To Dot Product Graphs and Beyond

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TO DOT PRODUCT GRAPHS
AND BEYOND
by
Sean Bailey
A dissertation submitted in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY
in
Mathematics
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Logan, Utah
2016
ABSTRACT

To Dot Product Graphs

and Beyond

by

Sean Bailey, Doctor of Philosophy

Utah State University, 2016

We will introduce three new classes of graphs; namely bipartite dot product graphs, probe dot product graphs, and combinatorial orthogonal graphs. All of these representations were inspired by a vector representation known as a dot product representation.

Given a bipartite graph $G = (X, Y, E)$, the bipartite dot product representation of $G$ is a function $f : X \cup Y \to \mathbb{R}^k$ and a positive threshold $t$ such that for any $x \in X$ and $y \in Y$, $xy \in E$ if and only if $f(x) \cdot f(y) \geq t$. The minimum $k$ such that a bipartite dot product representation exists for $G$ is the bipartite dot product dimension of $G$, denoted $bdp(G)$. We will show that such representations exist for all bipartite graphs as well as give an upper bound for the bipartite dot product dimension of any graph. We will also characterize the bipartite graphs of bipartite dot product dimension 1 by their forbidden subgraphs.

An undirected graph $G = (V, E)$ is a probe $\mathcal{C}$ graph if its vertex set can be partitioned into two sets, $N$ (nonprobes) and $P$ (probes) where $N$ is independent and there exists $E' \subseteq N \times N$ such that $G' = (V, E \cup E')$ is a $\mathcal{C}$ graph. In this dissertation we introduce probe $k$-dot product graphs and characterize (at least partially) probe 1-dot product graphs in terms of forbidden subgraphs and certain 2-SAT formulas. These characterizations are given for the very different circumstances: when the partition into probes and nonprobes is given, and when the partition is not given.
Vectors \( x = (x_1, x_2, \ldots, x_n)^T \) and \( y = (y_1, y_2, \ldots, y_n)^T \) are combinatorially orthogonal if \( |\{i : x_i y_i \neq 0\}| \neq 1 \). An undirected graph \( G = (V,E) \) is a combinatorial orthogonal graph if there exists \( f : V \rightarrow \mathbb{R}^n \) for some \( n \in \mathbb{N} \) such that for any \( u,v \in V \), \( uv \notin E \) iff \( f(u) \) and \( f(v) \) are combinatorially orthogonal. These representations can also be limited to a mapping \( g : V \rightarrow \{0,1\}^n \) such that for any \( u,v \in V \), \( uv \notin E \) iff \( g(u) \cdot g(v) \neq 1 \). We will show that every graph has a combinatorial orthogonal representation. We will also state the minimum dimension necessary to generate such a representation for specific classes of graphs.

(134 pages)
We will introduce three new vector representations of graphs. These representations are based on relationships between the vectors that are used. Specifically, we will examine scenarios where we ignore specific relationships, where we consider if information is missing, and where we look for when the information in common is not of a specified amount.
ACKNOWLEDGMENTS

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Sean Bailey
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CHAPTER 1
INTRODUCTION AND NOTATION

1.1 Prologue

The focus of this dissertation is on the representation of graphs using vectors as vertices and adjacency determined by algebraic, geometric, or combinatorial properties among the vectors. The representation of graphs by vectors is rife with open problems and is amenable to many applications. Perhaps the most well-defined open problem involves a worst-case scenario on the dimension (or number of components) of vectors needed to construct a representation for any graph. A less well-defined problem is the determination of said dimension of specific classes of graphs.

Among the first authors to consider vector representations were Erdős, Harary, and Tutte. They embedded a graph in Euclidean space $\mathbb{R}^d$ such that vertices are adjacent if and only if their Euclidean distance between their associated vectors is exactly 1 [23]. They asked the question of what was the minimum dimension $d$ such that a representation exists for a given graph $G$. Their results were later built on by Erdős and others [25].

Another vector representation that has been studied extensively is orthogonal vector representations. Lovász used this representation to compute the celebrated Shannon capacity of a graph [55]. Since that paper, orthogonal vector representations have been shown to be intimately related to the vertex packing polytope and have been used to design polynomial-time algorithms for finding maximum cliques and optimum colorings in perfect graphs [37, 38]. Other papers have examined linear algebra properties such as rank and graph properties such as connectivity and vector coloring using orthogonal representations [43, 56, 63, 65].

A seminal vector representation as far as this dissertation is concerned is dot product representations. Dot product representations involve the assignment of a vector to each vertex such that vertices are adjacent if and only if the dot product of their corresponding vectors is greater than a given value [28, 66]. It is variations on this
representation that this thesis studies.

In Chapter 2, we present a brief summary of results related to dot product representations. These concepts include classifications of graphs of specific dot product dimensions, as well as dot product dimensions of specific classes of graphs. Of particular interest to the principle work in this thesis is a conjecture about the maximum dot product dimension of a graph on \( n \) vertices. In this chapter, we also introduce some variations on dot product representations that have been studied by others.

In Chapter 3 we examine different hereditary classes of graphs. Among these classes, we examine dot product graphs of dimension 2. As a way of contextualizing our results the inclusion relationships between various graph classes (most of which are based on other kinds of representations) is determined. These relationships are concisely represented in a Hasse diagram which, at a glance, can tell what properties or theorems about a graph class in the diagram can be applied to other graph classes in the diagram.

In Chapter 4, we introduce a variation on dot product representations we call bipartite dot product graphs. We show that every bipartite graph has such a representation and that the property of having a representation is hereditary. With the fact that having a bipartite dot product representation of dimension \( k \) is hereditary in hand, we characterize bipartite graphs with bipartite dot product dimension of 1 by forbidden subgraphs.

In Chapter 5, probe graphs come into consideration. Inspired by the observation that dot product dimension appears to be unimodal with respect to the number of edges, we asked the question of which graphs would have a given dot product dimension if a set of edges is added. We will build on previous work on probe trivially perfect graphs. Our results include a characterization of probe 1-dot product graphs when the probes are known. When the partition of probes is unknown, a characterization is given when the graph is either a tree or bipartite. A partial characterization of unpartitioned probe 1-dot product graphs is also given. We will conjecture that this characterization is complete. We will also show that recognition of whether a graph is a probe 1-dot product graph can be done in polynomial time.

In Chapter 6, we will introduce a generalization of orthogonal vector representations called combinatorial orthogonal graphs. This representation is also a variation of dot product graphs by mapping the given vectors to \( \{0,1\}^k \) vectors. We will show that every graph has a combinatorial orthogonal representation. We also present a forbidden subgraph characterization of combinatorial orthogonal graphs of dimension 1 and 2.
Chapter 7 is the concluding chapter. We will summarize our results and their impact. We will also present future questions for each of the representations given, as well as future lines of inquiry for dot product graphs in general.

1.2 General Notation and Definitions

We will use notational conventions found in [9] and [80]. For example, there are two major ways to denote a graph and its constituent sets. One way is using the equation “$G = (V, E)$” which allows something like $x \in V$ to mean $x$ is a vertex of $G$. The other way is to state that $V(G)$ and $E(G)$ represent the vertex and edge sets of the graph $G$, respectively. Also an edge may be denoted by the juxtaposition of the symbols for the vertices that constitute it.

All graphs in this thesis are finite, simple, and undirected; meaning that the vertex and edge sets are finite, and the edge set is a set consisting of unordered pairs of distinct vertices. We may abuse notation and write $E \subset A \times B$, where $A$ and $B$ are (sub)sets of vertices to indicate that $E$ is a set of edges incident to vertices from $A$ and $B$.

Other notation with be introduced as needed.
CHAPTER 2
DOT PRODUCT REPRESENTATIONS OF GRAPHS

The primary vector representation that is the basis of this dissertation is the dot product representation. Let $G = (V, E)$ be a graph. A $k$-dot product representation of $G$ is a function $f : V \rightarrow \mathbb{R}^k$ along with a real number constant, $t$, such that for vertices $u, v \in V$ if and only if $f(u) \cdot f(v) \geq t$ [28, 66]. The minimum $k$ such that there exists a $k$-dot product representation for $G$ is the dot product dimension, written $\rho(G)$. If the dot product dimension of a graph $G$ is at most $k$, we may refer to $G$ as a $k$-dot product graph.

2.1 Motivation of Past Results

Reiterman, Rödl, and Šiřajová first introduced dot product representations of graphs in [66]. Reiterman et al. were motivated by the communication complexity applications considered in [1, 55, 64]. Additionally, they were looking to generalize threshold graphs as introduced by Chvátal and Hammer [17]. We will later show that Reiterman et al. were able to generalize threshold graphs.

Scheinerman, Fiduccia, Trenk, and Zito independently introduced dot product graphs in [28]. The motivation of Scheinerman et al. was also multi-faceted. A primary motivation was a generalization of intersection number. Let $G$ be a graph. An intersection representation of $G$ is an assignment of a set $S_v$ to each vertex $v$ such that, for vertices $u$ and $v$, $uv \in E$ if and only if $S_u \cap S_v \neq \emptyset$. The intersection number of $G$, denoted $\iota(G)$, is the smallest size of the union of the sets assigned to the vertices among all intersection representations for $G$. Scheinerman et al. were able to show that each set $S_v$ can be replaced by a characteristic $\{0, 1\}$-vector [66]. Suppose that $\mathcal{S} = \bigcup_{v \in V} S_v$, where $\{S_v : v \in V\}$ is an intersection representation of $G$. If $|\mathcal{S}| = k$, then without loss of generality $\mathcal{S} = \{1, 2, \cdots, k\}$ and to each vertex $y$ we may assign the characteristic vector of $S_y$ with respect to $\mathcal{S}$, and obtain a $k$-dot product representation of $G$. That is, if $y \in V$, and $f(y) = (y_1, y_2, \cdots, y_k)^T$ with $y_i = 1$ if and only if $i \in S_y$ and $y_i = 0$
otherwise, we have, for vertices \( x \) and \( y \), \( f(x) \cdot f(y) \geq 1 \) if and only if \( S_x \cap S_y \neq \emptyset \). Clearly if we relax the constraint \( f(v) \in \{0,1\}^k \) to \( f(v) \in \mathbb{R}^k \), we see \( \rho(G) \leq \eta(G) \).

Scheinerman et al. were also interested in generalizing other representations, such as threshold graphs and interval graphs. They proved that interval graphs can be represented by 2-dot product representations. A final motivation of Scheinerman et al. was to look for efficient data structures to represent a family of graphs. These data structures can then be used as the basis for efficient algorithms.

It should be noted that there was one noticeable difference between Reiterman et al. and Scheinerman et al., namely the threshold constant \( t \). Reiterman et al. allowed \( t \) to be any real number. Meanwhile, Scheinerman et al. restricted \( t \) to be strictly positive. Thus the class of graphs with a given dot product dimension is slightly larger for Reiterman et al.. In this thesis we will follow Scheinerman et al.. By restricting \( t > 0 \), we can scale all of the vectors such that \( t = 1 \). Unless otherwise stated, we will assume that \( t = 1 \) throughout this dissertation. This threshold tends to make most of the proofs simpler.

### 2.2 Past Results

Reiterman et al. and Scheinerman et al. both began their study of dot product graphs by asking what graphs have a dot product representation. Both groups were able to answer this question with Theorem 2.1. The proof of this theorem is constructive in nature. This tended to be a common method when examining dot product graphs. We will use similar constructive proofs for some of the theorems will be prove later in this dissertation.

**Theorem 2.1.** [28, 66] Let \( G \) be a graph. Then there exists dot product representation of \( G \).

**Proof.** Let \( G \) be a graph with \( n \) vertices and \( k \) edges. We arbitrarily label each edge from 1 to \( k \). We can define \( f : V \to \mathbb{R}^k \) such that the \( i^{th} \) component of \( f(u) \) is 1 if \( u \) is incident to the edge corresponding to \( i \), with 0 otherwise. Thus \( uv \in E(G) \) implies the \( i^{th} \) component of \( f(v) \) and \( f(u) \) is 1. So \( f(v) \cdot f(u) \geq 1 \). Similarly \( uv \notin E(G) \) implies that there is no common component in \( f(u) \) and \( f(v) \) that is 1. So \( f(v) \cdot f(u) = 0 < 1 \) in this case. Thus we have a dot product representation of \( G \). \( \Box \)
Because a dot product representation exists for every graph, the study of graphs has focused on two problems:

1. Given a set of vectors, what is the graph that is generated?

2. Given a graph $G$, what is the minimum dimension $k$ such that a dot product representation exists?

The second question is equivalent to “what is $\rho(G)$”, i.e. what is the dot product dimension of $G$?

We will consider both questions throughout this dissertation. The first question can be applied to application problems, but we will use it to ask generally what graphs are possible if the vectors are of dimension $k$. Part of our identification of what graphs are possible include what graphs are not possible with a given dimension. The second question will lead us to consider the bounds on the dot product dimension of a graph based on various global and local structures.

2.3 Bounds of Dot Product Dimension

2.3.1 General Bounds

General bounds on dot product dimension were of interest to both Reiterman et al. and Scheinerman et al.. The constructive proof of Theorem 2.1 leads to the following corollary.

Corollary 2.2. Let $G$ be a graph. Then $\rho(G) \leq |E|$.

The other bounds on the dot product dimension are primarily based on the hereditary property of dot product dimension. This property is based on induced subgraphs. $H$ is an induced subgraph of $G$ if $H$ can be obtained by deleting a set of vertices of $G$. Note that deleting a vertex entails deleting all edges incident to it. A graph representation or property is thus hereditary if the induced subgraphs of $G$ have a property when $G$ has that property. The hereditary property of dot product dimension is given in Theorem 2.3.

Theorem 2.3. [28, 66] If $H$ is an induced subgraph of $G$, then $\rho(H) \leq \rho(G)$.

Because dot product dimension is hereditary, dot product dimension is related to the number of vertices. A less intuitive graph parameter related to dot product
dimensions is the distance between vertices. The distance between two vertices, \( u \) and \( v \), is the minimum number of edges that would need to be traversed to get from \( u \) to \( v \) and is denoted \( d(u,v) \). The relationship of vertices and their distance to dot product dimension can be seen in Theorem 2.4 and its associate proof.

**Theorem 2.4.** [28] Let \( G = (V,E) \) be a graph and let \( v \) be any vertex of \( G \). Then \( \rho(G) \leq \rho(G-v) + 1 \). If \( v,w \in V \) and \( d(v,w) > 2 \), then \( \rho(G) \leq \rho(G-v-w) + 1 \).

**Proof.** For the first result, let \( k = \rho(G-v) \) and choose a \( k \)-dot product representation \( f \) of \( G-v \). Choose \( \epsilon \in (0,1) \) so that for all \( x,y \in V(G-v) \) with \( xy \notin E(G-v) \) we have \( f(x) \cdot f(y) \leq 1 - \epsilon \).

We now form a \((k+1)\)-dot product representation \( \hat{f} \) of \( G \) by adding an extra coordinate to the representation \( f \). Let

\[
\hat{f}(u) = \begin{cases} 
    \begin{bmatrix}
    f(u) \\
    \epsilon
    \end{bmatrix} & u \neq v \text{ and } uv \in E(G) \\
    \begin{bmatrix}
    f(u) \\
    0
    \end{bmatrix} & u \neq v \text{ and } uv \notin E(G) \\
    \begin{bmatrix}
    0 \\
    1
    \end{bmatrix} & u = v
\end{cases}
\]

It is simple to check that \( \hat{f} \) is a representation of \( G \).

Now suppose that \( x, y \in V(G) \) are at distance greater than 2. Let \( k = \rho(G-x-y) \) and let \( f \) be a strict \( k \)-dot product representation of \( G-x-y \). Choose \( \epsilon \in (0,1) \) so that for \( u,v \in V(G-x-y) \) if \( uv \notin E(G) \), then \( f(u) \cdot f(v) \leq 1 - \epsilon \), and if \( uv \in E(G) \) then \( f(u) \cdot f(v) \geq 1 + \epsilon \). We now form a \((k+1)\)-dot product representation \( f \) of \( G \) by
adding an extra coordinate to the representation $f$. Let

$$
\hat{f}(u) = \begin{cases} 
\begin{bmatrix} f(u) \\ \epsilon \\ f(u) \\ -\epsilon \\ f(u) \\ 0 \\ 0 \\ 1 \end{bmatrix} & \text{if } u \neq x, y, ux \in E(G), \text{ and } uy \notin E(G) \\
0 & \text{if } u = x \\
1 & u = y 
\end{cases}
$$

Since no vertex can be adjacent to both $x$ and $y$, we can simply check that $\hat{f}$ is a $(k+1)$-dot product representation of $G$. 

A general bound on the dot product dimension of a graph $G$ based on the number of vertices of $G$ can be derived from Theorem 2.4 via inductively removing all of the vertices of $G$. This bound is given in the following corollary.

**Corollary 2.5.** [28, 66] Let $G$ be a graph on $n$ vertices. Then $\rho(G) \leq n - 1$.

These bounds of dot product dimension in terms of $|V|$ and $|E|$ are also supplemented by bounds in terms of other global parameters of the graph. For example, Scheinerman et al. showed that the dot product dimension is bounded by the intersection number of a graph. This relationship was explained earlier in Section 2.1.

Dot product dimension can also be bounded by local structures of graphs, as suspected by Scheinerman et al.. The first local structure is connected components. A graph is *connected* if and only if there exists a path between any two vertices. The *components* of $G$ are the maximal induced subgraphs of $G$ that are each connected. We may write $\sum_{i=1}^{k} G_i$ to mean a graph with components $G_1, G_2, \cdots, G_k$. 
Theorem 2.6. [28, 66] Let $G$ be a graph with components $G_1, G_2, \ldots, G_k$. Then
\[ \rho \left( \sum_{i=1}^{k} G_i \right) \leq \max_i (\rho(G_i)) + 1. \]

This result combined with our Corollary 2.2 imply that dot product dimension is affected by adjacencies within the graph. Hence structures within a graph such as cliques and independent sets affect the dot product dimension.

A connected graph with every vertex adjacent to every other vertex is called complete. A complete graph on $n$ vertices is denoted $K_n$. Additionally, an induced subgraph that is complete is called a clique. While not every graph is complete, every graph does contain at least one clique.

The opposite of a clique is an independent or stable set of vertices. This is a set of vertices that are pairwise non-adjacent. The maximum size of an independent set in a graph $G$ is denoted $\alpha(G)$.

The following theorems are all based on cliques, complete graphs, and independent sets.

Proposition 2.7. [28, 66] $\rho(K_n) = 1$

Theorem 2.8. [66] Let $G$ be a graph on $n$ vertices. Then $\rho(G) \leq n - \alpha(G)$.

Theorem 2.9. [66] Let $G$ be a graph with $A \subset V$ and $K_A$ is a clique on $A$. Then $\rho(G \cup K_A) \leq \rho(G) + 1$.

Theorem 2.9 is of particular interest as it implies that adding or subtracting a specific set of edges will have minimal effect on the dot product dimension of a graph. In particular, if $A$ consisted of 2 vertices, then Theorem 2.9 yields the addition of any edge increases the dot product dimension by at most one. It is not clear, however, what effect exactly the size of $A$ has on $\rho(G \cup K_A)$. This precedent of adding edges will lead to probe dot product graphs in Chapter 5, where we investigate the addition and absence of certain subsets of edges in a formal way.

2.3.2 Bounds on Specific Graphs

In addition to general bounds, bounds on specific types of graphs have been studied. The graphs for which there have been definitive results include paths, cycles, trees, bipartite graphs, and planar graphs. We will introduce each of these graphs and dot
product bounds associated with them.

A path is a sequence of unique vertices each of which is adjacent to the preceding and following vertex in the sequence. We will denote a path on $n$ vertices as $P_n$.

**Lemma 2.10.** [28, 66] Let $G$ be a path on $n$ vertices with $n \geq 4$. Then $\rho(G) = 2$.

A cycle is a sequence of unique vertices each of which is adjacent to the preceding and following vertex in the sequence and the sequence begins and ends with the same vertex. The length of a cycle is the number of vertices in a cycle. We will refer to a cycle of length $k$ as a $k$-cycle, denoted $C_k$. Additionally, we define a chordless cycle in a graph $G$ to be a cycle such that non-adjacent vertices of the cycle are not adjacent in the graph $G$.

**Lemma 2.11.** [28, 66] Let $G$ be a cycle on $n$ vertices with $n \geq 4$. Then $\rho(G) = 2$.

A tree is a connected graph with no cycles. Trees have played a special role in graph theory and all areas where graph theory is applied. It is perhaps surprising how deep the theory of trees is. Paths are a subset of trees. Other subsets of trees are stars and caterpillars. A graph $G$ is a star if the distance between any pair of vertices is less than or equal to 2. A graph $G$ is a caterpillar if all vertices are within a distance of 1 of a central path.

**Theorem 2.12.** [28] Let $T$ be a tree. Then:

- $\rho(T) = 0$ if $T = K_1$
- $\rho(T) = 1$ if $T$ is a star (and not a $K_1$)
- $\rho(T) = 2$ if $T$ is a caterpillar (and not a star)
- $\rho(T) = 3$ if $T$ is none of the prior mentioned trees

The proof of Theorem 2.12 as given by Scheinerman et al. is constructive in nature. To show that there exist a tree with dot product dimension 3, Scheinerman et al. used a proof by contradiction to show that it was impossible to represent the graph in Figure 2.1 with a 2-dot product representation. The graph in Figure 2.1 can be noted to grow in three directions from the central vertex. This observation strengthens our statement.
earlier that dot product dimension is affected by distance.

A graph $G = (V, E)$ is a bipartite graph if $V = X \cup Y$ with $X \cap Y = \emptyset$ and for any vertices $x, y$ if $xy \in E(G)$, then $x \in X$ and $y \in Y$. We may write $G = (X, Y, E)$ to indicate that $G$ is bipartite and that $X$ and $Y$ are the (independent) sets of the vertex partitions. These graphs allow us to focus only on adjacencies between the two sets, $X$ and $Y$. Thus bipartite graphs are ideal for analyzing matching problems, or any situation in which there are relationships between two disjoint sets or all relationships among two distinguished sets are ignored.

Complete bipartite graphs are bipartite graphs $G = (X, Y, E)$ such that $xy \in E(G)$ for every $x \in X$ and $y \in Y$. We will use the notation $K_{n_1,n_2}$ for a complete bipartite graph, where $n_1$ and $n_2$ are the number of vertices in each subset, $X, Y$, respectively. An example of $K_{3,2}$ is shown in Figure 2.2.
Bipartite graphs can be extended to $k$-partite graphs with $k$ partite sets similar to the 2 partite sets in bipartite graphs. A complete $k$-partite graph is denoted $K_{n_1,n_2,\ldots,n_k}$.

