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Anti-Associative Systems

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ANTI-ASSOCIATIVE SYSTEMS

by

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To Dr. Charles H. Cunkle for introducing me to mathematical research and for encouragement and direction in that research.

LIST OF CHARTS

INTRODUCTION

A set of elements with a binary operation is called a system, or, more explicitly, a mathematical system. [2] The following discussion will involve systems with only one operation. This operation will be denoted by "$\cdot$" and will sometimes be referred to as a product.

A system, S, of n elements $(x_1, x_2, \ldots, x_n)$ is associative if

$$x_i \cdot (x_j \cdot x_k) = (x_i \cdot x_j) \cdot x_k$$

for all $i, j, k \leq n$.

In a modern algebra class the following problem was proposed. What is the least number of elements a system can have and be non-associative? A system, S, of n elements $(x_1, x_2, \ldots, x_n)$ is non-associative if it fails to be associative which implies that

$$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$$

for some $i, j, k \leq n$. It is obvious that a system of one element must be associative. Any binary operation could have but one result. A nonassociative system of two elements (a,b) can be constructed by letting $a \cdot a = b$, $b \cdot a = b$.

$$a \cdot (a \cdot a) = a \cdot b,$$

and $$(a \cdot a) \cdot a = b \cdot a = b.$$

If $a \cdot b = a$, then

$$a \cdot (a \cdot a) \neq (a \cdot a) \cdot a.$$

Thus, the system is nonassociative.

As is often the case this question leads to others. Are there systems of n elements $(x_1, x_2, \ldots, x_n)$ such that

$$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$$

for all $i, j, k \leq n$? If such systems exist, what are their characteristics? Such questions as these led to the development of this paper.

A system, S, of n elements $(x_1, x_2, \ldots, x_n)$ where

$$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$$

for all $i, j, k \leq n$ is called an anti-associative system.

The purpose of this paper is to establish the existence of anti-associative systems of n elements and to find characteristics of these systems in as much detail as possible.

Propositions will first be considered that apply to anti-associative systems in general. Then anti-associative systems of two, three, and four elements will be obtained. The general results that each of these special cases lead to will be developed. A special type of anti-associative system will be considered. These special anti-associative systems suggest a broader field. For a set of elements a group of classes of systems is defined. The operation may be associative, anti-associative, or neither. Many questions are left unanswered as to the characteristics of anti-associative systems, but this paper opens new avenues to attack a broader problem.

GENERAL PROPERTIES

There are several properties of anti-associative systems in general that will be of value in later discussions. These properties will be presented in the form of theorems.

Theorem 1. If a system, S, of n elements $(x_1, x_2, \ldots, x_n)$ is anti-associative, then S is not commutative.

Note: A system, S, of n elements $(x_1, x_2, \ldots, x_n)$ is commutative if $x_i \cdot x_j = x_j \cdot x_i$ for all $i, j \leq n$.

Proof: This proof is by contradiction. Assume that S is commutative. Let a be an element of S.

$a \cdot (a \cdot a) = (a \cdot a) \cdot a$.

This contradicts the hypothesis that S is anti-associative.

Theorem 2. If a system, S, is anti-associative, then $a \cdot a \neq a$ for each element a of S.

Proof: This proof is by contradiction.

Assume that $a \cdot a = a$ where a is an element of S.

$a \cdot (a \cdot a) = a \cdot a = a$.

$(a \cdot a) \cdot a = a \cdot a = a$.

$a \cdot (a \cdot a) = (a \cdot a) \cdot a$.

This contradicts the hypothesis that S is anti-associative.

Corollary. If a system, S, of n elements $(x_1, x_2, \ldots, x_n)$ is anti-associative, then it has no right or left identity.

Note: A system, S, of n elements $(x_1, x_2, \ldots, x_n)$ has a right identity, x_g, where g is a constant and $g \leq n$, if $x_i \cdot x_g = x_i$ for all $i \leq n$.

A system, S, of n elements $(x_1, x_2, \ldots, x_n)$ has a left identity, x_h, where h is a constant and $h \leq n$, if $x_h \cdot x_i = x_i$ for all $i \leq n$.

If a system has a right and left identity and they are the same element, then this element is called the identity of the system.

Proof: This proof is by contradiction.

Assume that S has a right identity, x_g.

$x_g \cdot x_g = x_g$.

Theorem 2 states that if a system has such an element it is not anti-associative.

This contradicts the original hypothesis that S is anti-associative.

Assume that S has a left identity, x_h.

$x_h \cdot x_h = x_h$.

Theorem 2 states that if a system has such an element, then the system is not anti-associative.

This contradicts the original hypothesis that S is anti-associative.

Anti-associative systems are not commutative, and any element a of the system is such that $a \cdot a \neq a$. The second condition implies that an anti-associative system has no right or left identity.

ANTI-ASSOCIATIVE SYSTEMS OF TWO ELEMENTS

In order to be anti-associative a system of two elements (a,b) must have $a \cdot a = b$ (Theorem 2), $b \cdot b = a$ (Theorem 2). Also $a \cdot b \neq b \cdot a$ as $a \cdot a$ commutes and $b \cdot b$ commutes; thus, by Theorem 1 $a \cdot b$ can not commute. If $a \cdot b = a$, then $b \cdot a = b$. If $a \cdot b = b$, then $b \cdot a = a$. This can be put in the form of operation tables. The two tables are:

$\cdot$	a	b
a	b	a
b	b	a

,

$\cdot$	a	b
a	b	b
b	a	a

.

These tables are the same form as the ordinary multiplication table:

x	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

All anti-associative systems of two elements must be of these forms, but different symbols may be used for the two elements. It must be shown that each of these systems is anti-associative.

The system, S, of two elements (a,b) such that

	a	b
a	b	a
b	b	a

can easily be shown to be anti-associative. Let x_i with $i \leq 2$ represent either element of the system.

$x_i \cdot (x_j \cdot a) = x_i \cdot b = a$ with $i, j \leq 2$.
$(x_i \cdot x_j) \cdot a = b$.
$x_i \cdot (x_j \cdot a) \neq (x_i \cdot x_j) \cdot a$.
$x_i \cdot (x_j \cdot b) = x_i \cdot a = b$.
$(x_i \cdot x_j) \cdot b = a$.
$x_i \cdot (x_j \cdot b) \neq (x_i \cdot x_j) \cdot b$.
$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all $i, j, k \leq 2$.
S is anti-associative.

In a like manner the system, S, of two elements (a,b) such that

$\cdot$	a	b
a	b	b
b	a	a

can be shown to be anti-associative. Let x_i with $i \leq 2$ represent either element of the system.

$a \cdot (x_i \cdot x_j) = b$ with $i, j \leq 2$.
$(a \cdot x_i) \cdot x_j = b \cdot x_i = a$.
$a \cdot (x_i \cdot x_j) \neq (a \cdot x_i) \cdot x_j$.
$b \cdot (x_i \cdot x_j) = a$.
$(b \cdot x_i) \cdot x_j = a \cdot x_j = b$.
$b \cdot (x_i \cdot x_j) \neq (b \cdot x_i) \cdot x_j$.
$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all $i, j, k \leq 2$.
S is anti-associative.

There is a special relationship between the two anti-associative systems of two elements. If, in either one of these systems, the columns become the corresponding rows, the second system is found. The $x_i \cdot x_j$ element of the original system becomes the $x_j \cdot' x_i$ element of the second system. These two systems are called <u>transposes</u> of each other. Thus, in a system of two elements the transpose of an anti-associative system is also anti-associative. Is this true of systems of more than two elements?

<u>Theorem 3</u>. If a system, S, of n elements $(x_1, x_2, \ldots, x_n)$ is anti-associative, then its transpose, S', is anti-associative.

Proof: $x_i \cdot' x_j = x_j \cdot x_i$ for all $i, j \leq n$.
$x_i \cdot' (x_j \cdot' x_k) = (x_k \cdot x_j) \cdot x_i$ for all $i, j, k \leq n$.
$(x_i \cdot' x_j) \cdot' x_k = x_k \cdot (x_j \cdot x_i)$.
$x_k \cdot (x_j \cdot x_i) \neq (x_k \cdot x_j) \cdot x_i$.
$x_i \cdot' (x_j \cdot' x_k) \neq (x_i \cdot' x_j) \cdot' x_k$.
Thus, the transpose of an anti-associative system is anti-associative.

