu}^{a}_{\ bc} = \mu^{d}_{\ ef}\Lambda^{a}_{\ d}\bar{\Lambda}^{e}_{\ b}\bar{\Lambda}^{f}_{\ c},
$$

$$\begin{aligned}\tilde{W}_c &= \bar{\Lambda}^b{}_c W_b,\\ \tilde{T}^{ab}{}_c &= T^{cd}{}_e \Lambda^a{}_c \Lambda^b{}_d \bar{\Lambda}^e{}_c.\end{aligned}$$

Transforming the momentum cross-term equation

Finally, we look at the cross-term from the momentum basis,

$$\begin{aligned}\mathbf{d}_{(x)}\mathbf{f}_a &= \boldsymbol{\sigma}^b{}_a \mathbf{f}_b + \frac{1}{2}\eta^{bc}\left(\mathbf{d}_{(x)}\eta_{ab}\right)\mathbf{f}_c\\ &\quad -\frac{1}{2}\left(\mathbf{d}_{(y)}\eta_{ab} - 2\eta_{ac}\boldsymbol{\rho}^c{}_b + 2W^c\mathbf{f}_c\eta_{ab}\right)\mathbf{e}^b + S_a^{\ b}{}_{\ c}\mathbf{f}_b\mathbf{e}^c.\end{aligned}$$

Changing the basis, this turns into

$$\begin{aligned}\mathbf{d}_{(x)}\left(\bar{\Lambda}^c{}_a\mathbf{f}_c\right) &= \tilde{\sigma}^b{}_{ad}\left(\Lambda^d{}_e\mathbf{e}^e\right)\left(\bar{\Lambda}^c{}_b\mathbf{f}_c\right) + \frac{1}{2}\eta^{bc}\Lambda^d{}_e\mathbf{e}^e\partial_d\eta_{ab}\left(\bar{\Lambda}^f{}_c\mathbf{f}_f\right)\\ &\quad -\frac{1}{2}\left(\bar{\Lambda}^f{}_c\mathbf{f}_f\right)\left(\Lambda^b{}_d\mathbf{e}^d\right)\partial^c\eta_{ab} + \eta_{ac}\tilde{\rho}^c{}_b{}^e\left(\bar{\Lambda}^f{}_e\mathbf{f}_f\right)\left(\Lambda^b{}_d\mathbf{e}^d\right)\\ &\quad -\tilde{W}^c\left(\bar{\Lambda}^f{}_c\mathbf{f}_f\right)\eta_{ab}\left(\Lambda^b{}_d\mathbf{e}^d\right) + \tilde{S}_a^{\ b}{}_{\ c}\left(\bar{\Lambda}^f{}_b\mathbf{f}_f\right)\left(\Lambda^c{}_d\mathbf{e}^d\right)\\ 0 &= \left(\Lambda^b{}_e\bar{\Lambda}^d{}_a\boldsymbol{\sigma}^e{}_d - \bar{\Lambda}^d{}_a\mathbf{d}_{(x)}\Lambda^b{}_d\right)\bar{\Lambda}^c{}_b\mathbf{f}_c - \left(\mathbf{d}_{(x)}\bar{\Lambda}^f{}_a\mathbf{f}_f\right) - \bar{\Lambda}^c{}_a\mathbf{d}_{(x)}\mathbf{f}_c\\ &\quad +\frac{1}{2}\eta^{bc}\Lambda^d{}_e\mathbf{e}^e\partial_d\eta_{ab}\left(\bar{\Lambda}^f{}_c\mathbf{f}_f\right) - \frac{1}{2}\left(\bar{\Lambda}^f{}_c\mathbf{f}_f\right)\left(\Lambda^b{}_d\mathbf{e}^d\right)\partial^c\eta_{ab}\\ &\quad +\eta_{ac}\tilde{\rho}^c{}_b{}^e\left(\bar{\Lambda}^f{}_e\mathbf{f}_f\right)\left(\Lambda^b{}_d\mathbf{e}^d\right) - \tilde{W}^c\left(\bar{\Lambda}^f{}_c\mathbf{f}_f\right)\eta_{ab}\left(\Lambda^b{}_d\mathbf{e}^d\right)\\ &\quad +\tilde{S}_a^{\ b}{}_{\ c}\left(\bar{\Lambda}^f{}_b\mathbf{f}_f\right)\left(\Lambda^c{}_d\mathbf{e}^d\right)\\ 0 &= \frac{1}{2}\eta^{bc}\Lambda^d{}_e\bar{\Lambda}^f{}_c\partial_d\eta_{ab}\mathbf{e}^e\mathbf{f}_f + \frac{1}{2}\bar{\Lambda}^f{}_c\Lambda^b{}_e\partial^c\eta_{ab}\mathbf{e}^e\mathbf{f}_f\\ &\quad -\eta_{ac}\tilde{\rho}^c{}_b{}^d\bar{\Lambda}^f{}_d\Lambda^b{}_e\mathbf{e}^e\mathbf{f}_f + \tilde{W}^c\bar{\Lambda}^f{}_c\eta_{ab}\Lambda^b{}_e\mathbf{e}^e\mathbf{f}_f\\ &\quad -\tilde{S}_a^{\ b}{}_{\ c}\bar{\Lambda}^f{}_b\Lambda^c{}_e\mathbf{e}^e\mathbf{f}_f - \frac{1}{2}\bar{\Lambda}^c{}_a\eta^{bf}\left(\partial_e\eta_{cb}\right)\mathbf{e}^e\mathbf{f}_f - \frac{1}{2}\bar{\Lambda}^c{}_a\partial^f\eta_{ce}\mathbf{e}^e\mathbf{f}_f\\ &\quad +\bar{\Lambda}^c{}_a\eta_{cd}\rho^d{}_e{}^f\mathbf{e}^e\mathbf{f}_f - \bar{\Lambda}^c{}_aW^f\eta_{ce}\mathbf{e}^e\mathbf{f}_f + \bar{\Lambda}^c{}_aS_c^{\ f}{}_{\ e}\mathbf{e}^e\mathbf{f}_f.\end{aligned}$$

Substituting the known transformations and collecting terms

$$\begin{aligned}0 &= \bar{\Lambda}^f{}_a\left(\Lambda^d{}_e\tilde{\partial}_d\varphi - \partial_e\varphi\right) + \bar{\Lambda}^b{}_a\eta_{eb}\left(\bar{\Lambda}^f{}_c\tilde{\partial}^c\varphi - \partial^f\varphi\right)\\ &\quad -\bar{\Lambda}^f{}_d\Lambda^b{}_e\eta_{ac}\left(\tilde{\rho}^c{}_b{}^d - \Lambda^c{}_g\rho^g{}_h{}^k\Lambda^d{}_k\bar{\Lambda}^h{}_b\right)\\ &\quad +\bar{\Lambda}^d{}_a\eta_{de}\left(\bar{\Lambda}^f{}_c\tilde{W}^c - W^f\right)\end{aligned}$$

$$-\bar{\Lambda}^f_{\ b}\Lambda^c_{\ e}\left(\tilde{S}_a^{\ b}{}_{\ c} - \bar{\Lambda}^c_{\ a}S_c^{\ g}{}_{\ h}\Lambda^b_{\ g}\bar{\Lambda}^h_{\ c}\right),$$

we have the transformations

$$\begin{aligned}
\tilde{\rho}^c_{\ b}{}^{\ d} &= \Lambda^c_{\ g}\rho^g_{\ h}{}^{\ k}\Lambda^d_{\ k}\bar{\Lambda}^h_{\ b}, \\
\tilde{W}^c &= \Lambda^c_{\ f}W^f, \\
\tilde{S}_a^{\ b}{}_{\ c} &= \bar{\Lambda}^c_{\ a}S_c^{\ g}{}_{\ h}\Lambda^b_{\ g}\bar{\Lambda}^h_{\ c}.
\end{aligned}$$

Summary of Lorentz transformations

Summarizing, we have the connection transformations

$$\begin{aligned}
\tilde{\boldsymbol{\sigma}}^a_{\ b} &= \Lambda^a_{\ e}\bar{\Lambda}^d_{\ b}\boldsymbol{\sigma}^e_{\ d} - \bar{\Lambda}^d_{\ b}\mathbf{d}_{(x)}\Lambda^a_{\ d}, \\
\tilde{\boldsymbol{\gamma}}^b_{\ a} &= \Lambda^b_{\ c}\bar{\Lambda}^d_{\ a}\boldsymbol{\gamma}^c_{\ d} - \bar{\Lambda}^c_{\ a}\mathbf{d}_{(y)}\Lambda^b_{\ c},
\end{aligned}$$

and tensors,

$$\tilde{T}^a_{bc} = \Lambda^a_{\ f}T^f_{de}\bar{\Lambda}^d_{\ b}\bar{\Lambda}^e_{\ c}, \tag{B.1}$$

$$\tilde{\mu}^a_{bc} = \mu^d_{\ ef}\Lambda^a_{\ d}\bar{\Lambda}^e_{\ b}\bar{\Lambda}^f_{\ c}, \tag{B.2}$$

$$\tilde{W}_c = \bar{\Lambda}^b_{\ c}W_b, \tag{B.3}$$

$$\tilde{T}^{ab}_{\ \ c} = T^{cd}_{\ \ e}\Lambda^a_{\ c}\Lambda^b_{\ d}\bar{\Lambda}^e_{\ c}, \tag{B.4}$$

$$\tilde{\rho}^c_{\ b}{}^{\ d} = \Lambda^c_{\ g}\rho^g_{\ h}{}^{\ k}\Lambda^d_{\ k}\bar{\Lambda}^h_{\ b}, \tag{B.5}$$

$$\tilde{W}^c = \Lambda^c_{\ f}W^f, \tag{B.6}$$

$$\tilde{S}_a^{\ bc} = \bar{\Lambda}^f_{\ a}S_f^{\ de}\Lambda^b_{\ d}\Lambda^c_{\ e}, \tag{B.7}$$

$$\tilde{S}_a^{\ b}{}_{\ c} = \bar{\Lambda}^c_{\ a}S_c^{\ g}{}_{\ h}\Lambda^b_{\ g}\bar{\Lambda}^h_{\ c}. \tag{B.8}$$

B.3. Dilatations

Conformal change of basis

Now consider a dilatation. Once again, start with the basis equations

$$\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d{}_c + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}_{(x)}\mathbf{e}^a + \mathbf{d}_{(y)}\mathbf{e}^a &= \mathbf{e}^b\left(\boldsymbol{\sigma}^a{}_b + \boldsymbol{\gamma}^a{}_b\right) + \frac{1}{2}\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac} + \mathbf{d}_{(y)}\eta^{ac}\right)\mathbf{e}^b \\
&\quad + \frac{1}{2}\left(\mathbf{d}_{(x)}\eta^{ab} + 2\eta^{ac}\boldsymbol{\mu}^b{}_c - 2W_c\mathbf{e}^c\eta^{ab}\right)\mathbf{f}_b \\
&\quad + T^{ab}{}_c\mathbf{f}_b\mathbf{e}^c + \frac{1}{2}T^a_{bc}\mathbf{e}^b\mathbf{e}^c, \\
\mathbf{d}\mathbf{f}_a &= \Theta^{bc}_{da}\boldsymbol{\tau}^d{}_c\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + \mathbf{S}_a \\
\mathbf{d}_{(x)}\mathbf{f}_a + \mathbf{d}_{(y)}\mathbf{f}_a &= \left(\boldsymbol{\sigma}^b{}_a + \boldsymbol{\gamma}^b{}_a\right)\mathbf{f}_b + \frac{1}{2}\eta^{bc}\left(\mathbf{d}_{(x)}\eta_{ab} + \mathbf{d}_{(y)}\eta_{ab}\right)\mathbf{f}_c \\
&\quad - \frac{1}{2}\left(\mathbf{d}_{(y)}\eta_{ab} - 2\eta_{ac}\boldsymbol{\rho}^c{}_b + 2W^c\mathbf{f}_c\eta_{ab}\right)\mathbf{e}^b \\
&\quad + S_a{}^b{}_c\mathbf{f}_b\mathbf{e}^c + \frac{1}{2}S_a{}^{bc}\mathbf{f}_b\mathbf{f}_c,
\end{aligned}$$

and let the new basis forms be

$$\begin{aligned}
\tilde{\mathbf{e}}^a &= e^{\phi}\mathbf{e}^a, \\
\tilde{\mathbf{f}}_a &= e^{-\phi}\mathbf{f}_a;
\end{aligned}$$

and therefore,

$$\begin{aligned}
\tilde{\eta}^{ab} &= \left\langle\tilde{\mathbf{e}}^a, \tilde{\mathbf{e}}^b\right\rangle \\
&= e^{2\phi}\left\langle\mathbf{e}^a, \mathbf{e}^b\right\rangle \\
&= e^{2\phi}\eta^{ab}, \\
\tilde{\eta}_{ab} &= e^{-2\phi}\eta_{ab}, \\
\tilde{W}_\mu &= W_\mu + \partial_\mu\phi, \\
\tilde{W}_a &= \tilde{e}_a{}^\mu\left(W_\mu + \partial_\mu\phi\right) \\
&= e^{-\phi}e_a{}^\mu\left(W_\mu + \partial_\mu\phi\right)
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\phi}\left(W_a + \partial_a\phi\right), \\
\tilde{W}^{\mu} &= W^{\mu} + \partial^{\mu}\phi, \\
\tilde{W}^{a} &= \tilde{e}_{\mu}{}^{a}\left(W^{\mu} + \partial^{\mu}\phi\right) \\
&= e^{\phi}\left(W^{a} + \partial^{a}\phi\right).
\end{aligned}
$$

Dilatation of the solder form structure equation

The transformed configuration equation becomes

$$
\begin{aligned}
\mathbf{d}_{(x)}\left(e^{\phi}\mathbf{e}^{a}\right) + \mathbf{d}_{(y)}\left(e^{\phi}\mathbf{e}^{a}\right) &= \left(e^{\phi}\mathbf{e}^{b}\right)\left(\tilde{\boldsymbol{\sigma}}^{a}_{\ b} + \tilde{\boldsymbol{\gamma}}^{a}_{\ b}\right) \\
&\quad + \frac{1}{2}\eta_{cb}e^{-2\phi}\left(\mathbf{d}_{(x)}\left(e^{2\phi}\eta^{ac}\right) + \mathbf{d}_{(y)}\left(e^{2\phi}\eta^{ac}\right)\right)e^{\phi}\mathbf{e}^{b} \\
&\quad + \frac{1}{2}\mathbf{d}_{(x)}\left(e^{2\phi}\eta^{ab}\right)e^{-\phi}\mathbf{f}_{b} + e^{2\phi}\eta^{ac}\tilde{\boldsymbol{\mu}}^{b}{}_{c}e^{-\phi}\mathbf{f}_{b} \\
&\quad - \tilde{W}_{c}e^{3\phi}\mathbf{e}^{c}\eta^{ab}e^{-\phi}\mathbf{f}_{b} + \tilde{T}^{ab}{}_{c}\mathbf{f}_{b}\mathbf{e}^{c} + \frac{1}{2}\tilde{T}^{a}_{\ bc}e^{2\phi}\mathbf{e}^{b}\mathbf{e}^{c} \\
0 &= e^{\phi}\mathbf{e}^{b}\tilde{\boldsymbol{\sigma}}^{a}_{\ b} + e^{\phi}\mathbf{e}^{b}\tilde{\boldsymbol{\gamma}}^{a}_{\ b} \\
&\quad + \eta_{cb}e^{\phi}\mathbf{d}_{(x)}\phi\eta^{ac}\mathbf{e}^{b} + \frac{1}{2}\eta_{cb}e^{\phi}\mathbf{d}_{(x)}\eta^{ac}\mathbf{e}^{b} \\
&\quad + e^{\phi}\delta^{a}_{b}\mathbf{d}_{(y)}\phi\mathbf{e}^{b} + \frac{1}{2}\eta_{cb}e^{\phi}\mathbf{d}_{(y)}\eta^{ac}\mathbf{e}^{b} \\
&\quad + e^{\phi}\eta^{ab}\mathbf{d}_{(x)}\phi\mathbf{f}_{b} + \frac{1}{2}e^{\phi}\mathbf{d}_{(x)}\eta^{ab}\mathbf{f}_{b} + e^{\phi}\eta^{ac}\tilde{\boldsymbol{\mu}}^{b}{}_{c}\mathbf{f}_{b} \\
&\quad - \left(W_{c} + \partial_{c}\phi\right)e^{\phi}\eta^{ab}\mathbf{e}^{c}\mathbf{f}_{b} + \tilde{T}^{ab}{}_{c}\mathbf{f}_{b}\mathbf{e}^{c} \\
&\quad + \frac{1}{2}\tilde{T}^{a}_{\ bc}e^{2\phi}\mathbf{e}^{b}\mathbf{e}^{c} \\
&\quad - e^{\phi}\mathbf{d}_{(x)}\phi\mathbf{e}^{a} - e^{\phi}\mathbf{d}_{(y)}\phi\mathbf{e}^{a} \\
&\quad - e^{\phi}\mathbf{e}^{b}\boldsymbol{\sigma}^{a}_{\ b} - e^{\phi}\mathbf{e}^{b}\boldsymbol{\gamma}^{a}_{\ b} \\
&\quad - \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}_{(x)}\eta^{ac}\mathbf{e}^{b} - \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}_{(y)}\eta^{ac}\mathbf{e}^{b} \\
&\quad - \frac{1}{2}e^{\phi}\mathbf{d}_{(x)}\eta^{ab}\mathbf{f}_{b} - e^{\phi}\eta^{ac}\boldsymbol{\mu}^{b}{}_{c}\mathbf{f}_{b} + e^{\phi}W_{c}\mathbf{e}^{c}\eta^{ab}\mathbf{f}_{b} \\
&\quad - e^{\phi}T^{ab}{}_{c}\mathbf{f}_{b}\mathbf{e}^{c} - e^{\phi}\frac{1}{2}T^{a}_{\ bc}\mathbf{e}^{b}\mathbf{e}^{c},
\end{aligned}
$$

so

$$
0 = e^{\phi}\mathbf{e}^{b}\left(\tilde{\boldsymbol{\sigma}}^{a}_{\ b} - \boldsymbol{\sigma}^{a}_{\ b}\right)
$$

$$
\begin{aligned}
&+e^{\phi}\eta^{ac}\left(\tilde{\boldsymbol{\mu}}^{b}{}_{c}-\boldsymbol{\mu}^{b}{}_{c}\right)\mathbf{f}_{b}+e^{\phi}\left(\mathbf{e}^{b}\tilde{\boldsymbol{\gamma}}^{a}{}_{b}-\mathbf{e}^{b}\boldsymbol{\gamma}^{a}{}_{b}\right)\\
&+\left(\tilde{T}^{ab}{}_{c}-e^{\phi}T^{ab}{}_{c}\right)\mathbf{f}_{b}\mathbf{e}^{c}+\frac{1}{2}e^{2\phi}\left(\tilde{T}^{a}{}_{bc}-e^{-\phi}T^{a}{}_{bc}\right)\mathbf{e}^{b}\mathbf{e}^{c};
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\tilde{\boldsymbol{\sigma}}^{a}{}_{b} &= \boldsymbol{\sigma}^{a}{}_{b},\\
\tilde{\boldsymbol{\mu}}^{b}{}_{c} &= \boldsymbol{\mu}^{b}{}_{c},\\
\tilde{\boldsymbol{\gamma}}^{a}{}_{b} &= \boldsymbol{\gamma}^{a}{}_{b},\\
\tilde{T}^{ab}{}_{c} &= e^{\phi}T^{ab}{}_{c},\\
\tilde{T}^{a}{}_{bc} &= e^{-\phi}T^{a}{}_{bc}.
\end{aligned}
$$

Dilatations of the cosolder structure equation

The cosolder equation,

$$
\begin{aligned}
\mathbf{d}_{(x)}\mathbf{f}_{a}+\mathbf{d}_{(y)}\mathbf{f}_{a} &= \left(\boldsymbol{\sigma}^{b}{}_{a}+\boldsymbol{\gamma}^{b}{}_{a}\right)\mathbf{f}_{b}+\frac{1}{2}\eta^{bc}\left(\mathbf{d}_{(x)}\eta_{ab}+\mathbf{d}_{(y)}\eta_{ab}\right)\mathbf{f}_{c}\\
&\quad-\frac{1}{2}\left(\mathbf{d}_{(y)}\eta_{ab}-2\eta_{ac}\boldsymbol{\rho}^{c}{}_{b}+2W^{c}\mathbf{f}_{c}\eta_{ab}\right)\mathbf{e}^{b}\\
&\quad+S_{a}{}^{b}{}_{c}\mathbf{f}_{b}\mathbf{e}^{c}+\frac{1}{2}S_{a}{}^{bc}\mathbf{f}_{b}\mathbf{f}_{c},
\end{aligned}
$$

transforms into

$$
\begin{aligned}
\mathbf{d}_{(x)}\left(e^{-\phi}\mathbf{f}_{a}\right)+\mathbf{d}_{(y)}\left(e^{-\phi}\mathbf{f}_{a}\right) &= \left(\tilde{\boldsymbol{\sigma}}^{b}{}_{a}+\tilde{\boldsymbol{\gamma}}^{b}{}_{a}\right)e^{-\phi}\mathbf{f}_{b}\\
&\quad+\frac{1}{2}e^{2\phi}\eta^{bc}\left(\mathbf{d}_{(x)}\left(e^{-2\phi}\eta_{ab}\right)\right)e^{-\phi}\mathbf{f}_{c}\\
&\quad+\frac{1}{2}e^{2\phi}\eta^{bc}\left(\mathbf{d}_{(y)}\left(e^{-2\phi}\eta_{ab}\right)\right)e^{-\phi}\mathbf{f}_{c}\\
&\quad-\frac{1}{2}\left(\mathbf{d}_{(y)}\left(e^{-2\phi}\eta_{ab}\right)-2\left(e^{-2\phi}\eta_{ac}\right)\tilde{\boldsymbol{\rho}}^{c}{}_{b}\right)e^{\phi}\mathbf{e}^{b}\\
&\quad-\frac{1}{2}\left(2e^{\phi}\left(W^{c}+\partial^{c}\phi\right)e^{-\phi}\mathbf{f}_{c}\left(e^{-2\phi}\eta_{ab}\right)\right)e^{\phi}\mathbf{e}^{b}\\
&\quad+\tilde{S}_{a}{}^{b}{}_{c}\mathbf{f}_{b}\mathbf{e}^{c}+\frac{1}{2}e^{-2\phi}\tilde{S}_{a}{}^{bc}\mathbf{f}_{b}\mathbf{f}_{c}\\
0 &= \tilde{\boldsymbol{\sigma}}^{b}{}_{a}e^{-\phi}\mathbf{f}_{b}+\tilde{\boldsymbol{\gamma}}^{b}{}_{a}e^{-\phi}\mathbf{f}_{b}-e^{-\phi}\mathbf{d}_{(x)}\phi\mathbf{f}_{a}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(x)}\eta_{ab}\mathbf{f}_c - e^{-\phi}\mathbf{d}_{(y)}\phi\mathbf{f}_a \\
&+\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(y)}\eta_{ab}\mathbf{f}_c + e^{-\phi}\mathbf{d}_{(x)}\phi\mathbf{f}_a \\
&-e^{-\phi}\mathbf{d}_{(x)}\mathbf{f}_a + e^{-\phi}\mathbf{d}_{(y)}\phi\mathbf{f}_a - e^{-\phi}\mathbf{d}_{(y)}\mathbf{f}_a \\
&+e^{-\phi}\eta_{ab}\mathbf{d}_{(y)}\phi\mathbf{e}^b - \frac{1}{2}e^{-\phi}\mathbf{d}_{(y)}\eta_{ab}\mathbf{e}^b \\
&+e^{-\phi}\eta_{ac}\tilde{\boldsymbol{\rho}}^c{}_b\mathbf{e}^b \\
&-e^{-\phi}\eta_{ab}W^c\mathbf{f}_c\mathbf{e}^b - e^{-\phi}\eta_{ab}\partial^c\phi\mathbf{f}_c\mathbf{e}^b \\
&+\tilde{S}_a{}^b{}_c\mathbf{f}_b\mathbf{e}^c + \frac{1}{2}e^{-2\phi}\tilde{S}_a{}^{bc}\mathbf{f}_b\mathbf{f}_c,
\end{aligned}
$$

so

$$
\begin{aligned}
0 \;=\; & \tilde{\boldsymbol{\sigma}}^b{}_a e^{-\phi}\mathbf{f}_b - e^{-\phi}\boldsymbol{\sigma}^b{}_a\mathbf{f}_b + \tilde{\boldsymbol{\gamma}}^b{}_a e^{-\phi}\mathbf{f}_b - e^{-\phi}\boldsymbol{\gamma}^b{}_a\mathbf{f}_b \\
& +e^{-\phi}\eta_{ac}\tilde{\boldsymbol{\rho}}^c{}_b\mathbf{e}^b - e^{-\phi}\eta_{ac}\boldsymbol{\rho}^c{}_b\mathbf{e}^b \\
& +\tilde{S}_a{}^b{}_c\mathbf{f}_b\mathbf{e}^c - e^{-\phi}S_a{}^b{}_c\mathbf{f}_b\mathbf{e}^c + \frac{1}{2}e^{-2\phi}\tilde{S}_a{}^{bc}\mathbf{f}_b\mathbf{f}_c - \frac{1}{2}e^{-\phi}S_a{}^{bc}\mathbf{f}_b\mathbf{f}_c;
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\tilde{\boldsymbol{\sigma}}^b{}_a &= \boldsymbol{\sigma}^b{}_a, \\
\tilde{\boldsymbol{\gamma}}^b{}_a &= \boldsymbol{\gamma}^b{}_a, \\
\tilde{\boldsymbol{\rho}}^c{}_b &= \boldsymbol{\rho}^c{}_b, \\
\tilde{S}_a{}^b{}_c &= e^{-\phi}S_a{}^b{}_c, \\
\tilde{S}_a{}^{bc} &= e^{\phi}S_a{}^{bc}.
\end{aligned}
$$

We conclude the only nontensors are $\boldsymbol{\sigma}^b{}_a$ and $\boldsymbol{\gamma}^b{}_a$ under Lorentz transformations, and W_a and W^a under dilatations.