**Theorem 2.13.** [28, 66] If $G$ is a bipartite graph on $n$ vertices, then $\rho(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

**Theorem 2.14.** [28, 66] $\rho(K_{n_1,n_2}) = \min\{n_1, n_2\}$.

**Corollary 2.15.** [28] Let $n_1 \geq n_2 \geq \ldots \geq n_p \geq 1$ with $n_i \in \mathbb{Z}$. Then $\rho(K_{n_1,n_2,\ldots,n_p}) = n_2$.

A *planar graph* is a graph $G$ that can be drawn such that edges of $G$ do not intersect except at the endpoints. Scheinerman et al. originally hypothesized that $\rho(G) \leq 3$ for any planar graph $G$. This bound was disproven and improved in [47].

**Theorem 2.16.** [47] Every planar graph is a 4-dot product graph, and there exist planar graphs that are not 3-dot product graphs.

### 2.3.3 Characterization of Graphs of a Specific Dot Product Dimension

These past bounds have been an attempt to determine the dot product dimension of a given graph. Yet we mentioned that there is also the question of which graphs are possible if the vectors are restricted to a specific dimension. The natural dimension to begin with is 1. Both Reiterman et al. and Scheinerman et al. were able to characterize graphs of dot product dimension 1. Their characterization fulfilled the motivation of Reiterman et al. to determine a generalization of threshold graphs.

**Theorem 2.17.** [28, 66] The following statements are equivalent:

- $\rho(G) \leq 1$
- $G$ has at most two non-trivial components, both of which are threshold graphs.
- Any induced subgraph of $G$ cannot include: $3K_2$, $P_4$, or $C_4$.

However, attempts to characterize graphs of dot product dimension $k$ with $k \geq 2$ have not been completed. Partial characterizations of graph with a 2-dot product representation have been given in [28] and [45]. The latter also attempted to identify what relationship 2-dot product graphs have with other well-known classes of graphs. A precise determination of those relationships was not accomplished; of course, if such a determination was made, a characterization of 2-dot product graphs would be had.
Johnson et al. were not able to determine a class of graphs that contains 2-dot product graphs.

### 2.3.4 Complexity and Conjecture

The apparent difficulty of computing the dot product dimension of a graph is what leads researchers to determine bounds and focus on restricted structures. In the midst of determining the dot product dimension in such circumstances and discovering no simple or unified way of doing it, explorations of complexity were born. Scheinerman et al. conjectured that for $k \geq 2$ determining if a graph is a $k$-dot product graph is NP-hard. Kang et al. were able to verify this conjecture in [48].

**Theorem 2.18.** [48] Let $G$ be a graph. Recognizing whether $G$ is a $k$-dot product graph is NP-hard for all $k > 1$.

Because the recognition is NP-hard, the bounds on dot product dimension are extremely useful. This led Scheinerman et al. to make a significant conjecture about the maximum dot product dimension of a graph on $n$ vertices. This conjecture gives a strict upper bound on the dot product dimension of any graph on $n$ vertices.

**Conjecture 2.19.** [28] Let $\rho(n)$ be the maximum dot product dimension of any graph on $n$ vertices. Then $\rho(n) = \left\lfloor \frac{n}{2} \right\rfloor$.

This bound would be strict because of Theorem 2.14. The conjecture has been verified for a few classes of graphs; the most recent progress and summary can be found in [53].

### 2.4 Variations on Dot Product Graphs

The study of dot product graphs has also included variations. Those variations have included variations on the value of the dot product and variations on the domain over which the components of the vectors are chosen.
2.4.1 Exact and Asymptotic Dot Product Graphs

A variation of dot product graphs is exact dot product graphs [78]. An exact $k$-dot product representation of a graph $G$ is a function $f : V \rightarrow \mathbb{R}^k$ where each vertex is assigned a vector, $f(u) = (x_1, x_2, \cdots, x_k)^T$, in $\mathbb{R}^k$ such that $f(u) \cdot f(v) = 1$ if $uv \in E$ and $f(u) \cdot f(v) = 0$ otherwise. These vectors are not required to be of a fixed length and it can be interpreted as all nonadjacent vertices have orthogonal vectors.

The results developed for exact dot product graphs were similar to general dot product graphs. The most similar results include the existence of exact dot product representation for all graphs and how the exact dot product representation is affected by the removal of a vertex. Both of these results are proven similarly to how they are with dot product graphs.

Additionally, the minimum dimension of exact dot product representations, denoted $dp(G)$, was considered for specific classes of graphs, including complete graphs, bipartite graphs, and cycles. The specific results include the following theorems.

**Theorem 2.20.** Let $G = K_n$ for $n \geq 2$, Then $dp(G) = 1$.

**Theorem 2.21.** Let $G$ be a bipartite graph with $n$ vertices and $c$ components. Then $dp(G) = n - c$.

**Theorem 2.22.** Let $n \geq 3$ be an integer. Then

$$dp(C_n) = \begin{cases} 
 n - 1 & \text{if } n \text{ is even} \\
 n - 2 & \text{if } n \text{ is odd}
\end{cases}$$

While these results pertain to similar graphs to the ones studied as dot product graphs, it can be noted that there are significant differences between these bounds and those for dot product graphs. The similarity in the graphs studied and the proofs (mostly algorithmic) will set the stage for the representations we will study in subsequent chapters.

A related variation is asymptotic dot product graphs. An asymptotic $k$-dot product representation of a graph $G$ is for all $\epsilon > 0$ a function $f : V \rightarrow \mathbb{R}^k$ where each vertex is assigned a vector, $f(u) = (x_1, x_2, \cdots, x_k)^T$, in $\mathbb{R}^k$ such that $f(u) \cdot f(v) \in (1 - \epsilon, 1 + \epsilon)$ if $uv \in E$ and $f(u) \cdot f(v) \in (-\epsilon, \epsilon)$ otherwise. The specific representation are based on $\epsilon$, so there exists a series of representations for any asymptotic dot product graph that
converge to an exact dot product graph representation. To understand this representation, consider the following case, namely the cycle on four vertices, $C_4$, which is pictured in Figure 2.3. We define a 2-asymptotic dot product representation of $C_4$ with $\epsilon = \frac{3}{N^2}$ as

$$
\begin{align*}
    f(x_1) &= \begin{bmatrix} \frac{N}{2} \\ 0 \end{bmatrix}, \\
    f(x_2) &= \begin{bmatrix} \frac{1}{N} \\ \frac{1}{N} \end{bmatrix}, \\
    f(x_3) &= \begin{bmatrix} 0 \\ N \end{bmatrix}, \\
    f(x_4) &= \begin{bmatrix} \frac{1}{N} \\ \frac{1}{N} \end{bmatrix}.
\end{align*}
$$

It can be noticed that some of the dot products are exact, e.g. $f(x_1) \cdot f(x_2) = 1$, while other dot products are within $\epsilon$ of the desired value, e.g. $f(x_2) \cdot f(x_4) = \frac{2}{N^2} < \frac{3}{N^2}$.

![Figure 2.3: The Graph Consisting of a Cycle on 4 Vertices.](image)

Similar to exact dot product graphs, these representations were examined for general bounds on asymptotic dot product dimension and bounds for specific classes of graphs. Again classes of graphs such as complete graphs, bipartite graphs, and cycles were examined. However many of the proofs relied on a general bound based on the vertex cover number. The vertex cover number, denoted $\beta(G)$ is the minimum size of a subset of $V, K$, such that every edge in $G$ is incident to at least one vertex in $K$.

One interesting result for exact dot product representations is their relationships between their dimension and their adjacency matrix. This result is given in the following theorem.

**Theorem 2.23.** Let $G$ be a graph, and let $A$ be its adjacency matrix. If $D$ is a diagonal matrix and $A + D$ is positive semidefinite ($A + D \succeq 0$), then $dp(G) \leq \text{rank}(A + D)$. Furthermore,

$$
    dp(G) = \min\{\text{rank}(A + D) : A + D \succeq 0, D \text{ diagonal}\}.
$$

This result can be extended to asymptotic dot product representations (with loss of equality) because every exact dot product representation is an asymptotic dot product.
representation. This result is also of interest as it ties these representations to linear algebra.

2.4.2 Dot Product Graphs Over Fields

The next variation is dot product graphs over any field. This variation was examined by [59]. The definition is the same as the general dot product graph except the components are from the field $F$ and $uv \in E$ if and only if $f(u) \cdot f(v) \geq 1_F$.

Minton [59] showed that such a representation over an arbitrary field exists for every graph. Additionally, it was shown that there exists an exact dot product representation over an arbitrary field for every graph.

2.4.3 Dot Product Representations Using Tropical Arithmetic

Another variation is tropical dot product graphs. This is a dot product graph over the tropical semi-ring, $\mathbb{T}$. The tropical semi-ring is the set of real numbers except tropical addition is real maximum and tropical multiplication is real addition. Thus dot product of vectors with tropical components can be viewed as the maximum of the sums of the same components.

Using this semi-ring, [2] defined the tropical dot product representation of a graph $G$ as a mapping $f : V \rightarrow \mathbb{T}^k$ such that for $u, v \in V$ $uv \in E$ if and only if $f(u) \odot f(v) \geq 1$, where $f(u) \odot f(v) = \min_{1 \leq i \leq k} \{f(u)_i + f(v)_i\}$. The minimum $k$ for which such a representation exists is the tropical dot product dimension, denoted $\rho_T(G)$.

It was shown that a representation exists for every graph in a constructive manner similar to Theorem 2.1. The authors also showed the bounds of specific classes of graphs, namely trees, cycles, and complete $k$-partite graphs.
CHAPTER 3
CONTAINMENT OF HEREDITARY GRAPH CLASSES

3.1 Introduction

Graphs can be examined based on which graph classes they belong to. A graph class is a set of graphs with a common property or parameter. Hereditary graph classes are of particular interest. Because hereditary graphs maintain a given property or parameter over induced subgraphs, they can be characterized by what are the minimal graphs that do not have the property or parameter. This minimal graphs are forbidden subgraphs and the characterization is known as forbidden subgraph characterization. Forbidden subgraph characterizations are useful characterizations as they can provide insight into the structure of graphs in the class and be helpful in designing algorithms related to the graphs [21].

Forbidden subgraph characterizations are unique and thus can be used to prove relationships between hereditary graph classes. This comparison is the motivation of this chapter and examined in part in [45] and [58].

3.2 Hereditary Graph Classes

We will specifically compare the following graph classes: chordal graphs, split graphs, interval graphs, unit interval graphs, interval overlap graphs, circular arc graphs, threshold graphs, and 2-dot product graphs. These classes are at issue because it turns out that these are the graph classes apparently closest in structure to 2-dot product graphs.

3.2.1 Chordal Graphs

A class of graphs that are of particular interest are chordal graphs. A graph $G$ is chordal if it does not contain any induced cycles of length 4 or greater. Thus the forbidden induced subgraphs of a chordal graph on $n$ vertices are $C_i$ for $4 \leq i \leq n$.

It should be noted that all trees are chordal graphs. In fact, it was shown in [32, 79]
that chordal graphs are the intersection graphs of subtrees of a tree. This containment of
trees in chordal graphs will be useful when we compare chordal graphs to 2-dot product
graphs.

3.2.2 Interval Graphs

An interval representation of a graph is a family of intervals on the real line such
that each vertex is assigned an interval and vertices are adjacent if and only if the cor-
responding intervals intersect. A graph that has such a representation is an interval
graph. The invention or discovery of interval graphs is credited to Benzer in 1959 [7],
in the course of his studies of the topology of the gene, and sometimes of Hajós in 1957
[39], with respect to his purely combinatorial question that asks basically which graphs
have an interval representation.

Interval graphs have been studied extensively and several useful properties have
been found to be associated with them. Some of the seminal papers on interval graphs
include [8],[29], and [33].

A graph related to interval graphs is an asteroidal triple. An asteroidal triple is
an independent set of three vertices such that each pair is joined by a path that avoids
the neighborhood of the third [20]. A graph is asteroidal triple-free, denoted AT-free, if
it contains no induced asteroidal triples. Interval graphs are precisely the AT-free and
chordal graphs. Furthermore, these graphs are characterized by the forbidden induced
subgraphs in Figure 3.1 [8].

As we previously noted, the development of interval graphs seemed to be based
on its application of gene sequencing [7, 85]. Interval graphs also were applied to math-
ematical models of population biology by Cohen, specifically food webs [18]. Other ap-
lications include using interval graphs to represent resource allocation problems [4, 19].

3.2.3 Unit Interval Graphs

A variation of interval graphs are unit interval graphs. Introduced by Roberts in
[67], these are interval graphs where every interval has length exactly 1. These graphs
can be characterized as interval graphs that do not contain an induced $K_{1,3}$, see Figure
3.2 [31, 67]. It has also been shown that unit interval graphs are equivalent to proper
interval graphs [67]. A graph $G$ is a proper interval graph if every vertex can be assigned
an interval of the real line such that no interval is properly contained in another, and
two vertices are adjacent if and only if their corresponding intervals intersect. Similar to interval graphs in general, unit interval graphs have been applied to physical mapping of DNA and genome reconstruction [34, 71].

3.2.4 Interval Overlap Graphs

Another representation of graphs related to interval graphs are interval overlap graphs. A graph is an interval overlap graph if there is a family of intervals such that
each vertex is assigned an interval and vertices are adjacent if and only if the corresponding intervals intersect and one interval is not contained the other. This representation is also equivalent to a class of graphs known as circle graphs [36]. A graph $G$ is a *circle graph* if the vertices correspond to a set of chords on a circle such that vertices are adjacent if and only if their chords intersect in the interior of the circle [26].

Circle graphs were introduced as a way to solve a computer science problem about queues and stacks that was introduced by Knuth [50]. Another application of circle graphs is the abstract representation of a special case for wire routing [69].

These graphs have been studied extensively, and various polynomial recognition algorithms have been developed [11, 30, 61, 70]. The set of forbidden subgraphs are shown in Figure 3.3[12].

![3.3: The Forbidden Subgraphs of Interval Overlap Graphs](image)

### 3.2.5 Circular Arc Graphs

Circular arc graphs are a generalization of interval graphs. A graph $G$ is a *circular arc graph* if there exists a family of arcs of a circle such that each vertex is assigned an arc and vertices are adjacent if and only if the corresponding arcs intersect. Since their origin, finding a forbidden subgraph characterization of circular-arc graphs has been a challenging open problem. Many partial results have been proposed, but a full answer remains elusive [3, 10, 27, 49, 54, 75–77]. Some of the characterized minimally non-circular-arc graphs are shown in Figure 3.4 [75].

As with other interval graphs, circular-arc graphs have been applied to modeling genetics [72]. Additional applications of circular-arc graphs include traffic control scheduling and all-optical ring networks [73, 74].
3.2.6 Threshold graphs

Another representation that we will reference is threshold graphs, introduced and defined by Chvátal and Hammer \cite{16, 17}. A graph $G = (V, E)$ is a threshold graph if a non-negative weight $w_v$ can be assigned to each $v \in V$ and a threshold $t$ chosen, such that $w(U) \leq t$ if and only if $U$ is a stable set where $U \subseteq V$ and $w(U) = \sum_{v \in U} w_v$. Another definition is that $G$ is a threshold graph if and only if there exists a hyperplane, called a separator, that strictly separates the characteristic vectors of the stable sets of $G$ from the characteristic vectors of the non-stable sets.

The structure of threshold graphs has been well studied. Two characteristics that are salients of threshold graphs are alternating 4-cycles and degree partitions. An alternating 4-cycle of a graph $G = (V, E)$ is a configuration consisting of distinct vertices $a, b, c, d$ such that $ab, cd \in E$ and $ac, bd \notin E$. By considering the presence or absence of edges $ad$ and $bc$, we see that the vertices of an alternating 4-cycle induce a path $P_4$, a 4-cycle $C_4$, or a matching $2K_2$ \cite{58}. An alternating 4-cycle is depicted in Figure 3.5.

Let $G = (V, E)$ be a graph whose distinct positive vertex-degrees are $d_1 < \cdots < d_m$ and let $d_0 = 0$ (even if no vertex of degree 0 exists), $d_{m+1} = |V| - 1$. Let $D_i = \{v \in V :$
Figure 3.5: An Alternating 4-cycle and the Possible Induced graphs. A solid line implies a present edge and a dashed line a optional edge.

\[ \text{deg}(v) = d_i \] for \( i = 0, \cdots, m \). The sequence \( D_0, \cdots, D_m \) is the degree partition of \( G \).

These characteristics are used to give the basic characterizations of threshold graphs in Theorem 3.1.

**Theorem 3.1.** [16, 17, 22, 35, 42] For a graph \( G = (V, E) \), the following are equivalent:

1. \( G \) is a threshold graph;

2. \( G \) does not have an alternating 4-cycle;

3. \( G \) can be constructed from the one-vertex graph by repeatedly adding an isolated vertex or a dominating vertex;
4. for each \( v \in D_k \),

\[
N(v) = \bigcup_{j=1}^{k} D_{m+1-j} \quad \text{for } k = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor
\]

\[
N[v] = \bigcup_{j=1}^{k} D_{m+1-j} \quad \text{for } k = \left\lfloor \frac{m}{2} \right\rfloor + 1, \ldots, m,
\]

in other words, for \( x \in D_i \) and \( y \in D_j \), \( xy \in E \) if and only if \( i + j > m \);

5. \( d_{k+1} = d_k + |D_{m-k}| \quad k = 0, \ldots, m, k \neq \left\lfloor \frac{m}{2} \right\rfloor \);

6. there exists non-negative real weights \( w_v, V \in V \) and \( t \) such that for distinct vertices \( u \) and \( v \), \( w_u + w_v > t \) if and only if \( uv \in E \).

Threshold graphs have been extensively studied and used for a variety of applications. Chvátal and Hammer studied threshold graphs for their application in set-packing problems [16, 17]. Henderson and Zalcstein studied applications of threshold graphs to parallel processing. Specifically they found that threshold graphs can be used to control the flow of information between processors [42]. Threshold graphs were also used by Ordman in resource allocation problems and Koop in modeling cyclic scheduling and personnel allocation [51, 62].

3.3 Containment Relationships Among Graph Classes

Many of the graph classes that have been discussed have similarities and relationships between one another. The relationship between these graph classes can be shown using a Hasse diagram where \( X \rightarrow Y \) if \( X \subset Y \). These relationships are proven in Theorem 3.2 and shown in Figure 3.6. Let \( TH \) denote the class of threshold graphs, \( IN \) the class of interval graphs, \( UI \) the class of unit interval graphs, \( IO \) the class of interval overlap graphs, \( CA \) the class of circular-arc graphs, \( CH \) the class of chordal graphs, and \( 2DP \) the class of graphs with dot product dimension less than or equal to 2.

**Theorem 3.2.** The Hasse diagram in Figure 3.6 is correct.

**Proof.** \( TH \rightarrow IN \): The proper containment can first be seen by examining a \( P_4 \). This graph is an interval graph, but not a threshold graph.

Let \( G \) be a threshold graph. Then there exists \( f : V(G) \rightarrow \mathbb{R} \) and \( t \in \mathbb{R} \) such that \( f(u) + f(v) \geq t \) if and only if \( uv \in E(G) \). Then define the interval for \( u \) as
$I_u = [f(u), t - f(u)]$.

$TH \rightarrow IO$: The proper containment can first be seen by examining a $C_4$. This graph is a circular-arc graph, but not a threshold graph.

Let $G$ be a threshold graph. Then there exists the degree partition $D_0, \ldots, D_m$ of $G$. For each $i = 0, \ldots, \lfloor m/2 \rfloor$ and for each vertex in $v \in D_i$, take $I(v) = (0, i)$; for each $i = \lfloor m/2 \rfloor + 1, \ldots, m$ and for each vertex in $v \in D_i$, take $I(v) = (-i + \frac{1}{2}, m - i + \frac{1}{2})$. This ensures that the vertices of the stable set $S = D_0 \cup \cdots \cup D_{\lfloor m/2 \rfloor}$, no two intervals overlap; for vertices of the clique $K = D_{\lfloor m/2 \rfloor + 1} \cup \cdots \cup D_m$, every two intervals overlap or coincide; and every interval of a vertex in $K$ overlaps the intervals of the correct vertices of $S$. It remains to perturb the intervals of the vertices of the same $D_i \subseteq K$ and make them overlap each other instead of coinciding. This can be achieved by translating the interval of the $k$-th vertex of $D_i$ by $\frac{k}{2n}$, where $n$ is the total number of vertices. The resulting representation is an interval overlap representation [58].

$UI \rightarrow IN$: The proper containment can first be seen by examining a claw. This graph is an interval graph, but not a unit interval graph.

By definition unit interval graphs are interval graphs.

$UI \rightarrow IO$: The proper containment can first be seen by examining a $C_4$. This graph is an interval overlap graph, but not a unit interval graph.

To prove that a unit interval graph is a interval overlap graph, assume without loss of generality that each vertex $v$ is associated with an open unit interval $I(v)$. For each interval $I(v)$ we do the following. Check if $I(v)$ coincides with $I(u)$ for some $u \neq v$. If
so, then assign

$$
\epsilon = \frac{1}{2} \min\{|(v) \cap I(u)| : u \neq v, I(v) \cap I(u) \neq \emptyset\} > 0
$$

and translate by $\epsilon$ the interval $I(v)$ as well as every interval that is completely to the right of $I(v)$. This operation does not change the intersection pattern of the intervals, and ensures that $I(v)$ does not coincide with any other interval. Therefore the resulting unit interval representation is also an interval overlap representation.

$IN \rightarrow CH$: The proper containment can first be seen by examining a bipartite claw (see Figure 3.1). This graph is a chordal graph, but not an interval graph.

Chordal graphs by definition only have one family of forbidden induced subgraphs, namely $C_n$ with $n \geq 4$. This family is also a subset of the forbidden induced subgraphs of interval graphs [8]. Thus interval graphs are contained in chordal graphs.

$IN \rightarrow CA$: The proper containment can first be seen by examining a $C_4$. This graph is a circular-arc graph, but not an interval graph.

Let $G$ be an interval graph. So there exists an interval representation of $G$ where each interval is contained in $[0, 2\pi)$. We can then project each of these intervals onto the unit circle, giving a circular arc representation.

$IN \rightarrow 2DP$: The proper containment can first be seen by examining a $C_4$. This graph is a graph with dot product dimension 2, but not an interval graph.

Interval graphs are shown to have dot product dimension of at most 2 [28].

$IN \nrightarrow IO$: Consider the graph in Figure 3.7. This graph is an interval graph that fails to have an interval overlap representation [58].

![Figure 3.7: Example of Interval Graph that is Not Interval Overlap Graph](image)

$IO \nrightarrow IN$: Consider $C_4$. This is a known interval overlap graph that is one of the
forbidden subgraphs for interval graphs.

