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An Investigation of the Properties of Join Geometry

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AN INVESTIGATION OF THE PROPERTIES OF JOIN GEOMETRY

by

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INTRODUCTION

This paper presents a proof that the classical geometry as stated by Karol Borsuk [1] follows from the join geometry of Walter Prenowitz [2] .

The approach taken is to assume the axioms of Prenowitz. Using these as the foundation, the theory of join geometry is then developed to include such ideas as "convex set", "linear set", the important concept of "dimension", and finally the relation of "betweenness". The development is in the form of definitions with the important extensions given in the form of theorems.

With a firm foundation of theorems in the join geometry, the axioms of classical geometry are examined, and then they are proved as theorems or modified and proved as theorems.

The basic notation to be used is that of set theory. No distinction is made between the set consisting of a single element and the element itself. Thus the notation for set containment is $\subset$, and is used to denote element containment also. The set containing no elements, or the empty set, is denoted by $\emptyset$. The set of points belonging to at least one of the sets under consideration is called union, denoted $\cup$. The set of points belonging to each of the sets under consideration is called the intersection and denoted by $\cap$. Any other notation used will be defined at the first usage.

DEVELOPMENT OF THE PROPERTIES OF JOIN GEOMETRY

Axioms and definitions

Consider a set G and a two term operation, $\cdot$, called "join", which associates with each pair of elements a and b of G a uniquely determined set called the join of a and b and denoted $a \cdot b$ or more simply ab. The system consisting of the set G and the operation join is called a join geometry if the system satisfies the following axioms, (J1) to (J8).

(J1) (Closure Law). If a, b $\subset$ G, then $\emptyset \neq ab \subset G$.

(J2) (Commutative Law). If a, b $\subset$ G, then ab = ba.

In order to state the associative law, it is necessary to extend the concept of join of elements to include the join of two sets. This is done in the following definition.

Definition 1. If A, B $\subset$ G, then AB is the set of elements obtained by joining each element of A to each element of B and aggregating the results. Formally stated:

$$AB = \bigcup_{a \subset A,\ b \subset B} (ab).$$

(J3) (Associative Law). If a, b, c $\subset$ G, then (ab)c = a(bc).

The operation of join is used to define the "inverse" operation which is the basis of the next axiom.

Definition 2. If $a, b \subset G$, then a/b is the set of all elements x such that $a \subset bx$. a/b is called the extension of a from b, and formally stated:

$$a/b = \{x \mid a \subset bx\}.$$

(J4) (Closure Law). If $a, b \subset G$, then $\emptyset \neq a/b \subset G$.

(J5) (Transposition Law). If $a, b, c, d \subset G$, and if $a/b \cap c/d \neq \emptyset$, then $ad \cap bc \neq \emptyset$.

(J6) (Idempotent Law). $aa = a = a/a$ for each $a \subset G$.

In the next definition, the operation of extension is generalized to extension for sets.

Definition 3. If $A, B \subset G$, then A/B is the set of elements obtained by extending each element of A from each element of B and aggregating the results. Formally stated:

$$A/B = \bigcup_{a \subset A,\ b \subset B} (a/b).$$

The definitions of convex set and linear set are necessary in order to state the remaining axioms.

Definition 4. A subset A of G is convex if $x, y \subset A$ implies $xy \subset A$.

Definition 5. A subset S of G is the "convex closure" of a subset A of G if S satisfies the following properties:

(a) S is convex;

(b) $A \subset S$;

(c) If X is convex and $A \subset X$ then $S \subset X$.

Definition 6. A subset A of G is linear if x, y $\subset$ A implies xy $\subset$ A and x/y $\subset$ A.

Definition 7. A subset S of G is the "linear closure" of a subset A of G if S satisfies the following properties:

(a) S is linear;

(b) A $\subset$ S;

(c) If X is linear and A $\subset$ X then S $\subset$ X.

The linear closure of A is denoted by $\{A\}$.

Definition 8. The linear closure of two subsets A and B of G, denoted $\{A,B\}$, is the set S which satisfies the following properties:

(a) S is linear;

(b) A, B $\subset$ S;

(c) If X is linear and A, B $\subset$ X, then S $\subset$ X.

Definition 9. If a, b $\subset$ G and a $\neq$ b, then line ab is the linear closure of a and b. Line ab = $\{a,b\}$.

The last two axioms of the join geometry are concepts involving linear sets.

(J7) If $b \subset \{a_1, a_2\}$, $b \neq a_2$, then $\{a_1, a_2\} = \{b, a_2\}$.

(J8) $\{a,b\} = a/b \cup ab \cup b/a \cup a \cup b$.

The linear closure of a set of elements has not been specifically defined, but it is readily seen that in the case where the set A consists of the elements $a_1, a_2, \ldots, a_n$ the linear closure of A is $\{a_1, a_2, \ldots, a_n\}$. This particular case is considered in the following definitions.

Definition 10. The set of elements $a_1, a_2, \ldots, a_n$ is linearly independent if the statement $a_1, \ldots, a_n \subset \{x_1, \ldots, x_{n-1}\}$ is false for every choice of $x_1, \ldots, x_{n-1}$.

Definition 11. If $A = \{a_1, \ldots, a_n\}$, and $a_1, \ldots, a_n$ are linearly independent, then $a_1, \ldots, a_n$ is a basis of A.

Definition 12. If $A = \{a_1, \ldots, a_n\}$ then the dimension of A is n. The dimension of A is denoted d(A). The dimension of $\emptyset$ is zero.

Definition 13. A plane is a linear set A such that $d(A) = 3$.

The Dimension Theorem

This section is devoted to the development of the dimension theorem which will prove of importance in establishing theorems and counterexamples in later sections.

Theorem 1. If $A \subset B$ then $AC \subset BC$ and $CA \subset CB$.

Proof. Let $x \subset AC$. By Definition 1, $x \subset ac$ where $a \subset A$, $c \subset C$. But $A \subset B$, so $a \subset B$ and $x \subset ac$ where $a \subset B$, $c \subset C$. Thus $x \subset BC$ and $AC \subset BC$.

Let $y \subset CA$. By Definition 1, $y \subset ca$ where $c \subset C$, $a \subset A$. But $A \subset B$, so $a \subset B$, and $y \subset ca$ where $c \subset C$, $a \subset B$. Hence $y \subset CB$ and $CA \subset CB$.

Corollary 1. If $A' \subset A$, $B' \subset B$ then $A'B' \subset AB$.

Proof. Since $A' \subset A$ then $A'B' \subset AB'$ by Theorem 1. Also, since $B' \subset B$, then $AB' \subset AB$. Thus $A'B' \subset AB' \subset AB$.

Theorem 2. $(AB)C = A(BC)$.

Proof. Let $x \subset (AB)C$. Then $x \subset yc$ where $y \subset AB$, $c \subset C$. Since $y \subset AB$, then $y \subset ab$ where $a \subset A$, $b \subset B$. By Definition 1, $yc \subset (ab)c$. But $(ab)c = a(bc)$ by (J3) so that $yc \subset a(bc)$. Thus $x \subset yc \subset a(bc) \subset A(BC)$ and $(AB)C \subset A(BC)$.

Similarly $A(BC) \subset (AB)C$ and thus $(AB)C = A(BC)$.

Theorem 3. $AB = BA$.

Proof. Let $x \subset AB$. Then $x \subset ab$ where $a \subset A$, $b \subset B$. But $ab = ba$ by (J2) so $x \subset ba \subset BA$. Thus $AB \subset BA$.

Substitution of B for A and A for B in the first part of the proof gives $BA \subset AB$. Thus $AB = BA$.

Theorem 4. If $A \subset B$, then $A/C \subset B/C$ and $C/A \subset C/B$.

Proof. Let $x \subset A/C$. Then $x \subset a/c$ where $a \subset A$, $c \subset C$ by Definition 3. But $A \subset B$, so $a \subset B$ and $x \subset a/c$ where $a \subset B$, $c \subset C$. Thus $x \subset B/C$ and $A/C \subset B/C$.