Appendix C

Homogeneous biconformal solution in the orthonormal basis

C.1. Making the known solution block diagonal

Orthogonal Lagrangian basis

We have the known solution

$$\begin{aligned}
\boldsymbol{\omega}^{\alpha}_{\beta} &= 2\Delta^{\alpha\mu}_{\nu\beta} s_{\mu} \mathbf{d}w^{\nu}, \\
\boldsymbol{\omega}^{\alpha} &= \mathbf{d}w^{\alpha}, \\
\boldsymbol{\omega}_{\alpha} &= \mathbf{d}s_{\alpha} + \frac{1}{2}\left(\delta_{\alpha\beta}s^2 - s_{\alpha}s_{\beta}\right)\mathbf{d}w^{\beta} \\
&= \mathbf{d}s_{\alpha} + \frac{1}{2}k_{\alpha\beta}\mathbf{d}w^{\beta}, \\
\boldsymbol{\omega} &= -s_{\beta}\mathbf{d}w^{\beta}.
\end{aligned}$$

Suppose we find linear combinations of these $\boldsymbol{\kappa}^{\beta}, \boldsymbol{\lambda}_{\alpha}$ that make the metric block diagonal, with $\boldsymbol{\lambda}_{\alpha} = 0$ and $\boldsymbol{\kappa}^{\beta} = 0$ giving Lagrangian submanifolds. Then any further transformation,

$$\begin{aligned}
\tilde{\boldsymbol{\kappa}}^{\alpha} &= A^{\alpha}_{\beta}\boldsymbol{\kappa}^{\beta}, \\
\tilde{\boldsymbol{\lambda}}_{\alpha} &= B^{\beta}_{\alpha}\boldsymbol{\lambda}_{\beta},
\end{aligned}$$

leaves these submanifolds unchanged and is therefore equivalent. Now suppose one of the linear combinations is

$$\begin{aligned}
\tilde{\boldsymbol{\lambda}}_{\alpha} &= \alpha A^{\beta}_{\alpha}\mathbf{d}s_{\beta} + \beta\tilde{C}_{\alpha\mu}\mathbf{d}w^{\mu} \\
&= A^{\beta}_{\alpha}\alpha\mathbf{d}s_{\beta} + \beta A^{\beta}_{\alpha}C_{\beta\mu}\mathbf{d}w^{\mu} \\
&= A^{\beta}_{\alpha}\left(\alpha\mathbf{d}s_{\beta} + \beta C_{\beta\mu}\mathbf{d}w^{\mu}\right),
\end{aligned}$$

where the constants are required to keep the transformation nondegenerate. Then

$$\boldsymbol{\lambda}_{\alpha} = \alpha\mathbf{d}s_{\alpha} + \beta C_{\alpha\beta}\mathbf{d}w^{\beta}$$

spans the same subspace. A similar argument holds for $\tilde{\boldsymbol{\kappa}}^\beta$, so if we can find a basis at all, there is also one of the form

$$\begin{aligned}\boldsymbol{\lambda}_\alpha &= \alpha \mathbf{d}s_\alpha + \beta C_{\alpha\beta}\mathbf{d}w^\beta,\\ \boldsymbol{\kappa}^\alpha &= \mu \mathbf{d}w^\alpha + \nu B^{\alpha\beta}\mathbf{d}s_\beta.\end{aligned}$$

Now check the symplectic condition,

$$\begin{aligned}\boldsymbol{\kappa}^\alpha\boldsymbol{\lambda}_\alpha &= \left(\mu \mathbf{d}w^\alpha + \nu B^{\alpha\beta}\mathbf{d}s_\beta\right)(\alpha \mathbf{d}s_\alpha + \beta C_{\alpha\mu}\mathbf{d}w^\mu)\\ &= \alpha\mu \mathbf{d}w^\alpha\mathbf{d}s_\alpha + \mu\beta C_{\alpha\mu}\mathbf{d}w^\alpha\mathbf{d}w^\mu + \nu\alpha B^{\alpha\beta}\mathbf{d}s_\beta\mathbf{d}s_\alpha + \nu\beta C_{\alpha\mu}B^{\alpha\beta}\mathbf{d}s_\beta\mathbf{d}w^\mu.\end{aligned}$$

To have $\boldsymbol{\kappa}^\alpha\boldsymbol{\lambda}_\alpha = \mathbf{d}w^\alpha\mathbf{d}s_\alpha$, $B^{\alpha\beta}$ and $C_{\alpha\beta}$ must be symmetric and

$$\begin{aligned}\alpha\mu 1 - \nu\beta B^t C &= 1,\\ B = B^t &= \frac{\alpha\mu - 1}{\nu\beta}C^{-1} \equiv \alpha\beta\bar{C}.\end{aligned}$$

We then have

$$\begin{aligned}\boldsymbol{\lambda}_\alpha &= \alpha \mathbf{d}s_\alpha + \beta C_{\alpha\beta}\mathbf{d}w^\beta,\\ \boldsymbol{\kappa}^\alpha &= \mu \mathbf{d}w^\alpha + \frac{\alpha\mu - 1}{\beta}\bar{C}^{\alpha\beta}\mathbf{d}s_\beta.\end{aligned}$$

Now look at the inner products. We know

$$\begin{bmatrix} \langle \mathbf{d}w^\alpha, \mathbf{d}w^\beta\rangle & \langle \mathbf{d}w^\alpha, \mathbf{d}s_\beta\rangle \\ \langle \mathbf{d}s_\alpha, \mathbf{d}w^\beta\rangle & \langle \mathbf{d}s_\alpha, \mathbf{d}s_\beta\rangle \end{bmatrix} \equiv \begin{bmatrix} 0 & \delta^\alpha_\beta \\ \delta^\beta_\alpha & -k_{\alpha\beta}\end{bmatrix},$$

so

$$\begin{aligned}0 &= \langle \boldsymbol{\kappa}^\alpha, \boldsymbol{\lambda}_\beta\rangle\\ &= \left\langle \mu \mathbf{d}w^\alpha + \frac{\alpha\mu - 1}{\beta}\bar{C}^{\alpha\mu}\mathbf{d}s_\mu, \alpha \mathbf{d}s_\beta + \beta C_{\beta\nu}\mathbf{d}w^\nu\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \mu \mathbf{d}w^\alpha, \alpha \mathbf{d}s_\beta \rangle + \left\langle \frac{\alpha\mu - 1}{\beta} \bar{C}^{\alpha\mu} \mathbf{d}s_\mu, \alpha \mathbf{d}s_\beta \right\rangle \\
&\quad + \left\langle \frac{\alpha\mu - 1}{\beta} \bar{C}^{\alpha\mu} \mathbf{d}s_\mu, \beta C_{\beta\nu} \mathbf{d}w^\nu \right\rangle \\
&= \mu\alpha\delta^\alpha_\beta - \frac{1}{\beta}\alpha\left(\alpha\mu - 1\right)\bar{C}^{\alpha\mu}k_{\mu\beta} + \left(\alpha\mu - 1\right)\bar{C}^{\alpha\mu}C_{\beta\mu} \\
&= \left(2\alpha\mu - 1\right)\delta^\alpha_\beta - \frac{1}{\beta}\alpha\left(\alpha\mu - 1\right)\bar{C}^{\alpha\mu}k_{\mu\beta} \\
\left(2\alpha\mu - 1\right)\delta^\alpha_\beta &= \frac{1}{\beta}\alpha\left(\alpha\mu - 1\right)\bar{C}^{\alpha\mu}k_{\mu\beta} \\
C_{\nu\beta} &= \frac{\alpha\left(\alpha\mu - 1\right)}{\beta\left(2\alpha\mu - 1\right)}k_{\nu\beta} \\
\bar{C}^{\alpha\beta} &= \frac{\beta\left(2\alpha\mu - 1\right)}{\alpha\left(\alpha\mu - 1\right)}k^{\alpha\beta}.
\end{aligned}
$$

Therefore, if the required basis exists, then there is an equivalent one of the form

$$
\begin{aligned}
\boldsymbol{\lambda}_\alpha &= \alpha \mathbf{d}s_\alpha + \frac{\alpha\left(\alpha\mu - 1\right)}{\left(2\alpha\mu - 1\right)}k_{\alpha\beta}\mathbf{d}w^\beta, \\
\boldsymbol{\kappa}^\alpha &= \mu \mathbf{d}w^\alpha + \frac{2\alpha\mu - 1}{\alpha}k^{\alpha\beta}\mathbf{d}s_\beta.
\end{aligned}
$$

The metric

The metric on the submanifolds, in the given coordinates, now follows from the remaining inner products,

$$
\begin{aligned}
\left\langle \boldsymbol{\kappa}^\alpha, \boldsymbol{\kappa}^\beta \right\rangle &= \left\langle \mu \mathbf{d}w^\alpha + \frac{2\alpha\mu - 1}{\alpha}k^{\alpha\mu}\mathbf{d}s_\mu, \mu \mathbf{d}w^\beta + \frac{2\alpha\mu - 1}{\alpha}k^{\beta\nu}\mathbf{d}s_\nu \right\rangle \\
&= \left\langle \mu \mathbf{d}w^\alpha, \frac{2\alpha\mu - 1}{\alpha}k^{\beta\nu}\mathbf{d}s_\nu \right\rangle + \left\langle \frac{2\alpha\mu - 1}{\alpha}k^{\alpha\mu}\mathbf{d}s_\mu, \mu \mathbf{d}w^\beta \right\rangle \\
&\quad + \left\langle \frac{2\alpha\mu - 1}{\alpha}k^{\alpha\mu}\mathbf{d}s_\mu, \frac{2\alpha\mu - 1}{\alpha}k^{\beta\nu}\mathbf{d}s_\nu \right\rangle \\
&= 2\mu\frac{2\alpha\mu - 1}{\alpha}k^{\alpha\beta} - k_{\mu\nu}\left(\frac{2\alpha\mu - 1}{\alpha}\right)^2 k^{\alpha\mu}k^{\beta\nu} \\
&= \frac{2\alpha\mu - 1}{\alpha^2}k^{\alpha\beta},
\end{aligned}
$$

and

$$\begin{aligned}
\langle \boldsymbol{\lambda}_\alpha, \boldsymbol{\lambda}_\beta \rangle &= \left\langle \alpha \mathbf{d}s_\alpha + \frac{\alpha\left(\alpha\mu - 1\right)}{\left(2\alpha\mu - 1\right)} k_{\alpha\mu} \mathbf{d}w^\mu, \alpha \mathbf{d}s_\beta + \frac{\alpha\left(\alpha\mu - 1\right)}{\left(2\alpha\mu - 1\right)} k_{\beta\nu} \mathbf{d}w^\nu \right\rangle \\
&= \langle \alpha \mathbf{d}s_\alpha, \alpha \mathbf{d}s_\beta \rangle + \left\langle \alpha \mathbf{d}s_\alpha, \frac{\alpha\left(\alpha\mu - 1\right)}{\left(2\alpha\mu - 1\right)} k_{\beta\nu} \mathbf{d}w^\nu \right\rangle \\
&\quad + \left\langle \frac{\alpha\left(\alpha\mu - 1\right)}{\left(2\alpha\mu - 1\right)} k_{\alpha\mu} \mathbf{d}w^\mu, \alpha \mathbf{d}s_\beta \right\rangle \\
&= -\alpha^2 k_{\alpha\beta} + \frac{\alpha^2\left(\alpha\mu - 1\right)}{\left(2\alpha\mu - 1\right)} k_{\beta\alpha} + \frac{\alpha^2\left(\alpha\mu - 1\right)}{\left(2\alpha\mu - 1\right)} k_{\alpha\beta} \\
&= -\frac{\alpha^2}{2\alpha\mu - 1} k_{\alpha\beta}.
\end{aligned}$$

This shows the metric on the Lagrangian submanifolds is proportional to $k_{\alpha\beta}$, and we normalize with

$$\begin{aligned}
\frac{2\alpha\mu - 1}{\alpha^2} &= k \equiv \pm 1 \\
2\alpha\mu - 1 &= \alpha^2 k \\
\mu &= \frac{1 + k\alpha^2}{2\alpha}.
\end{aligned}$$

Therefore, we have a block diagonalization of the form

$$\begin{aligned}
\boldsymbol{\kappa}^\alpha &= \frac{k}{2\alpha}\left(\left(\alpha^2 + k\right)\mathbf{d}w^\alpha + 2\alpha^2 k^{\alpha\beta}\mathbf{d}s_\beta\right), \\
\boldsymbol{\lambda}_\alpha &= \frac{1}{2\alpha}\left(2\alpha^2 \mathbf{d}s_\alpha + \left(\alpha^2 - k\right) k_{\alpha\beta}\mathbf{d}w^\beta\right).
\end{aligned}$$

Let $\alpha = k\beta$,

$$\begin{aligned}
\boldsymbol{\kappa}^\alpha &= \frac{k}{2\beta}\left(\left(k\beta^2 + 1\right)\mathbf{d}w^\alpha + 2k\beta^2 k^{\alpha\beta}\mathbf{d}s_\beta\right), \\
\boldsymbol{\lambda}_\alpha &= \frac{1}{2\beta}\left(2k\beta^2 \mathbf{d}s_\alpha + \left(k\beta^2 - 1\right) k_{\alpha\beta}\mathbf{d}w^\beta\right).
\end{aligned}$$

Now that we have established the metric

$$k_{\alpha\beta} = s^2\left(\delta_{\alpha\beta} - \frac{2}{s^2} s_\alpha s_\beta\right),$$

where $\delta_{\alpha\beta}$ is the Euclidean metric and $s^2 = \delta^{\alpha\beta}s_\alpha s_\beta > 0$, and have found one basis for the submanifolds, we may form the orthonormal basis for each. Since the metrics are inverse, the coefficient matrices will be inverse as well,

$$\begin{aligned}
\mathbf{e}^a &= \frac{k}{2\beta}h_\alpha^{\ a}\left(\left(1+k\beta^2\right)\mathbf{d}w^\alpha + 2k\beta^2 k^{\alpha\beta}\mathbf{d}s_\beta\right), \\
\mathbf{f}_a &= \frac{1}{2\beta}h_a^{\ \alpha}\left(2k\beta^2\mathbf{d}s_\alpha - \left(1-k\beta^2\right)k_{\alpha\beta}\mathbf{d}w^\beta\right).
\end{aligned}$$

This is a one-parameter class of allowed bases, and determining the orthonormal metric,

$$\begin{aligned}
\eta_{ab} &= h_a^{\ \alpha}h_b^{\ \beta}k_{\alpha\beta} \\
&= h_a^{\ \alpha}h_b^{\ \beta}s^2\left(\delta_{\alpha\beta} - \frac{2}{s^2}s_\alpha s_\beta\right),
\end{aligned}$$

which is clearly Lorentzian.

It is convenient to define

$$\begin{aligned}
\delta_{ab} &\equiv h_a^{\ \alpha}h_b^{\ \beta}\delta_{\alpha\beta} \\
s_a &\equiv h_a^{\ \alpha}s_\alpha.
\end{aligned}$$

We check the inner product of the orthonormal basis. The symplectic form is

$$\begin{aligned}
\mathbf{e}^a\mathbf{f}_a &= \frac{k}{2\beta}h_\alpha^{\ a}\left(\left(1+k\beta^2\right)\mathbf{d}w^\alpha + 2k\beta^2k^{\alpha\beta}\mathbf{d}s_\beta\right)\frac{1}{2\beta}h_a^{\ \beta}\left(2k\beta^2\mathbf{d}s_\beta - \left(1-k\beta^2\right)k_{\beta\mu}\mathbf{d}w^\mu\right) \\
&= \frac{k}{4\beta^2}\left(\left(1+k\beta^2\right)2k\beta^2\mathbf{d}w^\alpha\mathbf{d}s_\alpha + 2k\beta^2\left(1-k\beta^2\right)\mathbf{d}w^\beta\mathbf{d}s_\beta\right) \\
&= \frac{2k\beta^2k}{4\beta^2}\left(1+k\beta^2+1-k\beta^2\right)\mathbf{d}w^\alpha\mathbf{d}s_\alpha \\
&= \mathbf{d}w^\alpha\mathbf{d}s_\alpha,
\end{aligned}$$

and the inner products are

$$\left\langle \mathbf{e}^a, \mathbf{e}^b \right\rangle = \frac{1}{4\beta^2}h_\alpha^{\ a}h_\beta^{\ b}\left(\left\langle \left(1+k\beta^2\right)\mathbf{d}w^\alpha, 2k\beta^2k^{\beta\nu}\mathbf{d}s_\nu \right\rangle + \left\langle 2k\beta^2k^{\alpha\mu}\mathbf{d}s_\mu, \left(1+k\beta^2\right)\mathbf{d}w^\beta \right\rangle\right)$$

$$
\begin{aligned}
&+\frac{1}{4\beta^2}h_\alpha^{\ a}h_\beta^{\ b}\left(\left\langle 2k\beta^2k^{\alpha\mu}\mathbf{d}s_\mu, 2k\beta^2k^{\beta\nu}\mathbf{d}s_\nu\right\rangle\right)\\
&= \frac{1}{4\beta^2}h_\alpha^{\ a}h_\beta^{\ b}\left(2k\beta^2\left(1+k\beta^2\right)k^{\beta\alpha}+\left(1+k\beta^2\right)2k\beta^2k^{\alpha\beta}-2k\beta^2k^{\alpha\mu}k_{\mu\nu}2k\beta^2k^{\beta\nu}\right)\\
&= \frac{1}{4\beta^2}h_\alpha^{\ a}h_\beta^{\ b}\left(2k\beta^2\left(1+k\beta^2\right)+2k\beta^2\left(1+k\beta^2\right)-4\beta^4\right)k^{\alpha\beta}\\
&= \frac{1}{4\beta^2}h_\alpha^{\ a}h_\beta^{\ b}\left(2k\beta^2+2k\beta^2\right)k^{\alpha\beta}\\
&= kh_\alpha^{\ a}h_\beta^{\ b}k^{\alpha\beta}\\
&= k\eta^{ab},\\
\langle\mathbf{e}^a,\mathbf{f}_b\rangle &= \frac{k}{4\beta^2}h_\alpha^{\ a}h_b^{\ \beta}\left(2k\beta^2\left(1+k\beta^2\right)\delta_\beta^\alpha-4\beta^4\delta_\beta^\alpha-2k\beta^2\left(1-k\beta^2\right)\delta_\beta^\alpha\right)\\
&= \frac{1}{2}\left(1+k\beta^2-2k\beta^2-1+k\beta^2\right)\delta_b^a\\
&= 0,\\
\langle\mathbf{f}_a,\mathbf{f}_b\rangle &= \frac{1}{4\beta^2}h_a^{\ \alpha}h_b^{\ \beta}\left\langle 2k\beta^2\mathbf{d}s_\alpha-\left(1-k\beta^2\right)k_{\alpha\beta}\mathbf{d}w^\beta, 2k\beta^2\mathbf{d}s_\beta-\left(1-k\beta^2\right)k_{\beta\nu}\mathbf{d}w^\nu\right\rangle\\
&= \frac{2k\beta^2}{4\beta^2}h_a^{\ \alpha}h_b^{\ \beta}\left(-2k\beta^2-\left(1-k\beta^2\right)-\left(1-k\beta^2\right)\right)k_{\alpha\beta}\\
&= \frac{2k\beta^2}{4\beta^2}h_a^{\ \alpha}h_b^{\ \beta}\left(-2\right)k_{\alpha\beta}\\
&= -kh_a^{\ \alpha}h_b^{\ \beta}k_{\alpha\beta}\\
&= -k\eta_{ab},
\end{aligned}
$$

as expected.

Inverting for the coordinate differentials

We solve for the coordinate differentials in terms of $\mathbf{e}^a, \mathbf{f}_b$. Compute the linear combination

$$
\begin{aligned}
\mathbf{e}^a+\lambda\eta^{ab}\mathbf{f}_b &= \frac{k}{2\beta}h_\alpha^{\ a}\left(\left(1+k\beta^2\right)\mathbf{d}w^\alpha+2k\beta^2k^{\alpha\beta}\mathbf{d}s_\beta\right)\\
&\quad+\eta^{ab}\frac{\lambda}{2\beta}h_b^{\ \alpha}\left(2k\beta^2\mathbf{d}s_\alpha-\left(1-k\beta^2\right)k_{\alpha\beta}\mathbf{d}w^\beta\right)\\
&= \frac{1}{2\beta}\left(\left(k+\beta^2\right)h_\beta^{\ a}-\lambda\left(1-k\beta^2\right)\eta^{ab}h_b^{\ \alpha}k_{\alpha\beta}\right)\mathbf{d}w^\beta\\
&\quad+\beta h_\alpha^{\ a}\left(k^{\alpha\beta}+k\lambda k^{\alpha\beta}\right)\mathbf{d}s_\beta,
\end{aligned}
$$

so with $\lambda = -k$,

$$\begin{aligned}
\mathbf{e}^a - k\eta^{ab}\mathbf{f}_b &= \frac{1}{2\beta}\left(\left(k+\beta^2\right)h_\beta^{\ a} + k\left(1-k\beta^2\right)h_\mu^{\ a}\eta^{cb}h_c^{\ \mu}h_b^{\ \alpha}k_{\alpha\beta}\right)\mathbf{d}w^\beta \\
&= \frac{1}{2\beta}\left(k+\beta^2+k-\beta^2\right)h_\beta^{\ a}\mathbf{d}w^\beta \\
&= \frac{k}{\beta}h_\beta^{\ a}\mathbf{d}w^\beta.
\end{aligned}$$

To solve for $\mathbf{d}s_\alpha$, we use this in $\mathbf{f}_a$,

$$\begin{aligned}
\mathbf{f}_a &= \frac{1}{2\beta}h_a^{\ \alpha}\left(2k\beta^2\mathbf{d}s_\alpha - \left(1-k\beta^2\right)k_{\alpha\beta}\mathbf{d}w^\beta\right) \\
&= \frac{1}{2\beta}h_a^{\ \alpha}\left(2k\beta^2\mathbf{d}s_\alpha - \left(1-k\beta^2\right)k_{\alpha\beta}k\beta h_b^{\ \beta}\left(\mathbf{e}^b - k\eta^{bc}\mathbf{f}_c\right)\right) \\
&= k\beta h_a^{\ \alpha}\mathbf{d}s_\alpha - \left(1-k\beta^2\right)\frac{1}{2}\left(k\eta_{ab}\mathbf{e}^b - \mathbf{f}_a\right) \\
k\beta h_a^{\ \alpha}\mathbf{d}s_\alpha &= \mathbf{f}_a + \frac{1}{2}\left(1-k\beta^2\right)k\eta_{ab}\mathbf{e}^b - \frac{1}{2}\left(1-k\beta^2\right)\mathbf{f}_a \\
&= \frac{1}{2}\left(1+k\beta^2\right)\mathbf{f}_a + \frac{1}{2}\left(1-k\beta^2\right)k\eta_{ab}\mathbf{e}^b \\
\mathbf{d}s_\alpha &= \frac{k}{2\beta}\left(1+k\beta^2\right)h_\alpha^{\ a}\mathbf{f}_a + \frac{1}{2\beta}\left(1-k\beta^2\right)h_\alpha^{\ a}\eta_{ab}\mathbf{e}^b.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\mathbf{d}w^\alpha &= k\beta h_a^{\ \alpha}\left(\mathbf{e}^a - k\eta^{ab}\mathbf{f}_b\right), \\
\mathbf{d}s_\alpha &= \frac{1}{2\beta}h_\alpha^{\ a}\left(k\left(1+k\beta^2\right)\mathbf{f}_a + \left(1-k\beta^2\right)\eta_{ab}\mathbf{e}^b\right).
\end{aligned}$$

Finally, we check that these are independent. If they were not, linear dependence implies the existence of κ such that

$$\begin{aligned}
k^{\beta\alpha}\left(\frac{k}{2\beta}\left(1+k\beta^2\right)h_\alpha^{\ a}\mathbf{f}_a + \frac{1}{2\beta}\left(1-k\beta^2\right)h_\alpha^{\ a}\eta_{ab}\mathbf{e}^b\right) &= \kappa k\beta h_a^{\ \beta}\left(\mathbf{e}^a - k\eta^{ab}\mathbf{f}_b\right) \\
\frac{k}{2\beta}\left(1+k\beta^2\right)\eta^{ab}\mathbf{f}_a + \frac{1}{2\beta}\left(1-k\beta^2\right)\mathbf{e}^b &= \kappa k\beta\left(\mathbf{e}^b - k\eta^{bc}\mathbf{f}_c\right) \\
\left(k+\beta^2\right)\eta^{ab}\mathbf{f}_a + \left(1-k\beta^2\right)\mathbf{e}^b &= 2\kappa k\beta^2\left(\mathbf{e}^b - k\eta^{bc}\mathbf{f}_c\right) \\
\frac{k+\beta^2}{1-k\beta^2}\eta^{ab}\mathbf{f}_a + \mathbf{e}^b &= \frac{2\kappa k\beta^2}{\left(1-k\beta^2\right)}\left(\mathbf{e}^b - k\eta^{bc}\mathbf{f}_c\right)
\end{aligned}$$

$$\left(\frac{k+\beta^2}{1-k\beta^2}+\frac{2k\beta^2}{(1-k\beta^2)}\kappa k\right)\eta^{ab}\mathbf{f}_a+\left(1-\frac{2k\beta^2}{(1-k\beta^2)}\kappa\right)\mathbf{e}^b \;=\; 0.$$

Linear dependence requires both coefficients to vanish,

$$\begin{aligned}\kappa &= \frac{1-k\beta^2}{2k\beta^2},\\ \kappa &= -\frac{1+k\beta^2}{2k\beta^2},\end{aligned}$$

which is impossible. The orthonormal basis is therefore linearly independent, as required.

C.2. The spin connection

The entire spin connection is defined as

$$\boldsymbol{\tau}^a_{\ b} \;\equiv\; h_\alpha^{\ a}\boldsymbol{\omega}^\alpha_{\ \beta}h_b^{\ \beta}-h_b^{\ \alpha}\mathbf{d}h_\alpha^{\ a},$$

with antisymmetric and symmetric parts $\boldsymbol{\alpha}^a_{\ b}\equiv\Theta^{ad}_{cb}\boldsymbol{\tau}^c_{\ d}$ and $\boldsymbol{\beta}^a_{\ b}\equiv\Xi^{ad}_{cb}\boldsymbol{\tau}^c_{\ d}$. Each of these, as well as the Weyl vector, further subdivides between the $\mathbf{e}^a$ and $\mathbf{f}_a$ subspaces,

$$\begin{aligned}\boldsymbol{\alpha}^a_{\ b} &\equiv \boldsymbol{\sigma}^a_{\ b}+\boldsymbol{\gamma}^a_{\ b}=\sigma^a_{\ bc}\mathbf{e}^c+\gamma^{a\ c}_{\ b}\mathbf{f}_c,\\ \boldsymbol{\beta}^a_{\ b} &\equiv \boldsymbol{\mu}^a_{\ b}+\boldsymbol{\rho}^a_{\ b}=\mu^a_{\ bc}\mathbf{e}^c+\rho^{a\ c}_{\ b}\mathbf{f}_c,\\ \boldsymbol{\omega} &\equiv W_a\mathbf{e}^a+W^a\mathbf{f}_a.\end{aligned}$$

All quantities may be written in terms of the new basis. We will make use of $s_a\equiv h_a^{\ \alpha}s_\alpha$ and $\delta_{ab}\equiv h_a^{\ \alpha}h_b^{\ \beta}\delta_{\alpha\beta}$. In terms of these, we easily find

$$\begin{aligned}\eta_{ab} &= s^2\left(\delta_{ab}-\frac{2}{s^2}s_as_b\right),\\ \eta^{ab} &= \frac{1}{s^2}\left(\delta^{ab}-\frac{2}{s^2}\delta^{ac}\delta^{bd}s_cs_d\right),\\ \delta_{ab} &= \frac{1}{s^2}\left(\eta_{ab}+2s_as_b\right),\\ \delta^{ab} &= s^2\eta^{ab}+\frac{2}{s^2}\delta^{ac}\delta^{bd}s_cs_d\end{aligned}$$

$$= \quad s^2 \left(\eta^{ab} + 2\eta^{ac} s_c \eta^{ad} s_d \right).$$

The basis change from the known solution to a solution in terms of an orthonormal basis on Lagrangian submanifolds is

$$\begin{aligned} \mathbf{d}w^\beta &= k\beta h_a^{\ \beta} \left(\mathbf{e}^a - k\eta^{ab} \mathbf{f}_b \right), \\ \mathbf{d}s_\alpha &= \frac{1}{2\beta} h_\alpha^{\ a} \left(\left(1 - k\beta^2 \right) \eta_{ab} \mathbf{e}^b + k \left(1 + k\beta^2 \right) \mathbf{f}_a \right), \end{aligned}$$

where the solution for the spin connection and Weyl form is

$$\begin{aligned} \boldsymbol{\omega}^\alpha_{\ \beta} &= 2\Delta^{\alpha\mu}_{\nu\beta} s_\mu \mathbf{d}w^\nu, \\ \boldsymbol{\omega} &= -s_\alpha \mathbf{d}w^\alpha. \end{aligned}$$

These immediately become

$$\begin{aligned} \boldsymbol{\omega}^a_{\ b} &= 2\Delta^{ac}_{db} s_c k\beta \left(\mathbf{e}^d - k\eta^{de} \mathbf{f}_e \right), \\ \boldsymbol{\omega} &= -\beta s_a k \mathbf{e}^a + \eta^{ab} \beta s_a \mathbf{f}_b, \end{aligned}$$

and we easily expand the projection Δ^{ac}_{db} in terms of the new metric. Substituting to find $\boldsymbol{\tau}^a_{\ b}$ results in

$$\begin{aligned} \boldsymbol{\tau}^a_{\ b} &= 2\Delta^{ac}_{db} s_c k\beta \left(\mathbf{e}^d - k\eta^{de} \mathbf{f}_e \right) - h_b^{\ \alpha} \mathbf{d} h_\alpha^{\ a} \\ &= \left(\delta^a_d \delta^c_b - \delta^{ac} \delta_{db} \right) s_c k\beta \left(\mathbf{e}^d - k\eta^{de} \mathbf{f}_e \right) - h_b^{\ \alpha} \mathbf{d} h_\alpha^{\ a} \\ &= \left(\delta^a_d \delta^c_b - s^2 \left(\eta^{ac} + 2\eta^{af} s_f \eta^{cg} s_g \right) \left(\frac{1}{s^2} \left(\eta_{bd} + 2 s_b s_d \right) \right) \right) s_c k\beta \left(\mathbf{e}^d - k\eta^{de} \mathbf{f}_e \right) \\ &\quad - h_b^{\ \alpha} \mathbf{d} h_\alpha^{\ a} \\ &= \left(2\Theta^{ac}_{db} - 2\eta^{ac} s_b s_d - 2\eta^{af} s_f \eta^{cg} s_g \eta_{bd} - 4\eta^{af} s_f \eta^{cg} s_g s_b s_d \right) s_c k\beta \left(\mathbf{e}^d - k\eta^{de} \mathbf{f}_e \right) \\ &\quad - h_b^{\ \alpha} \mathbf{d} h_\alpha^{\ a} \\ &= \beta \left(2\Theta^{ac}_{db} s_c + 2\eta^{af} s_f \eta_{bd} + 2\eta^{ac} s_b s_c s_d \right) k\mathbf{e}^d - h_b^{\ \alpha} \mathbf{d} h_\alpha^{\ a} \end{aligned}$$

$$-\beta\left(2\Theta^{ac}_{db}s_c+2\eta^{af}\eta_{bd}s_f+2\eta^{ac}s_bs_cs_d\right)\eta^{de}\mathbf{f}_e,$$

so

$$\boldsymbol{\tau}^a_{\ b} \;=\; \beta\left(2\Theta^{ac}_{db}s_c+2\eta^{ae}\eta_{bd}s_e+2\eta^{ae}s_es_bs_d\right)\left(k\mathbf{e}^d-\eta^{dg}\mathbf{f}_g\right)-h_b^{\ \alpha}\mathbf{d}h_\alpha^{\ a}.$$