The anti-associative systems of two elements appear to conform to a pattern. Each column (or row) of the operation table is composed of one single element which differs from the operation element of the column (or row). Does this same pattern hold for anti-associative systems of n elements?

<u>Theorem 4.</u> If a system, S, of n elements $(x_1, x_2, \ldots, x_n)$ is such that $x_i \cdot x_g = x_h \neq x_g$ (or $x_g \cdot x_i = x_h \neq x_g$) where $i = 1, 2, \ldots, n$; g is an arbitrary constant, $g \leq n$, h is a constant; $h \neq g$ and $h \leq n$, then S is an anti-associative system.

Proof: $x_i \cdot (x_j \cdot x_g) = x_i \cdot x_h \neq x_h$ for all $i, j \leq n$.

$(x_i \cdot x_j) \cdot x_g = x_h$.

$x_i \cdot (x_j \cdot x_g) \neq (x_i \; x_j) \; x_g$.

$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all $i, j, k \leq n$.

Thus, S is anti-associative.

There exist anti-associative systems of any finite number of elements which are of the form of the anti-associative systems of two elements (see above).

Any extension of the operation table of an anti-associative system of two elements by repeating any column (or row) and its corresponding row (or column) gives an anti-associative system of the form of Theorem 4. It does not matter what symbol is used to represent the new element. As an example, the anti-associative system,

·	a	b
a	b	a
b	b	a ,

can be extended by repeating the first row and then the first column. Repeating the first row gives

·	a	b	c
a	b	a	
b	b	a	
c	b	a	,

and repeating the first column gives

	a	b	c
a	b	a	b
b	b	a	b
c	b	a	b

The resulting system is of the form of Theorem 4 and is therefore anti-associative. This system can be extended a second time by repeating a row (or column) and its corresponding column (or row) from the first extension. This may be done any number of times. The operation will only result in the original two elements. It can be shown that all anti-associative systems can be extended in the same way.

Theorem 5. If an element x_{n+1} is adjoined to an anti-associative system, S, of n elements $(x_1, x_2, \ldots, x_n)$ such that

$$x_{n+1} \cdot x_i = x_g \cdot x_i$$
$$x_j \cdot x_{n+1} = x_j \cdot x_g$$

Where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n, n+1$, g is a constant and $g \leq n$, then the extended system is anti-associative.

Proof: $x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all i, j, $k \leq n$

If $i = n+1$, then

$$x_{n+1} \cdot (x_j \cdot x_k) = x_g \cdot (x_j \cdot x_k), \text{ and}$$
$$(x_{n+1} \cdot x_j) \cdot x_k = (x_g \cdot x_j) \cdot x_k.$$

Thus, $x_{n+1} \cdot (x_j \cdot x_k) \neq (x_{n+1} \cdot x_j) \cdot x_k$.

If $j = n+1$ or $k = n+1$, the proof can be obtained in a similar manner.

If $i, j = n+1$, then

$$x_{n+1} \cdot (x_{n+1} \cdot x_k) = x_{n+1} \cdot (x_g \cdot x_k) = x_g \cdot (x_g \cdot x_k), \text{ and}$$
$$(x_{n+1} \cdot x_{n+1}) \cdot x_k = (x_g \cdot x_{n+1}) \cdot x_k = (x_g \cdot x_g) \cdot x_k.$$

Thus, $x_{n+1} \cdot (x_{n+1} \cdot x_k) \neq (x_{n+1} \cdot x_{n+1}) \cdot x_k$.

If $i, j = n+1$ or $j, k = n+1$, the proof can be obtained in a similar manner.

If $i, j, k = n+1$, then

$$x_{n+1} \cdot (x_{n+1} \cdot x_{n+1}) = x_{n+1} \cdot (x_g \cdot x_{n+1}) = x_{n+1} \cdot (x_g \cdot x_g) = x_g \cdot (x_g \cdot x_g),$$

and

$$(x_{n+1} \cdot x_{n+1}) \cdot x_{n+1} = (x_g \cdot x_{n+1}) \cdot x_{n+1} = (x_g \cdot x_g) \cdot x_{n+1} = (x_g \cdot x_g) \cdot x_g.$$

Thus, $x_{n+1} \cdot (x_{n+1} \cdot x_{n+1}) \neq (x_{n+1} \cdot x_{n+1}) \cdot x_{n+1}$.

Thus, $x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all i, j, $k \leq n+1$.

The extended system is anti-associative.

The extension can be repeated any number of times. Each extension gives an anti-associative system, and any anti-associative system can be extended in this way.

A two-element anti-associative system (x_1, x_2) can be extended to a n element $(x_1, x_2, \ldots, x_n)$ anti-associative system such that the element $x_i \cdot x_j = x_1$ or $x_i \cdot x_j = x_2$ for all i, $j \leq n$.

Although there were only two anti-associative systems of two elements, they opened the way to anti-associative systems of any finite number of elements and helped establish a way of extending all anti-associative systems.

ANTI-ASSOCIATIVE SYSTEMS OF THREE ELEMENTS

Some anti-associative systems of three elements have already been found. It must be determined whether there are others. The number of systems increases rapidly as the number of elements increases.

If the binary operation is called a product, there are four products in a system of two elements. Each of these products can result in either of the two elements. This gives a possibilty of 2^4 or 16 different systems of two elements.

The systems of three elements have nine different products, and each product can result in any of the three elements. There is a possibility of 3^9 or 19,683 different systems of three elements. Thus, many more difficulties will arise in working with three-element systems than in working with two-element systems.

A mapping of T into T' where T and T' are two sets of elements, is a correspondence between T and T' that associates with each element, a, of T a unique element, a', of T'. [4]

A mapping of T into T' is a mapping of T onto T' if each element, a', of T' corresponds to some element, a, of T. The mapping of T onto T' is a one-to-one mapping of T onto T' if for each pair of elements (a,b) of T, $a \neq b$ implies that in the corresponding pair of elements (a',b') of T', that $a' \neq b'$. [4]

The notation $\rightleftarrows$ will be used for a one-to-one mapping.

Let S and S' be systems with elements $x_1, x_2, \dots, x_n$ and $x_1', x_2', \dots, x_n'$ respectively. A one-to-one mapping $S \rightleftarrows S'$ of the elements of S onto those of S' is called an isomorphism if whenever $x_1 \rightleftarrows x_1'$ and $x_2 \rightleftarrows x_2'$, then $x_1 \cdot x_2 \rightleftarrows x_1' \cdot x_2'$. (See isomorphism between groups as defined by Hall. [3])

Theorem 6. If an isomorphism exists between a system, S, of n elememts $(x_1, x_2, \dots, x_n)$ and an anti-associative system, S', of n elements $(x_1', x_2', \dots, x_n')$, then the system, S is anti-associative.

Proof: $x_i' \cdot (x_j' \cdot x_k') \neq (x_i' \cdot x_j') \cdot x_k'$ for all i, j, $k \leq n$.

Let $x_i' \rightleftarrows x_i$ for all $i \leq n$.

$x_i' \cdot x_j' \rightleftarrows x_i \cdot x_j$.

$x_j' \cdot x_k' \rightleftarrows x_j \cdot x_k$.
$x_i' \cdot (x_j' \cdot x_k') \rightleftarrows x_i \cdot (x_j \cdot x_k)$.
$(x_i' \cdot x_j') \cdot x_k' \rightleftarrows (x_i \cdot x_j) \cdot x_k$.
An isomorphism is a one-to-one mapping.
Thus, $x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$.
The system, S, is anti-associative.

In order to find all anti-associative systems of three elements, a definite procedure must be set up. All possibilities will be covered by letting a·a = a, a·a = b, a·a = c respectively. In each case the other products need to be defined so as to give an anti-associative system.

Case 1. Let a·a = a. This could not possibly give an anti-associative system (Theorem 2).

Case 2. Let a·a = b. This will be divided into three subcases where a·b = a, a·b = b, a·b = c respectively.

Case 2A. Let a·a = b, a·b = a. The rest of the products must be defined in such a way that $x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all i, j, k ≤ 3. Each product will be defined so as to give each element of the system respectively. Upon each assumption it will be determined whether there exists i, j, k such that $x_i \cdot (x_j \cdot x_k) = (x_i \cdot x_j) \cdot x_k$. If such i, j, k exist then the resulting system could not be anti-associative. The combination must be rejected. In this way it will be determined whether an anti-associative system is possible when a·a = b, a·b = a.

Assume that b·a = a when a·a = b, a·b = a.

a·(a·a) = a·b = a.
(a·a)·a = b·a = a.
a·(a·a) = (a·a)·a.

