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ERROR STRUCTURE OF RANDOMIZED DESIGN UNDER BACKGROUND CORRELATION WITH A MISSING VALUE

by

Tseng-Chi Chang

A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Applied Statistics

Approved:

UTAH STATE UNIVERSITY
Logan, Utah

1965

ACKNOWLEDGMENTS

I wish to express my deep gratitude to Dr. Neeti R. Bohidar, my major professor, for suggesting this problem, offering me guidance throughout the research and preparation of the manuscript. Without him this thesis would not be possible.

Tseng-Chi Chang

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CHAPTER I

INTRODUCTION

The analysis of variance technique is probably the most popular statistical technique used for testing hypotheses and estimating parameters. Eisenhart (12) presents two classes of problems solvable by the analysis of variance and the assumption underlying each class. Cochran (9) lists the assumptions and also discusses the consequences when these assumptions are not met. It is evident that if all the assumptions are not satisfied, the confidence placed in any result obtained in this manner is adversly affected to varying degrees according to the extent of the violation.

One of the assumptions in the analysis of variance procedures is that of uncorrelated errors. The experimenter may not always meet this conditions because of economical or environmental reasons. In fact, Wilk (35) questions the validity of the assumption of uncorrelated errors in any physical situation. For example, consider an experiment over a sequence of years. A correlation due to years may exist, no matter what randomization technique is used, because the outcome of the previous year determines to a great extent the outcome of this year. Another example would be the case of selecting experimental units from the same source, such as, sampling students with the same background or selecting units from the same production process. This points out the fact that the condition such as background, or a defect in the production process may have forced a correlation among the experimental units. Problems of this nature frequently occur in Industrial, Biological and

Psychological experiments.

Another phenomenon which affects the analysis of variance is the missing data. From time to time certain observations are missing, through death of animals, destruction of crops, or failure to record. In the analysis of variance, two changes may be noted due to the missing observations. For example, in the Randomized Block Design the treatments and blocks sum of squares become entangled, so that the treatments sum of squares must be computed after allowing for block effects. Secondly, if 'a' observations are absent, the total number of degrees of freedom is reduced by 'a'. Unless one or more complete treatments or blocks is missing, the number of parameters required to describe these effects will be the same as before. Consequently, the missing degrees of freedom all come from the error sum of squares. To the experimenter it may be very difficult to analyze a set of incomplete data. For this reason Yates (40), following a suggestion by Fisher, considered inserting values for the missing observations so as to obtain a set of complete data. Suppose that only a single observation is missing, and a value X is needed to be substituted for this observation. In order to find a numerical value for X, Yates proposed to use the value that minimized the error sum of squares. If this value is inserted in place of the missing observation, and if the data are analyzed as if no observations were absent, Yates showed that several important properties hold: (a) The estimates of treatment and block effects are exactly the same as those obtained by the standard least squares procedure. (b) The error sum of squares is exactly the same as given by the standard least squares procedure. (c) To obtain the correct partition of the degrees of freedom, we subtract one from the total sum of squares and one from the error sum of squares. Yates also showed that the method of insertion fails to agree with the standard least squares procedure in two respects. The treatment sum of squares as obtained in the analysis of variance of the complete data is always slightly larger than the corrected treatment sum of squares for a F-test of the treatments. Unless an appreciable fraction of the total observation is missing, this overestimation is unlikely to be larger; further, the exact F-test can be obtained by means of some additional calculation. The second defect of the method of insertion is that it may not give proper t-tests. That this will happen is clear because in the analysis of variance of complete data 'r' replications are ascribed to the treatment that contains the missing observation, whereas there are only (r-1) replications.

When both correlation and missing data occur together in an experiment, the analysis of variance incurs more disturbance. The object of this thesis is to consider experiments under different conditions (with either correlation or a missing value, or both) and derive explicit expressions of the error structure under each condition. The correlation considered here is after randomization. Thus background correlation as used in this study is defined as any correlation not removed by randomization. By taking observations at random they have equal probability of selection and are independent in the probability sense. Moreover, in this thesis only the case of one missing value is considered. Two or more missing data require more complicated methods. Yates suggested an iterative scheme to be used for estimation, followed by special methods for calculating unbiased estimates of the sums of squares for treatments which would not be covered here.

Four different cases are compared below. They are: (a) without correlation and without missing value, (b) without correlation but with a missing value, (c) with correlation but without missing value, and

(d) with correlation and with a missing value. Each is considered in three analysis of variance models: random, fixed, and mixed. The presentation of the study of "The Derivation of Error Structure of Randomized Design under Background Correlation with a Missing Value", in this paper will consist of a review of literature in Chapter II and general theory in Chapter III. The subsequent chapters give the special results to the Randomized Block Design, Latin Square Design, and Graeco-Latin Square Design in Chapters IV, V and VI, respectively. The Chapter VII is a summary.

CHAPTER II

REVIEW OF LITERATURE

There is an abundance of articles in the statistical literature dealing with the estimation of variance of components, derivation of expected mean squares, missing data and many types of correlation. This thesis combines all of these studies. The structure of expected mean squares includes covariance as well as variance components. Background correlation is more general than most other correlations and includes many of them.

Allan and Wishart (1) were the first to present a formula for computing the value for one missing or extremely divergent value for a randomized block experiment and Latin Square Design. Yates showed that their formula resulted in minimizing the error sum of squares. He presented an iterative procedure for calculating the values for several missing experimental units. Bartlett (5) suggests the procedure of inserting a one for the missing value and zeros otherwise and performing a covariance analysis with the zeros and the one as the independent variate. If more than one experimental unit is missing, the same procedure is followed except that a multiple covariance is performed. Nair (25) used Bartlett's method for analyzing the results from a k X k latin square design with several missing values. A paper by Delury (11) summarizes most of the results for handling missing experimental units in latin squares or sets of latin squares. Yates and Hale (39) give a method of analysis for two or more missing rows, columns, or treatments in a latin square. For the latin square with two or more missing data, Kempthorne (22) outlines a method for making an unbiased test.

The first explicit mention of the subject of estimation of components of variance seems to have been made in 1935 by R. A. Fisher (17). In a discussion of the intraclass correlation coefficient, which for data of the kind described by equations of the form $Y_{ij} = \mu + a_i + b_{ij}$, is defined as $\rho = \sigma_a^2/(\sigma_a^2 + \sigma_b^2)$. Fisher showed that the among group mean squares from the analysis of variance of such data has an expected value equal to $\sigma_b^2 + n\sigma_a^2$. It being well known that the expected value of the within groups mean square is σ_b^2.

Essentially similar results were proved by Irwin (21) in 1931 from data from Randomized Block and Latin Square experiments. Tippet (31) has given a similar discussion which is more detailed in its explanation of how estimations may be obtained from the analysis of variance.

A brief, but clear, statement of the fundamental assumptions and concepts involved in the estimation and interpretation of variance components has been given in an appendix to a paper by Winsor and Clarke (37). It is unfortunate that this paper was not widely available since a clear understanding of its contents might have avoided some of the unprecise thinking which has been evidenced on the subject of variance components.

The distributions of estimates of components of variance formed from linear combinations of analysis of variance mean squares has been given in several forms. The usual analysis of variance mean squares are distributed as multiple X^2, and the distribution of such estimates are given in several ways. B. C. Bhattecharyya (8) has expressed this distribution in terms of Bessel Functions. Fisher (17) suggested an argument of the Behrens-Fisher type, which to data has not been utilized.

Although in this study, we are not particularly interested in the procedures of testing and estimation, they are a critical background to the development of the techniques of what can be done with the expected values and error structures in the analysis of variance.

Another aspect deserving scrutiny is just what portions of the analysis of randomized experiments depend on the strong distributional assumptions usually used. M. B. Wilk (35) makes a statement to the effect that the assumptions of independence and normality for random errors are always falsified in practice, and questions the corresponding results.

The various independence assumptions which are made by many writers often seem to hold no relationship to the physical situation. Criticism is made of this by Crump (10). As part of this same problem, there exist what appear to be contradictory view points on the analysis of fairly simple experiments among different writers, each of whom base their assumptions on an assumed linear model. For example, the expected mean square and the recommended "error term" for a two-factor, mixed model situation given by Mood (24), Hald (19), Mentzer (23) and Scheffe' (27) differ from that given by Kempthorne (22), Anderson and Bancroft (3) and Tukey (34). Wilk and Kempthorne (36) give a derivation of the expected mean squares with assumptions other than those of the analysis of variance and leave it up to the reader to decide which assumptions best fit his needs as to which results are used.

A method for the analysis of variance of multiple classification with unequal numbers other than by fitting of constants is given by Patterson (26). A method of adjusting is worked out and claimed to be similar to the fitting of constants, at least mathematically.

In 1954 Anscombe and Tukey (4) reviewed and proposed various methods of examining and testing data for non-additivity and also for non-consistency of variance and non-normality, including some graphical procedures. Tukey (33) gives as part of a paper of considerable scope an extensive discussion on choice of error terms. In another paper Tukey (32) describes the statistical test procedure based on the isolation of one degree of freedom in the analysis of variance, sensitive to the non-additivity of classifications.

Tate (30) explains the theory of two correlated variables where both are continuous and also where one is fixed. The error involved in biserial correlation is discussed by Soper (28).

In the simple case of correlation between two experimental units, the corresponding structures have been taken into account in the expected mean squares for special cases. These have been for isolates studies in psychology, genetics and plant science. Fisher, Finney and Robertson along with many others have worked on these isolated cases, but a unified approach under several different models has not been attempted.

Most recently Burnet (8) in his thesis "Error Structure under Background Correlation" here at Utah State University under the direction of Dr. Bohidar closely examined the change in error structure under background correlation of two types of multiple classification, namely, nested classification and cross classification, each divided into two main groups, the orthogonal and non-orthogonal.

CHAPTER III

GENERAL THEORY

In this chapter a number of fundamental lemmas and theorems will be developed. They are frames of this thesis in the sense that each is a general case and will be frequently referred to in chapters following.

First, it is necessary to make a statement about the model of analysis of variance because each model has different meaning and each component recognized in the model has different distributional properties. Roughly there are three kinds of models: random, fixed, and mixed. In each model any observation can be divided into four parts (assume without interaction):

1. an overall mean which is denoted as μ,
2. a treatment deviation,
3. all other restriction deviations, and
4. a random element which is denoted as e.

In algebraical, this becomes,

$$Y_{ijk\ldots} = \mu + t_i + a_j + b_k + c_l + \text{-----} + e_{ijk} \text{ ----}$$

where a, b, c ---- are restriction deviations.

In the random model, all components except the overall mean which is always fixed, are normally distributed with a zero mean and their own standard deviation.

In the fixed model, all components, except the random element, are fixed. They are no longer normally distributed, instead have a property that the sum of all deviation of each of the component in a model equals to zero, i.e.,

$$\Sigma_i t_i = \Sigma_j a_j = \Sigma_k b_k = \Sigma_l c_l = ----- = 0$$

Mixed model combines the above two cases so that it actually is a special case of either the random model or the fixed model. Throughout this thesis only the type of mixed model in which treatments are randomly selected but all restrictions are fixed is considered. However, the same concept could be extended to all possible mixed models which we are not able to cover here.

Next is to define "restriction." Its synonymous meaning is control. The essence of needing restrictions is that the treatments could be grouped into replications in different ways with the consequence that the effects of all restrictions are equalized for all treatments. A practical example is the Randomized Block Design. It has only one restriction because the experimenter exerts control upon blocks. Within each block all units are closely comparable or very similar. It is important to point here that treatments are always <u>free</u> from control. It is the purpose of the experiment to detect the differences among them. Likewise, the Latin Square Design has two restrictions because each treatment happens once in a row and once in a column. In Graeco-Latin Square Design each treatment appears not only once in each row and column, but also once with each Greek letter.

Notice that from Randomized Block Design to Latin Square Design and then to Graeco-Latin Square Design though number of restrictions have been increased, the total number of observed values remain unchanged. Graphically speaking, they are still on a two-dimension plane, even though restrictions imposed on have been increased.

As the number of restrictions change, the sources of variation in the analysis of variance change correspondingly. This could be easily seen from the construction of the model in which each restriction is contained as a component. Because it is frequently necessary to obtain the number of sources of variation associated with a certain number of restrictions, Fundamental Lemma I has been developed below and its proof is trival since adding one more restriction results in one more source of variation.

Fundamental Lemma I

The number of sources of variation in analysis of variance is equal to the number of restrictions plus two.

Let N = number of sources of variation

r = number of restrictions

$N = r + 2$

When dealing with correlated units, it is also necessary to know the relationship between the number of kinds of covariance and the number of restrictions imposed on an experiment. For example, in an one-restriction random model,

$Y_{ij} = \mu + a_i + b_j + c_{ij}$, where,

$$\left\{\begin{array}{l} \left.\begin{array}{l} a_i \text{ are distributed as } N\ (o, \sigma a) \\ b_j \text{ are distributed as } N\ (o, \sigma b) \end{array}\right\} \text{ either a or b is a restriction} \\ c_{ij} \text{ are distributed as } N\ (o, \sigma c) \end{array}\right.$$

The kinds of covariance introduced are:

1. $C_a = \underset{i \neq i'}{E} (a_i\, a_{i'})$

2. $C_b = \underset{j \neq j'}{E} (b_j\, b_{j'})$

3. $C_{ab'} = \underset{j \neq j'}{E} (c_{ij}\, c_{ij'})$

4. $C_{a'b} = \underset{i \neq i'}{E} (c_{ij}\, c_{i'j})$

and

$$5.\quad C_{a'b'} = \underset{j\neq j'}{\overset{E}{i\neq i'}} (c_{ij}\ c_{i'j'})$$

Graphically then, the Y_{ij}'s variable may be arranged as follows:

Assume $i = 1, 2, 3$

$j = 1, 2, 3, 4$, then

	b_1	b_2	b_3	b_4
a_1	Y_{11}	Y_{12}	Y_{13}	Y_{14}
a_2	Y_{21}	Y_{22}	Y_{23}	Y_{24}
a_3	Y_{31}	Y_{32}	Y_{33}	Y_{34}

where a's and b's are levels of A and B recognized in the model. When referring to C_a, it means the pair-wise grouping such as $[a_1\ a_2]$, $[a_1\ a_3]$, etc. A similar interpretation is easily extended to C_b which is covariance between two different b's levels such as $[b_1\ b_2]$, $[b_2\ b_3]$, etc. $C_{ab'}$ is defined as the covariance between two random elements, C_{ij}, $C_{jj'}$, both belonging to the same level of A but different levels of B. $C_{a'b}$ is defined as the covariance between two random elements C_{ij}, $C_{i'j}$, both belonging to the same level of B but different levels of A. Likewise $C_{a'b'}$ is defined as the covariance between two random elements, C_{ij}, $C_{i'j'}$, each belonging to different levels of A, as well as different levels of B.

As number of restrictions are increased, more kinds of covariance are added. In two-restriction case of a random model,

$$Y_{ijk} = \mu + a_i + b_j + c_k + l_{ijk} \quad \left\{ \begin{array}{ll} a_i & N\ (o, \sigma a) \\ b_j & N\ (o, \sigma b) \\ c_k & N\ (o, \sigma c) \\ l_{ijk} & N\ (o, \sigma 1) \end{array} \right.$$

there are:

C_a

C_b

C_c

$$C_{ab'c'} = \underset{\substack{j \neq j' \\ k \neq k'}}{E} \left[l_{ijk} \quad l_{ij'k'} \right] \left\{ \begin{array}{l} \text{Covariance between 2 random} \\ \text{elements of same level of A} \\ \text{but different levels of B, C.} \end{array} \right.$$

$$C_{a'bc'} = \underset{\substack{i \neq i' \\ k \neq k'}}{E} \left[l_{ijk} \; l_{i'jk'} \right]$$

$$C_{a'b'c} = \underset{\substack{i \neq i' \\ j \neq j'}}{E} \left[l_{ijk} \quad l_{i'j'k} \right]$$

$$C_{a'b'c'} = \underset{\substack{i \neq i' \\ j \neq j' \\ k \neq k'}}{E} \left[l_{ijk} \quad l_{i'j'k'} \right]$$

Graphically, all Y_{ijk}'s variable may be arranged as follows:

Assume $i = 1$ ------ 5

$j = 1$ ------ 5

$k = 1$ ------ 5

Y_{111}	Y_{123}	Y_{135}	Y_{142}	Y_{154}
Y_{212}	Y_{224}	Y_{231}	Y_{243}	Y_{255}
Y_{313}	Y_{325}	Y_{332}	Y_{344}	Y_{351}
Y_{414}	Y_{421}	Y_{433}	Y_{445}	Y_{452}
Y_{515}	Y_{522}	Y_{534}	Y_{541}	Y_{553}

Fundamental Lemma II

In a k X k experiment, if there are r restrictions the total kinds of covariance is twice the number of restrictions plus three. The proof is trivial.

The rest of this chapter is devoted to two subjects, the derivation of a missing data formula and computation of expected sum of squares (symbolically denoted as ESS) of each source of variation in analysis of variance associated with three different cases:

(i) with a missing value but without correlation

(ii) without missing value but with correlation

(iii) with a missing value, and with correlation.

Each case will be considered in three different ways, one restriction, two restrictions, and three restrictions (all in a k X k experiment). For each different restriction, a fundamental lemma is developed. And then, three theorems will be generalized to r restrictions for each of three different cases.

Moreover, the case to be examined below is restricted to a k X k experiment. Therefore, the Randomized Block Design is nothing but a special case of one restriction on a k X k experiment and will be discussed in detail in the next chapter.

Following is a theorem of a missing data formula generalized to r restrictions on a k X k experiment.

Theorem I

In a k X k experiment with r restrictions, the missing value is estimated by the following formula. Let the model be

$$Y_{abc ---- s} = \mu + t_a + r_{1b} + r_{2c} + ---- + r_{rs} + l_{abc, ---- s},$$

then

$$X = \frac{k\left(\sum_{i} R_{ii}' + T_t'\right) - rG'}{(k-1)\quad(k-r)} \qquad i = 1, 2 \text{ ---- } r$$

where the r restrictions to be R_1, R_2 ---- R_r.

Also assume that the missing value occurs in 1st level of R_1, 2nd level of R_2 ---- r^{th} level of R_r, and t^{th} treatment. Let

T_t' = the total for the (k-1) observations in t^{th} treatment

R_{11}' = the total for the (k-1) observations in 1st level of restriction 1

R_{22}' = the total for the (k-1) observations in 2nd level of restriction 2

.
.
.

R_{rr}' = the total for the (k-1) observations in r^{th} level of restriction r

G' = the total of the observed units not including the missing value.

Proof:

Table 3. Algebraic analysis of variance

Due to	Sum of Squares
Restriction 1	$\frac{1}{k}(R_{11}' + X)^2 + \frac{1}{k}\sum_{b\neq 1}^{k} R_{1b}^2 - \frac{(G'+X)^2}{k^2}$
Restriction 2	$\frac{1}{k}(R_{22}' + X)^2 + \frac{1}{k}\sum_{c\neq 2}^{k} R_{2c}^2 - \frac{(G'+X)^2}{k^2}$
⋮	⋮
Restriction r	$\frac{1}{k}(R_{rr}' + X)^2 + \frac{1}{k}\sum_{s\neq r}^{k} R_{rs}^2 - \frac{(G'+X)^2}{k^2}$
Treatment	$\frac{1}{k}(T_t' + X)^2 + \frac{1}{k}\sum_{a\neq t}^{k} T_a - \frac{(G'+X)^2}{k^2}$
Error	by subtraction
Total	$\sum_{(a\neq t} \text{ ---- } \sum_{s\neq r)} Y^2_{abc \text{ --- } s} + X^2 - \frac{(G'+X)^2}{k^2}$

The procedure is to substitute X for the missing value and perform the analysis of variance.

Lets denote Q as the error sum of squares,

$$Q = X^2 + \frac{r(G'+X)^2}{k^2} - \left[\frac{(T'_t+X)^2}{k}\right] - \frac{\sum_{i=1}^{r}(R'_{ii}+X)^2}{k} + \text{terms not involving } X.$$

This sum of squares is now to be minimized for variation in X, and this is simply done by equating the differental with regard to X to zero and solving for X.

$$\frac{dQ}{dX} = 2X + \frac{2r(G'+X)}{k^2} - \frac{2}{k}\left[(T_t' + X) + \sum_{i=1}^{r}(R'_{ii} + X)\right] = 0$$

$$\text{or,} \quad Xk^2 + rX + rG' - (r+1)\,Xk - k\left[T_t' + \sum_{i=1}^{r} R_{ii}'\right] = 0$$

$$\text{or,} \quad k\left[T'_t + \sum_{i=1}^{r} R_{ii}'\right] - rG' = X\left[k^2 - (r+1)\,k + r\right]$$

$$\text{or,} \quad X = \frac{k\left\{\sum_{i=1}^{r} R'_{ii} + T'_t\right\} - rG'}{(k-1)\ (k-r)}$$

The above completes the proof of the theorem.

After knowing the missing data formula, one can proceed to examine the error structure associated with different situations.