Let $y \subset C/A$. Then $y \subset c/a$ where $c \subset C$, $a \subset A$ by definition 3. But $A \subset B$, so $a \subset B$ and $x \subset c/a$ where $c \subset C$, $a \subset B$. Thus $x \subset C/B$ and $C/A \subset C/B$.

Corollary 4. If $A' \subset A$, $B' \subset B$, then $A'/B' \subset A/B$.

Proof. Since $A' \subset A$, then $A'/B' \subset A/B'$ by Theorem 4, and since $B' \subset B$, then $A/B' \subset A/B$ by Theorem 4. Thus $A'/B' \subset A/B$.

Theorem 5. $A \cap BC \neq \emptyset$ if and only if $A/B \cap C \neq \emptyset$.

Proof. Suppose $A \cap BC \neq \emptyset$. This means that there is a point a such that $a \subset A$ and $a \subset BC$. By Definition 1,

$a \subset BC$ means $a \subset bc$ where $b \subset B$, $c \subset C$. Thus $a \subset bc$ and by Definition 2, $c \subset a/b$. But $a/b \subset A/B$, so $c \subset A/B$. Since $c \subset C$, it follows that $A/B \cap C \neq \emptyset$.

Suppose $A/B \cap C \neq \emptyset$. Then there exists a point c such that $c \subset A/B$ and $c \subset C$. By Definition 3, $c \subset A/B$ means that $c \subset a/b$ where $a \subset A$, $b \subset B$. But $c \subset a/b$ means $a \subset bc$ by Definition 2. But $bc \subset BC$, so $a \subset BC$. Since $a \subset A$ it follows that $A \cap BC \neq \emptyset$.

Theorem 6. $a/bc = (a/b)/c$.

Proof. Let $x \subset a/bc$, that is, $x \cap a/bc \neq \emptyset$. By Theorem 5, $x(bc) \cap a \neq \emptyset$. Now $x(bc) = x(cb) = (xc)b$ by (J2) and (J3) so that $(xc)b \cap a \neq \emptyset$. Again using Theorem 5, one has $xc \cap a/b \neq \emptyset$, and $x \cap (a/b)/c \neq \emptyset$. Thus $x \subset (a/b)/c$ and $a/bc \subset (a/b)/c$.

Let $y \subset (a/b)/c$, that is, $y \cap (a/b)/c \neq \emptyset$. By Theorem 5, $yc \cap a/b \neq \emptyset$, and further $(yc)b \cap a \neq \emptyset$. But by (J2) and (J3), $(yc)b = y(cb) = y(bc)$ so that $y(bc) \cap a \neq \emptyset$. Using Theorem 5 this means that $y \cap a/bc \neq \emptyset$. Thus $y \subset a/bc$ and $(a/b)/c \subset a/bc$. Combining both parts gives $a/bc = (a/b)/c$.

Corollary 6. $A/BC = (A/B)/C$.

Proof. Let $x \subset A/BC$. Then $x \subset a/bc$ where $a \subset A$, $b \subset B$, $c \subset C$. By Theorem 6, $x \subset (a/b)/c$. Thus $x \subset (A/B)/C$ and $A/BC \subset (A/B)/C$.

Let $y \subset (A/B)/C$. Then $y \subset (a/b)/c$ where $a \subset A$, $b \subset B$, $c \subset C$. By Theorem 6, $y \subset a/bc$. Thus $y \subset A/BC$. From this it follows that $(A/B)/C \subset A/BC$ and that $A/BC = (A/B)/C$.

Theorem 7(a) $a/(b/c) \subset ac/b$ and (b) $a(b/c) \subset ab/c$.

Proof. (a) Let $x \subset a/(b/c)$. Then $x \cap a/(b/c) \neq \emptyset$. By Theorem 5, $x(b/c) \cap a \neq \emptyset$ and $b/c \cap a/x \neq \emptyset$. By (J5) this means that $bx \cap ac \neq \emptyset$. By Theorem 5, $x \cap ac/b \neq \emptyset$. Thus $x \subset ac/b$ and $a/(b/c) \subset ac/b$.

(b) Let $y \subset a(b/c)$. Then $y \cap a(b/c) \neq \emptyset$. By Theorem 5, $y/a \cap b/c \neq \emptyset$. Then $yc \cap ab \neq \emptyset$ by (J5). Thus by Theorem 5, $y \cap ab/c \neq \emptyset$, that is, $y \subset ab/c$. Thus $a(b/c) \subset ab/c$.

Corollary 7(a) $A/(B/C) \subset AC/B$ and (b) $A(B/C) \subset AB/C$.

Proof. (a) Let $x \subset A/(B/C)$. Then $x \subset a/(b/c)$ where $a \subset A$, $b \subset B$, $c \subset C$. By Theorem 7(a), $x \subset ac/b$. Thus $x \subset AC/B$ and $A/(B/C) \subset AC/B$.

(b) Let $y \subset A(B/C)$. Then $y \subset a(b/c)$ where $a \subset A$, $b \subset B$, $c \subset C$. By Theorem 7(b), $y \subset ab/c$. Thus $y \subset AB/C$ and this gives $A(B/C) \subset AB/C$.

Theorem 8. A set A is convex if and only if $AA \subset A$.

Proof. Suppose A is convex. Then $xy \subset A$ for $x, y \subset A$ by Definition 4. Thus $AA = \bigcup_{x \subset A,\, y \subset A} (xy) \subset A$. Conversely, if $AA \subset A$, then $xy \subset AA \subset A$ for $x, y \subset A$. Thus A is convex.

Theorem 9. If A, B are convex, then $A \cap B$, AB, and A/B are convex.

Proof. If $A \cap B = \emptyset$, then Definition 4 is vacuously satisfied and $A \cap B$ is convex.

If $A \cap B \neq \emptyset$, then let $x, y \subset A \cap B$. Then $xy \subset A$ and $xy \subset B$ since A and B are convex. Thus $xy \subset A \cap B$ and $A \cap B$

is convex by Definition 4.

Consider AB. By Theorem 2, (AB)(AB) = A(BA)B and by Theorem 3, A(BA)B = A(AB)B. But by Theorem 2, one has A(AB)B = (AA)(BB) and (AA)(BB) = AB by Theorem 8. Thus AB is convex by Theorem 8.

Now consider A/B. $(A/B)(A/B) \subset (A/B)A/B$ by Corollary 7(b). By Theorem 3, $(A/B)A/B \subset A(A/B)/B$ and again by Corollary 7(b), $A(A/B)/B \subset (AA/B)/B$. But by Corollary 6, $(AA/B)/B \subset AA/BB$. Now $AA/BB \subset A/B$ by Theorem 8, so $(A/B)(A/B) \subset A/B$ and A/B is convex by Theorem 8.

Theorem 10. A set A is linear if and only if A is convex and $A/A \subset A$.

Proof. Assume A is linear. Then A is convex since for every $x, y \subset A$, $xy \subset A$ by Definition 6. Also since A is linear, $x/y \subset A$ for every $x, y \subset A$. But this means that $\bigcup_{x \subset A,\, y \subset A} (x/y) \subset A$, and since $\bigcup_{x \subset A,\, y \subset A} (x/y) = A/A$, then $A/A \subset A$.

Assume A is convex and $A/A \subset A$. Since A is convex, $xy \subset A$ for every $x, y \subset A$. $A/A \subset A$ means $x/y \subset A$ where $x, y \subset A$. Thus by Definition 6, A is linear.

Corollary 10. If A is linear and $X, Y \subset A$, then $XY \subset A$ and $X/Y \subset A$.

Proof. Since $X, Y \subset A$, then $XY \subset AA$. But $AA \subset A$ by Theorem 8. Thus $XY \subset A$.

Similarly, $X/Y \subset A/A$. But $A/A \subset A$. Thus $X/Y \subset A$.

Theorem 11. If A and B are linear, then $A \cap B$ is linear.

Proof. If $A \cap B = \emptyset$, then Definition 6 is vacuously satisfied and $A \cap B$ is linear.

If $A \cap B \neq \emptyset$, then let $x, y \subset A \cap B$. Since A is linear, then by Definition 6, $x/y \subset A$. Since B is linear, then $x/y \subset B$. Thus $x/y \subset A \cap B$. By Theorem 9, $A \cap B$ is convex, so $xy \subset A \cap B$. By Definition 6, $A \cap B$ is linear.