Antisymmetric projection

The antisymmetric part is then

$$\begin{aligned}
\boldsymbol{\alpha}^a_{\ b} &\equiv \Theta^{ad}_{cb}\boldsymbol{\tau}^c_{\ d}\\
&= \Theta^{ad}_{cb}\left(\beta\left(2\Theta^{cm}_{nd}s_m+2\eta^{ce}\eta_{dn}s_e+2\eta^{ce}s_es_ds_n\right)\left(k\mathbf{e}^n-\eta^{ng}\mathbf{f}_g\right)-h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c}\right)\\
&= \beta\left(2\Theta^{am}_{nb}s_m+\left(\delta^a_c\delta^d_b-\eta^{ad}\eta_{bc}\right)\eta^{ce}\eta_{dn}s_e+\left(\delta^a_c\delta^d_b-\eta^{ad}\eta_{bc}\right)\eta^{ce}s_es_ds_n\right)\left(k\mathbf{e}^n\right)\\
&\quad -\beta\left(2\Theta^{am}_{nb}s_m+\left(\delta^a_c\delta^d_b-\eta^{ad}\eta_{bc}\right)\eta^{ce}\eta_{dn}s_e+\left(\delta^a_c\delta^d_b-\eta^{ad}\eta_{bc}\right)\eta^{ce}s_es_ds_n\right)\left(\eta^{ng}\mathbf{f}_g\right)\\
&\quad -\Theta^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c}\\
&= \beta\left(\delta^a_ns_b-\eta^{am}\eta_{bn}s_m+\eta_{bn}\eta^{ae}s_e-\delta^a_ns_b+\eta^{ae}s_es_bs_n-\eta^{ad}\eta_{bc}\eta^{ce}s_es_ds_n\right)\left(k\mathbf{e}^n\right)\\
&\quad -\beta\left(\delta^a_ns_b-\eta^{am}\eta_{bn}s_m+\eta_{bn}\eta^{ae}s_e-\delta^a_ns_b+\eta^{ae}s_es_bs_n-\eta^{ad}\eta_{bc}\eta^{ce}s_es_ds_n\right)\left(\eta^{ng}\mathbf{f}_g\right)\\
&\quad -\Theta^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c}\\
&= -\Theta^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c},
\end{aligned}$$

with the remaining terms cancelling identically. Furthermore, since $h_\alpha^{\ c}$ is a purely s_α-dependent rotation at each point, the remaining $h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c}$ term will be of the form

$$\begin{aligned}
h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c} &= \left(h_d^{\ \alpha}\frac{\partial}{\partial s_\beta}h_\alpha^{\ c}\right)\mathbf{d}s_\beta\\
&= \left(h_d^{\ \alpha}\frac{\partial}{\partial s_\beta}h_\alpha^{\ c}\right)\frac{1}{2\beta}h_\beta^{\ a}\left(\left(1-k\beta^2\right)\eta_{ab}\mathbf{e}^b+\left(k+\beta^2\right)\mathbf{f}_a\right),
\end{aligned}$$

giving the parts of $\boldsymbol{\alpha}^a_{\ b}$ as

$$\boldsymbol{\sigma}^a_{\ b} \;=\; -\frac{1-k\beta^2}{2\beta}\Theta^{ad}_{cb}\left(h_b^{\ \alpha}\frac{\partial}{\partial s_\beta}h_\alpha^{\ a}\right)h_\beta^{\ c}\eta_{cd}\mathbf{e}^d, \tag{C.1}$$

$$\boldsymbol{\gamma}^a_{\ b} \;=\; -\frac{k+\beta^2}{2\beta}\Theta^{ad}_{cb}\left(h_b^{\ \alpha}\frac{\partial}{\partial s_\beta}h_\alpha^{\ a}\right)h_\beta^{\ c}\mathbf{f}_c. \tag{C.2}$$

Notice we may make one or the other of these, but not both, equal to zero by choosing $\beta^2 = 1$ and either sign for k.

Symmetric projection

Continuing, we are particularly interested in the symmetric pieces of the connection since they constitute a new feature of the theory. Applying the symmetric projection to $\boldsymbol{\tau}^a_{\ b}$, we expand

$$\begin{aligned}
\boldsymbol{\beta}^a_{\ b} &\equiv \Xi^{ad}_{cb}\boldsymbol{\tau}^c_{\ d} \\
&= \beta\Xi^{ad}_{cb}\left(2\Theta^{cm}_{nd}s_m + 2\eta^{ce}\eta_{dn}s_e + 2\eta^{ce}s_e s_d s_n\right)\left(k\mathbf{e}^n - \eta^{ng}\mathbf{f}_g\right) - \Xi^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c} \\
&= \beta\left(\left(\delta^a_c\delta^d_b + \eta^{ad}\eta_{bc}\right)\eta^{ce}\eta_{dn}s_e + \left(\delta^a_c\delta^d_b + \eta^{ad}\eta_{bc}\right)\eta^{ce}s_e s_d s_n\right)\left(k\mathbf{e}^n - \eta^{ng}\mathbf{f}_g\right) \\
&\quad -\Xi^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c} \\
&= \beta\left(\eta^{ae}\eta_{bn}s_e + \delta^a_n s_b + \eta^{ae}s_e s_b s_n + \eta^{ad}\eta_{bc}\eta^{ce}s_e s_d s_n\right)\left(k\mathbf{e}^n - \eta^{ng}\mathbf{f}_g\right) - \Xi^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c} \\
&= \beta\left(\eta^{ac}\eta_{bd}s_c + \delta^a_d s_b + 2\eta^{ac}s_b s_c s_d\right)\left(k\mathbf{e}^d - \eta^{de}\mathbf{f}_e\right) - \Xi^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c}.
\end{aligned}$$

We need to express the symmetric part, $\Xi^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c}$, in terms of the metric. First we can show

$$\begin{aligned}
k^{\alpha\mu}\mathbf{d}k_{\mu\beta} &= k^{\alpha\mu}\mathbf{d}\left(h_\mu^{\ a}h_\beta^{\ b}\eta_{ab}\right) \\
&= h_c^{\ \alpha}h_d^{\ \mu}\eta^{cd}\left(\mathbf{d}h_\mu^{\ a}h_\beta^{\ b}\eta_{ab} + h_\mu^{\ a}\mathbf{d}h_\beta^{\ b}\eta_{ab}\right) \\
&= h_c^{\ \alpha}h_d^{\ \mu}h_\beta^{\ b}\eta^{cd}\eta_{ab}\mathbf{d}h_\mu^{\ a} + h_b^{\ \alpha}\mathbf{d}h_\beta^{\ b} \\
&= h_c^{\ \alpha}h_\beta^{\ b}\eta^{cd}\eta_{ab}\left(h_d^{\ \mu}\mathbf{d}h_\mu^{\ a}\right) + h_b^{\ \alpha}h_\beta^{\ c}\left(h_c^{\ \mu}\mathbf{d}h_\mu^{\ b}\right) \\
&= h_c^{\ \alpha}h_\beta^{\ b}\eta^{cd}\eta_{ab}\left(h_d^{\ \mu}\mathbf{d}h_\mu^{\ a}\right) + h_b^{\ \alpha}h_\beta^{\ c}\delta^d_c\delta^b_a\left(h_d^{\ \mu}\mathbf{d}h_\mu^{\ a}\right) \\
&= h_c^{\ \alpha}h_\beta^{\ b}\left(\eta^{cd}\eta_{ab} + \delta^d_b\delta^c_a\right)\left(h_d^{\ \mu}\mathbf{d}h_\mu^{\ a}\right) \\
&= 2h_c^{\ \alpha}h_\beta^{\ b}\Xi^{cd}_{ab}\left(h_d^{\ \mu}\mathbf{d}h_\mu^{\ a}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Xi^{ad}_{cb}\left(h_d^{\ \mu}\mathbf{d}h_\mu^{\ c}\right) &= \frac{1}{2}h_\alpha^{\ a}h_b^{\ \beta}k^{\alpha\mu}\mathbf{d}k_{\mu\beta}\\
&= \frac{1}{2}h_\alpha^{\ a}h_b^{\ \beta}k^{\alpha\mu}\mathbf{d}\left(s^2\delta_{\mu\beta}-2s_\mu s_\beta\right)\\
&= \frac{1}{2}h_\alpha^{\ a}h_b^{\ \beta}\frac{1}{s^2}\left(\delta^{\alpha\mu}-\frac{2}{s^2}\delta^{\alpha\sigma}s_\sigma\delta^{\mu\lambda}s_\lambda\right)\left(2\delta_{\mu\beta}\delta^{\nu\rho}s_\rho\mathbf{d}s_\nu-2\mathbf{d}s_\mu s_\beta-2s_\mu\mathbf{d}s_\beta\right)\\
&= h_\alpha^{\ a}h_b^{\ \beta}\frac{1}{s^2}\left(\delta^{\alpha\mu}-\frac{2}{s^2}\delta^{\alpha\sigma}s_\sigma\delta^{\mu\lambda}s_\lambda\right)\left(\delta_{\mu\beta}\delta^{\nu\rho}s_\rho-\delta^\nu_\mu s_\beta-\delta^\nu_\beta s_\mu\right)\mathbf{d}s_\nu\\
&= h_\alpha^{\ a}h_b^{\ \beta}\frac{1}{s^2}\left(\delta^\alpha_\beta\delta^{\nu\rho}s_\rho-\frac{2}{s^2}\delta^{\alpha\sigma}\delta^{\nu\rho}s_\sigma s_\beta s_\rho-\delta^{\alpha\nu}s_\beta\right)\mathbf{d}s_\nu\\
&\quad +h_\alpha^{\ a}h_b^{\ \beta}\frac{1}{s^2}\left(\frac{2}{s^2}\delta^{\alpha\sigma}\delta^{\nu\lambda}s_\sigma s_\lambda s_\beta-\delta^\nu_\beta\delta^{\alpha\mu}s_\mu+2\delta^\nu_\beta\delta^{\alpha\sigma}s_\sigma\right)\mathbf{d}s_\nu\\
&= h_\alpha^{\ a}h_b^{\ \beta}\frac{1}{s^2}\left(\delta^\alpha_\beta\delta^{\nu\rho}s_\rho-\delta^{\alpha\nu}s_\beta+\delta^\nu_\beta\delta^{\alpha\mu}s_\mu\right)\mathbf{d}s_\nu\\
&= \frac{1}{s^2}\left(\delta^a_b\delta^{cd}s_d-\delta^{ac}s_b+\delta^c_b\delta^{ad}s_d\right)h_c^{\ \nu}\mathbf{d}s_\nu\\
&= \frac{1}{2\beta}\frac{1}{s^2}\left(\delta^a_b\delta^{cd}s_d-\delta^{ac}s_b+\delta^c_b\delta^{ad}s_d\right)\left(\left(1-k\beta^2\right)\eta_{cf}\mathbf{e}^f+k\left(1+k\beta^2\right)\mathbf{f}_c\right)\\
&= \frac{1}{2\beta}\frac{1}{s^2}\left(\delta^a_b s^2\left(\eta^{cd}+2\eta^{ce}s_e\eta^{df}s_f\right)s_d\right)\left(\left(1-k\beta^2\right)\eta_{cf}\mathbf{e}^f\right)\\
&\quad -\frac{1}{2\beta}\frac{1}{s^2}\left(s^2\left(\eta^{ac}+2\eta^{ce}s_e\eta^{af}s_f\right)s_b\right)\left(\left(1-k\beta^2\right)\eta_{cf}\mathbf{e}^f\right)\\
&\quad +\frac{1}{2\beta}\frac{1}{s^2}\left(\delta^c_b s^2\left(\eta^{ad}+2\eta^{ae}s_e\eta^{df}s_f\right)s_d\right)\left(\left(1-k\beta^2\right)\eta_{cf}\mathbf{e}^f\right)\\
&\quad +\frac{1}{2\beta}\frac{1}{s^2}\left(\delta^a_b s^2\left(\eta^{cd}+2\eta^{ce}s_e\eta^{df}s_f\right)s_d\right)\left(k\left(1+k\beta^2\right)\mathbf{f}_c\right)\\
&\quad -\frac{1}{2\beta}\frac{1}{s^2}\left(s^2\left(\eta^{ac}+2\eta^{ce}s_e\eta^{af}s_f\right)s_b\right)\left(+k\left(1+k\beta^2\right)\mathbf{f}_c\right)\\
&\quad +\frac{1}{2\beta}\frac{1}{s^2}\left(\delta^c_b s^2\left(\eta^{ad}+2\eta^{ae}s_e\eta^{df}s_f\right)s_d\right)\left(k\left(1+k\beta^2\right)\mathbf{f}_c\right)\\
&= -\frac{1}{2\beta}\left(\delta^a_b\eta^{cd}s_d+\delta^c_b\eta^{ad}s_d+\eta^{ac}s_b+2\eta^{af}\eta^{ce}s_b s_e s_f\right)\left(1-k\beta^2\right)\eta_{cf}\mathbf{e}^f\\
&\quad -\frac{1}{2\beta}\left(\delta^a_b\eta^{cd}s_d+\delta^c_b\eta^{ad}s_d+\eta^{ac}s_b+2\eta^{af}\eta^{ce}s_b s_e s_f\right)k\left(1+k\beta^2\right)\mathbf{f}_c.
\end{aligned}$$

Substituting this back into the symmetric part of the spin connection we get

$$\begin{aligned}
\boldsymbol{\beta}^a_{\ b} &= \beta\left(\eta^{ac}\eta_{bd}s_c+\delta^a_d s_b+2\eta^{ac}s_b s_c s_d\right)\left(k\mathbf{e}^d-\eta^{de}\mathbf{f}_e\right)-\Xi^{ad}_{cb}h_d^{\ \alpha}\mathbf{d}h_\alpha^{\ c}\\
&= k\beta\left(\eta^{ac}\eta_{bd}s_c+\delta^a_d s_b+2\eta^{ac}s_b s_c s_d\right)\mathbf{e}^d
\end{aligned}$$

$$
\begin{aligned}
&+\left(\delta^a_b\eta^{cd}s_d+\delta^c_b\eta^{ad}s_d+\eta^{ac}s_b+2\eta^{ae}\eta^{cd}s_bs_ds_e\right)\frac{1}{2\beta}\left(1-k\beta^2\right)\eta_{cf}\mathbf{e}^f\\
&-\beta\left(\eta^{ac}\eta_{bd}s_c+\delta^a_ds_b+2\eta^{ac}s_bs_cs_d\right)\eta^{de}\mathbf{f}_e\\
&+\left(\delta^a_b\eta^{cd}s_d+\delta^c_b\eta^{ad}s_d+\eta^{ac}s_b+2\eta^{ae}\eta^{cd}s_bs_ds_e\right)\frac{k}{2\beta}\left(1+k\beta^2\right)\mathbf{f}_c\\
=\;&\left(-k\beta\delta^a_bs_c+\frac{1}{2\beta}\left(1+k\beta^2\right)\left(\delta^a_bs_c+\delta^a_cs_b+\eta^{ad}\eta_{bc}s_d+2\eta^{ad}s_bs_cs_d\right)\right)\mathbf{e}^c\\
&+\left(\beta\delta^a_b\eta^{cd}s_d+\frac{k}{2\beta}\left(1-k\beta^2\right)\left(\delta^a_b\eta^{cd}s_d+\delta^c_b\eta^{ad}s_d+\eta^{ac}s_b+2\eta^{ad}\eta^{ce}s_bs_ds_e\right)\right)\mathbf{f}_c.
\end{aligned}
$$

Now define the coefficients,

$$\gamma_\pm\equiv\frac{1}{2\beta}\left(1\pm k\beta^2\right),$$

so

$$
\begin{aligned}
\boldsymbol{\beta}^a{}_b\;=\;&\left(-k\beta\delta^a_bs_c+\gamma_+\left(\delta^a_bs_c+\delta^a_cs_b+\eta^{ad}\eta_{bc}s_d+2\eta^{ad}s_bs_cs_d\right)\right)\mathbf{e}^c\\
&+\left(\beta\delta^a_b\eta^{cd}s_d+k\gamma_-\left(\delta^a_b\eta^{cd}s_d+\delta^c_b\eta^{ad}s_d+\eta^{ac}s_b+2\eta^{ad}\eta^{ce}s_bs_ds_e\right)\right)\mathbf{f}_c.
\end{aligned}
$$

The independent parts are

$$
\begin{aligned}
\boldsymbol{\mu}^a{}_b&=\left(-k\beta\delta^a_bs_c+\gamma_+\left(\delta^a_bs_c+\delta^a_cs_b+\eta^{ad}\eta_{bc}s_d+2\eta^{ad}s_bs_cs_d\right)\right)\mathbf{e}^c,\\
\boldsymbol{\rho}^a{}_b&=\left(\beta\delta^a_b\eta^{cd}s_d+k\gamma_-\left(\delta^a_b\eta^{cd}s_d+\delta^c_b\eta^{ad}s_d+\eta^{ac}s_b+2\eta^{ad}\eta^{ce}s_bs_ds_e\right)\right)\mathbf{f}_c.
\end{aligned}
$$

C.3. Involution conditions

Finally, check the involution conditions,

$$
\begin{aligned}
0&=\boldsymbol{\mu}^a{}_b\mathbf{e}^b-\mathbf{v}_{(x)}\mathbf{e}^a,\\
0&=\boldsymbol{\rho}^b{}_a\mathbf{f}_b-\mathbf{u}_{(y)}\mathbf{f}_a.
\end{aligned}
$$

For the first,

$$
\begin{aligned}
0\;&=\;\boldsymbol{\mu}^a{}_b\mathbf{e}^b-\mathbf{v}_{(x)}\mathbf{e}^a\\
&=\;\left(-k\beta\delta^a_bs_c+\gamma_+\left(\delta^a_bs_c+\delta^a_cs_b+\eta^{ad}\eta_{bc}s_d+2\eta^{ad}s_bs_cs_d\right)\right)\mathbf{e}^c\mathbf{e}^b+\beta s_bk\mathbf{e}^b\mathbf{e}^a
\end{aligned}
$$

$$
\begin{aligned}
&= -k\beta\delta^a_b s_c \mathbf{e}^c\mathbf{e}^b + \beta s_b k\mathbf{e}^b\mathbf{e}^a \\
&= -k\beta s_c \mathbf{e}^c\mathbf{e}^a + k\beta s_b\mathbf{e}^b\mathbf{e}^a \\
&= 0,
\end{aligned}
$$

while for the second,

$$
\begin{aligned}
0 &= \left(\beta\delta^a_b\eta^{cd}s_d + k\gamma_-\left(\delta^a_b\eta^{cd}s_d + \delta^c_b\eta^{ad}s_d + \eta^{ac}s_b + 2\eta^{ad}\eta^{ce}s_b s_d s_e\right)\right)\mathbf{f}_c\mathbf{f}_a - \eta^{ac}\beta s_a\mathbf{f}_c\mathbf{f}_b \\
&= \beta\eta^{ac}s_a\mathbf{f}_c\mathbf{f}_b - \eta^{ac}\beta s_a\mathbf{f}_c\mathbf{f}_b \\
&= 0.
\end{aligned}
$$

The involution conditions are identically satisfied for all values of β and both signs.

C.4. Riemannian curvature of the Lagrangian submanifolds

Momentum submanifold

On the $\mathbf{e}^a = 0$ Lagrangian submanifold, the Θ^{ac}_{db} projection gives

$$
\begin{aligned}
0 &= \frac{1}{2}R^{a\ cd}_{\ b}\mathbf{f}_c\mathbf{f}_d - \boldsymbol{\rho}^c_{\ b}\boldsymbol{\rho}^a_{\ c} + k\Theta^{ac}_{db}\Delta^{df}_{ec}\eta^{eg}\mathbf{f}_f\mathbf{f}_g, \\
0 &= \mathbf{d}_{(y)}\mathbf{f}_b - \boldsymbol{\gamma}^a_{\ b}\mathbf{f}_a,
\end{aligned}
$$

while the symmetric projection is

$$
\begin{aligned}
\mathbf{D}^{(y)}\boldsymbol{\rho}^a_{\ b} &= -k\Xi^{ac}_{db}\Delta^{df}_{ec}\eta^{eg}\mathbf{f}_f\mathbf{f}_g, \\
\mathbf{d}_{(y)}\mathbf{u}_{(f)} &= 0,
\end{aligned}
$$

where we now have

$$
\begin{aligned}
\boldsymbol{\rho}^a_{\ b} &= \left(\beta\delta^a_b\eta^{cd}s_d + k\gamma_-\left(\delta^a_b\eta^{cd}s_d + \delta^c_b\eta^{ad}s_d + \eta^{ac}s_b + 2\eta^{ad}\eta^{ce}s_b s_d s_e\right)\right)\mathbf{f}_c, \\
\mathbf{u}_{(f)} &= \eta^{ac}\beta s_a\mathbf{f}_c,
\end{aligned}
$$

$$\boldsymbol{\gamma}^a{}_b = -\frac{k+\beta^2}{2\beta}\Theta^{ad}_{cb}\left(h_b{}^\alpha \frac{\partial}{\partial s_\beta} h_\alpha{}^a\right) h_\beta{}^c \mathbf{f}_c.$$

Rather than computing $R^a{}_b{}^{cd}$ directly from $\boldsymbol{\gamma}^a{}_b$, which requires the rather complicated local basis change $h_\alpha{}^a$, we find it using the RHS of the equation,

$$\begin{aligned}
\frac{1}{2}R^a{}_b{}^{cd}\mathbf{f}_c\mathbf{f}_d &= \boldsymbol{\rho}^c{}_b\boldsymbol{\rho}^a{}_c - k\Theta^{ac}_{db}\Delta^{df}_{ec}\eta^{eg}\mathbf{f}_f\mathbf{f}_g \\
&= \left(\beta\delta^c_b\eta^{fd}s_d + k\gamma_-\left(\delta^c_b\eta^{fd}s_d + \delta^f_b\eta^{cd}s_d + \eta^{cf}s_b + 2\eta^{cd}\eta^{ef}s_b s_d s_e\right)\right) \\
&\quad \times\left(\beta\delta^a_c\eta^{gh}s_h + k\gamma_-\left(\delta^a_c\eta^{gh}s_h + \delta^g_c\eta^{ah}s_h + \eta^{ag}s_c + 2\eta^{ah}\eta^{kg}s_c s_h s_k\right)\right)\mathbf{f}_f\mathbf{f}_g \\
&\quad -\frac{1}{4}\left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\eta^{eg}\left(\delta^d_e\delta^f_c\right)\mathbf{f}_f\mathbf{f}_g \\
&\quad +\frac{1}{4}\left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\eta^{eg}s^2\left(\eta^{df} + 2\eta^{dh}s_h\eta^{fk}s_k\right)\frac{1}{s^2}\left(\eta_{ec} + 2s_c s_e\right)\mathbf{f}_f\mathbf{f}_g,
\end{aligned}$$

where, again

$$\begin{aligned}
\eta_{ab} &= s^2\left(\delta_{ab} - \frac{2}{s^2}s_a s_b\right), \\
\eta^{ab} &= \frac{1}{s^2}\left(\delta^{ab} - \frac{2}{s^2}\delta^{ac}\delta^{bd}s_c s_d\right), \\
\delta_{ab} &= \frac{1}{s^2}\left(\eta_{ab} + 2s_a s_b\right), \\
\delta^{ab} &= s^2\eta^{ab} + \frac{2}{s^2}\delta^{ac}\delta^{bd}s_c s_d \\
&= s^2\left(\eta^{ab} + 2\eta^{ac}s_c\eta^{ad}s_d\right).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{2}R^a{}_b{}^{cd}\mathbf{f}_c\mathbf{f}_d &= k\beta\gamma_-\left(\eta^{ac}s_c\left(\delta^g_b\eta^{fd}s_d + \delta^f_b\eta^{gd}s_d\right) + \left(\eta^{ag}\eta^{fd}s_d + \eta^{af}\eta^{gd}s_d\right)s_b\right)\mathbf{f}_f\mathbf{f}_g \\
&\quad +\gamma_-^2\left(-\delta^f_b\eta^{ag} - \delta^f_b\eta^{gd}\eta^{ah}s_d s_h\right)\mathbf{f}_f\mathbf{f}_g \\
&\quad +\gamma_-^2\left(\eta^{af}\eta^{eg}s_b s_e - \eta^{ag}\eta^{ef}s_b s_e + \eta^{ag}\eta^{fd}s_d s_b\right)\mathbf{f}_f\mathbf{f}_g \\
&\quad +\frac{1}{2}k\left(\eta^{af}\delta^g_b - \eta^{ag}\delta^f_b + \eta^{ef}\delta^g_b\eta^{ac}s_c s_e - \eta^{eg}\delta^f_b\eta^{ac}s_c s_e\right)\mathbf{f}_f\mathbf{f}_g \\
&\quad +\frac{1}{2}k\left(\eta^{eg}\eta^{af}s_b s_e - \eta^{ef}\eta^{ag}s_b s_e\right)\mathbf{f}_f\mathbf{f}_g \\
&= \Theta^{ag}_{cb}\gamma_-^2\left(\eta^{cf} + 2\eta^{cd}\eta^{ef}s_d s_e\right)\mathbf{f}_f\mathbf{f}_g
\end{aligned}$$

$$
\begin{aligned}
& +\Theta^{ag}_{cb} k \left(\eta^{fc} + 2\eta^{cd}\eta^{ef} s_d s_e \right) \mathbf{f}_f \mathbf{f}_g \\
= \quad & \frac{1}{s^2} \Theta^{ag}_{cb} \left(k + \gamma_-^2 \right) \delta^{cf} \mathbf{f}_f \mathbf{f}_g,
\end{aligned}
$$

where the constant is

$$
\begin{aligned}
k + \gamma_-^2 &= k + \frac{1}{4\beta^2} \left(1 - 2k\beta^2 + \beta^4 \right) \\
&= k - \frac{k}{2} + \frac{1}{4\beta^2} + \frac{1}{4}\beta^2 \\
&= \frac{k}{2} + \frac{1}{4\beta^2} + \frac{1}{4}\beta^2.
\end{aligned}
$$

This may vanish if we have $k = -1$ and $\beta^2 = 1$. Let $\beta = e^{\lambda}$ so

$$
\begin{aligned}
k + \gamma_-^2 &= \frac{1}{4}\beta^2 + \frac{k}{2} + \frac{1}{4\beta^2} \\
&= \frac{e^{2\lambda} + 2k + e^{-2\lambda}}{4} \\
&= \left(\frac{e^{\lambda} + k e^{-\lambda}}{2} \right)^2 \\
&= \begin{cases} \cosh^2 \lambda & k = 1 \\ \sinh^2 \lambda & k = -1 \end{cases}.
\end{aligned}
$$

Curvature of the configuration submanifold

For the configuration space

$$
\begin{aligned}
\mathbf{R}^a_{\ b} &\equiv \mathbf{d}^{(x)} \boldsymbol{\sigma}^a_{\ b} - \boldsymbol{\sigma}^c_{\ b} \boldsymbol{\sigma}^a_{\ c} \\
&= -\mathbf{D}^{(x)} \boldsymbol{\mu}^a_{\ b} + \boldsymbol{\mu}^c_{\ b} \boldsymbol{\mu}^a_{\ c} + k \Delta^{ac}_{eb} \eta_{cd} \mathbf{e}^d \mathbf{e}^e,
\end{aligned}
$$

with antisymmetric and symmetric parts,

$$
\begin{aligned}
\mathbf{R}^a_{\ b} &= \boldsymbol{\mu}^c_{\ b} \boldsymbol{\mu}^a_{\ c} + k \Theta^{ac}_{db} \Delta^{df}_{ec} \eta_{fg} \mathbf{e}^g \mathbf{e}^e, \\
0 &= -\mathbf{D}^{(x)} \boldsymbol{\mu}^a_{\ b} + k \Xi^{ac}_{db} \Delta^{df}_{ec} \eta_{fg} \mathbf{e}^g \mathbf{e}^e,
\end{aligned}
$$

where

$$\boldsymbol{\mu}^a_{\ b} = \left(-k\beta\delta^a_b s_c + \gamma_+\left(\delta^a_b s_c + \delta^a_c s_b + \eta^{ad}\eta_{bc}s_d + 2\eta^{ad}s_b s_c s_d\right)\right)\mathbf{e}^c.$$