No anti-associative system can exist when a·a = b, a·b = a, b·a = a. Assume that b·a = b. Chart 1 develops the only possible anti-associative systems when a·a = b, a·b = a, b·a = b. These systems are:

(1)

·	a	b	c
a	b	a	a
b	b	a	a
c	b	a	a

,

(2)

·	a	b	c
a	b	a	a
b	b	a	a
c	c	a	a

,

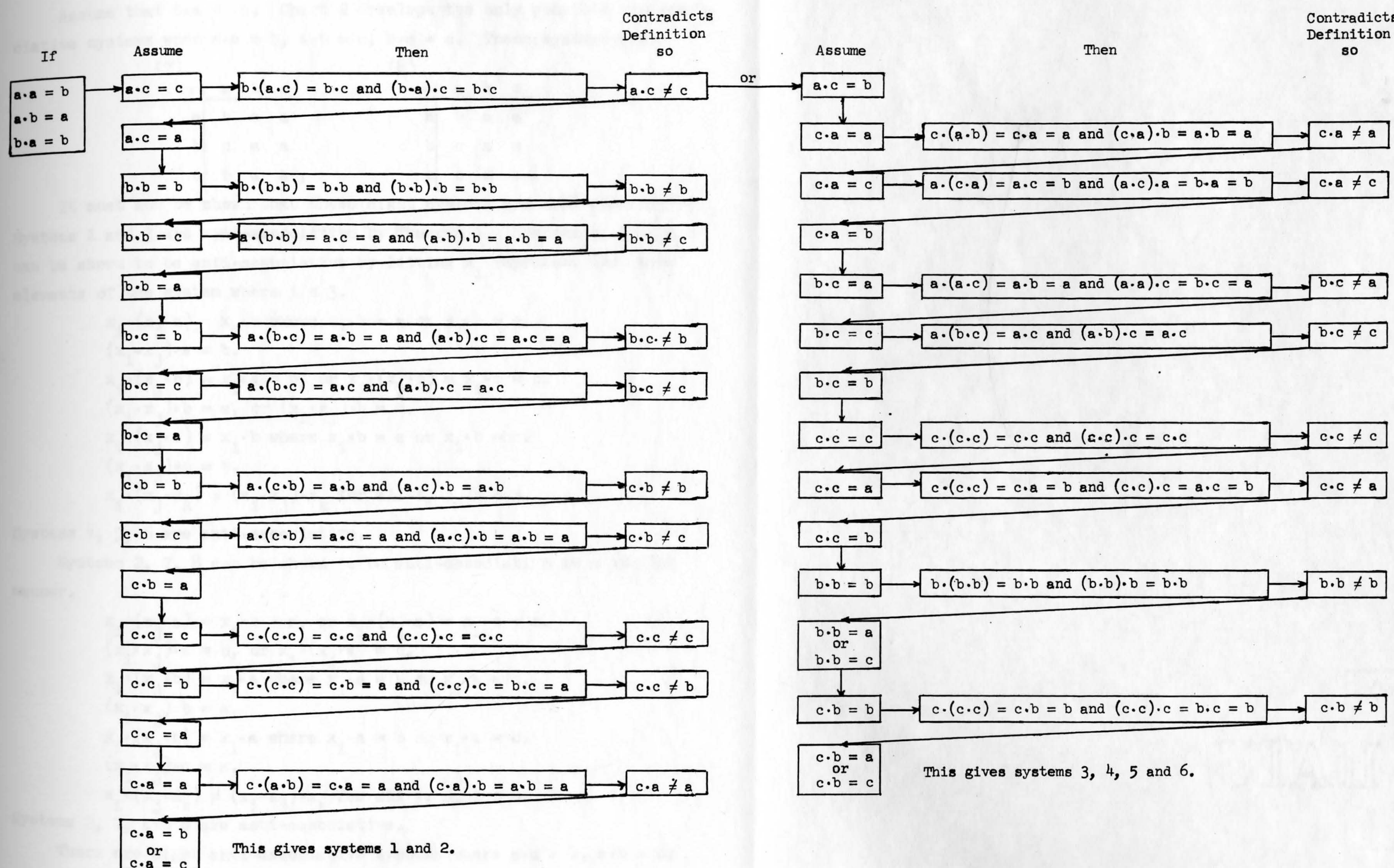
(3)

·	a	b	c
a	b	a	b
b	b	a	b
c	b	a	b

,

Chart 1

Possible Anti-associative Systems when a·a = b, a·b = a, b·a = b.



(4)

$\cdot$	a	b	c
a	b	a	b
b	b	a	b
c	b	c	b

,

(5)

$\cdot$	a	b	c
a	b	a	b
b	b	c	b
c	b	a	b

,

(6)

$\cdot$	a	b	c
a	b	a	b
b	b	c	b
c	b	c	b

.

Assume that $b \cdot a = c$. Chart 2 develops the only possible anti-associative systems when $a \cdot a = b$, $a \cdot b = a$, $b \cdot a = c$. These systems are:

(7)

$\cdot$	a	b	c
a	b	a	a
b	c	a	a
c	b	a	a

,

(8)

$\cdot$	a	b	c
a	b	a	a
b	c	a	a
c	c	a	a

.

It must now be shown that these eight systems are anti-associative. Systems 1 and 3 are anti-associative by Theorem 4. Systems 4, 5, and 6 can be shown to be anti-associative by letting x_i represent the three elements of the system where $i \leq 3$.

$x_i \cdot (x_j \cdot a) = x_i \cdot b$ where $x_i \cdot b = a$ or $x_i \cdot b = c$.
$(x_i \cdot x_j) \cdot a = b$.
$x_i \cdot (x_j \cdot b) = x_i \cdot a = b$, or $x_i \cdot (x_j \cdot b) = x_i \cdot c = b$.
$(x_i \cdot x_j) \cdot b = a$, or $(x_i \cdot x_j) \cdot b = c$.
$x_i \cdot (x_j \cdot c) = x_i \cdot b$ where $x_i \cdot b = a$ or $x_i \cdot b = c$.
$(x_i \cdot x_j) \cdot c = b$.
$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all $i, j, k \leq 3$.

Systems 4, 5, 6 are anti-associative.

Systems 2, 7, 8 can be shown to be anti-associative in a similar manner.

$x_i \cdot (x_j \cdot a) = x_i \cdot b = a$, or $x_i \cdot (x_j \cdot a) = x_i \cdot c = a$.
$(x_i \cdot x_j) \cdot a = b$, or $x_i \cdot (x_j \cdot a) = c$.
$x_i \cdot (x_j \cdot b) = x_i \cdot a$ where $x_i \cdot a = b$ or $x_i \cdot a = c$.
$(x_i \cdot x_j) \cdot b = a$.
$x_i \cdot (x_j \cdot c) = x_i \cdot a$ where $x_i \cdot a = b$ or $x_i \cdot a = c$.
$(x_i \cdot x_j) \cdot c = a$.
$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all $i, j, k \leq n$.

Systems 2, 7, and 8 are anti-associative.