Lemma 12. If B is a linear set and $A \subset B$, $A \neq \emptyset$, then $AB = B$ and AB is linear.

Proof. Let $x \subset AB$. Since $AB = \bigcup_{a \subset A \cap B,\ b \subset B} (ab)$,

and since B is linear, then for all $a \subset A \cap B$, $b \subset B$, it is true that $ab \subset B$. But this means that $x \subset B$. Thus $AB \subset B$.

Let $y \subset B$, $a \subset A \cap B$. Now $y/a \subset B$ since B is linear. This means that there exists an element $z \subset y/a$ such that $y \subset az$. But $a \subset A \cap B$, $z \subset B$, so $y \subset AB$ and $B \subset AB$. Thus $AB = B$ and since B is linear, AB is linear.

Lemma 13. If B is a linear set and $A \subset B$, $A \neq \emptyset$, then $A/B = B$, and A/B is linear.

Proof. Let $x \subset A/B$. Since $A/B = \bigcup_{a \subset A \cap B,\ b \subset B} (a/b)$

and B is linear, then for all $a, b \subset B$, it is true that $a/b \subset B$. Thus $x \subset B$ and $A/B \subset B$.

Let $y \subset B$. Then $a/y \subset A/B$ where $a \subset A \cap B$. Since B is linear, then for some $z \subset B$, $a \subset yz$. This means that $y \subset a/z$. Thus $y \subset A/B$ and $B \subset A/B$. Now it follows that

$B = A/B$ and since B is linear then A/B is linear.

Lemma 14. If A and B are linear and $A \cap B \neq \emptyset$, then $B \subset ((A \cap B)/(A \cap B))/B \subset A/B$.

Proof. By Lemma 13, $B = (A \cap B)/B$. By Theorem 11, $A \cap B$ is linear, so $B \subset ((A \cap B)/(A \cap B))/B$ by Theorem 10. But $(A \cap B) \subset A$, so $((A \cap B)/(A \cap B))/B \subset (A/A)/B$. But A is linear, so $A/A \subset A$ and $(A/A)/B \subset A/B$. Thus it follows that $B \subset ((A \cap B)/(A \cap B))/B \subset A/B$.

Theorem 15. If A and B are linear, $A \cap B \neq \emptyset$, then A/B is linear.

Proof. $(A/B)/(A/B) \subset ((A/B)B)/A$ by Corollary 7(a). By Lemma 14, $((A/B)B)/A \subset ((A/B)(A/B))/A$. But A/B is convex by Theorem 9, so $(A/B)(A/B) \subset A/B$. Thus it follows that $((A/B)(A/B))/A \subset (A/B)/A$. By Corollary 6, $(A/B)/A = A/(AB)$ and $A/(AB) = (A/A)/B$. But A is linear, so $A/A \subset A$ and $(A/A)/B \subset A/B$. Thus by Theorem 10, A/B is linear.

Theorem 16. If A and B are linear, $A \cap B \neq \emptyset$, then $A/B = \{A, B\}$.

Proof. By Lemma 14, $B \subset A/B$.

Let $a \subset A \cap B \subset A$. Then $aA \subset A$ since A is linear. This is true in the special case where $a \subset A \cap B$. If $a \subset A \cap B$, then $aA/a \subset A/a$, or $A \subset A/a$. But $a \subset A \cap B$, so $a \subset B$ and $A \subset A/a \subset A/B$. Thus A and B are subsets of A/B.

Since $A, B \subset \{A, B\}$, then $A/B \subset \{A, B\}$ by Corollary 10. But A/B is linear by Theorem 15, so by Definition 8 it is true that $A/B = \{A, B\}$.

Theorem 17. If A is a linear set with a dimension n, $b \subset A$, then there exists a subset, B of A with a dimension n-1 such that $b \not\subset B$.

Proof. Since A has a dimension n, then there exist n independent elements, say $x_1, \ldots, x_n$ such that $A = \{x_1, \ldots, x_n\}$. Consider the set of all subsets of A having dimension n-1, and having as a basis elements of the basis of A. That is, consider $B_i = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$, $1 \leq i \leq n$. Now $\bigcap_i B_i = \emptyset$, since $x_1, \ldots, x_n \not\subset B_i$ for any i. If b were an element of B_i for all i, then the intersection of the B_i's would not be empty. Thus there exists at least one B_i, say B, such that $b \not\subset B$ and $d(B) = n-1$.

Theorem 18. If $b \subset \{a_1, \ldots, a_n\}$, $b \not\subset \{a_2, \ldots, a_n\}$, then $\{a_1, \ldots, a_n\} = \{b, a_2, \ldots, a_n\}$.

Proof. Since $b \subset \{a_1, \ldots, a_n\}$, then by Definition 6, $\{b, a_2, \ldots, a_n\} \subset \{a_1, \ldots, a_n\}$. For the reverse inclusion, $b \subset \{a_1, \ldots, a_n\} \subset \{\{a_1, a_2\}, \{a_2, \ldots, a_n\}\} = \{a_1, a_2\}/\{a_2, \ldots, a_n\}$ by Theorem 16. This means that $b \subset p/q$ where $p \subset \{a_1, a_2\}$ and $q \subset \{a_2, \ldots, a_n\}$. Now $p \neq a_2$ since if $p = a_2$, then it would follow that $b \subset \{a_2, \ldots, a_n\}$. Since $p \neq a_2$, then by (J7), $\{a_1, a_2\} = \{p, a_2\}$. Thus $a_1 \subset \{a_1, a_2\}$, and $\{a_1, a_2\} = \{p, a_2\} \subset \{b, q, a_2\} \subset \{b, a_2, \ldots, a_n\}$. Thus $\{a_1, \ldots, a_n\} \subset \{b, a_2, \ldots, a_n\}$, and $\{a_1, \ldots, a_n\} = \{b, a_2, \ldots, a_n\}$.

Corollary 18, If $b \subset \{a_1, \ldots, a_n\}$, then $\{a_1, \ldots, a_n\} = \{b, a'_1, \ldots, a'_{n-1}\}$, where $a'_1, \ldots, a'_{n-1}$ are n-1 independent elements from the set $a_1, \ldots, a_n$.

Proof. By Theorem 17, there exists a subset of $\{a_1,\ldots,a_n\}$, say $B = \{a_1',\ldots,a_{n-1}'\}$, where $d(B) = n-1$ and $b \not\subset B$. By Theorem 18, since $b \subset \{a_1,\ldots,a_n\}$ and $b \not\subset \{a_1',\ldots,a_{n-1}'\}$, then $\{a_1,\ldots,a_n\} = \{b, a_1',\ldots,a_{n-1}'\}$.

Theorem 19. If $A = \{a_1,\ldots,a_n\}$, then any n independent elements of A form a basis of A.

Proof. Suppose $b_1,\ldots,b_n \subset \{a_1,\ldots,a_n\}$, where the b's are independent. Then by Theorem 17, one of the b's, say b_1, is not in $\{a_2,\ldots,a_n\}$. By Theorem 18, this means $\{a_1,\ldots,a_n\} = \{b_1,a_2,\ldots,a_n\} = \{a_2,\ldots,a_n,b_1\}$. Similarly, $b_2,\ldots,b_n \subset \{a_2,\ldots,a_n,b_1\}$ and $b_2,\ldots,b_n \not\subset \{a_3,\ldots,a_n,b_1\}$. Hence one of the remaining b's, say b_2, is not contained in $\{a_3,\ldots,a_n,b_1\}$ and by Theorem 18, this means that $\{a_2,\ldots,a_n,b_1\} = \{b_2,a_3,\ldots,a_n,b_1\} = \{a_3,\ldots,a_n,b_1,b_2\}$. Continuing to exchange b's for a's in this manner, it follows that $\{a_1,\ldots,a_n\} = \{b_1,\ldots,b_n\}$, and the theorem is proved.