For the antisymmetric part,

$$\begin{aligned}
\boldsymbol{\mu}^c_{\ b}\boldsymbol{\mu}^a_{\ c} &= \left(-k\beta\delta^e_b s_c + \gamma_+\left(\delta^e_b s_c + \delta^e_c s_b + \eta^{ed}\eta_{bc}s_d + 2\eta^{ed}s_b s_c s_d\right)\right) \\
&\quad \times\left(-k\beta\delta^a_e s_g + \gamma_+\left(\delta^a_e s_g + \delta^a_g s_e + \eta^{af}\eta_{eg}s_f + 2\eta^{af}s_e s_g s_f\right)\right)\mathbf{e}^c\mathbf{e}^g \\
&= \beta^2\delta^a_b s_c s_g \mathbf{e}^c\mathbf{e}^g \\
&\quad -k\beta\gamma_+\left(\delta^a_b s_c s_g + \delta^a_g s_c s_b + \eta^{af}\eta_{bg}s_c s_f + 2\eta^{af}s_c s_b s_g s_f\right)\mathbf{e}^c\mathbf{e}^g \\
&\quad -k\beta\gamma_+\left(\delta^a_b s_g s_c + \delta^a_c s_g s_b + \eta^{ad}\eta_{bc}s_g s_d + 2\eta^{ad}s_g s_b s_c s_d\right)\mathbf{e}^c\mathbf{e}^g \\
&\quad +\gamma_+^2\left(\delta^e_b s_c + \delta^e_c s_b + \eta^{ed}\eta_{bc}s_d + 2\eta^{ed}s_b s_c s_d\right) \\
&\quad \times\left(\delta^a_e s_g + \delta^a_g s_e + \eta^{af}\eta_{eg}s_f + 2\eta^{af}s_e s_g s_f\right)\mathbf{e}^c\mathbf{e}^g \\
&= -k\beta\gamma_+\left(\left(\delta^a_g s_c + \delta^a_c s_g\right)s_b + \eta^{ad}\left(\eta_{bg}s_c + \eta_{bc}s_g\right)s_d\right)\mathbf{e}^c\mathbf{e}^g \\
&\quad +\gamma_+^2\left(\delta^a_g s_c s_b + \eta^{af}\eta_{bg}s_c s_f + \eta^{ad}\eta_{bc}s_d s_g - \delta^a_g\eta_{bc}\right)\mathbf{e}^c\mathbf{e}^g \\
&\quad +\gamma_+^2\left(\eta^{af}\eta_{bc}s_g s_f - 2\eta^{af}\eta_{bc}s_g s_f - 2\delta^a_g s_b s_c\right)\mathbf{e}^c\mathbf{e}^g \\
&= \gamma_+^2\left(-\delta^a_d\eta_{bc} - \delta^a_d s_b s_c - \eta^{ae}\eta_{bc}s_d s_e\right)\mathbf{e}^c\mathbf{e}^d \\
&= -\gamma_+^2\left(\frac{1}{2}\left(\delta^a_d\delta^e_b - \eta^{ae}\eta_{bd}\right)\eta_{ce} + \frac{1}{2}\left(\delta^a_d\delta^e_b - \eta^{ae}\eta_{bd}\right)2s_c s_e\right)\mathbf{e}^c\mathbf{e}^d \\
&= -\gamma_+^2\Theta^{ae}_{db}\left(\eta_{ce} + 2s_c s_e\right)\mathbf{e}^c\mathbf{e}^d,
\end{aligned}$$

and

$$\begin{aligned}
k\Theta^{ac}_{db}\Delta^{df}_{ec}\eta_{fg}\mathbf{e}^g\mathbf{e}^e &= \frac{1}{4}k\left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{bd}\right)\left(\delta^d_e\delta^f_c - \delta_{ec}\delta^{df}\right)\eta_{fg}\mathbf{e}^g\mathbf{e}^e \\
&= -\frac{1}{2}k\left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{bd}\right)\left(\delta^d_g\eta_{ce} + \delta^d_g s_e s_c + \eta^{dh}\eta_{ce}s_h s_g\right)\mathbf{e}^g\mathbf{e}^e \\
&= -k\left(\frac{1}{2}\left(\delta^a_g\delta^c_b - \eta^{ac}\eta_{bg}\right)\eta_{ce} + \frac{1}{2}\left(\delta^a_g\delta^c_b - \eta^{ac}\eta_{bg}\right)2s_e s_c\right)\mathbf{e}^g\mathbf{e}^e \\
&= -k\Theta^{ac}_{db}\left(\eta_{ce} + 2s_e s_c\right)\mathbf{e}^d\mathbf{e}^e,
\end{aligned}$$

so

$$\begin{aligned}
\mathbf{R}^a_{\ b} &= \boldsymbol{\mu}^c_{\ b}\boldsymbol{\mu}^a_{\ c} + k\Theta^{ac}_{db}\Delta^{df}_{ec}\eta_{fg}\mathbf{e}^g\mathbf{e}^e \\
&= \left(k-\gamma_+^2\right)\Theta^{ae}_{db}\left(\eta_{ec}+2s_cs_e\right)\mathbf{e}^c\mathbf{e}^d \\
&= \left(\gamma_+^2-k\right)s^2\Theta^{ac}_{db}\delta_{ce}\mathbf{e}^d\mathbf{e}^e.
\end{aligned}$$

We have

$$\begin{aligned}
\gamma_+ &\equiv \frac{1}{2\beta}\left(1+k\beta^2\right) \\
\left(\gamma_+^2-k\right)s^2 &= \frac{1}{4\beta^2}\left(1+2k\beta^2+\beta^4-4\beta^2k\right)s^2 \\
&= \frac{1}{4\beta^2}\left(1-2k\beta^2+\beta^4\right)s^2,
\end{aligned}$$

so with $\beta = e^\lambda$,

$$\begin{aligned}
\left(\gamma_+^2-k\right) &= \frac{1}{4}\left(e^{-2\lambda}-2k+e^{2\lambda}\right) \\
&= \left(\frac{e^\lambda - ke^{-\lambda}}{2}\right)^2 \\
&= \begin{cases} \sinh^2\lambda & k=1 \\ \cosh^2\lambda & k=-1 \end{cases}.
\end{aligned}$$

Combining the two curvatures, we have

$$\begin{aligned}
\mathbf{R}^a_{(x)\,b} &= \left(\gamma_+^2-k\right)s^2\Theta^{ac}_{db}\delta_{ce}\mathbf{e}^d\mathbf{e}^e \\
R^a_{\ bde} &= \frac{1}{2}\left(\gamma_+^2-k\right)s^2\left(\Theta^{ac}_{db}\delta_{ce}-\Theta^{ac}_{eb}\delta_{cd}\right) \\
&= \frac{1}{2}\left(\gamma_+^2-k\right)s^2\left(\Theta^{ac}_{db}\frac{1}{s^2}\left(\eta_{ce}+2s_cs_e\right)-\Theta^{ac}_{eb}\frac{1}{s^2}\left(\eta_{cd}+2s_cs_d\right)\right) \\
&= \frac{1}{2}\left(\gamma_+^2-k\right)\left(\Theta^{ac}_{db}\delta^f_e-\Theta^{ac}_{eb}\delta^f_d\right)\left(\eta_{cf}+2s_cs_f\right), \\
R^{a\ de}_{\ b} &= \frac{1}{2}\frac{1}{s^2}\left(k+\gamma_-^2\right)\left(\Theta^{ae}_{cb}\delta^{cd}-\Theta^{ad}_{cb}\delta^{ce}\right) \\
\eta_{df}\eta_{eg}R^{a\ de}_{\ b} &= \frac{1}{2}\frac{1}{s^2}\left(k+\gamma_-^2\right)\left(\Theta^{ae}_{cb}\eta_{df}\eta_{eg}s^2\left(\eta^{cd}+2\eta^{ch}s_h\eta^{dk}s_k\right)\right)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}\frac{1}{s^2}\left(k+\gamma_-^2\right)\left(\Theta_{cb}^{ad}\eta_{df}\eta_{eg}s^2\left(\eta^{ce}+2\eta^{ch}s_h\eta^{ek}s_k\right)\right) \\
= \;& \frac{1}{2}\left(k+\gamma_-^2\right)\left(\Theta_{fb}^{ae}\left(\eta_{eg}+2s_es_g\right)-\Theta_{gb}^{ae}\left(\eta_{ef}+2s_es_f\right)\right) \\
\eta_{df}\eta_{eg}R_{\;b}^{a\;fg} = \;& \frac{1}{2}\left(k+\gamma_-^2\right)\left(\Theta_{db}^{ac}\delta_e^f-\Theta_{eb}^{ac}\delta_d^f\right)\left(\eta_{cf}+2s_cs_f\right).
\end{aligned}
$$

Therefore,

$$
\eta_{df}\eta_{eg}R_{\;b}^{a\;fg}-R_{\;bde}^{a} = \frac{1}{2}k\left(\Theta_{db}^{ac}\delta_e^f-\Theta_{eb}^{ac}\delta_d^f\right)\left(\eta_{cf}+2s_cs_f\right),
$$

so the difference of the configuration and momentum curvatures is independent of the linear combination of basis forms used.

For the Einstein tensors,

$$
\begin{aligned}
\eta_{ac}\eta_{bd}R_{\;e}^{c\;de}-\frac{1}{2}\eta_{ab}R_{(y)}-R_{\;acb}^{c}+\frac{1}{2}\eta_{ab}R_{(x)} &= \frac{1}{2}k\left(n-2\right)\left(\eta_{ab}+s_as_b\right)-\frac{1}{2}\eta_{ab}k \\
&= \frac{1}{2}k\left(\left(n-3\right)\eta_{ab}+\left(n-2\right)s_as_b\right).
\end{aligned}
$$

Symmetric curvature equation and constancy of the scale vector

The symmetric field equation is

$$
0=-\mathbf{D}^{(x)}\boldsymbol{\mu}_{\;b}^{a}+k\Xi_{db}^{ac}\Delta_{ec}^{df}\eta_{fg}\mathbf{e}^g\mathbf{e}^e.
$$

Computing the second term,

$$
\begin{aligned}
k\Xi_{db}^{ac}\Delta_{ec}^{df}\eta_{fg}\mathbf{e}^g\mathbf{e}^e &= k\Xi_{db}^{ac}\Delta_{ec}^{df}\eta_{fg}\mathbf{e}^g\mathbf{e}^e \\
&= \frac{k}{4}\left(\delta_d^a\delta_b^c+\eta^{ac}\eta_{bd}\right)\left(\delta_e^d\delta_c^f-\delta^{df}\delta_{ec}\right)\eta_{fg}\mathbf{e}^g\mathbf{e}^e \\
&= \frac{k}{4}\delta_d^a\delta_b^c\left(\delta_e^d\delta_c^f-s^2\left(\eta^{df}+2\eta^{dg}s_g\eta^{fh}s_h\right)\frac{1}{s^2}\left(\eta_{ce}+2s_es_c\right)\right)\eta_{fg}\mathbf{e}^g\mathbf{e}^e \\
&\quad +\frac{k}{4}\eta^{ac}\eta_{bd}\left(\delta_e^d\delta_c^f-s^2\left(\eta^{df}+2\eta^{dg}s_g\eta^{fh}s_h\right)\frac{1}{s^2}\left(\eta_{ce}+2s_es_c\right)\right)\eta_{fg}\mathbf{e}^g\mathbf{e}^e \\
&= \frac{k}{4}\left(-2s_b\left(\delta_e^as_k+\delta_k^as_e\right)-2\eta^{ac}s_c\left(\eta_{bk}s_e+\eta_{be}s_k\right)\right)\mathbf{e}^k\mathbf{e}^e \\
&= 0.
\end{aligned}
$$

Therefore, with $\mathbf{d}^{(x)}\mathbf{v} = 0$ and $\mathbf{D}^{(x)}\eta_{ab} = 0$,

$$\begin{aligned}
0 &= \mathbf{D}^{(x)}\boldsymbol{\mu}^a_{\ b} \\
&= \mathbf{D}^{(x)}\left[\left(-k\beta\delta^a_b s_c + \gamma_+\left(\delta^a_b s_c + \delta^a_c s_b + \eta^{ad}\eta_{bc}s_d + 2\eta^{ad}s_b s_c s_d\right)\right)\mathbf{e}^c\right] \\
&= -k\beta\delta^a_b s_{c;e}\mathbf{e}^e\mathbf{e}^c \\
&\quad +\gamma_+\left(\delta^a_b s_{c;e} + \delta^a_c s_{b;e} + \eta^{ad}\eta_{bc}s_{d;e}\right)\mathbf{e}^e\mathbf{e}^c \\
&\quad +\gamma_+\left(2\eta^{ad}s_{b;e}s_c s_d + 2\eta^{ad}s_b s_{c;e}s_d + 2\eta^{ad}s_b s_c s_{d;e}\right)\mathbf{e}^e\mathbf{e}^c \\
&= \gamma_+\left(\delta^a_c s_{b;e} + \eta^{ad}\eta_{bc}s_{d;e} + 2\eta^{ad}s_{b;e}s_c s_d + 2\eta^{ad}s_b s_c s_{d;e}\right)\mathbf{e}^e\mathbf{e}^c,
\end{aligned}$$

and we conclude

$$\begin{aligned}
0 &= \delta^a_c s_{b;e} + \eta^{ad}\eta_{bc}s_{d;e} + 2\eta^{ad}s_{b;e}s_c s_d + 2\eta^{ad}s_b s_c s_{d;e} \\
&\quad -\delta^a_e s_{b;c} - \eta^{ad}\eta_{be}s_{d;c} - 2\eta^{ad}s_{b;c}s_e s_d - 2\eta^{ad}s_b s_e s_{d;c}.
\end{aligned}$$

Contract with s_a,

$$\begin{aligned}
0 &= s_c s_{b;e} + \eta_{bc}\eta^{ad}s_a s_{d;e} - 2s_{b;e}s_c + 2s_b s_c s_a\eta^{ad}s_{d;e} \\
&\quad -s_e s_{b;c} - \eta_{be}\eta^{ad}s_a s_{d;c} + 2s_{b;c}s_e - 2s_a\eta^{ad}s_b s_e s_{d;c} \\
&= s_e s_{b;c} - s_{b;e}s_c + \eta_{bc}\frac{1}{2}\left(\eta^{ad}s_a s_d\right)_{;e} \\
&\quad +s_b s_c\left(\eta^{ad}s_a s_d\right)_{;e} - \eta_{be}\eta^{ad}s_a s_{d;c} - s_b s_e\left(s_a\eta^{ad}s_d\right)_{;c} \\
&= s_e s_{b;c} - s_{b;e}s_c,
\end{aligned}$$

and then with $\eta^{ae}s_a$,

$$\begin{aligned}
0 &= \eta^{ae}s_a s_e s_{b;c} - \eta^{ae}s_a s_{b;e}s_c \\
&= -s_{b;c} - \eta^{ae}s_a s_{b;e}s_c.
\end{aligned}$$

But $s_{a;b} - s_{b;a} = 0$, so the second term becomes

$$\begin{aligned} \eta^{ae} s_a s_{b;e} s_c &= \eta^{ae} s_a s_{e;b} s_c \\ &= \frac{1}{2} \left(\eta^{ae} s_a s_e\right)_{;b} s_c \\ &= 0 \end{aligned}$$

so we have

$$s_{a;b} = 0.$$

C.5. Components of the solder form

We are interested in the nature of the coefficients of the solder form, $h_a^{\ \alpha}$, which turn the coordinate basis into the orthonormal one,

$$\begin{aligned} \eta_{ab} &= h_a^{\ \alpha} h_b^{\ \beta} k_{\alpha\beta} \\ &= h_a^{\ \alpha} h_b^{\ \beta} s^2 \left(\delta_{\alpha\beta} - \frac{2}{s^2} s_\alpha s_\beta\right). \end{aligned}$$

Let $\tilde{h}_a^{\ \alpha} \equiv \sqrt{s^2} h_a^{\ \alpha}$ to remove the conformal factor. Recalling $\delta_{\alpha\beta}$ is the Euclidean metric, $diag\,(+1, +1, \ldots, +1)$, we see that on orthogonal transformation will preserve $\delta_{\alpha\beta}$, while rotating the vector, s_α. At any fixed point, s^0_α, let $\tilde{h}_a^{\ \alpha}$ be the orthogonal transformation $\tilde{h}_a^{\ \alpha} = O_a^{\ \alpha}\left(s^0_\alpha, x\right)$that takes s^0_α to some fixed direction, say $x = s_1$. In this rotated system, $s_\alpha = \left(\sqrt{s^2}, 0, \ldots, 0\right)$, so

$$\begin{aligned} h_a^{\ \alpha} h_b^{\ \beta} s^2 \left(\delta_{\alpha\beta} - \frac{2}{s^2} s_\alpha s_\beta\right) &= O_a^{\ \alpha}\left(s^0_\alpha, x\right) O_b^{\ \beta}\left(s^0_\alpha, x\right) \left(\delta_{\alpha\beta} - \frac{2}{s^2} s_\alpha s_\beta\right) \\ &= O_a^{\ \alpha}\left(s^0_\alpha, x\right) \delta_{\alpha\beta} O_b^{\ \beta}\left(s^0_\alpha, x\right) - \frac{2}{s^2} O_a^{\ \alpha}\left(s^0_\alpha, x\right) s_\alpha O_b^{\ \beta}\left(s^0_\alpha, x\right) s_\beta \\ &= \delta_{ab} - \frac{2}{s^2} s^2 \delta_{a1} \delta_{b1} \\ &= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \end{aligned}$$

Therefore, the required transformation at every point is just

$$h_a^{\ \alpha} = \frac{1}{\sqrt{s^2}} O_a^{\ \alpha}\left(s_\alpha, x\right).$$

The explicit form of such a rotation is written as follows. Let $n_\alpha = (1, 0, \ldots, 0)$ be the unit vector in the chosen direction. We want to rotate in the (n_α, s_β) plane. We need a unit vector which, together with $\mathbf{n}$, spans this plane. Let

$$\begin{aligned}
\mathbf{m} &= \frac{s_\alpha - \left(s_\beta n^\beta\right) n_\alpha}{\sqrt{\left(s_\mu - \left(s_\beta n^\beta\right) n_\mu\right)\left(s^\mu - \left(s_\beta n^\beta\right) n^\mu\right)}} \\
&= \frac{s_\alpha - \left(s_\beta n^\beta\right) n_\alpha}{\sqrt{s_\mu s^\mu - \left(s_\beta n^\beta\right) n_\mu s^\mu - \left(s_\beta n^\beta\right) n^\mu s_\mu + \left(s_\beta n^\beta\right)\left(s_\beta n^\beta\right)}} \\
&= \frac{\mathbf{s} - \left(s_\beta n^\beta\right)\mathbf{n}}{\sqrt{s^2 - \left(s_\beta n^\beta\right)^2}}.
\end{aligned}$$

Now, for a general vector $\mathbf{x}$, decompose so $\mathbf{x}$ is the sum of

$$\begin{aligned}
\mathbf{x}_n &= (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}, \\
\mathbf{x}_m &= (\mathbf{x}\cdot\mathbf{m})\,\mathbf{m}, \\
\mathbf{x}_\perp &= \mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n} - (\mathbf{x}\cdot\mathbf{m})\,\mathbf{m}.
\end{aligned}$$

The new $\mathbf{x}$ is given by

$$\begin{aligned}
\tilde{\mathbf{x}}_\perp &= \mathbf{x}_\perp, \\
\tilde{\mathbf{x}}_n &= \mathbf{x}_n\cos\theta - \mathbf{x}_m\sin\theta, \\
\tilde{\mathbf{x}}_m &= \mathbf{x}_n\sin\theta + \mathbf{x}_m\cos\theta,
\end{aligned}$$

where the angle of rotation is given by $\cos\theta = \frac{(\mathbf{s}\cdot\mathbf{n})}{\sqrt{s^2}}$, $\sin\theta = +\sqrt{1 - \frac{(\mathbf{s}\cdot\mathbf{n})^2}{s^2}}$.

Now expand this out to find the transformation

$$\tilde{\mathbf{x}}_\perp = \mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n} - (\mathbf{x}\cdot\mathbf{m})\,\mathbf{m},$$

$$\begin{aligned}
\tilde{\mathbf{x}}_{\perp}^2 &= \mathbf{x}\cdot\mathbf{x} - (\mathbf{x}\cdot\mathbf{n})^2 - (\mathbf{x}\cdot\mathbf{m})^2\,, \\
\tilde{\mathbf{x}}_n &= (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cos\theta - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{m}\sin\theta, \\
\tilde{\mathbf{x}}_m &= (\mathbf{x}\cdot\mathbf{m})\,\mathbf{n}\sin\theta + (\mathbf{x}\cdot\mathbf{m})\,\mathbf{m}\cos\theta.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{\mathbf{x}} &= \mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n} - (\mathbf{x}\cdot\mathbf{m})\,\mathbf{m} + (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\cos\theta \\
&\quad - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{m}\sin\theta + (\mathbf{x}\cdot\mathbf{m})\,\mathbf{n}\sin\theta + (\mathbf{x}\cdot\mathbf{m})\,\mathbf{m}\cos\theta \\
&= \mathbf{x} - \mathbf{n}\left((\mathbf{x}\cdot\mathbf{n}) - (\mathbf{x}\cdot\mathbf{n})\cos\theta - (\mathbf{x}\cdot\mathbf{m})\sin\theta\right) \\
&\quad - \mathbf{m}\left((\mathbf{x}\cdot\mathbf{m}) + (\mathbf{x}\cdot\mathbf{n})\sin\theta - (\mathbf{x}\cdot\mathbf{m})\cos\theta\right), \\
\tilde{x}^\alpha &= \left(\delta_\beta^\alpha - (1-\cos\theta)\,n^\alpha n_\beta + n^\alpha m_\beta\sin\theta - m^\alpha n_\beta\sin\theta - (1-\cos\theta)\,m^\alpha m_\beta\right)x^\beta.
\end{aligned}$$

Check the norm of $\mathbf{x}$,

$$\begin{aligned}
\tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}} &= \mathbf{x}\cdot\mathbf{x} - (\mathbf{x}\cdot\mathbf{n})^2(1-\cos\theta) + (\mathbf{x}\cdot\mathbf{n})(\mathbf{x}\cdot\mathbf{m})\sin\theta \\
&\quad - (\mathbf{x}\cdot\mathbf{m})^2(1-\cos\theta) - (\mathbf{x}\cdot\mathbf{m})(\mathbf{x}\cdot\mathbf{n})\sin\theta \\
&\quad - (\mathbf{x}\cdot\mathbf{n})^2(1-\cos\theta) + (\mathbf{x}\cdot\mathbf{n})(\mathbf{x}\cdot\mathbf{m})\sin\theta \\
&\quad + ((\mathbf{x}\cdot\mathbf{n})(1-\cos\theta) - (\mathbf{x}\cdot\mathbf{m})\sin\theta)((\mathbf{x}\cdot\mathbf{n})(1-\cos\theta) - (\mathbf{x}\cdot\mathbf{m})\sin\theta) \\
&\quad - (\mathbf{x}\cdot\mathbf{m})^2 - (\mathbf{x}\cdot\mathbf{n})(\mathbf{m}\cdot\mathbf{x})\sin\theta + (\mathbf{x}\cdot\mathbf{m})^2\cos\theta \\
&\quad + ((\mathbf{x}\cdot\mathbf{m})(1-\cos\theta) + (\mathbf{x}\cdot\mathbf{n})\sin\theta)((\mathbf{x}\cdot\mathbf{m})(1-\cos\theta) + (\mathbf{x}\cdot\mathbf{n})\sin\theta) \\
&= \mathbf{x}\cdot\mathbf{x} \\
&\quad - (\mathbf{x}\cdot\mathbf{n})^2\left((1-\cos\theta) + (1-\cos\theta) - (1-\cos\theta)^2 - \sin^2\theta\right) \\
&\quad + (\mathbf{x}\cdot\mathbf{n})(\mathbf{x}\cdot\mathbf{m})(\sin\theta - \sin\theta + \sin\theta - \sin\theta - (1-\cos\theta)\sin\theta) \\
&\quad + (\mathbf{x}\cdot\mathbf{n})(\mathbf{x}\cdot\mathbf{m})((1-\cos\theta)\sin\theta - (1-\cos\theta)\sin\theta + \sin\theta(1-\cos\theta)) \\
&\quad - (\mathbf{x}\cdot\mathbf{m})^2\left(1 - \cos\theta - \sin^2\theta + 1 - \cos\theta - (1-\cos\theta)^2\right) \\
&= \mathbf{x}\cdot\mathbf{x} \\
&\quad - (\mathbf{x}\cdot\mathbf{n})^2\left(2 - 2\cos\theta - 1 + 2\cos\theta - \cos^2\theta - \sin^2\theta\right)
\end{aligned}$$

$$
\begin{aligned}
&+ (\mathbf{x}\cdot\mathbf{n})(\mathbf{x}\cdot\mathbf{m})(\sin\theta - \sin\theta + \cos\theta\sin\theta - \sin\theta + \cos\theta\sin\theta) \\
&- (\mathbf{x}\cdot\mathbf{n})(\mathbf{x}\cdot\mathbf{m})(\sin\theta - \sin\theta + \cos\theta\sin\theta - \sin\theta + \cos\theta\sin\theta) \\
&- (\mathbf{x}\cdot\mathbf{m})^2\left(1 - \sin^2\theta - \cos\theta - \cos\theta + 2\cos\theta - \cos^2\theta\right) \\
= \;& \mathbf{x}\cdot\mathbf{x},
\end{aligned}
$$

so

$$
\tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}} = \mathbf{x}\cdot\mathbf{x}
$$

and the transformation is a rotation.