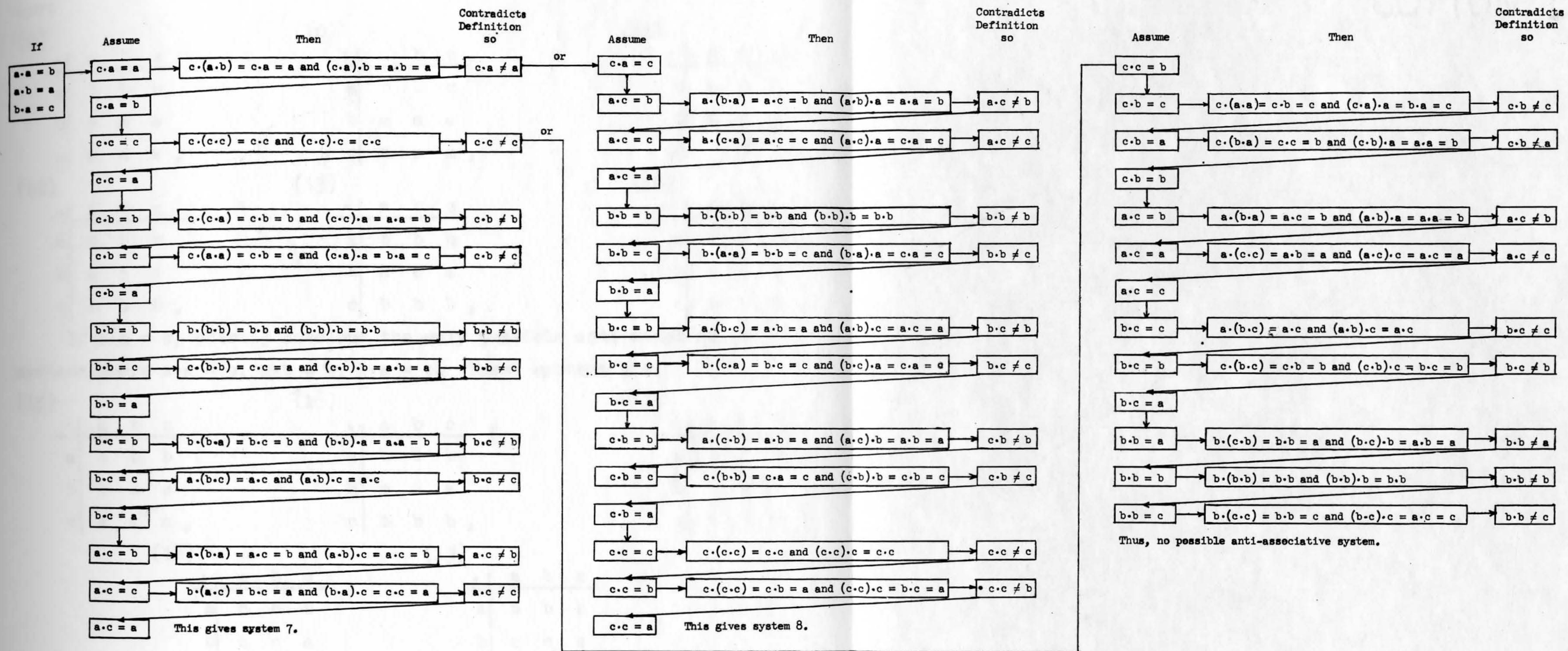
There are eight anti-associative systems where $a \cdot a = b$, $a \cdot b = b$.

Case 2B. Let $a \cdot a = b$, $a \cdot b = b$. Assume that $b \cdot a = b$.

$a \cdot (a \cdot a) = a \cdot b = b$.

Chart 2

Possible Anti-associative Systems when a·a = b, a·b = a, b·a = c.



$(a \cdot a) \cdot a = b \cdot a = b.$

$a \cdot (a \cdot a) = (a \cdot a) \cdot a.$

No anti-associative system can exist when $a \cdot a = b$, $a \cdot b = b$, $b \cdot a = b$.

Assume that $b \cdot a = a$. $a \cdot a = b$, $a \cdot b = b$, $b \cdot a = a$ is the transpose of the systems of chart 1 where $a \cdot a = b$, $a \cdot b = a$, $b \cdot a = b$. $a \cdot a = b$, $a \cdot b = b$, $b \cdot a = a$ will, thus, give anti-associative systems which are transposes of systems 1 through 6 by Theorem 3. The transpose of systems 1 through 6 are:

(9)

•	a	b	c
a	b	b	b
b	a	a	a
c	a	a	a

,

(10)

•	a	b	c
a	b	b	c
b	a	a	a
c	a	a	a

,

(11)

•	a	b	c
a	b	b	b
b	a	a	a
c	b	b	b

,

(12)

•	a	b	c
a	b	b	b
b	a	a	c
c	b	b	b

,

(13)

•	a	b	c
a	b	b	b
b	a	c	a
c	b	b	b

,

(14)

•	a	b	c
a	b	b	b
b	a	c	c
c	b	b	b

.

If $b \cdot a = c$, chart 3 develops the only possible anti-associative systems where $a \cdot a = b$, $a \cdot b = b$, $b \cdot a = c$. These systems are:

(15)

•	a	b	c
a	b	b	b
b	c	c	c
c	a	a	a

,

(16)

•	a	b	c
a	b	b	b
b	c	a	a
c	b	b	b

,

(17)

•	a	b	c
a	b	b	b
b	c	a	c
c	b	b	b

,

(18)

•	a	b	c
a	b	b	b
b	c	c	a
c	b	b	b

,

(19)

•	a	b	c
a	b	b	b
b	c	c	c
c	b	b	b

.

System 15 is anti-associative by Theorem 4. Systems 16 through 19 can be shown to be anti-associative by the following equations.

$a \cdot (x_i \cdot x_j) = b.$

$(a \cdot x_i) \cdot x_j = b \cdot x_j$ where $b \cdot x_j = c$ or $b \cdot x_j = a$.

$b \cdot (x_i \cdot x_j) = c$, or $(b \cdot x_i) \cdot x_j = a$.

$(b \cdot x_i) \cdot x_j = c \cdot x_j = b$ or $(b \cdot x_i) \cdot x_j = a \cdot x_j = b$.

$c \cdot (x_i \cdot x_j) = b$

$(c \cdot x_i) \cdot x_j = b \cdot x_j$ where $b \cdot x_j = c$ or $b \cdot x_j = a$.

Chart 3

Possible Anti-associative Systems when $a \cdot a = b$, $a \cdot b = b$, $b \cdot a = c$.

If $a \cdot a = b$, $a \cdot b = b$, $b \cdot a = c$

Assume	Then	Contradicts Definition so
$c \cdot a = c$	$c \cdot (a \cdot b) = c \cdot b$ and $(c \cdot a) \cdot b = c \cdot b$	$c \cdot a \neq c$
$c \cdot a = a$		
$b \cdot b = a$	$b \cdot (a \cdot a) = b \cdot b = a$ and $(b \cdot a) \cdot a = c \cdot a = a$	$b \cdot b \neq a$
$b \cdot b = b$	$b \cdot (b \cdot b) = b \cdot b$ and $(b \cdot b) \cdot b = b \cdot b$.	$b \cdot b \neq b$
$b \cdot b = c$		
$b \cdot c = a$	$b \cdot (b \cdot a) = b \cdot c = a$ and $(b \cdot b) \cdot a = c \cdot a = a$	$b \cdot c \neq a$
$b \cdot c = b$	$b \cdot (c \cdot a) = b \cdot a = c$ and $(b \cdot c) \cdot a = b \cdot a = c$	$b \cdot c \neq b$
$b \cdot c = c$		
$c \cdot b = b$	$c \cdot (a \cdot a) = c \cdot b = b$ and $(c \cdot a) \cdot a = a \cdot a = b$	$c \cdot b \neq b$
$c \cdot b = c$	$c \cdot (b \cdot c) = c \cdot c$ and $(c \cdot b) \cdot c = c \cdot c$	$c \cdot b \neq c$
$c \cdot b = a$		
$a \cdot c = a$	$a \cdot (c \cdot a) = a \cdot a$ and $(a \cdot c) \cdot a = a \cdot a$	$a \cdot c \neq a$
$a \cdot c = c$	$a \cdot (a \cdot c) = a \cdot c = c$ and $(a \cdot a) \cdot c = b \cdot c = c$	$a \cdot c \neq c$
$a \cdot c = b$		
$c \cdot c = c$	$c \cdot (c \cdot c) = c \cdot c$ and $(c \cdot c) \cdot c = c \cdot c$	$c \cdot c \neq c$
$c \cdot c = b$	$c \cdot (b \cdot c) = c \cdot c = b$ and $(c \cdot b) \cdot c = a \cdot c = b$	$c \cdot c \neq b$
$c \cdot c = a$	This gives system 15.	

or

Assume	Then	Contradicts Definition so
$c \cdot a = b$		
$a \cdot c = a$	$a \cdot (c \cdot a) = a \cdot b = b$ and $(a \cdot c) \cdot a = a \cdot a = b$	$a \cdot c \neq a$
$a \cdot c = c$	$a \cdot (c \cdot a) = a \cdot b = b$ and $(a \cdot c) \cdot a = c \cdot a = b$	$a \cdot c \neq c$
$a \cdot c = b$		
$c \cdot c = c$	$c \cdot (c \cdot c) = c \cdot c$ and $(c \cdot c) \cdot c = c \cdot c$	$c \cdot c \neq c$
$c \cdot c = a$	$c \cdot (c \cdot c) = c \cdot a = b$ and $(c \cdot c) \cdot c = a \cdot c = b$	$c \cdot c \neq a$
$c \cdot c = b$		
$c \cdot b = a$	$c \cdot (b \cdot a) = c \cdot c = b$ and $(c \cdot b) \cdot a = a \cdot a = b$	$c \cdot b \neq a$
$c \cdot b = c$	$c \cdot (a \cdot a) = c \cdot b = c$ and $(c \cdot a) \cdot a = b \cdot a = c$	$c \cdot b \neq c$
$c \cdot b = b$		
$b \cdot b = b$	$b \cdot (b \cdot b) = b \cdot b$ and $(b \cdot b) \cdot b = b \cdot b$	$b \cdot b \neq b$
$b \cdot b = a$ or $b \cdot b = c$		
$b \cdot c = b$	$c \cdot (c \cdot c) = c \cdot b = b$ and $(c \cdot c) \cdot c = b \cdot c = b$	$b \cdot c \neq b$
$b \cdot c = a$ or $b \cdot c = c$	This gives systems 16, 17, 18 and 19.	

$$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k \text{ for all } i,\ j,\ k \leq 3.$$

Systems 16 through 19 are anti-associative.