Corollary 19(a). Let $a_1,\ldots,a_n$ be independent and $a_{n+1} \not\subset \{a_1,\ldots,a_n\}$. Then $a_1,\ldots,a_n,a_{n+1}$ are independent.

Proof. Suppose $a_1,\ldots,a_{n+1}$ are not independent. Then $a_1,\ldots,a_{n+1} \subset \{x_1,\ldots,x_n\}$ for some $x_1,\ldots,x_n$ by Definition 10. Thus $a_{n+1} \subset \{x_1,\ldots,x_n\}$, and $a_1,\ldots,a_n \subset \{x_1,\ldots,x_n\}$. By Theorem 19, this means that $\{a_1,\ldots,a_n\} = \{x_1,\ldots,x_n\}$. But this is a contradiction of the hypothesis, since $a_{n+1} \not\subset \{a_1,\ldots,a_n\}$. Thus $a_1,\ldots,a_{n+1}$ are independent.

Corollary 19(b). $a_1,\ldots,a_n$ are independent if and only if $a_i \not\subset \{a_1,\ldots,a_{i-1}\}$, $1 \leq i \leq n$.

Proof. The proof is by induction.

Let $M = \left\{ i \mid a_i \not\subset \{a_1, \ldots, a_{i-1}\} \text{ and } a_1, \ldots, a_i \text{ are independent} \right\}$.

For $i = 1$, the case is trivial since a_1 is independent. Therefore $1 \subset M$.

Assume $k \subset M$, that is, $a_k \not\subset \{a_1, \ldots, a_{k-1}\}$ and $a_1, \ldots, a_k$ are independent.

For $i = k+1$, $a_{k+1} \not\subset \{a_1, \ldots, a_k\}$. But $a_1, \ldots, a_k$ are independent, so by Corollary 19(a), $a_1, \ldots, a_{k+1}$ are independent. Therefore $k+1 \subset M$ whenever $k \subset M$, and M is the set of all positive integers.

For the converse, consider the contrapositive. Suppose $a_i \subset \{a_1, \ldots, a_{i-1}\}$ for some i. Then $a_1, \ldots, a_n$ are contained in the linear set $\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$ and the elements $a_1, \ldots, a_n$ are not independent.

Corollary 19(c). A maximal independent subset of a linear set A is a basis of A.

Proof. Suppose $a_1, \ldots, a_n$ is a maximal independent subset of A, that is, if $a_1, \ldots, a_n, a_{n+1}$ are independent, then $a_{n+1} \not\subset A$.

Since A is linear, then $\{a_1, \ldots, a_n\} \subset A$.

Let $x \subset A$. If $x \not\subset \{a_1, \ldots, a_n\}$, then $a_1, \ldots, a_n, x$ are independent by Corollary 19(b). This contradicts the assumption that $a_1, \ldots, a_n$ is a maximal independent subset of A. Thus $x \subset \{a_1, \ldots, a_n\}$, and $A \subset \{a_1, \ldots, a_n\}$, and it follows that $A = \{a_1, \ldots, a_n\}$.

One might notice that a subset of an independent set whose elements are elements from the basis of the original set is independent for if it were not, it would be impossible to reconstruct the original set by adding the independent elements deleted to form the subset.

Corollary 19(d). If A and B are linear, $A \subset B$, and B has a dimension, then $d(A) \leq d(B)$. Further, $d(A) = d(B)$ only if $A = B$.

Proof. Suppose $a_1, \ldots, a_n$ is a maximal independent subset of B. Then by Corollary 19(c), $B = \{a_1, \ldots, a_n\}$. But $A \subset B$, so $A \subset \{a_1, \ldots, a_n\}$. Thus $A = \{a_1', \ldots, a_i'\}$, where $1 \leq i \leq n$, and $a_1', \ldots, a_i' \subset \{a_1, \ldots, a_n\}$, and the elements $a_1', \ldots, a_i'$ are independent.

Now, $d(B) = n$, since $\{a_1, \ldots, a_n\}$ has n independent elements, and $d(A) = i$, since $a_1', \ldots, a_i'$ has i independent elements. But $i \leq n$, so $d(A) \leq d(B)$.

If $d(A) = d(B) = n$, then by Theorem 19, since $A \subset B$, it follows that $A = B$.

Theorem 20. (Dimension Theorem). Let A and B be linear sets which have dimensions, and let $A \cap B \neq \emptyset$, then $d(\{A, B\}) + d(A \cap B) = d(A) + d(B)$.

Proof. Let $d(A) = m$, $d(B) = n$, and assume $m \geq n$. Since $A \cap B \neq \emptyset$, then $d(A \cap B) = c$, where $c \leq n$ by Corollary 19(d). Thus $A \cap B = \{z_1, \ldots, z_c\} \subset A = \{x_1, \ldots, x_m\}$, and $\{z_1, \ldots, z_c\} \subset B = \{y_1, \ldots, y_n\}$. By repeated use of Corollary 18, it follows that $\{x_1, \ldots, x_m\} = \{z_1, \ldots, z_c, x_1', \ldots, x_{m-c}'\}$, and $\{y_1,\ldots,y_n\} = \{z_1,\ldots,z_c,y'_1,\ldots,y'_{n-c}\}$. But it is also true that $\{A,B\} = \{\{z_1,\ldots,z_c,x'_1,\ldots,x'_{m-c}\},\{z_1,\ldots,z_c,y'_1,\ldots,y'_{n-c}\}\}$, and $\{\{z_1,\ldots,z_c,x'_1,\ldots,x'_{m-c}\},\{z_1,\ldots,z_c,y'_1,\ldots,y'_{n-c}\}\}$ $= \{z_1,\ldots,z_c,x'_1,\ldots,x'_{m-c},y'_1,\ldots,y'_{n-c}\}$.

Thus $d(\{A,B\}) + d(A \cap B) = c + (m-c) + (n-c) + c = m + n$. But $m + n = d(A) + d(B)$.

THE AXIOMS OF INCIDENCE

Taking the join geometry as developed in the previous sections, it is now possible to prove the classical axioms of incidence as theorems. The axioms do not follow directly as theorems however, and it will be necessary to modify some of them.

The modification, namely imposing the restriction that there exist a linear set of dimension four in the terminology of Prenowitz, does not essentially alter the classical geometry, but merely states a condition which is assumed in the classical treatment.

Consider the following theorems, which are the axioms of incidence as stated by Borsuk [1] .

Theorem I1. For any line, L, there exist two distinct points, a and b, such that $a, b \subset L$.

Proof. By Definition 9, a line is a linear set determined by two distinct points. Thus there exist two distinct points, say a and b, such that $a, b \subset L$. In fact, $L = \{a,b\}$.

The second axiom of incidence states that "for any two points a and b, there exists at least one line L, such that $a, b \subset L$". This does not follow in the join geometry. To see this, consider this counterexample.

The linear set $\{ a \}$ satisfies the axioms and definitions of join geometry, and yet in the case a = b, one cannot assert the existence of a line. That is, if a and b exist and a = b, then $\{a,b\} = \{a\} = a$, a point, not a line.

The theorem is true, however, when there exists a linear set which has dimension of four. Consider the following theorem.

Theorem I2. If there exists a linear set, S, where d(S) = 4, then for any points a and b, where a, b $\subset$ S, there exists at least one line, L, such that a, b $\subset$ L.

Proof. d(S) = 4 means $S = \{x_1, x_2, x_3, x_4\}$, where x_1, x_2, x_3, x_4 are independent points of S.

Since a, b $\subset$ S, and since S is linear, then by Definition 6, ab, a/b, b/a, a, b $\subset$ S. Thus the union of the sets belongs to S, and $ab \cup a/b \cup b/a \cup a \cup b = \{a,b\} \subset S$. But $\{a,b\}$ is a line if a $\neq$ b, by Definition 9, and the theorem is proved if a $\neq$ b.

If a = b, then since a $\subset$ S, by Corollary 18, it follows that $\{x_1, x_2, x_3, x_4\} = \{a, x_1', x_2', x_3'\}$, where x_1', x_2', x_3' are independent, and x_1', x_2', x_3' are three of the elements x_1, x_2, x_3, x_4. Clearly, $\{a, x_1'\} \subset S$, and a, b $\subset \{a, x_1'\}$. Since a $\neq x_1'$, then $\{a, x_1'\}$ is a line, and the theorem is proved.