Therefore,

$$
\begin{aligned}
h_\beta^{\ a} = \;& \frac{1}{\sqrt{s^2}}\delta_\alpha^a\left(\delta_\beta^\alpha - (1-\cos\theta)\,n^\alpha n_\beta\right) \\
& + \frac{1}{\sqrt{s^2}}\delta_\alpha^a\left(n^\alpha m_\beta\sin\theta - m^\alpha n_\beta\sin\theta - (1-\cos\theta)\,m^\alpha m_\beta\right),
\end{aligned}
$$

where

$$
\begin{aligned}
n_\alpha &= (1,0,\ldots,0), \\
\mathbf{m} &= \frac{\mathbf{s} - \left(s_\beta n^\beta\right)\mathbf{n}}{\sqrt{s^2 - \left(s_\beta n^\beta\right)^2}},
\end{aligned}
$$

and

$$
\begin{aligned}
\cos\theta &= \frac{(\mathbf{s}\cdot\mathbf{n})}{\sqrt{s^2}}, \\
\sin\theta &= +\sqrt{1 - \frac{(\mathbf{s}\cdot\mathbf{n})^2}{s^2}}.
\end{aligned}
$$

Appendix D

Christoffel symbol

We define Christoffel symbols (connections) as the symmetric part of the submanifold basis structure equations. For the biconformal submanifolds in the Lagrangian basis they are

$$\begin{aligned} e_c^{\ \mu}\partial_b e_\mu^{\ a} + \sigma^a_{\ cb} + \delta^b_a\partial_c\phi - \tilde{\Gamma}^a_{\ cb} &= 0, \\ f_\mu^{\ c}\partial^b f_a^{\ \mu} - \gamma_a^{\ c\,b} - \delta^c_a\partial^b\phi + \tilde{\Gamma}_a^{\ c\,b} &= 0. \end{aligned}$$

In the orthonormal gauge, we have

$$\begin{aligned} e_c^{\ \mu}\partial_b e_\mu^{\ a} + \sigma^a_{\ cb} - \tilde{\Gamma}^a_{\ cb} &= 0, \\ f_\mu^{\ c}\partial^b f_a^{\ \mu} - \gamma_a^{\ c\,b} + \tilde{\Gamma}_a^{\ c\,b} &= 0. \end{aligned}$$

This gives the following relationship between the Christoffel symbols and the submanifold torsion and cotorsion,

$$\begin{aligned} \tilde{\Gamma}^a_{\ bc} - \tilde{\Gamma}^a_{\ cb} &\equiv -T^a_{\ bc}, \\ \tilde{\Gamma}^{b\,c}_{\ a} - \tilde{\Gamma}^{c\,b}_{\ a} &\equiv S_a^{\ bc}. \end{aligned}$$

In a coordinate basis, we can solve for the connection

$$\begin{aligned} \tilde{\Gamma}^\sigma_{\ \beta\nu} &= \Gamma^\sigma_{\ \beta\nu} + \frac{1}{2}\left(T^\sigma_{\ \nu\beta} + g^{\gamma\sigma}g_{\nu\alpha}T^\alpha_{\ \gamma\beta} + g^{\gamma\sigma}g_{\beta\alpha}T^\alpha_{\ \gamma\nu}\right), \\ \tilde{\Gamma}^{\beta\,\gamma}_{\ \sigma} &= \Gamma^{\beta\,\gamma}_{\ \sigma} - \frac{1}{2}\left(S_\sigma^{\ \gamma\beta} + g_{\nu\sigma}g^{\beta\alpha}S_\alpha^{\ \gamma\nu} + g_{\nu\sigma}g^{\gamma\alpha}S_\alpha^{\ \beta\nu}\right), \end{aligned}$$

where the untilded Christoffel symbols are defined normally (as in a Riemannian geometry), with respect to the coordinate metric,

$$\begin{aligned} \Gamma^\sigma_{\ \beta\nu} &\equiv \frac{1}{2}g^{\gamma\sigma}\left(\partial_\beta g_{\nu\gamma} + \partial_\nu g_{\gamma\beta} - \partial_\gamma g_{\beta\nu}\right), \\ \Gamma^{\beta\,\gamma}_{\ \sigma} &\equiv -\frac{1}{2}g_{\nu\sigma}\left(\partial^\beta g^{\gamma\nu} + \partial^\gamma g^{\nu\beta} - \partial^\nu g^{\beta\gamma}\right). \end{aligned}$$

Appendix E

Extrinsic curvature of Riemannian submanifolds

E.1. Metric relation

Consider an n-dimensional C^k Riemannian manifold $\mathcal{M}$ with metric

$$ds^2 = g_{\alpha\beta} dX^\alpha dX^\beta.$$

For $m < n$, consider an m-dimensional C^k Riemannian manifold $\mathcal{N}$ with metric

$$dl^2 = h_{ij} dx^i dx^j,$$

which is isometrically immersed in $\mathcal{M}$ as a submanifold. This is defined by requiring the length of any curve in $\mathcal{N}$, evaluated using dl, should be equal to the length of the same curve evaluated using ds of the ambient space. Thus, for any infinitesimal displacement in $\mathcal{N}$, the differentials dx^i and dX^α are related by

$$g_{\alpha\beta} dX^\alpha dX^\beta = h_{ij} dx^i dx^j.$$

For points in $\mathcal{N}$ there are two ways to specify charts. Since $\mathcal{N}$ is a manifold, we may always introduce coordinate charts, x^i, in the usual way. On the other hand, since $\mathcal{N}$ is embedded in $\mathcal{M}$ we have the restriction of the coordinates X^α to $\mathcal{N}$. Concretely, if $\phi : U \to R^n$ is any chart on an open set U in $\mathcal{M}$ intersecting $V \subset \mathcal{N}$, and $\chi : V \to R^m$ any chart on a set V open in $\mathcal{N}$, then

$$\phi \circ \chi^{-1} : R^m \to R^n$$

is a 1 to 1 mapping from R^m into R^n via the neighborhood $U \cap V$, i.e., a collection of real-valued functions

$$X^\alpha|_{\mathcal{N}} = f^\alpha\left(x^i\right),$$

where $i = 1, \ldots, m < n$ define a submanifold $\mathcal{N} \subset \mathcal{M}$. These functions, f^α, are regular and of class C^k, and the rank of

$$\frac{\partial f^\alpha}{dx^i}$$

is equal to m. Denote this restriction of the coordinates X^α by $X^\alpha_{\mathcal{N}}$.

The length of any curve in $\mathcal{N}$ is given by the restriction of $g_{\alpha\beta}$ to $\mathcal{N}$; we say the metric on $\mathcal{N}$ is induced by the metric of the ambient space $\mathcal{M}$.

Since

$$dX^\alpha_{\mathcal{N}} = \frac{\partial f^\alpha}{\partial x^i} dx^i,$$

we have

$$g_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^m} = h_{ij}.$$

Since the f^α are functions on $\mathcal{M}$, these derivatives are covariant,

$$\frac{\partial f^\alpha}{\partial x^i} = D_i f^\alpha,$$

and we may write

$$g_{\alpha\beta} f^\alpha_{,i} f^\beta_{,j} = h_{ij}.$$

E.2. Curves

Now consider a curve $C : R \to \mathcal{N}$ in the submanifold, with coordinates $x^i(\lambda)$. This curve has tangent vectors

$$\frac{dx^i}{d\lambda} \frac{\partial}{\partial x^i}$$

and hence, components

$$\psi^i = \frac{dx^i}{d\lambda}.$$

This is also a curve in $\mathcal{M}$, with

$$X^\alpha(\lambda) = f^\alpha\left(x^i(\lambda)\right),$$

and its tangent vectors are tangent to $\mathcal{N}$. We have

$$\begin{aligned}\xi^\alpha &= \frac{dX^\alpha}{d\lambda} \\ &= \frac{\partial f^\alpha}{\partial x^i}\frac{dx^i}{d\lambda} \\ &= f^\alpha_{,i}\psi^i.\end{aligned}$$

Choose m such curves, such that their tangent vectors, $\psi^i_{(\sigma)}$, form an orthonormal basis,

$$h_{ij}\psi^i_{(\sigma)}\psi^j_{(\tau)} = \delta_{\sigma\tau},$$

and choose adapted coordinates such that $\psi^i_{(s)} = \delta^i_s$. Then, in terms of the full manifold, the vectors $\xi^\alpha_{(s)} = f^\alpha_{,i}\psi^i_{(s)} = f^\alpha_{,s}$, $s = 1, \ldots, m$ span the tangent space of the submanifold. We may also fix an orthonormal set of vectors orthogonal to these, $\zeta^\alpha_{(A)}$, $A = 1, \ldots, n-m$. Then

$$\begin{aligned}g_{\alpha\beta}\zeta^\alpha_{(A)}\zeta^\beta_{(B)} &= \delta_{AB}, \\ g_{\alpha\beta}\xi^\alpha_{(s)}\zeta^\beta_{(B)} &= 0.\end{aligned}$$

E.3. Second fundamental form

Now return to the relation between the metrics,

$$g_{\alpha\beta}f^\alpha_{,i}f^\beta_{,j} = h_{ij},$$

and differentiate using the covariant derivative of the submanifold,

$$\begin{aligned}0 &= D_k h_{ij} \\ &= D_k\left(g_{\alpha\beta}f^\alpha_{,i}f^\beta_{,j}\right) \\ &= D_k\left(g_{\alpha\beta}\xi^\alpha_{(i)}\xi^\beta_{(j)}\right) \\ &= \left(\xi^\mu_{(k)}D_\mu\right)\left(g_{\alpha\beta}\xi^\alpha_{(i)}\xi^\beta_{(j)}\right) \\ &= \xi^\mu_{(k)}g_{\alpha\beta}\left(D_\mu\xi^\alpha_{(i)}\xi^\beta_{(j)} + \xi^\alpha_{(i)}D_\mu\xi^\beta_{(j)}\right).\end{aligned}$$

Taking the sum-sum-difference,

$$\begin{aligned} 0 &= \xi^{\mu}_{(k)} g_{\alpha\beta} \left(D_\mu \xi^{\alpha}_{(i)} \xi^{\beta}_{(j)} + \xi^{\alpha}_{(i)} D_\mu \xi^{\beta}_{(j)} \right) + \xi^{\mu}_{(i)} g_{\alpha\beta} \left(D_\mu \xi^{\alpha}_{(j)} \xi^{\beta}_{(k)} + \xi^{\alpha}_{(j)} D_\mu \xi^{\beta}_{(k)} \right) \\ &\quad -\xi^{\mu}_{(j)} g_{\alpha\beta} \left(D_\mu \xi^{\alpha}_{(k)} \xi^{\beta}_{(i)} + \xi^{\alpha}_{(k)} D_\mu \xi^{\beta}_{(i)} \right) \\ &= \xi^{\beta}_{(j)} \xi^{\mu}_{(k)} g_{\alpha\beta} D_\mu \xi^{\alpha}_{(i)} + \xi^{\alpha}_{(i)} \xi^{\mu}_{(k)} g_{\alpha\beta} D_\mu \xi^{\beta}_{(j)} + \xi^{\mu}_{(i)} \xi^{\beta}_{(k)} g_{\alpha\beta} D_\mu \xi^{\alpha}_{(j)} \\ &\quad +\xi^{\alpha}_{(j)} \xi^{\mu}_{(i)} g_{\alpha\beta} D_\mu \xi^{\beta}_{(k)} - \xi^{\mu}_{(j)} \xi^{\beta}_{(i)} g_{\alpha\beta} D_\mu \xi^{\alpha}_{(k)} - \xi^{\alpha}_{(k)} \xi^{\mu}_{(j)} g_{\alpha\beta} D_\mu \xi^{\beta}_{(i)} \\ &= 2 \xi^{\alpha}_{(i)} \xi^{\mu}_{(k)} g_{\alpha\beta} D_\mu \xi^{\beta}_{(j)}. \end{aligned}$$

Then

$$g_{\alpha\beta} \xi^{\alpha}_{(i)} \left(\xi^{\mu}_{(k)} D_\mu \xi^{\beta}_{(j)} \right) = 0$$

shows the vector $\xi^{\mu}_{(k)} D_\mu \xi^{\beta}_{(j)}$ is orthogonal to the submanifold. Expanding in terms of the complementary basis, this shows

$$\xi^{\mu}_{(k)} D_\mu \xi^{\beta}_{(j)} = \sum_{A=1}^{n-m} L^{A}_{kj} \zeta^{\alpha}_{(A)}.$$

The tensor L^{A}_{ij} is the second fundamental form, or extrinsic curvature, of the submanifold.

E.4. Adapted coordinates

Now introduce adapted coordinates,

$$X^{\alpha} = \left(x^i, y^A \right),$$

such that

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dX^{\alpha} dX^{\beta} \\ &= g_{AB} dy^A dy^B + h_{ij} dx^i dx^j. \end{aligned}$$

In these coordinates, the specification of the submanifold is simply $y^A = 0$, and we can take the functions f^α to be

$$\begin{aligned} f^i &= x^i, \\ f^A &= 0, \end{aligned}$$

so

$$\xi^\mu_{(k)} = f^\alpha_{,k} = \delta^\alpha_k.$$

The second fundamental form is then

$$\begin{aligned} \xi^\mu_{(k)} D_\mu \xi^\beta_{(j)} &= D_i \delta^\beta_{(j)} \\ &= \partial_i \delta^\beta_{(j)} + \delta^\alpha_{(j)} \Gamma^\beta{}_{\alpha i} \\ &= \Gamma^\beta{}_{ji} \\ &= \Gamma^A{}_{ji} \zeta^\beta_{(A)}, \end{aligned}$$

where the last step follows from the preceding proof, with $\zeta^\beta_{(A)}$ now any convenient basis for the cospace. In this basis, the extrinsic curvature is simply given by certain of the connection components, Γ^A_{ji}.

Appendix F

Variation of the Wehner-Wheeler action

F.1. Curved structure equations, action, and variation

Here, we go through the variation of the Wehner-Wheeler action in the orthogonal (but not orthonormal) basis of [7]. Note, in this variation we have reverted to using Latin indices to represent that basis as in [7]. The main reason being that the variation requires the use of quite a few indices at a time; therefore, we use the Latin for ease of calculation. The final field equations given at the end of this appendix can then be transformed to the orthonormal basis that is used in Chapters 3 and 4. The structure equations are

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a_{\ b} &= \boldsymbol{\omega}^c_{\ b}\boldsymbol{\omega}^a_{\ c} + \Delta^{ac}_{db}\left(\mathbf{f}_c + h_{cf}\mathbf{e}^f\right)\left(\mathbf{e}^d - h^{de}\mathbf{f}_e\right) + \boldsymbol{\Omega}^a_b,\\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c\boldsymbol{\omega}^a_{\ c} + \boldsymbol{\omega}\mathbf{e}^a + \frac{1}{2}\mathbf{D}h^{ac}\left(\mathbf{f}_c + h_{cd}\mathbf{e}^d\right) + \mathbf{T}^a,\\
\mathbf{d}\mathbf{f}_a &= \boldsymbol{\omega}^b_{\ a}\mathbf{f}_b - \boldsymbol{\omega}\mathbf{f}_a - \frac{1}{2}\mathbf{D}h_{ab}\left(\mathbf{e}^b - h^{bc}\mathbf{f}_c\right) + \mathbf{S}_a,\\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a\mathbf{f}_a + \boldsymbol{\Omega},
\end{aligned}$$

and the action is

$$S = \int \left(\alpha\boldsymbol{\Omega}^a_b + \beta\delta^a_b\boldsymbol{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f.$$

We are able to separate the Levi-Civita tensor into two pieces because we demand that $\mathbf{e}^a, \mathbf{f}_b$ span submanifolds.

F.2. The variation of the Weyl vector

We begin with the variation of the Weyl vector, $\boldsymbol{\omega}$,

$$\begin{aligned}
0 &= \delta S\\
&= \beta\int \mathbf{d}\left(\delta\boldsymbol{\omega}\right)\varepsilon_{ac\ldots d}\varepsilon^{ae\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&= \beta\int \varepsilon_{ac\ldots d}\varepsilon^{ae\ldots f}\left[\mathbf{d}\left(\delta\boldsymbol{\omega}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\right) + \delta\boldsymbol{\omega}\mathbf{d}\left(\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\right)\right].
\end{aligned}$$

Dropping the surface term and using the structure equations,

$$\begin{aligned}
0 &= \beta\int\delta\boldsymbol{\omega}\left[\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\mathbf{d}\left(\mathbf{e}^c\mathbf{e}^d\ldots\mathbf{e}^e\right)\mathbf{f}_f\ldots\mathbf{f}_g\right]\\
&\quad+\beta\int\delta\boldsymbol{\omega}\left[(-1)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{d}\left(\mathbf{f}_e\mathbf{f}_f\ldots\mathbf{f}_g\right)\right]\\
&= \beta\int\delta\boldsymbol{\omega}\left[(n-1)\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\left(\mathbf{d}\mathbf{e}^c\right)\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right]\\
&\quad+\beta\int\delta\boldsymbol{\omega}\left[(-1)^{n-1}(n-1)\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\left(\mathbf{d}\mathbf{f}_e\right)\mathbf{f}_f\ldots\mathbf{f}_g\right]\\
&= \beta\int\delta\boldsymbol{\omega}(n-1)\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\left(\mathbf{e}^h\boldsymbol{\omega}^c{}_h+\boldsymbol{\omega}\mathbf{e}^c\right)\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\\
&\quad+\beta\int\delta\boldsymbol{\omega}(n-1)\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\left(\frac{1}{2}\mathbf{D}h^{ch}\left(\mathbf{f}_h+h_{hi}\mathbf{e}^i\right)+\mathbf{T}^c\right)\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\\
&\quad+\beta\int\delta\boldsymbol{\omega}(-1)^{n-1}(n-1)\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\left(\boldsymbol{\omega}^h{}_e\mathbf{f}_h-\boldsymbol{\omega}\mathbf{f}_e\right)\mathbf{f}_f\ldots\mathbf{f}_g\\
&\quad+\beta\int\delta\boldsymbol{\omega}(-1)^{n-1}(n-1)\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\left(-\frac{1}{2}\mathbf{D}h_{eh}\left(\mathbf{e}^h-h^{hi}\mathbf{f}_i\right)+\mathbf{S}_e\right)\mathbf{f}_f\ldots\mathbf{f}_g.
\end{aligned}$$

Defining

$$\begin{aligned}
\tilde{\mathbf{T}}^c &= \mathbf{T}^c+\frac{1}{2}\mathbf{D}h^{ch}\left(\mathbf{f}_h+h_{hi}\mathbf{e}^i\right),\\
\tilde{\mathbf{S}}_a &= \mathbf{S}_a-\frac{1}{2}\mathbf{D}h_{ab}\left(\mathbf{e}^b-h^{bc}\mathbf{f}_c\right),
\end{aligned}$$

and collecting like terms,

$$\begin{aligned}
0 &= (n-1)\beta\int\delta\boldsymbol{\omega}\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\mathbf{e}^h\boldsymbol{\omega}^c{}_h\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad(n-1)\beta\int\delta\boldsymbol{\omega}\left((-1)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\boldsymbol{\omega}^h{}_e\mathbf{f}_h\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad+(n-1)\beta\int\delta\boldsymbol{\omega}\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\boldsymbol{\omega}\mathbf{e}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad+(n-1)\beta\int\delta\boldsymbol{\omega}\left((-1)^n\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\boldsymbol{\omega}\mathbf{f}_e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad+(n-1)\beta\int\delta\boldsymbol{\omega}\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{\mathbf{T}}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad+(n-1)\beta\int\delta\boldsymbol{\omega}\left((-1)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_f\ldots\mathbf{f}_g\right).
\end{aligned}$$

For the Weyl vector terms we have

$$\begin{aligned}
\delta_1 &= (n-1)\,\beta \int \delta\boldsymbol{\omega}\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\boldsymbol{\omega}\mathbf{e}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (n-1)\,\beta \int \delta\boldsymbol{\omega}\left((-1)^n\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\boldsymbol{\omega}\mathbf{f}_e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (n-1)\,\beta \int \delta\boldsymbol{\omega}\left((-1)^q\,(n-1)!\delta^{f\ldots g}_{cd\ldots e}\boldsymbol{\omega}\mathbf{e}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (n-1)\,\beta \int \delta\boldsymbol{\omega}\left((-1)^n\,(-1)^{n-1}\,\boldsymbol{\omega}\,(-1)^q\,(n-1)!\delta^{ef\ldots g}_{c\ldots d}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (n-1)\,\beta \int \delta\boldsymbol{\omega}\,(\boldsymbol{\omega}-\boldsymbol{\omega})\,(-1)^q\,(n-1)!\delta^{ef\ldots g}_{c\ldots d}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\mathbf{f}_f\ldots\mathbf{f}_g\\
&= 0.
\end{aligned}$$

Now check the spin connection terms. Expand the spin connection as $\boldsymbol{\omega}^c{}_h = \omega^c{}_{hi}\mathbf{e}^i + \omega^{c\,i}_{h}\mathbf{f}_i$, and the variation as $\delta\boldsymbol{\omega} = A_j\mathbf{e}^j + B^j\mathbf{f}_j$, so

$$\begin{aligned}
\delta_2 &= (n-1)\,\beta \int \delta\boldsymbol{\omega}\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\mathbf{e}^h\boldsymbol{\omega}^c{}_h\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (n-1)\,\beta \int \delta\boldsymbol{\omega}\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\boldsymbol{\omega}^h{}_e\mathbf{f}_h\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (n-1)\,\beta \int \delta\boldsymbol{\omega}\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\mathbf{e}^h\omega^c{}_{hi}\mathbf{e}^i\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (n-1)\,\beta \int \delta\boldsymbol{\omega}\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\omega^h{}_{ei}\mathbf{e}^i\mathbf{f}_h\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (n-1)\,\beta \int \delta\boldsymbol{\omega}\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\mathbf{e}^h\omega^{c\,i}_{h}\mathbf{f}_i\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (n-1)\,\beta \int \delta\boldsymbol{\omega}\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\omega^{h\,i}_{e}\mathbf{f}_i\mathbf{f}_h\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (-1)^n\,(n-1)\,\beta \int B^j\omega^h_{ki}\left(\varepsilon_{ahd\ldots e}\mathbf{e}^k\mathbf{e}^i\mathbf{e}^d\ldots\mathbf{e}^e\varepsilon^{af\ldots g}\mathbf{f}_j\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (-1)^n\,(n-1)\,\beta \int B^j\omega^h_{ki}\left(\varepsilon_{ac\ldots d}\mathbf{e}^i\mathbf{e}^c\ldots\mathbf{e}^d\varepsilon^{akf\ldots g}\mathbf{f}_j\mathbf{f}_h\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (n-1)\,\beta \int A_j\omega^{h\,i}_{k}\left(\varepsilon_{ahd\ldots e}\varepsilon^{af\ldots g}\mathbf{e}^j\mathbf{e}^k\mathbf{f}_i\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad + (n-1)\,\beta \int A_j\omega^{h\,i}_{k}\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{akf\ldots g}\mathbf{e}^j\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_i\mathbf{f}_h\mathbf{f}_f\ldots\mathbf{f}_g\right).
\end{aligned}$$

Define the volume forms,

$$\mathbf{e}^a\ldots\mathbf{e}^b = \varepsilon^{a\ldots b}\boldsymbol{\Phi}^n,$$

$$\begin{aligned}\mathbf{f}_a \dots \mathbf{f}_b &= \varepsilon_{a\dots b}\mathbf{\Phi}_n,\\ \mathbf{\Phi}^n_n &= \mathbf{\Phi}^n\mathbf{\Phi}_n.\end{aligned}$$

Then

$$\begin{aligned}\delta_2 &= (-1)^n(n-1)\beta\int B^j\omega^h_{ki}\left(\varepsilon_{ahd\dots e}\mathbf{e}^k\mathbf{e}^i\mathbf{e}^d\dots\mathbf{e}^e\varepsilon^{af\dots g}\mathbf{f}_j\mathbf{f}_f\dots\mathbf{f}_g\right)\\
&\quad+(-1)^n(n-1)\beta\int B^j\omega^h_{ki}\left(\varepsilon_{ac\dots d}\mathbf{e}^i\mathbf{e}^c\dots\mathbf{e}^d\varepsilon^{akf\dots g}\mathbf{f}_j\mathbf{f}_h\mathbf{f}_f\dots\mathbf{f}_g\right)\\
&\quad+(-1)^n(n-1)\beta\int A_j\omega^{h\,i}_{k}\left(\varepsilon_{ahd\dots e}\varepsilon^{af\dots g}\mathbf{e}^j\mathbf{e}^k\mathbf{e}^d\dots\mathbf{e}^e\mathbf{f}_i\mathbf{f}_f\dots\mathbf{f}_g\right)\\
&\quad-(-1)^n(n-1)\beta\int A_j\omega^{h\,i}_{k}\left(\varepsilon_{ac\dots d}\varepsilon^{akf\dots g}\mathbf{e}^j\mathbf{e}^c\dots\mathbf{e}^d\mathbf{f}_i\mathbf{f}_h\mathbf{f}_f\dots\mathbf{f}_g\right)\\
&= (-1)^n(n-1)\beta\int B^j\omega^h_{ki}\left(\varepsilon_{ahd\dots e}\varepsilon^{kid\dots e}\mathbf{\Phi}^n\varepsilon^{af\dots g}\varepsilon_{jf\dots g}\mathbf{\Phi}_n\right)\\
&\quad+(-1)^n(n-1)\beta\int B^j\omega^h_{ki}\left(\varepsilon_{ac\dots d}\varepsilon^{ic\dots d}\mathbf{\Phi}^n\varepsilon^{akf\dots g}\varepsilon_{jhf\dots g}\mathbf{\Phi}_n\right)\\
&\quad+(-1)^n(n-1)\beta\int A_j\omega^{h\,i}_{k}\left(\varepsilon_{ahd\dots e}\varepsilon^{af\dots g}\varepsilon^{jkd\dots e}\mathbf{\Phi}^n\varepsilon_{if\dots g}\mathbf{\Phi}_n\right)\\
&\quad-(-1)^n(n-1)\beta\int A_j\omega^{h\,i}_{k}\left(\varepsilon_{ac\dots d}\varepsilon^{akf\dots g}\varepsilon^{jc\dots d}\mathbf{\Phi}^n\varepsilon_{ihf\dots g}\mathbf{\Phi}_n\right)\\
&= (-1)^n(n-1)\beta\int B^j\omega^h_{ki}\left(2(n-2)!\delta^{ki}_{ah}(n-1)!\delta^a_j+(n-1)!\delta^i_a 2(n-2)!\delta^{ak}_{jh}\right)\mathbf{\Phi}^n_n\\
&\quad+(-1)^n(n-1)\beta\int A_j\omega^{h\,i}_{k}\left(2(n-2)!\delta^{jk}_{ah}(n-1)!\delta^a_i-2(n-2)!\delta^{ak}_{ih}(n-1)!\delta^j_a\right)\mathbf{\Phi}^n_n\\
&= (-1)^n(n-1)(n-1)!(n-2)!\beta\int B^j\left(\omega^i_{ji}-\omega^k_{kj}+\omega^k_{kj}-\omega^h_{jh}\right)\mathbf{\Phi}^n_n\\
&\quad+(-1)^n(n-1)(n-1)!(n-2)!\beta\int A_j\left(\omega^{k\,j}_{k}-\omega^{j\,k}_{k}-\omega^{k\,j}_{k}+\omega^{j\,k}_{k}\right)\mathbf{\Phi}^n_n\\
&= 0.\end{aligned}$$

This leaves the field equation in terms of the torsion and cotorsion, as expected,

$$\begin{aligned}0 &= (n-1)\beta\int\delta\boldsymbol{\omega}\left(\varepsilon_{acd\dots e}\varepsilon^{af\dots g}\tilde{\mathbf{T}}^c\mathbf{e}^d\dots\mathbf{e}^e\mathbf{f}_f\dots\mathbf{f}_g\right)\\
&\quad+(n-1)\beta\int\delta\boldsymbol{\omega}\left((-1)^{n-1}\varepsilon_{ac\dots d}\varepsilon^{aef\dots g}\mathbf{e}^c\dots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_f\dots\mathbf{f}_g\right).\end{aligned}$$

Expanding the variation gives two equations,

$$\begin{aligned}
0 &= (n-1)\,\beta\mathbf{e}^j\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{\mathbf{T}}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad+(n-1)\,\beta\mathbf{e}^j\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_f\ldots\mathbf{f}_g\right),\\
0 &= (n-1)\,\beta\mathbf{f}_j\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{\mathbf{T}}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad+(n-1)\,\beta\mathbf{f}_j\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_f\ldots\mathbf{f}_g\right).
\end{aligned}$$

For the first,

$$\begin{aligned}
0 &= (n-1)\,\beta\mathbf{e}^j\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{\mathbf{T}}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g+(-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (n-1)\,\beta\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{\mathbf{T}}^c\mathbf{e}^j\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad+(n-1)\,\beta\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^j\mathbf{e}^c\ldots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (n-1)\,\beta\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{T}^{ch}{}_k\mathbf{f}_h\mathbf{e}^k\mathbf{e}^j\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&\quad+(n-1)\,\beta\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^j\mathbf{e}^c\ldots\mathbf{e}^d\frac{1}{2}\tilde{S}_e{}^{hk}\mathbf{f}_h\mathbf{f}_k\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (n-1)\,\beta\left((-1)^n\,\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{T}^{ch}{}_k\varepsilon^{kjd\ldots e}\varepsilon_{hf\ldots g}\right)\mathbf{\Phi}^n_n\\
&\quad+(n-1)\,\beta\left((-1)^{n-1}\,\frac{1}{2}\tilde{S}_e{}^{hk}\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\varepsilon^{jc\ldots d}\varepsilon_{hkf\ldots g}\right)\mathbf{\Phi}^n_n\\
&= 2\,(n-2)!\,(n-1)!\,(n-1)\,(-1)^n\,\beta\left(\delta^a_h\delta^{kj}_{ac}\tilde{T}^{ch}{}_k-\frac{1}{2}\tilde{S}_e{}^{hk}\delta^j_a\delta^{ae}_{hk}\right)\mathbf{\Phi}^n_n\\
&= (n-2)!\,(n-1)!\,(n-1)\,(-1)^n\,\beta\left(\tilde{T}^{jk}{}_k-\tilde{T}^{kj}{}_k-\frac{1}{2}\tilde{S}_e{}^{je}+\frac{1}{2}\tilde{S}_e{}^{ej}\right)\mathbf{\Phi}^n_n,
\end{aligned}$$

so

$$\beta\tilde{S}_k{}^{jk}=\beta\left(\tilde{T}^{jk}{}_k-\tilde{T}^{kj}{}_k\right).$$

