AN INVESTIGATION OF THE RANGE OF A BOOLEAN FUNCTION

by

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TABLE OF CONTENTS

INTRODUCTION TO BOOLEAN ALGEBRA ........................................ 1
DEFINITION OF A RELATIVE BOOLEAN ALGEBRA ....................... 9
BOOLEAN FUNCTIONS .......................................................................... 16
BOOLEAN FUNCTIONS IN A RELATIVE BOOLEAN ALGEBRA ............. 24
FUNCTIONS BETWEEN RELATIVE BOOLEAN ALGEBRAS ............... 31
RELATIVE BOOLEAN ALGEBRAS AND SUB-BOOLEAN ALGEBRAS ........ 37
INTRODUCTION TO BOOLEAN ALGEBRA

The purpose of this section is to define a boolean algebra and to determine some of the important properties of it.

A boolean algebra is a set $B$ with two binary operations, join and meet, denoted by $+$ and juxtaposition respectively, and a unary operation, complementation, denoted by $'$, which satisfy the following axioms:

(1) for all $a, b \in B$ (that is, for all $a, b$ elements of $B$) $a + b = b + a$ and $a \cdot b = b \cdot a$, (the commutative laws),

(2) for all $a, b, c \in B$, $a + b \cdot c = (a + b) \cdot (a + b)$ and $a \cdot (b + c) = a \cdot b + a \cdot c$, (the distributive laws),

(3) there exists $0 \in B$ such that for each $a \in B$, $a + 0 = a$, and there exists $1 \in B$ such that for each $a \in B$, $a \cdot 1 = a$,

(4) for each $a \in B$, $a + a' = 1$ and $a \cdot a' = 0$.

If $a + e = a$ for all $a$ in $B$ then $0 = 0 + e = e + 0 = e$, so that there is exactly one element in $B$ which satisfies the first half of axiom 3, namely $0$. Similarly there is exactly one element in $B$ which satisfies the second half of axiom 3, namely $1$.

The $0$ and $1$ as defined above will be called the distinguished elements.
If in the statement of any of the four axioms join, meet, 0, and 1 are replaced by meet, join, l, 0 respectively, the axiom remains unchanged. Thus it follows that if a statement can be proved from the axioms, then the statement with join, meet, 0, and 1 replaced by meet, join, l, and 0 respectively can also be proved. The two statements are called dual statements.

From the axioms it follows that: \(a a = a a + 0 = a a + a a' = a (a + a') = a l = a\); thus the idempotent laws follow:

(i) \(a a = a\) and its dual

(ii) \(a + a = a\)

Let \(c + x = 1\) and \(c x = 0\), then \(c' = c' 1 = c' (x + c) = c' x + c' c = x c' + x c = x (c' + c) = x l = x\). Thus axiom 4 defines the operation complementation in the respect that \(x = c'\) if and only if \(c + x = 1\) and \(c x = 0\).

The above property is also useful to show two elements are equal.

Since \(l = a + a' = a' + a\) and \(0 = a a' = a' a\) it follows from above that:

(ii) \((a')' = a\)

Notice that (ii) is a self-dual statement.

It follows that \(l = b + 1\), since \(b (l + b) = b l + b b = b + b = b\) and \(b' (l + b) = b' l + b' b = b' + 0 = b'\) implies that \(l = b + b' = b (l + b) + b' (b + l) = (b + b') (1 + b) = 1 (l + b) = l + b\), therefore:

(iii) \(l = b + 1\) and its dual
(iii') $0 = b \ 0$

The absorption laws thus follow: $a + a \ b = a \ l + a \ b = a \ (l + b) = a \ l = a$:

(iv) $a + a \ b = a$ and its dual

(iv') $a (a + b) = a$

Also: $a + b \ a' = (a + b) (a + a') = (a + b) \ a' = a + b$. Hence:

(v) $a + b = a + b \ a'$ and its dual

(v') $a b = a (b + a')$

Next the associative law for join will be established.

Using the fact $a' \ (b \ a) = a' \ (b \ a) + a \ a' = a' \ (b \ a + a) = a' \ a = 0$ in the form $x' [((y + z) \ x)] = 0$ it follows that:

$(x + y) + z = (x + x' \ y) + z = [x + (0 + x' \ y)] + z = [x + (y + z) + x + x' \ y] + z = [x + (y + z) \ (x + y) + x' \ y] + z = [x + (y + z)] \ (x + y) + [x + (y + z)] \ z = [x + (y + z)] \ ((x + y) + z]$. Thus $a + (b + c) = (c + b) + a = [c + (b + a)] \ [(c + b) + a] = [a + (b + c)] \ [(a + b) + c] = (a + b) \ c$, and the associative laws are proved:

(vi) $a + (b + c) = (a + b) + c$ and its dual

(vi') $a (b \ c) = (a \ b) \ c$

Define $a + b + c$ to be the common value of $a + (b + c)$ and $(a + b) + c$. Also define $a \ b \ c = a \ (b \ c) = (a \ b) \ c$. 

From (iii) and (iii') it follows that \( l = 0 + 1 \) and \( 0 = 1 \cdot 0 = 0 \cdot 1 \); hence:

(vii) \( l = 0' \) and its dual
(vii') \( 0 = 1' \)

Since \((a + b) a' b' = a a' b' + b a' b' = 0 + 0 = 0\) and \((a + b) + a' b' = (a + b + a') (a + b + b') = 1 \cdot l = l\), the De Morgan laws follow:

(viii) \( (a + b)' = a' b' \) and its dual
(viii') \( (a b)' = a' + b' \)

Define \( x < y \) to mean \( x + y = y \).

If \( x < y \) and \( y < z \) then \( x + y = y \) and \( y + z = z \), thus \( x + z = x + y + z = y + z = z \) and \( x < z \). Therefore the relation \( < \) is transitive.

If \( x < y \) and \( y < x \) then \( x + y = y \) and \( y + x = x \) thus \( x = y \). Hence the relation \( < \) is a partial order in a boolean algebra. A partial order is a reflexive, anti-symmetric, transitive relation.

Define \( x > y \) to mean \( x y = y \). In a sense \( > \) is the dual relation of \( < \).

If \( x < y \) then \( x + y = y \) and \( y x = (x + y) x = x \) thus \( y > x \). Similarly if \( x > y \) then \( y < x \).

Since \( x y = x \) is a necessary and sufficient condition that \( x + y = y \), either condition will be used for \( x < y \). Furthermore \( x < y \) and \( y > x \) will be used interchangeably.

By (iii) and (iii'), it follows that \( 0 < a < l \) for all elements \( a \) in \( B \).
Let \( x < y \). Then \( x + y = y \), or by De Morgan’s law \( x' y' = y' \) and \( y' < x' \). Also \( a (x + y) = a y \) or \( a x + a y = a y \) thus \( a x < a y \). Furthermore
\[ a + x + y = (a + x) + (a + y) = a + y \] thus \( a + x < a + y \).

If \( x < y \) then \( x y' = (x y) y' = 0 \), also if \( x y' = 0 \) then \( x y = x y + x y' = x (y + y') = x \) or \( x < y \). Similarly if \( x < y \) then by De Morgan’s law on the above result \( x' + y = 1 \), and if \( x' + y = 1 \) then \( x < y \).

Let \( x_1, \ldots, x_n \) be variables whose common domain is a subset \( D \) of a boolean algebra \( B \). A function \( f \) is called a boolean function if the rule for the function, \( f(x_1, \ldots, x_n) \), can be be built up from the variables \( x_1, \ldots, x_n \) and elements of \( B \) by a finite number of operations meet, join, and complementation.