One might notice that there are an infinite number of choices for the line when a = b, since the linear set containing a and any other point of S determines a line satisfying the given conditions.

Theorem I3. If a $\neq$ b, then there exists at most one line, L, such that a, b $\subset$ L.

Proof. By Definition 9, if $a \neq b$, then the linear set $\{a,b\}$ is a line, say L. Assume there exists another line, say $L' = \{c,d\}$, such that $a, b \subset L'$. Since $a \subset \{c,d\}$, then either $a = d$ or $a \neq d$.

If $a = d$, then $\{c,d\} = \{c,a\}$. But $b \subset \{c,d\}$, so that $b \subset \{c,a\}$. By (J7), since $a \neq b$, then $\{b,a\} = \{c,a\}$. Thus $\{b,a\} = \{a,b\} = \{c,d\}$.

If $a \neq d$, then $\{c,d\} = \{a,d\}$ by (J7). But $a \neq b$, so $\{a,d\} = \{a,b\}$ by (J7). Thus $\{a,b\} = \{c,d\}$, and the theorem is proved.

Theorem I4. For any plane, P, there exist three non-collinear points, say a, b, c, such that $a, b, c \subset P$.

Proof. By Definition 13, $d(P) = 3$. Thus $P = \{x_1, x_2, x_3\}$, where x_1, x_2, x_3 are independent. Since x_1, x_2, x_3 are independent, then by Definition 10, x_1, x_2, x_3 are not contained in $\{y_1, y_2\}$ for any choice of y_1 and y_2. Thus x_1, x_2, x_3 are non-collinear, and since P is a linear set, $x_1, x_2, x_3 \subset P$, and the theorem follows.

The fifth axiom of incidence asserts the existence of a plane, given any three points. The following counter-example shows that this is not true in the join geometry.

If $a, b, c \subset \{a,b\}$, $a \neq b$, then one can only assert the existence of a line. Also, if $a = b = c$, then only the existence of a point follows.

The fifth axiom follows as a theorem if, as in Theorem I2, there exists a linear set of dimension four.

Theorem 15. If there exists a linear set, S, where $d(S) = 4$, then for any three points a, b, c, where a, b, $c \subset S$, there exists at least one plane, P, such that a, b, $c \subset P$.

Proof. $d(S) = 4$ means $S = \{x_1, x_2, x_3, x_4\}$, where x_1, x_2, x_3, x_4 are independent points of S.

Since a, b, $c \subset S$, then a, b, $c \subset \{x_1, x_2, x_3, x_4\}$. Since S is linear, then $\{a, b, c, x_4\} \subset S$. If a, b, c are distinct and non-collinear, then a, b, c are independent, and $\{a, b, c\}$ is a plane containing a, b, c, and contained in S, and the theorem is proved.

If a, b, c are distinct, but contained in the same line, then consider $\{a, b, x_1'\}$, where x_1' is a point of S not contained in $\{a, b\}$. By Definition 10, there does exist such a point, so it follows that $\{a, b, x_1'\}$ is a plane. Since a, b, c, are contained in $\{a, b\}$ and $\{a, b\} \subset \{a, b, x_1'\}$, the theorem follows.

If any two of the points a, b, c are the same, then the plane constructed in the above discussion will satisfy the theorem.

If all of the points a, b, and c coincide, then by Corollary 18, $\{x_1, x_2, x_3, x_4\} = \{a, x_1', x_2', x_3'\}$ where x_1', x_2' and x_3' are elements from the set x_1, x_2, x_3, x_4. Clearly, the linear set, $\{a, x_1', x_2'\}$ is a plane satisfying the conditions of the theorem.

In each case, a plane is displayed satisfying the theorem, so the theorem is proved.

In the above theorem, there are many choices for the plane which satisfy the given conditions.

Theorem I6. If a, b, and c are three non-collinear points, then there exists at most one plane, P, such that a, b, $c \subset P$.

Proof. By the first part of Theorem I5, there exists a plane, namely $\{a,b,c\}$ satisfying the conditions of this theorem. Suppose there exists a plane, say $P' = \{d,e,f\}$, such that a, b, $c \subset P'$.

Since a, b, c are independent, then a, b, $c \not\subset \{e,f\}$, and one of the elements a, b, c is not in $\{e,f\}$. For simplicity, suppose $a \not\subset \{e,f\}$. Then by Theorem 18, it is so that $\{d,e,f\} = \{a,e,f\} = \{e,f,a\}$. Clearly, b and c are contained in $\{e,f,a\}$. Furthermore, b, $c \not\subset \{f,a\}$. Hence one of the elements b, c, say b, is not in $\{f,a\}$. Then $\{e,f,a\} = \{b,f,a\} = \{f,a,b\}$, and $\{f,a,b\} = \{d,e,f\}$. But $c \not\subset \{a,b\}$, so $\{f,a,b\} = \{c,a,b\} = \{a,b,c\}$. Thus $\{a,b,c\} = \{d,e,f\}$, and the theorem is proved.

Theorem I7. For any line, L, and any plane, P, if there exist two distinct points, a and b, such that a, $b \subset L$, and a, $b \subset P$, then $L \subset P$.

Proof. Since L is a linear set, and since a and b are independent, then $L = \{a,b\}$ by Theorem 19. Since P is linear and a, $b \subset P$, then $a \cup b \cup a/b \cup b/a \cup ab \subset P$. But by (J8), $a \cup b \cup a/b \cup b/a \cup ab = \{a,b\}$. Thus $\{a,b\} \subset P$ and $L \subset P$.

The next classical axiom restricts the classical geometry to three dimensions, and since join geometry is not restricted by any dimension, it is easy to produce a counterexample to the axiom.

The axiom states that for any planes, P and Q, if there exists a point, a, such that $a \subset P$ and $a \subset Q$, then there exists a point $b \neq a$, such that $b \subset P$ and $b \subset Q$. In other words, the intersection of two planes is a line.

In a Euclidean four space, if two intersecting planes are not contained in a Euclidean three space, then their intersection is a point. This can be shown with the use of the Dimension Theorem.

Let A and B be the planes and C the four space. Since C is linear, then $\{A,B\} \subset C$, thus $d(\{A,B\}) \leq d(C) = 5$ by Corollary 19(d). Also, $d(\{A,B\}) \geq 4$, since otherwise there would be a three space containing A and B. Thus $d(\{A,B\})$ is 5 and by the Dimension Theorem, the dimension of the intersection of A and B can be computed as follows;

$$d(\{A,B\}) + d(A \cap B) = d(A) + d(B), \text{ and}$$

$5 + d(A \cap B) = 3 + 3$, and $d(A \cap B) = 1$. Thus the intersection of A and B is a point.

The axiom follows as a theorem if the two planes are contained in a linear set of dimension four.

Theorem I8. If two planes A and B are contained in a linear set S, such that $d(S) = 4$, and if there exists a point, a, such that $a \subset A$ and $a \subset B$, then there exists a point, b, where $b \neq a$, and $b \subset A$ and $b \subset B$.

Proof. Since S is linear, then $\{A,B\} \subset S$, and by

Corollary 19(d), $d(\{A,B\}) \leq d(S) = 4$. Also, if $d(\{A,B\})$ is less than 3, then $d(\{A,B\}) < d(A)$, and by Corollary 19(d), this is a contradiction, so $d(\{A,B\}) \geq 3$.

If $d(\{A,B\}) = 3$, then $A = \{A,B\} = B$ and the theorem is proved. If $d(\{A,B\}) > 3$, that is, $d(\{A,B\}) = 4$, then by the Dimension Theorem, $4 + d(A \cap B) = 3 + 3$, or $d(A \cap B) = 2$. Thus the theorem follows.

The last axiom of incidence states that there exist four non-coplanar points. In join geometry, any linear set satisfies the axioms of join geometry. Thus one can have a valid model of join geometry with a point, a line, a plane, etc., and so it does not necessarily follow that there exist four non-coplanar points. However, if as before, the existence of a linear set, S, such that $d(S) = 4$ is a part of the hypothesis, then a valid theorem results.