Fundamental Lemma III A

The k X k experiment of random model with one restriction and one missing value and without correlation has three components recognized in the model. The model is,

$$Y_{ij} = \mu + a_i + b_j + e_{ij}$$

$$\begin{cases} i = 1, 2, ---- k \\ j = 1, 2, ---- k \end{cases}$$

$$\left\{\begin{array}{l} \left.\begin{array}{l} a_i \sim N(o, \sigma_a) \\ b_j \sim N(o, \sigma_b) \end{array}\right\} \text{ either one is a restriction} \\ e_{ij} \sim N(o, \sigma_e) \end{array}\right.$$

The ESS for each is,

$$ESS(A) = k(k-1)\,\sigma_a^2 + \left[(k-1) + \frac{1}{(k-1)}\right]\sigma_e^2$$

$$ESS(B) = k(k-1)\,\sigma_b^2 + \left[(k-1) + \frac{1}{(k-1)}\right]\sigma_e^2$$

$$ESS\ (ERROR) = \left[(k-1)\ (k-1)-1\right]\sigma_e^2$$

Proof:

In order not to lose generality, let us assume that the missing value occurs in h^{th} level of A, and l^{th} level of B.

For other levels of A where missing value does not occur,

$$Y_{i.} = \sum_j Y_{ij} = \sum_j (\mu + a_i + b_j + e_{ij}) = k\mu + k a_i + \sum_j b_j + \sum_j e_{ij}$$

$$Y_{i.}^2 = k^2\mu^2 + k^2 a_i^2 + \sum_j b_j^2 + \sum_j e_{ij}^2 + \text{cross product terms}$$

$$\sum_{i\neq h} Y_{i.}^2 = (k-1)k^2\mu^2 + k^2\sum_{i\neq h} a_i^2 + (k-1)\sum_j b_j^2 + \sum_{i\neq h}\sum_j e_{ij}^2 + \text{cross product terms}$$

$$\frac{\sum_{i\neq h} Y_{i.}^2}{k} = k(k-1)\,u^2 + k\sum_{i\neq h} a_i^2 + \frac{(k-1)}{k}\sum_j b_j^2 + \sum_{i\neq h}\sum_j e_{ij}^2 + \text{cross product terms}$$

$$E\left[\frac{\sum_{i\neq h} Y_{i.}^2}{k}\right] = k(k-1)\,\mu^2 + k(k-1)\,\sigma_a^2 + (k-1)\,\sigma_b^2 + (k-1)\,\sigma_e^2$$

For the h^{th} level of A, let X_{hl} denote the missing value which is derived by Theorem I.

Now we proceed to derive the ESS for $Y_{h.}$

$$Y_{h.} = \sum_{j\neq l} Y_{hj} + X_{hl}$$

$$= \sum_{j\neq l} (\mu + a_h + b_j + e_{hj}) + \frac{kY_{h.}' + kY_{.l}' - Y_{..}'}{(k-1)\ (k-1)}$$

$$= (k-1)\,\mu + (k-1)\,a_h + \sum_{j\neq l} b_j + \sum_{j\neq l} e_{hj}$$

$$+ \frac{k\left[(k-1)\,\mu + (k-1)\,a_h + \sum_{j\neq l} b_j + \sum_{j\neq l} e_{hj}\right] + k\left[(k-1)\mu + \sum_{i\neq h} a_i + (k-1)b_l + \sum_{i\neq h} e_{il}\right]}{(k-1)\ (k-1)}$$

$$= \frac{\left[(k^2-1)\mu + k\sum_{i\neq h} a_i + (k-1)a_h + k\sum_{j\neq l} b_j + (k-1)b_l + \sum_{i\neq h}\sum_{j\neq l} e_{ij}\right]}{(k-1)\,(k-1)}$$

$$= (k-1)\mu + (k-1)a_h + \sum_{j\neq l} b_j + \sum_{j\neq l} e_{hj}$$

$$+ \mu + a_h + b_l + \frac{(k-1)\sum_{j\neq l} e_{hj} + (k-1)\sum_{i\neq h} e_{il} - \sum_{i\neq h}\sum_{j\neq l} e_{ij}}{(k-1)\,(k-1)}$$

$$= k\mu + ka_h + \sum_j b_j + \frac{k(k-1)\sum_{j=l} e_{hj} + (k-1)\sum_{i\neq h} e_{il} - \sum_{i\neq h}\sum_{j\neq l} e_{ij}}{(k-1)\,(k-1)}$$

$$Y_{h\circ}^2 = k^2\mu^2 + k^2 a_h^2 + \sum_j b_j^2 + \frac{k^2(k-1)^2\sum_{j\neq l} e_{hj}^2 + (k-1)^2\sum_{i\neq h} e_{il}^2 + \sum_{i\neq h}\sum_{j\neq l} e_{ij}^2}{(k-1)^4} +$$

cross product terms

$$E\left[\frac{Y_{h\circ}^2}{k}\right] = k\mu^2 + k\sigma_a^2 + \sigma_b^2 + \frac{k^2(k-1)^3+(k-1)^3+(k-1)^2}{k(k-1)^4}\sigma_e^2$$

$$E\left[\frac{\sum_{i\neq h} Y_{i\circ}^2 + Y_{h\circ}^2}{k}\right] = k^2\mu^2 + k^2\sigma_a^2 + k\sigma_b^2 + \frac{(k-1)^3 + k(k-1) + 1}{(k-1)^2}\sigma_e^2$$

$$Y_{\circ\circ} = \sum_{i\neq h}\sum_j Y_{ij} + Y_{h\circ}$$

$$= k(k-1)\mu + k\sum_{i\neq h} a_i + (k-1)\sum_j b_j + \sum_{i\neq h}\sum_j e_{ij} + k\mu + ka_h + \sum_j b_j$$

$$+ \frac{k(k-1)\sum_{j\neq l} e_{hj} + (k-1)\sum_{i\neq h} e_{il} - \sum_{i\neq h}\sum_{j\neq l} e_{ij}}{(k-1)\,(k-1)}$$

$$= k^2\mu + k\sum_i a_i + k\sum_j b_j + \frac{k(k-1)\left[\sum_{j\neq l} e_{hj} + \sum_{i\neq h} e_{il}\right] + k(k-2)\sum_{i\neq h}\sum_{j\neq l} e_{ij}}{(k-1)\,(k-1)}$$

$$Y_{\circ\circ}^2 = k^4\mu^2 + k^2\sum_i a_i^2 + k^2\sum_j b_j^2 + \frac{k^2(k-1)^2\left[\sum_{j\neq l} e_{hj}^2 + \sum_{i\neq h} e_{il}^2\right] + k^2(k-2)^2\sum_{i\neq h}\sum_{j\neq l} e_{ij}^2}{(k-1)^4} +$$

cross product terms

$$\frac{Y_{\cdot\cdot}^2}{k^2} = k^2\mu^2 + \sum_i a_i^2 + \sum_j b_j^2 + \frac{(k-1)^2\left[\sum_{j\neq l} e_{jh}^2 + \sum_{i\neq h} e_{il}^2\right] + (k-2)^2 \sum_{i\neq h}\sum_{j\neq l} e_{ij}^2}{(k-1)^4} +$$

cross product terms

$$E\left[\frac{Y_{\cdot\cdot}^2}{k^2}\right] = k^2\mu^2 + k\sigma_a^2 + k\sigma_b^2 + \frac{2(k-1)^3 + (k-1)^2(k-2)^2}{(k-1)^4}\sigma_e^2$$

$$= k^2\mu^2 + k\sigma_a^2 + k\sigma_b^2 + \left[1 + \frac{1}{(k-1)2}\right]\sigma_e^2$$

$$E\left[\frac{\sum_{i\neq h} Y_{i\cdot}^2 + Y_{h\cdot}^2}{k} - \frac{Y_{\cdot\cdot}^2}{k^2}\right] = k(k-1)\sigma_a^2 + \left[(k-1) + \frac{1}{(k-1)}\right]\sigma_e^2$$

Since it is symmetrical,

$$E\left[\frac{\sum_{j\neq l} Y_{\cdot j}^2 + Y_{\cdot l}^2}{k} - \frac{Y_{\cdot\cdot}^2}{k^2}\right] = k(k-1)\sigma_b^2 + \left[(k-1) + \frac{1}{(k-1)}\right]\sigma_e^2$$

for the error sum of squares,

$$Y_{ij} = \mu + a_i + b_j + e_{ij}$$
$$Y_{ij}^2 = \mu^2 + a_i^2 + b_j^2 + e_{ij}^2 + \text{cross product terms}$$

$$E\left[\sum_{\substack{i\neq h\\ j\neq l}}\sum Y_{ij}^2 + X_{hl}^2\right] = (k^2-1)\mu^2 + (k^2-1)\sigma_a^2 + (k^2-1)\sigma_b^2 + (k^2-1)\sigma_e^2 + \mu^2 + \sigma_a^2 + \sigma_b^2$$

$$+ \frac{(k-1)^3 + (k-1)^3 + (k-1)^2}{(k-1)^4}\sigma_e^2$$

$$= k^2\mu^2 + k^2\sigma_a^2 + k^2\sigma_b^2 + \left[(k^2-1) + \frac{2k-1}{(k-1)^2}\right]\sigma_e^2$$

$$\text{ESS(ERROR)} = E\left[\sum_{\substack{i\neq h\\ j\neq l}}\sum Y_{ij}^2 + X_{hl}^2 - \left(\frac{\sum_{i\neq h} Y_{i\cdot}^2 + Y_{h\cdot}^2}{k}\right) - \left(\frac{\sum_{j\neq l} Y_{\cdot j}^2 + Y_{\cdot l}^2}{k}\right) + \frac{Y_{\cdot\cdot}^2}{k^2}\right] =$$

$$\left[(k-1)(k-1) - 1\right]\sigma_e^2$$

Fundamental Lemma III B

The k X k experiment of random model with two restrictions and one missing value and without correlation has four components recognized in the model.

The model is,

$$Y_{ijh} = \mu + a_i + b_j + c_h + e_{ijh} \qquad \begin{cases} i = 1, 2, ---- k \\ j = 1, 2, ---- k \\ h = 1, 2, ---- k \end{cases}$$

$$\left.\begin{cases} a_i \sim N(o, \sigma_a) \\ b_j \sim N(o, \sigma_b) \\ c_h \sim N(o, \sigma_c) \\ e_{ijh} \sim N(o, \sigma_e) \end{cases}\right\} \text{two of them are restrictions}$$

The ESS for each is,

$$ESS(A) = k(k-1)\,\sigma_a^2 + \left[(k-1) + \frac{1}{(k-2)}\right]\sigma_e^2$$

$$ESS(B) = k(k-1)\,\sigma_b^2 + \left[(k-1) + \frac{1}{(k-2)}\right]\sigma_e^2$$

$$ESS(C) = k(k-1)\,\sigma_c^2 + \left[(k-1) + \frac{1}{(k-2)}\right]\sigma_e^2$$

$$ESS(ERROR) = \left[(k-1)\ (k-2)-1\right]\sigma_e^2$$

Proof:

Also assume that the missing value occurs in 1th level of A, mth level of B, and nth level of C.

For other levels of A where missing value does not occur,

$$Y_{i..} = k\,\mu + ka_i + \sum_j b_j + \sum_h c_h + \sum_{(jh)} e_{ijh}$$

$$Y_{i..}^2 = k^2\mu^2 + k^2 a_i^2 + \sum_j b_j^2 + \sum_h c_h^2 + \sum_{(jh)} e_{ijh}^2 + \text{cross product terms}$$

$$\sum_{i\neq 1} Y_{i..}^2 = k^2(k-1)\mu^2 + k^2 \sum_{i\neq 1} a_i^2 + (k-1)\sum_j b_j^2 + (k-1)\sum_h c_h^2 + \sum_{i\neq 1}\sum_{(jh)} e_{ijh}^2 +$$

cross product terms

$$\frac{\sum\limits_{i\neq 1} Y_{i..}^2}{k} = k(k-1)\mu^2 + k\sum_{i\neq 1} a_i^2 + \frac{(k-1)}{k}\sum_j b_j^2 + \frac{(k-1)}{k}\sum_h c_h^2 + \frac{1}{k}\sum_{i\neq 1}\sum_{(jh)} e_{ijh}^2 +$$

cross product terms

$$E\left[\frac{\sum_{i\neq l} Y_{i..}^{\,2}}{k}\right] = k(k-1)\,\mu^2+k(k-1)\,\sigma_a^{\,2}+(k-1)\,\sigma_b^{\,2}+(k-1)\,\sigma_c^{\,2}+(k-1)\,\sigma_e^{\,2}$$

for the l^{th} level of A,

$$Y_{l..} = \sum_{\left(\substack{j\neq m\\ h\neq n}\right)} Y_{ljh}+X_{lmn} = \sum_{\left(\substack{j\neq m\\ h\neq n}\right)} (\mu+a_l+b_j+c_h+e_{ljh})+\frac{k(Y_{l..}'+Y_{.m.}'+Y_{..n}')-2\,Y_{...}'}{(k-1)\quad(k-2)}$$

(note X_{lmn} is directly derived from Theorem I)

$$= (k-1)\,\mu+(k-1)a_l+\sum_{j\neq m} b_j+\sum_{h\neq n} c_h+\sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh}$$

$$+\frac{k\left[(k-1)\mu+(k-1)a_l+\sum_{j\neq m} b_j+\sum_{h\neq n} c_n+\sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh}\right]}{(k-1)\quad(k-2)}$$

$$+\frac{k\left[(k-1)\mu+\sum_{i\neq l} a_i+(k-1)b_m+\sum_{h\neq n} c_h+\sum_{\left(\substack{i\neq l\\ h\neq n}\right)} e_{imh}\right]}{(k-1)\quad(k-2)}$$

$$+\frac{k\left[(k-1)\mu+\sum_{j\neq m} b_j+(k-1)c_n+\sum_{\left(\substack{i\neq l\\ j\neq m}\right)} e_{ijn}\right]}{(k-1)\quad(k-2)}$$

$$-\frac{2\left[(k^2-1)\mu+k\sum_{i\neq l} a_i+(k-1)a_l+k\sum_{j\neq m} b_j+(k-1)b_m+k\sum_{h\neq n} c_h+(k-1)c_n+\left(\sum_{i\neq l}\sum_{j\neq m}\sum_{h\neq n}\right)e_{ij}\right]}{(k-1)\quad(k-2)}$$

$$= (k-1)\mu+(k-1)a_l+\sum_{j\neq m} b_j+\sum_{h\neq n} c_h+\sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh}+\mu+a_l+b_m+c_n$$

$$+\frac{(k-2)\sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh}+(k-2)\sum_{\left(\substack{i\neq l\\ h\neq n}\right)} e_{imh}+(k-2)\sum_{\left(\substack{i\neq l\\ j\neq m}\right)} e_{ijn}-2\sum_{i\neq l}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}}{(k-1)\quad(k-2)}$$

$$=k\,\mu+ka_l+\sum_j b_j+\sum_h c_h+\frac{k(k-2)\sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh}+(k-2)\sum_{\left(\substack{i\neq l\\ h\neq n}\right)} e_{imh}+(k-2)\sum_{\left(\substack{i\neq l\\ j\neq m}\right)} e_{ijn}-2\sum_{i\neq l}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}}{(k-1)\quad(k-2)}$$

$$Y_{l..}^{\,2}=k^2\mu^2+k^2a_l^{\,2}+\sum_j b_j^{\,2}+\sum_h c_h^{\,2}+\frac{k^2(k-2)^2\sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh}^{\,2}+(k-2)^2\sum_{\left(\substack{i\neq l\\ h\neq n}\right)} e_{imh}^{\,2}+(k-2)^2}{(k-1)^2\quad(k-2)^2}$$

$$\frac{\sum_{\left(\substack{i\neq l\\ j\neq m}\right)} e_{ijn}^{\,2}+4\sum_{i\neq l}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}}{(k-1)^2\quad(k-2)^2}+\text{cross product terms}$$

$$\frac{Y_{1..}^{\;2}}{k} = k\,\mu^2 + ka_1^{\;2} + \frac{1}{k}\sum_j b_j^{\;2} + \frac{1}{k}\sum_h c_h^{\;2} + \frac{k^2(k-2)\left(\sum\limits_{\substack{j\neq m\\ h\neq n}}\right)e_{1jh}^{\;2} + (k-2)^2\left(\sum\limits_{\substack{i\neq 1\\ h\neq n}}\right)e_{imh}^{\;2}}{k(k-1)^2\;(k-2)^2}$$

$$+\frac{(k-2)^2\left(\sum\limits_{\substack{i\neq 1\\ j\neq m}}\right)e_{ijn} + 4\sum\limits_{i\neq 1}\sum\limits_{j\neq m}\sum\limits_{h\neq n} e_{ijh}}{k(k-1)^2\;(k-2)^2} + \text{cross product terms}$$

$$E\left[\frac{Y_{1..}^{\;2}}{k}\right] = k\,\mu^2 + k\sigma_a^{\;2} + \sigma_b^{\;2} + \sigma_c^{\;2} + \left[\frac{k^2(k-2)^2(k-1) + (k-2)^2(k-1) + (k-2)^2(k-1)}{k(k-1)^2\;(k-2)^2}\right.$$

$$\left.\frac{+4(k-1)(k-2)}{k(k-1)^2(k-2)^2}\right]\sigma_e^{\;2}$$

$$= k\,\mu^2 + k\sigma_a^{\;2} + \sigma_b^{\;2} + \sigma_c^{\;2} + \frac{k^2-2k+2}{(k-1)(k-2)}\,\sigma_e^{\;2}$$

$$E\left[\frac{\sum\limits_{i\neq 1} Y_{i..}^{\;2} + Y_{1..}^{\;2}}{k}\right] = k^2\mu^2 + k^2\sigma_a^{\;2} + k\sigma_b^{\;2} + k\sigma_c^{\;2} + \left[(k-1) + \frac{k^2-2k+2}{(k-1)\;(k-2)}\right]\sigma_e^{\;2}$$

$$Y_{...} = \sum_{i\neq 1} Y_{i..} + Y_{1..} = \sum_{i\neq 1}\Big(k\,\mu + ka_i + \sum_j b_j + \sum_h c_h + \sum_{(jh)} e_{ijh}\Big)$$

$$+k\,\mu + ka_1 + \sum_j b_j + \sum_h c_h + \frac{k(k-2)\left(\sum\limits_{\substack{j\neq m\\ h\neq n}}\right)e_{1jh} + (k-2)\left(\sum\limits_{\substack{i\neq 1\\ h\neq n}}\right)e_{imh} + (k-2)\left(\sum\limits_{\substack{i\neq 1\\ j\neq m}}\right)e_{ijn}}{(k-1)\;(k-2)}$$

$$-\frac{2\sum\limits_{i\neq 1}\sum\limits_{j\neq m}\sum\limits_{h\neq n} e_{ijh}}{(k-1)\;(k-2)}$$

$$= k^2\mu + k\sum_i a_i + k\sum_j b_j + k\sum_h c_h + \frac{k(k-2)\left[\left(\sum\limits_{\substack{j\neq m\\ h\neq n}}\right)e_{1jh} + \left(\sum\limits_{\substack{i\neq 1\\ h\neq n}}\right)e_{imh} + \left(\sum\limits_{\substack{i\neq 1\\ j\neq m}}\right)e_{ijn}\right]}{(k-1)\;(k-2)}$$

$$+\frac{k(k-3)\sum\limits_{i\neq 1}\sum\limits_{j\neq m}\sum\limits_{h\neq n} e_{ijh}}{(k-1)\;(k-2)}$$

$$Y_{...}^{\;2} = k^4\mu^2 + k^2\sum_i a_i^{\;2} + k^2\sum_j b_j^{\;2} + k^2\sum_h c_h^{\;2} + \frac{k^2(k-2)^2\left[\left(\sum\limits_{\substack{j\neq m\\ h\neq n}}\right)e_{1jh}^{\;2} + \left(\sum\limits_{\substack{i\neq 1\\ h\neq n}}\right)e_{imh}^{\;2} + \left(\sum\limits_{\substack{i\neq 1\\ j\neq m}}\right)e_{ijn}^{\;2}\right]}{(k-1)^2\;(k-2)^2}$$

$$+\frac{k^2(k-3)^2\sum\limits_{i\neq 1}\sum\limits_{j\neq m}\sum\limits_{h\neq n} e_{ijh}}{(k-1)^2\;(k-2)^2} + \text{cross product terms}$$

$$E\left[\frac{Y_{\cdots}^{2}}{k^{2}}\right]=k^{2}\mu^{2}+k\sigma_{a}^{2}+k\sigma_{b}^{2}+k\sigma_{c}^{2}+\frac{3(k-1)(k-2)^{2}+(k-1)(k-2)(k-3)^{2}}{(k-1)^{2}\ \ (k-2)^{2}}\sigma_{e}^{2}$$

$$=k^{2}\mu^{2}+k\sigma_{a}^{2}+k\sigma_{b}^{2}+k\sigma_{c}^{2}+\frac{k^{2}-3k+3}{(k-1)\ (k-2)}\sigma_{e}^{2}$$

$$ESS(A)=E\left[\frac{\sum_{i\neq l}Y_{i\cdot\cdot}^{2}+Y_{l\cdot\cdot}^{2}}{k}-\frac{Y_{\cdots}^{2}}{k^{2}}\right]=k(k-1)\sigma_{a}^{2}+\left[(k-1)+\frac{1}{(k-2)}\right]\sigma_{e}^{2}$$