The range of the boolean function \( f \) is the set \( R = \{r : r \in B, \text{ and there exist } d_1, \ldots, d_n \in D \text{ for which } f(d_1, \ldots, d_n) = r\} \). \( \{x : P(x)\} \) means the set of all \( x \) such that \( x \) has property \( P \); hence the set \( R \) is the set of all elements \( r \) of \( B \) for which there exist \( d_1, \ldots, d_n \) elements of \( D \) and \( f(d_1, \ldots, d_n) = r \).

**Theorem I.1**

Every boolean function of one variable in \( B \) has a rule of the form \( f(x) = f(1) x + f(0) x' \).

**Proof:** Since \( a = a (x + x') = a x + a x' \), and \( x = 1 x + 0 x' \), the statement is true if \( f(x) = a \) or \( f(x) = x \).
If \( g(x) \) and \( h(x) \) are of the required form, that is if \( g(x) = g(1) x + g(0) x' \) and \( h(x) = h(1) x + h(0) x' \), then \([g(x)]', g(x) + h(x), \) and \( g(x) h(x) \) are also of the required form, since:

\[[g(x)]' = [g(1) x + g(0) x']' =
\]
\[[[g(1)]' + x'] [g(0)]' + x] =
\[[g(1)]' [g(0)]' + [g(1)]' x + [g(0)]' x' =
\[[g(1)]' [g(0)]' x + [g(1)]' [g(0)]' x' + [g(1)]' x +
\]
\[[g(0)]' x' =
\]
\[[g(1)]' x + [g(0)]' x'. Also, \( g(x) + h(x) =
\)
\[g(1) x + g(0) x' + h(1) x + h(0) x' =
\]
\[[g(1) + h(1)] x + [g(0) + h(0)] x'. And, \( g(x) h(x) =
\]
\[[g(1) x + g(0) x'] [h(1) x + h(0) x'] =
\[g(1) h(1) x + g(0) h(0) x'.
\]

Since a boolean function of one variable is a finite number of applications of meet, join, and complementation on \( x \) and elements of \( B \), it follows by induction that all boolean functions of one variable are in the form \( f(x) = f(1) x + f(0) x' \).

Define, in a boolean algebra \( B \), \( \sum_{(e) \in B} f(e_1, \ldots, e_n) \) and \( \prod_{(e) \in B} f(e_1, \ldots, e_n) \) to be the join and meet respectively over all combinations such that either \( e_i = 0 \) or \( e_i = 1 \), where \( 0 \) and \( 1 \) are the distinguished elements of \( B \).
Example: \[ \sum_{(e)} f(e_1, e_2) = f(1,1) + f(1,0) + f(0,1) + f(0,0) \]

and \[ \prod_{(e)} f(e_1, e_2) = f(1,1) f(1,0) f(0,1) f(0,0). \]

Let e equal 0 or 1, define \( x^e \) to be \( x \) if \( e = 1 \) and \( x' \) if \( e = 0 \).

It then follows that a boolean function of one variable is in the form \( f(x) = f(1) x + f(0) x' = \sum_{(e)} f(e_1) x^e. \)

**Theorem I.2**

If \( f \) is a boolean function of \( n \) variables then

\[ f(x_1, \ldots, x_n) = \sum_{(e)} f(e_1, \ldots, e_n) x_1^{e_1} \cdots x_n^{e_n}. \]

**Proof:** The theorem will be proved by induction.

If \( n=1 \), then from above the statement is true. Assume that for \( n=k \) the statement is true. Then since

\[ f(x_1, \ldots, x_k, x_{k+1}) \]

can be thought of as a boolean function of one variable \( x_{k+1} \), \( f(x_1, \ldots, x_k, x_{k+1}) = f(x_1, \ldots, x_k, 1) x_{k+1} + f(x_1, \ldots, x_k, 0) x_{k+1}' \). By the induction hypothesis, \( f(x_1, \ldots, x_k, x_{k+1}) = \)

\[ \sum_{(e)} f(e_1, \ldots, e_k, 1) x_1^{e_1} \cdots x_k^{e_k} x_{k+1} + \sum_{(e)} f(e_1, \ldots, e_k, 0) x_1^{e_1} \cdots x_k^{e_k} x_{k+1}'. \]
Thus the statement is true for \( k+1 \) whenever it is true for \( k \), and hence true for all positive integers.

**Theorem I.3**

\[
\sum_{(e)_{B}} a_1^{e_1} \cdots a_n^{e_n} = 1, \quad a_i \in B.
\]

**Proof:** This also will be proved by induction.

If \( n=1 \) then \( \sum_{(e)_{B}} a_1^{e_1} = a_1 + a'_1 = 1. \) Assume the statement is true for \( n=k \). Then

\[
\sum_{(e)_{B}} a_1^{e_1} \cdots a_k^{e_k} a_{k+1}^{e_{k+1}} = \left[ \sum_{(e)_{B}} a_1^{e_1} \cdots a_k^{e_k} \right] a_{k+1} + \left[ \sum_{(e)_{B}} a_1^{e_1} \cdots a_k^{e_k} \right] a'_{k+1} = a_{k+1} + a'_{k+1} = 1.
\]
DEFINITION OF A RELATIVE BOOLEAN ALGEBRA

This section will define the concept of a relative boolean algebra of a boolean algebra, and give some properties of them.

Let a boolean algebra \( B \) and \( a, b \) elements of \( B \), be given. Define the set \( S_{a,b} \) by \( S_{a,b} = \{ x : x \in B, a < x < b \} \). Hence \( S_{a,b} \) is the set of all elements \( x \) of \( B \) where \( a < x < b \).

Note that if \( a < b \) then \( a, b \in S_{a,b} \). Let \( x \in S_{a,b} \), by transitivity \( a < b \). Thus \( S_{a,b} \neq \emptyset \), the empty set, if and only if \( a < b \). In section I it was shown that for all \( x \in B, 0 < x < 1 \); thus \( S_{0,1} = B \).

Let \( S_{a,b} \neq \emptyset \), that is \( a < b \). Since \( a (b + a) = a \) and \( b (b + a) = b x + a b = b x + a, a < b x + a < b \). Thus for all \( x \in B, b x + a \in S_{a,b} \). If \( x \in S_{a,b} \), then \( a < x < b \) or \( b x + a = x + a = x \).

Throughout this section it will be assumed that a boolean algebra \( B \) has been given and that all elements are elements of \( B \).

Theorem II.1

\( S_{a,b} \) is closed under meet and join of \( B \); that is if \( x, y \in S_{a,b} \), then \( x + y, x y \in S_{a,b} \).
Proof: Let \( x, y \in S_{a, b} \). From above \( a < x, y < b \)

and it follows that \( (x + y) + b = (x + b) + (y + b) = b, \)

and \( (x + y) a = x a + y a = a; \) therefore \( a < x + y < b. \)

Similarly, \( x y + b = (x + b) (y + b) = b, x y a = (x a) (y a) = a \) and \( a < xy < b \). Hence \( x + y \) and \( xy \) are elements of \( S_{a, b} \).

If \( S_{a, b} = \emptyset \) then \( S_{a, b} \) is closed under meet

and join.

From above note that \( C(x) = b x' + a \in S_{a, b} \) if \( S_{a, b} \neq \emptyset \). The element \( C(x) \) will be called the relative complement in \( B \) with respect to \( S_{a, b} \), or simply the relative complement if no confusion will result. \( C(x) \) is defined, of course, for each element \( x \) of \( B. \)

For convenience, \( x^o \) or \( x^+ \) will be used to denote \( C(x) \). \( C(x) \) will be shown to have properties of complementation in the set \( S_{a, b} \).