There are 11 anti-associative systems that result from letting $a \cdot a = b$, $a \cdot b = b$.

Case 2C. Let $a \cdot a = b$, $a \cdot b = c$. Assume that $b \cdot a = c$.

$$a \cdot (a \cdot a) = a \cdot b = c.$$
$$(a \cdot a) \cdot a = b \cdot a = c.$$
$$a \cdot (a \cdot a) = (a \cdot a) \cdot a.$$

No anti-associative system can exist when $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = c$.

Assume that $b \cdot a = a$. $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = a$ is the transpose of the systems of chart 2 where $a \cdot a = b$, $a \cdot b = a$, $b \cdot a = c$. $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = a$ will, thus, give anti-associative systems which are transposes of systems 7 and 8 by Theorem 3. The transposes of systems 7 and 8 are:

(20)

$\cdot$	a	b	c
a	b	c	b
b	a	a	a
c	a	a	a ,

(21)

$\cdot$	a	b	c
a	b	c	c
b	a	a	a
c	a	a	a .

Assume that $b \cdot a = b$. $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = b$ is the transpose of the systems of chart 3 where $a \cdot a = b$, $a \cdot b = b$, $b \cdot a = c$. $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = b$ will, thus, give anti-associative systems which are transposes of systems 15 through 19 by Theorem 3. The transposes of systems 15 through 19 are:

(22)

$\cdot$	a	b	c
a	b	c	a
b	b	c	a
c	b	c	a ,

(23)

$\cdot$	a	b	c
a	b	c	b
b	b	a	b
c	b	a	b ,

(24)

$\cdot$	a	b	c
a	b	c	b
b	b	a	b
c	b	c	b ,

(25)

$\cdot$	a	b	c
a	b	c	b
b	b	c	b
c	b	a	b ,

(26)

$\cdot$	a	b	c
a	b	c	b
b	b	c	b
c	b	c	b .

There are seven anti-associative systems where $a \cdot a = b$, $a \cdot b = c$.

Altogether there are 26 anti-associative systems where $a \cdot a = b$.

Case 3. Let $a \cdot a = c$. This case will be divided into three subcases where $a \cdot c = a$, $a \cdot c = b$, $a \cdot c = c$ respectively.

Case 3A. Let a•a = c, a•c = a, Case 2A can be applied here by the interchange of b and c. The new systems (27, 28, 29, 30, 31, 32, 33, 34) are isomorphic to the systems of case 2A. The one-to-one mapping is $a \rightleftarrows a$, $b \rightleftarrows c$, $c \rightleftarrows b$. Thus systems 27 through 34 are anti-associative by Theorem 6.

(27)

•	a	b	c
a	c	a	a
b	c	a	a
c	c	a	a

,

(28)

•	a	b	c
a	c	a	a
b	b	a	a
c	c	a	a

,

(29)

•	a	b	c
a	c	c	a
b	c	c	a
c	c	c	a

,

(30)

•	a	b	c
a	c	c	a
b	c	c	b
c	c	c	a

,

(31)

•	a	b	c
a	c	c	a
b	c	c	a
c	c	c	b

,

(32)

•	a	b	c
a	c	c	a
b	c	c	b
c	c	c	b

,

(33)

•	a	b	c
a	c	a	a
b	c	a	a
c	b	a	a

,

(34)

•	a	b	c
a	c	a	a
b	b	a	a
c	b	a	a

.

There are eight anti-associative systems where a•a = c, a•c = a.

Case 3B. Let a•a = c, a•c = b. Case 2C can be applied here by the interchange of b and c. The new systems (35, 36, 37, 38, 39, 40, 41) are isomorphic to the systems of case 2C. The one-to-one mapping is $a \rightleftarrows a$, $b \rightleftarrows c$, $c \rightleftarrows b$. Thus systems 35 through 41 are anti-associative by Theorem 6.

(35)

•	a	b	c
a	c	c	b
b	a	a	a
c	a	a	a

,

(36)

•	a	b	c
a	c	b	b
b	a	a	a
c	a	a	a

,

(37)

•	a	b	c
a	c	a	b
b	c	a	b
c	c	a	b

,

(38)

•	a	b	c
a	c	c	b
b	c	c	a
c	c	c	a

,

(39)

•	a	b	c
a	c	c	b
b	c	c	b
c	c	c	a

,

(40)

•	a	b	c
a	c	c	b
b	c	c	a
c	c	c	b

,

(41)

	a	b	c
a	c	c	b
b	c	c	b
c	c	c	b.

There are seven anti-associative systems where $a \cdot a = c$, $a \cdot c = b$.

Case 3C. Let $a \cdot a = c$, $a \cdot c = c$. Case 2B can be applied here by the interchange of b and c. The new systems (42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52) are isomorphic to the systems of case 2B. The one-to-one mapping is $a \rightleftarrows a$, $b \rightleftarrows c$, $c \rightleftarrows b$. Thus systems 42 through 52 are anti-associative by Theorem 6.

(42)

·	a	b	c
a	c	c	c
b	a	a	a
c	a	a	a,

(43)

·	a	b	c
a	c	b	c
b	a	a	a
c	a	a	a,

(44)

·	a	b	c
a	c	c	c
b	c	c	c
c	a	a	a,

(45)

·	a	b	c
a	c	c	c
b	c	c	c
c	a	b	a,

(46)

·	a	b	c
a	c	c	c
b	c	c	c
c	a	a	b,

(47)

·	a	b	c
a	c	c	c
b	c	c	c
c	a	b	b,

(48)

·	a	b	c
a	c	c	c
b	a	a	a
c	b	b	b,

(49)

·	a	b	c
a	c	c	c
b	c	c	c
c	b	a	a,

(50)

·	a	b	c
a	c	c	c
b	c	c	c
c	b	b	a,

(51)

·	a	b	c
a	c	c	c
b	c	c	c
c	b	a	b,

(52)

·	a	b	c
a	c	c	c
b	c	c	c
c	b	b	b.

There are 11 anti-associative systems where $a \cdot a = c$, $a \cdot c = c$.

There are 26 anti-associative systems where $a \cdot a = c$.

Altogether there are 52 anti-associative systems of three elements. These systems can be divided into two classes. The first class of systems (1, 3, 9, 11, 15, 19, 22, 26, 27, 29, 37, 41, 42, 44, 48, 52) is such that

$$x_i \cdot x_g = x_h \neq x_g \text{ (or } x_g \cdot x_i = x_h \neq x_g\text{)},$$

the form of Theorem 4. Each column (or row) contains only one element which differs from the operation element of the column (or row). The second class of systems (2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 16, 17, 18, 20, 21, 23, 24, 25, 28, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 43, 45, 46, 47, 49, 50, 51) has a column (or row) which contains two elements, neither being the same as the operation element of the column (or row). The other two columns (or rows) consist of the operation element of the aforementioned column (or row).

Any of these anti-associative systems can be extended any number of times by the principle of Theorem 5.

ANTI-ASSOCIATIVE SYSTEMS OF FOUR ELEMENTS

The system

$\cdot$	a	b	c	d
a	b	c	a	a
b	b	c	a	a
c	b	c	a	a
d	b	c	a	b

is anti-associative. This is verified by the following equations.