Since it is symmetrical,

$$ESS(B)=E\left[\frac{\sum_{j\neq m}Y_{\cdot j\cdot}^{2}+Y_{\cdot m\cdot}^{2}}{k}-\frac{Y_{\cdots}^{2}}{k^{2}}\right]=k(k-1)\sigma_{b}^{2}+\left[(k-1)+\frac{1}{(k-2)}\right]\sigma_{e}^{2}$$

$$ESS(C)=E\left[\frac{\sum_{h\neq n}Y_{\cdot\cdot h}^{2}+Y_{\cdot\cdot n}^{2}}{k}-\frac{Y_{\cdots}^{2}}{k^{2}}\right]=k(k-1)\sigma_{c}^{2}+\left[(k-1)+\frac{1}{(k-2)}\right]\sigma_{e}^{2}$$

For error sum of squares,

$$Y_{ijh}=\mu+a_{i}+b_{j}+c_{h}+e_{ijh}$$

$$Y_{ijh}^{2}=\mu^{2}+a_{i}^{2}+b_{j}^{2}+c_{h}^{2}+e_{ijh}^{2}+\text{ cross product terms}$$

$$E\left[\left(\sum_{i\neq l}\ \sum_{j\neq m}\ \sum_{h\neq n}\right)Y_{ijh}^{2}\right]=(k^{2}-1)\ \mu^{2}+(k^{2}-1)\sigma_{a}^{2}+(k^{2}-1)\sigma_{b}^{2}+(k^{2}-1)\sigma_{c}^{2}+(k^{2}-1)\sigma_{e}^{2}$$

$$E\left[\left(\sum_{i\neq l}\ \sum_{j\neq m}\ \sum_{h\neq n}\right)Y_{ijh}+X_{lmn}^{2}\right]=k^{2}\mu^{2}+k^{2}\sigma_{a}^{2}+k^{2}\sigma_{b}^{2}+k^{2}\sigma_{c}^{2}+\left[(k^{2}-1)+\frac{3k-2}{(k-1)(k-2)}\right]\sigma_{e}^{2}$$

$$ESS(ERROR)=E\left[\left(\sum_{i\neq l}\ \sum_{j\neq m}\ \sum_{h\neq n}\right)Y_{ijh}^{2}+X_{lmn}^{2}-\left(\frac{\sum_{i\neq l}Y_{i\cdot\cdot}^{2}+Y_{l\cdot\cdot}^{2}}{k}\right)-\left(\frac{\sum_{j\neq m}Y_{\cdot j\cdot}^{2}+Y_{\cdot m\cdot}^{2}}{k}\right)-\left(\frac{\sum_{h\neq n}Y_{\cdot\cdot h}^{2}+Y_{\cdot\cdot n}^{2}}{k}\right)+\frac{2Y_{\cdot\cdot}^{2}}{k^{2}}\right]$$

$$=\left[(k-1)\ (k-2)-1\right]\sigma_{e}^{2}$$

The proof is complete.

Fundamental Lemma III C

The k X k experiment with three restrictions and one missing value and without correlation has five components recognized in the model. The model is,

$$Y_{ijhl} = \mu + a_i + b_j + c_h + d_l + e_{ijhl} \qquad \begin{cases} i = 1, 2, ---- k \\ j = 1, 2, ---- k \\ h = 1, 2, ---- k \\ l = 1, 2, ---- k \end{cases}$$

$$\left\{ \begin{array}{l} \left. \begin{array}{l} a_i \sim N(o, \sigma_a) \\ b_j \sim N(o, \sigma_b) \\ c_h \sim N(o, \sigma_c) \end{array} \right\} \text{three of them are restrictions} \\ d_l \sim N(o, \sigma_d) \end{array} \right.$$

The ESS for each is,

$$ESS(A) = k(k-1)\,\sigma_a^2 + \left[(k-1) + \frac{1}{(k-3)}\right]\sigma_e^2$$

$$ESS(B) = k(k-1)\,\sigma_b^2 + \left[(k-1) + \frac{1}{(k-3)}\right]\sigma_e^2$$

$$ESS(C) = k(k-1)\,\sigma_c^2 + \left[(k-1) + \frac{1}{(k-3)}\right]\sigma_e^2$$

$$ESS(D) = k(k-1)\,\sigma_d^2 + \left[(k-1) + \frac{1}{(k-3)}\right]\sigma_e^2$$

$$ESS(ERROR) = \left[(k-1)\ (k-3)-1\right]\sigma_e^2$$

Proof:

Also assume the missing value occurs in m^{th} level of A, n^{th} level of B, o^{th} level of C, and p^{th} level of D.

For other levels of A where missing value does not occur,

$$Y_{i\cdot\cdot\cdot} = k\,\mu + k a_i + \sum_j b_j + \sum_h c_h + \sum_l d_l + \sum_{(jhl)} e_{ijhl}$$

$$Y_{i\cdot\cdot\cdot}^2 = k^2\mu^2 + k^2 a_i^2 + \sum_j b_j^2 + \sum_h c_h^2 + \sum_l d_l^2 + \sum_{(jhl)} e_{ijhl}^2 + \text{cross product terms}$$

$$\sum_{i\neq m} Y_{i\ldots}^{\ 2} = (k-1)k^2\mu^2 + k^2\sum_{i\neq m} a_i^{\ 2} + (k-1)\sum_j b_j^{\ 2} + (k-1)\sum_h c_h^{\ 2} + (k-1)\sum_l d_l^{\ 2} + \sum_{i\neq m}\ \sum_{(jhl)} e_{ijhl}^{\ 2} +$$

cross product terms

$$E\left[\frac{\sum_{i\neq m} Y_{i\ldots}^{\ 2}}{k}\right] = k(k-1)\mu^2 + k(k-1)\sigma_a^{\ 2} + (k-1)\sigma_b^{\ 2} + (k-1)\sigma_c^{\ 2} + (k-1)\sigma_d^{\ 2} +$$

$$(k-1)\sigma_e^{\ 2}$$

for the m^{th} level of A,

$$Y_{m\ldots} = \sum_{(j\neq n\ h\neq o\ l\neq p)} Y_{mjhl} + X_{mnop} = (k-1)\mu + (k-1)a_m + \sum_{j\neq n} b_j + \sum_{h\neq o} c_h + \sum_{l\neq p} d_l +$$

$$\sum_{(j\neq n\ h\neq o\ l\neq p)} e_{mjhl} + \frac{k(Y'_{m\ldots} + Y'_{.n..} + Y'_{..o.} + Y'_{\ldots p}) - 3Y'_{\ldots .}}{(k-1)\quad (k-3)}$$

(Note: X_{mnop} is derived directly from Theorem I)

$$= (k-1)\mu + (k-1)a_m + \sum_{j\neq n} b_j + \sum_{h\neq o} c_h + \sum_{l\neq p} d_l + \sum_{\substack{j\neq n\\ h\neq o\\ l\neq p}} e_{mjhl}$$

$$+ \frac{k\left[(k-1)\mu + (k-1)a_m + \sum_{j\neq n} b_j + \sum_{h\neq o} c_h + \sum_{l\neq p} d_l + \sum_{\substack{j\neq n\\ h\neq o\\ l\neq p}} e_{mjhl}\right]}{(k-1)\quad (k-3)}$$

$$+ \frac{k\left[(k-1)\mu + \sum_{i\neq m} a_i + (k-1)b_n + \sum_{h\neq o} c_h + \sum_{l\neq p} d_l + \sum_{\substack{i\neq m\\ h\neq o\\ l\neq p}} e_{inhl}\right]}{(k-1)\quad (k-3)}$$

$$+ \frac{k\left[(k-1)\mu + \sum_{i\neq m} a_i + \sum_{j\neq o} b_j + (k-1)c_o + \sum_{l\neq p} d_l + \sum_{\substack{i\neq m\\ j\neq n\\ l\neq p}} e_{ijol}\right]}{(k-1)\quad (k-3)}$$

$$+ \frac{k\left[(k-1)\mu + \sum_{i\neq m} a_i + \sum_{j\neq n} b_j + \sum_{h\neq o} c_h + (k-1)d_p + \sum_{\substack{i\neq m\\ j\neq n\\ h\neq o}} e_{ijhp}\right] - 3\left[(k^2-1)\mu + k\sum_{i\neq m} a_i + (k-1)a_m\right.}{(k-1)\quad (k-3)}$$

$$+\frac{k\sum_{j\neq n} b_j+(k-1)b_n+k\sum_{h\neq o} c_h+(k-1)c_o+k\sum_{l\neq p} d_l+(k-1)d_p+\sum_{(i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p)} e_{ijhl}}{(k-1)\quad(k-3)}\Bigg]$$

$$=(k-1)\mu+(k-1)a_m+\sum_{j\neq n} b_j+\sum_{h\neq o} c_h+\sum_{l\neq p} d_l+\sum_{\left(\substack{j\neq n\\ h\neq o\\ l\neq p}\right)} e_{mjhl}$$

$$+\mu+a_m+b_n+c_o+d_p+\frac{(k-3)\left[\sum_{\left(\substack{j\neq n\\ h\neq o\\ l\neq p}\right)} e_{mjhl}+\sum_{\left(\substack{i\neq m\\ h\neq o\\ l\neq p}\right)} e_{inhl}+\sum_{\left(\substack{i\neq m\\ j\neq o\\ l\neq p}\right)} e_{ijol}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ h\neq o}\right)} e_{ijlp}\right]-3\sum_{(i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p)} e_{ijh}}{(k-1)\quad(k-3)}$$

$$=k\,\mu+ka_m+\sum_j b_j+\sum_h c_h+\sum_l d_l$$

$$+\frac{k(k-3)\sum_{\left(\substack{j\neq n\\ h\neq o\\ l\neq p}\right)} e_{mjhl}+(k-3)\left[\sum_{\left(\substack{i\neq m\\ h\neq o\\ l\neq p}\right)} e_{inhl}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ l\neq p}\right)} e_{ijol}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ h\neq o}\right)} e_{ijhp}\right]-3\sum_{(i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p)} e_{ijhl}}{(k-1)\quad(k-3)}$$

$$Y_{m\cdots}^{\,2}=k^2\mu^2+k^2a_m^{\,2}+\sum_j b_j^{\,2}+\sum_h c_h^{\,2}+\sum_l d_l^{\,2}$$

$$+\frac{k^2(k-3)^2\sum_{\left(\substack{j\neq n\\ h\neq o\\ l\neq p}\right)} e_{mjhl}^{\,2}+(k-3)^2\left[\sum_{\left(\substack{i\neq m\\ h\neq o\\ l\neq p}\right)} e_{inhl}^{\,2}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ l\neq p}\right)} e_{ijol}^{\,2}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ h\neq o}\right)} e_{ijhp}^{\,2}\right]+9\sum_{(i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p)} e_{ijhl}^{\,2}+}{(k-1)^2\quad(k-3)^2}$$

cross product terms

$$E\left[\frac{Y_{m\cdots}^{\,2}}{k}\right]=k\,\mu^2+k\sigma_a^{\,2}+\sigma_b^{\,2}+\sigma_c^{\,2}+\sigma_d^{\,2}+\frac{k^2(k-3)^2(k-1)+3(k-1)(k-3)^2}{k(k-1)^2\quad(k-3)^2}$$

$$+\frac{9\,(k-1)\,(k-3)}{k(k-1)^2(k-3)^2}\sigma_e^{\,2}$$

$$=k\,\mu^2+k\sigma_a^{\,2}+\sigma_b^{\,2}+\sigma_c^{\,2}+\sigma_d^{\,2}+\frac{k(k-3)+3}{(k-1)\quad(k-3)}\sigma_e^{\,2}$$

$$E\left[\frac{\sum_{i\neq m}Y_{i...}^{2}+Y_{m...}^{2}}{k}\right]=k^{2}\mu^{2}+k^{2}\sigma_{a}^{2}+k\sigma_{b}^{2}+k\sigma_{c}^{2}+k\sigma_{d}^{2}+\left[(k-1)+\frac{k(k-3)+3}{(k-1)(k-3)}\right]\sigma_{e}^{2}$$

$$Y_{....}=\sum_{i\neq m}Y_{i...}+Y_{m...}=\sum_{i\neq m}(k\mu+ka_{i}+\sum_{j}b_{j}+\sum_{h}c_{h}+\sum_{l}d_{l}+\sum_{(jhl)}e_{ijhl})$$

$$+k\mu+ka_{m}+\sum_{j}b_{j}+\sum_{h}c_{h}+\sum_{l}d_{l}$$

$$+\frac{k(k-3)\sum_{\left(\substack{j\neq n\\ h\neq o\\ l\neq p}\right)}e_{mjhl}+(k-3)\left[\sum_{\left(\substack{i\neq m\\ h\neq o\\ l\neq p}\right)}e_{inhl}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ l\neq p}\right)}e_{ijol}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ h\neq o}\right)}e_{ijhp}\right]-3\sum_{i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p}e_{ijhl}}{(k-1)\quad(k-3)}$$

$$=k^{2}\mu+k\sum_{i}a_{i}+k\sum_{j}b_{j}+k\sum_{h}c_{h}+k\sum_{l}d_{l}$$

$$+\frac{k(k-3)\left[\sum_{\left(\substack{j\neq n\\ h\neq o\\ l\neq p}\right)}e_{mjhl}+\sum_{\left(\substack{i\neq m\\ h\neq o\\ l\neq p}\right)}e_{inhl}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ l\neq p}\right)}e_{ijol}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ h\neq o}\right)}e_{ijhp}\right]+k(k-4)\sum_{i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p}e_{ijhl}}{(k-1)\quad(k-3)}$$

$$Y_{....}^{2}=k^{4}\mu^{2}+k^{2}\sum_{i}a_{i}^{2}+k^{2}\sum_{j}b_{j}^{2}+k^{2}\sum_{h}c_{h}^{2}+k^{2}\sum_{l}d_{l}^{2}$$

$$+\frac{k^{2}(k-3)^{2}\left[\sum_{\left(\substack{j\neq n\\ h\neq o\\ l\neq p}\right)}e_{mjhl}^{2}+\sum_{\left(\substack{i\neq m\\ h\neq o\\ l\neq p}\right)}e_{inhl}^{2}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ l\neq p}\right)}e_{ijol}^{2}+\sum_{\left(\substack{i\neq m\\ j\neq n\\ h\neq o}\right)}e_{ijhp}^{2}\right]+k^{2}(k-4)^{2}\sum_{i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p}e_{ijhl}^{2}}{(k-1)^{2}\quad(k-3)^{2}}+$$

cross product terms

$$E\left[\frac{Y_{....}^{2}}{k^{2}}\right]=k^{2}\mu^{2}+k\sigma_{a}^{2}+k\sigma_{b}^{2}+k\sigma_{c}^{2}+k\sigma_{d}^{2}+\frac{(k-2)^{2}}{(k-1)\quad(k-3)}\sigma_{e}^{2}$$

$$ESS(A)=E\left(\frac{\sum_{i\neq m}Y_{i...}^{2}+Y_{m...}^{2}}{k}-\frac{Y_{....}^{2}}{k^{2}}\right)=k(k-1)\sigma_{a}^{2}+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_{e}^{2}$$

Since it is symmetrical,

$$ESS(B)=E\left(\frac{\sum_{j\neq n}Y_{.j..}^{2}+Y_{.n..}^{2}}{k}-\frac{Y_{....}^{2}}{k^{2}}\right)=k(k-1)\sigma_{b}^{2}+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_{e}^{2}$$

$$\mathrm{ESS(C)}=E\left(\frac{\sum_{h\neq o} Y_{..h.}^{2}+Y_{..o.}^{2}}{k}-\frac{Y_{....}^{2}}{k^{2}}\right)=k(k-1)\sigma_c^{2}+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^{2}$$

$$\mathrm{ESS(D)}=E\left(\frac{\sum_{l\neq p} Y_{...l}^{2}+Y_{...p}^{2}}{k}-\frac{Y_{....}^{2}}{k^{2}}\right)=k(k-1)\sigma_d^{2}+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^{2}$$

for the error sum of squares

$$Y_{ijhl}=\mu+a_i^{2}+b_j^{2}+c_h^{2}+d_l^{2}+e_{ijhl}$$

$$Y_{ijhl}^{2}=\mu^{2}+a_i^{2}+b_j^{2}+c_h^{2}+d_l^{2}+e_{ijhl}^{2}+\text{cross product terms}$$

$$E\left[\sum_{(i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p)} Y_{ijhl}^{2}\right]=(k^{2}-1)\mu^{2}+(k^{2}-1)\sigma_a^{2}+(k^{2}-1)\sigma_b^{2}+(k^{2}-1)\sigma_c^{2}+$$

$$(k^{2}-1)\sigma_d^{2}+(k^{2}-1)\sigma_e^{2}$$

$$E\left[\sum_{(i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p)} Y_{ijhl}^{2}+X_{mnop}^{2}\right]=k^{2}\mu^{2}+k^{2}\sigma_a^{2}+k^{2}\sigma_b^{2}+k^{2}\sigma_c^{2}+k^{2}\sigma_d^{2}+$$

$$\left[(k^{2}-1)+\frac{4k-3}{(k-1)\ (k-3)}\right]\sigma_e^{2}$$

$$\mathrm{ESS(ERROR)}=E\left[\sum_{(i\neq m}\sum_{j\neq n}\sum_{h\neq o}\sum_{l\neq p)} Y_{ijhl}^{2}+X_{mnop}^{2}-\left(\frac{\sum_{i\neq m} Y_{i...}^{2}+Y_{m...}^{2}}{k}\right)-\left(\frac{\sum_{j\neq n} Y_{.j..}^{2}+Y_{.n..}^{2}}{k}\right)-\right.$$

$$\left.\left(\frac{\sum_{h\neq o} Y_{..h.}^{2}+Y_{..o.}^{2}}{k}\right)-\left(\frac{\sum_{l\neq p} Y_{...l}^{2}+Y_{...p}^{2}}{k}\right)+3\ \frac{Y_{....}^{2}}{k^{2}}\right]$$

$$=\ \left[(k-1)\ (k-3)-1\right]\sigma_e^{2}$$

The proof is complete.

The above completes the proof of three lemmas. In order to generalize for restrictions, the following theorem is developed.

Theorem II

The k X k experiment of random model with r restrictions and one missing value and without correlation has (r+2) components recognized in the model.

The model is,

$$Y_{abcd----s} = \mu + t_a + r_{1b} + r_{2c} + ---- + r_{rs} + e_{abc----s} \quad \begin{cases} a = 1, 2, ---- k \\ b = 1, 2, ---- k \\ \vdots \\ s = 1, 2, ---- k \end{cases}$$

$$\begin{cases} t_a \backsim N(o, \sigma_t) \\ \left.\begin{array}{l} r_{1b} \backsim N(o, \sigma_{r1}) \\ \vdots \\ r_{rs} \backsim N(o, \sigma_{rr}) \end{array}\right\} \text{these are r restrictions} \\ e_{abc----s} \backsim N(o, \sigma_e) \end{cases}$$

Proof:

The ESS for each is,

$$ESS(T) = k(k-1)\sigma_t^2 + \left[(k-1) + \frac{1}{(k-r)}\right]\sigma_e^2$$

$$ESS(R_i) = k(k-1)\sigma_{ri}^2 + \left[(k-1) + \frac{1}{(k-r)}\right]\sigma_e^2 \qquad i = 1, 2, ---- r$$

$$ESS(ERROR) = \left[(k-1)\ (k-r)-1\right]\sigma_e^2$$

In order not to lose generality, assume the missing value occurs in t^{th} level of T, first level of R_1, second level of R_2, etc.