\( CH \rightarrow IO \): If this containment was true, then it would imply that interval graphs are contained in interval overlap graphs, which would be a contradiction.

\( IO \rightarrow CH \): Consider \( C_4 \). This is a known interval overlap graph that is a forbidden subgraph for chordal graphs by definition.

\( CH \rightarrow CA \): Consider the bipartite claw. This is a chordal graph that is one of the forbidden subgraphs for circular-arc graphs.

\( CA \rightarrow CH \): Consider \( C_4 \). This is a circular-arc graph that is a forbidden subgraph for chordal graphs by definition.

\( CH \rightarrow 2DP \): Consider the bipartite claw. This is a chordal graph with dot product dimension 3 [28].

\( 2DP \rightarrow CH \): Consider \( C_4 \). This graph has dot product dimension 2 and is a forbidden subgraph for chordal graphs by definition.

\( CA \rightarrow 2DP \): Consider the graph in Figure 3.8. This graph is a circular-arc graph that has dot product dimension 3 [45].

\( 2DP \rightarrow CA \): Consider \( C_4 \cup K_2 \). This graph has dot product dimension 2 and is a forbidden induced subgraph for circular-arc graphs.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example_graph.png}
\caption{Example of A Circular-Arc Graph with Dot Product Dimension 3}
\end{figure}
CHAPTER 4
BIPARTITE DOT PRODUCT GRAPHS

4.1 Definitions

A bipartite graph, \( G = (V, E) \), is an undirected graph where the set of vertices \( V \) can be partitioned into two sets, \( X \) and \( Y \), such that \( X \) and \( Y \) are disjoint independent sets. The relation of bipartite graphs to hypergraphs and directed graphs, as well as their applications to matching problems, leads us to consider bipartite dot product representations.

Let \( G = (V, E) \) be a bipartite graph. As we previously mentioned, we may write \( G = (X, Y, E) \) if \( V = X \cup Y \) with \( X \cap Y = \emptyset \) and for any vertices \( x, y \) if \( xy \in E(G) \), then \( x \in X \) and \( y \in Y \). A bipartite dot product representation of \( G \) is a function \( f : \{X, Y\} \to \mathbb{R}^k \) and a threshold \( t > 0 \) such that for any \( x \in X \) and \( y \in Y \) \( xy \in E(G) \) if and only if \( f(x) \cdot f(y) \geq t \). Since \( t > 0 \), we can use \( t = 1 \) without loss of generality. The minimum \( k \) such that a bipartite dot product representation exists for a bipartite graph \( G \) is the bipartite dot product dimension of \( G \), denoted \( bpd(G) \).

This definition of bipartite dot product graphs is similar to dot product graphs. This new definition allows us to examine classes of graphs where a relaxation on the minimum dimension can be given due to ignoring possible adjacencies. This definition also led to several general results for bipartite dot product graphs.

4.2 General Results for Bipartite Dot Product Dimension

The relaxation of dot product dimension that motivated our examination of bipartite dot product representations allows us to use the dot product dimension as an upper bound on the bipartite dot product dimension, as proven in Theorem 4.1. This bound also means that for any bipartite graph \( G \) there is bipartite dot product representation, since there is a dot product representation by Theorem 2.1.

**Theorem 4.1.** Let \( G \) be a bipartite graph. Then \( bpd(G) \leq \rho(G) \).
**Proof.** Let $G$ be a bipartite graph and $\rho(G) = k$. Then there exists $f : V \to \mathbb{R}^k$ and such that for any $x, y \in V$ $xy \in E$ if and only if $f(x) \cdot f(y) \geq 1$. But since $V = X \cup Y$ and $xy \in E$ if and only if $x \in X$ and $y \in Y$, then $f$ is also a bipartite dot product representation of $G$. Thus $bpd(G) \leq \rho(G)$.

This upper bound for the bipartite dot product dimension can also be found using graph structures, namely bicliques. A **biclique** of a graph $G$ is a subgraph $H$ of $G$ with $V(H) = X \cup Y$ where $X$ and $Y$ are each independent sets and every vertex $x \in X$ is adjacent to every vertex $y \in Y$. Thus the minimum number of bicliques of $G$ such that each edge of $G$ is contained in at least one biclique of $G$ is the **biclique cover number** of $G$, denoted $bc(G)$. Biclique cover number has been extensively studied, as seen in [1, 14, 15, 24, 46, 60].

Theorem 4.2 proves that the biclique cover number of $G$ bounds the bipartite dot product dimension of $G$.

**Theorem 4.2.** Let $G$ be a bipartite graph. Then $bpd(G) \leq bc(G)$.

**Proof.** Suppose $bc(G) = k$. Without loss of generality, we can label the bicliques $B_1, \ldots, B_k$. Define $f : X \cup Y \to \mathbb{R}^k$ such that for any $v \in X \cup Y$ $f(v)_i = 1$ if $v \in B_i$ and 0 otherwise. Then for any $x \in X$ and $y \in Y$, $f(x) \cdot f(y) \geq 1$ if and only if $x, y \in B_i$ for some $i$. Thus $f$ is a bipartite dot product dimension of $G$.

One of the key characteristics of dot product representations is their hereditary property. As shown in Theorem 4.3, bipartite dot product representations also have the hereditary property. This property allows us to characterize graphs with bipartite dot product dimension $k$ via forbidden induced subgraphs.

**Theorem 4.3.** Let $G = (X, Y, E)$ be a bipartite graph and $G'$ be an induced subgraph of $G$. Then $bpd(G') \leq bpd(G)$.

**Proof.** Let $bpd(G) = k$. Let $f : X \cup Y \to \mathbb{R}^k$ be a $k$-dot product representation of $G$ and $G' = (X', Y', E')$ be an induced subgraph of $G$. Then $f' : X' \cup Y' \to \mathbb{R}^k$ defined by $f$ restricted to $X' \cup Y'$ is still a $k$-bipartite dot product representation.
4.3 Characterization of 1-Bipartite Dot Product Graphs

In addition to the general results, bipartite dot product dimension can be determined for specific graphs. In particular, we will characterize the graphs of bipartite dot product dimension 1 by their forbidden subgraphs. We will show that our list is both necessary and sufficient.

The subsequent subsections will accomplish this characterization. We will first identify the structures that prohibit bipartite dot product dimension of 1. Then we will algorithmically show that, provided that there are no forbidden substructures in $G$, we can create a 1-bipartite dot product representation of $G$.

4.3.1 Forbidden Subgraphs of 1-Bipartite Dot Product Graphs

Our characterization will begin by showing a list of two graphs that have bipartite dot product dimension of 2. These graphs are $3K_2$ and $P_5$. These forbidden subgraphs can be seen in Figure 4.1. The proof that their bipartite dot product dimension is not 1 is shown in Lemmas 4.4 and 4.5.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4_1.png}
\caption{Forbidden Subgraphs of 1-Bipartite Dot Product Graphs.}
\end{figure}

\textbf{Lemma 4.4.} Let $G = 3K_2$. Then the $bpd(G) = 2$.

\textit{Proof.} Suppose that $bpd(G) = 1$. Then there exists a function $f : X \cup Y \rightarrow \mathbb{R}$ such that $f(x) \cdot f(y) \geq 1$ if and only if $xy \in E(G)$. Label the vertices of $G$ as shown in Figure 4.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4_2.png}
\caption{A Labeling of $3K_2$.}
\end{figure}

By the definition of $f$, $f(x_i) \cdot f(y_j) \geq 1$ if $i = j$ and $f(x_i) \cdot f(y_j) < 1$ if $i \neq j$.

Without loss of generality, we assume that $f(x_1) > 0$ and $f(x_1) \geq f(x_k)$ for $k \in \{2, 3\}$. This also implies that $f(y_1) \geq \frac{1}{f(x_1)} > 0$.

If $f(x_2) > 0$, then $f(y_2) \geq \frac{1}{f(x_2)} > 0$. But $f(x_1) > f(x_2)$ so $f(y_2) \geq \frac{1}{f(x_2)} \geq 1$. This implies that $f(x_2) \cdot f(y_2) \geq 1$, which is a contradiction since $x_1y_2 \notin E(G)$. This implies that $f(x_2), f(y_2) < 0$. Similarly it can be shown that $f(x_3), f(y_3) < 0$.

Without loss of generality, we now assume that $f(x_2) \geq f(x_3)$. So $f(y_2) \leq \frac{1}{f(x_2)} \leq \frac{1}{f(x_3)}$, which implies that $f(y_2) \cdot f(x_3) \geq 1$. But this is another contradiction of $x_3y_2 \notin E(G)$. Thus our initial assumption that $bpd(G) = 1$ is false, so $bpd(G) \geq 2$.

To prove that $bpd(G) \leq 2$, define $g : X \cup Y \to \mathbb{R}^2$ such that

$$
x_1, y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
x_2, y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
x_3, y_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
$$

A brief examination shows that this bipartite representation is valid.

Thus $bpd(G) = 2$. □

**Lemma 4.5.** Let $G = P_5$. Then the $bpd(G) = 2$.

**Proof.** Suppose that $bpd(G) = 1$. Then there exists a function $f : X \cup Y \to \mathbb{R}$ such that $f(x) \cdot f(y) \geq 1$ if and only if $xy \in E(G)$. Label the vertices of $G$ as shown in Figure 4.3.

![Figure 4.3: P_5 Labeling](image)

Let $f(y_1) = \alpha$. Then $f(x_1), f(x_2) \geq \frac{1}{\alpha}$ in order for $f(x_1) \cdot f(y_1), f(x_2) \cdot f(y_1) \geq 1$. Similarly since $x_3y_1 \notin E$, then $f(x_3) < \frac{1}{\alpha}$. For $f(y_2) \cdot f(x_2) \geq 1$, it is necessary that
To prove that $bpd(G) \leq 2$, we just need to note that the induced subgraphs on $\{x_1, x_2, y_2\}$ and $\{x_2, x_2, y_2\}$ are both $K_{1,2}$. Therefore the biclique cover number of a $P_5$ is 2, and thus $bpd(G) \leq 2$ by Theorem 4.2.

Thus $bpd(G) = 2$.

### 4.3.2 Constructive Algorithm for 1-Bipartite Dot Product Graphs

Based on the forbidden subgraphs shown in Lemmas 4.4 and 4.5, we developed Algorithm 1. This algorithm takes a bipartite graph $G = (X, Y, E)$ along with the vertex degrees and returns a 1-bipartite dot product representation of $G$, namely $F$. The returned representation will fail to be valid only if $bpd(G) \geq 2$.

Algorithm 1 is built on identifying the vertex of $G$ with maximum degree. This vertex is assigned a value equivalent to its degree. The other vertices in the same partite set as the vertex of maximum degree are assigned values relative to the cardinality of the intersection of the neighborhood of each vertex and the neighborhood of the vertex of maximum degree. The vertices in the other partite set are assigned values based on the values assigned the vertices adjacent to each one.

The validity of Algorithm 1 is shown in Theorem 4.6. An example of how Algorithm 1 works and an example of how Algorithm 1 fails if $P_5$ is present are given after the proof of Theorem 4.6.

**Theorem 4.6.** Let $G$ be a bipartite graph. If $G$ does not contain $3K_2$ or $P_5$ as induced subgraph, then Algorithm 1 returns a 1-bipartite dot product representation of $G$.

**Proof.** Let $G = (X, Y, E)$ be a bipartite graph with no induced $3K_2$ or $P_5$.

First we will consider when $\hat{x}\hat{y} \in E(G)$ for some $\hat{x} \in X$ and $\hat{y} \in Y$. There are three cases to consider:

**Case 1:** Suppose $\hat{y} \in N(x_1)$.

Then $N(x_1) \cap N(\hat{x}) \neq \emptyset$. So $F(\hat{y}) = \frac{1}{\min(F(x_j))}$ such that $\hat{y} \in N(x_1) \cap N(x_j)$. But since $F(\hat{x}) \geq \min(F(x_j))$, $F(\hat{y}) \geq \frac{1}{F(\hat{x})}$. Thus $F(\hat{x}) \cdot F(\hat{y}) \geq 1$.

**Case 2:** Suppose $\hat{y} \notin N(x_1)$ and $N(x_1) \cap N(\hat{x}) \neq \emptyset$.

Since $N(x_1) \cap N(\hat{x}) \neq \emptyset$, there exists $y_i \in Y$ such that $y_i \in N(x_1) \cap N(\hat{x})$. By definition of $x_1$, $deg(x_1) \geq deg(\hat{x})$. That implies that $|N(x_1)| \geq |N(x_1) \cap N(\hat{x})| + 1$. So
Algorithm 1 Returns a 1-bipartite dot product representation of $G$ (F), and fails if $\text{bpd}(G) \geq 2$.

1: **INPUT:** $G=(X,Y,E)$ and $\text{deg}(v)$ for all $v \in X \cup Y$
2: Vertices begin with no labels.
3: Define $F$ as a 1-bipartite dot product representation of $G$.
4: if A vertex $v$ has $\text{deg}(v) = 0$ then
5: Define $F(v) = 0$.
6: else
7: for An unlabeled vertex of maximum degree do
8: Label this vertex as $x_1$ and the independent set $X$.
9: Label the remaining vertices in $X$ such that for $i > j$,
10: $|N(x_i) \cap N(x_1)| \geq |N(x_j) \cap N(x_1)|$.
11: if There exists $k$ such that for $i \geq k$, $N(x_i) \cap N(x_1) = \emptyset$, and $\text{deg}(x_i) > 0$ then
12: Label the vertex of maximum degree of such vertices as $x_k$
13: Label the vertices in $X$ with $N(x_i) \cap N(x_k) \neq \emptyset$ such that for $i > j$,
14: $|N(x_i) \cap N(x_k)| \geq |N(x_j) \cap N(x_k)|$.
15: end if
16: for All vertices in $Y$ do
17: Label the vertices $y_1, y_2, \ldots, y_n$ where $n = |Y|$.
18: end for
19: for Each $x \in X$ do
20: Define $F(x_i) = \text{deg}(x_i)$ and $F(x_i) = |N(x_i) \cap N(x_1)|$ for $i < k$.
21: Define $F(x_k) = -\text{deg}(x_k)$ and $F(x_j) = -|N(x_k) \cap N(x_j)|$ for $j \geq k$.
22: end for
23: for $y \in Y$ do
24: for $y_i \in N(x_1)$ do
25: Define $F(y_i) = \frac{1}{\min(F(x_j))}$ where $y_i \in N(x_j) \cap N(x_1)$.
26: end for
27: for $y_m \in N(x_k)$ do
28: Define $F(y_m) = \frac{1}{\max(F(x_j))}$ where $y_m \in N(x_j) \cap N(x_k)$.
29: end for
30: end if
31: return $F(v)$ for all $v \in V$

there exists $y_k \in Y$ such that $y_k \in N(x_1)$ and $y_k \notin N(\hat{x})$. Thus $y_k x_1 y_i \hat{x} y$ is a $P_5$. So $G$ has a $P_5$ as an induced subgraph, which is a contradiction.

Case 3: Suppose that $\hat{y} \notin N(x_1)$ and $N(x_1) \cap N(\hat{x}) = \emptyset$.

If $\hat{y} \notin N(x_1)$, then there exists an $x_k$ explained in Lines 10-12 such that $\hat{y} \in N(x_k)$. So $F(\hat{y}) = \frac{1}{\max(F(x_j))}$ such that $\hat{y} \in N(x_k) \cap N(x_j)$. But since $\hat{F}(\hat{x}) \geq \max(F(x_j))$, $F(\hat{y}) = \frac{1}{\hat{F}(\hat{x})}$. Thus $F(\hat{x}) \cdot F(\hat{y}) \geq 1$.

Now consider when $\hat{x} y \notin E(G)$ for some $\hat{x} \in X$ and $\hat{y} \in Y$. 

Case 1: Suppose either $\hat{y}$ or $\hat{x}$ is an isolated vertex.

In either case, $F(\hat{x}) \cdot F(\hat{y}) = 0 < 1$.

Case 2: Suppose $\hat{y} \in N(x_1)$ and $N(x_1) \cap N(\hat{x}) \neq \emptyset$.

First suppose that there exists $x_j \in X$ such that $\hat{y} \in N(x_j)$ and $|N(x_1) \cap N(x_j)| \leq |N(x_1) \cap N(\hat{x})|$. Then there exists $y_k \in N(x_1) \cap N(\hat{x})$ such that $y_k \notin N(x_j)$. Then $x_jy_1y_k\hat{x}$ is $P_3$ that is an induced subgraph of $G$. This is a contradiction.

So for any $x_j \in X$ such that $\hat{y} \in N(x_j)$, $|N(x_1) \cap N(x_j)| > |N(x_1) \cap N(\hat{x})|$. Then $F(\hat{y}) \leq \frac{1}{|N(x_1) \cap N(x_j)|} < \frac{1}{|N(x_1) \cap N(\hat{x})|}$. Thus $F(\hat{x})F(\hat{y}) < 1$.

Case 3: Suppose $\hat{y} \in N(x_1)$ and $N(x_1) \cap N(\hat{x}) = \emptyset$.

This involves the reordering from Lines 10-12 in the algorithm. If that reordering occurred at least twice, there exists non-isolated vertices $x_{k_1}, x_{k_2} \in X$ such that $N(x_1) \cap N(x_{k_1}) = N(x_1) \cap N(x_{k_2}) = N(x_{k_1}) \cap N(x_{k_2}) = \emptyset$. Then $3K_2$ is an induced subgraph of $G$, which would be a contradiction. Thus the reordering can be done at most once.

In this case, $F(\hat{x}) < 0$ and $F(\hat{y}) > 0$. So $F(\hat{x}) \cdot F(\hat{y}) < 0 < 1$.

Case 4: Suppose $\hat{y} \notin N(x_1)$ and $N(x_1) \cap N(\hat{x}) \neq \emptyset$.

Since $\hat{y}$ is not adjacent to either $x_1$ or $\hat{x}$ and $\hat{y}$ is not an isolated vertex, there exists $x_i \in X$ such that $\hat{y}x_i \in E$. Similarly there exists $y_i \in N(x_1) \cap N(\hat{x})$ because $N(x_1) \cap N(\hat{x}) \neq \emptyset$.

If $N(x_1) \cap N(x_i) \neq \emptyset$ or $N(x_i) \cap N(\hat{x}) \neq \emptyset$, then $P_3$ is an induced subgraph of $G$, which would be a contradiction. Thus $N(x_i) \cap (N(x_1) \cup N(\hat{x})) = \emptyset$. Then we can label $x_i$ as $x_k$ and $F(\hat{y}) < 0$ by the lines 20 and 27 of the algorithm. But $F(\hat{x}) > 0$ since $N(x_1) \cap N(\hat{x}) \neq \emptyset$. Therefore $F(\hat{x}) \cdot F(\hat{y}) < 0 < 1$.

Case 5: Suppose $\hat{y} \notin N(x_1)$ and $N(x_1) \cap N(\hat{x}) = \emptyset$.

Since $\hat{y}$ is not isolated, there exists $x_k$ such that $\hat{y} \in N(x_k)$. Similarly there exists $\hat{y} \in N(\hat{x})$ since $\hat{x}$ is not isolated. It can also be noted that since $x_1$ is the vertex of maximum degree there exists $y_1 \in N(x_1)$ such that $y_1 \notin N(x_k)$.

If $\hat{y} \notin N(x_k)$, then $3K_2$ is an induced subgraph of $G$, which would be a contradiction. Thus $\hat{y} \in N(x_k)$. In this case, we need to consider the $\text{deg}(x_k)$ and $\text{deg}(\hat{x})$. If $\text{deg}(\hat{x}) \geq \text{deg}(x_k)$, then there exists $y_2 \in N(\hat{x})$ and $y_2 \in N(x_k)$. But this means that $P_3$ is an induced subgraph of $G$, which would be a contradiction. Thus $\text{deg}(\hat{x}) < \text{deg}(x_k)$. We can then assume that $x_k$ is the $x_k$ in line 8 of the algorithm.

Thus $F(\hat{x}) = -|N(x_k) \cap N(\hat{x})| > -\text{deg}(x_k)$ and $F(\hat{y}) = \frac{1}{F(x_k)} = -\frac{1}{\text{deg}(x_k)}$. Therefore
$$F(\hat{x}) \cdot F(\hat{y}) = -|N(x_k) \cap N(\hat{x})| \cdot \frac{1}{\deg(x_k)} \leq \frac{\deg(x_k)}{\deg(x_k)} = 1. \quad \square$$

4.3.3 Example and Nonexample of How Algorithm 1 Works

For an example of how Algorithm 1 works, we will let $G = (X, Y, E)$ be the bipartite graph $H_1$ in Figure 4.4. In $H_1$, the grey vertices are in $X$ and the black vertices are in $Y$.

First, the isolated vertices can be assigned vectors $[0]$, as designated in Lines 4 and 5. We will also label the vertices, as designated Lines 7-16. Figure 4.5 shows these assignments and labels. It can be noted that $x_3$ is $x_k$.

Next we will assign the vertices in $x \in X$ the vectors as explained in Lines 19-20. These assignments are seen in Figure 4.6.

Next we will assign the vertices in $y \in Y$ the vectors as explained in Lines 22-28. These assignments are seen in Figure 4.7.

A brief examination of these vectors shows that this 1-dot product representation of $H_1$ is valid.
For a nonexample of how Algorithm 1 works, we will let $G = (X, Y, E)$ be the bipartite graph $H_2$ in Figure 4.8. In $H_1$, the grey vertices are in $X$ and the black vertices are in $Y$. An examination of $H_2$ shows that $3K_2$ and $P_5$ are both induced subgraphs.

There are no isolated vertices in $H_2$ so we can skip to Line 7. We will label the
vertices in $X$, as designated Lines 7-16. Figure 4.9 shows these assignments and labels. It can be noted that $x_4$ is $x_k$.

![Figure 4.9: A Nonexample Graph for Algorithm 1. Vertices labeled.](image)

Next we will assign the vertices in $x \in X$ the vectors as explained in Lines 19-20. These assignments are seen in Figure 4.10. For $x_5$, we can assign $F(x_5) = 0$ because $|N(x_4) \cap N(x_5)|$ and $5 > 4$.

![Figure 4.10: A Nonexample Graph for Algorithm 1. Vectors assigned for $X$.](image)

Next we will assign the vertices in $y \in Y$ the vectors as explained in Lines 22-28. These assignments are seen in Figure 4.11. However, there is no assignment for $y_5$ since it is not in $N(x_1)$ or $N(x_4)$.

$F$ fails first because $y_5$ has no assignment. Further, any assignment of $y_5$ will not result in a dot product with [0] to be greater than or equal to 1. Next it can be noted that
it $F(x_3) \cdot F(y_1) = 1$, which is a contradiction of the nonadjacency of those vertices. Thus $F$ is not a valid representation when either a $3K_2$ or $P_5$ are induced subgraphs of the graph.