Theorem II.2

\( S_{a, b} \) is a boolean algebra, with distinguished elements \( a, b \), where the meet and join are the same as in \( B \) and complementation being the relative complement in \( B \) with respect to \( S_{a, b} \), if and only if \( S_{a, b} \neq \emptyset \).

Proof: Assume \( S_{a, b} \neq \emptyset \). From above, the three operations, meet, join, and relative complement, are
closed. Meet and join are commutative and distributive since they are the operations in \( B \). If \( x \in S_{a,b} \), then \( x + a = x \) and \( x b = x \), thus \( a \) and \( b \) are the distinguished elements of \( S_{a,b} \) as defined by axiom 3.

Let \( x^0 = b x^1 + a \), the relative complement of \( x \).

For each \( s \) of \( S_{a,b} \), \( x x^0 = x (b x^1 + a) = b x x^1 + a x = a \) and \( x + x^0 = x + b x^1 + a = x + b = b \). Thus axiom 4 is satisfied. Therefore the set \( S_{a,b} \) together with the operations meet, join, and the relative complement in \( B \) with respect to \( S_{a,b} \) is a boolean algebra.

If \( S_{a,b} \) together with the three operations form a boolean algebra, then \( S_{a,b} \neq \emptyset \).

Since the meet and join in the boolean algebra \( S_{a,b} \) are the same as in \( B \), the partial order, \( < \), is also the same, that is if \( x, y \in S_{a,b} \) and if \( x < y \) in \( B \) then \( x < y \) in \( S_{a,b} \).

A relative boolean algebra of a boolean algebra \( B \) is a subset, \( S_{a,b} \), of \( B \), \( a < b \), together with the operations meet and join of \( B \) and the relative complement in \( B \) with respect to \( S_{a,b} \). From theorem II.2, a relative boolean algebra is a boolean algebra.

Theorem II.3

Let \( S_{a,b} \) be a relative boolean algebra of \( B \) and \( S_{c,d} \) be a relative boolean algebra of \( S_{a,b} \).
Then $\overline{S}_{c,d}$ is a relative boolean algebra of $B$.

Proof: Since $\overline{S}_{c,d}$ is a relative boolean algebra of $S_{a,b}$, $a < c < d < b$. If $x \in B$ and $c < x < d$ then $x \in S_{a,b}$, thus $\overline{S}_{c,d} = \{x : x \in S_{a,b}, c < x < d\} = \{x : x \in B, c < x < d\}$. The meet and join are the same operations in both $S_{a,b}$ and $\overline{S}_{c,d}$, and in $B$. Thus it remains to be shown that the relative complement in $S_{a,b}$ with respect to $\overline{S}_{c,d}$ is the same as the relative complement in $B$ with respect to $\overline{S}_{c,d}$.

Let $x^0 = b'x + a$ be the complement in $S_{a,b}$. Then the relative complement in $S_{a,b}$ with respect to $\overline{S}_{c,d}$ is $d x^0 + c = d(bx' + a) + c = d b x' + d a + c = d x' + c$ since $a < c < d < b$. Thus the complement in $\overline{S}_{c,d}$ is the relative complement in $B$ with respect to $\overline{S}_{c,d}$.

Theorem II.4

If $S_{a,b}$ and $S_{c,d}$ are relative boolean algebras of $B$ and $S_{c,d} \subseteq S_{a,b}$, [$C$ means is a subset of], then $S_{c,d}$ is a relative boolean algebra of $S_{a,b}$.

Proof: Since $S_{a,b}$ and $S_{c,d}$ are relative boolean algebras of $B$, the meet and join are the same as in $B$. Thus, since $S_{a,b} \subseteq S_{c,d}$, the meet and join in $S_{c,d}$ is the same as in $S_{a,b}$. Also $S_{c,d} = \{x : x \in S_{a,b}\}$.
\[ c < x < d \]. Hence it remains to show that the complement in \( S_c,d \) is the relative complement in \( S_{a,b} \) with respect to \( S_c,d \); that is \( d (b x' + a) + c = d x' + c \). This is immediate since \( a < c < d < b \).

The next theorem gives a connection between relative boolean algebras in \( B \).

**Theorem II.5**

Let \( S_{a,b} \) be a relative boolean algebra of \( B \) and \( x^o = b x' + a \). If \( x \in S_{c,d} \), then \( x^o \in S_{c,d} \).

**Proof:** Let \( x \in S_{c,d} \). Then \( c < d \), or by section I, \( d' < x' < c' \), and \( b d' + a < b x' + a < b c' + a \). Thus \( 1^o < x^o < c^o \) or \( x^o \in S_{c,d} \).

**Theorem II.6**

Let \( S_{a,b} \) be a relative boolean algebra of \( B \) and \( x^o = b x' + a \). If \( S_{h,k} \) is a relative boolean algebra of \( B \) and \( x^+ \) is the complement of \( x \) in \( S_{h,k} \), then \( k^o, h^o \) is a relative boolean algebra with the complement of \( x \) being \([x^o]^+ \).

**Proof:** Since \( S_{h,k} \neq \emptyset \), it follows from theorem II.5 that \( S_{k^o, h^o} \neq \emptyset \), and thus \( S_{k^o, h^o} \) is a relative boolean algebra with the complement of \( x \) being \( k^o x' \cdot h^o \). But \([x^o]^+ \) = \([k (b x' + a) + h]^o \) =
The function $f$ defined by $f(x) = (((x^+)^+)^+)^+$, where '+' and '+' are defined as in theorem II.6, will be seen later to be a homomorphism from $S_{h,k}$ onto $S_{k_0,h_0}$.

Theorem II.7

$x \in S_{a,b} \cap S_{c,d}$, where $\cap$ is the set intersection, if and only if $a + c < x < b d$; that is, $S_{a,b} \cap S_{c,d} = S_{a + c,t,d}$.

Proof: If $x \in S_{a,b} \cap S_{c,d}$, then $x < b$ and $x < d$. Therefore $x (b d) = (x b) (x d) = x$ or $x < b d$.

Also $x > a$ and $x > c$, and it follows that $x > a + c$.

Hence $a + c < x < b d$.

If $a + c < x < b d$ then $a < x < b$ and $c < x < d$ or $x \in S_{a,b}$ and $x \in S_{c,d}$.

Theorem II.8

The following are equivalent if $S_{a,b} \neq \emptyset$ and $S_{c,d} \neq \emptyset$.

(a) $S_{a,b} \cap S_{c,d} \neq \emptyset$

(b) $a + c < b d$

(c) $a < d$ and $c < b$

(d) $a + c \in S_{a,b} \cap S_{c,d}$

(e) $b d \in S_{a,b} \cap S_{c,d}$
(f) \( S_{a,b} \cap S_{c,d} \) is a relative boolean algebra.

Proof: Let \( S = S_{a,b} \cap S_{c,d} \).

(a) implies (b): Let \( S \neq \emptyset \). Then there exists \( x \in S \), and by theorem II.7, \( a + c < x < b \cdot d \). Hence \( a + c < b \cdot d \).

(b) implies (c): Let \( a + c < b \cdot d \). Then \( C = (a + c) \cdot (b \cdot d)' = (a + c) \cdot (b' + d') = a \cdot b' + c \cdot b' + a \cdot d' + c \cdot d' = c \cdot b' + a \cdot d' \). Hence \( c \cdot b' = 0 \) and \( a \cdot d' = 0 \), or \( a < d \) and \( c < b \).

(c) implies (d): Let \( a < d \) and \( c < b \). Since \( a < b \) and \( c < d \) it follows that \( (a + c) + b \cdot d = (a + c + b) \cdot (a + c + d) = (b + c) \cdot (a + d) = b \cdot d \), or \( a + c < b \cdot d \). By theorem II.7, \( a + c \in S \).