For other levels of T where missing value does not occur,

$$Y_{a\Delta} = k\,\mu + kt_a + \sum_b r_{1b} + \sum_c r_{2c} + ---- + \sum_s r_{rs} + \sum_{(b----s)} e_{abc----s}$$

(Δ is defined as sum over all other subscripts. Ex. $Y_{a\circ\circ\circ} = Y_{a\Delta}$)

$$E\left[\frac{\sum_{a\neq t} Y_{a\Delta}}{k}\right]^2 = k(k-1)\ \mu^2 + k(k-1)\ \sigma_t^2 + (k-1)\ \sigma_{r1}^2 + (k-1)\sigma_{r2}^2 + ---- + (k-1)\ \sigma_{rr}^2 +$$

$$(k-1)\,\sigma_e^2 \qquad\qquad (1 \leq t \leq k)$$

$$Y_{t\Delta} = \sum_{\begin{pmatrix} b\neq 1 \\ c\neq 2 \\ s\neq r \end{pmatrix}} Y_{tbc----s} + X_{t12----r} = k\mu + kt_t + \sum_b r_{1b} + \sum_c r_{2c} + ---- + \sum_s r_{rs}$$

$$+\frac{k(k-r)\sum\limits_{\left(\substack{b\neq 1\\ s\neq r}\right)} e_{tbc--s} + (k-r)\left[\sum\limits_{\substack{a\neq t\\ s\neq r}} e_{a1--s} + ---- + \sum\limits_{\substack{a\neq t\\ b\neq 1}} e_{abc--r}\right] - r\sum\limits_{a\neq t}\cdots\sum\limits_{s\neq r} e_{abc--s}}{(k-1)\quad(k-r)}$$

$$E\left[\frac{Y_{t\Delta}^{2}}{k}\right] = k\mu^2 + k\sigma_t^2 + \sigma_{r1}^2 + \sigma_{r2}^2 + ---- + \sigma_{rr}^2 + \frac{k^2(k-1)(k-r)^2 + r(k-1)(k-r)^2}{k(k-1)^2\;(k-r)^2}$$

$$+\frac{r^2(k-1)(k-r)}{k(k-1)^2(k-r)^2}\sigma_e^2$$

$$= k\mu^2 + k\sigma_t^2 + \sigma_{r1}^2 + \sigma_{r2}^2 + ---- + \sigma_{rr}^2 + \frac{k(k-r)+r}{(k-1)(k-r)}\sigma_e^2$$

$$Y_\Delta = \sum_{a\neq t} Y_{a\Delta} + Y_{t\Delta} = k^2\mu + k\sum_a t_a + k\sum_b r_{1b} + k\sum_c r_{2c} + ---- + k\sum_s r_{rs}$$

$$+\frac{k(k-r)\left[\sum e_{tbc--s} + \sum e_{a1c--s} + ---- + \sum e_{abc--r}\right] + k(k-r-1)\sum\ldots\sum e_{abc--s}}{(k-1)\quad(k-r)}$$

$$E\left[\frac{Y_\Delta^2}{k^2}\right] = k^2\mu^2 + k\sigma_t^2 + k\sigma_{r1}^2 + k\sigma_{r2}^2 + ---- + k\sigma_{rr}^2 + \frac{k^2-k-kr+r+1}{(k-1)\;(k-r)}\sigma_e^2$$

$$ESS(T) = E\left[\frac{\sum\limits_{a\neq t} Y_{a\Delta}^2 + Y_{t\Delta}^2}{k} - \frac{Y_\Delta^2}{k^2}\right] = k(k-1)\sigma_t^2 + \left[(k-1) + \frac{1}{(k-r)}\right]\sigma_e^2$$

Similarly,

$$ESS(R_i) = k(k-1)\sigma_{ri}^2 + \left[(k-1) + \frac{1}{(k-r)}\right]\sigma_e^2 \qquad (i = 1,\ 2,\ ----\ r)$$

For the error sum of squares,

$$Y_{abc--s} = \mu + t_a + r_{1b} + r_{2c} + ---- + r_{rs} + e_{abc--s}$$

$$Y_{abc--s}^2 = \mu^2 + t_a^2 + r_{1b}^2 + r_{2c}^2 + ---- + r_{rs}^2 + e_{abc--s}^2 + \text{cross product terms}$$

$$E(Y_{abc--s}^2) = \mu^2 + {}_t^2 + \sum_{i=1}^{r} {}_{ri}^2 + {}_e^2$$

$$E\left[\sum_{\left(\substack{a\neq t\\ s\neq r}\right)}\cdots\sum Y_{abc--s}^2\right] = (k^2-1)\left[\mu^2 + \sigma_t^2 + \sum_{i=1}^{r}\sigma_{ri}^2 + \sigma_e^2\right]$$

about the missing value,

$$E(X_{t12--r}^2) = \mu + \sigma_t^2 + \sum_{i=1}^{r}\sigma_{ri}^2 + \frac{(k-r)^2(r+1)(k-1) + r^2(k-1)(k-r)}{(k-1)^2\;(k-r)^2}\sigma_e^2$$

$$E\left[\sum_{\substack{a\neq t \\ s\neq r}}\cdots\sum Y_{abc--s}^{2}+X_{t123--r}^{2}\right]=k^{2}\mu^{2}+k^{2}\sigma_{t}^{2}+k^{2}\sum_{i=1}^{r}\sigma_{ri}^{2}+\left[\frac{k(r+1)-r}{(k-1)(k-r)}+(k^{2}-1)\right]\sigma_{e}^{2}$$

$$\mathrm{ESS(ERROR)}=E\left[\sum_{\substack{a\neq t \\ s\neq r}}\cdots\sum Y_{abc--s}+X_{t123--r}-\left(\frac{\sum_{a} Y_{a\Delta}}{k}^{2}\right)-\cdots\cdots-\frac{\sum_{s} Y_{s\Delta}^{2}}{k}+r\,\frac{Y_{\Delta}^{2}}{k^{2}}\right]$$

$$=\left[(k-1)\ (k-r)-1\right]\sigma_{e}^{2}$$

Hence, the proof is complete.

When the background correlation is taken into consideration, the model remains the same. However, the error structure is modified by the addition of covariance terms.

Fundamental Lemma IV A

The k X k experiment of random model with one restriction and correlation but without missing value has three components recognized in the model. The model is,

$$Y_{ij}=\mu+a_{i}+b_{j}+e_{ij}\qquad\begin{cases} i=1,\ 2,\ ----\ k \\ j=1,\ 2,\ ----\ k\end{cases}$$

$$\begin{cases}\left.\begin{matrix} a_{i}\backsim N(o,\sigma_{a}) \\ b_{j}\backsim N(o,\sigma_{b})\end{matrix}\right\}\text{ either one is a restriction} \\ e_{ij}\backsim N(o,\sigma_{e})\end{cases}$$

$$\mathrm{ESS(A)}=k(k-1)\sigma_{a}^{2}-k(k-1)C_{a}+(k-1)\sigma_{e}^{2}+(k-1)^{2}C_{ab'}-(k-1)C_{a'b}-(k-1)^{2}C_{a'b'}$$

$$\mathrm{ESS(B)}=k(k-1)\sigma_{b}^{2}-k(k-1)C_{b}+(k-1)\sigma_{e}^{2}-(k-1)C_{ab'}+(k-1)^{2}C_{a'b}-(k-1)^{2}C_{a'b'}$$

$$\mathrm{ESS(ERROR)}=(k-1)^{2}\sigma_{e}^{2}-(k-1)^{2}C_{ab'}-(k-1)^{2}C_{a'b}+(k-1)^{2}C_{a'b'}$$

Proof:

The derivation requires the use of Lemma I and II.

$$Y_{i.}=k\mu+ka_{i}+\sum_{j}b_{j}+\sum_{j}e_{ij}$$

$$Y_{i\cdot}^2 = k^2\mu^2 + k^2 a_i^2 + \sum_j b_j^2 + \sum_{j\neq j'} b_j b_{j'} + \sum_j e_{ij}^2 + \sum_{j\neq j'} e_{ij} e_{ij'} + \text{cross product terms}$$

$$\sum_i Y_{i\cdot}^2 = k^3\mu^2 + k^2 \sum_i a_i^2 + k \sum_j b_j^2 + k \sum_{j\neq j'} b_j b_{j'} + \sum_i \sum_j e_{ij}^2 + \sum_i \sum_{j\neq j'} e_{ij} e_{ij'} +$$

cross product terms

$$E\left[\frac{\sum_i Y_{i\cdot}^2}{k}\right] = k^2\mu^2 + k^2\sigma_a^2 + k\sigma_b^2 + k(k-1)C_b + k\sigma_e^2 + k(k-1)C_{ab'}$$

$$Y_{\cdot\cdot} = k^2\mu + k\sum_i a_i + k\sum_j b_j + \sum_i \sum_j e_{ij}$$

$$Y_{\cdot\cdot}^2 = k^4\mu^2 + k^2\sum_i a_i^2 + k^2\sum_{i\neq i'} a_i a_{i'} + k^2\sum_j b_j^2 + k^2\sum_{j\neq j'} b_j b_{j'} + \sum\sum_{ij} e_{ij}^2 + \sum_i \sum_{j\neq j'} e_{ij} e_{ij'}$$

$$+ \sum_{i\neq i'} \sum_j e_{ij} e_{i'j} + \sum_{i\neq i'} \sum_{j\neq j'} e_{ij} e_{i'j'} + \text{cross product terms}$$

$$E\left[\frac{Y_{\cdot\cdot}^2}{k^2}\right] = k^2\mu^2 + k\sigma_a^2 + k(k-1)C_a + k\sigma_b^2 + k(k-1)C_b + \sigma_e^2 + (k-1)C_{ab'} + (k-1)C_{a'b}$$

$$+ (k-1)\,(k-1)C_{a'b'}$$

$$ESS(A) = E\left[\frac{\sum_i Y_{i\cdot}^2}{k} - \frac{Y_{\cdot\cdot}^2}{k^2}\right] = k(k-1)\sigma_a^2 - k(k-1)C_a + (k-1)\sigma_e^2 + (k-1)(k-1)C_{ab'}$$

$$- (k-1)C_{a'b} - (k-1)^2 C_{a'b'}$$

Similarly,

$$ESS(B) = E\left[\frac{\sum_j Y_{\cdot j}^2}{k} - \frac{Y_{\cdot\cdot}^2}{k^2}\right] = k(k-1)\sigma_b^2 - k(k-1)C_b + (k-1)\sigma_e^2 - (k-1)C_{ab'} +$$

$$(k-1)^2 C_{a'b} - (k-1)^2 C_{a'b'}$$

For the error sum of squares,

$$Y_{ij} = \mu + a_i + b_j + e_{ij}$$

$$Y_{ij}^2 = \mu^2 + a_i^2 + b_j^2 + e_{ij}^2 + \text{cross product terms}$$

$$E\ (\sum_i \sum_j Y_{ij}^{\ 2}) = k^2\mu^2+k^2\sigma_a^{\ 2}+k^2\sigma_b^{\ 2}+k^2\sigma_e^{\ 2}$$

$$\mathrm{ESS(ERROR)}=E\ (\sum_i \sum_j Y_{ij}^{\ 2} - \frac{\sum_i Y_{i.}^{\ 2}}{k} - \frac{\sum_j Y_{.j}^{\ 2}}{k} + \frac{Y_{..}^{\ 2}}{k^2}) =(k-1)^2\sigma_e^{\ 2}-(k-1)^2C_{a'b}-(k-1)^2$$

$$C_{ab'} + (k-1)^2C_{a'b'}$$

The proof is complete.

Fundamental Lemma IV B

The k X k experiment of random model with two restrictions and correlation but without missing value has four components recognized in the model. The model is,

$$Y_{ijh} = \mu + a_i + b_j + c_h + e_{ijh} \qquad \begin{cases} i = 1,\ 2,\ ---- k \\ j = 1,\ 2,\ ---- k \\ h = 1,\ 2,\ ---- k \end{cases}$$

$$\left\{\begin{array}{l} \left.\begin{array}{l} a_i \backsim N(o, \sigma_a) \\ b_j \backsim N(o, \sigma_b) \\ c_h \backsim N(o, \sigma_c) \end{array}\right\} \text{two of them are restrictions} \\ e_{ijh} \backsim N(o, \sigma_e) \end{array}\right.$$

The ESS for each is,

$$\mathrm{ESS(A)}=k(k-1)\sigma_a^{\ 2}-k(k-1)C_a+(k-1)\sigma_e^{\ 2}+(k-1)^2C_{ab'c'}-(k-1)C_{a'bc'}-(k-1)$$

$$C_{a'b'c}-(k-1)\ (k-2)C_{a'b'c'}$$

$$\mathrm{ESS(B)}=k(k-1)\ \sigma_b^{\ 2}-k(k-1)C_b+(k-1)\ \sigma_e^{\ 2}-(k-1)C_{ab'c'}+(k-1)^2C_{a'bc'}-(k-1)$$

$$C_{a'b'c}-(k-1)\ (k-2)C_{a'b'c'}$$

$$\mathrm{ESS(C)}=k(k-1)\sigma_c^{\ 2}-k(k-1)C_c+(k-1)\ \sigma_e^{\ 2}-(k-1)C_{ab'c'}-(k-1)C_{a'bc'}+(k-1)^2$$

$$C_{a'b'c}-(k-1)\ (k-2)C_{a'b'c'}$$

$$\mathrm{ESS(ERROR)}=(k-1)\ (k-2)\sigma_e^{\ 2}-(k-1)(k-2)C_{ab'c'}-(k-1)(k-2)C_{a'bc'}-(k-1)(k-2)$$

$$C_{a'b'c}+2(k-1)(k-2)C_{a'b'c'}$$

Proof:

Again the derivation requires the use of Lemma I and II.

$$Y_{i..} = k\mu + ka_i + \sum_j b_j + \sum_h c_h + \sum_{(jh)} e_{ijh}$$

$$Y_{i..}^2 = k^2\mu^2 + k^2 a_i^2 + \sum_j b_j^2 + \sum_{j\neq j'} b_j b_{j'} + \sum_h c_h^2 + \sum_{h\neq h} c_h c_{h'} + \sum_{(jh)} e_{ijh}^2 + \sum_{\substack{j\neq j'\\ h\neq h'}} e_{ijh} e_{ij'h'} + \text{cross product terms}$$

$$\sum_i Y_{i..}^2 = k^3\mu^2 + k^2\sum_i a_i^2 + k\sum_j b_j^2 + k\sum_{j\neq j'} b_j b_{j'} + k\sum_h c_h^2 + k\sum_{h\neq h'} c_h c_{h'} + \sum_i \sum_{(jh)} e_{ijh}^2 + \sum_i \sum_{\substack{j\neq j'\\ h\neq h'}} e_{ijh} e_{ij'h'} + \text{cross product terms}$$

$$E\left(\frac{\sum_i Y_{i..}^2}{k}\right) = k^2\mu^2 + k^2\sigma_a^2 + k\sigma_b^2 + k(k-1)C_b + k\sigma_c^2 + k(k-1)C_c + k\sigma_e^2 + k(k-1)C_{ab'c'}$$

$$Y_{...} = k^2\mu + k\sum_i a_i + k\sum_j b_j + k\sum_h c_h + \sum_i \sum_{(jh)} e_{ijh}$$

$$Y_{...}^2 = k^4\mu^2 + k^2\sum_i a_i^2 + k^2\sum_{i\neq i'} a_i a_{i'} + k^2\sum_j b_j^2 + k^2\sum_{j\neq j'} b_j b_{j'} + k^2\sum_h c_h^2 + k^2\sum_{h\neq h'} c_h c_{h'}$$

$$+ \sum_i \sum_{(jh)} e_{ijh}^2 + \sum_i \sum_{\substack{j\neq j'\\ h\neq h'}} e_{ijh} e_{ij'h'} + \sum_j \sum_{\substack{i\neq i'\\ h\neq h'}} e_{ijh} e_{i'jh'} + \sum_h \sum_{\substack{i\neq i'\\ j\neq j'}} e_{ijh} e_{i'j'h} + \sum\sum_{\substack{i\neq i'\\ j\neq j'\\ h\neq h'}} e_{ijh} e_{i'j'h'} + \text{cross product terms}$$

$$E\left(\frac{Y_{...}^2}{k^2}\right) = k^2\mu^2 + k\sigma_a^2 + k(k-1)C_a + k\sigma_b^2 + k(k-1)C_b + k\sigma_c^2 + k(k-1)C_c$$

$$+ \sigma_e^2 + (k-1)C_{ab'c'} + (k-1)C_{a'bc'} + (k-1)C_{a'b'c} + (k-1)(k-2)C_{a'b'c'}$$

$$ESS(A) = E\left(\frac{\sum_i Y_{i..}^2}{k} - \frac{Y_{...}^2}{k^2}\right) = k(k-1)\sigma_a^2 - k(k-1)C_a + (k-1)\sigma_e^2 + (k-1)^2 C_{ab'c'}$$

$$-(k-1)C_{a'bc'} - (k-1)C_{a'b'c} - (k-1)(k-2)C_{a'b'c'}$$

Similarly,

$$ESS(B)=E\left(\frac{\sum_j Y_{.j.}^2}{k}-\frac{Y_{...}^2}{k^2}\right)=k(k-1)\sigma_b^2-k(k-1)C_b+(k-1)\sigma_e^2-(k-1)C_{ab'c'}$$

$$+(k-1)^2C_{a'bc'}-(k-1)C_{a'b'c}-(k-1)(k-2)C_{a'b'c'}$$

$$ESS(C)=E\left(\frac{\sum_h Y_{..h}^2}{k}-\frac{Y_{...}^2}{k^2}\right)=k(k-1)\sigma_c^2-k(k-1)C_c+(k-1)\sigma_e^2-(k-1)C_{ab'c'}$$

$$-(k-1)C_{a'bc'}+(k-1)^2C_{a'b'c}-(k-1)(k-2)C_{a'b'c'}$$

For the error sum of squares,

$$Y_{ijh}=\mu+a_i+b_j+c_h+e_{ijh}$$

$$Y_{ijh}^2=\mu^2+a_i^2+b_j^2+c_h^2+e_{ijh}^2+\text{cross product terms}$$

$$E\left(\sum_i\sum_{(jh)}Y_{ijh}^2\right)=k^2\mu^2+k^2\sigma_a^2+k^2\sigma_b^2+k^2\sigma_c^2+k^2\sigma_e^2$$

$$ESS(ERROR)=E\left(\sum_i\sum_{(jh)}Y_{ijh}^2-\frac{\sum_i Y_{i..}^2}{k}-\frac{\sum_j Y_{.j.}^2}{k}-\frac{\sum_h Y_{..h}^2}{k}+\frac{Y_{...}^2}{k^2}\right)$$

$$=(k-1)(k-2)\sigma_e^2-(k-1)(k-2)C_{ab'c'}-(k-1)(k-2)C_{a'bc'}-(k-1)(k-2)$$

$$C_{a'b'c}+2(k-1)(k-2)C_{a'b'c'}$$

The proof is complete.

Fundamental Lemma IV C

The k X k experiment of random model with three restrictions and correlation but without missing value has five components recognized in the model. The model is,

$$Y_{ijhl}=\mu+a_i+b_j+c_h+d_l+e_{ijhl}\qquad\left\{\begin{array}{l}i=1,\ 2,\ ----\ k\\ j=1,\ 2,\ ----\ k\\ h=1,\ 2,\ ----\ k\\ l=1,\ 2,\ ----\ k\end{array}\right.$$

$$\left.\begin{cases} a_i \backsim N(o, \sigma_a) \\ b_j \backsim N(o, \sigma_b) \\ c_h \backsim N(o, \sigma_c) \\ d_l \backsim N(o, \sigma_d) \\ e_{ijhl} \backsim N(o, \sigma_e) \end{cases}\right\} \text{three of them are restrictions}$$

The ESS for each is,

$$ESS(A)=k(k-1)\sigma_a^2-k(k-1)C_a+(k-1)\sigma_e^2+(k-1)^2C_{ab'c'd'}-(k-1)\left[C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]-(k-1)(k-3)C_{a'b'c'd'}$$

$$ESS(B)=k(k-1)\sigma_b^2-k(k-1)C_b+(k-1)\sigma_e^2+(k-1)^2C_{a'bc'd'}-(k-1)\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd}\right]-(k-1)(k-3)C_{a'b'c'd'}$$

$$ESS(C)=k(k-1)\sigma_c^2-k(k-1)C_c+(k-1)\sigma_e^2+(k-1)^2C_{a'b'cd'}-(k-1)\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd}\right]-(k-1)(k-3)C_{a'b'c'd'}$$

$$ESS(D)=k(k-1)\sigma_d^2-k(k-1)C_d+(k-1)\sigma_e^2+(k-1)^2C_{a'b'c'd}-(k-1)\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}\right]-(k-1)(k-3)C_{a'b'c'd'}$$

$$ESS(ERROR)=(k-1)(k-3)\sigma_e^2-(k-1)(k-3)\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]+3(k-1)(k-3)C_{a'b'c'd'}$$

Proof:

Again the derivation requires the use of Lemmas I and II.