Theorem I9. If there exists a linear set S such that $d(S) = 4$, then there exist four non-coplanar points, say a, b, c, d.

Proof. Since $d(S) = 4$, then $S = \{x_1, x_2, x_3, x_4\}$, where x_1, x_2, x_3, x_4 are independent. Thus by Definition 10, x_1, x_2, x_3, $x_4 \not\subset \{y_1, y_2, y_3\}$ for any choice of y_1, y_2 and y_3. But for some choice of y_1, y_2, y_3 the linear set $\{y_1, y_2, y_3\}$ is a plane. Thus x_1, x_2, x_3, x_4 are not contained in a plane. But this is what is meant by points being non-coplanar, and the theorem is proved.

THE AXIOMS OF ORDER

In studying the notion of order in join geometry, one is concerned basically with the question of how collinear points are related. This relationship between collinear points is also the basis for studying order in classical geometry. Since the concept of order is based on the relationship "betweenness" in both geometries, there is little trouble in proving the classical axioms of order as theorems in join geometry.

To do this, the idea of betweenness must be defined in terms of the operation join.

Definition B. If $x \subset ab$, and $a \neq b$, then x is between a and b, and one writes (axb), or in the notation of Borsuk, B(axb).

Theorem 01. If (abc), then a, b, and c are collinear and distinct.

Proof. Since (abc), then $b \subset ac$ by Definition B. But $ac \subset \{a,c\}$, so $a, b, c \subset \{a,c\}$ and a, b, c are collinear.

By hypothesis, $b \subset ac$ and $a \neq c$. Suppose $a = b$. Then $a \cap ac \neq \emptyset$, so that $a/a \cap c \neq \emptyset$ by (J5), and $a \cap c \neq \emptyset$ by (J6). This means $a = c$, which is a contradiction. Thus $a \neq b$. By making the substitution of c for a, it follows that $c \neq b$. Thus $a \neq b \neq c$ and a, b, c are distinct.

Theorem 02. If (abc), then (cba).

Proof. (abc) means that $b \subset ac$, $a \neq c$. But if $b \subset ac$, then $b \subset ca$ by (J2). Thus $b \subset ca$, $a \neq c$, and by Definition B, it follows that (cba).

Theorem 03. If (abc), then not (bac).

Proof. Assume (bac). This means $a \subset bc$, $b \neq c$. If $a \subset bc$, then $b \not\subset ac$ by Definition 2. Thus $b \not\subset ac$, and it is not true that (abc). Since the contrapositive of the theorem is true, then the theorem is true.

Theorem 04. If a, b, and c are collinear and distinct, then (abc) or (bca) or (cab).

Proof. Since a, b, and c are collinear and distinct, then $a, b, c \subset \{a,b\}$.

But $\{a,b\} = ab \cup a/b \cup b/a \cup a \cup b$ by (J8). Since $c \subset \{a,b\}$, then $c \subset ab \cup a/b \cup b/a \cup a \cup b$. $c \neq a$, and $c \neq b$, so $c \subset ab \cup a/b \cup b/a$.

If $c \subset ab$, then (acb).

If $c \subset a/b$, then $a \subset bc$, and (cab).

If $c \subset b/a$, then $b \subset ac$, and (abc).

Thus it follows that (abc) or (bca) or (cab).

Theorem 05. If points a and b are distinct, then there exists a point, c, such that (abc).

Proof. Since a and b are distinct, and since $b/a \neq \emptyset$ by (J4), then there exists a point, say c, such that $c \subset b/a$. Thus $b \subset ac$ and (abc).

Theorem 06. If a and b are distinct, then there exists a point, c, such that (acb).

Proof. Since a and b are distinct, and ab $\neq \emptyset$ by (J1), then there exists a point, say c, such that c $\subset$ ab. Thus (acb).

Theorem O7. If (abc) and (bcd), then (abd).

Proof. Since (abc), then b $\subset$ ac, and since (bcd), then c $\subset$ bd. Thus b $\subset$ abd, and b/b $\subset$ ad by Theorem 5. But b/b = b by (J6), so b $\subset$ ad and (abd).

Theorem O8. If (abd) and (bcd), then (abc).

Proof. Since (abd), then b $\subset$ ad, and since (bcd), then c $\subset$ bd. This means c $\cap$ bd $\neq \emptyset$, and d $\cap$ c/b $\neq \emptyset$. Thus d $\subset$ c/b. Thus b $\subset$ a(c/b) $\subset$ ac/b by Theorem 7(b). Since b $\subset$ ac/b, then b $\cap$ ac/b $\neq \emptyset$, and bb $\cap$ ac $\neq \emptyset$. But bb = b, so b $\cap$ ac $\neq \emptyset$, and b $\subset$ ac. This means (abc).

The last axiom of order, namely Pasch's Postulate, does not follow directly from the theorems previously developed, and a further extension of the theory is needed to proceed.

The next section will be devoted to the development of the necessary material and to the proof of Pasch's Postulate.

PASCH'S POSTULATE

In order to prove this last axiom of order in classical geometry, it is necessary to extend some of the results in previous sections, and to state some new definitions. The concept of the relationship of points in a set to some linear subset of the set will be developed to the extent needed for the proof of Pasch's Postulate.

Definition 14. If M is a non-empty linear set, then a is congruent to b modulo M means that $aM \cap bM \neq \emptyset$, written $a \equiv b(\text{mod } M)$.

Definition 15. If M is a non-empty linear set, then $A \equiv B(\text{mod } M)$ means that for each $x \subset A$ there is a $y \subset B$ such that $x \equiv y(\text{mod } M)$, and for each $y \subset B$ there is an $x \subset A$ such that $y \equiv x(\text{mod } M)$.

Definition 16. The set of x which satisfies $x \equiv a(\text{mod } M)$ is called the congruence set modulo M determined by a, and is denoted by $(a)_M$. Briefly, $(a)_M$ is called the coset of M determined by a.

Definition 17. Cosets A and B of M are opposite if there exist elements $a \subset A$, $b \subset B$ such that $ab \cap M \neq \emptyset$.

Theorem 21(a) $a \equiv a(\text{mod } M)$; (b) if $a \equiv b(\text{mod } M)$, then $b \equiv a(\text{mod } M)$; (c) if $a \equiv b(\text{mod } M)$, and $b \equiv c(\text{mod } M)$, then $a \equiv c(\text{mod } M)$.

Proof. (a) and (b) are immediate from Definition 14. To prove (c), $a \equiv b(\text{mod } M)$ means $aM \cap bM \neq \emptyset$, and from Theorem 5, it follows that $b \cap aM/M \neq \emptyset$. Similarly, since $b \equiv c(\text{mod } M)$, it follows that $b \cap cM/M \neq \emptyset$. Thus $b \subset aM/M$ and $b \subset cM/M$ and so $aM/M \cap cM/M \neq \emptyset$. By Theorem 5, this means $aMM \cap cMM \neq \emptyset$. Since M is a linear set, then M is convex and $MM = M$ by Theorem 8. Now it follows that $aM \cap cM \neq \emptyset$, and $a \equiv c(\text{mod } M)$.

Theorem 22. If $a \equiv b(\text{mod } M)$, then $ca \equiv cb(\text{mod } M)$.

Proof. Since $a \equiv b(\text{mod } M)$, then by Definition 14, $aM \cap bM \neq \emptyset$, and $a \cap bM/M \neq \emptyset$. Thus $a \subset bM/M$. By Theorem 1, $ac \subset (bM/M)c$, and by Corollary 7(b), $(bM/M)c \subset bcM/M$, so $ac \subset bcM/M$ and $ac \cap bcM/M \neq \emptyset$. Thus $acM \cap bcM \neq \emptyset$ and $ac \equiv bc(\text{mod } M)$.

Corollary 22. If $a \equiv b(\text{mod } M)$ and $c \equiv d(\text{mod } M)$, then $ac \equiv bd(\text{mod } M)$.