(d) implies (e): If \( a + c \in S \) then by theorem II.7, \( a + c < b \cdot d \) and hence \( b \cdot d \in S \) by theorem II.7.

(e) implies (f): Since \( b \cdot d \in S \), \( S \neq \emptyset \), and by theorems II.7 and II.2, \( S \) is a relative boolean algebra.

(f) implies (a): Since \( S \) is a relative boolean algebra, by theorem II.2, \( S \neq \emptyset \).
In this section the connection between boolean functions and the sets of the form $S_{a,b}$ will be shown; namely that the range of a boolean function is a relative boolean algebra.

In this and the following sections $f$ will denote a boolean function of $n$ variables, that is $f(x_1, \cdots, x_n) = \sum_{(e) \in B} f(e_1, \cdots, e_n) x_1^{e_1} \cdots x_n^{e_n}$.

**Theorem III.1**

If $f$ is a function in $B$ such that $f(a_1, \cdots, a_n) = a$ and $f(b_1, \cdots, b_n) = b$, then there exist $c_1, \cdots, c_n$ such that $f(c_1, \cdots, c_n) = a + b c'$ for any $c$ an element of $B$, and furthermore $c_i \in S_{a_i b_i, a_i + b_i}$.

**Proof:** Let $c_i = a_i c + b_i c'$. Since $a_i b_i c_i = a_i b_i$ and $a_i + b_i + c_i = a_i + b_i$, it follows that $a_i b_i < c_i < a_i + b_i$.

Since $c_i' = (a_i c + b_i c')' = a_i' c + b_i' c'$, it follows that $c_i' = a_i' c + b_i' c'$. Thus $f(c_1, \cdots, c_n) = \sum_{(e) \in B} f(e_1, \cdots, e_n) (a_1 c + b_1 c')^{e_1} \cdots (a_n c + b_n c')^{e_n}$.
\[
\sum_{(e)B} f(e_1, \ldots, e_n) (a_1^e c + b_1^e c') \cdots (a_n^e c + b_n^e c') =
\]
\[
c \sum_{(e)B} f(e_1, \ldots, e_n) a_1^{e_1} \cdots a_n^{e_n} + c' + \sum_{(e)B} f(e_1, \ldots, e_n) b_1^{e_1} \cdots b_n^{e_n} =
\]
a \ c + b \ c'.

The elements \( c_1, \ldots, c_n \) exhibited have the properties required in the statement of the theorem, and thus the theorem is proved.

The next two theorems are immediate results of theorem III.1.

Theorem III.2

If \( f(a_1, \ldots, a_n) = a \) and \( f(b_1, \ldots, b_n) = b \) then there exist \( c_1, \ldots, c_n \) and \( d_1, \ldots, d_n \) such that \( f(c_1, \ldots, c_n) = a + b \) and \( f(d_1, \ldots, d_n) = ab \) where \( c_i, d_i \in S_{a_i b_i} a_i + b_i \).

Proof: In theorem III.1 let \( c \) be equal to \( a \) then to \( b \).

The next theorem follows by induction on theorem III.2.

Theorem III.3

If \( f(p_1, \ldots, p_n) = p_i \) for \( i=1, \ldots, m \), then
there exist \( r_1, \ldots, r_n \) and \( s_1, \ldots, s_n \) such that

\[
f(r_1, \ldots, r_n) = \sum_{i=1}^{m} p_i \quad \text{and} \quad f(s_1, \ldots, s_n) = \prod_{i=1}^{m} p_i.\]

Furthermore, \( r_j, s_j \in S_{c_j}, d_j \) where \( c_j = \prod_{j=1}^{m} p_i \) and \( d_j = \sum_{j=1}^{m} p_j i \).

The following theorems in this section are concerned with the restriction of the domain of \( f \).

**Theorem III.4**

Let \( x_i \in S_{a,b} \neq \emptyset \) and \( f \) a boolean function. Then

\[
f(e_1, \ldots, e_n) x_1 \ldots x_n = f(\eta_1, \ldots, \eta_n) x_1 \ldots x_n ,
\]

where \( \eta_i = a \) if \( e_i = 0 \) and \( \eta_i = b \) if \( e_i = 1 \).

**Proof:** Let \( x_i \in S_{a,b} \) and \( \eta_i = a \) if \( e_i = 0 \)

and \( \eta_i = b \) if \( e_i = 1 \). Consider \( p_i \cdot e_i \), where

\( p_i \)

is either 0 or 1. There are four cases to consider:

1. if \( p_i = 0 \) and \( e_i = 0 \)
2. if \( p_i = 0 \) and \( e_i = 1 \)
3. if \( p_i = 1 \) and \( e_i = 0 \)
4. if \( p_i = 1 \) and \( e_i = 1 \).

Case (1), since \( \eta_i = a \), \( \eta_i x_i = a' x_i = x_i = x_i \).

Case (2), since \( \eta_i = b \), \( \eta_i x_i = b' x_i = 0 \).

Case (3), since \( \eta_i = a \), \( \eta_i x_i = a' x_i = 0 \).
Case (4), since \( \eta_1 = b, \eta_1 x_i = b x_i = x_1 = x_i \).

It then follows that if \( p_i = e_i \) then \( \eta_1 x_i \), and if \( p_i \neq e_i \) then \( \eta_1 x_i = 0 \). Hence, if for each \( i, p_i = e_i \), it follows that \( \eta_1 x_1 \cdots \eta_n x_n = x_1 \cdots x_n \), otherwise \( \eta_1 x_1 \cdots \eta_n x_n = 0 \). Thus

\[
\sum_{(p)} \frac{f(p_1, \cdots, p_n)}{B} \eta_1 x_1 \cdots \eta_n x_n =
\]

\[
f(e_1, \cdots, e_n) x_1 \cdots x_n, \text{ since there is only one combination such that } p_i = e_i \text{ for each } i.
\]

Let \( c = \prod_{(\eta)} \frac{f(\eta_1, \cdots, \eta_n)}{B} \) and \( d = \sum_{(\eta)} \frac{f(\eta_1, \cdots, \eta_n)}{B} \),

where the meet and join extend over all combinations such that either \( \eta_1 = a \) or \( \eta_1 = b \).

The next theorems show that if \( f \) is restricted to \( S_{a,b} \neq \emptyset \), then the range is \( S_c, d \), where \( c \) and \( d \) are defined above.
Theorem III.5

Let \( f \) be a boolean function with \( a, b, c, \) and \( d \) defined above. Then there exist \( c_1, \cdots, c_n \) and \( d_1, \cdots, d_n \) such that \( f(c_1, \cdots, c_n) = c \) and \( f(d_1, \cdots, d_n) = d, \)
where \( c_i, d_i \in S_{a,b}. \)

Proof: Application of theorem III.3.