4.3.4 Primary Theorem

Our results combine to given us a forbidden induced subgraph characterization of 1-bipartite dot product graphs, as seen in Theorem 4.7.

**Theorem 4.7.** A bipartite graph $G$ is a 1-bipartite dot product graph if and only if $G$ has no induced $3K_2$ or $P_5$.

*Proof.* The necessity of the forbidden subgraphs are given from Lemmas 4.4 and 4.5. The sufficiency of the forbidden subgraphs is given by Theorem 4.6. 

4.4 Linear Algebra Relation to Bipartite Dot Product Dimension

Any bipartite graph can be thought of as a rectangular $(0, 1)$-matrix. Let $G = (X, Y, E)$ be a bipartite graph with $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_m\}$. Then the bipartite adjacency matrix of $G$, denoted $B(G)$ or simply $B$, is an $n \times m$ $(0, 1)$-matrix, where $b_{ij}$ is 1 if and only if $x_i y_j \in E(G)$. This perspective allows for combinatorial analysis of linear algebraic or other properties of $(0, 1)$-matrices. To understand these matrices, consider the bipartite graph in Figure 4.12.
The bipartite adjacency matrix of Figure 4.12 is

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\]

Because our bipartite graphs can be viewed as matrices, linear algebra and its theory can be used to analyze these graphs. We will use \( M_{m,n}(\mathbb{R}) \) to denote the set of all \( m \times n \) matrices with real entries.

We will utilize the linear algebra parameter real rank to analyze the bipartite adjacency matrix. If \( A \in M_{m,n}(\mathbb{R}) \), the real rank of \( A \), denoted \( \text{rank}(A) \), is the largest number of columns of \( A \) that constitute a linearly independent set [44]. This set of columns is not unique, but the cardinality of this set is unique.

The real rank of a matrix is also equivalent to the factor rank of the matrix. The factor rank of \( A \in M_{m,n}(\mathbb{R}) \) is the minimum integer \( k \) such that \( A = CF \), where \( C \in M_{m,k}(\mathbb{R}) \) and \( F \in M_{k,n}(\mathbb{R}) \). The comparison of factor rank and the bipartite dot product dimension leads to the following theorem.

**Theorem 4.8.** Let \( G \) be a bipartite graph and \( B \) the bipartite adjacency matrix of \( G \). Then \( \text{bpd}(G) \leq \text{rank}(B) \).
Proof. Suppose that the bipartite sets of $G$ are $X$ and $Y$, with $|X| = m$ and $|Y| = n$. Suppose that $\text{rank}(B) = k$. Then there exists real matrices $R \in M_{m,k}(\mathbb{R})$ and $S \in M_{k,n}(\mathbb{R})$ such that

$$B = RS = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,k} \\
 r_{2,1} & r_{2,2} & \cdots & \vdots \\
 \vdots & \vdots & \ddots & \vdots \\
 r_{m,1} & \cdots & \cdots & r_{m,k} \end{bmatrix} \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\
 s_{2,1} & s_{2,2} & \cdots & \vdots \\
 \vdots & \vdots & \ddots & \vdots \\
 s_{k,1} & \cdots & \cdots & s_{k,n} \end{bmatrix}.$$ 

Since $B$ requires an arbitrary assignment of the vertices such that $X = x_1, \cdots, x_m$ and $Y = y_1, \cdots, y_n$, we can assign the vertices $x_i \in X$ the vector $\vec{x}_i = (r_{i,1}, \cdots, r_{i,k})^T$ and the vertices $y_i \in X$ the vector $\vec{y}_i = (s_{1,i}, \cdots, s_{k,i})^T$. By definition of matrix multiplication, $\vec{x}_i \cdot \vec{y}_j = B_{i,j}$, which is 1 if $x_iy_j \in E$ and 0 otherwise. Thus there exists a $k$-bipartite dot product representation of $G$ and $\text{bpd}(G) \leq k$.

This bound however is not tight for all graphs. An example of this is the graph $2K_2$. This graph and a 1-bipartite dot product representation of it can be seen in Figure 4.13.

![Figure 4.13: A 2K2 and its 1-Bipartite Dot Product Representation.](image)

However the bipartite adjacency matrix of $2K_2$ is

$$\begin{bmatrix} 1 & 0 \\
 0 & 1 \end{bmatrix}.$$ 

The real rank of $B$ is 2, which is greater than the bipartite dot product dimension.
CHAPTER 5
PROBE DOT PRODUCT GRAPHS

5.1 Introduction

Probe interval graphs were first introduced by Zhang in the context of genome research[84]. A graph is a probe interval graph if its vertex set can be partitioned into two sets, probes $P$ and nonprobes $N$, such that $N$ is independent and edges can be added between vertices of $N$ in such a way that the resulting graph is an interval graph. This definition can of course readily be generalized to any graph property $\mathcal{C}$: A graph $G$ is probe $\mathcal{C}$ if its vertex set can be partitioned into two sets, probes $P$ and nonprobes $N$, such that $N$ is independent and edges can be added between a subset of $N$ in such a way that the resulting graph $G'$ has property $\mathcal{C}$. An extension of $G = (P \cup N, E)$ is a graph $G' = (V', E')$ where $V' = P \cup N$ and $E' = E \cup E^*$, where $E^*$ is a set of edges whose endpoints are members of $N$. A valid $\mathcal{C}$ extension of $G$ is an extension of $G$ that has property $\mathcal{C}$. If a partition into probes and nonprobes is given, we talk about a partitioned probe class, otherwise about an unpartitioned one.

As we consider the graphs on $n$ vertices, we notice that the dot product dimension appears to increase as edges are added until some point where adding edges causes the dot product dimension to decrease. Consider the example of this for graphs on 4 vertices. If there is only one edge, then the graph is a $K_2$ with two isolated vertices, which has dot product dimension 1. If there are exactly two edges, then the two possible graphs are $2K_2$ and $P_3$ with an isolated vertex, both of which have dot product dimension 1. If there are exactly three edges, then the two possible graphs are $K_3$ and $P_4$. The maximum dot product dimension of these is 2, for the $P_4$. If there are exactly four edges edges, then the two possible graphs are $E_1$ in Figure 5.1 and $C_4$. The maximum dot product dimension of these is 2, for the $C_4$. If there are exactly five edges, then the only possible graph is $E_2$ in Figure 5.1 which has dot product dimension 1. If there are exactly 6 edges, then the only possible graph is $K_4$, which has dot product dimension 1.
This action of adding edges to gain a specific dot product dimension leads us to consider probe dot product graphs. This approach of considering dot product dimension by adding specific edges is completely different than other approaches to dot product dimension. To date, the study of dot product representations has focused on the hereditary nature of dot product dimension, specifically considering the deletion and addition of vertices. By instead considering adding edges, we are instead looking on how the density of a graph impacts the dot product dimension. This approach may provide additional insights into Conjecture 2.19.

5.2 Complete Bipartite Graphs

We consider complete bipartite graphs, which have been shown to achieve the bound in Conjecture 2.19 [28] if the partite sets are equal. We ask for “what dimensions $k$ is a $K_{m,n}$ a probe $k$-dot product graph?” we prove that via the addition of certain edges among vertices of smaller partite sets, a $K_{m,n}$ is a probe $k$-dot product graph for $k$ such that $1 \leq k \leq \min m, n$, main result of this section, Theorem 5.2. This theorem is the first of its kind.

The proof of this theorem utilizes Theorem 2.14 and Lemma 5.1, which focuses on the presence of universal vertices and their impact of dot product dimension. A universal vertex of a graph $G$ is a vertex which is adjacent to every other vertex in $G$.

Lemma 5.1. Let $G = (V, E)$ with a universal vertex $v \in V$. If $\rho(G - v) = k$ and there exists a strictly positive $k$-dot product representation of $G - v$, then $\rho(G) = k$.

Proof. Suppose that $G - v$ has a strictly positive $k$-dot product representation, namely $f(V \setminus v)$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{example_graphs.png}
\caption{Example Graphs on Four Vertices.}
\end{figure}
An alternate definition of dot product is \( \vec{x} \cdot \vec{y} = ||\vec{x}|| \cdot ||\vec{y}|| \cdot \cos(\theta) \), where \( \theta \) is the angle between the vectors. Because the \( k \)-dot product representation of \( G - v \) is strictly positive, all vertices are within \( \pi/4 \) of the vector \( \vec{v} = \{1,1,\cdots,1\}^T \).

Let \( f(u) \) be the vector in the representation of \( G - v \) with minimum Euclidean length. Then \( \vec{v} \cdot f(u) \geq k \cdot ||f(u)|| \cdot \frac{\sqrt{2}}{2} \). We now form a \( (k) \)-dot product representation \( F \) of \( G \) based on the representation \( f \). Let

\[
F(x) = \begin{cases} f(x) & x \neq v \\ \sqrt{2} / k \cdot ||f(u)||, & u = v \\ \end{cases}
\]

A simple check shows that this representation is a \( k \)-dot product representation of \( G \). Since \( \rho(G) = \rho(G - v) = k \), \( \rho(G) = k \).

**Theorem 5.2.** Let \( G = K_{n,m} \). Then \( G \) is a probe \( k \)-dot product graph for all \( 1 \leq k \leq \min\{n,m\} \).

**Proof.** Suppose \( G = (X \cup Y, E) \) is a complete bipartite graph with \( |X| = n \) and \( |Y| = m \). Without loss of generality, we may assume that \( n \leq m \). It will suffice to show that there exists a set of edges in \( \tilde{G} \) such that when added to \( G \) there exists a \( k \)-dot product representation of \( G \).

Based on Theorem 2.14, \( \rho(G) = n \). Thus \( G \) is trivially a probe \( n \)-dot product graph since no edges need to be added to produce a \( n \)-dot product representation of \( G \).

We can label the vertices of \( X \) as \( x_1, x_2, \cdots, x_n \). Since \( X \) is an independent set of vertices, we can add any subset of edges between the vertices in \( X \).

If \( E' = E \cup \{x_1x_i|i = 2,3,\cdots,n\} \), then \( x_1 \) is a universal vertex in \( G' \). Since there exists a positive \( n_1 \)-dot product representation for any \( K_{n_1,n_2} \) with \( n_1 \leq n_2 \), the dot product dimension of \( G' \) is \( \rho(G') = n - 1 \).

This process can be repeated for \( x_i \) to reduce the dot product dimension by 1 each time until \( G' \) is a complete split graph with \( X \) a clique. Thus \( G \) is a probe \( k \)-dot product graph for all \( 1 \leq k \leq n \). \qed
Because complete bipartite graphs are among the graphs with the conjectured bound on dot product dimension from Conjecture 2.19, the study of probe dot product graphs may provide additional insight into proving the conjecture.

This focus on probe \( k \)-dot product graphs has led to multiple results that we will present in this chapter. A forbidden induced subgraph characterization of partitioned probe 1-dot product graphs will be given, as will a theorem of some of the properties of unpartitioned probe 1-dot product graphs. We will also give a forbidden induced subgraph characterization of unpartitioned probe 1-dot product graphs when the initial graph is either a forest or bipartite. Finally, we will show in this chapter that probe 1-dot product graphs can be recognized in polynomial time.

5.3 Trivially Perfect Graphs

In 1978, Golumbic [35] introduced trivially perfect graphs. A graph \( G \) is \textit{trivially perfect} if for each subgraph \( H \) of \( G \), the size of a maximal independent set in \( H \) equals the number of maximal cliques in \( H \). The trivially perfect graphs are also called quasi-threshold graphs [83] or comparability graphs of trees [81, 82]. Several characterizations of trivially perfect graphs are known, some of which are given below.

\textbf{Theorem 5.3.} [35, 83] The following are equivalent for a graph \( G = (V,E) \)

1. \( G \) is a trivially perfect graph;
2. \( G \) is \((C_4,P_4)\)-free;
3. for all edges \( xy \in E : N[x] \subset N[y] \) or \( N[y] \subset N[x] \);
4. \( G \) is the intersection graph of a set of intervals on the straight line such that every two intervals either are disjoint or one contains the other;
5. There is a rooted directed forest \( F = (V,H) \) such that \( x,y \in E \) if and only if in \( F \) there is either a path from \( x \) to \( y \) or vice versa;
6. Each connected induced subgraph of \( G \) has a universal vertex.

From the forbidden subgraph characterization of 1-dot product graphs, it is clear that 1-dot product graphs are exactly the \( 3K_2 \)-free trivially perfect graphs. This is also true for the probe version.
Theorem 5.4. A graph $G$ is a probe 1-dot product graph if and only if it is $3K_2$-free and a probe trivially perfect graph.

Proof. The only if-part is obvious because any induced $3K_2$ in $G$ is an induced $3K_2$ or an induced $P_4$ in any extension of $G$.

For the if-part, let $G = (V,E)$ be $3K_2$-free and probe trivially perfect. Note that every induced subgraph of $G$ is also $3K_2$-free and probe trivially perfect. We claim that for any trivially perfect extension $G' = (P \cup N, E \cup E')$ of $G$, there exists $E'' \subset N \times N$ such that $G'' = (P \cup N, E \cup E'')$ is a 1-dot product extension of $G$.

If $G$ is connected, then $G'$ is connected and by Theorem 5.3, $G'$ has a universal vertex $v$. By induction, applied to $H = G - v$ and $H' = G' - v$, there exists $F'' \subseteq (N-v) \times (N-v)$ such that $H'' = (\{ P - v \} \cup \{ N - v \}, E(G-v) \cup E'')$ is a 1-dot product extension of $G - v$. Then, clearly, $G'' = (P \cup N, E \cup E'')$ with $E = F''$ if $v \in P$ and $E'' = F'' \cup \{ vx | x \in N - v \}$ if $v \in N$ is a 1-dot product extension of $G$. \qed

In 2009, H.N. de Ridder et al characterized probe trivially perfect graphs [5]. Because Theorem 5.4 shows a relationship between probe trivially perfect graphs and probe 1-dot product graphs, these characterizations will be used throughout our chapter. The characterization of partitioned probe trivially perfect graphs is in Theorem 5.5 and the characterization of the unpartitioned case in Theorem 5.6.

Theorem 5.5. [5] Let $G = (P \cup N, E)$ be an arbitrary partitioned graph with $N$ independent. The following statements are equivalent:

1. $G$ is a probe trivially perfect;

2. In $G$ every induced $C_4$ has two vertices, every $P_4$ has a midpoint and an endpoint, and every induced $P_5$ has three vertices in $N$;

3. $G$ is free of the induced subgraphs in Figure 5.2

4. Each connected induced subgraph $H$ of $G$ has a universal vertex or a vertex in $N \cap V(H)$ adjacent to all vertices in $P \cap V(H)$.

Theorem 5.6. [5] The following statements are equivalent for any graph $G$:
Figure 5.2: Forbidden Induced Subgraphs of Partitioned Probe Trivially Perfect Graphs. The white nodes are non-probes and the black nodes are probes.

1. \( G \) is probe trivially perfect;

2. \( G \) admits an independent set \( N \) such that in \( G \) every induced \( C_4 \) has two vertices in \( N \), every induced \( P_4 \) has a midpoint and an endpoint in \( N \), and every induced \( P_5 \) has three vertices in \( N \);

3. \( G \) admits an independent set \( N \) such that each connected induced subgraph \( H \) has a universal vertex or a vertex in \( N \cap V(H) \) adjacent to all vertices in \( P \cap V(H) \);

4. \( G \) admits an independent set such that \( G^* \) (with respect to \( N \)) is trivially perfect.

### 5.4 Probe 1-Dot Product Graphs

We will now characterize the partitioned probe 1-dot product graphs. We will do this utilizing a Boolean 2-SAT formula rather than a constructive algorithm as we did in Chapter 4 for 1-bipartite dot product graphs. A significant benefit of using a Boolean 2-SAT formula is that it can be used to create a polynomial (at worst) time recognition algorithm.

Boolean 2-SAT formulas use Boolean variables, \( TRUE \) or \( FALSE \), and Boolean operators, \( \lor \) and \( \land \), to create Boolean clauses. The conjunction of Boolean clauses is the joining of Boolean clauses using \( \land \). Thus the conjunction of multiple clauses is \( TRUE \) if and only if each of the clauses is \( TRUE \).

Our 2-SAT formula is based on some given partition of the vertices of a graph \( G \) into \( N \) and \( P \). The formula will check for the validity of the partition, as well as examine what vertices from each induced \( P_4 \), \( C_4 \), and \( P_5 \) are in \( N \). The formula (DPG1) is given as follows.

- For any vertex \( v \in V \) of \( G \), create a Boolean variable \( x_v \),

- for each edge \( ab \) of \( G \), \( (\bar{x}_a \lor \bar{x}_b) \) is a clause, the edge-clause for \( ab \),
• for each $C_4 = abcd$ of $G$, $(x_a \lor x_b)$ and $(x_c \lor x_d)$ are two clauses, the $C_4$-clauses for that $C_4$,

• for each $P_4 = abcd$ of $G$, $(x_a \lor x_d)$ and $(x_b \lor x_c)$ are two clauses, the $P_4$-clauses for that $P_4$,

• for each $P_5 = abcde$ of $G$, $(x_c)$ is a clause, the $P_5$-clause for that $P_5$.

The formula $DP1(G)$ is the conjunction of all edge-clauses, all $C_4$-clauses, all $P_4$-clauses, and all $P_5$-clauses. We will show that $G$ is a partitioned probe 1-dot product graph if and only if $DP1(G)$ is satisfiable and $G$ does not contain any induced $3K_2$. Note that $DP1(G)$ has at most $O(|V| \cdot |E|^2)$ clauses and can be constructed in time $O(|V| + |V| \cdot |E|^2)$.

If $G = (P \cup N, E)$ is a partitioned probe 1-dot product graph and $G' = (P, N, E \cup E')$ is a 1-dot product extension of $G$, then for every two nonprobes $x, y \in N$ belonging to an induced $C_4$ or $P_4$ in $G$, clearly $xy \in E'$. We define a particular extension we will use in what follows.

**Definition 5.7.** Let $G = (P \cup N, E)$ be a partitioned graph with $N$ independent. Then $G^* = (P \cup N, E \cup E^*)$ is the extension of $G$ with $xy \in E^* \setminus E$ if and only if $x, y \in N$ and $x, y$ belong to an induced $C_4$ or $P_4$ in $G$.

If $G = (P \cup N, E)$ is a probe 1-dot product graph and if $G^*$ is defined as a 1-dot product graph, then $G^*$ is a valid extension of $G$. Indeed, one of our main results is that $G$ is a partitioned probe 1-dot product graph if $G^*$ is a 1-dot product graph.

*Figure 5.3: Forbidden Subgraphs of Partitioned 1-Dot Product Probe Graphs.*
The white nodes are non-probes and the black nodes are probes.
Theorem 5.8. Let $G = (P \cup N, E)$ be an arbitrary partitioned graph with $N$ independent. The following statements are equivalent:

(a) $G$ is a partitioned probe $1$-dot product graph;

(b) $G$ is $3K_2$-free, and $DP1(G)$ is satisfied by assigning $x_v := \text{true}$ if $v \in N$ and $x_v := \text{false}$ otherwise;

(c) $G$ is $3K_2$-free and each induced $P_4$ must have a midpoint and an endpoint in $N$, each induced $C_4$ must have two vertices in $N$, and each induced $P_5$ must have $3$ vertices in $N$;

(d) $G$ does not contain any of the graphs in Figure 5.3;

(e) $G$ has at most two connected components of any induced subgraphs and each connected induced subgraph $H$ of $G$ has a universal vertex or a vertex in $N \cap V(H)$ adjacent to all vertices in $P \cap V(H)$.

(f) The extension $G^*$ of $G$ is a $1$-dot product graph.

Proof. (a) $\Rightarrow$ (b) Assume that $G$ is a partitioned probe $1$-dot product graph. Then by Theorem 5.4, $G$ is $3K_2$-free and a probe trivially perfect graph. By Theorem 5.6, in every partitioned trivially perfect graph each induced $P_4$ must have a midpoint and an endpoint in $N$, each induced $C_4$ must have two vertices in $N$, and each induced $P_5$ must have $3$ vertices in $N$. Therefore, these properties must also be satisfied for any probe $1$-dot product graph. Since $N$ is independent, every edge in $G$ must contain a vertex in $P$. Thus every clause of $DP1(G)$ is satisfied by the assignment given in (b).

(b) $\Rightarrow$ (d) It is easy to check that $DP1(G)$ is not satisfied by the assignment given in (b) for any of the graphs in Figure 5.3. Thus there is no such induced subgraph in $G$.

(c) $\Leftrightarrow$ (d) It can first be noted that the different possible partitions of a $3K_2$ into probes and nonprobes $N$, with $N$ independent, are given in Figure 5.4.

Examining these partitioned graphs, it can be noted that their corresponding extensions have either a $3K_2$ or $P_4$. Either case would be a forbidden subgraph of a 1-dot product graph. Thus any partition of $3K_2$ is a forbidden subgraph of a probe 1-dot product graph. Therefore by Theorem 5.6 and Theorem 5.4, a graph is a partitioned probe 1-dot product graph if and only if it is a partitioned $3K_2$ or a partitioned probe trivially perfect graph. Thus the forbidden subgraphs of partitioned probe 1-dot product
Figure 5.4: Partitions of $3K_2$. The white node are non-probes and the black nodes are probes.

graphs are the graphs in Figure 5.4 and Figure 5.2. The union of these graphs is Figure 5.3.

$(d) \Rightarrow (e)$ The first condition is trivially proved based on the forbidden $3K_2$ among the forbidden subgraphs in Figure 5.3.

The proof of the second condition of $(e)$ is by induction on $|V| = |P| + |N|$. Assume that $(d) \Rightarrow (e)$ for all graphs with strictly fewer vertices than $G$. Then in particular all proper connected induced subgraphs $H$ of $G$ have a universal vertex or a vertex in $N \cap V(H)$ adjacent to all vertices in $P \cap V(H)$. So we only need to show that when $G$ is connected, it has a universal vertex or a nonprobe adjacent to all probes. Assume that $G$ is connected, if not then we may argue for each component.

