Derivation of Transfer Function of a Two-Degree-of-Freedom Damped System with Base Excitation

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ABSTRACT

The derivation of a closed-form expression for the acceleration transfer function of a two-degree-of-freedom (2DOF) spring-damper system with a base excitation vibration input is presented in this paper. An analytical example was solved with a finite element model to verify the closed-form expression. In the current small satellite industry, dynamic coupling between the launch vehicle and the deployer/satellite assembly is that of interest. This coupling can amplify the launch loads into the deployer/satellite assembly. In this paper, analytical method is used to derive the two-degree-of-freedom system response of a mass, spring, and damper system to a base excitation. After achieving the analytical response of each degree of freedom response, one can calculate the transfer function (an input multiplier to get the output) as a function of two-degree-of-freedom system parameters m (mass), k (stiffness) and c (damping).

Nomenclature

A	=	base acceleration, harmonic form
c	=	damping constant
F	=	force
k	=	spring constant
m	=	mass
TF	=	transfer function, acceleration
R	=	ratio of input frequency over resonant frequency of system 2
x	=	absolute displacement
$\dot{x}$	=	absolute velocity
$\ddot{x}$	=	absolute acceleration
X	=	absolute displacement
y	=	base displacement
$\dot{y}$	=	base velocity
$\ddot{y}$	=	base acceleration
z	=	displacement relative to base
$\dot{z}$	=	velocity relative to base
$\ddot{z}$	=	acceleration relative to base
Z	=	displacement relative to base, harmonic form
$\ddot{Z}$	=	acceleration relative to base, harmonic form
β	=	ratio of resonant frequency of system 1 over resonant frequency of system 2
ζ	=	critical damping ratio
μ	=	ratio of mass of system 1 over mass of system 2
ω	=	input frequency [rad/s]
ω_i	=	resonant frequency of system i [rad/s]
Subscripts		
1	=	system 1, connected to the base
2	=	system 2, connected to system 1

1. Introduction

In the current small satellite industry, dynamic coupling between the launch vehicle and the deployer/satellite assembly amplifies launch loads into the deployer/satellite assembly. Characterization of this dynamic coupling is required for first-pass estimates of those loads beyond a simple enveloping of loads provided in the launch vehicle user manuals or a simple one-degree-of-freedom amplification.

The deployer/satellite assembly is thus modeled as a 2DOF damped system, with the satellite's mass connected to the deployer's, which in turn is connected to "ground." Literature reviews typically show derivations only up to the matrix formulation of the system, but is usually not expressed in a closed form that can be simply programmed into a spreadsheet program for ease of use. This paper addresses the need for a simple closed-form expression for use in first-pass estimates of 2DOF transmissibilities and coupling effects given mass and uncoupled frequencies of the individual systems.

II. Literature Review

Literature review of the topic at hand resulted in extensive theory, but no exact derivations of the closed-form analytical expression of the 2DOF damped base excitation explicitly exist or were internally obfuscated.

Li [1] presents the closed form transfer function of a damped 2DOF system with direct force input on the mass not connected to the base, or "ground." The closed form derivation presented in this paper is almost identical to that of Li's derivation, with the relevant changes in derivation for a base excitation input to the mass connected to the base, or "ground." Irvine [2] presents the most detailed derivation of the closed-form expression of an undamped 2DOF with base excitation input. Irvine's derivations for a similar problem have been used as a guide for the derivation of this specific system.

III. Derivation of the Transfer Function

Figure 1 shows the 2DOF damped system with base excitation. The equations of motion for each body in the mechanical system are created for absolute displacement by way of free-body diagrams.

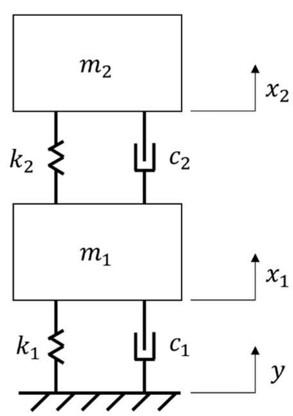

Figure 1: Diagram of 2DOF damped system with base excitation *y*

For the first body, the free-body diagram is created and shown in Fig. 2.