Similarly, for the second,

$$\begin{aligned}
0 &= (n-1)\,\beta\mathbf{f}_j\left(\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{\mathbf{T}}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g+(-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (n-1)\,\beta\left((-1)^n\,\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{\mathbf{T}}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_j\mathbf{f}_f\ldots\mathbf{f}_g+\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_j\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&= (n-1)\,\beta\left((-1)^n\,\frac{1}{2}\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{T}^c{}_{hk}\mathbf{e}^h\mathbf{e}^k\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_j\mathbf{f}_f\ldots\mathbf{f}_g\right)
\end{aligned}$$

$$
\begin{aligned}
&+(n-1)\,\beta\left(\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\tilde{S}_e{}^h{}_k\mathbf{f}_h\mathbf{e}^k\mathbf{f}_j\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
=\;&(n-1)\,\beta\left((-1)^n\frac{1}{2}\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\tilde{T}^c{}_{hk}\mathbf{e}^h\mathbf{e}^k\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_j\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
&-(n-1)\,\beta\left(\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\tilde{S}_e{}^h{}_k\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{e}^k\mathbf{f}_h\mathbf{f}_j\mathbf{f}_f\ldots\mathbf{f}_g\right)\\
=\;&2\,(n-2)!\,(n-1)!\,(n-1)\,\beta\left((-1)^n\frac{1}{2}\varepsilon_{acd\ldots e}\varepsilon^{af\ldots g}\frac{1}{2}\tilde{T}^c{}_{hk}\varepsilon^{hkd\ldots e}\varepsilon_{jf\ldots g}\right)\boldsymbol{\Phi}_n^n\\
&-2\,(n-2)!\,(n-1)!\,(n-1)\,\beta\left(\varepsilon_{ac\ldots d}\varepsilon^{aef\ldots g}\tilde{S}_e{}^h{}_k\varepsilon^{c\ldots dk}\varepsilon_{hjf\ldots g}\right)\boldsymbol{\Phi}_n^n\\
=\;&(n-2)!\,(n-1)!\,(n-1)\,\beta\left((-1)^n\left(\delta_a^h\delta_c^k-\delta_c^h\delta_a^k\right)\delta_j^a\frac{1}{2}\tilde{T}^c{}_{hk}\right)\boldsymbol{\Phi}_n^n\\
&-(n-2)!\,(n-1)!\,(n-1)\,\beta\left(\left(\delta_h^a\delta_j^e-\delta_h^e\delta_j^a\right)(-1)^{n-1}\delta_a^k\tilde{S}_e{}^h{}_k\right)\boldsymbol{\Phi}_n^n\\
=\;&(n-2)!\,(n-1)!\,(n-1)\,(-1)^n\,\beta\left(\tilde{T}^c{}_{jc}+\tilde{S}_j{}^k{}_k-\tilde{S}_h{}^h{}_j\right)\boldsymbol{\Phi}_n^n,
\end{aligned}
$$

so

$$\beta\tilde{T}^c{}_{jc}=\beta\left(\tilde{S}_h{}^h{}_j-\tilde{S}_j{}^k{}_k\right).$$

Check

Assume the tensorial character, so from the action

$$S=\int\left(\alpha\boldsymbol{\Omega}_b^a+\beta\delta_b^a\boldsymbol{\Omega}+\gamma\mathbf{e}^a\mathbf{f}_b\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f,$$

we have

$$
\begin{aligned}
0\;=\;&\delta S\\
=\;&\beta\int\mathbf{d}\left(\delta\boldsymbol{\omega}\right)\varepsilon_{ac\ldots d}\varepsilon^{ae\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
=\;&\beta\int\varepsilon_{ac\ldots d}\varepsilon^{ae\ldots f}\left[\mathbf{d}\left(\delta\boldsymbol{\omega}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\right)+\delta\boldsymbol{\omega}\mathbf{d}\left(\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\right)\right]\\
=\;&\beta\int\varepsilon_{ac\ldots d}\varepsilon^{ae\ldots f}\delta\boldsymbol{\omega}\mathbf{d}\left(\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\right)\\
=\;&(n-1)\,\beta\int\delta\boldsymbol{\omega}\left(\varepsilon_{acg\ldots d}\varepsilon^{ae\ldots f}\tilde{\mathbf{T}}^c\mathbf{e}^g\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\right)\\
&+(n-1)\,\beta\int\delta\boldsymbol{\omega}\left((-1)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{aeg\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\tilde{\mathbf{S}}_e\mathbf{f}_g\ldots\mathbf{f}_f\right).
\end{aligned}
$$

With $\delta\boldsymbol{\omega} = A_k \mathbf{e}^k + B^k \mathbf{f}_k$,

$$
\begin{aligned}
0 &= (n-1)\,\beta \int \delta\boldsymbol{\omega} \left(\varepsilon_{acg\ldots d} \varepsilon^{ae\ldots f} \tilde{\mathbf{T}}^c \mathbf{e}^g \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f + (-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{aeg\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \tilde{\mathbf{S}}_e \mathbf{f}_g \ldots \mathbf{f}_f \right) \\
&= (n-1)\,\beta \int A_k \left(\varepsilon_{acg\ldots d} \varepsilon^{ae\ldots f} \tilde{\mathbf{T}}^c \mathbf{e}^k \mathbf{e}^g \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\beta \int A_k \left((-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{aeg\ldots f} \mathbf{e}^k \mathbf{e}^c \ldots \mathbf{e}^d \tilde{\mathbf{S}}_e \mathbf{f}_g \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\beta \int B^k \left((-1)^{n} \varepsilon_{acg\ldots d} \varepsilon^{ae\ldots f} \tilde{\mathbf{T}}^c \mathbf{e}^g \ldots \mathbf{e}^d \mathbf{f}_k \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\beta \int B^k \left(\varepsilon_{ac\ldots d} \varepsilon^{aeg\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \tilde{\mathbf{S}}_e \mathbf{f}_k \mathbf{f}_g \ldots \mathbf{f}_f \right) \\
&= (n-1)\,\beta \int A_k \left((-1)^{n} \varepsilon_{acg\ldots d} \varepsilon^{ae\ldots f} \tilde{T}^{cm}{}_n \mathbf{e}^n \mathbf{e}^k \mathbf{e}^g \ldots \mathbf{e}^d \mathbf{f}_m \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\beta \int A_k \left((-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{aeg\ldots f} \mathbf{e}^k \mathbf{e}^c \ldots \mathbf{e}^d \frac{1}{2} \tilde{S}_e{}^{mn} \mathbf{f}_m \mathbf{f}_n \mathbf{f}_g \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\beta \int B^k \left((-1)^{n} \varepsilon_{acg\ldots d} \varepsilon^{ae\ldots f} \frac{1}{2} \tilde{T}^c{}_{mn} \mathbf{e}^m \mathbf{e}^n \mathbf{e}^g \ldots \mathbf{e}^d \mathbf{f}_k \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\beta \int B^k \left((-1)^{n} \varepsilon_{ac\ldots d} \varepsilon^{aeg\ldots f} \mathbf{e}^n \mathbf{e}^c \ldots \mathbf{e}^d \tilde{S}_e{}^m{}_n \mathbf{f}_m \mathbf{f}_k \mathbf{f}_g \ldots \mathbf{f}_f \right).
\end{aligned}
$$

Therefore, the field equations are

$$
\begin{aligned}
0 &= (n-1)\,\beta \left((-1)^{n} \varepsilon_{acg\ldots d} \varepsilon^{ae\ldots f} \tilde{T}^{cm}{}_n \varepsilon^{nkg\ldots d} \varepsilon_{me\ldots f} \right) \\
&\quad + (n-1)\,\beta \left((-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{aeg\ldots f} \varepsilon^{kc\ldots d} \frac{1}{2} \tilde{S}_e{}^{mn} \varepsilon_{mng\ldots f} \right) \\
&= 2\,(n-1)!\,(n-1)!\,(-1)^n\,\beta \left(\delta^a_m \delta^{nk}_{ac} \tilde{T}^{cm}{}_n - \delta^k_a \delta^{ae}_{mn} \frac{1}{2} \tilde{S}_e{}^{mn} \right) \\
&= (n-1)!\,(n-1)!\,(-1)^n\,\beta \left(\tilde{T}^{km}{}_m - \tilde{T}^{nk}{}_n - \tilde{S}_e{}^{ke} \right), \\
0 &= 2\,(n-1)!\,(n-1)!\,(-1)^n\,\beta \left(\delta^a_k \delta^{mn}_{ac} \frac{1}{2} \tilde{T}^c{}_{mn} + \delta^n_a \delta^{ae}_{mk} \tilde{S}_e{}^m{}_n \right) \\
&= (n-1)!\,(n-1)!\,(-1)^n\,\beta \left(\tilde{T}^c{}_{kc} + \tilde{S}_k{}^m{}_m - \tilde{S}_m{}^m{}_k \right),
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\beta \left(\tilde{T}^{km}{}_m - \tilde{T}^{nk}{}_n \right) &= \beta \tilde{S}_e{}^{ke}, \\
\beta \left(\tilde{S}_k{}^m{}_m - \tilde{S}_m{}^m{}_k \right) &= -\beta \tilde{T}^c{}_{kc}.
\end{aligned}
$$

These check.

F.3. Spin connection variation

Now vary the spin connection,

$$
\begin{aligned}
0 &= \delta S \\
&= \alpha \int \left(\delta \boldsymbol{\Omega}^a_{\ b}\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&= \alpha \int \left(\mathbf{d}\delta\boldsymbol{\omega}^a_{\ b} - \delta\boldsymbol{\omega}^g_{\ b}\boldsymbol{\omega}^a_{\ g} - \boldsymbol{\omega}^g_{\ b}\delta\boldsymbol{\omega}^a_{\ g}\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&= \alpha \int \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{d}\left(\delta\boldsymbol{\omega}^a_{\ b} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f\right) + \alpha \int \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \delta\boldsymbol{\omega}^a_{\ b} \mathbf{d}\left(\mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f\right) \\
&\quad -\alpha \int \left(\delta\boldsymbol{\omega}^g_{\ b}\boldsymbol{\omega}^a_{\ g} + \boldsymbol{\omega}^g_{\ b}\delta\boldsymbol{\omega}^a_{\ g}\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&= \alpha \int \delta\boldsymbol{\omega}^a_{\ b} \left((n-1)\, \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{d}\mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f\right) \\
&\quad +\alpha \int \delta\boldsymbol{\omega}^a_{\ b} \left((-1)^{n-1} (n-1)\, \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{d}\mathbf{f}_e \ldots \mathbf{f}_f\right) \\
&\quad -\alpha \int \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \left(\delta\boldsymbol{\omega}^g_{\ b}\boldsymbol{\omega}^a_{\ g} + \boldsymbol{\omega}^g_{\ b}\delta\boldsymbol{\omega}^a_{\ g}\right) \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f.
\end{aligned}
$$

Now, substitute from the structure equations and expand the spin connection and its variation as

$$
\begin{aligned}
\boldsymbol{\omega}^a_{\ g} &= \omega^a_{\ gi} \mathbf{e}^i + \omega^{a\ i}_{\ g} \mathbf{f}_i, \\
\delta\boldsymbol{\omega}^h_{\ k} &= A^h_{\ ki} \mathbf{e}^i + B^{h\ i}_{\ k} \mathbf{f}_i.
\end{aligned}
$$

Then

$$
\begin{aligned}
0 &= \alpha \int \delta\boldsymbol{\omega}^a_{\ b} \left((n-1)\, \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{d}\mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f\right) \\
&\quad +\alpha \int \delta\boldsymbol{\omega}^a_{\ b} \left((-1)^{n-1} (n-1)\, \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{d}\mathbf{f}_e \ldots \mathbf{f}_f\right) \\
&\quad -\alpha \int \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \left(\delta\boldsymbol{\omega}^g_{\ b}\boldsymbol{\omega}^a_{\ g} + \boldsymbol{\omega}^g_{\ b}\delta\boldsymbol{\omega}^a_{\ g}\right) \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&= \alpha \int \left(A^a_{\ bi} \mathbf{e}^i + B^{a\ i}_{\ b} \mathbf{f}_i\right) \left((n-1)\, \varepsilon_{ach\ldots d} \varepsilon^{be\ldots f} \left(\mathbf{e}^g \boldsymbol{\omega}^c_{\ g} + \boldsymbol{\omega}\mathbf{e}^c + \tilde{\mathbf{T}}^c\right) \mathbf{e}^h \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f\right) \\
&\quad +\alpha \int \left(A^a_{\ bi} \mathbf{e}^i + B^{a\ i}_{\ b} \mathbf{f}_i\right) \left((-1)^{n-1} (n-1)\, \varepsilon_{ac\ldots d} \varepsilon^{beh\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \left(\boldsymbol{\omega}^g_{\ e} \mathbf{f}_g - \boldsymbol{\omega}\mathbf{f}_e + \tilde{\mathbf{S}}_e\right) \mathbf{f}_h \ldots \mathbf{f}_f\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\alpha\int \varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\left(\left(A^{g}_{\ bi}\mathbf{e}^{i}+B^{g\ i}_{\ b}\mathbf{f}_{i}\right)\left(\omega^{a}_{\ gj}\mathbf{e}^{j}+\omega^{a\ j}_{\ g}\mathbf{f}_{j}\right)\right)\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&-\alpha\int \varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\left(\left(\omega^{g}_{\ bj}\mathbf{e}^{j}+\omega^{g\ j}_{\ b}\mathbf{f}_{j}\right)\left(A^{a}_{\ gi}\mathbf{e}^{i}+B^{a\ i}_{\ g}\mathbf{f}_{i}\right)\right)\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
=\ &\alpha\int (n-1)\,\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}A^{a}_{\ bi}\mathbf{e}^{i}\left(\mathbf{e}^{g}\boldsymbol{\omega}^{c}_{\ g}+\boldsymbol{\omega}\mathbf{e}^{c}+\tilde{\mathbf{T}}^{c}\right)\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\alpha\int (-1)^{n-1}(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}A^{a}_{\ bi}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\left(\boldsymbol{\omega}^{g}_{\ e}\mathbf{f}_{g}-\boldsymbol{\omega}\mathbf{f}_{e}+\tilde{\mathbf{S}}_{e}\right)\mathbf{f}_{h}\ldots\mathbf{f}_{f}\\
&-\alpha\int A^{a}_{\ bi}\left(\varepsilon_{mc\ldots d}\varepsilon^{be\ldots f}\omega^{m\ j}_{\ a}\mathbf{e}^{i}\mathbf{f}_{j}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}+\varepsilon_{ac\ldots d}\varepsilon^{me\ldots f}\omega^{b\ j}_{\ m}\mathbf{f}_{j}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\right)\\
&+\alpha\int (n-1)\,\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}B^{a\ i}_{\ b}\mathbf{f}_{i}\left(\mathbf{e}^{g}\boldsymbol{\omega}^{c}_{\ g}+\boldsymbol{\omega}\mathbf{e}^{c}+\tilde{\mathbf{T}}^{c}\right)\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\alpha\int (-1)^{n-1}(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}B^{a\ i}_{\ b}\mathbf{f}_{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\left(\boldsymbol{\omega}^{g}_{\ e}\mathbf{f}_{g}-\boldsymbol{\omega}\mathbf{f}_{e}+\tilde{\mathbf{S}}_{e}\right)\mathbf{f}_{h}\ldots\mathbf{f}_{f}\\
&-\alpha\int B^{a\ i}_{\ b}\left(\varepsilon_{mc\ldots d}\varepsilon^{be\ldots f}\mathbf{f}_{i}\left(\omega^{m}_{\ aj}\mathbf{e}^{j}+\omega^{m\ j}_{\ a}\mathbf{f}_{j}\right)\right)\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&-\alpha\int B^{a\ i}_{\ b}\left(\varepsilon_{ac\ldots d}\varepsilon^{me\ldots f}\left(\omega^{b}_{\ mj}\mathbf{e}^{j}+\omega^{b\ j}_{\ m}\mathbf{f}_{j}\right)\mathbf{f}_{i}\right)\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}.
\end{aligned}
$$

So we have two equations,

$$
\begin{aligned}
0\ =\ &\alpha\,(n-1)\,\Delta^{ma}_{bn}\left(\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{i}\tilde{\mathbf{T}}^{c}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\right)\\
&+\alpha\,(n-1)\,\Delta^{ma}_{bn}\left((-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\tilde{\mathbf{S}}_{e}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\right)\\
&+\alpha\,(n-1)\,\Delta^{ma}_{bn}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{i}\boldsymbol{\omega}\mathbf{e}^{c}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+(-1)^{n}\,\alpha\,(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\boldsymbol{\omega}\mathbf{f}_{e}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\\
&+\alpha\,(n-1)\,\Delta^{ma}_{bn}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{i}\mathbf{e}^{g}\omega^{c\ j}_{\ g}\mathbf{f}_{j}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+(-1)^{n-1}\,\alpha\,(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\omega^{g\ j}_{\ e}\mathbf{f}_{j}\mathbf{f}_{g}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\\
&-\alpha\,(-1)^{n-1}\,\Delta^{ma}_{bn}\varepsilon_{mc\ldots d}\varepsilon^{be\ldots f}\omega^{m\ j}_{\ a}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{j}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&-\alpha\,(-1)^{n}\,\varepsilon_{ac\ldots d}\varepsilon^{me\ldots f}\omega^{b\ j}_{\ m}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{j}\mathbf{f}_{e}\ldots\mathbf{f}_{f},\\
0\ =\ &\alpha\,(n-1)\,\Delta^{ma}_{bn}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{f}_{i}\tilde{\mathbf{T}}^{c}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\alpha\,(-1)^{n-1}\,(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{f}_{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\tilde{\mathbf{S}}_{e}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\\
&+\alpha\,(n-1)\,\Delta^{ma}_{bn}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{f}_{i}\boldsymbol{\omega}\mathbf{e}^{c}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\alpha\,(-1)^{n}\,(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{f}_{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\boldsymbol{\omega}\mathbf{f}_{e}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\\
&+\alpha\,(n-1)\,\Delta^{ma}_{bn}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{f}_{i}\mathbf{e}^{g}\omega^{c}_{\ gj}\mathbf{e}^{j}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha\left(-1\right)^{n-1}\left(n-1\right)\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{f}_i\mathbf{e}^c\ldots\mathbf{e}^d\omega^g_{\ ej}\mathbf{e}^j\mathbf{f}_g\mathbf{f}_h\ldots\mathbf{f}_f\\
&-\alpha\Delta^{ma}_{bn}\varepsilon_{mc\ldots d}\varepsilon^{be\ldots f}\mathbf{f}_i\omega^m_{\ aj}\mathbf{e}^j\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&-\alpha\varepsilon_{ac\ldots d}\varepsilon^{me\ldots f}\omega^b_{\ mj}\mathbf{e}^j\mathbf{f}_i\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f.
\end{aligned}
$$

Look at the Weyl vector pieces,

$$
\begin{aligned}
\delta_1 &= \alpha\left(n-1\right)\Delta^{ma}_{bn}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^i\boldsymbol{\omega}\mathbf{e}^c\mathbf{e}^h\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&\quad+\left(-1\right)^n\alpha\left(n-1\right)\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^i\mathbf{e}^c\ldots\mathbf{e}^d\boldsymbol{\omega}\mathbf{f}_e\mathbf{f}_h\ldots\mathbf{f}_f\\
&= \alpha\left(n-1\right)\Delta^{ma}_{bn}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^iW^j\mathbf{f}_j\mathbf{e}^c\mathbf{e}^h\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&\quad+\left(-1\right)^n\alpha\left(n-1\right)\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^i\mathbf{e}^c\ldots\mathbf{e}^dW^j\mathbf{f}_j\mathbf{f}_e\mathbf{f}_h\ldots\mathbf{f}_f\\
&= \alpha\left(n-1\right)\Delta^{ma}_{bn}W^j\left(\left(-1\right)^{n-1}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^i\mathbf{e}^c\mathbf{e}^h\ldots\mathbf{e}^d\mathbf{f}_j\mathbf{f}_e\ldots\mathbf{f}_f\right)\\
&\quad\alpha\left(n-1\right)\Delta^{ma}_{bn}W^j\left(\left(-1\right)^n\varepsilon_{ac\ldots d}\mathbf{e}^i\mathbf{e}^c\ldots\mathbf{e}^d\varepsilon^{beh\ldots f}\mathbf{f}_j\mathbf{f}_e\mathbf{f}_h\ldots\mathbf{f}_f\right)\\
&= \left(-1\right)^n\alpha\left(n-1\right)\Delta^{ma}_{bn}W^j\left(-\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\varepsilon^{ich\ldots d}\varepsilon_{je\ldots f}+\varepsilon_{ac\ldots d}\varepsilon^{ic\ldots d}\varepsilon^{beh\ldots f}\varepsilon_{jeh\ldots f}\right)\boldsymbol{\Phi}^n_n\\
&= \left(-1\right)^n\alpha\left(n-1\right)\Delta^{ma}_{bn}W^j\left(-\left(n-1\right)!\left(n-1\right)!\delta^b_j\delta^i_a+\left(n-1\right)!\left(n-1\right)!\delta^b_j\delta^i_a\right)\boldsymbol{\Phi}^n_n\\
&= 0,\\
\delta'_1 &= \alpha\left(n-1\right)\Delta^{ma}_{bn}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{f}_i\boldsymbol{\omega}\mathbf{e}^c\mathbf{e}^h\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&\quad+\alpha\left(-1\right)^n\left(n-1\right)\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{f}_i\mathbf{e}^c\ldots\mathbf{e}^d\boldsymbol{\omega}\mathbf{f}_e\mathbf{f}_h\ldots\mathbf{f}_f\\
&= \alpha\left(n-1\right)\left(-1\right)^n\Delta^{ma}_{bn}\left(\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}W_j\mathbf{e}^j\mathbf{e}^c\mathbf{e}^h\ldots\mathbf{e}^d\mathbf{f}_i\mathbf{f}_e\ldots\mathbf{f}_f\right)\\
&\quad-\alpha\left(n-1\right)\left(-1\right)^n\Delta^{ma}_{bn}\left(\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^j\mathbf{e}^c\ldots\mathbf{e}^dW_j\mathbf{f}_i\mathbf{f}_e\mathbf{f}_h\ldots\mathbf{f}_f\right)\\
&= \alpha\left(n-1\right)\left(-1\right)^nW_j\Delta^{ma}_{bn}\left(\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\varepsilon^{jch\ldots d}\varepsilon_{ie\ldots f}-\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\varepsilon^{jc\ldots d}\varepsilon_{ieh\ldots f}\right)\boldsymbol{\Phi}^n_n\\
&= \left(n-1\right)!\left(n-1\right)!\alpha\left(n-1\right)\left(-1\right)^n\Delta^{ma}_{bn}W_j\left(\delta^j_a\delta^b_i-\delta^j_a\delta^b_i\right)\boldsymbol{\Phi}^n_n\\
&= 0.
\end{aligned}
$$

Here are the spin connection terms, dropping the full volume form,

$$
\begin{aligned}
\sigma &= \alpha\left(n-1\right)\left(-1\right)^n\Delta^{ma}_{bn}\omega^{c\ j}_{\ g}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\varepsilon^{igh\ldots d}\varepsilon_{je\ldots f}\\
&\quad+\left(-1\right)^{n-1}\alpha\left(n-1\right)\omega^{g\ j}_{\ e}\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\varepsilon^{ic\ldots d}\varepsilon_{jgh\ldots f}\Delta^{ma}_{bn}
\end{aligned}
$$

$$
\begin{aligned}
& -\Delta_{bn}^{ma}\alpha\left(-1\right)^{n-1}\omega_{a}^{h\ j}\varepsilon_{hc\ldots d}\varepsilon^{be\ldots f}\varepsilon^{ic\ldots d}\varepsilon_{je\ldots f}\\
& -\alpha\left(-1\right)^{n}\omega_{h}^{b\ j}\varepsilon_{ac\ldots d}\varepsilon^{he\ldots f}\varepsilon^{ic\ldots d}\varepsilon_{je\ldots f}\Delta_{bn}^{ma}\\
= \ & \alpha\left(n-1\right)\left(-1\right)^{n}2\left(n-1\right)!\left(n-2\right)!\Delta_{bn}^{ma}\left(\omega_{g}^{c\ j}\delta_{ac}^{ig}\varepsilon_{j}^{b}-\omega_{e}^{g\ j}\delta_{jg}^{be}\delta_{a}^{i}\right)\\
& -\alpha\left(-1\right)^{n-1}\left(n-1\right)!\left(n-1\right)!\Delta_{bn}^{ma}\left(\omega_{a}^{m\ j}\delta_{j}^{b}\delta_{m}^{i}-\omega_{m}^{b\ j}\delta_{j}^{m}\delta_{a}^{i}\right)\\
= \ & \alpha\left(n-1\right)\left(n-1\right)!\left(n-2\right)!\Delta_{bn}^{ma}\left(\left(-1\right)^{n}\left(\omega_{c}^{c\ b}\delta_{a}^{i}-\omega_{a}^{i\ b}-\omega_{e}^{e\ b}\delta_{a}^{i}+\omega_{e}^{b\ e}\delta_{a}^{i}\right)\right)\\
& -\alpha\left(n-1\right)\left(n-1\right)!\left(n-2\right)!\Delta_{bn}^{ma}\left(\left(-1\right)^{n-1}\left(\omega_{a}^{i\ b}-\omega_{m}^{b\ m}\delta_{a}^{i}\right)\right)\\
= \ & \alpha\left(n-1\right)\left(n-1\right)!\left(n-2\right)!\Delta_{bn}^{ma}\left(\left(-1\right)^{n}\left(\omega_{c}^{c\ b}\delta_{a}^{i}-\omega_{a}^{i\ b}-\omega_{e}^{e\ b}\delta_{a}^{i}+\omega_{e}^{b\ e}\delta_{a}^{i}\right)\right)\\
& +\alpha\left(n-1\right)\left(n-1\right)!\left(n-2\right)!\Delta_{bn}^{ma}\left(\left(-1\right)^{n}\left(\omega_{a}^{i\ b}-\omega_{m}^{b\ m}\delta_{a}^{i}\right)\right)\\
= \ & \alpha\left(n-1\right)\left(n-1\right)!\left(n-2\right)!\left(-1\right)^{n}\Delta_{bn}^{ma}\left(-\omega_{a}^{i\ b}+\omega_{e}^{b\ e}\delta_{a}^{i}+\omega_{a}^{i\ b}-\omega_{m}^{b\ m}\delta_{a}^{i}\right)\\
= \ & 0,\\
\sigma' \ = \ & \alpha\left(n-1\right)\left(-1\right)^{n}\Delta_{bn}^{ma}\omega_{gj}^{c}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\varepsilon^{gjh\ldots d}\varepsilon_{ie\ldots f}\\
& +\alpha\left(n-1\right)\left(-1\right)^{n}\Delta_{bn}^{ma}\omega_{gj}^{a}\omega_{ej}^{g}\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\varepsilon^{jc\ldots d}\varepsilon_{igh\ldots f}\\
& -\left(-1\right)^{n}\alpha\Delta_{bn}^{ha}\omega_{aj}^{h}\varepsilon_{mc\ldots d}\varepsilon^{be\ldots f}\varepsilon^{jc\ldots d}\varepsilon_{ie\ldots f}\\
& -\alpha\left(-1\right)^{n-1}\Delta_{bn}^{ma}\omega_{hj}^{b}\varepsilon_{ac\ldots d}\varepsilon^{he\ldots f}\varepsilon^{jc\ldots d}\varepsilon_{ie\ldots f}\\
= \ & \alpha2\left(n-1\right)!\left(n-1\right)!\left(-1\right)^{n}\Delta_{bn}^{ma}\omega_{gj}^{c}\delta_{i}^{b}\delta_{ac}^{gj}+\alpha2\left(n-1\right)!\left(n-1\right)!\left(-1\right)^{n}\Delta_{bn}^{ma}\omega_{ej}^{g}\delta_{a}^{j}\delta_{ig}^{be}\\
& -\left(-1\right)^{n}\left(n-1\right)!\left(n-1\right)!\alpha\Delta_{bn}^{ma}\omega_{aj}^{m}\delta_{i}^{b}\delta_{m}^{j}-\alpha\left(n-1\right)!\left(n-1\right)!\left(-1\right)^{n-1}\Delta_{bn}^{ma}\omega_{mj}^{b}\delta_{i}^{m}\delta_{a}^{j}\\
= \ & \alpha\left(n-1\right)!\left(n-1\right)!\left(-1\right)^{n}\Delta_{bn}^{ma}\left(\omega_{ac}^{c}\delta_{i}^{b}-\omega_{ga}^{g}\delta_{i}^{b}+\omega_{ea}^{e}\delta_{i}^{b}-\omega_{ia}^{b}-\omega_{am}^{m}\delta_{i}^{b}+\omega_{ia}^{b}\right)\\
= \ & 0.
\end{aligned}
$$

Therefore, we correctly find two relationships among the curvatures,

$$
\begin{aligned}
0 \ &= \ \alpha\Delta_{bn}^{ma}\left(\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{i}\tilde{\mathbf{T}}^{c}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}+\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\tilde{\mathbf{S}}_{e}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\right),\\
0 \ &= \ \alpha\Delta_{bn}^{ma}\left(\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\mathbf{f}_{i}\tilde{\mathbf{T}}^{c}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}+\alpha\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{f}_{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\tilde{\mathbf{S}}_{e}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\right).
\end{aligned}
$$