Theorem III.6

Let \( f \) be a boolean function with \( a, b, c, \) and \( d \) defined above. Then there exist \( h_1, \cdots, h_n \) such that \( f(h_1, \cdots, h_n) = h \) and \( h_i \in S_{a,b}. \)

Proof: Let \( c_i \) and \( d_i \) be defined as in theorem III.5. By theorem III.5 and III.4 there exist \( h_1, \cdots, h_n \) such that \( f(h_1, \cdots, h_n) = d \ h + c h' = h \) and \( h_i \subseteq S_{c_i d_i, c_i + d_i} \) which is a subset of \( S_{a,b}. \)

Theorem III.7

Let \( f \) be a boolean function and let

\[ c = \prod_{(\eta)} f(\eta_1, \cdots, \eta_n) \quad \text{and} \quad d = \sum_{(\eta)} f(\eta_1, \cdots, \eta_n). \]

If \( x_i \in S_{a,b} \) then \( f(x_1, \cdots, x_n) \in S_{c,d}. \)

Proof: Let \( x_i \in S_{a,b}. \) Recall that \( c' = \sum_{(e)} [f(\eta_1, \cdots, \eta_n)]^e \) and that \( \eta_i = a \) if \( e_i = 0 \)
and \( \eta_i = b \) if \( e_i = 1. \) It then follows by theorem III.4.
that: \( f(x_1, \ldots, x_n) + c' = \)

\[
\sum_{(e)B} f(e_1, \ldots, e_n)x_1^{e_1} \cdots x_n^{e_n} + c' 
\]

\[
\sum_{(e)B} f(\eta_1, \ldots, \eta_n)x_1^{\eta_1} \cdots x_n^{\eta_n} + \sum_{(e)B} [f(\eta_1, \ldots, \eta_n)]' = 
\]

\[
\sum_{(e)B} x_1^{\eta_1} \cdots x_n^{\eta_n} + \sum_{(e)B} [f(\eta_1, \ldots, \eta_n)]' = 1 + c' = 1. 
\]

Therefore \( f(x_1, \ldots, x_n) > c \).

Also it follows that: \( f(x_1, \ldots, x_n) + d = \)

\[
\sum_{(e)B} f(e_1, \ldots, e_n)x_1^{e_1} \cdots x_n^{e_n} + d = 
\]

\[
\sum_{(e)B} f(\eta_1, \ldots, \eta_n)x_1^{\eta_1} \cdots x_n^{\eta_n} + \sum_{(e)B} f(\eta_1, \ldots, \eta_n) = 
\]

\[
\sum_{(e)B} f(\eta_1, \ldots, \eta_n) = d. \text{ Hence } f(x_1, \ldots, x_n) < d. 
\]

Thus \( f(x_1, \ldots, x_n) \) is an element of \( S_{c,d} \) whenever \( x_i \in S_{a,b} \). Hence the theorem is proved.

Theorem III.8

The range of a function, \( f \), in \( B \) when the domain is restricted to \( S_{a,b} \neq \varnothing \) is \( S_{c,d} \), where

\[
\bigotimes_{(\eta)S_{a,b}} f(\eta_1, \ldots, \eta_n) \text{ and } d = \sum_{(\eta)S_{a,b}} f(\eta_1, \ldots, \eta_n) \text{ where the } 
\]

join and meet are over all combinations such that either \( \eta_i = a \) or \( \eta_i = b \).
Some immediate results of theorem III.8 are:

Theorem III.9

The range of a function, $f$, in $b$ is $S_{h,k}$ where

\[ h = \prod_{(e)B} f(e_1, \ldots, e_n) \quad \text{and} \quad k = \sum_{(e)B} f(e_1, \ldots, e_n). \]

Theorem III.10

Let $f$ be a boolean function whose domain is $B$.

The range of $f$ is $B$ if and only if $\prod_{(e)B} f(e_1, \ldots, e_n) = 0$ and $\sum_{(e)B} f(e_1, \ldots, e_n) = 1$.

Another important theorem is:

Theorem III.11

If $f(0, \ldots, 0) < b$ and $f(1, \ldots, 1) > a$ then $f(x_1, \ldots, x_n) \in S_{a,b}$, whenever $x_i \in S_{a,b}$.

Proof: Let $x_i \in S_{a,b}$ and $f(0, \ldots, 0) < b$ and $f(1, \ldots, 1) > a$. It follows that $f(x_1, \ldots, x_n) = f(1, \ldots, 1) x_1 \cdots x_n + f(x_1, \ldots, x_n) > a$.

Whenever there is an $i$ such that $e_i \neq 0$ then

\[ b x_1 \cdots x_n = x_1 \cdots x_n. \]

Since $f(0, \ldots, 0) b = f(1, \ldots, 0)$, it follows that $b f(x_1, \ldots, x_n) =$

\[ \sum_{(e)B} f(e_1, \ldots, e_n) b x_1 \cdots x_n = \]
\[ \sum_{(e)_{B}} f(e_1, \ldots, e_n) x_1^{e_1} \cdots x_n^{e_n} = f(x_1, \ldots, x_n). \] Thus

\[ f(x_1, \cdots, x_n) < b. \]

Therefore \( f(x_1, \cdots, x_n) \in S_{a, b}. \)
BOOLEAN FUNCTIONS IN A RELATIVE BOOLEAN ALGEBRA

In this section it will be understood that $\eta_i = a$ whenever $e_i = 0$ and $\eta_i = b$ whenever $e_i = 1$. This section will consider the relationship between functions in $B$ and functions in a relative boolean algebra of $B$.

Let $S_{a,b}$ be a relative boolean algebra of $B$ and $x^o = b x^i + a$.

A useful theorem is:

Theorem IV.1

If $x_i \in S_{a,b}$ then $b x_1^0 \cdots x_n^0 + a = x_1^a \cdots x_n^a$, where $x_i^0 = x_i$ if $\eta_i = b$ and $x_i^0 = x^o$ if $\eta_i = a$.

Proof: Let $x_i \in S_{a,b}$. If $e_i = 0$ for some $i$, then it follows that there exists a permutation, $i_1, i_2, \ldots, i_j, i_{j+1}, \ldots, i_n$, of the first $n$ positive integers such that $e_i^k = 1$ if $1 \leq k \leq j$ and $e_i^k = 0$ if $j+1 \leq k \leq n$. Hence: $b x_1^e \cdots x_n^e + a = b x_1^i \cdots x_j^i \cdots x_{j+1}^i \cdots x_n^i + a x_1^i \cdots x_i^i = x_1^i \cdots x_i^i (b x_i^i + a) \cdots (b x_{i+1}^i + a) = x_1^0 \cdots x_j^0 x_{j+1}^0 \cdots x_n^0 = x_1^a \cdots x_n^a$.
If $e_i = 1$ for all $i$, then $b x_1^e_1 \cdots x_n^e_n = x_1^{\eta_1} \cdots x_n^{\eta_n}$.

Hence in all cases the statement is true.

An alternate statement of theorem IV.1 follows.

Theorem IV.1a

If $x_i \in S_{a,b}$ then $a' b x_1^e_1 \cdots x_n^e_n = a' x_1^{\eta_1} \cdots x_n^{\eta_n}$.

A function, $g$, of $n$ variables of a relative boolean algebra, $S_{a,b}$, of $B$ will have the form $g(x_1, \cdots, x_n) = \sum \frac{g(\eta_1, \cdots, \eta_n) x_1^{\eta_1} \cdots x_n^{\eta_n}}{S_{a,b}}$, where the join extends over all combinations such that either $\eta_i = a$ or $\eta_i = b$.

$x^b = x$ and $x^a = x^u = b x^t + a$, the complement in $S_{a,b}$.

The next theorem connects functions in $B$ and functions in $S_{a,b}$.

If $f$ is a function in $B$, then the statement "$f$ is a function in $S_{a,b}$" means there exists a $g$ in $S_{a,b}$ such that $f(x_1, \cdots, x_n) = g(x_1, \cdots, x_n)$ whenever $x_i \in S_{a,b}$.
Theorem IV.2

If \( f \) is a function in \( B \), and \( x_i \in S_{a,b} \) implies that \( f(x_1, \ldots, x_n) \in S_{a,b} \), then \( f \) is a function in \( S_{a,b} \) when the domain of the variables is restricted to \( S_{a,b} \).