$x_i \cdot (x_j \cdot a) = x_i \cdot b = c.$

$(x_i \cdot x_j) \cdot a = b.$

$x_i \cdot (x_j \cdot b) = x_i \cdot c = a.$

$(x_i \cdot x_j) \cdot b = c.$

$x_i \cdot (x_j \cdot c) = x_i \cdot a = b.$

$(x_i \cdot x_j) \cdot c = a.$

$x_i \cdot (x_j \cdot d) = x_i \cdot a = b$, or $x_i \cdot (x_j \cdot d) = x_i \cdot b = c.$

$(x_i \cdot x_j) \cdot d = a \cdot d = a$, $(x_i \cdot x_j) \cdot d = b \cdot d = a$, or $(x_i \cdot x_j) \cdot d = c \cdot d = a.$

$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all $i, j, k \leq 4.$

The first three elements form an anti-associative system as can be seen by the previous equations. Thus, this is an example of an extension of an anti-associative system, but is not the same type of extension as that of Theorem 5. Thus, all anti-associative systems of four elements cannot be found just by extending the anti-associative systems of three elements.

Interesting anti-associative systems of four elements can be found. Using the previous methods of this paper, it becomes a long and tedious task to find all anti-associative systems of four elements. New methods are needed to continue this line of development.

CYCLIC ANTI-ASSOCIATIVE SYSTEMS

A special type of anti-associative system will be defined that will make it possible to work with equalities. A system, S, of n elements $(x_1, x_2, \ldots, x_n)$ such that $(x_i \cdot x_j) \cdot x_k = x_{h+m}$ whenever $x_i \cdot (x_j \cdot x_k) = x_h$ for all i, j, $k \leq n$ and $0 < m < n$ will be cyclic anti-associative. It is understood that $x_{i+n} = x_i$.

A two-element system that is anti-associative is obviously cyclic anti-associative with m = 1. See work on two-element anti-associative systems pages 5 and 6. A system which is not anti-associative cannot be cyclic anti-associative. There are two cyclic anti-associative systems of two elements.

Cyclic anti-associative systems of three elements can have m = 1 or m = 2. All of the cyclic anti-associative systems of three elements (a, b, c) with m = 1 can be found by taking the cases where $a \cdot a = a$, $a \cdot a = b$, $a \cdot a = c$ respectively.

Case 1. Let $a \cdot a = a$. An anti-associative system is not possible (Theorem 2). A system which is not anti-associative cannot be cyclic anti-associative. A cyclic anti-associative system is not possible when $a \cdot a = a$.

Case 2. Let $a \cdot a = b$. Subcases will be used where $a \cdot b = a$, $a \cdot b = b$, $a \cdot b = c$ respectively.

Case 2A. Let $a \cdot a = b$, $a \cdot b = a$.

$$a \cdot (a \cdot a) = a \cdot b = a.$$
$$(a \cdot a) \cdot a = b \cdot a.$$

By the definition of cyclic anti-associative system with m = 1, it is seen that $b \cdot a = b$.

$$a \cdot (a \cdot b) = a \cdot a = b.$$
$$(a \cdot a) \cdot b = b \cdot b.$$

Thus, $b \cdot b = c$.

Then

$$b \cdot (a \cdot a) = b \cdot b = c.$$
$$(b \cdot a) \cdot a = b \cdot a = b.$$

This shows that the system where $a \cdot a = b$, $a \cdot b = a$ cannot be cyclic anti-associative.

Case 2B. Let $a \cdot a = b$, $a \cdot b = b$. Chart 4 develops the only possible cyclic anti-associative system in which $a \cdot a = b$, $a \cdot b = b$. The operation table for this system is:

(1)

$\cdot$	a	b	c
a	b	b	b
b	c	c	c
c	a	a	a .

The following equations verify that this system is cyclic anti-associative.

$$a \cdot (x_i \cdot x_j) = b.$$
$$(a \cdot x_i) \cdot x_j = b \cdot x_i = c.$$
$$b \cdot (x_i \cdot x_j) = c.$$
$$(b \cdot x_i) \cdot x_j = c \cdot x_j = a.$$
$$c \cdot (x_i \cdot x_j) = a.$$
$$(c \cdot x_i) \cdot x_j = a \cdot x_j = b.$$

The system is cyclic anti-associative.

Case 2C. Let $a \cdot a = b$, $a \cdot b = c$.

$$a \cdot (a \cdot a) = a \cdot b = c.$$
$$(a \cdot a) \cdot a = b \cdot a.$$

$b \cdot a = a$.

$$b \cdot (a \cdot a) = b \cdot b.$$
$$(b \cdot a) \cdot a = a \cdot a = b.$$

$b \cdot b = a$.

Then

$$b \cdot (b \cdot b) = b \cdot a = a.$$
$$(b \cdot b) \cdot b = a \cdot b = c.$$

This shows that the system where $a \cdot a = b$, $a \cdot b = c$ cannot be cyclic anti-associative.

Case 3. Let $a \cdot a = c$. Subcases will be used where $a \cdot c = a$, $a \cdot c = b$, $a \cdot c = c$ respectively.

Case 3A. Let $a \cdot a = c$, $a \cdot c = a$.

$$a \cdot (a \cdot a) = a \cdot c = a.$$
$$(a \cdot a) \cdot a = c \cdot a.$$

$c \cdot a = b$.

$$a\ (a\ c) = a\ a = c.$$
$$(a \cdot a) \cdot c = c \cdot c.$$

Chart 4

The Cyclic Anti-associative System where $a \cdot a = b$, $a \cdot b = b$, $m = 1$.

If	Then	By Definition
$a \cdot a = b$ $a \cdot b = b$	$a \cdot (a \cdot a) = a \cdot b = b$ and $(a \cdot a) \cdot a = b \cdot a$	$b \cdot a = c$
	$a \cdot (a \cdot b) = a \cdot b = b$ and $(a \cdot a) \cdot b = b \cdot b$	$b \cdot b = c$
	$a \cdot (b \cdot a) = a \cdot c$ and $(a \cdot b) \cdot a = b \cdot a = c$	$a \cdot c = b$
	$a \cdot (a \cdot c) = a \cdot b = b$ and $(a \cdot a) \cdot c = b \cdot c$	$b \cdot c = c$
	$b \cdot (a \cdot a) = b \cdot b = c$ and $(b \cdot a) \cdot a = c \cdot a$	$c \cdot a = a$
	$b \cdot (a \cdot b) = b \cdot b = c$ and $(b \cdot a) \cdot b = c \cdot b$	$c \cdot b = a$
	$b \cdot (a \cdot c) = b \cdot b = c$ and $(b \cdot a) \cdot c = c \cdot c$	$c \cdot c = a$

$c \cdot c = a.$

Then

$$c \cdot (c \cdot c) = c \cdot a = b.$$
$$(c \cdot c) \cdot c = a \cdot c = a.$$

This shows that the system where $a \cdot a = c$, $a \cdot c = a$ cannot be cyclic anti-associative.

Case 3B. Let $a \cdot a = c$, $a \cdot c = b$. Chart 5 develops the only possible cyclic anti-associative system in which $a \cdot a = c$, $a \cdot c = b$. The operation table for this system is:

(2)

$\cdot$	a	b	c
a	c	a	b
b	c	a	b
c	c	a	b

The following equations verify that this system is cyclic anti-associative.

$$x_i \cdot (x_j \cdot a) = x_i \cdot c = b.$$
$$(x_i \cdot x_j) \cdot a = c.$$
$$x_i \cdot (x_j \cdot b) = x_i \cdot a = c.$$
$$(x_i \cdot x_j) \cdot b = a.$$
$$x_i \cdot (x_j \cdot c) = x_i \cdot b = a.$$
$$(x_i \cdot x_j) \cdot c = b.$$

The system is cyclic anti-associative.

Case 3C. Let $a \cdot a = c$, $a \cdot c = c$.

$$a \cdot (a \cdot a) = a \cdot c = c.$$
$$(a \cdot a) \cdot a = c \cdot a.$$

$c \cdot a = a.$

$$a \cdot (a \cdot c) = a \cdot c = c.$$
$$(a \cdot a) \cdot c = c \cdot c.$$

$c \cdot c = a.$

Then

$$c \cdot (a \cdot a) = c \cdot c = a.$$
$$(c \cdot a) \cdot a = a \cdot a = c.$$

This shows that the system where $a \cdot a = c$, $a \cdot c = c$ cannot be cyclic anti-associative.

There are only two cyclic anti-associative systems of three elements where $m = 1$.

Chart 5

The Cyclic Anti-associative System where $a \cdot a = c$, $a \cdot c = b$, $m = 1$.