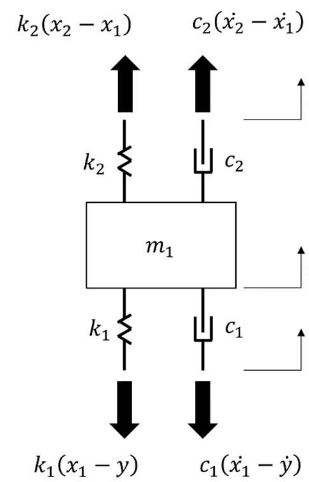

Figure 2: Free-body diagram of forces for body 1

The equations of motion for the first body are created.

$$\sum F_1 = m_1\ddot{x}_1 \quad (1)$$

$$-k_1(x_1 - y) + k_2(x_2 - x_1) - c_1(\dot{x}_1 - \dot{y}) + c_2(\dot{x}_2 - \dot{x}_1) = m_1\ddot{x}_1 \quad (2)$$

$$-k_1x_1 + k_1y + k_2x_2 - k_2x_1 - c_1\dot{x}_1 + c_1\dot{y} + c_2\dot{x}_2 - c_2\dot{x}_1 = m_1\ddot{x}_1 \quad (3)$$

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + k_1x_1 + k_2x_1 - k_2x_2 = c_1\dot{y} + k_1y \quad (4)$$

Similarly for the second body, the free-body diagram is shown in Fig. 3.

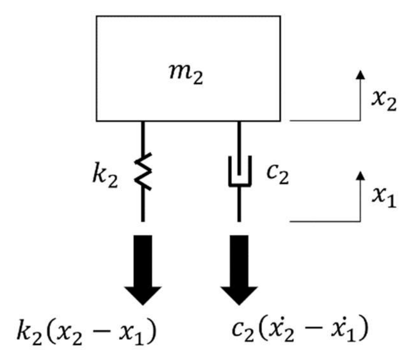

Figure 3: Free-body diagram of forces for body 2

The equations of motion for the second body are created.

$$\sum F_2 = m_2\ddot{x}_2 \quad (5)$$

$$-k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = m_2\ddot{x}_2 \quad (6)$$

$$-k_2x_2 + k_2x_1 - c_2\dot{x}_2 + c_2\dot{x}_1 = m_2\ddot{x}_2 \quad (7)$$

$$m_2\ddot{x}_2 - c_1\dot{x}_1 + c_2\dot{x}_2 - k_2x_1 + k_2x_2 = 0 \quad (8)$$

Rewriting the equations of motion in matrix form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1\dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} k_1 y \\ 0 \end{bmatrix} \quad (9)$$

To describe motion relative to the base displacement as z, let $z_i = x_i - y$, that is,

$$x_1 = z_1 + y \quad (10)$$

$$x_2 = z_2 + y \quad (11)$$

$$\dot{x}_1 = \dot{z}_1 + \dot{y} \quad (12)$$

$$\dot{x}_2 = \dot{z}_2 + \dot{y} \quad (13)$$

$$\ddot{x}_1 = \ddot{z}_1 + \ddot{y} \quad (14)$$

$$\ddot{x}_2 = \ddot{z}_2 + \ddot{y} \quad (15)$$

Substitute into the mechanical system of Eq. 9, expand, and simplify.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1 & 0 \\ 0 & -m_2 \end{bmatrix}\begin{bmatrix} \ddot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -m_1 \\ -m_2 \end{bmatrix}\ddot{y} \quad (16)$$

For harmonic motion, assume

$$z_1 = Z_1 e^{j\omega t} \quad (17)$$

$$z_2 = Z_2 e^{j\omega t} \quad (18)$$

$$\dot{z}_1 = j\omega \cdot Z_1 e^{j\omega t} \quad (19)$$

$$\dot{z}_2 = j\omega \cdot Z_2 e^{j\omega t} \quad (20)$$

$$\ddot{z}_1 = -\omega^2 \cdot Z_1 e^{j\omega t} \quad (21)$$

$$\ddot{z}_2 = -\omega^2 \cdot Z_2 e^{j\omega t} \quad (22)$$

$$\ddot{y} = A e^{j\omega t} \quad (23)$$

Substitute into Eq. 16, expand, and simplify.