Expanding the torsion and cotorsion and simplifying the first,

$$
0 \ = \ \alpha\Delta_{bn}^{ma}\left(\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}T^{\tilde{ck}}{}_{m}\mathbf{f}_{k}\mathbf{e}^{m}\mathbf{e}^{i}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\right)
$$

$$
\begin{aligned}
&+\alpha\Delta_{bn}^{ma}\left(\frac{1}{2}\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^{i}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\tilde{S}_{e}{}^{mk}\mathbf{f}_{m}\mathbf{f}_{k}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\right)\\
= \;& \alpha\Delta_{bn}^{ma}\left(\left(-1\right)^{n}\tilde{T}^{ck}{}_{m}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\varepsilon^{mih\ldots d}\varepsilon_{ke\ldots f}\right)\mathbf{\Phi}_{n}^{n}\\
&+\alpha\Delta_{bn}^{ma}\left(\frac{1}{2}\left(-1\right)^{n-1}\tilde{S}_{e}{}^{mk}\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\varepsilon^{ic\ldots d}\varepsilon_{mkh\ldots f}\right)\mathbf{\Phi}_{n}^{n}\\
= \;& \alpha\left(n-1\right)!2\left(n-2\right)!\Delta_{bn}^{ma}\left(\left(-1\right)^{n}\tilde{T}^{ck}{}_{m}\delta_{k}^{b}\delta_{ac}^{mi}+\frac{1}{2}\left(-1\right)^{n-1}\tilde{S}_{e}{}^{mk}\delta_{a}^{i}\delta_{mk}^{be}\right)\mathbf{\Phi}_{n}^{n}\\
= \;& \alpha\left(n-1\right)!\left(n-2\right)!\left(-1\right)^{n}\Delta_{bn}^{ma}\left(\tilde{T}^{ib}{}_{a}-\tilde{T}^{cb}{}_{c}\delta_{a}^{i}-\delta_{a}^{i}\tilde{S}_{e}{}^{be}\right)\mathbf{\Phi}_{n}^{n},
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\alpha\Delta_{bn}^{ma}\left(\tilde{T}^{ib}{}_{a}-\tilde{T}^{cb}{}_{c}\delta_{a}^{i}\right) &= \alpha\Delta_{bn}^{mi}\tilde{S}_{e}{}^{be}\\
\tilde{T}^{im}{}_{n}-\delta^{ma}\delta_{bn}\tilde{T}^{ib}{}_{a}-\delta_{n}^{i}\tilde{T}^{cm}{}_{c}+\delta^{mi}\delta_{bn}\tilde{T}^{cb}{}_{c} &= \delta_{n}^{i}\tilde{S}_{e}{}^{me}-\delta^{mi}\delta_{bn}\tilde{S}_{e}{}^{be}.
\end{aligned}
$$

There is only one independent trace. Taking the trace over ni gives

$$
\begin{aligned}
\alpha\Delta_{bn}^{ma}\left(\tilde{T}^{nb}{}_{a}-\tilde{T}^{cb}{}_{c}\delta_{a}^{n}\right) &= \alpha\Delta_{bn}^{mn}\tilde{S}_{e}{}^{be}\\
\frac{\alpha}{2}\tilde{T}^{am}{}_{a}-\frac{\alpha}{2}\delta^{ma}\delta_{bc}\tilde{T}^{bc}{}_{a}-\frac{\alpha}{2}\left(n-1\right)\tilde{T}^{cm}{}_{c} &= \frac{\alpha}{2}\left(n-1\right)\tilde{S}_{a}{}^{ma}\\
\delta_{bc}\tilde{T}^{bc}{}_{a} &= -\left(n-1\right)\delta_{ab}\tilde{S}_{c}{}^{bc}-\left(n-2\right)\delta_{ab}\tilde{T}^{cb}{}_{c}.
\end{aligned}
$$

So we have two results,

$$
\begin{aligned}
2\Delta_{mb}^{an}\tilde{T}^{cm}{}_{n} &= \delta_{n}^{c}\tilde{S}_{e}{}^{me}-\delta^{mc}\delta_{bn}\tilde{S}_{e}{}^{be}+\delta_{n}^{c}\tilde{T}^{cm}{}_{c}-\delta^{mc}\delta_{bn}\tilde{T}^{cb}{}_{c},\\
\tilde{S}_{e}{}^{be} &= -\frac{1}{n}\left(n-1\right)\tilde{T}^{ab}{}_{a}.
\end{aligned}
$$

Now, looking at the second,

$$
\begin{aligned}
0 \;=\; & \alpha\left(\frac{1}{2}\left(-1\right)^{n}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\tilde{T}^{c}{}_{km}\mathbf{e}^{k}\mathbf{e}^{m}\mathbf{e}^{h}\ldots\mathbf{e}^{d}\mathbf{f}_{i}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\right)\\
&+\alpha\left(\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\tilde{S}_{e}{}^{k}{}_{m}\mathbf{f}_{k}\mathbf{e}^{m}\mathbf{f}_{i}\mathbf{f}_{h}\ldots\mathbf{f}_{f}\right)\\
= \;& \alpha\left(\frac{1}{2}\left(-1\right)^{n}\varepsilon_{ach\ldots d}\varepsilon^{be\ldots f}\tilde{T}^{c}{}_{km}\varepsilon^{kmh\ldots d}\varepsilon_{ie\ldots f}\right)\mathbf{\Phi}_{n}^{n}
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha\left((-1)^n\,\tilde{S}_e{}^k{}_{\ m}\varepsilon_{ac\ldots d}\varepsilon^{beh\ldots f}\varepsilon^{mc\ldots d}\varepsilon_{kih\ldots f}\right)\mathbf{\Phi}_n^n\\
=\;&(-1)^n\,2\,(n-1)!\,(n-2)!\alpha\left(\frac{1}{2}\tilde{T}^c{}_{\ km}\delta^{km}_{ac}\delta^b_i+\tilde{S}_e{}^k{}_{\ m}\delta^m_a\delta^{be}_{ki}\right)\mathbf{\Phi}_n^n\\
=\;&(-1)^n\,(n-1)!\,(n-2)!\alpha\left(\frac{1}{2}\tilde{T}^c{}_{\ km}\left(\delta^k_a\delta^m_c-\delta^k_c\delta^m_a\right)\delta^b_i+\tilde{S}_e{}^k{}_{\ a}\left(\delta^b_k\delta^e_i-\delta^b_i\delta^e_k\right)\right)\mathbf{\Phi}_n^n\\
=\;&(-1)^n\,(n-1)!\,(n-2)!\alpha\left(\delta^b_i\tilde{T}^c{}_{\ ac}+\tilde{S}_i{}^b{}_{\ a}-\tilde{S}_e{}^e{}_{\ a}\delta^b_i\right)\mathbf{\Phi}_n^n.
\end{aligned}
$$

Therefore,

$$\alpha\left(\delta^b_i\tilde{T}^c{}_{\ ac}+\tilde{S}_i{}^b{}_{\ a}-\tilde{S}_e{}^e{}_{\ a}\delta^b_i\right)=0.$$

Taking the trace over bi,

$$
\begin{aligned}
n\tilde{T}^c{}_{\ ac}&=(n-1)\,\tilde{S}_b{}^b{}_{\ a}\\
\tilde{T}^c{}_{\ ac}&=\frac{1}{n}\,(n-1)\,\tilde{S}_b{}^b{}_{\ a}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\alpha\left(-\frac{1}{n}\tilde{S}_c{}^c{}_{\ a}\delta^b_i+\tilde{S}_i{}^b{}_{\ a}\right)&=0\\
\tilde{S}_c{}^b{}_{\ a}&=\frac{1}{n}\delta^b_c\tilde{S}_e{}^e{}_{\ a}.
\end{aligned}
$$

So we get two relations,

$$
\begin{aligned}
\tilde{S}_i{}^b{}_{\ a}&=\frac{1}{n}\delta^b_i\tilde{S}_c{}^c{}_{\ a},\\
\tilde{T}^c{}_{\ ac}&=\frac{1}{n}\,(n-1)\,\tilde{S}_b{}^b{}_{\ a}.
\end{aligned}
$$

Check

Varying the action we have,

$$
\begin{aligned}
0&=\delta S\\
&=\alpha\int\left(\delta\mathbf{\Omega}^a_b\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha \int \left(\mathbf{d}\delta\boldsymbol{\omega}_b^a - \delta\boldsymbol{\omega}_b^g \boldsymbol{\omega}_g^a - \boldsymbol{\omega}_b^g \delta\boldsymbol{\omega}_g^a \right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&= \alpha \int \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \delta\boldsymbol{\omega}_b^a \mathbf{D} \left(\mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&= (n-1)\,\alpha \int \delta\boldsymbol{\omega}_b^a \left(\varepsilon_{agc\ldots d} \varepsilon^{be\ldots f} \tilde{\mathbf{T}}^g \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\alpha \int \delta\boldsymbol{\omega}_b^a \left((-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \tilde{\mathbf{S}}_g \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&= (n-1)\,\alpha \int \left(A_{bk}^a \mathbf{e}^k + B_b^{ak} \mathbf{f}_k \right) \left(\varepsilon_{agc\ldots d} \varepsilon^{be\ldots f} \tilde{\mathbf{T}}^g \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\alpha \int \left(A_{bk}^a \mathbf{e}^k + B_b^{ak} \mathbf{f}_k \right) \left((-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \tilde{\mathbf{S}}_g \mathbf{f}_e \ldots \mathbf{f}_f \right).
\end{aligned}
$$

Then

$$
\begin{aligned}
0 &= (n-1)\,\alpha \int A_{bk}^a \left(\varepsilon_{agc\ldots d} \varepsilon^{be\ldots f} \tilde{\mathbf{T}}^g \mathbf{e}^k \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\alpha \int A_{bk}^a \left((-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \mathbf{e}^k \mathbf{e}^c \ldots \mathbf{e}^d \tilde{\mathbf{S}}_g \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\alpha \int B_b^{ak} \left((-1)^n \varepsilon_{agc\ldots d} \varepsilon^{be\ldots f} \tilde{\mathbf{T}}^g \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_k \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\alpha \int B_b^{ak} \left(\varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \tilde{\mathbf{S}}_g \mathbf{f}_k \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&= (n-1)\,\alpha \int A_{bk}^a \left((-1)^n \varepsilon_{agc\ldots d} \varepsilon^{be\ldots f} \tilde{T}^{gm}{}_n \mathbf{e}^n \mathbf{e}^k \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_m \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\alpha \int A_{bk}^a \left((-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \mathbf{e}^k \mathbf{e}^c \ldots \mathbf{e}^d \frac{1}{2} \tilde{S}_g{}^{mn} \mathbf{f}_m \mathbf{f}_n \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\alpha \int B_b^{ak} \left((-1)^n \varepsilon_{agc\ldots d} \varepsilon^{be\ldots f} \frac{1}{2} \tilde{T}^g{}_{mn} \mathbf{e}^m \mathbf{e}^n \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_k \mathbf{f}_e \ldots \mathbf{f}_f \right) \\
&\quad + (n-1)\,\alpha \int B_b^{ak} \left((-1)^n \varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \mathbf{e}^n \mathbf{e}^c \ldots \mathbf{e}^d \tilde{S}_g{}^m{}_n \mathbf{f}_m \mathbf{f}_k \mathbf{f}_e \ldots \mathbf{f}_f \right),
\end{aligned}
$$

so the field equations are

$$
\begin{aligned}
0 &= (n-1)\,\alpha \Delta_{bq}^{pa} \left(\varepsilon_{agc\ldots d} \varepsilon^{be\ldots f} \tilde{T}^{gm}{}_n \varepsilon^{nkc\ldots d} \varepsilon_{me\ldots f} \right) \\
&\quad - (n-1)\,\alpha \Delta_{bq}^{pa} \left(\varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \varepsilon^{kc\ldots d} \frac{1}{2} \tilde{S}_g{}^{mn} \varepsilon_{mne\ldots f} \right) \\
&= (n-1)!\,(n-1)! \alpha \Delta_{bq}^{pa} \left(\tilde{T}^{kb}{}_a - \delta_a^k \tilde{T}^{nb}{}_n - \delta_a^k \tilde{S}_g{}^{bg} \right) \\
0 &= 2\,(n-1)!\,(n-1)! \alpha \Delta_{bq}^{pa} \left(\delta_k^b \frac{1}{2} \tilde{T}^c{}_{ac} + \delta_a^n \delta_{mk}^{bg} \tilde{S}_g{}^m{}_n \right)
\end{aligned}
$$

$$= (n-1)!\,(n-1)!\alpha\Delta^{pa}_{bq}\left(\delta^b_k\tilde{T}^c{}_{ac}+\tilde{S}_k{}^b{}_a-\delta^b_k\tilde{S}_m{}^m{}_a\right),$$

and therefore,

$$\begin{aligned}\alpha\Delta^{pa}_{bq}\left(\delta^b_k\tilde{T}^c{}_{ac}+\tilde{S}_k{}^b{}_a-\delta^b_k\tilde{S}_m{}^m{}_a\right) &= 0,\\ \alpha\Delta^{pa}_{bq}\left(\tilde{T}^{kb}{}_a-\delta^k_a\tilde{T}^{nb}{}_n-\delta^k_a\tilde{S}_g{}^{bg}\right) &= 0.\end{aligned}$$

These agree with the previous results.

F.4. Solder form variation

Now, vary with respect to the solder form,

$$\begin{aligned}0 &= \delta S\\ &= \int(\alpha\delta\mathbf{\Omega}^a_b+\beta\delta^a_b\delta\mathbf{\Omega}+\gamma\delta\mathbf{e}^a\mathbf{f}_b)\,\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\ &\quad+\int(n-1)\,(\alpha\mathbf{\Omega}^a_b+\beta\delta^a_b\mathbf{\Omega}+\gamma\mathbf{e}^a\mathbf{f}_b)\,\varepsilon_{acd\ldots e}\varepsilon^{bf\ldots g}\delta\mathbf{e}^c\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g\\ &= \int\left(\alpha\left(-\Delta^{ac}_{db}\left(\delta h_{cf}\mathbf{e}^f+h_{ck}\delta\mathbf{e}^k\right)\left(\mathbf{e}^d-h^{de}\mathbf{f}_e\right)\right)\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\ &\quad+\int\left(-\Delta^{ac}_{kb}\left(\mathbf{f}_c+h_{cf}\mathbf{e}^f\right)\left(\delta\mathbf{e}^k-\delta h^{ke}\mathbf{f}_e\right)\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\ &\quad+\int\left(-\beta\delta^a_b\delta\mathbf{e}^k\mathbf{f}_k+\gamma\delta\mathbf{e}^k\delta^a_k\mathbf{f}_b\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\ &\quad+\int(n-1)\,(\alpha\mathbf{\Omega}^a_b+\beta\delta^a_b\mathbf{\Omega}+\gamma\mathbf{e}^a\mathbf{f}_b)\,\varepsilon_{akd\ldots e}\varepsilon^{bf\ldots g}\delta\mathbf{e}^k\mathbf{e}^d\ldots\mathbf{e}^e\mathbf{f}_f\ldots\mathbf{f}_g.\end{aligned}$$

Expand the variations as

$$\begin{aligned}\delta\mathbf{e}^k &= A^k{}_m\mathbf{e}^m+B^{km}\mathbf{f}_m,\\ \delta h^{ke} &= \left\langle\delta\mathbf{e}^k,\mathbf{e}^e\right\rangle+\left\langle\mathbf{e}^k,\delta\mathbf{e}^e\right\rangle\\ &= h^{me}A^k{}_m+h^{km}A^e{}_m,\\ h_{ck}\delta h^{ke} &= h_{ck}h^{me}A^k{}_m+h_{ck}h^{km}A^e{}_m,\\ -\delta h_{ck}h^{ke}h_{ef} &= h_{ck}h^{me}h_{ef}A^k{}_m+h_{ck}h^{km}h_{ef}A^e{}_m,\\ \delta h_{cf} &= -h_{ck}A^k{}_f-h_{fk}A^k{}_c.\end{aligned}$$

This becomes

$$\begin{aligned}
0 &= \int \alpha A^k{}_m \left(-\Delta^{ac}_{db} h_{ck} \mathbf{e}^m h^{de} \mathbf{f}_e - \Delta^{am}_{db} h_{fk} \mathbf{e}^f h^{de} \mathbf{f}_e\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int \alpha A^k{}_m \left(\Delta^{ac}_{db} h_{ck} h^{de} \mathbf{e}^m \mathbf{f}_e - \Delta^{ac}_{kb} \mathbf{f}_c \mathbf{e}^m\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int \alpha A^k{}_m \left(\Delta^{ac}_{kb} h^{me} h_{cf} \mathbf{e}^f \mathbf{f}_e + \Delta^{ac}_{eb} h^{em} h_{cf} \mathbf{e}^f \mathbf{f}_k\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int A^k{}_m \left(-\beta \delta^a_b \delta^i_k + \gamma \delta^a_k \delta^i_b\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^m \mathbf{f}_i \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int (n-1) A^k{}_m \left(\alpha \Omega^{ai}_b{}_j + \beta \delta^a_b \Omega^i{}_j - \gamma \delta^a_j \delta^i_b\right) \varepsilon_{akd\ldots e} \varepsilon^{bf\ldots g} \mathbf{f}_i \mathbf{e}^j \mathbf{e}^m \mathbf{e}^d \ldots \mathbf{e}^e \mathbf{f}_f \ldots \mathbf{f}_g \\
&+ \int \alpha B^{km} \left(-\Delta^{ac}_{db} h_{ck} \mathbf{f}_m \mathbf{e}^d - \Delta^{ac}_{kb} h_{cf} \mathbf{e}^f \mathbf{f}_m\right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int (n-1) B^{km} \left(\alpha \frac{1}{2} \Omega^a_{bij} + \beta \delta^a_b \frac{1}{2} \Omega_{ij}\right) (-1)^n \varepsilon_{akd\ldots e} \varepsilon^{bf\ldots g} \mathbf{e}^i \mathbf{e}^j \mathbf{e}^d \ldots \mathbf{e}^e \mathbf{f}_m \mathbf{f}_f \ldots \mathbf{f}_g,
\end{aligned}$$

resulting in two field equations,

$$\begin{aligned}
0 &= (-1)^{n-1} \alpha \left(-\Delta^{ac}_{db} h_{ck} h^{dj} \delta^m_i - \Delta^{am}_{db} h_{ik} h^{dj} + \Delta^{ac}_{db} h_{ck} h^{dj} \delta^m_i\right) \delta^i_a \delta^b_j \\
&\quad + (-1)^{n-1} \alpha \left(\Delta^{aj}_{kb} \delta^m_i + \Delta^{ac}_{kb} h^{mj} h_{ci} + \Delta^{ac}_{eb} h^{em} h_{ci} \delta^j_k\right) \delta^i_a \delta^b_j \\
&\quad + (-1)^{n-1} \left(-\beta \delta^a_b \delta^i_k + \gamma \delta^a_k \delta^i_b\right) \delta^b_i \delta^m_a - 2 \left(\alpha \Omega^{ai}_b{}_j + \beta \delta^a_b \Omega^i{}_j - \gamma \delta^a_j \delta^i_b\right) \delta^{jm}_{ak} \delta^b_i, \\
0 &= \alpha \left(\Delta^{an}_{ib} h_{nk} \delta^j_m - \Delta^{an}_{kb} h_{ni} \delta^j_m\right) \delta^b_j \delta^i_a - 2 \left(\alpha \frac{1}{2} \Omega^a_{bij} + \beta \delta^a_b \frac{1}{2} \Omega_{ij}\right) \delta^{ij}_{ak} \delta^b_m.
\end{aligned}$$

Expand the first,

$$\begin{aligned}
0 &= -\frac{1}{2} \alpha \delta^m_k + \frac{1}{2} \alpha \delta^{mc} h_{ck} \left(\delta_{db} h^{db}\right) - \frac{1}{2} \alpha \delta^m_k \\
&\quad + \frac{1}{2} \alpha \delta^{am} h_{ak} \left(\delta_{db} h^{db}\right) + \frac{1}{2} \alpha \delta^m_k - \frac{1}{2} \alpha \delta^{mc} h_{ck} \left(\delta_{db} h^{db}\right) \\
&\quad + \frac{1}{2} \alpha n \delta^m_k - \frac{1}{2} \alpha \delta^m_k + \frac{1}{2} \alpha \delta^m_k - \frac{1}{2} \alpha \delta_{kb} h^{mb} \left(\delta^{ac} h_{ca}\right) + \frac{1}{2} \alpha \delta^m_k - \frac{1}{2} \alpha \delta_{ek} h^{em} \left(\delta^{ac} h_{ca}\right) \\
&\quad - \beta \delta^m_k + n \gamma \delta^m_k - \alpha \Omega^{ab}_b{}_a \delta^m_k + \alpha \Omega^{mb}_b{}_k - \beta \Omega^a{}_a \delta^m_k + \beta \Omega^m{}_k + \gamma \left(n^2 - n\right) \delta^m_k \\
&= \alpha \Omega^{mb}_b{}_k - \alpha \Omega^{ab}_b{}_a \delta^m_k + \beta \Omega^m{}_k - \beta \Omega^a{}_a \delta^m_k \\
&\quad + \frac{1}{2} \left(\alpha n - 2\beta + 2n^2 \gamma\right) \delta^m_k + \frac{1}{2} \alpha \delta^{mc} h_{ck} \left(\delta_{db} h^{db}\right) - \alpha \delta_{kb} h^{mb} \left(\delta^{ac} h_{ca}\right).
\end{aligned}$$

The field equation is, therefore,

$$\begin{aligned} 0 &= \alpha\Omega_b^{mb}{}_k - \alpha\Omega_b^{ab}{}_a\delta_k^m + \beta\Omega^m{}_k - \beta\Omega^a{}_a\delta_k^m \\ &\quad + \frac{1}{2}\left(\alpha n - 2\beta + 2n^2\gamma\right)\delta_k^m + \frac{1}{2}\alpha\delta^{mc}h_{ck}\left(\delta_{db}h^{db}\right) - \alpha\delta_{kb}h^{mb}\left(\delta^{ac}h_{ca}\right). \end{aligned}$$

Check the trace,

$$\begin{aligned} 0 &= -(n-1)\left(\alpha\Omega_b^{ab}{}_a + \beta\Omega^a{}_a\right) \\ &\quad + \frac{1}{2}n\left(\alpha n - 2\beta + 2n^2\gamma\right) - \frac{1}{2}\alpha\left(\delta_{mb}h^{mb}\right)\left(\delta^{ac}h_{ca}\right). \end{aligned}$$

Therefore, we may write

$$\begin{aligned} 0 &= \alpha\Omega_b^{mb}{}_k + \beta\Omega^m{}_k - \left(\alpha\Omega_b^{ab}{}_a + \beta\Omega^a{}_a\right)\delta_k^m \\ &\quad + \frac{1}{2}\left(\alpha n - 2\beta + 2n^2\gamma\right)\delta_k^m + \frac{1}{2}\alpha\delta^{mc}h_{ck}\left(\delta_{db}h^{db}\right) - \alpha\delta_{kb}h^{mb}\left(\delta^{ac}h_{ca}\right) \\ &= \alpha\Omega_b^{mb}{}_k + \beta\Omega^m{}_k - \frac{1}{n-1}\left(\frac{1}{2}n\left(\alpha n - 2\beta + 2n^2\gamma\right) - \frac{1}{2}\alpha\left(\delta_{mb}h^{mb}\right)\left(\delta^{ac}h_{ca}\right)\right)\delta_k^m \\ &\quad + \frac{1}{2}\left(\alpha n - 2\beta + 2n^2\gamma\right)\delta_k^m + \frac{1}{2}\alpha\delta^{mc}h_{ck}\left(\delta_{db}h^{db}\right) - \alpha\delta_{kb}h^{mb}\left(\delta^{ac}h_{ca}\right) \\ &= \alpha\Omega_b^{mb}{}_k + \beta\Omega^m{}_k \\ &\quad + \frac{1}{n-1}\left(\beta - n^2\gamma + \frac{1}{2}\alpha\left(\left(\delta_{mb}h^{mb}\right)\left(\delta^{ac}h_{ca}\right) - n\right)\right)\delta_k^m \\ &\quad + \frac{1}{2}\alpha\delta^{mc}h_{ck}\left(\delta_{db}h^{db}\right) - \alpha\delta_{kb}h^{mb}\left(\delta^{ac}h_{ca}\right). \end{aligned}$$

The second equation is

$$0 = \frac{1}{2}\alpha\left(\left(n-2\right)h_{mk} + \delta_{km}\left(\delta^{an}h_{na}\right)\right) - \alpha\Omega_{mak}^a - \beta\Omega_{mk}.$$

F.5. Variation of the cosolder form

Finally, we vary with respect to $\mathbf{f}_a$,

$$0 = \delta S$$

$$
\begin{aligned}
&= \int \left(\alpha\delta\mathbf{\Omega}_b^a + \beta\delta_b^a\delta\mathbf{\Omega} + \gamma\mathbf{e}^a\delta\mathbf{f}_b\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f \\
&\quad + \int \left(\alpha\mathbf{\Omega}_b^a + \beta\delta_b^a\mathbf{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{bge\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\delta\mathbf{f}_g\mathbf{f}_e\ldots\mathbf{f}_f \\
&= \int \left(\alpha\left(-\Delta_{jb}^{ag}\left(\delta\mathbf{f}_g + \delta h_{gi}\mathbf{e}^i\right)\left(\mathbf{e}^j - h^{jk}\mathbf{f}_k\right)\right)\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f \\
&\quad + \int \left(\alpha\left(-\Delta_{jb}^{ag}\left(\mathbf{f}_g + h_{gi}\mathbf{e}^i\right)\left(-\delta h^{jk}\mathbf{f}_k - h^{jk}\delta\mathbf{f}_k\right)\right)\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f \\
&\quad + \int \left(-\beta\delta_b^a\mathbf{e}^g\delta\mathbf{f}_g + \gamma\mathbf{e}^a\delta\mathbf{f}_b\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f \\
&\quad + \int \left(\alpha\mathbf{\Omega}_b^a + \beta\delta_b^a\mathbf{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{bge\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\delta\mathbf{f}_g\mathbf{f}_e\ldots\mathbf{f}_f.
\end{aligned}
$$

To find how h_{ab} varies we have

$$
\langle\delta\mathbf{f}_a,\mathbf{f}_b\rangle + \langle\mathbf{f}_a,\delta\mathbf{f}_b\rangle \;=\; -\delta h_{ab},
$$

so with $\delta\mathbf{f}_a = C_{ab}\mathbf{e}^b + D_a{}^b\mathbf{f}_b$,

$$
D_a{}^c h_{cb} + D_b{}^c h_{ac} \;=\; \delta h_{ab};
$$

and therefore,

$$
\begin{aligned}
h^{na}D_a{}^c h_{cb} + D_b{}^c h^{na}h_{ac} &= h^{na}\delta h_{ab} \\
h^{na}D_a{}^c h_{cb} + D_b{}^n &= -\delta h^{na}h_{ab} \\
-h^{na}D_a{}^m - h^{mb}D_b{}^n &= \delta h^{nm}.
\end{aligned}
$$

Now substitute for the variations and collect terms.

$$
\begin{aligned}
0 &= \int C_{gr}\left(\alpha\Delta_{jb}^{ag}h^{js} - \alpha h^{jg}\Delta_{jb}^{as}\right)(-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^r\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_s\mathbf{f}_e\ldots\mathbf{f}_f \\
&\quad + \int C_{gm}\left(\frac{1}{2}\alpha\Omega_b^{ars} + \frac{1}{2}\beta\delta_b^a\Omega^{rs}\right)(-1)^{n-1}(n-1)\,\varepsilon_{ac\ldots d}\varepsilon^{bge\ldots f}\mathbf{e}^m\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_r\mathbf{f}_s\mathbf{f}_e\ldots\mathbf{f}_f \\
&\quad + \int \left(\alpha\Delta_{rb}^{ag}D_g{}^s + \alpha\Delta_{jb}^{ag}D_g{}^m h_{mr}h^{js}\right)(-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^r\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_s\mathbf{f}_e\ldots\mathbf{f}_f \\
&\quad + \int \left(\alpha\Delta_{jb}^{ag}D_r{}^m h_{gm}h^{js}\right)(-1)^{n-1}\,\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^r\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_s\mathbf{f}_e\ldots\mathbf{f}_f
\end{aligned}
$$

$$
\begin{aligned}
&- \int \left(\alpha \Delta^{ag}_{jb} h_{gr} h^{sm} D_m{}^{j} \right) (-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^r \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_s \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int \left(-\alpha \Delta^{ag}_{jb} h_{gr} h^{jm} D_m{}^{s} \right) (-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^r \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_s \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int \left(\alpha \Delta^{ag}_{jb} h_{gr} h^{jk} D_k{}^{s} \right) (-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^r \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_s \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int \left(-\beta \delta^a_b D_r{}^{s} + \gamma D_b{}^{s} \delta^a_r \right) (-1)^{n-1} \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^r \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_s \mathbf{f}_e \ldots \mathbf{f}_f \\
&+ \int D_g{}^{m} \left(\alpha \Omega^{as}_b{}_r + \beta \delta^a_b \Omega^s{}_r \right) (n-1) (-1)^n \varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \mathbf{e}^r \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_s \mathbf{f}_m \mathbf{f}_e \ldots \mathbf{f}_f \\
&- \int D_g{}^{m} \left(\gamma \delta^a_r \delta^s_b \right) (n-1) (-1)^n \varepsilon_{ac\ldots d} \varepsilon^{bge\ldots f} \mathbf{e}^r \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_s \mathbf{f}_m \mathbf{f}_e \ldots \mathbf{f}_f
\end{aligned}
$$