If $|N| \leq 1$, then $G$ is $(C_4, P_4, 2K_2)$-free. Thus $G$ is a threshold graph and contains a universal vertex. Let $|N| \geq 2$. First, we need to prove that there is some $x \in N$ such that $G - x$ is connected. Assume the contrary that for all $n \in N$, $G - n$ is disconnected. Consider $x, y \in N$ with $x \neq y$. Let $A$ be the connected component of $G - x$ containing $y$ and let $B \neq A$ be another connected component of $G - x$. It can be noted that both $A$ and $B$ contain probes or $G$ would be disconnected. By induction, $G[A]$ has a universal vertex $z$ or a vertex $z \in N \cap A$ adjacent to all vertices in $P \cap A$. By our assumption, $G - y$, and hence $G[A] - y$, is disconnected. Therefore it must be that $z = y$. Let $D$ be the connected component of $G - y$ containing $x$ and let $C \neq D$ be another connected component of $G - y$. Again both $C$ and $D$ must contain probes and $x$ is adjacent to all
vertices in \( D \cap P \). It can be noted that \( B \subseteq D \setminus A \) and \( C \subseteq A \setminus D \). So \( x \) and \( y \) together with any probe \( c \in C \), \( a \in A \cap N(x) \), \( b \in B \) induce a \( P_5 \) with two midpoints in \( N \), which is one of the forbidden subgraphs in Figure 5.3. This is a contradiction. So we may assume a vertex \( x \in N \) exists such that \( G - x \) is connected. By our induction hypothesis, \( G - x \) has a vertex \( v \) such that \( v \) is universal or \( v \in N - x \) and \( v \) is adjacent to all probes. If \( v \in N - x \), then we are done. So let \( v \) be universal in \( G - x \) and \( v \in P \). If \( v \) is also adjacent to \( x \) then we are again done. So suppose that \( v \) and \( x \) are nonadjacent. Then we claim that every vertex in \( N(x) \) is a universal vertex. First, \( N(x) \) is a clique because otherwise \( v, x \), and two nonadjacent vertices in \( N(x) \) would induce a \( C_4 \) with only one vertex in \( N \), which is one of the forbidden subgraphs in Figure 5.3. This is a contradiction. Next every vertex \( u \in N(x) \) is adjacent to every vertex \( y \in N - x \), otherwise \( x, u, v, y \) would induce a \( P_4 \) with the endpoints in \( N \), which is one of the forbidden subgraphs in Figure 5.3. This is a contradiction. Finally, if a vertex \( u \in N(x) \) is nonadjacent to a vertex \( w \in P \setminus N(x) \) then \( w, v, u, x \) would induce a \( P_4 \) with one endpoint in \( N \), which is one of the forbidden subgraphs in Figure 5.3. This is also a contradiction. Thus every vertex in \( N(x) \) is universal in \( G \), as claimed.

\((e) \Rightarrow (f)\) Again we will use a proof by induction on \(|V| = |P| + |N|\). Assume that \((e) \Rightarrow (f)\) for all graphs with strictly fewer vertices than \( G \).

If \( G \) is disconnected, by induction, \( H^* \) is 1-dot product for any connected component \( H \) of \( G \). Therefore \( G^* = H^*_1 \cup H^*_2 \) is 1-dot product, since \( G \) has at most two connected components. Additionally, if \(|N| \leq 1\) then \((e)\) implies that \( G \) is \((C_4, P_4, 3K_2)\)-free; thus \( G^* = G \) is a 1-dot product graph.

So, let \( G \) be connected and \(|N| \geq 2\). If \( G \) has a universal vertex \( v \), then \( v \in P \) and \( v \) is a universal vertex in \( G^* \). So \( G^* \) is equivalent to \((G - v)^*\) combined with \( v \) and all of the edges incident to \( v \) in \( G \). By induction, \((G - v)^*\) is a 1-dot product graph, so \( G^* \) is a 1-dot product graph. Now we can consider the case where \( G \) has a vertex \( v \in N \) adjacent to all vertices in \( P \). Since \( v \in N \), \( G^* - v = (G - v)^* \) and \((G - v)^* \) is a 1-dot product graph by induction. We can note that for all vertices \( x, y \in N - v \):

\[
xy \in E^* \Rightarrow vx, vy \in E^* \quad (5.1)
\]

\[
N_G(x) \setminus N_G(y) \neq \emptyset \land N_G(y) \setminus N_G(x) \neq \emptyset \Rightarrow vx, vy \in E^* \quad (5.2)
\]

\[
N_G(x) \subseteq N_G(y) \land vx \in E^* \Rightarrow vy \in E^* \quad (5.3)
\]

(5.1) follows directly from the definition of \( G^* \). For (5.2), let \( a \in N_G(x) \setminus N_G(y) \) and
If \( b \in N_G(y) \setminus N_G(x) \). If \( ab \in E^* \), then \( G[x,a,b,y] \) would be one of the forbidden subgraphs in Figure 5.3. So \( ab \notin E^* \). This means that \( xavb \) and \( ybva \) are induced \( P_4 \)'s in \( G \). Hence \( xv, vy \in E^* \). And (5.3) holds because if \( v, x \) belong to a \( C_4 \) in \( G \) then \( v, y \) also belong to the same \( C_4 \). If \( v, x \) belong to a \( P_4 \) in \( G \), then \( v, y \) also belong to the same \( P_4 \). In either case, \( vy \in E^* \).

Thus \( G^* \) cannot contain an induced \( C_4 \). So if \( vxyz \) is a \( C_4 \) in \( G^* \), then \( y \in N \) because \( v \) is adjacent to all of the vertices in \( P \). Therefore, \( x, z \in P \) by (5.1). But then \( vxyz \) is an induced \( C_4 \) in \( G \) and \( vy \) would be an edge in \( G^* \). Similarly, \( G^* \) cannot contain an induced \( P_4 \). So if \( vxyz \) is a \( P_4 \) in \( G^* \), then \( y, z \in N \) because \( v \) is adjacent to all of the vertices in \( P \). Therefore, \( yz \in E^* \). But then \( vy, vz \) would be edges in \( G^* \) by (5.1). Also if \( vxyz \) is a \( P_4 \) in \( G^* \), then \( z \in N \) because \( v \) is adjacent to all of the vertices in \( P \). Therefore, \( y \in P \). Additionally, \( x \in N \) or else \( xvyz \) would be an induced \( P_4 \) in \( G \) and \( vz \) would be an edge in \( G^* \). So by (2) and \( vz \notin E^* \), \( N_G(x) \subset N_G(y) \) and then by (5.3), \( vz \in E^* \). But this is a contradiction. Thus \( G^* \) is a 1-dot product graph.

(f) \( \Rightarrow \) (a) This is given by definition. \( \square \)

**Corollary 5.9.** A partitioned graph \( G = (P \cup N, E) \) can be recognized as a probe 1-dot product graph in polynomial time.

Now for the unpartitioned case.

**Theorem 5.10.** The following statements are equivalent for any graph \( G \):

(a) \( G \) is a probe 1-dot product graph;

(b) \( G \) is 3\( K_2 \)-free, and \( DP_1(G) \) is satisfiable;

(c) \( G \) is 3\( K_2 \)-free and admits an independent set \( N \) such that in \( G \) each induced \( P_4 \) must have a midpoint and an endpoint in \( N \), each induced \( C_4 \) must have two vertices in \( N \), and each induced \( P_5 \) must have 3 vertices in \( N \);

(d) \( G \) has at most two connected components of any induced subgraph and admits an independent set \( N \) such that each connected induced subgraph \( H \) of \( G \) has a universal vertex or a vertex in \( N \cap V(H) \) adjacent to all vertices in \( P \cap V(H) \);

(e) \( G \) admits an independent set \( N \) such that \( G^* \) (with respect to \( N \)) is a 1-dot product graph.
Proof. (a) ⇒ (b) Assume that $G$ is a probe 1-dot product graph. By Theorem 5.4, $G$ is 3$K_2$-free and probe trivially perfect. By Theorem 5.6, $G$ admits an independent set $N$ such that in $G$ every induced $C_4$ has two vertices in $N$, every induced $P_4$ has a midpoint and an endpoint in $N$, and every induced $P_5$ has three vertices in $N$.

We will first show that the independence of $N$ satisfies the conjunction of edges clauses $DP1(G)$. It should first be noted that each of the edge clauses $(\overline{x_a} \vee \overline{x_b})$ is true if and only if at least one of the boolean variables, $x_a$ or $x_b$ is false. This means that at least one vertex in each edge must be a probe. Thus the conjunction of edge clauses is true if and only if $N$ is an independent set.

We will next show that the $C_4$-clauses are satisfied. Since each induced $C_4 = abcd$ has two vertices in $N$, then $x_a = x_c = true$ and $x_b = x_d = false$ or $x_a = x_c = false$ and $x_b = x_d = true$. Otherwise some edge in $C_4$ would include two non-probes, which is a contradiction of $N$ being independent. (These two cases are shown in Figure 5.5.) In either case, the conjunction of the two $C_4$ clauses in $DP1(G)$, $(x_a \lor x_b) \land (x_c \lor x_d)$, is true.

![Figure 5.5: The $C_4$ Cases for Unpartitioned Probe 1-Dot Product Graphs.](image)

The white nodes are non-probes and the black nodes are probes.

Satisfaction of the $P_4$-clauses when every induced $P_4$ has a midpoint and an endpoint in $N$ is our next focus. Since each induced $P_4 = abcd$ has a midpoint and an endpoint in $N$, then $x_a = x_c = true$ and $x_b = x_d = false$ or $x_a = x_c = false$ and $x_b = x_d = true$. Otherwise some edge in $P_4$ would include two non-probes, which is a contradiction of $N$ being independent. (These cases are shown in Figure 5.6.) In either case, the conjunction of the two $P_4$ clauses in $DP1(G)$, $(x_a \lor x_d) \land (x_b \lor x_c)$, is true.

Finally, we will show that the $P_5$ clauses are satisfied. Since each induced $P_5 = abcede$ has three vertices in $N$, then $x_a = x_c = x_e = true$ and $x_b = x_d = false$. Otherwise some edge in $P_5$ would include two non-probes, which is a contradiction of $N$ being independent. (These cases are shown in Figure 5.7.) Then the conjunction of $P_5$ clause and the $P_4$ clauses for the induced $P_4$’s, $abcd$ and $bcde$, in $DP1(G)$,
Figure 5.6: The $P_4$ Cases for Unpartitioned Probe 1-Dot Product Graphs. The white nodes are non-probes and the black nodes are probes.

\[(x_c) \land (x_a \lor x_d) \land (x_b \lor x_c) \land (x_c \lor x_d),\] is true.

Because all of the clauses of $DP1(G)$ are true, the conjunction of said clauses would also be true. Thus $DP1(G)$ is satisfied.

(b) $\Rightarrow$ (c) Suppose that $G$ is 3$K_2$-free and admits some $N$ such that $DP1(G)$ is satisfied. It can be noted that $DP1(G)$ is satisfied if and only if conjunction of all edge-clauses, all $C_4$-clauses, all $P_4$-clauses, and all $P_5$-clauses is true for some vertex set $N$ of $G$. Since a conjunction is true if and only if each of those clauses is true, we will examine each clause to show what is required for that clause to be true.

Let $N$ be a vertex set of $G$ such that $DP1(G)$ is satisfied.

First, we will show the independence of $N$. It can be noted that each of the edge clauses $(x_a \lor x_b)$ is true is and only if at least one of the boolean variables, $x_a$ or $x_b$ is false. This means that at least one vertex in each edge must be a probe. This in turn implies that no edge is incident to two non-probes. Thus $N$ is an independent set of vertices.

Next we can consider the $C_4$-clauses for each induced $C_4 = abcd$. The conjunction of the two $C_4$ clauses in $DP1(G)$, $(x_a \lor x_b) \land (x_c \lor x_d)$, is true if and only if $(x_a \lor x_b) = true$ and $(x_c \lor x_d) = true$. These clauses and the edge clauses are true if and only if $x_a = x_c = true$ and $x_b = x_d = false$ or $x_a = x_c = false$ and $x_b = x_d = true$. Thus each induced $C_4$ must have two vertices in $N$.

We can now consider the $P_4$-clauses for each induced $P_4 = abcd$. The conjunction
of the two $P_4$ clauses in $DP1(G)$, $(x_a \lor x_d) \land (x_b \lor x_c)$, is true if and only if $(x_a \lor x_d) = \text{true}$ and $(x_b \lor x_c) = \text{true}$. These clauses and the edge clauses are true if and only if $x_a = x_c = \text{true}$ and $x_b = x_d = \text{false}$ or $x_a = x_c = \text{false}$ and $x_b = x_d = \text{true}$. Thus each induced $P_4$ must have a midpoint and an endpoint in $N$

Finally we can consider the $P_5$-clause for each induced $P_5 = abcd e$. The clause $(x_c)$ is true if and only if $c \in N$. But when considering $P_5$, we need to consider the two induced $P_4$’s, $abcd$ and $bcde$. As we have already shown, the conjunction of these $P_4$ clauses is $(x_a \lor x_d) \land (x_b \lor x_c) \land (x_b \lor x_e) \land (x_c \lor x_d)$. This conjunction is true, along with the $P_5$-clause and edge clauses, if any only if $x_a = x_c = x_e = \text{true}$ and $x_b = x_d = \text{false}$. Thus each induced $P_5$ must have 3 vertices in $N$.

Therefore, $G$ is $3K_2$-free and admits an independent set $N$ such that in $G$ each induced $P_4$ must have a midpoint and an endpoint in $N$, each induced $C_4$ must have two vertices in $N$, and each induced $P_5$ must have 3 vertices in $N$.

(c) $\Rightarrow$ (d) Suppose that $G$ is $3K_2$-free and admits an independent set $N$ such that in $G$ each induced $P_4$ must have a midpoint and an endpoint in $N$, each induced $C_4$ must have two vertices in $N$, and each induced $P_5$ must have 3 vertices in $N$. We will define $N$ as that independent set.

The first condition is trivially proved because $G$ is $3K_2$-free by our assumptions.

The proof of the second condition of (d) is by induction on $|V| = |P| + |N|$. Assume that (c) $\Rightarrow$ (d) for all graphs with strictly fewer vertices than $G$. Then in particular all proper connected induced subgraphs $H$ of $G$ have a universal vertex or a vertex in $N \cap V(H)$ adjacent to all vertices in $P \cap V(H)$. So we only need to show that when $G$ is connected, it has a universal vertex or a nonprobe adjacent to all probes. Assume that $G$ is connected.

If $|N| \leq 1$, then $G$ is $(C_4, P_4, 2K_2)$-free. Thus $G$ is a threshold graph and contains a universal vertex.

Let $|N| \geq 2$. First, we need to prove that there is some $x \in N$ such that $G - x$ is connected. Assume the contrary that for all $n \in N$, $G - n$ is disconnected. Consider $x, y \in N$ with $x \neq y$. Let $A$ be the connected component of $G - x$ containing $y$ and let $B \neq A$ be another connected component of $G - x$. It can be noted that both $A$ and $B$ contain probes or $G$ would be disconnected. By induction, $G[A]$ has a universal vertex $z$ or a vertex $z \in N \cap A$ adjacent to all vertices in $P \cap A$. By our assumption, $G - y$, and hence $G[A] - y$, is disconnected. Therefore it must be that $z = y$. Let $D$ be the connected component of $G - y$ containing $x$ and let $C \neq D$ be another connected
component of $G - y$. Again both $C$ and $D$ must contain probes and $x$ is adjacent to all vertices in $D \cap P$. It can be noted that $B \subseteq D \setminus A$ and $C \subseteq A \setminus D$. So $x$ and $y$ together with any probe $c \in C$, $a \in A \cap N(x)$, $b \in B$ induce a $P_3$ with two midpoints in $N$, which is a contradiction that each $P_3$ has three vertices in $N$. So we may assume a vertex $x \in N$ exists such that $G - x$ is connected. By our induction hypothesis, $G - x$ has a vertex $v$ such that $v$ is universal or $v \in N - x$ and $v$ is adjacent to all probes. If $v \in N - x$, then we are done. So let $v$ be universal in $G - x$ and $v \in P$. If $v$ is also adjacent to $x$ then we are again done. So suppose that $v$ and $x$ are nonadjacent. Then we claim that every vertex in $N(x)$ is a universal vertex. First, $N(x)$ is a clique because otherwise $v$, $x$, and two nonadjacent vertices in $N(x)$ would induce a $C_4$ with only one vertex in $N$, which is a contradiction that each $C_4$ has two vertices in $N$. Next every vertex $u \in N(x)$ is adjacent to every vertex $y \in N - x$, otherwise $x, u, v, y$ would induce a $P_4$ with the endpoints in $N$, which is a contradiction that each $P_4$ has an endpoint and midpoint in $N$. Finally, if a vertex $u \in N(x)$ is nonadjacent to a vertex $w \in P \setminus N(x)$ then $w, v, u, x$ would induce a $P_4$ with one endpoint in $N$, which is a contradiction that each $P_4$ has an endpoint and midpoint in $N$. Thus every vertex in $N(x)$ is universal in $G$, as claimed.

$(d) \Rightarrow (e)$ Again we will use a proof by induction on $|V| = |P| + |N|$. Assume that $(d) \Rightarrow (e)$ for all graphs with strictly fewer vertices than $G$.

If $G$ is disconnected, by induction, $H^*$ is 1-dot product for any connected component $H$ of $G$. Therefore $G^* = H_1^* \cup H_2^*$ is 1-dot product, since $G$ has at most two connected components. Additionally, if $|N| \leq 1$ then $(d)$ implies that $G$ is $(C_4, P_4, 3K_2)$-free; thus $G^* = G$ is a 1-dot product graph.

So, let $G$ be connected and $|N| \geq 2$. If $G$ has a universal vertex $v$, then $v \in P$ and $v$ is a universal vertex in $G^*$. So $G^*$ is equivalent to $(G_v)^*$ combined with $v$ and all of the edges incident to $v$ in $G$. By induction, $(G - v)^*$ is a 1-dot product graph, so $G^*$ is a 1-dot product graph. Now we can consider the case where $G$ has a vertex $v \in N$ adjacent to all vertices in $P$. Since $v \in N$, $G^* - v = (G - v)^*$ and $(G - v)^*$ is a 1-dot product graph by induction. We can note that for all vertices $x, y \in N - v$:

\[ xy \in E^* \Rightarrow vx, vy \in E^* \quad (5.4) \]
\[ N_G(x) \setminus N_G(y) \neq \emptyset \land N_G(y) \setminus N_G(x) \neq \emptyset \Rightarrow vx, vy \in E^* \quad (5.5) \]
\[ N_G(x) \subseteq N_G(y) \land vx \in E^* \Rightarrow vy \in E^* \quad (5.6) \]
(5.4) follows directly from the definition of $G^*$. For (5.5), let $a \in N_G(x) \setminus N_G(y)$ and $b \in N_G(y) \setminus N_G(x)$. If $ab \in E^*$, then $G[x, a, b, y]$ would be one of the forbidden subgraphs in Figure 5.3. So $ab \notin E^*$. This means that $xavb$ and $yvba$ are induced $P_4$'s in $G$. Hence $xv, vy \in E^*$. And (5.6) holds because if $v, x$ belong to a $C_4$ in $G$ then $v, y$ also belong to the same $C_4$. If $v, x$ belong to a $P_4$ in $G$, then $v, y$ also belong to the same $P_4$. In either case, $vy \in E^*$.

Now we show that $G^*$ cannot contain an induced $C_4$. So if $vxyz$ is a $C_4$ in $G^*$, then $y \in N$ because $v$ is adjacent to all of the vertices in $P$. Therefore, $x, z \in P$ by (5.4). But then $vxyz$ is an induced $C_4$ in $G$ and $vy$ would be an edge in $G^*$. Thus $G^*$ cannot contain an induced $C_4$.

Similarly, we will show that $G^*$ cannot contain an induced $P_4$. So if $vxyz$ is a $P_4$ in $G^*$, then $y, z \in N$ because $v$ is adjacent to all of the vertices in $P$. Therefore, $yz \in E^*$. But then $vy, vz$ would be edges in $G^*$ by (5.4). Also if $xvyz$ is a $P_4$ in $G^*$, then $z \in N$ because $v$ is adjacent to all of the vertices in $P$. Therefore, $y \in P$. Additionally, $x \in N$ or else $xvyz$ would be an induced $P_4$ in $G$ and $vz$ would be an edge in $G^*$. But $x, z \in N$ implies that $xz \in E^*$ so $xvyz$ is not an induced $P_4$ in $G^*$. Therefore $G^*$ cannot contain an induced $P_4$. Thus $G^*$ is a 1-dot product graph.

$(e) \Rightarrow (a)$ This is given by definition.

This theorem can lead to a partial forbidden induced subgraph characterization of unpartitioned probe 1-dot product graphs. This partial characterization is given in Corollary 5.11.

**Corollary 5.11.** A graph $G$ is a unpartitioned probe 1 dot product graph if it does not contain any of the graphs in Figure 5.8.

**Proof.** It is trivial to show that $3K_2$ is forbidden since the graphs are $3K_2$-free. The rest of the proof is based on part $(c)$ of Theorem 5.10. We will show the necessity of the graphs in Figure 5.8 to not admit an independent set $N$ such that in $G$ each induced $P_4$ must have a midpoint and an endpoint in $N$, each induced $C_4$ must have two vertices in $N$, and each induced $P_5$ must have 3 vertices in $N$. We will do this by first showing that each of the maximal independent sets $N$ of $G$ has a $P_-, C_4$, or $P_5$ that fails the desired partition. A maximal independent set of vertices is a set of independent vertices such that each of the other vertices in $G$ is adjacent to at least one of the vertices in the
Then we will show that each graph is minimal by examining all of the induced subgraphs with exactly one fewer vertex along with a partition $N$ that satisfies the desired conditions. A brief examination of the partition of each of the subgraphs will show that each satisfies the desired conditions. We will utilize symmetry in the graph to reduce the number of cases and subgraphs to consider.

Consider $C_5$. The maximal independent set of $C_5$ can be seen in Figure 5.9, with white vertices being non-probes and black vertices being probes. There is only one case because of symmetry. In this case, the path $abcd$ has two vertices in $N$, but both are endpoints.
Figure 5.9: The Probe Partitions of $C_5$. The white nodes are non-probes and the black nodes are probes.

In Figure 5.10, we show each of the induced subgraphs of $C_5$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there is only one subgraph to consider.

Figure 5.10: The Subgraphs of $C_5$ and Partitions of those subgraphs. The white nodes are non-probes and the black nodes are probes.

Consider $C_6$. The maximal independent set of $C_6$ can be seen in Figure 5.11, with white vertices being non-probes and black vertices being probes. There is only one case because of symmetry. In this case, the path $b c d e f$ has only two vertices in $N$.

In Figure 5.12, we show each of the induced subgraphs of $C_6$, and an independent

Figure 5.11: The Probe Partitions of $C_6$. The white nodes are non-probes and the black nodes are probes.
set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there is only one subgraph to consider.

Consider $P_6$. The maximal independent set of $P_6$ can be seen in Figure 5.13, with

![Figure 5.12: The Subgraphs of $C_6$ and Partitions of those subgraphs. The white nodes are non-probes and the black nodes are probes.](image)

white vertices being non-probes and black vertices being probes. There is only one case because of symmetry. In this case, the path $bcdef$ has only two vertices in $N$.

![Figure 5.13: The Probe Partitions of $P_6$. The white nodes are non-probes and the black nodes are probes.](image)

In Figure 5.14, we show each of the induced subgraphs of $P_6$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only three subgraphs to consider.