Proof: Since \( x_i \in S_{a,b} \) and \( f(x_1, \ldots, x_n) \in S_{a,b} \), it follows from theorem III.4 and IV.1 that

\[
f(x_1, \ldots, x_n) + a = \sum_{(e)B} f(\eta_1, \ldots, \eta_n) \eta_1 \ldots \eta_n + a =
\]

\[
\sum_{(e)B} f(\eta_1, \ldots, \eta_n) (b \eta_1 \ldots \eta_n + a) + a =
\]

\[
\sum_{(\eta)S_{a,b}} f(\eta_1, \ldots, \eta_n) (x_1 \ldots x_n + a) + a =
\]

\[
\sum_{(\eta)S_{a,b}} f(\eta_1, \ldots, \eta_n) \eta_1 \ldots \eta_n + a =
\]

\[
\sum_{(\eta)S_{a,b}} f(\eta_1, \ldots, \eta_n) \eta_1 \ldots \eta_n.
\]

Thus, \( f(x_1, \ldots, x_n) \) has the form of a function in \( S_{a,b} \) whenever \( x_i \in S_{a,b} \).

The importance of theorem IV.2 is not obvious from its statement. One result is that if a boolean algebra \( B \) is imbedded as a relative boolean algebra in another boolean algebra \( \overline{B} \), that is, the given algebra
B is a relative boolean algebra of a larger boolean algebra \( \mathbb{B} \), no new functions from \( B \) into \( B \) are formed by rules with coefficients from \( \mathbb{B} \).

Next it is of interest to examine the extension of a boolean function, that is, given a function \( g \) in \( S_{a,b} \) is there a function \( f \) in \( B \) such that whenever \( x_i \in S_{a,b} \) then \( f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \)?

It will be shown that a necessary and sufficient condition that a function \( g \) in \( S_{a,b} \) and a function \( f \) in \( B \) have the property that \( f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \), \( x_i \in S_{a,b} \), is that the following conditions hold:

1. \( a' \ g(\eta_1, \ldots, \eta_n) < f(e_1, \ldots, e_n) < g(\eta_1, \ldots, \eta_n) + b' \)
2. \( g(b, \ldots, b) < f(1, \ldots, 1) < g(b, \ldots, b) + b' \)
3. \( a' \ g(a, \ldots, a) < f(0, \ldots, 0) < g(a, \ldots, a) \).

Theorem IV.3

Let \( g \) be a function in \( S_{a,b} \), and \( f \) a function in \( B \), such that \( x_i \in S_{a,b} \) implies \( g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \). Then \( a' \ g(\eta_1, \ldots, \eta_n) < f(e_1, \ldots, e_n) < g(\eta_1, \ldots, \eta_n) + b' \).

Proof: By theorem IV.1a, \( a' \ g(\eta_1, \ldots, \eta_n) = a' \ b \ g(\eta_1, \ldots, \eta_n) = a' \ b \ f(\eta_1, \ldots, \eta_n) = \sum f(\overline{e}_1, \ldots, \overline{e}_n) \ a' \ b \ \eta_1 \ldots \eta_n \).
\sum_{(\bar{e})_B} f(\bar{e}_1, \ldots, \bar{e}_n) a' \eta_1 \cdots \eta_n, \text{ where } \bar{\eta}_i = a \text{ whenever } \bar{e}_i = 0 \text{ and } \bar{\eta}_i = b \text{ whenever } \bar{e}_i = 1. \text{ If } \bar{\eta}_i \neq \eta_i,

then \bar{\eta}_i = a \text{ and } \eta_i a' = 0. \text{ Also if } \bar{\eta}_i = \eta_i \text{ then } \bar{\eta}_i = b \text{ and } \bar{e}_i = e_i. \text{ Thus } a' g(\eta_1, \ldots, \eta_n) = f(e_1, \ldots, e_n) a' b. \text{ Thus } a' g(\eta_1, \ldots, \eta_n) < f(e_1, \ldots, e_n).

Again, by theorem IV.1, \( g(\eta_1, \ldots, \eta_n) + b' = \)

\[ \sum_{(\bar{e})_B} f(\bar{e}_1, \ldots, \bar{e}_n) b \eta_1 \cdots \eta_n + a + b' = \]

\[ \sum_{(\bar{e})_B} f(\bar{e}_1, \ldots, \bar{e}_n) \eta_1 \cdots \eta_n + a + b', \text{ where } \bar{\eta}_i = a \]

whenever \( \bar{e}_i = 0 \) and \( \bar{\eta}_i = b \) whenever \( \bar{e}_i = 1. \) Since \( a < b \) and \( \bar{\eta}_i \neq \eta_i \) implies \( \bar{\eta}_i = a, \) it follows that whenever \( i \) exists such that \( \eta_i \neq \bar{\eta}_i \) then \( \bar{\eta}_1 \cdots \bar{\eta}_n = a, \) and \( f(\bar{e}_1, \ldots, \bar{e}_n) \eta_1 \cdots \eta_n + a = \)

\[ f(\bar{e}_1, \ldots, \bar{e}_n) a + a = a. \text{ Thus, since } \eta_i = b \text{ whenever } \eta_i = \bar{\eta}_i, \text{ } g(\eta_1, \ldots, \eta_n) + b' = f(e_1, \ldots, e_n) b + a + b' = f(e_1, \ldots, e_n) + a + b'. \text{ Hence } g(\eta_1, \ldots, \eta_n) + b' > f(e_1, \ldots, e_n).

Thus the theorem has been established.
Theorem IV.4

Let $g$ be a function in $S_{a,b}$ and $f$ a function in $B$, such that $g(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)$ whenever $x_1 \in S_{a,b}$. Then $f(1,\ldots,1) > g(b,\ldots,b)$ and $f(0,\ldots,0) < g(a,\ldots,a)$.

Proof: By definition of a boolean function

$$f(b,\ldots,b) = f(1,\ldots,1) b + f(0,\ldots,0) b'.$$

Since $f(b,\ldots,b) = g(b,\ldots,b) \in S_{a,b}$, $f(b,\ldots,b) < b$. Therefore $0 = b' f(b,\ldots,b) = f(0,\ldots,0) b'$.

Hence

$$g(b,\ldots,b) = f(b,\ldots,b) = f(1,\ldots,1) b,$$

and $g(b,\ldots,b) < f(1,\ldots,1)$.

By theorem IV.3, $f(0,\ldots,0) [g(a,\ldots,a) + b'] = f(0,\ldots,0)$. From above, $f(0,\ldots,0) b' = 0$. Hence $f(0,\ldots,0) = f(0,\ldots,0) [g(a,\ldots,a) + b'] = f(0,\ldots,0) g(a,\ldots,a)$, or $f(0,\ldots,0) < g(a,\ldots,a)$.

Theorem IV.25

Let $g$ be a function in $S_{a,b}$ and $f$ a function in $B$. A necessary and sufficient condition that $f$ be the same as $g$ whenever the domain of $f$ is restricted to $S_{a,b}$ is that the following conditions be satisfied:

(i) $a' g(\eta_1,\ldots,\eta_n) < f(e_1,\ldots,e_n) < g(\eta_1,\ldots,\eta_n) + b'$

(ii) $g(b,\ldots,b) < f(1,\ldots,1) < g(b,\ldots,b) + b'$

(iii) $a' g(a,\ldots,a) < f(0,\ldots,0) < g(a,\ldots,a)$.