Proof. Since $a \equiv b(\text{mod } M)$, then by Theorem 22, $ac \equiv bc(\text{mod } M)$. Similarly, $bc \equiv bd(\text{mod } M)$. By Theorem 21(c) it follows that $ac \equiv bd(\text{mod } M)$.

Theorem 23. If M is a non-empty linear set and $m \subset M$, then $am \equiv a(\text{mod } M)$.

Proof. Let $x \subset am$. Then $xM \subset amM = a(mM) = aM$. Thus $xM \subset aM$ and $xM \cap aM \neq \emptyset$ so that $x \equiv a(\text{mod } M)$. By Definition 15, $am \equiv a(\text{mod } M)$.

Corollary 23. Let M be a non-empty linear set and

$m \subset M$. Then $x \equiv m(\text{mod } M)$ if and only if $x \subset M$.

Proof. $x \equiv m(\text{mod } M)$ means $xM \cap mM \neq \emptyset$. Since $mM = M$, then $(xM \cap mM) \subset M$. Thus $xM \subset M$ or $x \subset M/M = M$.

Conversely, suppose $x \subset M$. By Theorem 23, $xm \equiv x(\text{mod } M)$ and $xm \equiv m(\text{mod } M)$. Thus $x \equiv m(\text{mod } M)$ by Theorem 21(c).

Theorem 24. The cosets of a linear set M form a partition of G, where G is the entire set.

Proof. If $(a)_M$ and $(b)_M$ contain an element c in common, then $c \equiv a(\text{mod } M)$ and $c \equiv b(\text{mod } M)$. By Theorem 21(b) and (c) it follows that $a \equiv b(\text{mod } M)$. Thus $a \subset (b)_M$ and $b \subset (a)_M$, so $(a)_M = (b)_M$. Consequently, if $(a)_M \neq (b)_M$, then $(a)_M \cap (b)_M = \emptyset$.

Finally, every element c of G is in the particular coset $(c)_M$, since by Theorem 21(a), $c \equiv c(\text{mod } M)$.

Theorem 25. $(a)_M = aM/M$.

Proof. Let $x \subset (a)_M$. Then by Definition 16, one has $x \equiv a(\text{mod } M)$. By Definition 14, this means $xM \cap aM \neq \emptyset$, and so $x \cap aM/M \neq \emptyset$. Thus $x \subset aM/M$ and $(a)_M \subset aM/M$.

Let $y \subset aM/M$. Then $y \cap aM/M \neq \emptyset$ and $yM \cap aM \neq \emptyset$. By Definition 14, it follows that $y \equiv a(\text{mod } M)$ and $y \subset (a)_M$. Thus $aM/M \subset (a)_M$ and $(a)_M = aM/M$.

Theorem 26. Let A and B be opposite cosets of M. If $x \subset A$ and $y \subset B$, then $xy \cap M \neq \emptyset$.

Proof. By Definition 17, there exist $a \subset A$, $b \subset B$ such that $ab \cap M \neq \emptyset$. Then A and $(a)_M$ have a in common and $A = (a)_M$ by Theorem 24. Similarly, $B = (b)_M$. Consequently, there exist elements $x \subset (a)_M$ and $y \subset (b)_M$ such that $x \equiv a(\text{mod } M)$ and $y \equiv b(\text{mod } M)$. By Corollary 22, one has $xy \equiv ab(\text{mod } M)$. Let $m \subset (ab \cap M)$. By Definition 15, there exists $m' \subset xy$ such that $m' \equiv m(\text{mod } M)$. By Corollary 23, $m' \subset M$. Thus $xy \cap M \neq \emptyset$.

Corollary 26(a). If A and B are opposite cosets of M, then $A \cap B = \emptyset$, or $A = B = M$.

Proof. Suppose $A \cap B \neq \emptyset$. Then $A = B$ by Theorem 24. Let $p \subset A$. Then $p \subset B$ and by Theorem 26, $pp \cap M \neq \emptyset$. But $pp = p$, so $p \subset M$ and $A = B = M$ by Theorem 24.

Corollary 26(b). M is the only coset of M which is its own opposite.

Proof. M is opposite to M by Definition 17, since if $m \subset M$, then $mm = m \subset M$. Uniqueness follows from Corollary 26(a).

Theorem 27. A coset $(a)_M$ of M has a unique opposite coset of M, namely M/a.

Proof. Choose a' such that $a'a \cap M \neq \emptyset$. Then $(a')_M$ is opposite to $(a)_M$ by Definition 17.

To prove uniqueness, suppose $(b)_M$ is opposite to $(a)_M$. Let $x \subset (b)_M$. By Theorem 26, $xa \cap M \neq \emptyset$, so that one has $x \cap M/a \neq \emptyset$. Thus $x \subset M/a$ and $(b)_M \subset M/a$.

Conversely, suppose $x \subset M/a$. Then $x \cap M/a \neq \emptyset$ and $xa \cap M \neq \emptyset$. This means $a \cap M/x \neq \emptyset$ and so $a \subset M/x$. But by Theorem 26, $ab \cap M \neq \emptyset$, so $a \cap M/b \neq \emptyset$ and $a \subset M/b$. Since $a \subset M/x$ and $a \subset M/b$, then $M/x \cap M/b \neq \emptyset$. From (J5), it follows that $Mx \cap Mb \neq \emptyset$ and so $x \cap Mb/M \neq \emptyset$. This means $x \subset Mb/M$. But $Mb/M = (b)_M$ by Theorem 25, so $x \subset (b)_M$ and $M/a \subset (b)_M$. Thus $(b)_M = M/a$ and the theorem is proved.

Corollary 27. Let M be a non-empty linear set. Then any coset A of M is expressible in the form M/b; conversely, M/b is always a coset of M.

Proof. Let $(b)_M$ be opposite to A. Then A is opposite to $(b)_M$ and A = M/b by Theorem 27. Conversely, let M/b be given. By Theorem 27, M/b is a coset, namely the opposite coset to $(b)_M$.

Theorem 28. Let G and M be non-empty linear sets and $M \subset G$. If $d(M) < d(G)$, then elements of G determine at least three cosets of M.

Proof. Let $a \subset G$, $a \not\subset M$. Then $(a)_M$, $(a')_M$, where a' is an element such that $aa' \cap M \neq \emptyset$, and M are three cosets of M.

Suppose two of the cosets coincide. Then since M is the only coset of M which is its own opposite by Corollary 26(b), all three must coincide. Since $a \not\subset M$, this is impossible, and the theorem holds.

Lemma 29. Let A and B be linear sets and $A \cap B \neq \emptyset$. Then A/B = A/(A/B).

Proof. $\{A,B\}$ = A/B by Theorem 16, and $\{A,A/B\}$ = A/(A/B) by Theorem 16. But $\{A,B\} = \{A,A/B\}$ by Definition 8, and so A/B = A/(A/B).

Lemma 30. Let A and B be linear sets and $A \cap B \neq \emptyset$. Then $A/(A/B) = BA/A$.

Proof. $A/(A/B) \subset BA/A$ by Corollary 7(a).

$BA/A \subset A(A/B)/A$ by Theorem 16, and $A(A/B)/A \subset (AA/B)/A$ by Corollary 7(a). But $(AA/B)/A = (A/B)/A$ since A is linear. Also, $(A/B)/A = A/BA$ by Corollary 6, and $A/BA = (A/A)/B$. Since A is linear, $(A/A)/B = A/B$, and by Lemma 29, it is true that $A/B = A/(A/B)$. Thus $BA/A \subset A/(A/B)$ and it follows that $A/(A/B) = BA/A$.

Lemma 31. Let A and B be linear sets and $A \cap B \neq \emptyset$. Then $\{A,B\} = \bigcup_{b \subset B} (b)_A$.

Proof. $\{A,B\} = A/B$ by Theorem 16. By Lemma 29, $A/B = A/(A/B)$, and $A/(A/B) = BA/A$ by Lemma 30. But $BA/A = \bigcup_{b \subset B} bA/A$, and since $bA/A = (b)_A$ by Theorem 25, it follows that $BA/A = \bigcup_{b \subset B} (b)_A$ and so $\{A,B\} = \bigcup_{b \subset B} (b)_A$.