$$Y_{i\cdots}=k\mu+ka_i+\sum_j b_j+\sum_h c_h+\sum_l d_l+\sum_{(jhl)} e_{ijhl}$$

$$Y_{i\cdots}^2=k^2\mu^2+k^2a_i^2+\sum_j b_j^2+\sum_{j\neq j'} b_jb_{j'}+\sum_h c_h^2+\sum_{h\neq h'} c_hc_{h'}+\sum_l d_l^2+\sum_{l\neq l'} d_ld_{l'}+\sum_{(jhl)} e_{ijhl}^2+\sum_{\begin{pmatrix} j\neq j' \\ h\neq h' \\ l\neq l' \end{pmatrix}} e_{ijhl}e_{ij'h'l'}+\text{cross product terms}$$

$$\sum_i Y_{i\cdot\cdot\cdot}^2 = k^3\mu^2 + k^2 \sum_i a_i^2 + k \sum_j b_j^2 + k \sum_{j\neq j'} b_j b_{j'} + k \sum_h c_h^2 + k \sum_{h\neq h'} c_h c_{h'} + k \sum_l d_l^2 +$$

$$k \sum_{l\neq l'} d_l d_{l'} + \sum_i \sum_{(jhl)} e_{ijhl}^2$$

$$+ \sum_i \sum_{\begin{pmatrix} j\neq j' \\ h\neq h' \\ l\neq l' \end{pmatrix}} e_{ijhl} e_{ij'h'l'} + \text{cross product terms}$$

$$E\left(\frac{\sum_i Y_{i\cdot\cdot\cdot}^2}{k} \right) = k^2\mu^2 + k^2 \sigma_a^2 + k \sigma_b^2 + k(k-1)C_b + k \sigma_c^2 + k(k-1)C_c + k\sigma_d^2 + k(k-1)$$

$$C_d + k \sigma_e^2 + k(k-1)C_{ab'c'd'}$$

$$Y_{\cdot\cdot\cdot\cdot} = k^2\mu + k \sum_i a_i + k \sum_j b_j + k \sum_h c_h + k \sum_l d_l + \sum_i \sum_{(jh)} e_{ijhl}$$

$$Y_{\cdot\cdot\cdot\cdot}^2 = k^4\mu^2 + k^2 \sum_i a_i^2 + k^2 \sum_{i\neq i'} a_i a_{i'} + k^2 \sum_j b_j^2 + k^2 \sum_{j\neq j'} b_j b_{j'} + k^2 \sum_h c_h^2 + k^2 \sum_{h\neq h'}$$

$$c_h c_{h'} + k^2 \sum_l d_l^2 + k^2 \sum_{l\neq l'} d_l d_{l'}$$

$$+ \sum_i \sum_{(jhl)} e_{ijhl}^2 + \sum_i \sum_{\begin{pmatrix} j\neq j' \\ h\neq h' \\ l\neq l' \end{pmatrix}} e_{ijhl} e_{ij'h'l'} + \sum_j \sum_{\begin{pmatrix} i\neq i' \\ h\neq h' \\ l\neq l' \end{pmatrix}} e_{ijhl} e_{i'jh'l'} + \sum_h$$

$$\sum_{\begin{pmatrix} i\neq i' \\ j\neq j' \\ l\neq l' \end{pmatrix}} e_{ijhl} e_{i'j'hl'}$$

$$+ \sum_l \sum_{\begin{pmatrix} i\neq i' \\ j\neq j' \\ h\neq h' \end{pmatrix}} e_{ijhl} e_{i'j'h'l} + \sum \sum_{\begin{pmatrix} i\neq i' \\ j\neq j' \\ h\neq h' \\ l\neq l' \end{pmatrix}} e_{ijhl} e_{i'j'h'l'} + \text{cross product terms}$$

$$E\left(\frac{Y_{\cdot\cdot\cdot\cdot}^2}{k^2} \right) = k^2\mu^2 + k \sigma_a^2 + k(k-1)C_a + k \sigma_b^2 + k(k-1)C_b + k \sigma_c^2 + k(k-1)C_c + k\sigma_d^2 +$$

$$k(k-1)C_d + \sigma_e^2 + (k-1) \left[C_{ab'c'd'} + C_{a'b'cd'} + C_{a'b'c'd} \right] + (k-1)(k-3)$$

$$C_{a'b'c'd'}$$

$$ESS(A)=E\left(\frac{\sum_{i} Y_{i\cdots}^{2}}{k}-\frac{Y_{\cdots\cdot}^{2}}{k^2}\right)=k(k-1)\sigma_a^{2}-k(k-1)C_a+(k-1)\sigma_e^{2}+(k-1)^2C_{ab'c'd'}$$

$$-(k-1)\left[C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]-(k-1)(k-3)C_{a'b'c'd'}$$

Similarly,

$$ESS(B)=E\left(\frac{\sum_{j} Y_{\cdot j\cdot\cdot}^{2}}{k}-\frac{Y_{\cdots\cdot}^{2}}{k^2}\right)=k(k-1)\sigma_b^{2}-k(k-1)C_b+(k-1)\sigma_e^{2}+(k-1)^2C_{a'bc'd'}$$

$$-(k-1)\left[C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]-(k-1)(k-3)C_{a'b'c'd'}$$

$$ESS(C)=E\left(\frac{\sum_{h} Y_{\cdot\cdot h\cdot}^{2}}{k}-\frac{Y_{\cdots\cdot}^{2}}{k^2}\right)=k(k-1)\sigma_c^{2}-k(k-1)C_c+(k-1)\sigma_e^{2}+(k-1)^2C_{a'b'cd'}$$

$$-(k-1)\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd}\right]-(k-1)(k-3)C_{a'b'c'd'}$$

$$ESS(D)=E\left(\frac{\sum_{l} Y_{\cdots l}^{2}}{k}-\frac{Y_{\cdots\cdot}^{2}}{k^2}\right)=k(k-1)\sigma_d^{2}-k(k-1)C_d+(k-1)\sigma_e^{2}+(k-1)^2C_{a'b'c'd}$$

$$-(k-1)\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}\right]-(k-1)(k-3)C_{a'b'c'd'}$$

For the error sum of squares,

$$Y_{ijhl}=\mu+a_i+b_j+c_h+d_l+e_{ijhl}$$

$$Y_{ijhl}^{2}=\mu^2+a_i^{2}+b_j^{2}+c_h^{2}+d_l^{2}+e_{ijhl}^{2}+\text{cross product terms}$$

$$E\left(\sum_{i}\sum_{(jhl)}Y_{ijhl}^{2}\right)=k^2\mu^2+k^2\sigma_a^{2}+k^2\sigma_b^{2}+k^2\sigma_c^{2}+k^2\sigma_d^{2}+k^2\sigma_e^{2}$$

$$ESS(ERROR)=E\left[\sum_{i}\sum_{(jhl)}Y_{ijhl}^{2}-\frac{\sum_{i}Y_{i\cdots}^{2}}{k}-\frac{\sum_{j}Y_{\cdot j\cdot\cdot}^{2}}{k}-\frac{\sum_{h}Y_{\cdot\cdot h\cdot}^{2}}{k}-\frac{\sum_{l}Y_{\cdots l}^{2}}{k}+3\frac{Y_{\cdots\cdot}^{2}}{k^2}\right]$$

$$=(k-1)(k-3)\sigma_e^{2}-(k-1)(k-3)\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]+3$$

$$(k-1)(k-3)C_{a'b'c'd'}$$

The above completes the proof of three lemmas. In order to generalize to r restrictions, the following theorem is developed.

Theorem III

The k X k experiment of random model with r restrictions and correlation but without missing value has (r+2) components recognized in the model. The model is,

$$Y_{tabc--r} = \mu + t_t + r_{1a} + r_{2b} + ---- + r_{rr} + e_{tab--r}$$

$$\begin{cases} t = 1,\ 2,\ ----\ k \\ a = 1,\ 2,\ ----\ k \\ \vdots \qquad\qquad \vdots \\ r = 1,\ 2,\ ----\ k \end{cases}$$

$$\begin{cases} t_t \backsim N(o, \sigma_t) \\ \left.\begin{array}{l} r_{1a} \backsim N(o, \sigma_{r1}) \\ r_{2b} \backsim N(o, \sigma_{r2}) \\ \vdots \qquad \vdots \\ r_{rr} \backsim N(o, \sigma_{rr}) \end{array}\right\} \text{these are r restrictions} \\ e_{tab--r} \backsim N(o, \sigma_e) \end{cases}$$

Then, the ESS for each is,

$$ESS(T)=k(k-1)\sigma_t^2-k(k-1)C_t+(k-1)\sigma_e^2+(k-1)^2C_{\pi(j)}-(k-1)\sum_{i\neq t}C_{\pi(i)}-(k-1)$$

$$(k-r)C_{t'a'b'--r'}$$

$$\begin{cases} J = 1,\ 2,\ ----\ r \\ j = a,\ b,\ c,\ ----\ r \end{cases}$$

$$ESS(R_J) = k(k-1)\sigma_{rj}^2-k(k-1)C_{rj}+(k-1)\sigma_e^2+(k-1)^2C_{\pi(j)}-(k-1)\sum_{i\neq j}C_{\pi(i)}-(k-1)$$

$$(k-r)C_{t'a'b'--r'}$$

$$ESS(ERROR)=(k-1)(k-r)\sigma_e^2-(k-1)(k-r)\sum_i C_{\pi(i)}+r(k-1)(k-r)C_{t'a'b'--r'}$$

Proof:

Again the proof of this Theorem requires the use of Lemmas I and II. Let Δ denote as the sum of all rest subscripts. For example,

$Y_{a..}$ could be written as $Y_{a\Delta}$

$Y_{...}$ could be written as Y_{Δ}

Also defines $\pi(t)$ as an operator which assigns primes to all subscripts except those appearing in the parenthesis. In example, $C_{ab'c'}$ could be written as $C_{\pi(a)}$.

$$Y_{t\Delta} = k\mu + kt_t + \sum_a r_{1a} + \sum_b r_{2b} + ---- + \sum_r r_{rr} + \sum_{(ab--r)} e_{tab--r}$$

$$E\left(\frac{\sum_t Y_{t\Delta}^2}{k}\right) = k^2\mu^2 + k^2\sigma_t^2 + k\sum_j \sigma_{rj}^2 + k(k-1)\sum_{i\neq t} C_i + k\sigma_e^2 + k(k-1)C_{\pi(t)}$$

$$\begin{cases} J = 1, 2, ---- r \\ i = t, a, b, ---- r \end{cases}$$

$$Y_{\Delta} = k^2\mu + k\sum_t t_t + k\sum_a r_{1a} + k\sum_b r_{2b} + ---- + k\sum_r r_{rr} + \sum_t \sum_{(ab-rr)} e_{tab--r}$$

$$E\left(\frac{Y_{\Delta}^2}{k^2}\right) = k^2\mu^2 + k\sum_j \sigma_{rj}^2 + k\sigma_t^2 + k(k-1)\sum_i C_i + \sigma_e^2 + (k-1)\sum_i C_{\pi(i)} + (k-1)$$

$$(k-r)\, C_{t'a'b'--r'}$$

$$ESS(T) = E\left[\frac{\sum_t Y_{t\Delta}^2}{k} - \frac{Y_{\Delta}^2}{k^2}\right] = k(k-1)\sigma_t^2 - k(k-1)C_t + (k-1)\sigma_e^2 + (k-1)^2 C_{\pi(t)} - (k-1)$$

$$\sum_{i\neq t} C_{\pi(i)} - (k-1)(k-r)C_{ta'b'c'--r'}$$

The proof for $ESS(R_j)$ ($j = 1, 2, -- r$) are same, thus will be omitted.

$$ESS(ERROR) = E\left[\sum .. \sum Y_{tabc--r}^2 - \frac{\sum_t Y_{t\Delta}^2}{k} ---- + \frac{Y_{r\Delta}^2}{k^2}\right] = (k-1)(k-r)\sigma_e^2 -$$

$$(k-1)(k-r)\sum_i C_{\pi(i)} + r(k-1)(k-r)C_{t'a'b'--r'}$$

Hence, the proof is complete.

When dealing a case with both correlation and a missing value, the error structure becomes even more complicated. This can be seen from the following.

Fundamental Lemma V A

The k X k experiment of random model with one restriction and both a missing value and correlation has three components recognized in the model. The model is,

$$Y_{ij} = \mu + a_i + b_j + e_{ij} \qquad \begin{cases} i = 1, 2, ---- k \\ j = 1, 2, ---- k \end{cases}$$

$$\left.\begin{cases} a_i \backsim N(o, \sigma_a) \\ b_j \backsim N(o, \sigma_b) \\ e_{ij} \backsim N(o, \sigma_e) \end{cases}\right\} \text{ either one is a restriction}$$

The ESS for each is,

$$ESS(A)=k(k-1)\sigma_a^2-k(k-1)C_a+\left[(k-1)+\frac{1}{(k-1)}\right]\sigma_e^2+\left[(k-1)^2-\frac{1}{(k-1)}\right]C_{ab'}$$

$$-\frac{[1+(k-1)(k-1)]}{(k-1)}C_{a'b}+\frac{1-(k-1)(k-1)^2}{(k-1)}C_{a'b'}$$

$$ESS(B)=k(k-1)\sigma_b^2-k(k-1)C_b+\left[(k-1)+\frac{1}{(k-1)}\right]\sigma_e^2-\frac{[1+(k-1)(k-1)]}{(k-1)}C_{ab'}$$

$$+\left[(k-1)^2-\frac{1}{(k-1)}\right]C_{a'b}+\frac{1-(k-1)(k-1)^2}{(k-1)}C_{a'b'}$$

$$ESS(ERROR)=\left[(k-1)(k-1)-1\right]\sigma_e^2-\left[(k-1)(k-1)-1\right]\left[C_{ab'}+C_{a'b}\right]+\left[(k-1)\right.$$

$$\left.(k-1)-1\right]C_{a'b'}$$

Proof:

Again in order not to lose the generality, assume the missing value occurs in h^{th} level of A and 1^{th} level of B.

For other levels of A where missing value does not occur,

$$Y_{i.} = k\mu + ka_i + \sum_j b_j + \sum_j e_{ij}$$

$$Y_{i.}^{\;2} = k^2\mu^2+k^2a_i^{\;2} + \sum_j b_j^{\;2}+ \sum_{j\neq j'} b_jb_{j'} + \sum_j e_{ij}^{\;2}+ \sum_{j\neq j'} e_{ij}e_{ij'} + \text{cross product terms}$$

$$\sum_{i\neq h} Y_{i.}^{\;2}=(k-1)k^2\mu^2+k^2 \sum_{i\neq h} a_i^{\;2}+(k-1) \sum_j b_j^{\;2}+(k-1) \sum_{j\neq j'} b_jb_{j'}+\sum_{i\neq h} \sum_j e_{ij}^{\;2}+ \sum_{i\neq h} \sum_{j\neq j'}$$

$$e_{ij}e_{ij'} + \text{cross product terms}$$

$$E\left(\frac{\sum_{i\neq h} Y_{i.}^{\;2}}{k}\right) = k(k-1)\, \mu^2+k(k-1)\, \sigma_a^{\;2}+(k-1)\sigma_b^{\;2}+(k-1)^2C_b+(k-1)\, \sigma_e^{\;2}+(k-1)^2C_{ab'}$$

For the h^th^ level of **A**,

$$Y_{h.} = \sum_{j\neq 1} Y_{hj} + X_{ht} \qquad \text{(see Theorem I)}$$

$$= k\mu+ka_h+\sum_j b_j + \frac{k(k-1) \sum_{j\neq 1} e_{hj} +(k-1) \sum_{i\neq h} e_{i1} - \sum_{i\neq h} \sum_{j\neq 1} e_{ij}}{(k-1) \quad (k-1)}$$

$$Y_{h.}^{\;2} = k^2 \mu^2 + k^2 a_h^{\;2} + \sum_j b_j^{\;2} + \sum_{j\neq j'} b_jb_{j'}$$

$$+ \frac{k^2(k-1)^2 \left[\sum_{j\neq 1} e_{hj}^{\;2} + \sum_{\substack{j\neq 1\\ j\neq 1'}} e_{hj}\, e_{hj'}\right] +(k-1)^2\left[\sum_{i\neq h} e_{i1}^{\;2} + \sum_{\substack{i\neq h\\ i\neq i'}} e_{i1}\, e_{i'1}\right]}{(k-1)^4}$$

$$+\frac{\sum_{i\neq h} \sum_{j\neq 1} e_{ij}^{\;2}+ \sum_{i\neq h} \sum_{\substack{j\neq 1\\ j\neq j'}} e_{ij}e_{ij'} + \sum_{\substack{i\neq h\\ i\neq i'}} \sum_{j\neq 1} e_{ij}e_{i'j} + \sum_{\substack{i\neq h\\ i\neq i'}} \sum_{\substack{j\neq 1\\ j\neq j'}} e_{ij}e_{i'j'} +2k(k-1)^2\sum_{j\neq 1} e_{hj}}{(k-1)^2 \quad (k-1)^2}$$

$$\frac{\sum_{i\neq h} e_{i1}-2k(k-1) \sum_{j\neq 1} e_{hj} \sum_{i\neq h} \sum_{j\neq 1} e_{ij}-2(k-1) \sum_{i\neq h} e_{i1} \sum_{i\neq h} \sum_{j\neq 1} e_{ij}}{(k-1)^2 \quad (k-1)^2} + \text{cross product terms}$$

$$E\left(\frac{Y_{h.}^{\;2}}{k}\right) = k\, \mu^2 + k\,\sigma_a^{\;2} +\sigma_b^{\;2} +(k-1)\, C_b$$

$$+ \frac{k^2(k-1)^2 \left[(k-1)\, \sigma_e^{\;2}+(k-1)(k-2)C_{ab'}\right] +(k-1)^2 \left[(k-1)\, \sigma_e^{\;2} +(k-1)(k-2)\right.}{k(k-1)^2 \quad (k-1)^2}$$

$$\frac{\left.C_{a'b}\right] +(k-1)(k-1)\, \sigma_e^{\;2}+(k-1)(k-1)(k-2)C_{ab'}+(k-1)(k-1)(k-2)C_{a'b}+(k-1)^2}{k(k-1)^2 \quad (k-1)^2}$$

$$\frac{(k-2)^2 C_{a'b'} + 2k(k-1)^2(k-1)^2 C_{a'b'} - 2k(k-1)\left[(k-1)(k-1)C_{a'b} + (k-1)(k-1)(k-2)\right.}{k(k-1)^2 \quad (k-1)^2}$$

$$\frac{\left. C_{a'b'}\right] - 2(k-1)\left[(k-1)(k-1)C_{ab'} + (k-1)(k-1)(k-2)C_{a'b'}\right]}{k(k-1)^2 \quad (k-1)^2}$$

$$= k\mu^2 + k\sigma_a^2 + \sigma_b^2 + (k-1)C_b + \frac{k(k-1)+1}{(k-1)(k-1)}\sigma_e^2$$

$$+ \frac{k(k-1)(k-2)-1}{(k-1)\ (k-1)} C_{ab'} - \frac{k}{(k-1)\ (k-1)} C_{a'b} + \frac{k}{(k-1)\ (k-1)} C_{a'b'}$$

$$E\left(\frac{\sum_{i\neq h} Y_{i\circ}^2 + Y_{h\circ}^2}{k}\right) = k^2\mu^2 + k^2\sigma_a^2 + k\sigma_b^2 + k(k-1)C_b + \frac{(k-1)^3 + k(k-1)+1}{(k-1)\quad(k-1)}\sigma_e^2$$

$$+ \frac{(k-1)^4 + k(k-1)(k-2)-1}{(k-1)\qquad(k-1)} C_{ab'} - \frac{k}{(k-1)\ (k-1)} C_{a'b} + \frac{k}{(k-1)\ (k-1)} C_{a'b'}$$

$$Y_{\circ\circ} = \sum_{i=h} Y_{i\circ} + Y_{h\circ} = k(k-1)\mu + k\sum_{i\neq h} a_i + (k-1)\sum_j b_j + \sum_{i\neq h}\sum_j e_{ij} + k\mu + ka_h + \sum_j b_j$$

$$+ \frac{k(k-1)\sum_{j\neq 1} e_{hj} + (k-1)\sum_{i\neq h} e_{i1} - \sum_{i\neq h}\sum_{j\neq 1} e_{ij}}{(k-1)\quad(k-1)}$$

$$= k^2\mu^2 + k\sum_i a_i + k\sum_j b_j + \frac{k(k-1)\sum_{j\neq 1} e_{hj} + k(k-1)\sum_{i\neq h} e_{i1} + k(k-2)\sum_{i\neq h}\sum_{j\neq 1} e_{ij}}{(k-1)\quad(k-1)}$$

$$Y_{\circ\circ}^2 = k^4\mu^2 + k^2\sum_i a_i^2 + k^2\sum_{i\neq i'} a_i a_{i'} + k^2\sum_j b_j^2 + k^2\sum_{j\neq j'} b_j b_{j'}$$

$$+ \frac{k^2(k-1)^2\left[\sum_{j\neq 1} e_{hj}^2 + \sum_{\substack{j\neq 1\\ j\neq j'}} e_{hj}e_{hj'}\right] + k^2(k-1)\left[\sum_{i\neq h} e_{i1}^2 + \sum_{\substack{i\neq h\\ i\neq i'}} e_{i1}e_{i'1}\right]}{(k-1)^2 \quad (k-1)^2}$$

$$\frac{+k^2(k-2)^2\left[\sum_{i\neq h}\sum_{j\neq 1} e_{ij}^2 + \sum_{i\neq h}\sum_{\substack{j\neq 1\\ j\neq j'}} e_{ij}e_{ij'} + \sum_{\substack{i\neq h\\ i\neq i'}}\sum_{j\neq 1} e_{ij}e_{i'j} + \sum_{\substack{i\neq h\\ i\neq i'}}\sum_{\substack{j\neq 1\\ j\neq j'}} e_{ij}e_{i'j'}\right] + ----}{(k-1)^2 \quad (k-1)^2}$$