Proof: The necessity follows from theorems IV.3 and IV.4.
Assume the conditions hold. Since \( f(1, \ldots, 1) > g(b, \ldots, b) > a \) and \( f(0, \ldots, 0) < g(a, \ldots, a) < b \), it follows by theorem III.11 that \( f(x_1, \ldots, x_n) \in S_{a,b} \), whenever \( x_i \in S_{a,b} \). Thus \( f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + a = \sum_{(e)B} f(e_1, \ldots, e_n) x_1 \cdots x_n + a > \)

\[
\sum_{(e)B} a^i g(\eta_1, \ldots, \eta_n) x_1 \cdots x_n + a = \]

\[
\sum_{(e)B} a^i g(\eta_1, \ldots, \eta_n) b x_1 \cdots x_n + a. \text{ By theorem IV.1,}
\]

\[
f(x_1, \ldots, x_n) > \sum_{(\eta)S_{a,b}} a^i g(\eta_1, \ldots, \eta_n) x_1 \cdots x_n + a = \]

\[
a^i g(x_1, \ldots, x_n) + a = g(x_1, \ldots, x_n). \]

Also \( f(x_1, \ldots, x_n) = b f(x_1, \ldots, x_n) + a \) whenever \( x_i \in S_{a,b} \) from above. Thus, by theorem IV.1, \( f(x_1, \ldots, x_n) = 
\]

\[
a + b \sum_{(e)B} f(e_1, \ldots, e_n) x_1 \cdots x_n < \]

\[
a + b \sum_{(e)B} [g(\eta_1, \ldots, \eta_n) + b^i] x_1 \cdots x_n = \]

\[
a + \sum_{(e)B} g(\eta_1, \ldots, \eta_n) [b x_1 \cdots x_n + a] = \]

\[
a + \sum_{(\eta)S_{a,b}} g(\eta_1, \ldots, \eta_n) x_1 \cdots x_n = a + g(x_1, \ldots, x_n). \]

Thus \( f(x_1, \ldots, x_n) < g(x_1, \ldots, x_n) \).

Therefore \( f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \) if \( x_i \in S_{a,b} \).
FUNCTIONS BETWEEN RELATIVE BOOLEAN ALGEBRAS

A homomorphism between boolean algebras $\mathbb{B}$ and $\mathbb{B}'$ is a function $h$ of one variable such that:

1. $h(p + q) = h(p) + h(q)$
2. $h(p \cdot q) = h(p) \cdot h(q)$
3. $h(p') = [h(p)]'$

whenever $p$ and $q$ are elements of $\mathbb{B}$.

If $h$ is a homomorphism then $h(0) = h(p \cdot p') = h(p) [h(p)]' = 0$ and $h(1) = h(p + p') = h(p) + [h(p)]' = 1$. Hence the distinguished elements are preserved by the mapping $h$.

A boolean homomorphism is a homomorphism which can be expressed as a boolean function.

This section deals with functions between relative boolean algebras of $\mathbb{B}$ where the functions are boolean functions.

Let $S_{a,b}$ and $S_{c,d}$ be relative boolean algebras of $\mathbb{B}$. Let $h$ be a boolean function of one variable in $\mathbb{B}$ whose domain is $S_{a,b}$ and whose range is $S_{c,d}$. Let $h(x + y) = h(x) + h(y)$ and $h(x \cdot y) = h(x) \cdot h(y)$, whenever $x$ and $y$ are elements of $S_{a,b}$. For all $q \in S_{a,b}$, $h(q) = h(q + a) = h(q) + h(a)$. Since the range of $h$ is $S_{c,d}$, if $p \in S_{c,d}$ then there exists a $q \in S_{a,b}$ such that $h(q) = p$. Hence $p = p + h(a)$ for
all p in $S_c,d$. Since the distinguished elements are unique and $h(a)$ satisfies axiom 3, $h(a) = c$. Similarly it follows that $h(b) = d$. Let $x^o = b \cdot x' + a$ and $x^+ = d \cdot x' + c$, the relative complements of $S_a,b$ and $S_c,d$ respectively. Also $h(x^o) + h(x) = h(x + x^o) = h(b) = d$ and $h(x^o) h(x) = h(x^o x) = h(a) = c$, thus $h(x^o) = [h(x)]^+$. Therefore if $h$ is a boolean function with domain $S_a,b$ and range $S_c,d$ and $h$ preserves meet and join, then $h$ is a boolean homomorphism.

If $h$ is a boolean homomorphism with domain $S_a,b$ and $h(a) = c$, $h(b) = d$, then by theorem III.8 the range of $h$ is $S_c,d$.

Theorem V.1

Let $S_a,b$ and $S_c,d$ be relative boolean algebras of $B$. The following are equivalent:

(i) $a \cdot c = a \cdot d$ and $b' \cdot c = b' \cdot d$

(ii) there exists a boolean homomorphism between $S_a,b$ and $S_c,d$

(iii) there exists a boolean function, $h$, in $B$ whose domain is $S_a,b$ and range is $S_c,d$.

Proof:

(i) implies (ii): Consider the function $f$ described by $f(x) = d \cdot x + c$, $x \in S_a,b$. Since $f(a) = d \cdot a + c = a \cdot c + c = c$ and $f(b) = d \cdot b + c = b \cdot d + b' \cdot c + c = d \cdot b + b' \cdot d + c = d + c = d$, it follows from theorem III.8
that the range of f is $S_{c,d}$. Since $f(x + y) = d(x + y) + c = (d x + c) + (d x + c) = f(x) + f(y)$ and $f(x y) = d x y + c = (d x + c) (d y + c) = f(x) f(y)$ the binary operations are preserved, and f is a boolean homomorphism from $S_{a,b}$ to $S_{c,d}$.

(ii) implies (iii): Since all boolean homomorphisms have as a range a relative boolean algebra it follows that there exists a boolean function in B with domain $S_{a,b}$ and $S_{c,d}$.

(iii) implies (i): Let f be a function with domain $S_{a,b}$ and range $S_{c,d}$. Then by theorem III.8

$$f(a) f(b) = c \quad \text{and} \quad f(a) + f(b) = d \quad \text{or}$$

$$c = f(1) a + f(0) b' + f(1) f(0) a' b \quad \text{and}$$

$$d = f(1) b + f(0) a'. \quad \text{Thus} \quad c a = f(1) a = a [f(1) b + f(0) a'] = a d \quad \text{and} \quad c b' = f(0) b' = b' [f(1) b + f(0) a'] = b' d.$$

Note in theorem V.1 that if there is any boolean function with domain $S_{a,b}$ and range $S_{c,d}$, then there exists a boolean homomorphism with that domain and range. Furthermore the constants a, b, c, and d completely determine the existence of a homomorphism.

If $S_{c,d}$ is any relative boolean algebra of B, then since $S_{0,1} = B$ and $0 c = 0 d$ and $1' c = 1' d$, there is a boolean homomorphism from B to $S_{c,d}$, namely, $f(x) = d x + c$. 
Theorem V.2

Let \( S_{a,b} \) and \( S_{c,d} \) be relative boolean algebras of \( B \). \( S_{c,d} \) is a homomorphic image of \( S_{a,b} \), that is there exists a boolean homomorphism from \( S_{a,b} \) to \( S_{c,d} \), if and only if there exists a relative boolean algebra, \( S_{k,h} \), of \( B \) with complementation \( \circ \), such that \( a^\circ = d \) and \( b^\circ = c \).

**Proof:** Let \( S_{c,d} \) be a homomorphic image of \( S_{a,b} \). Then by the previous theorem, \( a c = a d \) and \( b' c = b' d \). Thus \( d a' + c = d a' + a c + c = d a' + d a + c = d + c = d \) and \( d b' + c = c b' + c = c \). Hence if \( a^\circ = d \) and \( b^\circ = c \), as was to be shown.

Let \( S_{a,b} \), \( S_{c,d} \), and \( S_{h,k} \) be relative boolean algebras of \( B \). Furthermore let \( a^\circ = k a' + h = d \) and \( b^\circ = k b' + h = c \). Consider \( f(x) = x^\circ = k x' + h \). Since \( f(a) + f(b) = d \) and \( f(a) f(b) = c \) it follows from theorem III.8 that \( f \) has a range \( S_{c,d} \) when the domain of \( f \) is \( S_{a,b} \). By theorem V.1, \( S_{c,d} \) is a homomorphic image of \( S_{a,b} \).