If	Then	By Definition
$a \cdot a = c$ $a \cdot c = b$	$a \cdot (a \cdot a) = a \cdot c = b$ and $(a \cdot a) \cdot a = c \cdot a$	$c \cdot a = c$
	$c \cdot (a \cdot a) = c \cdot c$ and $(c \cdot a) \cdot a = c \cdot a = c$	$c \cdot c = b$
	$a \cdot (c \cdot a) = a \cdot c = b$ and $(a \cdot c) \cdot a = b \cdot a$	$b \cdot a = c$
	$a \cdot (a \cdot c) = a \cdot b$ and $(a \cdot a) \cdot c = c \cdot c = b$	$a \cdot b = a$
	$a \cdot (a \cdot b) = a \cdot a = c$ and $(a \cdot a) \cdot b = c \cdot b$	$c \cdot b = a$
	$a \cdot (c \cdot b) = a \cdot a = c$ and $(a \cdot c) \cdot b = b \cdot b$	$b \cdot b = a$
	$a \cdot (c \cdot c) = a \cdot b = a$ and $(a \cdot c) \cdot c = b \cdot c$	$b \cdot c = b$

Let $m = 2$. Three cases will again be considered where $a \cdot a = a$, $a \cdot a = b$, $a \cdot a = c$ respectively.

Case 1. Let $a \cdot a = a$. No cyclic anti-associative system exists (see case 1 where $m = 1$).

Case 2. Let $a \cdot a = b$. Subcases will be used where $a \cdot b = a$, $a \cdot b = b$, $a \cdot b = c$ respectively.

Case 2A. Let $a \cdot a = b$, $a \cdot b = a$.

$$a \cdot (a \cdot a) = a \cdot b = a.$$
$$(a \cdot a) \cdot a = b \cdot a.$$

$b \cdot a = c$.

$$a \cdot (a \cdot b) = a \cdot a = b.$$
$$(a \cdot a) \cdot b = b \cdot b.$$

$b \cdot b = a$.

Then

$$b \cdot (b \cdot b) = b \cdot a = c.$$
$$(b \cdot b) \cdot b = a \cdot b = a.$$

This shows that the system where $a \cdot a = b$, $a \cdot b = a$ cannot be cyclic anti-associative.

Case 2B. Let $a \cdot a = b$, $a \cdot b = b$.

$$a \cdot (a \cdot a) = a \cdot b = b.$$
$$(a \cdot a) \cdot a = b \cdot a.$$

$b \cdot a = a$.

$$a \cdot (a \cdot b) = a \cdot b = b.$$
$$(a \cdot a) \cdot b = b \cdot b.$$

$b \cdot b = a$.

Then

$$b \cdot (b \cdot b) = b \cdot a = a.$$
$$(b \cdot b) \cdot b = a \cdot b = b.$$

This shows that the system where $a \cdot a = b$, $a \cdot b = b$ cannot be cyclic anti-associative.

Case 2C. Let $a \cdot a = b$, $a \cdot b = c$. Chart 6 develops the only possible cyclic anti-associative system in which $a \cdot a = b$, $a \cdot b = c$. The operation table for this system is:

Chart 6

The Cyclic Anti-associative System where $a \cdot a = b$, $a \cdot b = c$, $m = 2$.

If	Then	By Definition
$a \cdot a = b$ $a \cdot b = c$	$a \cdot (a \cdot a) = a \cdot b = c$ and $(a \cdot a) \cdot a = b \cdot a$	$b \cdot a = b$
	$b \cdot (a \cdot a) = b \cdot b$ and $(b \cdot a) \cdot a = b \cdot a = b$	$b \cdot b = c$
	$a \cdot (a \cdot b) = a \cdot c$ and $(a \cdot a) \cdot b = b \cdot b = c$	$a \cdot c = a$
	$a \cdot (a \cdot c) = a \cdot a = b$ and $(a \cdot a) \cdot c = b \cdot c$	$b \cdot c = a$
	$a \cdot (b \cdot c) = a \cdot a = b$ and $(a \cdot b) \cdot c = c \cdot c$	$c \cdot c = a$
	$a \cdot (b \cdot a) = a \cdot b = c$ and $(a \cdot b) \cdot a = c \cdot a$	$c \cdot a = b$
	$a \cdot (b \cdot b) = a \cdot c = a$ and $(a \cdot b) \cdot b = c \cdot b$	$c \cdot b = c$

(3)

$\cdot$	a	b	c
a	b	c	a
b	b	c	a
c	b	c	a .

The following equations verify that this system is cyclic anti-associative.

$x_i\cdot(x_j\cdot a) = x_i\cdot b = c.$

$(x_i\cdot x_j)\cdot a = b.$

$x_i\cdot(x_j\cdot b) = x_i\cdot c = a.$

$(x_i\cdot x_j)\cdot b = c.$

$x_i\cdot(x_j\cdot c) = x_i\cdot a = b.$

$(x_i\cdot x_j)\cdot c = a.$

The system is cyclic anti-associative.

Case 3. Let $a\cdot a = c$. Subcases will be used where $a\cdot c = a$, $a\cdot c = b$, $a\cdot c = c$ respectively.

Case 3A. Let $a\cdot a = c$, $a\cdot c = a$.

$a\cdot(a\cdot a) = a\cdot c = a.$

$(a\cdot a)\cdot a = c\cdot a.$

$c\cdot a = a.$

$a\cdot(a\cdot c) = a\cdot a = c.$

$(a\cdot a)\cdot c = c\cdot c.$

$c\cdot c = b.$

Then

$c\cdot(a\cdot a) = c\cdot c = b.$

$(c\cdot a)\cdot a = c\cdot a = c.$

This shows that the system where $a\cdot a = c$, $a\cdot c = a$ cannot be cyclic anti-associative.

Case 3B. Let $a\cdot a = c$, $a\cdot c = b$.

$a\cdot(a\cdot a) = a\cdot c = b.$

$(a\cdot a)\cdot a = c\cdot a.$

$c\cdot a = a.$

$c\cdot(a\cdot a) = c\cdot c.$

$(c\cdot a)\cdot a = a\cdot a = c.$

$c\cdot c = a.$

$c\cdot(c\cdot c) = c\cdot a = a.$

$(c\cdot c)\cdot c = a\cdot c = b.$

This shows that the system where $a \cdot a = c$, $a \cdot c = b$ cannot be cyclic anti-associative.

Case 3C. Let $a \cdot a = c$, $a \cdot c = c$. Chart 7 develops the only possible cyclic anti-associative system for which $a \cdot a = c$, $a \cdot c = c$. The operation table for this system is:

(4)

$\cdot$	a	b	c
a	c	c	c
b	a	a	a
c	b	b	b .

The following equations verify that this system is cyclic anti-associative.

$$a \cdot (x_i \cdot x_j) = c.$$
$$(a \cdot x_i) \cdot x_j = c \cdot x_j = b.$$
$$b \cdot (x_i \cdot x_j) = a.$$
$$(b \cdot x_i) \cdot x_j = a \cdot x_j = c.$$
$$c \cdot (x_i \cdot x_j) = b.$$
$$(c \cdot x_i) \cdot x_j = b \cdot x_j = a.$$

The system is cyclic anti-associative.

There are only two cyclic anti-associative systems of three elements where $m = 2$.

All the products of system 1 can be represented by $x_i \cdot x_j = x_{i+1}$. All of the products of system 2 can be represented by $x_i \cdot x_j = x_{j-1}$. All of the products of system 3 can be represented by $x_i \cdot x_j = x_{j-2}$. All of the products of system 4 can be represented by $x_i \cdot x_j = x_{i+2}$. This shows that cyclic anti-associative systems of three elements have a definite pattern. Either $x_i \cdot x_j = x_{j-g}$ or $x_i \cdot x_j = x_{i+g}$ where $0 < g < 3$. Are there cyclic anti-associative systems of n elements that follow this pattern?

<u>Theorem 7</u>. A system, S, of n elements $(x_1, x_2, \dots, x_n)$ is cyclic anti-associative if $x_i \cdot x_j = x_{i+g}$ (or $x_i \cdot x_j = x_{j-g}$) where $i, j \leq n$, g is a constant, and $0 < g < n$.

Proof: $x_i \cdot (x_j \cdot x_k) = x_i \cdot x_{j+g} = x_{i+g}$.
$(x_i \cdot x_j) \cdot x_k = x_{i+g} \cdot x_k = x_{i+2g}$.
Thus, S is cyclic anti-associative.

However, this is not the only form of cyclic anti-associative systems. The system

Chart 7

The Cyclic Anti-associative System where $a \cdot a = c$, $a \cdot c = c$, $m = 2$.