+ cross product terms

$$E\left(\frac{Y_{\circ\circ}^2}{k^2}\right) = k^2\mu^2 + k\sigma_a^2 + k(k-1)C_a + k\sigma_b^2 + k(k-1)C_b$$

$$+\frac{k^2(k-1)^2\left[(k-1)\sigma_e{}^2+(k-1)(k-2)C_{ab'}\right]+k^2(k-1)^2\left[(k-1)\quad{}_e{}^2+(k-1)(k-2)C_{a'b}\right]}{k^2(k-1)^2\ (k-1)^2}$$

$$+\frac{k^2(k-2)^2\left[(k-1)^2\sigma_e{}^2+(k-1)^2(k-2)C_{ab'}+(k-1)^2(k-2)C_{a'b}+(k-1)^2(k-2)^2\right.}{k^2(k-1)^2\ (k-1)^2}$$

$$\frac{\left.C_{a'b'}\right]+2k^2(k-1)^2(k-1)^2C_{a'b'}+2k^2(k-1)(k-2)\left[(k-1)^2C_{a'b}+(k-1)^2(k-2)C_{a'b'}\right]}{k^2(k-1)^2\ (k-1)^2}$$

$$+\frac{2k^2(k-1)(k-2)\left[(k-1)^2C_{ab'}+(k-1)^2(k-2)C_{a'b'}\right]}{k^2(k-1)^2\ (k-1)^2}$$

$$=k^2\mu^2+k\sigma_a{}^2+k(k-1)C_a+k\sigma_b{}^2+k(k-1)C_b+\left[1+\frac{1}{(k-1)(k-1)}\right]\sigma_e{}^2$$

$$+\frac{(k-1)^3-1}{(k-1)(k-1)}\ C_{ab'}+\frac{(k-1)^3-1}{(k-1)(k-1)}\ C_{a'b}+\frac{(k-1)^4+1}{(k-1)(k-1)}\ C_{a'b'}$$

$$ESS(A)=E\left[\frac{\sum\limits_{i\neq h}Y_{i.}{}^2+Y_{h.}{}^2}{k}-\frac{Y_{..}{}^2}{k^2}\right]=k(k-1)\sigma_a{}^2-k(k-1)C_a+\left[(k-1)+\frac{1}{(k-1)}\right]\sigma_e{}^2$$

$$+\left[(k-1)^2-\frac{1}{(k-1)}\right]C_{ab'}-\frac{\left[1+(k-1)(k-1)\right]}{(k-1)}C_{a'b}+\frac{1-(k-1)(k-1)^2}{(k-1)}C_{a'b'}$$

Similarly,

$$ESS(B)=E\left[\frac{\sum\limits_{j\neq l}Y_{.j}{}^2+Y_{.l}{}^2}{k}-\frac{Y_{..}{}^2}{k^2}\right]=k(k-1)\sigma_b{}^2-k(k-1)C_b+\left[(k-1)+\frac{1}{(k-1)}\right]\sigma_e{}^2$$

$$-\frac{\left[1+(k-1)(k-1)\right]}{(k-1)}C_{a'b}+\left[(k-1)^2-\frac{1}{(k-1)}\right]C_{ab'}+\frac{1-(k-1)(k-1)^2}{(k-1)}C_{a'b'}$$

For the error sum of squares,

$$Y_{ij}=\mu+a_i+b_j+e_{ij}$$

$$Y_{ij}{}^2=\mu^2+a_i{}^2+b_j{}^2+e_{ij}{}^2+\text{cross product terms}$$

$$E\left(\sum_{\substack{i\neq h\\ j\neq l}}\sum Y_{ij}{}^2\right)=(k^2-1)\mu^2+(k^2-1)\sigma_a{}^2+(k^2-1)\sigma_b{}^2+(k^2-1)\sigma_e{}^2$$

For the missing value,

$$X_{hl} = \frac{kY_{h.}'+kY_{.l}'-Y_{..}'}{(k-1)\ (k-1)} = \mu + a_h + b_l + \frac{(k-1)\sum_{j\neq l} e_{hj}+(k-1)\sum_{i\neq h} e_{il}-\sum_{i\neq h}\sum_{j\neq l} e_{ij}}{(k-1)\ (k-1)}$$

$$E(X_{hl}^2) = \mu^2+\sigma_a^2+\sigma_b^2+\frac{(k-1)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)C_{ab'}\right]+(k-1)^2\left[(k-1)\right.}{(k-1)^4}$$

$$\frac{\left.\sigma_e^2+(k-1)(k-2)C_{a'b}\right]+(k-1)^2\sigma_e^2+(k-1)^2(k-2)C_{a'b}+(k-1)^2(k-2)C_{ab'}+(k-1)^2}{(k-1)^4}$$

$$\frac{(k-2)^2C_{a'b'}-2(k-1)\left[(k-1)^2C_{a'b}+(k-1)^2(k-2)C_{a'b'}\right]}{(k-1)^4}$$

$$\frac{-2(k-1)\left[(k-1)^2C_{ab'}+(k-1)^2(k-2)C_{a'b'}\right]+2(k-1)^4C_{a'b'}}{(k-1)^4}$$

$$= \mu^2+\sigma_a^2+\sigma_b^2+\frac{2k-1}{(k-1)(k-1)}\sigma_e^2+\frac{(k-1)(k-2)-k}{(k-1)(k-1)}C_{ab'}$$

$$+\frac{(k-1)(k-2)-k}{(k-1)(k-1)}C_{a'b}-\frac{(k^2-4k+2)}{(k-1)(k-1)}C_{a'b'}$$

$$E\left(\sum\sum_{\substack{(i\neq h)\\(j\neq l)}} Y_{ij}^2+X_{hl}^2\right) = k^2\mu^2+k^2\sigma_a^2+k^2\sigma_b^2+\left[k^2-\frac{(k^2-4k+2)}{(k-1)(k-1)}\right]\sigma_e^2$$

$$+\frac{(k-1)^2(k-2)-k}{(k-1)\ (k-1)}C_{ab'}+\frac{(k-1)(k-2)-k}{(k-1)(k-1)}C_{a'b}-\frac{(k^2-4k+2)}{(k-1)(k-1)}C_{a'b'}$$

$$E\left((\sum_{i\neq h}\sum_{j\neq l}) Y_{ij}^2+X_{hl}^2-\left(\frac{\sum_{i\neq h} Y_{i.}^2+Y_{h.}^2}{k}\right)-\left(\frac{\sum_{j\neq l} Y_{.j}^2+Y_{.l}^2}{k}\right)+\frac{Y_{..}^2}{k^2}\right) = \left[(k-1)(k-1)-1\right]$$

$$\sigma_e^2-\left[(k-1)(k-1)-1\right]C_{ab'}$$

$$-\left[(k-1)(k-1)-1\right]C_{a'b}+\left[(k-1)(k-1)-1\right]C_{a'b'}$$

The proof is complete.

Fundamental Lemma V B

The k X k experiment of random model with two restrictions and both a missing value and correlation has four components recognized in the model. The model is,

$$Y_{ijh} = \mu + a_i + b_j + c_h + e_{ijh} \qquad \begin{cases} i = 1, 2, \text{----} k \\ j = 1, 2, \text{----} k \\ h = 1, 2, \text{----} k \end{cases}$$

$$\begin{cases} \left.\begin{array}{l} a_i \sim N(o, \sigma_a) \\ b_j \sim N(o, \sigma_b) \\ c_h \sim N(o, \sigma_c) \end{array}\right\} \text{ two of them are restrictions} \\ e_{ijh} \sim N(o, \sigma_e) \end{cases}$$

The ESS for each is,

$$ESS(A) = k(k-1)\sigma_a^2 - k(k-1)C_a + \left[(k-1) + \frac{1}{(k-2)}\right]\sigma_e^2 + \left[(k-1)^2 - \frac{1}{(k-2)}\right] C_{ab'c'} -$$

$$\frac{[1+(k-1)(k-2)]}{(k-2)}\left[C_{a'b'c} + C_{a'bc'}\right] + \frac{(k-2)^2(k-1)-2}{(k-2)} \quad C_{a'b'c'}$$

$$ESS(B) = k(k-1)\sigma_b^2 - k(k-1)C_b + \left[(k-1) + \frac{1}{(k-2)}\right]\sigma_e^2 + \left[(k-1)^2 - \frac{1}{(k-2)}\right] C_{a'bc'} -$$

$$\frac{[1+(k-1)(k-2)]}{(k-2)}\left[C_{ab'c'} + C_{a'b'c}\right] + \frac{(k-2)^2(k-1)-2}{(k-2)} \quad C_{a'b'c'}$$

$$ESS(C) = k(k-1)\sigma_c^2 - k(k-1)C_c + \left[(k-1) + \frac{1}{(k-2)}\right]\sigma_e^2 + \left[(k-1)^2 - \frac{1}{(k-2)}\right] C_{a'b'c} -$$

$$\frac{[1+(k-1)(k-2)]}{(k-2)}\left[C_{ab'c'} + C_{a'bc'}\right] + \frac{(k-2)^2(k-1)-2}{(k-2)} \quad C_{a'b'c'}$$

$$ESS(ERROR) = \left[(k-1)(k-2)-1\right]\sigma_e^2 - \left[(k-1)(k-2)-1\right]\left[C_{ab'c'} + C_{a'bc'} + C_{a'b'c}\right]$$

$$+ 2\left[(k-1)(k-2)-1\right] \quad C_{a'b'c'}$$

Proof:

Again the derivation requires the use of Fundamental Lemmas I and II. Also assume the missing value occurs in 1^{th} level of A, m^{th} level of B, and n^{th} level of C.

For other levels of A where missing value does not occur,

$$Y_{i..} = k\mu + ka_i + \sum_j b_j + \sum_h c_h + \sum_{(jh)} e_{ijh}$$

$$Y_{i..}^2 = k^2\mu^2 + k^2 a_i^2 + \sum_j b_j^2 + \sum_{j\neq j'} b_j b_{j'} + \sum_h c_h^2 + \sum_{h\neq h'} c_h c_{h'} + \sum_{(jh)} e_{ijh}^2 + \sum_{\left(\substack{j\neq j'\\ h\neq h'}\right)}$$

$$e_{ijh}\, e_{ij'h'} + \text{cross product terms}$$

$$\sum_{i\neq l} Y_{i..}^2 = (k-1)k^2\mu^2 + k^2 \sum_{i\neq l} a_i^2 + (k-1)\sum_j b_j^2 + (k-1)\sum_{j\neq j'} b_j b_{j'} + (k-1)\sum_h c_h^2 +$$

$$(k-1)\sum_{h\neq h'} c_h c_{h'} + \sum_{i\neq l}\ \sum_{(jh)} e_{ijh}^2 + \sum_{i\neq l}\ \sum_{\left(\substack{j\neq j'\\ h\neq h'}\right)} e_{ijh} e_{ij'h'} + \text{cross product terms}$$

$$E\left(\frac{\sum_{i\neq l} Y_{i..}^2}{k}\right) = k(k-1)\mu^2 + k(k-1)\,\sigma_a^2 + (k-1)\,\sigma_b^2 + (k-1)^2 C_b + (k-1)\,\sigma_c^2 +$$

$$(k-1)^2 C_c + (k-1)\,\sigma_e^2 + (k-1)^2 C_{ab'c'}$$

for the l^{th} level of A,

$$Y_{l..} = \sum_{\left(\substack{j\neq m\\ h\neq n}\right)} Y_{ljh} + X_{lmn}$$

$$= (k-1)\mu + (k-1)a_l + \sum_{j\neq m} b_j + \sum_{h\neq n} c_h + \sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh} + \frac{kY_{l..}' + kY_{.m.}' + kY_{..n}' - 2Y_{...}'}{(k-1)(k-2)}$$

$$= (k-1)\mu + (k-1)a_l + \sum_{j\neq m} b_j + \sum_{h\neq n} c_h + \sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh}$$

$$+ \frac{k\left[(k-1)\mu + (k-1)a_l + \sum_{j\neq m} b_j + \sum_{h\neq n} c_h + \sum_{\left(\substack{j\neq m\\ h\neq n}\right)} e_{ljh}\right] + k\left[(k-1)\mu + \sum_{i\neq l} a_i + (k-1)b_m + \sum_{h\neq n} c_h +\right.}{(k-1)\ (k-2)}$$

$$\underline{\left.\sum_{\left(\substack{i\neq l\\ h\neq n}\right)} e_{imh}\right]}$$

$$+ \frac{k\left[(k-1)\mu + \sum_{i\neq l} a_i + \sum_{j\neq m} b_j + (k-1)c_h + \sum_{\left(\substack{i\neq l\\ j\neq m}\right)} e_{ijn}\right] - 2\left[(k^2-1)\mu + k\sum_{i\neq l} a_i + (k-1)a_l + k\sum_{j\neq m}\right.}{(k-1)\ (k-2)}$$

$$\frac{\left.b_j + (k-1)b_m + k\sum_{h\neq n} c_h + (k-1)c_h + \left(\sum_{i\neq l}\ \sum_{j\neq m}\ \sum_{h\neq n}\right) e_{ijh}\right]}{(k-1)\ (k-2)}$$

$$= (k-1)\mu+(k-1)a_1+\sum_{j\neq m} b_j+\sum_{h\neq n} c_h+\sum_{(\substack{j\neq m\\ h\neq n})} e_{1jh}+\mu+a_1+b_m+c_h+\frac{(k-2)\sum_{(\substack{j\neq m\\ h\neq n})} e_{1jh}+(k-2)\sum_{(\substack{i\neq 1\\ h\neq n})}}{(k-1)(k-2)}$$

$$\frac{e_{imh}+(k-2)\sum_{(\substack{i\neq 1\\ j\neq m})} e_{ijn}-2\sum_{i\neq 1}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}}{(k-1)(k-2)}$$

$$=k\mu+ka_1+\sum_j b_j+\sum_h c_h+\frac{k(k-2)\sum_{(\substack{j\neq m\\ h\neq n})} e_{1jh}+(k-2)\sum_{(\substack{i\neq 1\\ h\neq n})} e_{imh}+(k-2)\sum_{(\substack{i\neq 1\\ j\neq m})} e_{ijn}-2\sum_{i\neq 1}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}}{(k-1)\ \ (k-2)}$$

$$Y_{1..}^2 = k^2\mu^2+k^2a_i^2+\sum_j b_j^2+\sum_{j\neq j'} b_j b_{j'}+\sum_h c_h^2+\sum_{h\neq h'} c_h c_{h'}$$

$$+\frac{k^2(k-2)^2\left[\sum_{(\substack{j\neq m\\ h\neq n})} e_{1jh}^2+\sum_{(\substack{j\neq m\neq j'\\ h\neq n\neq h'})} e_{1jh}e_{1j'h'}\right]+(k-2)^2\left[\sum_{(\substack{i\neq 1\\ h\neq n})} e_{imh}^2+\sum_{(\substack{i\neq 1\neq i'\\ h\neq n\neq h})} e_{imh}e_{i'mh'}\right]+(k-2)}{(k-1)^2\ \ (k-2)^2}$$

$$\frac{\left[\sum_{(\substack{i\neq 1\\ j\neq m})} e_{ijn}^2+\sum_{(\substack{i\neq 1\neq i'\\ j\neq m\neq j'})} e_{ijn}e_{i'j'n}\right]+4\left[\sum_{i\neq 1}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}^2+\sum_{i\neq 1}\sum_{j\neq m\neq j'}\sum_{h\neq n\neq h'} e_{ijh}e_{ij'h'}+\right.}{(k-1)^2(k-2)^2}$$

$$\frac{\sum_{i\neq 1\neq i'}\sum_{j\neq m}\sum_{h\neq n\neq h'} e_{ijh}e_{i'jh'}+\sum_{i\neq 1\neq i'}\sum_{j\neq m\neq j'}\sum_{h\neq n} e_{ijh}e_{i'j'h}+\sum_{i\neq 1\neq i'}\sum_{j\neq m\neq i'}\sum_{h\neq n\neq h'} e_{ijh}}{(k-1)^2(k-2)^2}$$

$$\frac{\left.e_{i'j'h'}\right]-4k(k-2)\sum_{(\substack{j\neq m\\ h\neq n})} e_{1jh}\sum_{i\neq 1}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}-4(k-2)\sum_{(\substack{i\neq 1\\ h\neq n})} e_{imh}\sum_{i\neq 1}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}-4(k-2)\sum_{(\substack{i\neq 1\\ j\neq m})}}{(k-1)^2(k-2)^2}$$

$$\frac{e_{ijn}\sum_{i\neq 1}\sum_{j\neq m}\sum_{h\neq n} e_{ijh}+2k(k-2)^2\sum_{(\substack{j\neq m\\ h\neq n})} e_{1jh}\sum_{(\substack{i\neq 1\\ h\neq n})} e_{imh}+2k(k-2)^2\sum_{(\substack{j\neq m\\ h\neq n})} e_{1jh}\sum_{(\substack{i\neq 1\\ j\neq m})} e_{1jn}+2(k-2)^2}{(k-1)^2(k-2)^2}$$

$$\frac{\sum_{(\substack{i\neq 1\\ h\neq n})} e_{imh}\sum_{(\substack{i\neq 1\\ j\neq m})} e_{1jn}}{(k-1)^2(k-2)^2}$$

$$E\left(\frac{Y_{1..}^2}{k}\right) = k\,\mu^2+k\sigma_a^2+\sigma_b^2+(k-1)C_b+\sigma_c^2+(k-1)C_c$$

$$+\frac{k^2(k-2)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)C_{ab'c'}\right]+(k-2)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)\right.}{k(k-1)^2\quad(k-2)^2}$$

$$\frac{\left.C_{a'bc'}\right]+(k-2)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)C_{a'b'c}\right]+4\left[(k-1)(k-2)\sigma_e^2+(k-1)(k-2)\right.}{k(k-1)^2\quad(k-2)^2}$$

$$\frac{(k-3)C_{ab'c'}+(k-1)(k-2)(k-3)C_{a'bc'}+(k-1)(k-2)(k-3)C_{a'b'c}+(k-1)(k-2)(k^2-6k+10)}{k(k-1)^2\quad(k-2)^2}$$

$$\left.\underline{C_{a'b'c'}}\right]$$

$$\frac{-4k(k-2)\left[(k-1)(k-2)C_{a'bc'}+(k-1)(k-2)C_{a'b'c}+(k-1)(k-2)(k-3)C_{a'b'c'}\right]}{k(k-1)^2\quad(k-2)^2}$$

$$\frac{-4(k-2)\left[(k-1)(k-2)C_{ab'c'}+(k-1)(k-2)C_{a'b'c}+(k-1)(k-2)(k-3)C_{a'b'c'}\right]}{k(k-1)^2\quad(k-2)^2}$$

$$\frac{-4(k-2)\left[(k-1)(k-2)C_{a'bc'}+(k-1)(k-2)C_{ab'c'}+(k-1)(k-2)(k-3)C_{a'b'c'}\right]}{k(k-1)^2\quad(k-2)^2}$$

$$\frac{+2k(k-2)^2\left[(k-1)C_{a'b'c}+(k-1)(k-2)C_{a'b'c'}\right]+2k(k-2)^2\left[(k-1)C_{a'bc'}+(k-1)(k-2)\right.}{k(k-1)^2\quad(k-2)^2}$$

$$\frac{\left.C_{a'b'c'}\right]+2(k-2)^2\left[(k-1)C_{ab'c'}+(k-1)(k-2)C_{a'b'c'}\right]}{k(k-1)^2\quad(k-2)^2}$$

$$=k\mu^2+k\sigma_a^2+\sigma_b^2+(k-1)C_b+\sigma_c^2+(k-1)C_c$$

$$+\frac{k^2-2k+2}{(k-1)(k-2)}\sigma_e^2+\frac{k(k-2)2-2}{(k-1)(k-2)}C_{ab'c'}-\frac{1}{(k-1)(k-2)}\left[C_{a'bc'}+C_{a'b'c}\right]+$$

$$\frac{2k}{(k-1)(k-2)}C_{a'b'c'}$$

$$E\left(\frac{\sum_{i\neq 1}Y_{i..}^{2}+Y_{1..}^{2}}{k}\right)=k^2\mu^2+k^2\sigma_a^2+k\sigma_b^2+k(k-1)C_b+k\sigma_c^2+k(k-1)C_c+\left[(k-1)+\frac{k^2-2k+2}{(k-1)(k-2)}\right]\sigma_e$$

$$+\left[\frac{k(k-2)^2-2}{(k-1)(k-2)}+(k-1)^2\right]C_{ab'c'}-\frac{k}{(k-1)(k-2)}\left[C_{a'bc'}+C_{a'b'c}\right]+\frac{2k}{(k-1)(k-2)}\;C_{a'b'c'}$$

$$Y_{\cdot\cdot\cdot}=\sum_{i\neq 1}Y_{i\cdot\cdot}+Y_{1\cdot\cdot}=k^2u+k\sum_i a_i+k\sum_j b_j+k\sum_h c_h$$

$$+\frac{k(k-2)\sum\limits_{\binom{j\neq m}{h\neq n}}e_{1jh}+k(k-2)\sum\limits_{\binom{i\neq 1}{h\neq n}}e_{imh}+k(k-2)\sum\limits_{\binom{i\neq 1}{j\neq m}}e_{ijn}+k(k-3)\sum\limits_{i\neq 1}\sum\limits_{j\neq m}\sum\limits_{h\neq n}e_{ijh}}{(k-1)\;(k-2)}$$