Consider $G_1$. The maximal independent sets of $G_1$ can be seen in Figure 5.15, with white vertices being non-probes and black vertices being probes. There are only three cases because of symmetry. In Case 1, note that the central path is a $P_4$ with no non-probes. In Case 2, note that that there an induced $C_4$ with no vertices in $N$. In Case 3, the central path is a $P_4$ with two vertices in $N$. However those vertices are both endpoints.

In Figure 5.16, we show each of the induced subgraphs of $G_1$, and an independent
Figure 5.14: The Subgraphs of $P_6$ and Partitions of those subgraphs. The white nodes are non-probes and the black nodes are probes.

Figure 5.15: The Probe Partitions of $G_1$. The white nodes are non-probes and the black nodes are probes.

set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only three subgraphs to consider.

Consider $G_2$. The maximal independent sets of $G_2$ can be seen in Figure 5.17, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In both Case 1 and Case 2, note that the probe vertices form $C_4$'s with no non-probes.
In Figure 5.16, we show each of the induced subgraphs of \( G_1 \), and an independent set \( N \) for each subgraph that satisfies the desired conditions. Because of symmetry, there are only three subgraphs to consider.

Consider \( G_3 \). The maximal independent sets of \( G_3 \) can be seen in Figure 5.19, with white vertices being non-probes and black vertices being probes. There are six cases to examine. In Case 1, the path \( eafg \) only has one vertex in \( N \). For Cases 2 and 3, the path \( edfg \) has two vertices in \( N \), but they are both endpoints. In Cases 4 and 5, the path \( bcde \) does not have any vertices in \( N \). Finally, in Case 6, the path \( bcde \) only
Figure 5.18: The Subgraphs of $G_2$ and Partitions of those subgraphs. The white nodes are non-probes and the black nodes are probes.

has one vertex in $N$.

Figure 5.19: The Probe Partitions of $G_3$. The white nodes are non-probes and the black nodes are probes.

In Figure 5.20, we show each of the induced subgraphs of $G_3$, and an independent set $N$ for each subgraph that satisfies the desired conditions. There are seven subgraphs
Figure 5.20: The Subgraphs of $G_3$ and Partitions of those subgraphs. The white nodes are non-probes and the black nodes are probes.

Consider $G_4$. The maximal independent sets of $G_4$ can be seen in Figure 5.21, with white vertices being non-probes and black vertices being probes. In Case 1, note that the path $eafg$ only has one vertex in $N$. For Case 2, the path $edfg$ has two vertices in $N$, but they are both endpoints. In Cases 3 and 5, the cycle $bcde$ does not have any vertices in $N$. Finally, in Case 4, the cycle $bcde$ only has one vertex in $N$.

In Figure 5.22, we show each of the induced subgraphs of $G_4$, and an independent set $N$ for each subgraph that satisfies the desired conditions. There are seven subgraphs to consider.

Consider $G_5$. The maximal independent sets of $G_5$ can be seen in Figure 5.23, with white vertices being non-probes and black vertices being probes. There are only three cases because of symmetry. In Cases 1 and 2, the path $abgh$ has two vertices in $N$, but they are both endpoints. In Case 3, the path $cdef$ does not have any vertices in $N$.

In Figure 5.24, we show each of the induced subgraphs of $G_5$, and an independent
Figure 5.21: The Probe Partitions of $G_4$. The white nodes are non-probes and the black nodes are probes.

set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only four subgraphs to consider.

Consider $G_6$. The maximal independent sets of $G_6$ can be seen in Figure 5.25, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In Case 1, the path $abgh$ has two vertices in $N$, but they are both endpoints. In Case 2, the cycle $cdef$ does not have any vertices in $N$.

In Figure 5.26, we show each of the induced subgraphs of $G_6$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only four subgraphs to consider.

Consider $G_7$. The maximal independent sets of $G_7$ can be seen in Figure 5.27, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In both cases, the path $adfh$ has two vertices in $N$, but they are both endpoints.

In Figure 5.28, we show each of the induced subgraphs of $G_7$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only two subgraphs to consider.

Consider $G_8$. The maximal independent sets of $G_8$ can be seen in Figure 5.29, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In Case 1, the path $ecdf$ only has one vertex in $N$. In
Case 2, the path $ecdf$ has two vertices in $N$, but both are endpoints.

In Figure 5.30, we show each of the induced subgraphs of $G_8$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only two subgraphs to consider.

Consider $G_9$. The maximal independent set of $G_9$ can be seen in Figure 5.31, with
white vertices being non-probes and black vertices being probes. There is only one case because of symmetry. In this case, the cycle \( ecdf \) has only one vertex in \( N \).

In Figure 5.32, we show each of the induced subgraphs of \( G_9 \), and an independent set \( N \) for each subgraph that satisfies the desired conditions. Because of symmetry,
there are only two subgraphs to consider.

Consider $G_{10}$. The maximal independent sets of $G_{10}$ can be seen in Figure 5.33, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In Case 1, the cycle $bcde$ has only one vertex in $N$. In Case 2, the path $abcd$ has only one vertex in $N$.

In Figure 5.34, we show each of the induced subgraphs of $G_{10}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only three subgraphs to consider.

Consider $G_{11}$. The maximal independent sets of $G_{11}$ can be seen in Figure 5.35, with white vertices being non-probes and black vertices being probes. There are only three cases because of symmetry. In Case 1, the cycle $abdg$ has only one vertex in $N$. In Case 2, the cycle $bcge$ has no vertices in $N$. In Case 3, the path $cdef$ has only one
Figure 5.27: The Probe Partitions of $G_7$. The white nodes are non-probes and the black nodes are probes.

Figure 5.28: The Subgraphs of $G_7$ and Partitions of those subgraphs. The white nodes are non-probes and the black nodes are probes.

Figure 5.29: The Probe Partitions of $G_8$. The white nodes are non-probes and the black nodes are probes.

vertex in $N$.

In Figure 5.36, we show each of the induced subgraphs of $G_{11}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. There are six subgraphs to consider.

Consider $G_{12}$. The maximal independent sets of $G_{12}$ can be seen in Figure 5.37, with white vertices being non-probes and black vertices being probes. There are only three cases because of symmetry. In Case 1, the path $gacd$ has only one vertex in $N$. 
In Case 2, the path \( dcfe \) has two vertices in \( N \), but both are endpoints. In Case 3, the path \( bcfg \) has no vertices in \( N \).

In Figure 5.38, we show each of the induced subgraphs of \( G_{12} \), and an independent set \( N \) for each subgraph that satisfies the desired conditions. Because of symmetry, there are only four subgraphs to consider.

Consider \( G_{13} \). The maximal independent sets of \( G_{13} \) can be seen in Figure 5.39, with white vertices being non-probes and black vertices being probes. There are only three cases because of symmetry. In Case 1, the path \( gacd \) has only one vertex in \( N \). In Case 2, the cycle \( bcfg \) has only one vertex in \( N \). In Case 3, the cycle \( bcfg \) has no vertices in \( N \).

In Figure 5.40, we show each of the induced subgraphs of \( G_{13} \), and an independent set \( N \) for each subgraph that satisfies the desired conditions. Because of symmetry, there are only four subgraphs to consider.

Consider \( G_{14} \). The maximal independent sets of \( G_{14} \) can be seen in Figure 5.41, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In Case 1, the path \( bcef \) has two vertices in \( N \), but
both are end points. In Case 2, the path $bcef$ has no vertices in $N$.

In Figure 5.42, we show each of the induced subgraphs of $G_{14}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only two subgraphs to consider.

Consider $G_{15}$. The maximal independent sets of $G_{15}$ can be seen in Figure 5.43, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In Case 1, the cycle $bdgf$ has no vertices in $N$. In Case 2, the cycle $cdef$ has only one vertex in $N$.

In Figure 5.44, we show each of the induced subgraphs of $G_{15}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only three subgraphs to consider.

Consider $G_{16}$. The maximal independent sets of $G_{16}$ can be seen in Figure 5.45,
Figure 5.34: The Subgraphs of $G_{10}$ and Partitions of those subgraphs. The white nodes are non-probes and the black nodes are probes.

Figure 5.35: The Probe Partitions of $G_{11}$. The white nodes are non-probes and the black nodes are probes.

with white vertices being non-probes and black vertices being probes. There are five cases because of symmetry. In Case 1, the path $fbc$ has only one vertex in $N$. In Case 2, the path $afe$ has only one vertex in $N$. In Cases 3, 4, and 5, the path $afe$ has only two vertices in $N$.

In Figure 5.46, we show each of the induced subgraphs of $G_{16}$, and an independent
set $N$ for each subgraph that satisfies the desired conditions. There are six subgraphs to consider.

Consider $G_{17}$. The maximal independent sets of $G_{17}$ can be seen in Figure 5.47, with white vertices being non-probes and black vertices being probes. There are only three cases because of symmetry. In Case 1, the path $abcdf$ has only two vertices in $N$. 
Figure 5.37: The Probe Partitions of $G_{12}$. The white nodes are non-probes and the black nodes are probes.

In Cases 2 and 3, the path $abcde$ has only two vertex in $N$.

In Figure 5.48, we show each of the induced subgraphs of $G_{17}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only five subgraphs to consider.

Consider $G_{18}$. The maximal independent sets of $G_{18}$ can be seen in Figure 5.49, with white vertices being non-probes and black vertices being probes. There are only three cases because of symmetry. In Case 1, the path $abfde$ has only two vertices in $N$. In Cases 2 and 3, the path $abcde$ has only two vertices in $N$.

In Figure 5.50, we show each of the induced subgraphs of $G_{18}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only three subgraphs to consider.

Consider $G_{19}$. The maximal independent sets of $G_{19}$ can be seen in Figure 5.51, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In Case 1, the path $fbce$ has only one vertex in $N$. In Case 2, the path $fbce$ has no vertices in $N$.

In Figure 5.52, we show each of the induced subgraphs of $G_{19}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only two subgraphs to consider.

Consider $G_{20}$. The maximal independent sets of $G_{20}$ can be seen in Figure 5.53, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In both cases, the path $dabcf$ has only two vertices in $N$.

In Figure 5.54, we show each of the induced subgraphs of $G_{20}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only two subgraphs to consider.
Consider $G_{21}$. The maximal independent sets of $G_{21}$ can be seen in Figure 5.55, with white vertices being non-probes and black vertices being probes. There are only two cases because of symmetry. In both cases, the path $efgcd$ has only two vertices in $N$.

Figure 5.38: The Subgraphs of $G_{12}$ and Partitions of those subgraphs. The white nodes are non-probes and the black nodes are probes.

Figure 5.39: The Probe Partitions of $G_{13}$. The white nodes are non-probes and the black nodes are probes.
In Figure 5.56, we show each of the induced subgraphs of $G_{21}$, and an independent set $N$ for each subgraph that satisfies the desired conditions. Because of symmetry, there are only two subgraphs to consider.

Thus each of the graphs is minimal and a forbidden induced subgraph of probe 1-dot product graphs.

The proof shows the graphs in Figure 5.8 are necessary forbidden induced subgraphs of unpartitioned probe 1-dot product graphs, but the sufficiency of this list remains to be proven. The complexity of this proof is based on having to consider all connected graphs which can be constructed from some combination of $C_4$'s, $P_4$'s, and $P_5$'s. Despite this complexity, we conjecture that the graphs in Figure 5.8 are a complete list of all forbidden induced subgraphs.
**Conjecture 5.12.** Let $G$ be a graph. Then $G$ is a probe 1-dot product graph if and only if $G$ contains none of the graphs in Figure 5.8 as induced subgraphs.

### 5.5 Cycle-Free Probe 1-Dot Product Graphs

We previously stated that all cycle-free graphs are forests. A forest is a graph whose connected components are all trees. Forests are a graph class that has been studied extensively and is used in a variety of applications. In Conjecture 5.12, we mentioned the difficulty of characterizing all unpartitioned probe 1-dot product graphs. Thus we will focus first on a characterization of unpartitioned probe 1-dot product graphs when the graphs are cycle-free. This characterization is given in Theorem 5.13.

**Theorem 5.13.** $G$ is a cycle-free probe 1-dot probe graph if and only if it does not have $3K_2$ or $P_6$ as an induced subgraph.
Proof. The necessity of $3K_2$ and $P_6$ is established by Corollary 5.11.

To prove sufficiency, we will refer to Theorem 2.12. Since every 1-dot product graph is trivially a probe 1-dot product graph, we will focus on trees that contain caterpillars as induced subgraphs. Every tree contains a central path. A central path is an induced path of maximum length. The branches and leaves of the tree will be considered. Branches are induced paths such that one of their endpoints is a vertex on the central path. Leaves are branches with only one vertex not on the central path. Since our graph must contain a $P_4$ in order to not be a star and $H$ cannot have an induced $P_6$, we only need to consider the two cases where the central path is a $P_4$ or a $P_5$. 
Let the central path of $G$ be a $P_4$. If we add any leaves to the endpoints of the $P_4$, then the central path will increase in length and thus not be this case. Similarly, if branches that are more than leaves are added to either midpoint, then there exists a central path longer than the $P_4$, which is a contradiction. Thus the only possible graph is a caterpillar with $P_4$ as the central path. This caterpillar can be seen in Figure 5.57.

In Figure 5.57, we also give a partition of the vertices such that there is an independent set $N$ such that each induced $P_4$ must have a midpoint and an endpoint in $N$, each induced $C_4$ must have two vertices in $N$, and each induced $P_5$ must have 3 vertices in $N$. Thus by Theorem 5.10, all caterpillars with a $P_4$ as a central path are probe 1-dot product graphs.

Let the central path of $G$ be a $P_5$. Again, we cannot add any leaves to the endpoints without lengthening the central path. Similarly, consider what happens if we add a branch with two additional vertices to the midpoints. Because of symmetry, there are only be two graphs to consider. These graphs are in Figure 5.58.

A brief examination of $F_1$ shows that is has an induced $P_6$ and thus is already contained in our forbidden induced subgraph characterization. In a similar manner, $F_2$ has an induced $3K_2$ and is also contained in our forbidden induced subgraph characterization. So the only possible forbidden induced subgraph is $F_3$, a caterpillar with $P_5$ as the central path, as seen in Figure 5.59, or its induced subgraphs. However, the partition of $F_5$ given in Figure 5.59 shows the existence of an independent set $N$ such
that each induced $P_4$ must have a midpoint and an endpoint in $N$, each induced $C_4$ must have two vertices in $N$, and each induced $P_5$ must have 3 vertices in $N$. Thus $G_3$ and all of its induced subgraphs are probe 1-dot product graphs.

Thus every tree that does not have $3K_2$ or $P_6$ as an induced subgraph is a probe 1-dot product graph.
5.6 Bipartite Probe 1-Dot Product Graphs

Our motivation for considering probe dot product graphs was Conjecture 2.19. We have already noted that this bound on dot product dimension can be achieved with complete bipartite graphs. Additionally, we showed that every complete bipartite graph is a probe 1-dot product graph. This leaves the question of what other bipartite graphs
are probe 1-dot product graphs.

We will show that a bipartite graph is a probe 1-dot product graph if and only if it does not have any of the graphs in Figure 5.60 as an induced subgraph. This forbidden induced subgraph characterization is given in Theorem 5.14.

**Theorem 5.14.** Let $G = (X \cup Y, E)$ be a bipartite graph. Then $G$ is a probe 1-dot product graph if and only if it does not contain any of the graphs in Figure 5.60 as an induced subgraph.

*Proof.* The necessity of the graphs in Figure 5.60 is established by Corollary 5.11.
The sufficiency of these graphs will be proven by contradiction. So suppose that there exists graph $H$ such that $H$ is a bipartite graph that is not a probe 1-dot product graph and does not contain any of the graph in Figure 5.60 as induced subgraphs.

By Theorem 5.10 (c), $H$ does not have an independent set $N$ such that each induced $P_4$ must have a midpoint and an endpoint in $N$, each induced $C_4$ must have two vertices in $N$, and each induced $P_5$ must have 3 vertices in $N$. We will consider each of these substructures.

First let us consider each $P_4$ in $G$. By definition of bipartite, one endpoint and one midpoint of every $P_4$ is in each of the partite sets. Thus if we define $N$ to be one of the partite sets ($X$ without loss of generality), then every $P_4$ has one endpoint and one midpoint in $N$.

When we consider each $C_4$ in $G$, it can again be noted that each $C_4$ must have two vertices in each partite set. If $N$ is defined to be $X$ still, we can again choose two
vertices from every $C_4$ that are in $X$.

For each $C_4$ and $P_4$, we simply choose either of the partite sets because each partite set is independent and there is a desired pair of vertices for each $C_4$ and $P_4$ in each partite set. Thus without loss of generality, we may choose $X$ as $N$.

This leaves us to consider $P_5$. If the endpoints of $P_5$ are in $X$, then the central
Figure 5.57: The Caterpillar on a $P_4$ and A Partitions of its vertices. The white nodes are non-probes and the black nodes are probes.

Figure 5.58: The Forbidden Branches on a $P_5$. The white nodes are non-probes and the black nodes are probes.

point is as well. Therefore $P_5$ has three vertices in $N$. If there is only one induced $P_5$, we can simply choose the partite set such that the endpoints of $P_5$ are contained in that partite set. What remains to consider is if there are at least two different induced $P_5$’s.

If all of the induced $P_5$’s all have the endpoints in the same partite set, then $G$ either contains a $3K_2$ or it is a probe 1-dot product graph. Both cases are contradictions. Therefore, we will consider the cases when there are at least two $P_5$’s where each $P_5$ has its endpoints in different partite sets. Since each $P_5$ must be induced, the only edges possible besides those in each $P_5$ will be between vertices in different $P_5$’s and different partite sets. For clarity, we will define $P_5$ as the path with three vertices in $X$ and $\hat{P}_5$ as the path with three vertices in $Y$. Since $P_5$ has only two vertices in $Y$, the maximum vertices that are in both paths is 4. We will consider the cases when the two paths have 4, 3, 2, 1, and 0 vertices in common.
Suppose that $P_5$ and $\hat{P}_5$ have four vertices in common. There are two subcases possible, namely that when only one endpoint of $\hat{P}_5$ is in $P_5$ and when both of the endpoints of $\hat{P}_5$ are in $P_5$. Both of these subcases are shown in Figure 5.61 with the dotted lines representing possible edges. In the graphs, the grey nodes represent the vertices in $X$ and the black nodes represent the vertices in $Y$. For Subcase 1, $P_5 = v_1v_2v_3v_4v_5$ and $\hat{P}_5 = v_2v_3v_4v_5v_6$. In Subcase 1, if $e_1$ is not present, then $v_1v_2v_3v_4v_5v_6$ is an induced $P_6$. If $e_1$ is present then $v_1v_2v_3v_4v_5v_6$ is an induced $C_6$. Both scenarios are contradictions.

For Subcase 2, $P_5 = v_3v_2v_1v_6v_5$ and $\hat{P}_5 = v_2v_3v_4v_5v_6$. In Subcase 2, $v_1v_2v_3v_4v_5v_6$ is an induced $C_6$ if $e_1$ is not present and $v_1v_2v_3v_4v_5v_6$ is an induced $G_{20}$ if $e_2$ is present. Again these scenarios are contradictions of our assumptions.

Suppose that $P_5$ and $\hat{P}_5$ have three vertices in common. We need to consider the situations when no endpoints of $\hat{P}_5$, when only one endpoint of $\hat{P}_5$ is in $P_5$, and when both of the endpoints of $\hat{P}_5$ are in $P_5$. These situations and the possible graphs are
Subcase 2

the grey nodes represent the vertices in X shown in Figure 5.62 with the dotted lines representing possible edges. In the graphs, the grey nodes represent the vertices in X and the black nodes represent the vertices in Y.

Suppose that there are no endpoints of $\hat{P}_5$ in $P_5$. Graph $H_1$ demonstrates the possible graphs with $P_5 = v_2 v_3 v_4 v_5 v_7$ and $\hat{P}_5 = v_1 v_2 v_3 v_4 v_5$. The induced subgraph on $\{v_1, v_2, v_3, v_4, v_6, v_7\}$ is an induced $P_6$ when $e_1$ is not present and is an induced $C_6$ when $e_1$ is present. Therefore this situation provides contradictions regardless of whether or not $e_2$ is present.

Suppose that there is exactly one endpoint of $\hat{P}_5$ in $P_5$. The possible graphs are $H_2$, $H_3$, $H_4$, and $H_5$. In $H_2$, $P_5 = v_1 v_2 v_3 v_4 v_5$ and $\hat{P}_5 = v_2 v_3 v_4 v_5 v_7$. The induced subgraph of $H_2$ on $\{v_1, v_2, v_3, v_4, v_5, v_7\}$ is an induced $P_6$ when $e_4$ is not present and is an induced $C_6$ when $e_4$ is present. Therefore this case yields a contradiction regardless of whether or not $e_3$ is present. In $H_3$, $P_5 = v_1 v_2 v_3 v_4 v_5$ and $\hat{P}_5 = v_2 v_1 v_7 v_3 v_6$. The induced subgraph of $H_2$ on $\{v_1, v_2, v_3, v_4, v_5, v_7\}$ is an induced $C_6$ when $e_5$ is not present and is an induced $G_{20}$ when $e_5$ is present. Therefore this case yields a contradiction regardless of whether or not $e_6$ is present. In $H_4$, $P_5 = v_1 v_2 v_3 v_4 v_5$ and $\hat{P}_5 = v_2 v_3 v_4 v_5 v_7$. The induced subgraph of $H_4$ on $\{v_1, v_2, v_3, v_4, v_5, v_7\}$ is an induced $P_6$ when $e_7$ is not present and is an induced $C_6$ when $e_7$ is present. Therefore this case yields a contradiction regardless of whether or not $e_8$ is present. In $H_5$, $P_5 = v_1 v_2 v_3 v_4 v_5$ and $\hat{P}_5 = v_2 v_6 v_4 v_3 v_7$. The induced subgraph of $H_5$ on $\{v_1, v_2, v_6, v_4, v_5, v_7\}$ is an induced $P_6$ when $e_9$ is not present and is an induced $C_6$ when $e_9$ is present. Therefore this case yields a contradiction regardless of whether or not $e_{10}$ is present.

Suppose that there are two endpoints of $\hat{P}_5$ in $P_5$. Graph $H_6$ demonstrates the possible graphs with $P_5 = v_2 v_3 v_4 v_5 v_6$ and $\hat{P}_5 = v_3 v_2 v_1 v_7 v_5$. The induced subgraph on

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**Figure 5.61:** The Subcases with Four Vertices in Both $P_5$'s. The grey nodes are in $X$ and the black nodes are in $Y$. Dotted lines are possible edges.
Possible Graphs when No Endpoints of \( \hat{P}_5 \) are in \( P_5 \)

Possible Graphs when Only One Endpoint of \( \hat{P}_5 \) are in \( P_5 \)

Possible Graphs when Two Endpoints of \( \hat{P}_5 \) are in \( P_5 \)

\( \{v_1, v_2, v_3, v_4, v_5, v_7\} \) is an induced \( C_6 \) when \( e_{11} \) is not present and is an induced \( G_{20} \) when \( e_{11} \) is present. Therefore this situation provides contradictions regardless of whether or not \( e_{12} \) is present.