The two field equations are

$$
\begin{aligned}
0 &= \alpha \Omega^{kbi}_b + \beta \Omega^{ki} - \frac{1}{2} (n-2) \alpha h^{ki} - \frac{1}{2} \alpha \delta^{ki} \left(\delta_{jb} h^{jb} \right), \\
0 &= \alpha \Omega^{ar}_s{}_a - \alpha \Omega^{nm}_m{}_n \delta^r_s + \beta \Omega^r{}_s - \beta \Omega^a{}_a \delta^r_s \\
&\quad + \frac{1}{2} \left(n\alpha - 2\beta + 2n^2 \gamma \right) \delta^r_s - \alpha \delta^{rc} h_{sc} \left(\delta_{ab} h^{ab} \right) + \frac{1}{2} \alpha \delta_{sc} h^{rc} \left(\delta^{ab} h_{ab} \right).
\end{aligned}
$$

Check

From the action,

$$
\begin{aligned}
0 &= \delta S \\
&= \delta \int (\alpha \mathbf{\Omega}^a_b + \beta \delta^a_b \mathbf{\Omega} + \gamma \mathbf{e}^a \mathbf{f}_b) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&= \int (\alpha \delta \mathbf{\Omega}^a_b + \beta \delta^a_b \delta \mathbf{\Omega} + \gamma \mathbf{e}^a \delta \mathbf{f}_b) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&\quad + \int (n-1) (\alpha \mathbf{\Omega}^a_b + \beta \delta^a_b \mathbf{\Omega} + \gamma \mathbf{e}^a \mathbf{f}_b) \varepsilon_{ac\ldots d} \varepsilon^{bef\ldots g} \mathbf{e}^c \ldots \mathbf{e}^d \delta \mathbf{f}_e \mathbf{f}_f \ldots \mathbf{f}_g \\
&= \int \left(-\alpha \Delta^{ai}_{jb} \left(\delta \mathbf{f}_i + \delta h_{ik} \mathbf{e}^k \right) \left(\mathbf{e}^j - h^{jm} \mathbf{f}_m \right) \right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&\quad + \int \left(-\alpha \Delta^{ai}_{jb} \left(\mathbf{f}_i + h_{ik} \mathbf{e}^k \right) \left(-\delta h^{jm} \mathbf{f}_m - h^{jm} \delta \mathbf{f}_m \right) \right) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&\quad + \int (-\beta \delta^a_b \mathbf{e}^g \delta \mathbf{f}_g + \gamma \mathbf{e}^a \delta \mathbf{f}_b) \varepsilon_{ac\ldots d} \varepsilon^{be\ldots f} \mathbf{e}^c \ldots \mathbf{e}^d \mathbf{f}_e \ldots \mathbf{f}_f \\
&\quad + \int (n-1) (\alpha \mathbf{\Omega}^a_b + \beta \delta^a_b \mathbf{\Omega} + \gamma \mathbf{e}^a \mathbf{f}_b) \varepsilon_{ac\ldots d} \varepsilon^{bef\ldots g} \mathbf{e}^c \ldots \mathbf{e}^d \delta \mathbf{f}_e \mathbf{f}_f \ldots \mathbf{f}_g,
\end{aligned}
$$

where

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a_{\ b} &= \boldsymbol{\omega}^c_{\ b}\boldsymbol{\omega}^a_{\ c} + \Delta^{ac}_{db}\left(\mathbf{f}_c + h_{cf}\mathbf{e}^f\right)\left(\mathbf{e}^d - h^{de}\mathbf{f}_e\right) + \boldsymbol{\Omega}^a_b,\\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a\mathbf{f}_a + \boldsymbol{\Omega}.
\end{aligned}$$

Now substitute in the variations,

$$\begin{aligned}
\delta\mathbf{f}_a &= C_{ab}\mathbf{e}^b + D_a\ {}^b\mathbf{f}_b,\\
\delta h_{ab} &= \left\langle \mathbf{f}_a, C_{bd}\mathbf{e}^d + D_b\ {}^d\mathbf{f}_d\right\rangle + \langle C_{ac}\mathbf{e}^c + D_a\ {}^c\mathbf{f}_c, \mathbf{f}_b\rangle\\
&= D_b\ {}^c h_{ac} + D_a\ {}^c h_{cb},\\
\delta h^{de} &= -h^{ae}D_a\ {}^d - h^{ad}D_a\ {}^e,
\end{aligned}$$

to get

$$\begin{aligned}
0 &= \int\left(-\alpha\Delta^{ai}_{jb}\left(C_{ik}\mathbf{e}^k + D_i\ {}^k\mathbf{f}_k\right)\left(\mathbf{e}^j - h^{jm}\mathbf{f}_m\right)\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&+\int\left(-\alpha\Delta^{ai}_{jb}\left(D_k\ {}^m h_{mi}\mathbf{e}^k + D_i\ {}^m h_{mk}\mathbf{e}^k\right)\left(\mathbf{e}^j - h^{jm}\mathbf{f}_m\right)\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&+\int\left(-\alpha\Delta^{ai}_{jb}\left(\mathbf{f}_i + h_{ik}\mathbf{e}^k\right)\left(h^{nj}D_n\ {}^m\mathbf{f}_m + h^{nm}D_n\ {}^j\mathbf{f}_m\right)\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&+\int\left(-\alpha\Delta^{ai}_{jb}\left(\mathbf{f}_i + h_{ik}\mathbf{e}^k\right)\left(-h^{jm}C_{mk}\mathbf{e}^k - h^{jm}D_m\ {}^k\mathbf{f}_k\right)\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&+\int\left(-\beta\delta^a_b\mathbf{e}^g C_{gk}\mathbf{e}^k - \beta\delta^a_b\mathbf{e}^g D_g\ {}^k\mathbf{f}_k\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&+\int\left(\gamma\mathbf{e}^a C_{bk}\mathbf{e}^k + \gamma\mathbf{e}^a D_b\ {}^k\mathbf{f}_k\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&+\int(n-1)\left(\alpha\boldsymbol{\Omega}^a_b + \beta\delta^a_b\boldsymbol{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)\varepsilon_{ac\ldots d}\varepsilon^{bef\ldots g}\mathbf{e}^c\ldots\mathbf{e}^d\left(C_{ek}\mathbf{e}^k + D_e\ {}^k\mathbf{f}_k\right)\mathbf{f}_f\ldots\mathbf{f}_g.
\end{aligned}$$

Then

$$\begin{aligned}
0 &= \int\left(\alpha\Delta^{ai}_{jb}C_{ik}h^{jm}\mathbf{e}^k\mathbf{f}_m + \alpha\Delta^{ai}_{jb}D_i\ {}^k\mathbf{e}^j\mathbf{f}_k\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&+\int\left(\alpha\Delta^{ai}_{jb}D_k\ {}^m h_{mi}h^{jn}\mathbf{e}^k\mathbf{f}_n + \alpha\Delta^{ai}_{jb}D_i\ {}^m h_{mk}h^{jn}\mathbf{e}^k\mathbf{f}_n\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f\\
&+\int\left(-\alpha\Delta^{ai}_{jb}h^{nj}D_n\ {}^m h_{ik}\mathbf{e}^k\mathbf{f}_m - \alpha\Delta^{ai}_{jb}h^{nm}D_n\ {}^j h_{ik}\mathbf{e}^k\mathbf{f}_m\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^c\ldots\mathbf{e}^d\mathbf{f}_e\ldots\mathbf{f}_f
\end{aligned}$$

$$
\begin{aligned}
&+\int\left(-\alpha\Delta_{jb}^{ai}h^{jm}C_{mk}\mathbf{e}^{k}\mathbf{f}_{i}+\alpha\Delta_{jb}^{ai}h^{jm}D_{m}\ {}^{k}h_{in}\mathbf{e}^{n}\mathbf{f}_{k}\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int\left(-\beta\delta_{b}^{a}D_{g}\ {}^{k}+\gamma D_{b}\ {}^{k}\delta_{g}^{a}\right)\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{g}\mathbf{f}_{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int C_{ek}\left(n-1\right)\left(\frac{1}{2}\alpha\Omega_{b}^{amn}+\frac{1}{2}\beta\delta_{b}^{a}\Omega^{mn}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{bef\ldots g}\mathbf{e}^{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{m}\mathbf{f}_{n}\mathbf{f}_{f}\ldots\mathbf{f}_{g}\\
&+\int D_{e}\ {}^{k}\left(n-1\right)\left(\alpha\mathbf{\Omega}_{b}^{a}+\beta\delta_{b}^{a}\mathbf{\Omega}+\gamma\mathbf{e}^{a}\mathbf{f}_{b}\right)\varepsilon_{ac\ldots d}\varepsilon^{bef\ldots g}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{k}\mathbf{f}_{f}\ldots\mathbf{f}_{g}
\end{aligned}
$$

so

$$
\begin{aligned}
0 &= \int\left(\alpha\Delta_{jb}^{ai}C_{ik}h^{jn}+\alpha\Delta_{kb}^{ai}D_{i}\ {}^{n}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{n}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int\left(\alpha\Delta_{jb}^{ai}D_{k}\ {}^{m}h_{mi}h^{jn}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{n}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int\left(\alpha\Delta_{jb}^{ai}D_{i}\ {}^{m}h_{mk}h^{jn}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{n}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int\left(-\alpha\Delta_{jb}^{ai}h^{nj}D_{n}\ {}^{m}h_{ik}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{m}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int\left(-\alpha\Delta_{jb}^{ai}h^{nm}D_{n}\ {}^{j}h_{ik}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{m}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int\left(-\alpha\Delta_{jb}^{am}h^{ji}C_{ik}+\alpha\Delta_{jb}^{ai}h^{jn}D_{n}\ {}^{m}h_{ik}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{m}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int\left(-\beta\delta_{b}^{a}D_{g}\ {}^{k}+\gamma D_{b}\ {}^{k}\delta_{g}^{a}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{be\ldots f}\mathbf{e}^{g}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{k}\mathbf{f}_{e}\ldots\mathbf{f}_{f}\\
&+\int C_{ek}\left(n-1\right)\left(\frac{1}{2}\alpha\Omega_{b}^{amn}+\frac{1}{2}\beta\delta_{b}^{a}\Omega^{mn}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{bef\ldots g}\mathbf{e}^{k}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{m}\mathbf{f}_{n}\mathbf{f}_{f}\ldots\mathbf{f}_{g}\\
&+\int D_{e}\ {}^{k}\left(n-1\right)\left(-\alpha\Omega_{b}^{am}\ {}_{n}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{bef\ldots g}\mathbf{e}^{n}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{m}\mathbf{f}_{k}\mathbf{f}_{f}\ldots\mathbf{f}_{g}\\
&+\int D_{e}\ {}^{k}\left(n-1\right)\left(-\beta\delta_{b}^{a}\Omega^{m}\ {}_{n}+\gamma\delta_{n}^{a}\delta_{b}^{m}\right)\left(-1\right)^{n-1}\varepsilon_{ac\ldots d}\varepsilon^{bef\ldots g}\mathbf{e}^{n}\mathbf{e}^{c}\ldots\mathbf{e}^{d}\mathbf{f}_{m}\mathbf{f}_{k}\mathbf{f}_{f}\ldots\mathbf{f}_{g}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
0 &= \alpha\Delta_{jb}^{ai}h^{jn}\delta_{n}^{b}\delta_{a}^{k}-\alpha\Delta_{jb}^{am}h^{ji}\delta_{m}^{b}\delta_{a}^{k}+2\left(\frac{1}{2}\alpha\Omega_{b}^{amn}+\frac{1}{2}\beta\delta_{b}^{a}\Omega^{mn}\right)\delta_{a}^{k}\delta_{mn}^{bi},\\
0 &= \left(\alpha\Delta_{kb}^{ar}\delta_{s}^{n}+\alpha\Delta_{jb}^{ai}\delta_{k}^{r}\delta_{s}^{m}h_{mi}h^{jn}+\alpha\Delta_{jb}^{ar}h_{sk}h^{jn}\right)\delta_{n}^{b}\delta_{a}^{k}\\
&\quad+\left(-\alpha\Delta_{jb}^{ai}h^{rj}\delta_{s}^{m}h_{ik}-\alpha\Delta_{sb}^{ai}h^{rm}h_{ik}+\alpha\Delta_{jb}^{ai}\delta_{s}^{m}h^{jr}h_{ik}\right)\delta_{m}^{b}\delta_{a}^{k}\\
&\quad+\left(-\beta\delta_{b}^{a}\delta_{g}^{r}+\gamma\delta_{b}^{r}\delta_{g}^{a}\right)\delta_{s}^{b}\delta_{a}^{g}\\
&\quad+2\left(-\alpha\Omega_{b}^{am}\ {}_{n}-\beta\delta_{b}^{a}\Omega^{m}\ {}_{n}+\gamma\delta_{n}^{a}\delta_{b}^{m}\right)\delta_{ms}^{br}\delta_{a}^{n},
\end{aligned}
$$

so

$$
\begin{aligned}
0 &= \alpha\Omega_b^{kbi} + \beta\Omega^{ki} - \frac{1}{2}(n-2)\,\alpha h^{ki} - \frac{1}{2}\alpha\delta^{ki}\left(\delta_{jb}h^{jb}\right), \\
0 &= \alpha\Omega_s^{ar}{}_a - \alpha\Omega_m^{nm}{}_n\delta_s^r + \beta\Omega^r{}_s - \beta\Omega^a{}_a\delta_s^r \\
&\quad + \frac{1}{2}\left(n\alpha - 2\beta + 2n^2\gamma\right)\delta_s^r \\
&\quad - \alpha\delta^{rc}h_{sc}\left(\delta_{ab}h^{ab}\right) + \frac{1}{2}\alpha\delta_{sc}h^{rc}\left(\delta^{ab}h_{ab}\right).
\end{aligned}
$$

F.6. Summary of field equations

Collecting the field equations we have:

$$
\begin{aligned}
\beta\tilde{S}_k{}^{jk} &= \beta\left(\tilde{T}^{jk}{}_k - \tilde{T}^{kj}{}_k\right), \\
\beta\tilde{T}^c{}_{jc} &= \beta\left(\tilde{S}_h{}^h{}_j - \tilde{S}_j{}^k{}_k\right), \\
0 &= \alpha\Delta_{bq}^{pa}\left(\delta_k^b\tilde{T}^c{}_{ac} + \tilde{S}_k{}^b{}_a - \delta_k^b\tilde{S}_m{}^m{}_a\right), \\
0 &= \alpha\Delta_{bq}^{pa}\left(\tilde{T}^{kb}{}_a - \delta_a^k\tilde{T}^{nb}{}_n - \delta_a^k\tilde{S}_g{}^{bg}\right), \\
0 &= \alpha\Omega_b^{mb}{}_k - \alpha\Omega_b^{ab}{}_a\delta_k^m + \beta\Omega^m{}_k - \beta\Omega^a{}_a\delta_k^m \\
&\quad + \frac{1}{2}\left(\alpha n - 2\beta + 2n^2\gamma\right)\delta_k^m - \alpha\delta_{kb}h^{mb}\left(\delta^{ac}h_{ca}\right) + \frac{1}{2}\alpha\delta^{mc}h_{ck}\left(\delta_{db}h^{db}\right), \\
0 &= \alpha\Omega_{mak}^a + \beta\Omega_{mk} - \frac{1}{2}\alpha\left(\left(n-2\right)h_{mk} + \delta_{km}\left(\delta^{an}h_{na}\right)\right), \\
0 &= \alpha\Omega_b^{mbk} + \beta\Omega^{mk} - \frac{1}{2}\alpha\left(\left(n-2\right)h^{mk} + \delta^{mk}\left(\delta_{ab}h^{ab}\right)\right), \\
0 &= \alpha\Omega_m^{ag}{}_a - \alpha\Omega_b^{ab}{}_a\delta_m^g + \beta\Omega^g{}_m - \beta\Omega^a{}_a\delta_m^g \\
&\quad + \frac{1}{2}\left(n\alpha - 2\beta + 2n^2\gamma\right)\delta_m^g - \alpha\delta^{ag}h_{ma}\left(\delta_{jb}h^{jb}\right) + \frac{1}{2}\alpha\delta_{mb}h^{bg}\left(\delta^{ak}h_{ka}\right).
\end{aligned}
$$

Appendix G

Conditions on the metric from cotorsion field equations

In Chapter 4, we investigated a simplified version of general, curved biconformal geometry. Specifically, we chose the simplest possible ansatz for the form of the symmetric spin connection and set all torsions to zero. We are left with a number of field equations involving the Cartan curvatures of the Weyl and $SO(4)$ connections and one field equation involving the cotorsions. This last field equation sets a number of conditions on the form of the metric.

The mixed cotorsion field equation gives the following conditions on the metric derivatives.

1. Letting $\nu\pi\beta = ijk$,

$$0 = \partial_i g_{jk} - \partial_k g_{ij} - g_{jk}s_i + g_{ij}s_k.$$

2. Letting $\nu\pi\beta = i0k$[1],

$$0 = \partial_i g_{k0} - \partial_k g_{i0} - g_{0k}s_i + g_{i0}s_k.$$

3. Letting $\nu\pi\beta = ij0$,

$$\begin{aligned} 0 &= \frac{1}{2}g_{ij}s_0 - g_{ij}s_0 \\ 0 &= s_0. \end{aligned}$$

4. Letting $\nu\pi\beta = 0jk$,

$$\begin{aligned} 0 &= \frac{1}{2}g_{jk}s_0 \\ s_0 &= 0. \end{aligned}$$

[1] This is the condition such that we can choose coordinates where $g_{i0} = 0$, which we do from 2. onward.

5. Letting $\nu\pi\beta = 00k$,

$$\begin{aligned} 0 &= \frac{1}{2}g_{00}s_k - g_{00}s_k \\ s_k &= 0. \end{aligned}$$

6. Letting $\nu\pi\beta = i00$,

$$\begin{aligned} 0 &= -\frac{1}{2}g_{00}s_i \\ s_i &= 0. \end{aligned}$$

7. Letting $\nu\pi\beta = 0j0$,

$$\begin{aligned} 0 &= \frac{1}{2}\left(\partial_0 g_{0j} - \partial_0 g_{0j}\right) - \frac{1}{2}\left(g_{j0}s_0 - g_{0j}s_0\right) \\ 0 &= 0. \end{aligned}$$

8. Let $\nu\pi\beta = 000$

$$\begin{aligned} 0 &= \partial_0 g_{00} - \frac{1}{2}\partial_0 g_{00} - \frac{1}{2}\partial_0 g_{00} \\ 0 &= 0. \end{aligned}$$

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  "The effects of negative-energy shells on Schwarzschild spacetime"
  Advisor: Tevian Dray
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RESEARCH INTERESTS

General relativity, conformal symmetry, alternative theories of gravity, multi-messenger astronomy with gravitational waves, cosmology, Cartan geometry

PUBLICATIONS

1. **C7 multi-messenger astronomy of GW sources**

   Jeffrey S. Hazboun and Shane L. Larson

   General Relativity and Gravitation Volume 46 (2014) 1771,

   DOI: 10.1007s10714-014-1771-6

2. **Limiting bimetric theories of gravity using gravitational wave observations**

   Jeffrey S. Hazboun, Manuel Pichardo Marcano and Shane L. Larson

   *arXiv:*1311.3153 *Submitted to Classical & Quantum Gravity*

3. **Time and dark matter from the conformal symmetries of Euclidean space**

   Jeffrey S. Hazboun and James T. Wheeler

   *arXiv:*1305.6972 *Accepted by Classical & Quantum Gravity*

4. **A systematic construction of curved phase space:**

   A gravitational gauge theory with symplectic form

   Jeffrey S. Hazboun and James T. Wheeler

   Journal of Physics Conference Series 360 (2012) 012013,

   DOI: 10.1088/1742-6596/360/1/012013

5. **The Effect of Negative-Energy Shells on the Schwarzschild Black Hole**

   Jeffrey S. Hazboun and Tevian Dray

   General Relativity and Gravitation Volume 42 (2010) 1457-1467,

   DOI: 10.1007/s10714-009-0916-5

6. **General relativity and time from conformal symmetry** *In Preparation*

   Jeffrey S. Hazboun and James T. Wheeler

ACADEMIC POSITIONS and SERVICE

- **Research Assistant**, for Shane L. Larson,

  Center for Interdisciplinary Exploration and Research in Astrophysics,

  May 2014- August 2014
- **Visiting Scholar**, Center for Relativistic Astrophysics, Georgia Tech

  Atlanta, Georgia, June 2012-May 2013
- **Referee for Classical and Quantum Gravity**
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- **Graduate Teaching Instructor**, Utah State University

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ADVANCED GRADUATE SCHOOLS

- **53rd Cracow School of Theoretical Physics** Zakopane, Poland. June 2013

  Eight day school. Topics: conformal symmetry, AdS/CFT, and quantum gravity

- **Higher Gauge Theory, Topological Quantum Field Theory and Quantum Gravity School and Workshop** Lisbon, Portugal, February 2011

  Seven day school. Topics: topology, category theory and quantum gravity

- **PASI Quantum Gravity Summer School** Morelia, Mexico, June 2010

  Ten day school. Topics: various approaches to quantum gravity lectured

RESEARCH TALKS PRESENTED

1. **Midwest Gravity Meeting** Milwaukee, Wisconsin, October 2013

   "Time from the conformal symmetries of a Euclidean space"

   Jeffrey S. Hazboun and James T. Wheeler

2. **Loops 13: International Conference on Quantum Gravity** Waterloo, Canada, July 2013

   "Lorentzian geometry from the conformal symmetries of a Euclidean space"

   Jeffrey S. Hazboun and James T. Wheeler

3. **GR20/AMALDI 10** Warsaw, Poland, July 2013

   "Testing Bimetric and Massive Gravity Theories using Multi-Messenger Astronomy"

   Jeffrey S. Hazboun and Shane L. Larson

4. **53rd Cracow School of Theoretical Physics** Zakopane, Poland, June 2013

   "Lorentzian spin connection from the conformal symmetries of a Euclidean space"

   Jeffrey S. Hazboun and James T. Wheeler

5. **Pacific Coast Gravity Meeting** Davis, California, March 2013

   "General relativity in signature changing phase space"

   Jeffrey S. Hazboun and James T. Wheeler

6. **Pacific Coast Gravity Meeting** Santa Barbara, California, March 2012

   "General relativity in phase space with a natural notion of time"

   Jeffrey S. Hazboun and James T. Wheeler

7. **Loops 11: International Conference on Quantum Gravity** Madrid, Spain, May 2011

   "A systematic construction of curved phase space: A gravitational gauge theory with symplectic form"

   Jeffrey S. Hazboun and James T. Wheeler

8. **Intermountain Graduate Research Symposium** Logan, Utah, March 2010

   "Quantum gravity in relativistic phase space"

   Jeffrey S. Hazboun and James T. Wheeler

9. **12th Marcel Grossman Gravity Meeting** Paris, France, July 2009

   "Multiple Spherical Shells in Schwarzschild Spacetime" (MS Work)

   Jeffrey S. Hazboun and Tevian Dray

10. **TEXAS Symposium on Relativistic Astrophysics** Vancouver, Canada, December 2008

    "Multiple Spherical Shells in Schwarzschild Spacetime" (Poster, MS Work)

    Jeffrey S. Hazboun and Tevian Dray

11. **Pacific Coast Gravity Meeting** Eugene, Oregon, March 2009

    "Single Spherical Shells in Schwarzschild Spacetime" (MS Work)

    Jeffrey S. Hazboun and Tevian Dray

GRANTS, HONORS and AWARDS

1. **USU College of Science Graduate Teacher of the Year,** 2013
2. **Graduate Student Senate: Graduate Enhancement Award,** *$4000*, 2013
3. **Travel Grant and Support, Loops '13,** *$1000*, 2013
4. **Travel Grant and Support, 53rd Cracow School of Theoretical Physics,** *$3000*, 2013
5. **Gene Adams Endowed Scholarship,** *$400*, Spring 2011
6. **Travel Grant and Support, Higher Gauge Theory, Topological Quantum Field Theory and**

   Quantum Gravity School and Workshop, *$1000*, 2013
7. **Travel Grant and Support, PASI Quantum Gravity Summer School,** *$2000*, 2013
8. **National Geographic Explorer's Grant,** Kamchatka Project Summer, *$25,000*, 2010
9. **Howard L. Blood Scholarship,** *$4000*, Summer 2010
10. **Sigma Pi Sigma Physics Honor Society,** April 2010
11. **Vice-President for Research Fellowship,** *$10,000*, Fall 2009
12. **NSF Student Travel Grant,** *$2000*, Summer 2009

13. **Finalist Fulbright Scholarship,** Spring 2009
14. **Fontana Travel Award,** *$375*, December 2008
15. **Best Picture, National Paddling Film Festival, Amateur Category,** *Lemonade*, 2006

TEACHING EXPERIENCE

- **Astronomy Instructor**, Utah State University, Logan, Utah, Fall 2014
  Instructor of record for a 300+ person astronomy class. Supervise two teaching assistants
- **Research Coadvisor**, Utah State University, Logan, Utah, Fall 2013-Spring 2014
  Mentored an undergraduate researcher working on problems of multi-messenger astronomy using gravitational waves
- **Online Physics Course Developer & Instructor**, Utah State University, Logan, Utah, 2012-Present
  The Universe: Proposed, developed and taught a for-credit, online cosmology class for nonscience students
  Continuously offered for the last 7 semesters. Over 400 students have taken this class
- **Physics Instructor**, Utah State University, Logan, Utah, Summer 2011
  General Physics I: Instructor of Record
- **Teaching Assistant**, Utah State University, Logan, Utah, Fall 2009-Spring 2012
  General Physics I: Recitation Leader and Lab Instructor
  General Physics II: Recitation Leader and Lab Instructor
- **Teaching Assistant**, Oregon State University, Corvallis, Oregon Fall 2006-Spring 2009
  Paradigms in Physics TA: NSF funded higher division class reform project
  Facilitated group work and took part in curriculum meetings
  Physics for the Life Sciences: Recitation Leader and Lab Instructor
  General Physics II: Lab Instructor
- **MCAT Physics Instructor** Princeton Review, Portland, Oregon, Summer 2007
  Developed curriculum to help students review for physics portion of the MCAT exam
  Taught students test-taking strategies to prepare for a stressful and fast-paced exam

INVITED LECTURES

- **Professional**
  - **Center for Relativistic Astrophysics** Departmental Colloquium, March, 2013
    Biconformal Space & Testing Alternative Theories of Gravity using Multi-Messenger Astronomy
  - **USU Physics Colloquium** Logan, Utah, February, 2013
    Best Practices for the Online Classroom
  - **USU Physics Colloquium** Logan, Utah, September 2010
    Curved Phase Space from conformal symmetry
  - **Oregon State Physics Colloquium** Corvallis, Oregon, March 2009
    Spherical Shells in a Schwarzschild Background
- **Public**
  - **Science Unwrapped** Utah State University, Logan, Utah, February 2013
    Explore to Conserve (500 person public lecture)
  - **Conservation Club Talk** Weber State Conservation Club, Ogden, Utah, February 2012
    A Scientist's Role in Conservation
  - **Science Unwrapped** Swaner Ecocenter, Park City, Utah, February 2011
    A Scientist's Role in Modern Exploration
  - **Science Interview for National Geographic**, November 2009
    Interviewed for *Phenomena: A science salon hosted by National Geographic* about the physics in the movie "Men Who Stare at Goats".
  - **Cache Valley Stargazers Talk** Logan, Utah, November 2009
    Black Holes: Ninjas of the Night Sky

RESEARCH REFERENCES

- **Dr. James T. Wheeler** *PhD Advisor, Utah State University*

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  Utah State University, Logan, UT 84322

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- **Dr. Shane Larson** *Collaborator*

  Research Associate Professor

  Center for Interdisciplinary Exploration and Research in Astrophysics (CIERA)

  Northwestern University, Evanston, IL 60208

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- **Dr. Tevian Dray** *MS Advisor, Oregon State University*

  Professor of Mathematics

  Oregon State University, Corvallis, OR 97331

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TEACHING REFERENCES

- **Dr. Jan Sojka**

  Department Head & Professor of Physics

  Utah State University, Logan, UT 84322

  Phone: 435-797-2857 email: `jan.sojka@usu.edu`

- **Dr. Corinne Manogue**

  PI NSF Paradigms in Physics Project & Professor of Physics

  Oregon State University, Corvallis, OR 97331

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