$$E\left(\frac{Y_{\cdot\cdot\cdot}}{k^2}\right)^2=k^2\mu^2+k\sigma_a^2+k(k-1)C_a+k\sigma_b^2+k(k-1)C_b+k\sigma_c^2+k(k-1)C_c$$

$$+\frac{k^2(k-2)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)C_{ab'c'}\right]+k^2(k-2)^2\left[(k-1)\sigma_e^2+(k-1)\right.}{k^2(k-1)^2(k-2)^2}$$

$$\frac{\left.(k-2)C_{a'bc'}\right]+k^2(k-2)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)C_{a'b'c}\right]+k^2(k-3)^2\left[(k-1)(k-2)\right.}{k^2(k-1)^2(k-2)^2}$$

$$\frac{\sigma_e^2+(k-1)(k-2)(k-3)C_{ab'c'}+(k-1)(k-2)(k-3)C_{a'bc'}+(k-1)(k-2)(k-3)C_{a'b'c}}{k^2(k-1)^2(k-2)^2}$$

$$\frac{\left.+(k-1)(k-2)(k^2-6k+10)C_{a'b'c'}\right]+2k^2(k-2)(k-3)\left[2(k-1)(k-2)C_{a'bc'}+2(k-1)(k-2)\right.}{k^2(k-1)^2(k-2)^2}$$

$$\frac{\left.C_{a'b'c}+2(k-1)(k-2)C_{ab'c'}+3(k-1)(k-2)(k-3)C_{a'b'c'}\right]+2k^2(k-2)^2\left[(k-1)C_{ab'c'}\right.}{k^2(k-1)^2\;(k-2)^2}$$

$$\frac{\left.+(k-1)C_{a'bc'}+(k-1)C_{a'b'c}+3(k-1)(k-2)C_{a'b'c'}\right]}{k^2(k-1)^2\;(k-2)^2}$$

$$=k^2\mu^2+k\sigma_a^2+k(k-1)C_a+k\sigma_b^2+k(k-1)C_b+k\sigma_c^2+k(k-1)C_c+\frac{k^2-3k+3}{(k-1)(k-2)}\;\sigma_e^2$$

$$+\frac{k^3-4k^2+5k-3}{(k-1)(k-2)}\left[C_{ab'c'}+C_{a'bc'}+C_{a'b'c}\right]+\frac{(k-3)^2(k^2-2)+6(k-2)^2}{(k-1)\;(k-2)}\;C_{a'b'c'}$$

$$ESS(A)=E\left(\frac{\sum_{i\neq l} Y_{i\circ\circ}^2+Y_{l\circ\circ}^2}{k}-\frac{Y_{\circ\circ\circ}^2}{k^2}\right)=k(k-1)\sigma_a^2-k(k-1)C_a+\left[(k-1)+\frac{1}{(k-2)}\right]\sigma_e^2$$

$$+\left[(k-1)^2-\frac{1}{(k-2)}\right]C_{ab'c'}-\frac{[1+(k-1)(k-2)]}{(k-2)}\left[C_{a'bc'}+C_{a'b'c}\right]+\frac{(k-2)^2(k-1)-2}{(k-2)}$$

$$C_{a'b'c'}$$

$$ESS(B)=E\left(\frac{\sum_{j\neq m} Y_{\circ j\circ}^2+Y_{\circ m\circ}^2}{k}-\frac{Y_{\circ\circ\circ}^2}{k^2}\right)=k(k-1)\sigma_b^2-k(k-1)C_b+\left[(k-1)+\frac{1}{(k-2)}\right]\sigma_e^2$$

$$+\left[(k-1)^2-\frac{1}{(k-2)}\right]C_{a'bc'}-\frac{[1+(k-1)(k-2)]}{(k-2)}\left[C_{ab'c'}+C_{a'b'c}\right]+\frac{(k-2)^2(k-1)-2}{(k-2)}$$

$$C_{a'b'c'}$$

$$ESS(C)=E\left(\frac{\sum_{h\neq n} Y_{\circ\circ h}^2+Y_{\circ\circ n}^2}{k}-\frac{Y_{\circ\circ\circ}^2}{k^2}\right)=k(k-1)\sigma_c^2-k(k-1)C_c+\left[(k-1)+\frac{1}{(k-2)}\right]\sigma_e^2$$

$$+\left[(k-1)^2-\frac{1}{(k-2)}\right]C_{a'b'c}-\frac{[1+(k-1)(k-2)]}{(k-2)}\left[C_{ab'c'}+C_{a'bc'}\right]+\frac{(k-2)^2(k-1)-2}{(k-2)}$$

$$C_{a'b'c'}$$

For the error sum of squares,

$$Y_{ijh}=\mu+a_i+b_j+c_h+e_{ijh}$$

$$Y_{ijh}^2=\mu^2+a_i^2+b_j^2+c_h^2+e_{ijh}^2 + \text{cross product terms}$$

$$E\left((\sum_{i\neq l}\sum_{j\neq m}\sum_{h\neq n})Y_{ijh}^2\right)=(k^2-1)\mu^2+(k^2-1)\sigma_a^2+(k^2-1)\sigma_b^2+(k^2-1)\sigma_c^2+(k^2-1)\sigma_e^2$$

$$E\left((\sum_{i\neq l}\sum_{j\neq m}\sum_{h\neq n})Y_{ijh}^2+X_{lmn}^2\right)=(k^2-1)(\mu^2+\sigma_a^2+\sigma_b^2+\sigma_c^2+\sigma_e^2)+\mu^2+\sigma_a^2+$$

$$\sigma_b^2+\sigma_c^2$$

$$\frac{+(k-2)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)C_{ab'c'}\right]+(k-2)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)C_{a'bc'}\right]}{(k-1)^2(k-2)^2}$$

$$\frac{+(k-2)^2\left[(k-1)\sigma_e^2+(k-1)(k-2)C_{a'b'c}\right]}{(k-1)^2(k-2)^2}$$

$$\frac{+4\left[(k-1)(k-2)\sigma_e^2+(k-1)(k-2)(k-3)C_{ab'c'}+(k-1)(k-2)(k-3)C_{a'bc'}+(k-1)(k-2)\right.}{(k-1)^2\ (k-2)^2}$$

$$\frac{\left.(k-3)C_{a'b'c}+(k-1)(k-2)(k^2-6k+10)C_{a'b'c'}\right]}{(k-1)^2\ (k-2)^2}$$

$$\frac{-4(k-2)\left[2(k-1)(k-2)C_{a'b'c}+2(k-1)(k-2)C_{a'bc'}+2(k-1)(k-2)C_{ab'c'}+3(k-1)(k-2)\right.}{(k-1)^2\ (k-2)^2}$$

$$\frac{\left.(k-3)C_{a'b'c'}\right]+2(k-2)^2\left[(k-1)C_{a'b'c}+(k-1)C_{a'bc'}+(k-1)C_{ab'c'}+3(k-1)(k-2)C_{a'b'c'}\right]}{(k-1)^2\ (k-2)^2}$$

$$= k^2\ \mu^2 +k^2\sigma_a^2+k^2\sigma_b^2+k^2\sigma_c^2+(k^2-1)\sigma_e^2+\frac{3k-2}{(k-1)(k-2)}\sigma_e^2$$

$$+\frac{k^2-6k+4}{(k-1)(k-2)}\left[C_{a'b'c}+C_{a'bc'}+C_{ab'c'}\right]-\frac{2(k^2-6k+4)}{(k-1)(k-2)}C_{a'b'c'}$$

$$\text{ESS(ERROR)}=E\left[\sum_{(i\neq l}\ \sum_{j\neq m}\ \sum_{h\neq n)}Y_{ijh}^2+X_{lmn}^2-\left(\frac{\sum_{i\neq l}Y_{i..}^2+Y_{l..}^2}{k}\right)-\left(\frac{\sum_{j\neq m}Y_{.j.}^2+Y_{.m.}^2}{k}\right)-\right.$$

$$\left.\left(\frac{\sum_{h\neq n}Y_{..h}^2+Y_{..n}^2}{k}\right)+\frac{2Y_{...}^2}{k^2}\right]$$

$$=\left[(k-1)(k-2)-1\right]\sigma_e^2-\left[(k-1)(k-2)-1\right]\left[C_{ab'c'}+C_{a'bc'}+C_{a'b'c'}\right]+2$$

$$\left[(k-1)(k-2)-1\right]\ C_{a'b'c'}$$

The proof is complete.

Fundamental Lemma V C

The k X k experiment of random model with three restrictions and both a missing value and correlation has five components recognized in the model. The model is,

$$Y_{ijh}=\mu+a_i+b_j+c_h+d_l+e_{ijhl}\qquad\left\{\begin{array}{l} i=1,\ 2,\ ----\ k\\ j=1,\ \approx,\ ----\ k\\ h=1,\ 2,\ ----\ k\\ l=1,\ 2,\ ----\ k\end{array}\right.$$

$$\left\{\begin{array}{l} a_i \backsim N(o, \sigma_a) \\ b_j \backsim N(o, \sigma_b) \\ c_h \backsim N(o, \sigma_c) \\ d_l \backsim N(o, \sigma_d) \\ e_{ijhl} \backsim N(o, \sigma_e) \end{array}\right.$$

(the first four bracketed:) three of them are restrictions

The ESS for each is,

$$ESS(A)=k(k-1)\sigma_a^2-k(k-1)C_a+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^2+\left[(k-1)^2-\frac{1}{(k-3)}\right]C_{a b' c' d'}$$
$$-\frac{[1+(k-1)(k-3)]}{(k-3)}\left[C_{a' b c' d'}+C_{a' b' c d'}+C_{a' b' c' d}\right]+\frac{3-(k-3)^2(k-1)}{(k-3)}C_{a' b' c' d'}$$

$$ESS(B)=k(k-1)\sigma_b^2-k(k-1)C_b+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^2+\left[(k-1)^2-\frac{1}{(k-3)}\right]C_{a' b c' d'}$$
$$-\frac{[1+(k-1)(k-3)]}{(k-3)}\left[C_{a b' c' d'}+C_{a' b' c d'}+C_{a' b' c' d'}\right]+\frac{3-(k-3)^2(k-1)}{(k-3)}C_{a' b' c' d'}$$

$$ESS(C)=k(k-1)\sigma_c^2-k(k-1)C_c+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^2+\left[(k-1)^2-\frac{1}{(k-3)}\right]C_{a' b' c d'}$$
$$-\frac{[1+(k-1)(k-3)]}{(k-3)}\left[C_{a b' c' d'}+C_{a' b c' d'}+C_{a' b' c' d'}\right]+\frac{3-(k-3)^2(k-1)}{(k-3)}C_{a' b' c' d'}$$

$$ESS(D)=k(k-1)\sigma_d^2-k(k-1)C_d+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^2+\left[(k-1)^2-\frac{1}{(k-3)}\right]C_{a' b' c' d}$$
$$-\frac{[1+(k-1)(k-3)]}{(k-3)}\left[C_{a b' c' d'}+C_{a' b c' d'}+C_{a' b' c d'}\right]+\frac{3-(k-3)^2(k-1)}{(k-3)}C_{a' b' c' d'}$$

$$ESS(ERROR)=\left[(k-1)(k-3)-1\right]\sigma_e^2-\left[(k-1)(k-3)-1\right]\left[C_{a b' c' d'}+C_{a' b c' d'}+C_{a' b' c d'}+\right.$$
$$\left.C_{a' b' c' d}\right]+3\left[(k-1)(k-3)-1\right]C_{a' b' c' d'}$$

Proof:

Again the derivation requires the use of Lemmas I and II. Also, assume the missing value occurs in p^{th} level of A, o^{th} level of B, r^{th} level of C, and s^{th} level of D.

For other levels of A,

$$Y_{i\ldots} = k\mu + ka_i + \sum_j b_j + \sum_h c_h + \sum_l d_l + \sum_{(jhl)} e_{ijhl}$$

$$Y_{i\ldots}^2 = k^2\mu^2 + k^2a_i^2 + \sum_j b_j^2 + \sum_{j\neq j'} b_jb_{j'} + \sum_h c_h^2 + \sum_{h\neq h'} c_hc_{h'} + \sum_l d_l^2 + \sum_{l\neq l'} d_ld_{l'} +$$

$$\sum_{(jhl)} e_{ijhl}^2 + \sum_{(j\neq j'} \sum_{h\neq h'} \sum_{l\neq l')} e_{ijhl}\, e_{ij'h'l'} + \text{cross product terms}$$

$$E\left(\frac{\sum_{i\neq p} Y_{i\ldots}}{k}\right)^2 = k(k-1)\mu^2 + k(k-1)\sigma_a^2 + (k-1)\sigma_b^2 + (k-1)^2C_b + (k-1)\sigma_c^2 + (k-1)^2$$

$$C_c + k\sigma_d^2 + (k-1)^2C_d + (k-1)\sigma_e^2 + (k-1)^2C_{ab'c'd'}$$

for the pth level of A,

$$Y_{p\ldots} = (k-1)\mu + (k-1)a_p + \sum_{j\neq o} b_j + \sum_{h\neq r} c_h + \sum_{l\neq s} d_l + \sum_{(jhl)} e_{pjhl} + X_{pors}$$

$$= k\mu + ka_p + \sum_j b_j + \sum_h c_h + \sum_l d_l$$

$$+ \frac{k(k-3)\sum_{(jhl)} e_{pijhl} + (k-3)\sum_{(ihl)} e_{iohl} + (k-3)\sum_{(ijl)} e_{ijrl} + (k-3)\sum_{(ijh)} e_{ijhs} - 3}{(k-1)\ (k-3)}$$

$$\frac{\sum_{i\neq p} \sum_{j\neq o} \sum_{h\neq r} \sum_{l\neq s} e_{ijhl}}{(k-1)\ (k-3)}$$

$$E\left(\frac{Y_{p\ldots}}{k}\right)^2 = k\mu^2 + k\sigma_a^2 + \sigma_b^2 + (k-1)C_b + \sigma_c^2 + (k-1)C_c + \sigma_d^2 + (k-1)C_d$$

$$+ \frac{k^2(k-1)(k-3)^2 + 3(k-1)(k-3)^2 + 9(k-1)(k-3)}{k(k-1)^2\ (k-3)^2}\ \sigma_e^2$$

$$+ \frac{k^2(k-1)(k-2)(k-3)^2 + 9(k-1)(k-3)(k-4) - 18(k-1)(k-3)^2 + 6(k-1)(k-3)^2}{k(k-1)^2\ (k-3)^2}\ C_{ab'c'd'}$$

$$+ \frac{(k-1)(k-2)(k-3)^2 + 9(k-1)(k-3)(k-4) - 6k(k-1)(k-3)^2 - 12(k-1)(k-3)^2 + 4k(k-1)}{k(k-1)^2\ (k-3)^2}$$

$$\frac{(k-3)^2 + 2(k-1)(k-3)^2}{k(k-1)^2(k-3)^2}\ C_{a'bc'd'}$$

$$+\frac{(k-1)(k-2)(k-3)^2+9(k-1)(k-3)(k-4)-6k(k-1)(k-3)^2-12(k-1)(k-3)^2+4k(k-1)}{k(k-1)^2\ (k-3)^2}$$

$$\frac{(k-3)^2+2(k-1)(k-3)^2}{k(k-1)^2\ (k-3)^2}\ C_{a'b'cd'}$$

$$+\frac{(k-1)(k-2)(k-3)^2+9(k-1)(k-3)(k-4)-6k(k-1)(k-3)^2-12(k-1)(k-3)^2+4k(k-1)}{k(k-1)^2\ (k-3)^2}$$

$$\frac{(k-3)^2+2(k-1)(k-3)^2}{k(k-1)^2\ (k-3)^2}\ C_{a'b'c'd}$$

$$+\frac{9(k-1)(k-3)(k^2-8k+18)-6k(k-1)(k-3)^2(k-4)-18(k-1)(k-3)^2(k-4)+6k(k-1)}{k(k-1)^2\ (k-3)^2}$$

$$\frac{(k-3)^3+6(k-1)(k-3)^3}{k(k-1)^2\ (k-3)^2}\ C_{a'b'c'd'}$$

$$= k\,\mu^2+k\,\sigma_a^2+\sigma_b^2+(k-1)C_b+\sigma_c^2+(k-1)C_c+\sigma_d^2+(k-1)C_d+\frac{k(k-3)+3}{(k-1)(k-3)}\,\sigma_e^2$$

$$+\frac{k(k-2)(k-3)-3}{(k-1)\ (k-3)}\ C_{ab'c'd'}-\frac{k}{(k-1)(k-3)}\left[C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]+$$

$$\frac{3k}{(k-1)(k-3)}\ C_{a'b'c'd'}$$

$$E\left(\frac{\sum_{i\neq p}Y_{i\ldots}^{\;2}+Y_{p\ldots}^{\;2}}{k}\right)=k^2\mu^2+k^2\,\sigma_a^2+k\,\sigma_b^2+k(k-1)C_b+k\,\sigma_c^2+k(k-1)C_c+k\,\sigma_d^2+k(k-1)C_d$$

$$+\left[(k-1)+\frac{k(k-3)+3}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)^2+\frac{k(k-2)(k-3)-3}{(k-1)\ (k-3)}\right]C_{ab'c'd'}$$

$$-\frac{k}{(k-1)(k-3)}\left[C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]+\frac{3k}{(k-1)(k-3)}\ C_{a'b'c'd'}$$

$$Y_{\cdot\cdot\cdot\cdot} = k^2\mu^2 + k\sum_i a_i + k\sum_j b_j + k\sum_h c_h + k\sum_l d_l$$

$$+\frac{k(k-3)\left[\sum_{(jhl)} e_{pjhl}+\sum_{(ihl)} e_{iohl}+\sum_{(ijl)} e_{ijrl}+\sum_{(ijh)} e_{ijhs}\right]+k(k-4)\sum_{i\neq p}\sum_{j\neq o}\sum_{h\neq r}\sum_{l\neq s} e_{ijhl}}{(k-1)(k-3)}$$

$$E\left(\frac{Y_{\circ\circ\circ\circ}^{2}}{k^2}\right)=k^2\mu^2+k\sigma_a^2+k(k-1)C_a+k\sigma_b^2+k(k-1)C_b+k\sigma_c^2+k(k-1)C_c+k\sigma_d^2+k(k-1)C_d$$

$$+\frac{4k^2(k-1)(k-3)^2+k^2(k-1)(k-3)(k-4)^2}{k^2(k-1)^2\ (k-3)^2}\sigma_e^2$$

$$+\frac{k^2(k-1)(k-2)(k-3)^2+k^2(k-1)(k-3)(k-4)^3+6k^2(k-1)(k-3)^2(k-4)+6k^2(k-1)(k-3)^2}{k^2(k-1)^2\ (k-3)^2}$$

$C_{ab'c'd}$;

$$+\frac{k^2(k-1)(k-2)(k-3)^2+k^2(k-1)(k-3)(k-4)^3+6k^2(k-1)(k-3)^2(k-4)+6k^2(k-1)(k-3)^2}{k^2(k-1)^2\ (k-3)^2}$$

$C_{a'bc'd'}$

$$+\frac{k^2(k-1)(k-2)(k-3)^2+k^2(k-1)(k-3)(k-4)^3+6k^2(k-1)(k-3)^2(k-4)+6k^2(k-1)(k-3)^2}{k^2(k-1)^2\ (k-3)^2}$$

$C_{a'b'cd'}$

$$+\frac{k^2(k-1)(k-2)(k-3)^2+k^2(k-1)(k-3)(k-4)^3+6k^2(k-1)(k-3)^2(k-4)+6k^2(k-1)(k-3)^2}{k^2(k-1)^2\ (k-3)^2}$$

$C_{a'b'c'd}$

$$+\frac{k^2(k-1)(k-3)(k-4)^2(k^2-8k+18)+8k^2(k-1)(k-3)^2(k-4)^2+12k^2(k-1)(k-3)^3}{k^2(k-1)^2\ (k-3)^2}$$