Thus this scenario provides contradictions regardless of the three vertices in common.

Suppose that \( P_5 \) and \( \hat{P}_5 \) have two vertices in common. We need to consider the situations when no endpoints of \( \hat{P}_5 \), when only one endpoint of \( \hat{P}_5 \) is in \( P_5 \), and when both of the endpoints of \( \hat{P}_5 \) are in \( P_5 \). These situations and the possible graphs are shown in Figure 5.63 with the dotted lines representing possible edges. In the graphs, the grey nodes represent the vertices in \( X \) and the black nodes represent the vertices in
Suppose that there are no endpoints of $\hat{P}_5$ in $P_5$. The graphs $H_1$ and $H_2$ demonstrate the possible graphs. In $H_1$, $P_5 = v_2v_3v_4v_5v_8$ and $\hat{P}_5 = v_1v_2v_6v_4v_7$. The induced subgraph of $H_1$ on $\{v_1, v_2, v_3, v_4, v_5, v_8\}$ is an induced $P_6$ when $e_1$ is not present and is an induced $C_6$ when $e_1$ is present. Therefore this case yields a contradiction regardless of whether or not $e_2$ and/or $e_3$ are present. In $H_2$, $P_5 = v_2v_6v_7v_8v_4$ and $\hat{P}_5 = v_1v_2v_3v_4v_5$. The induced subgraph of $H_2$ on $\{v_2, v_3, v_4, v_7, v_6\}$ is an induced $C_6$ when $e_5$ is not present and is an induced $G_{20}$ when $e_5$ is present. Therefore this case yields a contradiction regardless of whether or not $e_4$ and/or $e_6$ are present.

Suppose that there is exactly one endpoint of $\hat{P}_5$ in $P_5$. The possible graphs are $H_3$, $H_4$, $H_5$, and $H_6$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure563.pdf}
\caption{The Possible Graphs with Two Vertices in Both $P_5$’s. The grey nodes are in $X$ and the black nodes are in $Y$. Dotted lines are possible edges.}
\end{figure}
In $H_3$, $P_3 = v_1v_2v_3v_4v_5$ and $\hat{P}_3 = v_2v_1v_6v_7v_8$. For this graph, there are two subcases to consider - whether or not $e_7$ is present. If $e_7$ is not present, then the induced subgraph of $H_3$ on $v_6v_1v_2v_3v_4v_5$ is an induced $P_6$ when $e_8$ is not present and is an induced $C_6$ when $e_8$ is present. Therefore this case yields a contradiction regardless of whether or not any combination of $e_9, e_{10}, e_{11}$ are present. So we need to consider when $e_7$ is present. If $e_7$ is present and at least one of the edges $e_8, e_9, e_{10}$ is present, then $G_{20}$ is an induced subgraph of $H_3$ regardless of whether or not $e_{11}$ is present. If $e_7$ is present and $e_8, e_9, e_{10}$ are all not present, then the induced subgraph of $H_3$ on $v_8v_7v_6v_3v_4v_5$ is an induced $P_6$ when $e_{11}$ is not present and is an induced $C_6$ when $e_{11}$ is present. Therefore this case yields a contradiction.

In $H_4$, $P_3 = v_1v_2v_3v_4v_5$ and $\hat{P}_3 = v_2v_1v_6v_7v_8$. For this graph, there are two subcases to consider - whether or not $e_{13}$ is present. If $e_{13}$ is not present, then the induced subgraph of $H_4$ on $\{v_1, v_2, v_3, v_6, v_7, v_8\}$ is an induced $P_6$ when $e_{12}$ is not present and is an induced $C_6$ when $e_{12}$ is present. Therefore this case yields a contradiction regardless of whether or not any combination of $e_{14}, e_{15}, e_{16}$ are present. So we need to consider when $e_{13}$ is present. If $e_{13}$ is present and at least one of the edges $e_{12}, e_{14}, e_{15}$ is present, then $G_{20}$ is an induced subgraph of $H_4$ regardless of whether or not $e_{16}$ is present. If $e_{13}$ is present and $e_{14}$ and $e_{15}$ are both not present, then the induced subgraph of $H_4$ on $\{v_8, v_7, v_6, v_3, v_4, v_5\}$ is an induced $P_6$ when $e_{16}$ is not present and is an induced $C_6$ when $e_{16}$ is present. Therefore this case yields a contradiction.

In $H_5$, $P_3 = v_1v_2v_3v_4v_5$ and $\hat{P}_3 = v_2v_6v_7v_5v_8$. For this graph, there are multiple subcases to consider. The first is whether or not $e_{17}$ is present. If $e_{17}$ is not present, then the induced subgraph of $H_5$ on $\{v_1, v_2, v_6, v_7, v_5, v_8\}$ is an induced $P_6$ when $e_{18}$ is not present and is an induced $C_6$ when $e_{18}$ is present. Therefore this case yields a contradiction regardless of whether or not any combination of $e_{19}, e_{20}, e_{21}$ are present. So we need to consider when $e_{17}$ is present. If $e_{17}$ is present, then the induced subgraph of $H_5$ on $\{v_1, v_2, v_3, v_4, v_5, v_7\}$ is an induced $C_6$ when $e_{19}$ is not present and is an induced $G_{20}$ when $e_{19}$ is present. Therefore this case yields a contradiction regardless of whether or not any combination of $e_{18}, e_{20}, e_{21}$ are present. Therefore this case yields a contradiction.

In $H_6$, $P_3 = v_1v_2v_3v_4v_5$ and $\hat{P}_3 = v_2v_6v_4v_7v_8$. The induced subgraph of $H_6$ on $\{v_1, v_2, v_6, v_4, v_7, v_8\}$ is an induced $P_6$ when $e_{22}$ is not present and is an induced $C_6$ when $e_{22}$ is present. Therefore this case yields a contradiction regardless of whether or not any combination of $e_{23}$ and/or $e_{24}$ are present.
Suppose that there are two endpoints of \( \hat{P}_5 \) in \( P_5 \). This case has already been considered. It is \( H_2 \), except \( P_5 \) and \( \hat{P}_5 \) are switched.

Thus this scenario provides contradictions regardless of the two vertices in common.

Suppose that \( P_5 \) and \( \hat{P}_5 \) have exactly one vertex in common. We need to consider the situations when no endpoints of \( \hat{P}_5 \) and when only one endpoint of \( \hat{P}_5 \) is in \( P_5 \). These situations and the possible graphs are shown in Figure 5.64 with the dotted lines representing possible edges. In the graphs, the grey nodes represent the vertices in \( \mathbb{X} \) and the black nodes represent the vertices in \( \mathbb{Y} \).

Suppose that there are no endpoints of \( \hat{P}_5 \) in \( P_5 \). The graph \( H_1 \) demonstrates the possible graphs with \( P_5 = v_2v_3v_4v_5v_7 \) and \( \hat{P}_5 = v_6v_7v_8v_9 \). For this graph, there are multiple subcases to consider. The first is whether or not \( e_2 \) and/or \( e_3 \) are present. If both \( e_2 \) and \( e_3 \) are not present, then the induced subgraph of \( H_3 \) on \( \{v_1, v_2, v_3, v_7, v_8, v_9\} \) is an induced \( P_6 \) when \( e_1 \) is not present and is an induced \( C_6 \) when \( e_1 \) is present. Therefore this case yields a contradiction regardless of whether or not any combination of \( e_4, e_5, e_6, e_7, e_8 \) are present. So we need to consider when either \( e_2 \) or \( e_3 \) is present but not both. If this case, then the induced subgraph of \( H_1 \) on \( \{v_1, v_2, v_3, v_7, v_8, v_9\} \) is an induced \( G_{20} \). Therefore this case yields a contradiction regardless of whether or not any combination of \( e_4, e_5, e_6, e_7, e_8 \) are present. Next we need to consider when both \( e_2 \) and \( e_3 \) are present. If either \( e_6, e_7, e_8 \), or both are present, then \( G_{20} \) is an induced subgraph of \( H_1 \), regardless of whether or not \( e_1, e_4, e_5, e_8 \) are present. If both \( e_6 \) and \( e_7 \) are not present, then by symmetry we can relabel the vertices to have the same case as when \( e_2 \) and \( e_3 \) are missing. Therefore this case yields a contradiction.

Suppose that there is exactly one endpoint of \( \hat{P}_5 \) in \( P_5 \). The possible graph is \( H_2 \) with \( P_5 = v_1v_2v_3v_4v_5 \) and \( \hat{P}_5 = v_4v_6v_7v_8v_9 \). For this graph, there are multiple subcases to consider. The first is whether or not \( e_{14} \) is present. If \( e_{14} \) is not present, then the induced subgraph of \( H_2 \) on \( v_5v_4v_6v_7v_8v_9 \) is an induced \( P_6 \) when \( e_{15} \) is not present and is an induced \( C_6 \) when \( e_{15} \) is present. Therefore this case yields a contradiction regardless of whether or not any combination of \( e_{10}, e_{11}, e_{12}, e_{13} \) are present. So we need to consider when \( e_{14} \) is present. If \( e_{13} \) is also present, then the induced subgraph of \( H_2 \) on \( \{v_2, v_3, v_4, v_5, v_7, v_8\} \) is \( G_{20} \) regardless of whether or not any of the other edges are present. If \( e_{13} \) is present, then the induced subgraph of \( H_2 \) on \( \{v_1, v_2, v_3, v_4, v_6, v_7\} \) is an induced \( P_6 \) when \( e_{11} \) is not present and is an induced \( C_6 \) when \( e_{11} \) is present regardless of whether or not any of the other edges are present. Therefore this case yields a contradiction.
Possible Graphs when No Endpoints of $\hat{P}_5$ are in $P_5$

Possible Graphs when Exactly One Endpoint of $\hat{P}_5$ are in $P_5$

Figure 5.64: The Possible Graphs with Only One Vertices in Both $P_5$’s. The grey nodes are in $X$ and the black nodes are in $Y$. Dotted lines are possible edges.

Thus this scenario provides contradictions regardless of the single vertex in common.

The final scenario to consider is when $P_5$ and $\hat{P}_5$ have no vertices in common. These situations and the possible graph are shown in Figure 5.65 with the dotted lines representing possible edges. In $H_1$, with $P_5 = v_1v_2v_3v_4v_5$ and $\hat{P}_5 = v_6v_7v_8v_9v_{10}$. In the graphs, the grey nodes represent the vertices in $X$ and the black nodes represent the
For this graph, there are multiple subcases to consider.

The first is whether or not $e_1$ is present. If $e_1$ is present and $e_2$ and $e_6$ are not present, then the induced subgraph on $\{v_4, v_3, v_2, v_1, v_6, v_7, v_8, v_9\}$ is the same graph as $H_3$ in Figure 5.63, which we have already shown to be a contradiction.

Suppose that $e_1$ is present and exactly one of $e_2$ or $e_6$ is also present. Without loss of generality because of symmetry, we will assume $e_2$ is present. If $e_7$ is not present, then the induced subgraph on $v_3, v_6, v_7, v_8, v_9, v_{10}$ is an induced $P_6$ if $e_{12}$ is not present and is an induced $C_6$ if $e_{12}$ is present. So $e_7$ is present. Now if $e_{12}$ is present, then the induced subgraph on $\{v_3, v_6, v_7, v_8, v_9, v_{10}\}$ is an induced $G_{20}$. Therefore $e_{12}$ is not present. Thus the induced subgraph on $\{v_1, v_6, v_7, v_8, v_9, v_{10}\}$ is an induced $P_6$ if $e_{10}$ is not present and an induced $C_6$ if $e_{10}$ is present. In either case there is a contradiction.

Suppose that $e_1$ is present and both $e_2$ and $e_6$ are also present. If $e_7$ is not present, then the induced subgraph on $\{v_3, v_6, v_7, v_8, v_9, v_{10}\}$ is an induced $P_6$ if $e_{12}$ is not present and is an induced $C_6$ if $e_{12}$ is present. So $e_7$ is present. If $e_{12}$ is present, then the induced subgraph on $\{v_3, v_6, v_7, v_8, v_9, v_{10}\}$ is an induced $G_{20}$. Therefore $e_{12}$ is not present. By symmetry, $e_8$ is also not present. Now if either $e_3$ or $e_{10}$ are present, then the induced subgraph on $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\{v_1, v_6, v_7, v_8, v_9, v_{10}\}$ respectively, is an induced $G_{20}$. So $e_3$ and $e_{10}$ are not present. If $e_{11}$ is not present, then the induced

**Figure 5.65:** The Possible Graphs with No Vertices in Both $P_5$’s. The grey nodes are in $X$ and the black nodes are in $Y$. Dotted lines are possible edges.
subgraph on \{v_5, v_4, v_3, v_8, v_9, v_{10}\} is an induced $P_6$ if $e_{13}$ is not present and is an induced $C_6$ if $e_{13}$ is present. So $e_{11}$ is present. If $e_{13}$ is present, then the induced subgraph on \{v_3, v_4, v_3, v_8, v_9, v_{10}\} is an induced $G_{20}$. So $e_{13}$ is not present. If $e_8$ is not present, then the induced subgraph on \{v_3, v_4, v_8, v_7, v_6\} is an induced $G_{20}$. So $e_8$ is present. Similarly it can be shown that $e_9$ must be present as well.

If $e_4$ is not present, then the induced subgraph on \{v_1, v_2, v_3, v_4, v_7, v_6\} is an induced $G_{20}$. So $e_4$ is present. But this means that the induced graph on \{v_1, v_2, v_4, v_6, v_7, v_9\} is an induced $G_{20}$, which is a contradiction.

We have thus shown that if $e_1$ is present, then there is a contradiction. So $e_1$, and $e_{13}$ by symmetry, are not present. It can then be noted that if $v_1$ and $v_2$ are permuted with $v_5$ and $v_4$, respectively, then $e_3$ is isomorphic to $e_1$ and $e_{10}$ to $e_{13}$. Therefore, $e_3$ and $e_{10}$ are also not present. The resulting possible graph is seen in Figure 5.66.

For the graphs represented in Figure 5.66, we will begin by considering if $e_4$ is not present. If $e_5$ is not present, then the induced subgraph on \{v_1, v_2, v_4, v_5, v_6, v_7\} is an induced $3K_2$, which is a contradiction. There is a similar contradiction when $e_9$ is not present. So $e_5$ and $e_9$ are present. If $e_2$ is not present, then the induced subgraph on \{v_1, v_2, v_3, v_4, v_6, v_7\} is an induced $P_6$, which is a contradiction. There is a similar contradiction when $e_6$, $e_8$, or $e_{12}$ are not present. So $e_2$, $e_6$, $e_8$, $e_{12}$ are present. But then the induced graph on \{v_1, v_2, v_3, v_5, v_6, v_7\}, seen in Figure 5.67, is a $C_6$, which is a contradiction. Thus $e_4$ is present, and by symmetry $e_{11}$ is also present.
If both $e_5$ and $e_9$ are not present, then the induced subgraph on \{$v_2, v_3, v_4, v_8, v_7$\} is an induced $C_6$ if $e_7$ is not present and is an induced $G_{20}$ if $e_7$ is present. Therefore, at least one of $e_5$ or $e_9$ are present. Without loss of generality, we will assume that $e_5$ is present and $e_9$ is not. If $e_7$ is not present, then the induced subgraph on \{$v_2, v_3, v_4, v_9, v_8, v_7$\} is an induced $G_{20}$, which is a contradiction. So $e_7$ is present. If $e_6$ is not present, then the induced subgraph on \{$v_1, v_2, v_7, v_8, v_9, v_10$\} is an induced $P_6$, which is a contradiction. So $e_6$ is present. Thus the induced subgraph on \{$v_1, v_2, v_3, v_4, v_9, v_8$\} is an induced $G_{20}$, which is a contradiction. So both $e_5$ or $e_9$ are present. Then the induced subgraph on \{$v_1, v_2, v_4, v_5, v_6, v_7, v_9, v_10$\} is an induced $G_{21}$, which is a contradiction.

Thus all subcases when the two $P_5$’s are disjoint are contradictions.

We have thus shown that it is a contradiction that there exists a forbidden induced subgraph of bipartite probe 1-dot product graphs besides the graph in Figure 5.60.  

5.7 Recognition of Probe 1-Dot Product Graphs

By Theorem 5.4, it is possible to determine whether a given graph (partitioned or unpartitioned) is a probe 1-dot product graph by first determining whether it is probe trivially perfect and then checking whether it is $3K_2$-free. The latter can be done easily in linear time on a probe trivially perfect graph $G$ by noting that $G$ is $3K_2$-free if and only if the associated forest of $G^*$ either contains three trees on more than one vertex or contains a vertex with at least three children that are not leaves. Since determining if a graph $G$ is a probe trivially perfect graph [5], recognizing if a graph is a probe
1-dot-product graph can be done in linear time.
6.1 Introduction

Combinatorial orthogonality was first introduced by Beasley, Brualdi, and Shader [6]. They defined vectors \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) to be combinatorial orthogonal if

\[
|\{i : x_i y_i \neq 0\}| \neq 1.
\]

This definition means that combinatorial orthogonality of two vectors is only dependent on the positions of the nonzero coordinates. An alternate definition is vectors \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) with \( x_i, y_i \in \{0, 1\} \) for \( 1 \leq i \leq n \) are combinatorial orthogonal if \( x \cdot y \neq 1 \).

This definition can be extended to matrices such that a matrix \( A \) is combinatorial row-orthogonal if its rows are pairwise combinatorial orthogonal. Similarly \( A \) is combinatorial column-orthogonal if its columns are pairwise combinatorial orthogonal. If \( A \) is a square matrix that is both combinatorial row-orthogonal and combinatorial column-orthogonal, it is called a combinatorial orthogonal matrix. Beasley et al. used this definition to determine the minimum number of nonzero entries in an orthogonal matrix of order \( n \) that cannot be decomposed into two smaller orthogonal matrices. Some work has also been done on the combinatorial orthogonality of the digraph of orthogonal matrices [6, 57, 68].

6.2 Combinatorial Orthogonal Graphs

Let \( G = (V, E) \) be a simple undirected graph. Then we say \( G \) has a \( k \)-combinatorial orthogonal representation if there exists a function \( f : V \to \mathbb{R}^k \) such that for any \( u, v \in V \) \( uv \notin E \) if and only if \( f(u) \) and \( f(v) \) are combinatorial orthogonal. An alternate definition that we will utilize is \( G \) has a \( k \)-combinatorial orthogonal representation if there exists \( g : V \to \mathbb{R}^k \) with \( g_i(v) \in \{0, 1\} \) for \( v \in V \) and \( 1 \leq i \leq n \) such that for any \( u, v \in V \)
Proposition 6.1. The $k$-combinatorial orthogonal representations using real vectors with adjacency when associated vectors are combinatorial orthogonal are equivalent to the $k$-combinatorial orthogonal representations using \{0,1\}-vectors with adjacency when associated vectors have dot product not equal to 1.

Proof. Let $G = (V,E)$ be a graph.

Suppose that $G$ has a $k$-combinatorial orthogonal representation $f : V \rightarrow \mathbb{R}^k$ such that for any $u,v \in V$ $uv \notin E$ if and only if $f(u)$ and $f(v)$ are combinatorial orthogonal. There is an injective mapping $F : f(V) \rightarrow \{0,1\}^k$ such that $F(f(u))_i = 1$ if $f(u)_i \neq 0$ and 0 otherwise. The vectors $f(u)$ and $f(v)$ are combinatorial orthogonal if and only if the number of positions in common with nonzero entries is not equal to one. The mapping of these vectors, $F(f(u))$ and $F(f(v))$, has the same number of positions in common with nonzero entries. But two $\{0,1\}$-vectors with the number of positions in common with nonzero entries not equal to one will have dot product not equal to one. Thus $F \circ f$ is a $k$-combinatorial orthogonal representation using $\{0,1\}$-vectors with adjacency when associated vectors have dot product not equal to 1.

Suppose that $G$ has a $k$-combinatorial orthogonal representations $g : V \rightarrow \mathbb{R}^k$ with $g_i(v) \in \{0,1\}$ for $v \in V$ and $1 \leq i \leq n$ such that for any $u,v \in V$ $uv \in E$ if and only if $g(u) \cdot g(v) \neq 1$. Vectors $g(v)$ and $g(u)$ have dot product equal to one if and only if $g(v)_i = g(u)_i = 1$ for exactly one $i$. This means that $g(v)$ and $g(u)$ are combinatorial orthogonal if and only if $g(v) \cdot g(u) \neq 1$. Thus $g$ is a $k$-combinatorial orthogonal representation using real vectors with adjacency when associated vectors are combinatorial orthogonal.

Since both of the representations are equivalent, we will primarily use the representations with $\{0,1\}$-vectors. In either case, the question arises whether there exists a combinatorial orthogonal representation for every graph. Theorem 6.2 shows that such every graph has a combinatorial orthogonal representation.

Theorem 6.2. For every graph $G = (V,E)$, there exists an integer $k$ such that $G$ has a $k$-combinatorial orthogonal representation.
Proof. Let $\tilde{G} = (V, \tilde{E})$ be the complement of $G$ and $k = |\tilde{E}|$. Label the edges of $\tilde{G}$ $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k\} = \tilde{E}$. Define $f : V \to \mathbb{R}^k$ such that for $v \in V$ $f_i(v) = 1$ if $\tilde{e}_i$ is incident to $v$ and 0 otherwise. Note that $uv \notin E$ if and only if $f(u) \cdot f(v) = 1$. Similarly, $uv \notin E$ if and only if $f(u) \cdot f(v) = 0$. Thus $G$ is a $k$-combinatorial orthogonal graph.

We also define the \textit{combinatorial orthogonal dimension} of $G$, denoted $\rho_{co}(G)$, to be the minimum $k$ such that there exists a combinatorial orthogonal representation of $G$ with vectors in $\mathbb{R}^k$. If the combinatorial orthogonal dimension of a graph $G$ is at most $k$, we may refer to $G$ as a $k$-combinatorial orthogonal graph. From Theorem 6.2, we can see that $\rho_{co}(G) \leq |\tilde{E}|$. This bound is interesting in that it is based upon the non-adjacencies of $G$, which is opposite of the general bound for dot product graphs which is based on adjacencies.

There are still similarities between dot product graphs and combinatorial orthogonal graphs. The property of being a $k$ combinatorial orthogonal graph is hereditary (see Theorem 6.3), as are dot product graphs.