Lemma 32. If A, B and C are non-empty linear sets, where $B \subset C$ and $A \cap C \neq \emptyset$, then $(A/B) \cap C = (A \cap C)/B$.

Proof. Let $x \subset (A/B) \cap C$. Then $x \subset A/B$ and $x \subset C$. But $B \subset C$, so $x \subset A/(B \cap C)$ and $x \subset a/b$ where $a \subset A$ and $b \subset B$. But $x = c$ for some $c \subset C$, so $c \subset a/b$ and $a \subset bc$. Since b, $c \subset C$, then $a \subset C$. Thus $a \subset (A \cap C)$ and one has $a/b \subset (A \cap C)/b$. $(A \cap C)/b \subset (A \cap C)/B$, so $a/b \subset (A \cap C)/B$. But $x \subset a/b$, so $x \subset (A \cap C)/B$ and $(A/B) \cap C \subset (A \cap C)/B$.

Now, $A \cap C \subset A$, $B \subset B$, so $(A \cap C)/B \subset A/B$. Also, $A \cap C \subset C$ and $B \subset C$, so $(A \cap C)/B \subset C$. Thus it follows that

$(A \cap C)/B \subset (A/B) \cap C$ and $(A/B) \cap C = (A \cap C)/B$.

Theorem 33. Let M be a non-empty linear set. Then $(a)_M = (b)_M$ if and only if $M/a = M/b$.

Proof. Assume $(a)_M = (b)_M$. Then $a \subset (b)_M = bM/M$, and by Theorem 4, $M/a \subset M/(bM/M)$. $M/(bM/M) \subset MM/bM$ by Corollary 7(a), and M is linear, so $MM/bM = M/bM$. By Theorem 6, it follows that $M/bM = (M/M)/b$, and again, since M is linear, $M/M = M$ and $(M/M)/b = M/b$. Thus $M/a \subset M/b$. Similarly, it can be shown that $M/b \subset M/a$ and so $M/a = M/b$.

Assume $M/a = M/b$. Then by Corollary 27, $(a)_M = (b)_M$.

Theorem 34. Let A and B be linear sets and $A \cap B \neq \emptyset$. Then there is a one to one correspondence between the cosets of A determined by elements of $\{A, B\}$ and the cosets of $A \cap B$ determined by elements of B.

Proof. In view of Lemma 31, any coset of A determined by elements of $\{A, B\}$ is expressible in the form $(b)_A$, where $b \subset B$.

Suppose $(b_1)_A = (b_2)_A$ where $b_1, b_2 \subset B$. Then by Theorem 33, $A/b_1 = A/b_2$. It is also true that $A/b_1 \cap B = A/b_2 \cap B$. By Lemma 32 it follows that $(A \cap B)/b_1 = (A \cap B)/b_2$. Thus $(b_1)_{A \cap B} = (b_2)_{A \cap B}$, and so, corresponding to a coset of A determined by elements of $\{A, B\}$, is a unique coset of $A \cap B$ determined by elements of B.

Suppose $(b_1)_{A \cap B} = (b_2)_{A \cap B}$, where $b_1, b_2 \subset B$. Then $b_2 \subset (b_1)_{A \cap B} = b_1(A \cap B)/(A \cap B) \subset b_1A/A = (b_1)_A$. Since $b_2 \subset (b_1)_A$, then $(b_2)_A = (b_1)_A$ by Theorem 24. Thus corresponding to a coset of $A \cap B$ determined by elements of B is a unique coset of A determined by elements of $\{A, B\}$. Hence, a one to one correspondence is established.

Theorem 35. Elements of $\{a,b\}$, $b \neq a$, determine exactly three cosets of b.

Proof. By Theorem 28, elements of $\{a,b\}$ determine at least three cosets of b.

Let $(x)_b$ be any coset of b, where $x \subset \{a,b\}$. Since $\{a,b\} = a \cup b \cup a/b \cup b/a \cup ab$, then it follows that $x \subset a \cup b \cup a/b \cup b/a \cup ab$.

If $x = a$, then $x \subset (a)_b$ and $(x)_b = (a)_b$.

If $x = b$, then $x \subset (b)_b = b$, and $(x)_b = b$.

If $x \subset a/b$, then $x \cap a/b \neq \emptyset$ and $xb \cap a \neq \emptyset$ so that $xb \cap ab \neq \emptyset$. Thus $x \cap ab/b \neq \emptyset$ and $x \subset ab/b = (a)_b$. Hence $(x)_b = (a)_b$.

If $x \subset b/a$, then $(x)_b = b/a$.

If $x \subset ab$, then $x \cap ab \neq \emptyset$ and $xb \cap ab \neq \emptyset$. Thus $x \cap ab/b \neq \emptyset$ and $x \cap ab/b = (a)_b$. Hence $(x)_b = (a)_b$.

There are no other possibilities. Since $(x)_b$ must be $(a)_b$ or b or b/a, then exactly three cosets of b are determined by elements of $\{a,b\}$.

Theorem 36. If $C \subset B \subset A$, where A, B, C are linear and $C \neq \emptyset$, then the number of cosets of B determined by elements of A does not exceed the number of cosets of C determined by elements of A.

Proof. By Theorem 25, for $a \subset A$, $(a)_C = aC/C \subset aB/B$, and $aB/B = (a)_B$. Hence the theorem follows.

Theorem 37. If L is a line contained in a plane P, then the elements of P determine three cosets of L.

Proof. Let $a \subset P$, $a \not\subset L$ and let $b \subset L$. Now, $P = \{L, a\}$ and $\{L, a\} = \{L, \{a, b\}\}$. By Theorem 34, there is a one to one correspondence between the cosets of L determined by elements of $\{L, \{a, b\}\}$ and the cosets of $L \cap \{a, b\}$ determined by elements of $\{a, b\}$. But $b \subset L \cap \{a, b\}$ and $L \cap \{a, b\} \subset \{a, b\}$; also $\{a, b\} \neq L \cap \{a, b\}$. Hence the number of cosets of $L \cap \{a, b\}$ determined by elements of $\{a, b\}$ does not exceed the number of cosets of b determined by elements of $\{a, b\}$ by Theorem 36, which is three by Theorem 35. However, by Theorem 28, there are at least three cosets. Thus elements of $\{a, b\}$ determine exactly three cosets of $L \cap \{a, b\}$, and by the one to one correspondence, there are three cosets of L determined by elements of $\{L, \{a, b\}\} = P$. Hence, if a line is contained in a plane, the elements of the plane determine exactly three cosets of the line.

Theorem 38. Suppose L is a line contained in a plane P. If $a \subset P$, $a \not\subset L$ and $aa' \cap L \neq \emptyset$, then $P = L \cup L/a \cup L/a'$.

Proof. By Theorem 24, the element a determines a coset of L, namely $(a)_L$, and a' determines a coset of L, namely $(a')_L$. But $aa' \cap L \neq \emptyset$, so $(a)_L$ and $(a')_L$ are opposite cosets by Definition 17. By Theorem 27, $(a)_L = L/a'$ and $(a')_L = L/a$. A third coset of L is L. Hence, since there are only three cosets, these must be the three, and it follows from Theorem 24 that $P = L \cup L/a \cup L/a'$.

Theorem 39. (Pasch's Postulate). Let a line L be contained in a plane $\{a,b,c\}$ and a, b, c $\not\subset$ L. If $L \cap ab \neq \emptyset$, then $L \cap bc \neq \emptyset$ or $L \cap ac \neq \emptyset$ but not both.

Proof. By Theorem 38, $P = L \cup L/a \cup L/b$. Since $c \subset P$, then $c \subset L \cup L/a \cup L/b$. But $c \not\subset L$. so either $c \subset L/a$ or $c \subset L/b$, but not both, since the cosets are disjoint by Theorem 24.

If $c \subset L/a$, then $c \cap L/a \neq \emptyset$ and $ac \cap L \neq \emptyset$.

If $c \subset L/b$, then $c \cap L/b \neq \emptyset$ and $bc \cap L \neq \emptyset$.

Hence either $L \cap ac \neq \emptyset$ or $L \cap bc \neq \emptyset$ but not both.

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