$$\begin{bmatrix} -m_1\omega^2 + j\omega(c_1 + c_2) + (k_1 + k_2) & -j\omega c_2 - k_2 \\ -j\omega c_2 - k_2 & -m_2\omega^2 + j\omega c_2 + k_2 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} -m_1 \\ -m_2 \end{bmatrix} A \quad (24)$$

Using Cramer’s rule, solve for Z_1.

$$Z_1 = \frac{det\begin{bmatrix} -m_1 & -j\omega c_2 - k_2 \\ -m_2 & -m_2\omega^2 + j\omega c_2 + k_2 \end{bmatrix}}{det\begin{bmatrix} -m_1\omega^2 + j\omega(c_1 + c_2) + (k_1 + k_2) & -j\omega c_2 - k_2 \\ -j\omega c_2 - k_2 & -m_2\omega^2 + j\omega c_2 + k_2 \end{bmatrix}} A \quad (25)$$

Simplifying the denominator,

$$det\begin{bmatrix}-m_1\omega^2 + j\omega(c_1+c_2) + (k_1+k_2) & -j\omega c_2 - k_2 \\ -j\omega c_2 - k_2 & -m_2\omega^2 + j\omega c_2 + k_2\end{bmatrix} = \left(m_1 m_2 \omega^4 + \left(-m_1 k_2 - c_1 c_2 - m_2(k_1+k_2)\right)\omega^2 + k_1 k_2\right) + j\cdot\left(\left(-m_1 c_2 - m_2(c_1+c_2)\right)\omega^3 + (k_2 c_1 + k_1 c_2)\omega\right) \tag{26}$$

Simplifying the numerator,

$$det\begin{bmatrix}-m_1 & -j\omega c_2 - k_2 \\ -m_2 & -m_2\omega^2 + j\omega c_2 + k_2\end{bmatrix} = (m_1 m_2 \omega^2 - m_1 k_2 - m_2 k_2) + j\cdot(-m_1 c_2 - m_2 c_2)\omega \tag{27}$$

Substituting Eq. 26 and Eq. into Eq. 25 yields,

$$Z_1 = \frac{(m_1 m_2 \omega^2 - m_1 k_2 - m_2 k_2) + j\cdot(-m_1 c_2 - m_2 c_2)\omega}{\left(m_1 m_2 \omega^4 + \left(-m_1 k_2 - c_1 c_2 - m_2(k_1+k_2)\right)\omega^2 + k_1 k_2\right) + j\cdot\left(\left(-m_1 c_2 - m_2(c_1+c_2)\right)\omega^3 + (k_2 c_1 + k_1 c_2)\omega\right)} A \tag{28}$$

Similarly, using Cramer's rule, solve for Z_2.

$$Z_2 = \frac{det\begin{bmatrix}-m_1\omega^2 + j\omega(c_1+c_2) + (k_1+k_2) & -m_1 \\ -j\omega c_2 - k_2 & -m_2\end{bmatrix}}{det\begin{bmatrix}-m_1\omega^2 + j\omega(c_1+c_2) + (k_1+k_2) & -j\omega c_2 - k_2 \\ -j\omega c_2 - k_2 & -m_2\omega^2 + j\omega c_2 + k_2\end{bmatrix}} A \tag{26}$$

The denominator is identical to that solve for Z_1.
Simplifying the numerator,

$$det\begin{bmatrix}-m_1\omega^2 + j\omega(c_1+c_2) + (k_1+k_2) & -m_1 \\ -j\omega c_2 - k_2 & -m_2\end{bmatrix} = \left(m_1 m_2 \omega^2 - m_1 k_2 - m_2(k_1+k_2)\right) + j\cdot\left(-m_1 c_2 - m_2(c_1+c_2)\right)\omega \tag{30}$$

Substituting Eq. 26 and Eq. 30 into Eq. 29 yields,

$$Z_2 = \frac{\left(m_1 m_2 \omega^2 - m_1 k_2 - m_2(k_1+k_2)\right) + j\cdot\left(-m_1 c_2 - m_2(c_1+c_2)\right)\omega}{\left(m_1 m_2 \omega^4 + \left(-m_1 k_2 - c_1 c_2 - m_2(k_1+k_2)\right)\omega^2 + k_1 k_2\right) + j\cdot\left(\left(-m_1 c_2 - m_2(c_1+c_2)\right)\omega^3 + (k_2 c_1 + k_1 c_2)\omega\right)} A \tag{31}$$