Let \( S_{a,b} \) and \( S_{h,k} \) be relative boolean algebras in \( B \) and \( x^\circ \) be the relative complement of \( x \) with respect to \( S_{h,k} \). By theorems II.6 and V.2, \( S_{b^\circ,a^\circ} \) is a homomorphic image of \( S_{a,b} \).
There may be many equations for boolean functions in $B$ which are homomorphisms with domain $S_{a,b}$ and range $S_{c,d}$, but the next theorem shows that if $f$ and $g$ are boolean homomorphisms from $S_{a,b}$ to $S_{c,d}$ then $f(x) = g(x)$ whenever $x \in S_{a,b}$.

**Theorem V.3**

Let $S_{c,d}$ and $S_{a,b}$ be relative boolean algebras $B$. If $f$ is a boolean homomorphism with domain $S_{a,b}$ and range $S_{c,d}$ then $f(x) = d \cdot x + c$ whenever $x \in S_{a,b}$.

**Proof:** Since $f$ is a homomorphism from $S_{a,b}$ to $S_{c,d}$, $f(b) = f(1) \cdot b + f(0) \cdot b' = d$ or $f(1) \cdot b = d \cdot b$. Also $f(a) = f(1) \cdot a + f(0) \cdot a' = c$ or $f(0) \cdot a' = c \cdot a'$. Let $x \in S_{a,b}$. Then $f(x) = f(1) \cdot x + f(0) \cdot x' = f(1) \cdot b \cdot x + f(0) \cdot a' \cdot x' = d \cdot b \cdot x + c \cdot a' \cdot x' = d \cdot x + c \cdot x'$.

If $S_{a,b}$ and $S_{c,d}$ are relative boolean algebras of $B$, and there is a boolean homomorphism from $S_{a,b}$ to $S_{c,d}$, then the function $f$, $f(x) = d \cdot x + c$, is a boolean homomorphism, as was shown in the proof of theorem V.1.

Theorem V.3 does not give the form of the boolean function in $B$, but if $B$ is the domain then there is a unique boolean function which is a homomorphism from
B to $S_{c,d}$. Theorem V.4 follows from theorem V.3

**Theorem V.4**

If $B$ is the domain and $S_{c,d}$ is the range of a boolean homomorphism $f$ then $f(x) = d x + c$.

Furthermore, $f$ is the only boolean homomorphism from $B$ onto $S_{c,d}$.

Let $S_{a,b}$ and $S_{c,d}$ be relative boolean algebras in $B$ and $\land, \lor$ be the complementation operations in $S_{a,b}$ and $S_{c,d}$ respectively. If $S_{c,d}$ is the homomorphic image of $S_{a,b}$, then the function $f$ defined by $f(x) = (x^\circ)^+ \lor$, will be shown to be a boolean homomorphism from $S_{a,b}$ to $S_{c,d}$. $f(x) = (x^\circ)^+ = (x b' + a)^+$ = $d (b x' + a)' + c = d a' x + d b' + c = d a' x + c = d a' x + c a x + c$. By theorem V.1 $c a = d a$. Thus $f(x) = d a' x + d a x + c = d x + c$, and therefore $f$ is a boolean homomorphism.

Let $S_{a,b}$ and $S_{h,k}$ be relative boolean algebras of $B$ and $x^\circ = b x' + a$, $x^+ = k x' + h$. On page 14 it was stated that $f(x) = (((x^+)^\circ)^+)\circ$ is a homomorphism; this will now be shown. By theorem II.6, $S_{k^\circ,h^\circ}$ is a relative boolean algebra with the complement of $x$ being $((x^\circ)^+)^\circ$. Thus, from above, $f(x) = (((x^+)^\circ)^+)\circ$ is a boolean homomorphism from $S_{h,k}$ to $S_{k^\circ,h^\circ}$. 
RELATIVE BOOLEAN ALGEBRAS AND SUB-BOOLEAN ALGEBRAS

A sub-boolean algebra of a boolean algebra $B$ is a subset of $B$ which is closed under meet, join and complementation of $B$.

Let $S$ be a sub-boolean algebra of $B$. Let $x \in S$ then $x' \in S$ and $x + x' = 1 \in S$, also $xx' = 0 \in S$. Thus $S$ is a boolean algebra with distinguished elements 0, 1.

This section deals with the relationship between relative boolean algebras and sub-boolean algebras.

Let SRBA mean "sub-boolean algebra of a relative boolean algebra", and RSBA mean "relative boolean algebra of a sub-boolean algebra".

It will be shown that if $S$ is a SRBA of $B$ then $S$ is a RSBA of $B$ and if $S$ is a RSBA of $B$ then $S$ is a SRBA of $B$, where $B$ is a boolean algebra.

Theorem VI.1

If $\bar{S}$ is a sub-boolean algebra of $B$ and $\bar{S}_{a,b}$ is a relative boolean algebra of $\bar{S}$, then $\bar{S}_{a,b}$ is a sub-boolean algebra of the relative boolean algebra $S_{a,b}$ of $B$.

That is, a RSBA is a SRBA.

Proof: Since $\bar{S}_{a,b} = \{ x : a, b, x \in \bar{S}, \text{ and } a < x < b \}$, it follows that $\bar{S}_{a,b}$ is a subset of $S_{a,b}$. The comple-
ment in the RSBA, $\overline{S}_{a,b}$, is defined by $b x' + a$, the same as in $S_{a,b}$. Also the meet and join in both $\overline{S}_{a,b}$ and $S_{a,b}$ are the same as in $B$ and thus the same. Hence $\overline{S}_{a,b}$ is a sub-boolean algebra of $S_{a,b}$.

Theorem VI.2

Let $S_{a,b}$ be a relative boolean algebra of $B$ and $S$ a sub-boolean algebra of $S_{a,b}$. Then $S$ is a relative boolean algebra of some sub-boolean algebra of $B$. That is, a SRBA is a RSBA.

Proof: Let $\overline{S} = \{ x : b x + a \in S \}$. If $x, y \in \overline{S}$ then $x + y$ and $x y$ are elements of $\overline{S}$ since $b (x + y) + a = (b x + a) (b y + a)$ and $b x y + a = (b x + a) (b y + a)$ are elements of $S$. Let $x \in \overline{S}$. Then $b x' + a$ is the complement of $b x + a$ with respect to $S_{a,b}$. Thus $b x' + a \in S$ and $x' \in \overline{S}$. Hence $\overline{S}$ is a sub-algebra of $B$. Note that, since $b a + a \in S$ and $b b + a \in S$, then $a$ and $b$ are elements of $\overline{S}$.

Let $\overline{S}_{a,b} = \{ x : x \in \overline{S}, a < x < b \}$. $\overline{S}_{a,b}$ is a relative boolean algebra of $\overline{S}$. If $x \in \overline{S}_{a,b} \subseteq \overline{S}$, then $x = b x + a$, and by definition of $\overline{S}$, $b x + a \in S$. Hence $x \in S$ and $\overline{S}_{a,b} \subseteq S$. If $x \in S \subseteq S_{a,b}$, then $a < x < b$ and $x = a x + b$.

Thus $x \in \overline{S}_{a,b}$. Therefore $S = \overline{S}_{a,b}$. Since the complement of $x$ in $\overline{S}$ and $S$ is $b x' + a$, and since the meet and join are the same as in $B$, the boolean algebras $\overline{S}_{a,b}$ and $S$ are identical. Therefore $S$ is a RSBA.