\textbf{Theorem 6.3.} Let $H$ be an induced subgraph of $G$, then $\rho_{co}(H) \leq \rho_{co}(G)$

\textbf{Proof.} Let $f : V(G) \to \mathbb{R}^k$ be a function such that $G$ is a $k$-combinatorial orthogonal graph. It can easily be noted that $f$ restricted to $V(H)$ is a $k$-combinatorial orthogonal representation. \hfill $\square$

Theorem 6.3 allows the characterization of $k$-combinatorial orthogonal graphs, for any fixed $k$, by forbidden induced subgraphs or substructures.

Since we may use representations of the form $f : V \to \{0, 1\}^k$, there are $2^k$ different vectors to choose from for each vertex of the graph being represented. Therefore, a $k$-combinatorial orthogonal representations of a graph $G = (V, E)$ is tantamount to a partition of $V$ into $2^k$ classes, each class characterized by a behavior. For example, if $f(v) = \mathbf{0}$, then $v$ is a universal vertex. We record this observation, in a form useful to us, as the following lemma.

\textbf{Lemma 6.4.} Let $G$ be a graph and $v$ is a universal vertex. Then $\rho_{co}(G) = \rho_{co}(G - v)$.

\textbf{Proof.} Let $\rho_{co}(G) = k$. By Theorem 6.3, $\rho_{co}(G - v) \leq \rho_{co}(G)$.

We will now use a proof by contradiction. Suppose that $\rho_{co}(G - v) < \rho_{co}(G)$. 


Then there exists a \( k - 1 \)-combinatorial orthogonal representation of \( G - v \), namely \( f : V(G - v) \to \{0, 1\}^{k-1} \). Now consider the combinatorial orthogonal representation of \( G \) given as follows for any \( u \in V(G) \)

\[
F(u) = \begin{cases} 
\overrightarrow{f(u)} & \text{if } u \neq v \\
\overrightarrow{0} & \text{if } u = v 
\end{cases}
\]

A brief examination shows that this representation holds for \( G \). But this is a contradiction that \( \rho_{co}(G) = k \) since this is a \( k - 1 \)-combinatorial orthogonal representation of \( G \).

Therefore, \( \rho_{co}(G - v) = \rho_{co}(G) \). \( \square \)

### 6.3 Characterizations of Combinatorial Orthogonal Graphs of Dimension \( k \)

In this section, we will examine the combinatorial orthogonal graphs of a given dimension. The first characterization to consider is for dimension 1.

**Theorem 6.5.** Let \( G \) be a graph. Then \( \rho_{co}(g) = 1 \) if and only if \( G \) is a complete split graph.

**Proof.** (\( \Rightarrow \)) Since \( \rho_{co}(G) = 1 \), there are only two possible vectors for the representation of \( G \), namely \([1]\) and \([0]\). Every vertex that is associated with \([1]\) is not adjacent to every other vertex with the same vector. Thus these vertices form an independent set. Similarly the vertices that are associated with \([0]\) are universal vertices. By definition, a graph whose vertices can be partitioned into a set of universal vertices and an independent set is a complete split graph.

(\( \Leftarrow \)) Let \( G \) be a complete split graph. Then the vertices can be partitioned a clique \( V_1 \) and an independent set \( V_2 \) with every vertex in \( V_1 \) adjacent to every vertex in \( V_2 \). We can assign each vertex in \( V_1 \) the vector \([0]\) and each vertex in \( V_2 \) the vector \([1]\). Since \([1] \cdot [1] = 1\), the vertices in \( V_2 \) do preserve their independence. Similarly, \([0] \cdot [0] = 0\) and \([0] \cdot [1] = 0\) preserving the universal nature of the vertices in \( V_1 \). \( \square \)

This characterization of \(1\)-combinatorial orthogonal graphs as complete split graphs includes a characterization of forbidden subgraphs. This characterization of forbidden subgraphs is given in Theorem 6.6.
Theorem 6.6. [52] A graph is a complete split graph if and only if the graph is free of the graphs in Figure 6.1.

\[ K_2 \cup K_1 \quad C_4 \]

**Figure 6.1:** Forbidden Subgraphs of Complete Split Graphs and 1-Combinatorial Orthogonal Graphs

The forbidden $C_4$ subgraph of 1-combinatorial orthogonal graphs leads to the fact that the complete bipartite graph $K_{m,n}$ with $\min\{m,n\} \geq 2$ has $\rho_{co}(G) > 1$. But we can show the exact combinatorial orthogonal dimension of complete bipartite graphs and complete $k$-partite graphs.

**Lemma 6.7.** $\rho_{co}(K_{m,n}) = 2$ if $\min\{m,n\} \geq 2$.

*Proof.* Let $K_{m,n}$ be a complete bipartite graph with partite sets $X$ and $Y$ and $\min\{m,n\} \geq 2$. Without loss of generality, let $|X| = m$ and $|Y| = n$.

We will first show that 2 is an upper bound for $\rho(K_{m,n})$. We can assign all of the vertices in $X$ the vector $e_1 = [1,0]^T$ and all of the vertices in $Y$ the vector $e_2 = [0,1]^T$. It can be observed that $e_1 \cdot e_1 = 1$ and $e_2 \cdot e_2 = 1$. Therefore the independence of $X$ and of $Y$ is preserved. Similarly, $e_1 \cdot e_2 = 0$, which maintains the adjacency between $X$ and $Y$. This implies that $\rho_{co}(K_{m,n}) \leq 2$.

To show the lower bound, we can consider $C_4 \cong K_{2,2}$. Any $K_{m,n}$ with $\min m,n \geq 2$ has $K_{2,2}$ as a subgraph. By Theorem 6.6 and Theorem 6.3, $\rho_{co}(K_{m,n}) \geq 2$.

Thus $\rho_{co}(K_{m,n}) = 2$. \qed

**Theorem 6.8.** $\rho(K_{n_1,n_1,\ldots,n_k}) = k$ if $\min\{n_1,n_2,\ldots,n_k\} \geq 2$. 

Proof. Suppose $G = (V_1 \cup \cdots \cup V_k, E) \cong K_{n_1,n_2,\ldots,n_k}$ and $|V_i| = n_i$.

We will first show that $k$ is an upper bound for $\rho(K_{n_1,n_2,\ldots,n_k})$. We can assign all of the vertices in the $i^{th}$ partite set $e_i$, where $e_i$ is the $\{0,1\}^k$-vector with one 1 in the $i^{th}$ component and 0 in the other components. The properties of complete $k$-partite graphs are preserved since $e_i \cdot e_j$ is 1 if $i = j$ and 0 otherwise. This representation implies that $\rho(K_{n_1,n_2,\ldots,n_k}) \leq k$.

To show the lower bound, we will use a proof by contradiction. So suppose that $\rho(K_{n_1,n_2,\ldots,n_k}) < k$. This implies that there is a $k-1$ combinatorial orthogonal representation of $K_{n_1,n_2,\ldots,n_k}$. There are at most $k-1$ standard unit vectors. We can assign $k-1$ of the partite sets a standard unit vector. The $k^{th}$ partite set, $V_k$, is represented by vectors that have zero 1’s or two or more 1’s. If the all zero vector for any vertex in $V_k$, then that vertex is adjacent to every other vertex in $V_k$. This is a contradiction of the independence of $V_k$. So we are limited to vectors with at least two 1’s. But any such vector has a dot product of 1 with at least two of the other partite sets. This is a contradiction of the completeness. Further the vertices in $V_k$ need different vectors since the dot product of any vector with at least two 1’s with itself is 2 or greater, which implies adjacency contradictory to the independence of $V_k$.

Thus $\rho(K_{n_1,n_2,\ldots,n_k}) = k$. \qed

These theorems lay the foundation of characterizing combinatorial orthogonal graphs of dimension 2 and beyond. There characterizations are given in the following theorems.

**Theorem 6.9.** Let $G$ be a graph. Then $\rho_{co}(G) \leq 2$ if and only if $V$ can be partitioned into $V = \bigcup_{i=1}^{4} V_i$, with $V_i = \emptyset$ for $1 \leq i \leq 4$ possible such that:

- $V_1$ contains the universal vertices in $G$.
- $V_2$ is a clique that is not adjacent to any of the vertices in $V_3$ or $V_4$.
- $V_3$ and $V_4$ are each independent sets that form a complete bipartite graph.

(These relationships are represented in Figure 6.2.)
Proof. \((\Rightarrow)\) If \(\rho_{\text{co}}(G) \leq 2\), our vectors are of dimension at most 2. Therefore there are four possible vectors, namely

\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

It can be observed that the vertices corresponding to \(\vec{0}\) must be a clique of universal vertices since the dot product of the vector with any other vector is equal 0. Thus those vertices are adjacent to every other vertex. Similarly the vertices that are assigned \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\) form a clique and are adjacent to the vertices with the universal vertices but not to the vertices assigned \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\). Finally it has been shown that the vertices assigned \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\) form a complete bipartite graph. Thus the possible vectors correspond to the vertex partitions described.

\((\Leftarrow)\) Suppose that the vertices of \(G\) can be partitioned as described. We can assign the vertices in \(V_1\) with the vector \(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\), the vertices in \(V_2\) the vector \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\), and the vertices in \(V_3\) and \(V_4\) the vectors \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\), respectively. A brief examination of the dot products of these vectors shows that the representation maintains the properties of the vertex sets. Thus \(\rho_{\text{co}}(G) \leq 2\).

\(\square\)

**Figure 6.2:** Representation of the Relationship Between 2-Combinatorial Orthogonal Graphs Vertex Partitions
This characterization of the vertex structure can be used to identify forbidden subgraphs of 2-combinatorial orthogonal graphs. These forbidden subgraphs are given in Figure 6.4 and proven in Lemma 6.10.

**Lemma 6.10.** Let $G$ be a graph and $\rho_{co}(G) \leq 2$. Then none of the graphs in Figure 6.4 are induced subgraphs of $G$.

**Proof.** We will consider each graph to explain why is it forbidden. As we do so, it can be noted that no minimal forbidden subgraph will contain a universal vertex. This restriction is based on Lemma 6.4.

Consider $2K_2 \cup K_1$. This graph does not contain any universal vertices. Therefore the vertices need to be partitioned into a clique and a complete bipartite graph. This partitioning accounts for $2K_2$, but it does not account for the $K_1$. Similarly $K_2 \cup kK_1$ as we consider the $K_2$ as the clique $V_2$ and $kK_1$ as the independent set $V_3$. Therefore $2K_2 \cup K_1$ fails to satisfy the vertex partition properties of 2-combinatorial orthogonal graphs and is minimal.

Consider $P_4$. This graph does not contain any universal vertices. Therefore the vertices need to be partitioned into a clique and a complete bipartite graph. However, a $P_4$ is the minimal example of a graph that is neither a clique or a complete bipartite graph. Thus $P_4$ is a minimal forbidden subgraph of 2-combinatorial orthogonal graphs.

Consider $K_{2,2,2}$. This graph does not contain any universal vertices. We showed in Theorem 6.8 that the combinatorial orthogonal dimension of this graph is 3. Further, if we remove any vertex it can be noted that we have a universal vertex that is adjacent to a complete bipartite graph. Thus, $K_{2,2,2}$ is a minimal forbidden subgraph of 2-combinatorial orthogonal graphs.

Consider $2K_3$. This graph does not contain any universal vertices. Therefore the vertices need to be partitioned into a clique and a complete bipartite graph. While $K_3$ is a clique, $K_3$ is not a bipartite graph. Additionally if a vertex is removed, we have a $K_3 \cup K_2$. In that case, we label $K_3$ as the clique $(V_3)$ and $K_2$ as the complete bipartite graph $(V_3$ and $V_4)$. Thus, $2K_3$ is a minimal forbidden subgraph of 2-combinatorial orthogonal graphs.

Consider $2K_1 \cup P_3$. This graph does not contain any universal vertices. Therefore the vertices need to be partitioned into a clique and a complete bipartite graph. Since $P_3$ is the minimal graph that is not a clique, the $2K_1$ needs to be a clique, which is a
contradiction. Further if we remove one of the $K_1$’s, the remaining $K_1$ is a clique. If we removed an endpoint in the $P_3$, we have $K_2 \cup 2K_1$, which we have already shown has combinatorial orthogonal dimension 2. If we removed the midpoint in the $P_3$, we have $4K_1$, which we have already shown has combinatorial orthogonal dimension 1. Thus, $2K_1 \cup P_3$ is a minimal forbidden subgraph of 2-combinatorial orthogonal graphs.

Consider $G_1$. This graph does not contain any universal vertices. Therefore the vertices need to be partitioned into a clique and a complete bipartite graph. However this graph is connected, so the graph needs to be a clique or a complete bipartite graph. Since no vertex is a universal vertex, $G_1$ cannot be a clique (every vertex in a clique is universal for the clique). Similarly the largest independent sets of $G_1$ contains only two vertices and for $G_1$ to be a bipartite graph there needs to be an independent set of at least three vertices. To show that this graph is minimal, consider the labelling of $G_1$ as shown in Figure 6.3.

![Figure 6.3: Labelling of $G_1$](image)

If vertex $v_1$ is removed, then $v_2$ and $v_5$ are universal vertices of the induced subgraph with $v_3$ and $v_4$ forming an independent set. This implies that the induced subgraph has combinatorial orthogonal dimension 1. If vertex $v_2$ (or $v_5$) is removed, then the induced subgraph is a $K_{2,2}$, which has been shown to have combinatorial orthogonal dimension 2. If vertex $v_3$ (or $v_4$) is removed, then the induced subgraph has $v_4$ (or $v_3$) as a universal vertex. Removing that universal vertex then results in $K_2 \cup K_1$ which is a complete bipartite graph ($K_2$) and a clique ($K_1$). This induced graph has combinatorial orthogonal dimension 2. Thus, $G_1$ is a minimal forbidden subgraph of 2-combinatorial
orthogonal graphs.

\begin{figure}
\centering
\begin{tabular}{ccc}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill,inner sep=2pt] {};
\node (v2) at (-0.5,0.5) [circle,fill,inner sep=2pt] {};
\node (v3) at (0.5,0.5) [circle,fill,inner sep=2pt] {};
\node (v4) at (0,1) [circle,fill,inner sep=2pt] {};
\end{tikzpicture} & \begin{tikzpicture}
\node (v1) at (0,0) [circle,fill,inner sep=2pt] {};
\node (v2) at (-0.5,0.25) [circle,fill,inner sep=2pt] {};
\node (v3) at (0.5,0.25) [circle,fill,inner sep=2pt] {};
\end{tikzpicture} & \begin{tikzpicture}
\node (v1) at (0,0) [circle,fill,inner sep=2pt] {};
\node (v2) at (-0.5,0.5) [circle,fill,inner sep=2pt] {};
\node (v3) at (0.5,0.5) [circle,fill,inner sep=2pt] {};
\node (v4) at (0,1) [circle,fill,inner sep=2pt] {};
\node (v5) at (0,1.5) [circle,fill,inner sep=2pt] {};
\end{tikzpicture} \\
2K_2 \cup K_1 & P_4 & K_{2,2,2}
\end{tabular}
\caption{Minimally Forbidden Subgraphs of 2-Combinatorial Orthogonal Graphs}
\end{figure}

**Theorem 6.11.** Let $G$ be a graph. Then $\rho_{co}(G) \leq 3$ if and only if $V$ can be partitioned into $V = \bigcup_{i=1}^{8} V_i$, with $V_i = \emptyset$ for $1 \leq i \leq 8$ possible such that:

- $V_1$ contains the universal vertices in $G$.
- $V_2$ is a clique that is not adjacent to any of the vertices in $V_6$, $V_7$, or $V_8$.
- $V_3$, $V_4$, and $V_5$ are each cliques that are not adjacent to each other.
- $V_6$, $V_7$ and $V_8$ are each independent sets that form a complete tripartite graph.
- $V_2$ is adjacent to each of the cliques $V_3$, $V_4$, and $V_5$.
- Each of the cliques $V_3$, $V_4$, and $V_5$ are adjacent to one and only one of the independent set $V_6$, $V_7$, and $V_8$ with each clique adjacent to a different independent set.
Proof. (⇒) If \( \rho_{\co}(G) \leq 3 \), our vectors are of dimension at most 3. Therefore there are eight possible vectors, namely

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix},
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix},
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix},
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}.
\]

It can observed that the vertices corresponding to \( \overrightarrow{\mathbf{o}} \) must be a clique of universal vertices since the dot product of the vector with any other vector equals 0. Thus those vertices are adjacent to every other vertex. Similarly the vertices that are assigned \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) form a clique. Additionally these vertices are adjacent to the vertices assigned the vectors \( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) but not to the vertices assigned \( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) each form cliques but each clique is not adjacent to the other two cliques. The vertices assigned \( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) are adjacent to the vertices assigned \( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \), the vertices assigned \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) are adjacent to the vertices assigned \( \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \), and the vertices assigned \( \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) are adjacent to the vertices assigned \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \). Finally it has been shown that the vertices assigned \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) form a complete bipartite graph. Thus the possible vectors correspond to the vertex partitions described.

(⇐) Suppose that the vertices of \( G \) can be partitioned as described. We can assign the relationships are represented in Figure 6.5.)
vertices in $V_1$ with the vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, the vertices in $V_2$ the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and the vertices in $V_3, V_4,$ and $V_5$ the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, respectively, and the vertices in $V_6$, $V_7$, and $V_8$ the vectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, respectively. A brief examination of the dot products of these vectors shows that the representation maintains the properties of the vertex sets. Thus $\rho_{co}(G) \leq 3$. 

\[ \square \]

![Figure 6.5: Representation of the Relationships Between 3-Combinatorial Orthogonal Graphs Vertex Partitions](image)

These characterizations lead to an observation that for any combinatorial orthogonal graph of dimension $k$ the vertices can be partitioned into $2^k$ distinct vertex sets, since there are $2^k$ possible $\{0,1\}^k$ vectors. Further, these vector (and the associated vertices) can be partitioned into sets based on the number $i$ ($0 \leq i \leq k$) of nonzero entries. For each $i$ there are $\binom{k}{i}$ vectors which will all have similar behaviors. (Since $i = 0, \cdots, k$, this partitioning of vertices is verified by the known combinatorics identity, $2^k = \sum_{i=0}^{k} \binom{k}{i}$.) For example, let $G = (V,E)$ be a graph and $f : V \to \{0,1\}^k$ a combinatorial orthogonal representation of $G$. If $v \in V$ has $i \neq 1$ nonzero components in $f(v)$, then all of such vertices with the same vector will form cliques. If $v \in V$ has 1 nonzero components in $f(v)$, the vertices with the same vector form an independent set.
Similarly it can be noted that all vectors with $i$ nonzero components where $i \geq \left\lceil \frac{k}{2} \right\rceil + 1$ will be adjacent to all of the other vectors with the same $i$. Also for each of the vectors with $i$ nonzero components are all adjacent to the vectors with at least $k - i + 2$ nonzero components.
CHAPTER 7
CONCLUDING REMARKS AND FUTURE DIRECTIONS

As we studied the representation in this dissertation, we often found that solving one question led to several more questions. These open problems are given in this final chapter to direct future lines of research of these representations. Problems are given for bipartite dot product graphs, probe $k$-dot product graphs, and combinatorial orthogonal graphs.

7.1 Bipartite Dot Product Graphs

We showed the forbidden subgraph characterization of 1-bipartite dot product graphs in Theorem 4.7. We propose using this characterization to develop a 2-SAT recognition algorithm for 1-bipartite dot product graphs. This algorithm could then be used to determine the complexity of determining if a given graph is a 1-bipartite dot product graph.

We also propose finding a forbidden induced subgraph characterization of 2-bipartite dot product graphs. Since a characterization of 2-dot product graphs has yet to be found, this particular characterization may also be elusive [45]. If such is the case, we propose finding a partial characterization of 2-bipartite dot product graphs similar to the proof that a bipartite claw has dot product dimension 3 [28].

We also propose determining the dot product dimension of other classes of bipartite graphs, such as interval bigraphs [41]. We propose creating a representation similar to the cap-capture graphs used by Scheinerman to show that interval bigraphs have bipartite dot product dimension 2 [28].

Another class of graphs to compare bipartite dot product graphs with is difference graphs. A graph $G = (V, E)$ is a difference graph if there exists $f : V \rightarrow \mathbb{R}$ with $|f(v)| < T$ for each $v \in V$ such that $uv \in E$ if and only if $|f(u) - f(v)| > T$. It has been shown that difference graphs are bipartite [40]. This representation is related to threshold graphs, which were generalized by dot product graphs [66]. Thus we believe
that bipartite dot product graphs are a generalization of difference graphs.

In Theorem 4.8, we showed that the real rank of the bipartite adjacency matrix of a graph $G$ is an upper bound on the bipartite dot product dimension of $G$. But the multiple ranks of matrices. We propose considering alternate ranks such as non-negative integer rank and boolean rank.

Finally, we propose finding the maximum bipartite dot product dimension of a graph on $n$ vertices, similar to the dot product dimension bound in Conjecture 2.19. A related question is what graphs on $n$ vertices have attain this maximum bipartite dot product dimension.

7.2 Probe 1-Dot Product Graphs

We gave a partial characterization of unpartitioned probe 1-dot product graphs in Corollary 5.11. Conjecture 5.12 proposed that this characterization was complete. The proof of this conjecture is left as an open problem.

We also propose determining the dot product dimension of all of the forbidden subgraphs of probe 1-dot product graphs in Figure 5.8. We conjecture that the dot product dimension of each graph is 2. We similarly conjecture that any forbidden subgraphs of probe $k$-dot product graphs would be of dot product dimension $k + 1$. We believe that proving these conjectures will strengthen, if not prove, Conjecture 2.19.

7.3 Combinatorial Orthogonal Graphs

We gave a partial forbidden induced subgraph characterization of 2-combinatorial orthogonal graph in Figure 6.4. We conjecture that this characterization is complete.

Similarly, we propose finding the forbidden induced subgraph characterizations of 3- and 4-combinatorial orthogonal graphs. Based on those characterizations, we hope to find a sequence of numbers that correspond to the number of forbidden subgraphs for each dimension.

In addition to determining the characterizations of specific combinatorial orthogonal dimension, we propose determining the combinatorial orthogonal dimension bounds for specific classes of graphs, such as interval graphs, cycles, and trees.

We finally make the following conjecture about $k$-combinatorial orthogonal representations.

**Conjecture** 7.1. Let $X$ be a $k$-combinatorial orthogonal representation of a graph $G$,
and let $U$ be a $k \times k$ combinatorial orthogonal matrix. Then $UX$ is also a $k$-combinatorial orthogonal representation of $G$.

The proof of this conjecture will likely focus on the linear algebra properties related to combinatorial orthogonal matrices.
BIBLIOGRAPHY


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