Having assumed harmonic motion, the relative accelerations can be calculated directly from the relative displacements.

$$\ddot{Z}_1 = -\omega^2 \cdot Z_1 \tag{27}$$

$$\ddot{Z}_2 = -\omega^2 \cdot Z_2 \tag{28}$$

Substituting in Eq. 32 and Eq. 33 into Eq. 30 and Eq. 31 yields respectively the following for relative accelerations.

$$\ddot{Z}_1 = \frac{(-m_1m_2\omega^4 + (m_1k_2 + m_2k_2)\omega^2) + j\cdot(m_1c_2 + m_2c_2)\omega^3}{(m_1m_2\omega^4 + (-m_1k_2 - c_1c_2 - m_2(k_1+k_2))\omega^2 + k_1k_2) + j\cdot\left((-m_1c_2 - m_2(c_1+c_2))\omega^3 + (k_2c_1 + k_1c_2)\omega\right)}A \tag{34}$$

$$\ddot{Z}_2 = \frac{(-m_1m_2\omega^4 + (m_1k_2 + m_2(k_1+k_2))\omega^2) + j\cdot(m_1c_2 + m_2(c_1+c_2))\omega^3}{(m_1m_2\omega^4 + (-m_1k_2 - c_1c_2 - m_2(k_1+k_2))\omega^2 + k_1k_2) + j\cdot\left((-m_1c_2 - m_2(c_1+c_2))\omega^3 + (k_2c_1 + k_1c_2)\omega\right)}A \tag{35}$$

The transmissibilities are defined as,

$$TF = \left|\frac{Output\ acceleration}{Input\ acceleration}\right| \tag{29}$$

Recall that for absolute accelerations,

$$\ddot{x}_1 = \ddot{z}_1 + \ddot{y} \tag{30}$$
$$\ddot{x}_2 = \ddot{z}_2 + \ddot{y} \tag{31}$$

Thus,

$$\ddot{X}_1 = \ddot{Z}_1 + A \tag{32}$$
$$\ddot{X}_2 = \ddot{Z}_2 + A \tag{33}$$

Defining the output acceleration as $\ddot{X}_1$ and the input acceleration as *A*, we arrive at TF_1,

$$TF_1 = \left|\frac{\ddot{X}_1}{A}\right| = \left|\frac{\ddot{Z}_1 + A}{A}\right| \tag{34}$$

Substituting in Eq. 34 into Eq. 41 yields,

$$TF_1 = \frac{\sqrt{(k_1k_2 - (c_1c_2 + m_2k_1)\omega^2)^2 + \left((k_2c_1 + k_1c_2)\omega - m_2c_1\omega^3\right)^2}}{\sqrt{\left(m_1m_2\omega^4 + (-m_1k_2 - c_1c_2 - m_2(k_1+k_2))\omega^2 + k_1k_2\right)^2 + \left((-m_1c_2 - m_2(c_1+c_2))\omega^3 + (k_2c_1 + k_1c_2)\omega\right)^2}} \tag{42}$$

Similarly, for TF_2,

$$TF_2 = \left|\frac{\ddot{X}_2}{A}\right| = \left|\frac{\ddot{Z}_2 + A}{A}\right| \tag{35}$$

Substituting in Eq. 35 into Eq. 41 yields,

$$TF_2 = \frac{\sqrt{(k_1k_2 - c_1c_2\omega^2)^2 + \left((k_2c_1 + k_1c_2)\omega\right)^2}}{\sqrt{\left(m_1m_2\omega^4 + \left(-m_1k_2 - c_1c_2 - m_2(k_1 + k_2)\right)\omega^2 + k_1k_2\right)^2 + \left(\left(-m_1c_2 - m_2(c_1 + c_2)\right)\omega^3 + (k_2c_1 + k_1c_2)\omega\right)^2}} \quad (44)$$