If	Then	By Definition
$a \cdot a = c$ $a \cdot c = c$	$a \cdot (a \cdot a) = a \cdot c = c$ and $(a \cdot a) \cdot a = c \cdot a$	$c \cdot a = b$
	$a \cdot (a \cdot c) = a \cdot c = c$ and $(a \cdot a) \cdot c = c \cdot c$	$c \cdot c = b$
	$a \cdot (c \cdot a) = a \cdot b$ and $(a \cdot c) \cdot a = c \cdot a = b$	$a \cdot b = c$
	$a \cdot (a \cdot b) = a \cdot c = c$ and $(a \cdot a) \cdot b = c \cdot b$	$c \cdot b = b$
	$c \cdot (a \cdot a) = c \cdot c = b$ and $(c \cdot a) \cdot a = b \cdot a$	$b \cdot a = a$
	$c \cdot (a \cdot b) = c \cdot c = b$ and $(c \cdot a) \cdot b = b \cdot b$	$b \cdot b = a$
	$c \cdot (a \cdot c) = c \cdot c = b$ and $(c \cdot a) \cdot c = b \cdot c$	$b \cdot c = a$

$\cdot$	a	b	c	d
a	b	d	b	b
b	b	d	b	b
c	b	d	b	b
d	b	d	b	b

is cyclic anti-associative with $m = 2$. This is verified by the following equations.

$$x_i \cdot (x_j \cdot a) = x_i \cdot b = d.$$
$$(x_i \cdot x_j) \cdot a = b.$$
$$x_i \cdot (x_j \cdot b) = x_i \cdot d = b.$$
$$(x_i \cdot x_j) \cdot b = d.$$
$$x_i \cdot (x_j \cdot c) = x_i \cdot b = d.$$
$$(x_i \cdot x_j) \cdot c = b.$$
$$x_i \cdot (x_j \cdot d) = x_i \cdot b = d.$$
$$(x_i \cdot x_j) \cdot d = b.$$

Not only is this system cyclic anti-associative, but it is not of the form of Theorem 7.

SEMI-ASSOCIATIVE SYSTEMS

A permutation is a one-to-one mapping of a finite set onto itself.

[1]

If $x_i \rightleftarrows x_{i+g}$ where g is a constant; $0 \le g < n$; $x_{i+n} = x_i$, then the permutation is cyclic. A cyclic permutation which maps each element onto itself, $g = 0$, is called an identity permutation. The cyclic permutations where $0 < g < n$ were used in defining cyclic anti-associative systems.

All permutations which are not cyclic are called noncyclic permutations.

Let $x_i^{(P)}$ be the element corresponding to x_i with $i \le n$ in a permutation, P of n elements. If

$$(x_j \cdot x_k) \cdot x_h = x_i^{(P)} \text{ whenever}$$
$$x_j \cdot (x_k \cdot x_h) = x_i \text{ for all } i,\ j,\ k,\ h \le n,$$

then the system is semi-associative.

The identity permutation, $g = 0$, will cause $x_j \cdot (x_k \cdot x_h) = (x_j \cdot x_k) \cdot x_h$. All such systems will be associative. The cyclic permutations with $0 < g < n$ will cause $(x_j \cdot x_k) \cdot x_h = x_{i+g}$ whenever $x_j \cdot (x_k \cdot x_h) = x_i$. All such systems will be cyclic anti-associative.

The noncyclic permutations result in systems that are neither associative nor anti-associative and, in special cases, associative and anti-associative systems. If the element $(x_j \cdot x_k) \cdot x_h$ maps onto itself in the noncyclic permutation for all h, j, k, then $x_j \cdot (x_k \cdot x_h) = x_j \cdot (x_k \cdot x_h)$. The result is an associative system. If the element $x_j \cdot (x_k \cdot x_h)$ does not map onto itself in the noncyclic permutation for all h, j, k, then $x_j \cdot (x_k \cdot x_h) \ne x_j \cdot (x_k \cdot x_h)$. The result is an anti-associative system.

The system

·	a	b	c
a	b	b	b
b	b	b	b
c	b	b	a

is semi-associative by the noncyclic permutation ($a \rightleftarrows c$, $b \rightleftarrows b$, $c \rightleftarrows a$), and is associative. The following equations verify the previous statements about the system.

$x_i \cdot (x_j \cdot a) = x_i \cdot b = b.$
$(x_i \cdot x_j)\ a = b.$
$x_i \cdot (x_j \cdot b) = x_i \cdot b = b.$
$(x_i \cdot x_j) \cdot b = b.$
$x_i \cdot (x_j \cdot c) = x_i \cdot a = b$, or $x_i \cdot (x_j \cdot c) = x_i \cdot b = b.$
$(x_i \cdot x_j) \cdot c = a \cdot c = b$, or $(x_i \cdot x_j) \cdot c = b \cdot c = b.$
$x_i \cdot (x_j \cdot x_k) = (x_i \cdot x_j) \cdot x_k$ for all $i, j, k \leq 3.$

The system

$\cdot$	a	b	c
a	c	c	a
b	c	c	a
c	c	c	a

is semi-associative by the noncyclic permutation ($a \rightleftarrows c$, $b \rightleftarrows b$, $c \rightleftarrows a$), and is anti-associative. The following equations verify the previous statements about the system.

$x_i \cdot (x_j \cdot a) = x_i \cdot c = a.$
$(x_i \cdot x_j) \cdot a = c.$
$x_i \cdot (x_j \cdot b) = x_i \cdot c = a.$
$(x_i \cdot x_j) \cdot b = c.$
$x_i \cdot (x_j \cdot c) = x_i \cdot a = c.$
$(x_i \cdot x_j) \cdot c = a.$
$x_i \cdot (x_j \cdot x_k) \neq (x_i \cdot x_j) \cdot x_k$ for all $i, j, k \leq 3.$

Semi-associative systems are general systems that include associative, anti-associative, and neither associative nor anti-associative systems.

CONCLUSIONS

Semi-associative systems are directly linked with the permutations of the elements. The permutations on n elements form a group with respect to permutation multiplication. [5] Permutation multiplication is one permutation followed by another.

The study of groups [5] is basic to all modern algebra. This connection between groups and semi-associative systems should provide ample opportunity for further research.

SEMI-ASSOCIATIVE SYSTEMS

A permutation is a one-to-one mapping of a finite set onto itself.

[1]

If $x_i \rightleftarrows x_{i+g}$ where g is a constant; $0 \le g < n$; $x_{i+n} = x_i$, then the permutation is cyclic. A cyclic permutation which maps each element onto itself, $g = 0$, is called an identity permutation. The cyclic permutations where $0 < g < n$ were used in defining cyclic anti-associative systems.

All permutations which are not cyclic are called noncyclic permutations.

Let $x_i^{(P)}$ be the element corresponding to x_i with $i \le n$ in a permutation, P of n elements. If

$$(x_j \cdot x_k) \cdot x_h = x_i^{(P)} \text{ whenever}$$
$$x_j \cdot (x_k \cdot x_h) = x_i \text{ for all } i,\ j,\ k,\ h \le n,$$

then the system is semi-associative.

The identity permutation, $g = 0$, will cause $x_j \cdot (x_k \cdot x_h) = (x_j \cdot x_k) \cdot x_h$. All such systems will be associative. The cyclic permutations with $0 < g < n$ will cause $(x_j \cdot x_k) \cdot x_h = x_{i+g}$ whenever $x_j \cdot (x_k \cdot x_h) = x_i$. All such systems will be cyclic anti-associative.

The noncyclic permutations result in systems that are neither associative nor anti-associative and, in special cases, associative and anti-associative systems. If the element $(x_j \cdot x_k) \cdot x_h$ maps onto itself in the noncyclic permutation for all h, j, k, then $x_j \cdot (x_k \cdot x_h) = x_j \cdot (x_k \cdot x_h)$. The result is an associative system. If the element $x_j \cdot (x_k \cdot x_h)$ does not map onto itself in the noncyclic permutation for all h, j, k, then $x_j \cdot (x_k \cdot x_h) \ne x_j \cdot (x_k \cdot x_h)$. The result is an anti-associative system.

The system

·	a	b	c
a	b	b	b
b	b	b	b
c	b	b	a

is semi-associative by the noncyclic permutation ($a \rightleftarrows c$, $b \rightleftarrows b$, $c \rightleftarrows a$), and is associative. The following equations verify the previous statements about the system.