Consider the resonant frequency of each mass m_i, attached to "ground" by a spring k_i, and a damper c_i, make an uncoupled 1DOF system with an uncoupled frequency ω_i. Thus,

$$k_1 = m_1\omega_1^2 \quad (36)$$

$$k_2 = m_2\omega_2^2 \quad (37)$$

Introducing the terms ζ_1 and ζ_2 for the critical damping ratios of the uncoupled systems, let,

$$c_1 = 2m_1\omega_1\zeta_1 \quad (38)$$

$$c_2 = 2m_2\omega_2\zeta_2 \quad (39)$$

To make the final transfer function equations concise and unitless, let,

$$R_1 = \frac{\omega}{\omega_1} \quad (49)$$

$$R_2 = \frac{\omega}{\omega_2} \quad (50)$$

$$\mu = \frac{m_1}{m_2} \quad (51)$$

$$\beta = \frac{\omega_1}{\omega_2} \quad (52)$$

$$R_2 = \frac{\omega}{\omega_2} = \beta\frac{\omega}{\omega_1} \quad (53)$$

Substituting in Eq. 45 through Eq. 53 into Eq. 42, we arrive at the following closed-form expression for TF_1,

$$TF_1 = \frac{\sqrt{(1 - \beta(\beta + 4\zeta_1\zeta_2)R_1^2)^2 + 4R_1^2\left((\zeta_1 + \zeta_2\beta) - \zeta_1\beta^2R_1^2\right)^2}}{\sqrt{\left(1 - \left(4\zeta_1\zeta_2\beta + \beta^2 + \frac{1}{\mu} + 1\right)R_1^2 + \beta^2R_1^4\right)^2 + 4R_1^2\left((\zeta_1 + \zeta_2\beta) - \beta\left(\zeta_1\beta + \zeta_2 + \frac{\zeta_2}{\mu}\right)R_1^2\right)^2}} \quad (54)$$

Similarly, for TF_2,

$$TF_2 = \frac{\sqrt{(1 - (4\zeta_1\zeta_2\beta)R_1^2)^2 + 4R_1^2(\zeta_1 + \zeta_2\beta)^2}}{\sqrt{\left(1 - \left(4\zeta_1\zeta_2\beta + \beta^2 + \frac{1}{\mu} + 1\right)R_1^2 + \beta^2R_1^4\right)^2 + 4R_1^2\left((\zeta_1 + \zeta_2\beta) - \beta\left(\zeta_1\beta + \zeta_2 + \frac{\zeta_2}{\mu}\right)R_1^2\right)^2}} \quad (55)$$

IV. Verification Example

An example from Steinberg [3] was used to verify the calculations. Steinberg's example uses the following inputs: $k_1 = 490{,}900$, $k_2 = 4091$, $c_1 = 19.5$, $c_2 = 0.231$, $m_1 = 0.0777$, and $m_2 = 0.00259$. Steinberg provides the plots of the coupled transmissibilities for the example problem. Results calculated using the derived analytical equations were compared to the graphs provided by Steinberg, verifying that the results match.

A simple finite element model was created and solved using Nastran to further verify the derived expressions. The model consists of three coincident nodes, with the two masses modeled as CONM2 elements, and the two spring-dampers modeled using CELAS2 elements. The results from the FEA are identical to those calculated with the derived analytical expressions, as shown in Fig. 4.

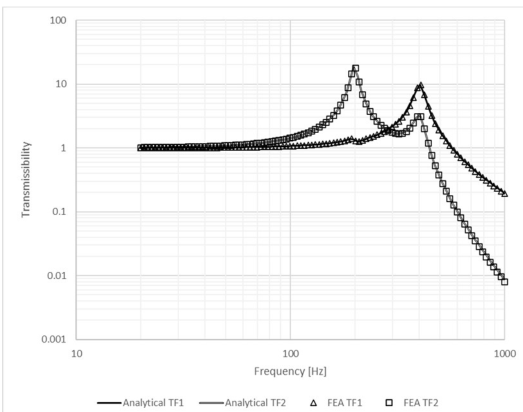

Figure 4: Comparison of analytical results with finite element analysis results

V. Conclusion

The derivation of the closed-form expression of a 2DOF damped system with base excitation input has been presented in this paper. This closed-form expression will be useful in first-pass estimates of the effects of dynamic coupling between two systems, especially in quick iterations with a spreadsheet program.

References

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[3] Steinberg, D., *Vibration Analysis for Electronic Equipment*, 2nd ed., John Wiley & Sons, Inc., New York, 1988.