$C_{a'b'c'd'}$

$$=k^2\mu^2+k(\sigma_a^2+\sigma_b^2+\sigma_c^2+\sigma_d^2)+k(k-1)(C_a+C_b+C_c+C_d)$$

$$+\frac{(k-2)^2}{(k-1)(k-3)}\sigma_e^2+\frac{k^3-5k^2+7k-4}{(k-1)\ (k-3)}\left[C_{ab'c'd'}+C_{a'bc'd'}-C_{a'b'cd'}+C_{a'b'c'd}\right]$$

$$+\frac{(k-4)^2(k^2-6)+12(k-3)^2}{(k-1)\ (k-3)}\ C_{a'b'c'd'}$$

$$ESS(A)=E\left(\frac{\sum_{i\neq p}Y_{i\cdot\cdot\cdot}^2+Y_{p\cdot\cdot\cdot}^2}{k}-\frac{Y_{\cdot\cdot\cdot\cdot}^2}{k^2}\right)=k(k-1)\sigma_a^2-k(k-1)C_a+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^2$$

$$+\left[(k-1)^2-\frac{1}{(k-3)}\right]C_{ab'c'd'}-\frac{[1+(k-1)(k-3)]}{(k-3)}\left[C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]$$

$$+\frac{3-(k-3)^2(k-1)}{(k-3)}\ C_{a'b'c'd'}$$

Similarly,

$$ESS(B)=E\left(\frac{\sum_{j\neq q}Y_{\cdot j\cdot\cdot}^2+Y_{\cdot q\cdot\cdot}^2}{k}-\frac{Y_{\cdot\cdot\cdot\cdot}^2}{k^2}\right)=k(k-1)\sigma_b^2-k(k-1)C_b+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^2$$

$$+\left[(k-1)^2-\frac{1}{(k-3)}\right]C_{a'bc'd'}-\frac{[1+(k-1)(k-3)]}{(k-3)}\left[C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]$$

$$+\frac{3-(k-3)^2(k-1)}{(k-3)}\ C_{a'b'c'd'}$$

$$ESS(C)=E\left(\frac{\sum_{h\neq r}Y_{\cdot\cdot h\cdot}^2+Y_{\cdot\cdot r\cdot}^2}{k}-\frac{Y_{\cdot\cdot\cdot\cdot}^2}{k^2}\right)=k(k-1)\sigma_c^2-k(k-1)C_c+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^2$$

$$+\left[(k-1)^2-\frac{1}{(k-3)}\right]C_{a'b'cd'}-\frac{[1+(k-1)(k-3)]}{(k-3)}\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd}\right]$$

$$+\frac{3-(k-3)^2(k-1)}{(k-3)}\ C_{a'b'c'd'}$$

$$ESS(D)=E\left(\frac{\sum_{l\neq s}Y_{\cdot\cdot\cdot l}^2+Y_{\cdot\cdot\cdot s}^2}{k}-\frac{Y_{\cdot\cdot\cdot\cdot}^2}{k^2}\right)=k(k-1)\sigma_d^2-k(k-1)C_d+\left[(k-1)+\frac{1}{(k-3)}\right]\sigma_e^2$$

$$\left[(k-1)^2-\frac{1}{(k-3)}\right]C_{a'b'c'd}$$

$$-\frac{[1+(k-1)(k-3)]}{(k-3)}\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}\right]+\frac{3-(k-3)^2(k-1)}{(k-3)}\ C_{a'b'c'd'}$$

For the error sum of squares,

$$Y_{ijhl} = \mu + a_i + b_j + c_h + d_l + e_{ijhl}$$

$$Y_{ijhl}^2 = \mu^2 + a_i^2 + b_j^2 + c_h^2 + d_l^2 + e_{ijhl}^2 + \text{cross product terms}$$

$$E\left(\sum_{(i\neq p}\ \sum_{j\neq o}\ \sum_{h\neq r}\ \sum_{l\neq s)} Y_{ijhl}^2\right) = (k^2-1)\left[\mu^2+\sigma_a^2+\sigma_b^2+\sigma_c^2+\sigma_d^2+\sigma_e^2\right]$$

For the missing value X_{pors},

$$X_{pors} = \mu + a_p + b_o + c_r + d_s + \frac{(k-3)\left[\sum_{(jhl)} e_{pjhl} + \sum_{(ihl)} e_{iohl} + \sum_{(ijl)} e_{ijrl} + \sum_{(ijh)} e_{ijhs}\right]}{(k-1)\ (k-3)}$$

$$\frac{-3 \sum_{i\neq p}\ \sum_{j\neq o}\ \sum_{h\neq r}\ \sum_{l\neq s} e_{ijhl}}{(k-1)(k-3)}$$

$$X_{pors}^2 = \mu^2 + a_p^2 + b_o^2 + c_r^2 + d_s^2 + \frac{(k-3)^2\left[\sum_{(jhl)} e_{pjhl}^2 + \sum_{\begin{pmatrix} j\neq j' \\ h\neq h' \\ l\neq l'\end{pmatrix}} e_{pjhl}\, e_{pj'h'l'}\right]}{(k-1)^2\ (k-3)^2}$$

$$\frac{+ (k-3)^2\left[\sum_{(ihl)} e_{iohl}^2 + \sum_{\begin{pmatrix} i\neq i' \\ h\neq h' \\ l\neq l'\end{pmatrix}} e_{iohl}\, e_{i'oh'l'}\right] + (k-3)^2\left[\sum_{(ijl)} e_{ijrl}^2 + \sum_{\begin{pmatrix} i\neq i' \\ j\neq j' \\ l\neq l'\end{pmatrix}} e_{ijrl}\, e_{i'j'rl'}\right]}{(k-1)^2\quad (k-3)^2}$$

$$\frac{+ (k-3)^2\left[\sum_{(ijh)} e_{ijhs}^2 + \sum_{\begin{pmatrix} i\neq i' \\ j\neq j' \\ h\neq h'\end{pmatrix}} e_{ijhs}\, e_{i'j'h's}\right] + 9\left[\sum_{i\neq p}\ \sum_{j\neq o}\ \sum_{h\neq r}\ \sum_{l\neq s} e_{ijhl}^2 + \sum_{i\neq p}\ \sum_{j\neq o\neq j'}\right.}{(k-1)^2 (k-3)^2}$$

$$\frac{\sum_{h\neq r\neq h'}\ \sum_{l\neq s\neq l'} e_{ijhl}\, e_{ij'h'l'} + \sum_{i\neq p\neq i'}\ \sum_{j\neq o}\ \sum_{h\neq r\neq h'}\ \sum_{l\neq s\neq l'} e_{ijhl}\, e_{i'jh'l'} + \sum_{i\neq p\neq i'}\ \sum_{j\neq o\neq j'}}{(k-1)^2\quad (k-3)^2}$$

$$\frac{\sum_{h\neq r}\ \sum_{l\neq s\neq l'} e_{ijhl}\, e_{i'j'hl'} + \sum_{i\neq p\neq i'}\ \sum_{j\neq o\neq j'}\ \sum_{h\neq r\neq h'}\ \sum_{l\neq s} e_{ijhl}\, e_{i'j'h'l} + \sum_{i\neq p\neq i'}\ \sum_{j\neq o\neq j'}}{(k-1)^2\quad (k-3)^2}$$

$$\frac{\left.\sum_{h\neq r\neq h'}\ \sum_{l\neq s\neq l'} e_{ijhl}\, e_{i'j'h'l'}\right] - 6(k-3)\left[\sum_{(jhl)} e_{pjhl} + \sum_{(ihl)} e_{iohl} + \sum_{(ijl)} e_{ijrl} + \sum_{(ijh)}\right.}{(k-1)^2\quad (k-3)^2}$$

$$\frac{e_{ijhs}\Big] \sum_{i\neq p}\sum_{j\neq o}\sum_{h\neq r}\sum_{l\neq s} e_{ijhl} + 2(k-3)^2\Big[\sum_{(jhl)} e_{pjhl} \sum_{(ihl)} e_{iohl} + \sum_{(jhl)} e_{pjhl} \sum_{(ijl)}}{(k-1)^2\ (k-3)^2}$$

$$\frac{e_{ijrl} + \sum_{(jhl)} e_{pjhl} \sum_{(ijh)} e_{ijhs} + \sum_{(ihl)} e_{iohl} \sum_{(ijl)} e_{ijrl} + \sum_{(ihl)} e_{iohl} \sum_{(ijh)} e_{pjhl} + \sum_{(ijl)}}{(k-1)^2\ (k-3)^2}$$

$$\underline{e_{ijrl} \sum_{(ijh)} e_{ijhs}\Big]} + \text{cross product terms}$$

$$E\left(X_{pors}{}^2\right) = \mu^2 + \sigma_a{}^2 + \sigma_b{}^2 + \sigma_c{}^2 + \sigma_d{}^2 + \frac{4(k-1)(k-3)^2+9(k-1)(k-3)}{(k-1)^2\ (k-3)^2}\sigma_e{}^2$$

$$+ \frac{(k-3)^2(k-1)(k-2)+9(k-1)(k-3)(k-4)-18(k-1)(k-3)^2+6(k-1)(k-3)^2}{(k-1)^2\ (k-3)^2}\Big[C_{ab'c'd'} +$$

$$C_{a'bc'd'} + C_{a'b'cd'} + C_{a'b'c'd}\Big]$$

$$+ \frac{9(k-1)(k-3)(k^2-8k+18)-24(k-1)(k-3)^2(k-4)+12(k-1)(k-3)^3}{(k-1)^2\ (k-3)^2}\ C_{a'b'c'd'}$$

$$= \mu^2 + \sigma_a{}^2 + \sigma_b{}^2 + \sigma_c{}^2 + \sigma_d{}^2 + \frac{4k-3}{(k-1)(k-3)}\sigma_e{}^2 + \frac{k^2-8k+6}{(k-1)(k-3)}\Big[C_{ab'c'd'} +$$

$$C_{a'bc'd'} + C_{a'b'cd'} + C_{a'b'c'd}\Big] - \frac{3k^2-24k+18}{(k-1)(k-3)}\ C_{a'b'c'd'}$$

$$E\left(\Big(\sum_{i\neq p}\sum_{j\neq o}\sum_{h\neq r}\sum_{l\neq s}\Big) Y_{ijhl}{}^2 + X_{pors}{}^2\right) = k^2\mu^2 + k^2\sigma_a{}^2 + k^2\sigma_b{}^2 + k^2\sigma_c{}^2 + k^2\sigma_d{}^2$$

$$\left[(k^2-1) + \frac{4k-3}{(k-1)(k-3)}\right]\sigma_e{}^2$$

$$+ \frac{k^2-8k+6}{(k-1)(k-3)}\Big[C_{ab'c'd'} + C_{a'bc'd'} + C_{a'b'cd'} + C_{a'b'c'd}\Big] - \frac{3k^2-24k+18}{(k-1)(k-3)}\ C_{a'b'c'd'}$$

$$\text{ESS(ERROR)} = E\Bigg[\Big(\sum_{i\neq p}\sum_{j\neq o}\sum_{h\neq r}\sum_{l\neq s}\Big) Y_{ijhl}{}^2 + X_{pors}{}^2 - \left(\frac{\sum_{i\neq p} Y_{i\cdots}{}^2 + Y_{p\cdots}{}^2}{k}\right) - \left(\frac{\sum_{j\neq o} Y_{\cdot j\cdot\cdot}{}^2 + Y_{\cdot o\cdot\cdot}{}^2}{k}\right) -$$

$$\left(\frac{\sum_{h\neq r} Y_{\cdot\cdot h\cdot}{}^2 + Y_{\cdot\cdot r\cdot}{}^2}{k}\right) - \left(\frac{\sum_{l\neq s} Y_{\cdots l}{}^2 + Y_{\cdots s}{}^2}{k}\right) + 3\,\frac{Y_{\cdots\cdot}{}^2}{k^2}\Bigg] = \Big[(k-1)(k-3)-1\Big]\sigma_e{}^2 - \Big[(k-1)(k-3)-1\Big]$$

$$\left[C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd}\right]+3\left[(k-1)(k-3)-1\right]C_{a'b'c'd'}$$

This completes the proof. In order to generalize to r restrictions, the following theorem is developed.

Theorem IV

The k X k experiment of random model with r restrictions and both a missing value and correlation has (r+2) components recognized in the model. The model is,

$$Y_{tabc--r} = \mu+t_t + r_{1a} + r_{2b} +----+ r_{rr} + e_{tabc--r}$$

$$\begin{cases} t = 1,\ 2,\ ----\ k \\ a = 1,\ 2,\ ----\ k \\ b = 1,\ 2,\ ----\ k \\ \\ r = 1,\ 2,\ ----\ k \end{cases}$$

$$\begin{cases} t_t \backsim N(o, \sigma_t) \\ \left.\begin{array}{l} r_{1a} \backsim N(o, \sigma_{r1}) \\ r_{2b} \backsim N(o, \sigma_{r2}) \\ \\ r_{rr} \backsim N(o, \sigma_{rr}) \end{array}\right\} \quad \text{these are r restrictions} \\ e_{tab--r} \backsim N(o, \sigma_e) \end{cases}$$

Assume the missing value occurs in t^{th} level of T where $1 \leq t \leq k$ and 1st level of R, 2nd level of R_2 ---- etc.

Then the ESS for each is,

$$ESS(T)=k(k-1)\sigma_t^2-k(k-1)C_t+\left[(k-1)+\frac{1}{(k-r)}\right]\sigma_e^2+\left[(k-1)^2-\frac{1}{(k-r)}\right]C_{\pi(t)}$$

$$-\frac{\left[1+(k-1)(k-r)\right]}{(k-r)}\sum_{i\neq t} C_{\pi(i)}+(-1)^r\frac{(k-r)^2(k-1)-r}{(k-r)}C_{t'a'b'--r'}\quad (i = t,\ a,\ b;--r)$$

$$ESS(R_J)=k(k-1)\sigma_{rJ}^{2}-k(k-1)C_{rJ}+\left[(k-1)+\frac{1}{(k-r)}\right]\sigma_e^{2}+\left[(k-1)^2-\frac{1}{(k-r)}\right]C_{\pi(j)}$$

$$-\frac{[1+(k-1)(k-r)]}{(k-r)}\sum_{i\neq j}C_{\pi(j)}+(-1)^r\frac{(k-r)^2(k-1)-r}{(k-r)}C_{t'a'b'--r'}$$

$$\begin{cases} J = 1, 2, ---- r \\ j = a, b, ---- r \end{cases}$$

$$ESS(ERROR)= (k-1)(k-r)-1 \quad {}_e^{2}- (k-1)(k-r)-1 \; \Sigma \; C_{\pi} + r \; (k-1)(k-r)-1$$

$$C_{t'a'b'--r'}$$

The proof of this theorem involves only lengthy algebraical manipulation and cumbersome notations, thus will be omitted here. However, the reader can verify it by following exactly the same way of those of Theorem II and III.

CHAPTER IV

RANDOMIZED BLOCK DESIGN

Randomized Block Design differs from the k X k experiment with one restriction in that its number of blocks is not necessarily equal to its number of treatments. Graphically, it is still on a two-dimension plane with only one modification that it is on a rectangular plane instead of on a square plane. This modification results in a slight change in the error structure of analysis of variance which will be examined in detail in this chapter.

This chapter is divided into four sections. The first three sections, one for each of the random, fixed and mixed model, give the expected mean squares (symbolically denoted as EMS) for each of the following cases: (1) without correlation and without missing value, (2) without correlation but with a missing value, (3) with correlation but without a missing value, and (4) with correlation and with a missing value. The fourth section is to examine the analysis of variance for exact test of significance of a Randomized Block Design with a missing value.

Most of the notations used below are the same as those in the previous chapters. Let the model be

$$Y_{ij} = \mu + a_i + b_j + e_{ij} \qquad \text{where} \begin{cases} i = 1, 2, \text{-----} a \\ j = 1, 2, \text{----} b \end{cases}$$

a_i is treatment effect

b_j is block effect

e_{ij} is random element associated with i^{th} treatment and j^{th} block

The missing data formula which we can derive from Theorem I by considering block as the only restriction has been given by almost all the standard textbook of Statistics as follows:

$$X = \frac{bY'_{.j} + aY'_{i.} - Y'_{..}}{(b-1)(a-1)}$$ where assuming the value in i^{th} treatment and j^{th} block is missing.

(A) Random Model

The model is $Y_{ij} = \mu + a_i + b_j + e_{ij}$ where $\begin{cases} i = 1, 2, ---- a \\ j = 1, 2, ---- b \end{cases}$ and μ is fixed, a_i's and b_j's and e_{ij}'s are random variables with the following distribution:

$a_i \backsim N(o, \sigma_a)$

$b_j \backsim N(o, \sigma_b)$

$e_{ij} \backsim N(o, \sigma_e)$

Without correlation and without missing value. The EMS for each of the three components recognized in the model are:

EMS (Blocks) $= a\sigma_b^2 + \sigma_e^2$

EMS (Treatments) $= b\sigma_a^2 + \sigma_e^2$

EMS (Error) $= \sigma_e^2$

The analysis of variance (from here on denoted as ANOVA) is then as shown in Table 4.1.1.

Table 4.1.1

Due To	df	EMS
Blocks	b-1	$a\sigma_b^2 + \sigma_e^2$
Treatments	a-1	$b\sigma_a^2 + \sigma_e^2$
Error	(a-1)(b-1)	σ_e^2
Total	ab-1	

Proof:

This has been given by most of the standard textbooks of Statistics. Here we just cite in order to be comparable.

Wtihout correlation but with a missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = a\sigma_b^2 + \left[1 + \frac{1}{(b-1)(a-1)}\right]\sigma_e^2$$

$$\text{EMS (Treatments)} = b\sigma_a^2 + \left[1 + \frac{1}{(b-1)(a-1)}\right]\sigma_e^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 4.1.2.

Table 4.1.2

Due To	df	EMS
Blocks	b-1	$a\sigma_b^2 + \left[1 + \frac{1}{(b-1)(a-1)}\right]\sigma_e^2$
Treatments	a-1	$b\sigma_a^2 + \left[1 + \frac{1}{(b-1)(a-1)}\right]\sigma_e^2$
Error	(a-1)(b-1)-1	σ_e^2
Total	ab-2	

Proof:

See Theorem II. Since the derivation is almost the same as that of Theorem II, the detailed proof is omitted.

With correlation but without missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = a(\sigma_b^2 - C_b) + \sigma_e^2 + (a-1)(C_{a'b} - C_{a'b'}) - C_{ab'}$$

$$\text{EMS (Treatments)} = b(\sigma_a^2 - C_a) + \sigma_e^2 - C_{a'b} + (b-1)(C_{ab'} - C_{a'b'})$$

$$\text{EMS (Error)} = \sigma_e^2 - C_{ab'} - C_{a'b} + C_{a'b'}$$

The ANOVA is then as shown in Table 4.1.3.

Table 4.1.3

Due To	df	EMS
Blocks	b-1	$a(\sigma_b^2-C_b)+\sigma_e^2+(a-1)(C_{a'b}-C_{a'b'})$ $-C_{ab'}$
Treatments	a-1	$b(\sigma_a^2-C_a)+\sigma_e^2-C_{a'b}+(b-1)(C_{ab'}-C_{a'b'})$
Error	(a-1)(b-1)	$\sigma_e^2-C_{ab'}-C_{a'b}+C_{a'b'}$
Total	ab-1	

Proof:

See Theorem III.

With correlation and with a missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS(Blocks)} = a(\sigma_b{}^2 - C_b) + \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e{}^2 - \left[1 - \frac{1}{(a-1)(b-1)}\right]$$

$$C_{ab'} + \frac{(a-1)^2(b-2)+a(a-2)}{(a-1)\ (b-1)}\, C_{a'b} - \left[(a-1) - \frac{1}{(a-1)(b-1)}\right] C_{a'b'}$$

$$\text{EMS(Treatments)} = b(\sigma_a{}^2 - C_a) + \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e{}^2 + \frac{(b-1)^2(a-2)+b(b-2)}{(a-1)\ (b-1)}$$

$$C_{ab'} - \left[1 - \frac{1}{(a-1)(b-1)}\right] C_{a'b} - \left[(b-1) - \frac{1}{(a-1)(b-1)}\right] C_{a'b'}$$

$$\text{EMS(Error)} = \sigma_e{}^2 - C_{ab'} - C_{a'b} + C_{a'b'}$$

Proof:

See Theorem IV.

The ANOVA is then as shown in Table 4.1.4.

Table 4.1.4

Due To	df	EMS
Blocks	b-1	$a(\sigma_b{}^2 - C_b) + \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e{}^2$ $\left[1 - \frac{1}{(a-1)(b-1)}\right] C_{ab'} + \frac{(a-1)^2(b-2)+a(a-2)}{(a-1)\ (b-1)}$ $C_{a'b} - \left[(a-1) - \frac{1}{(a-1)(b-1)}\right] C_{a'b'}$
Treatments	a-1	$b(\sigma_a{}^2 - C_a) + \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e{}^2 +$ $\frac{(b-1)^2(a-2)+b(b-2}{(a-1)\ (b-1)}\, C_{ab'} - \left[1 - \frac{1}{(a-1)(b-1)}\right]$ $C_{a'b} - \left[(b-1) - \frac{1}{(a-1)(b-1)}\right] C_{a'b'}$
Error	(a-1)(b-1)-1	$\sigma_e{}^2 - C_{ab'} - C_{a'b} + C_{a'b'}$
Total	ab-2	

This completes the presentation of all cases for the random model. There are many other cases that might be of interest for specific reasons which we are not able to include in this paper. In general, one finds that random model is rarely met in practice. Instead, most the models occurring are fixed or mixed models.

(B) Fixed Model

As it has been mentioned before the fixed model is a special case of the random model. In the derivation, the fixed factors are treated as constants. The only factor not treated as fixed is the random element which is always assumed to be a random variable. Our model is now,

$$Y_{ij} = \mu + a_i + b_j + e_{ij} \qquad \text{where} \quad \begin{cases} i = 1, 2, ---- a \\ j = 1, 2, ---- b \end{cases}$$

$$\underset{i}{E}\, a_i = o, \ \sum_j b_j = o, \text{ and } e_{ij} \qquad N(o, \quad _e)$$

Without correlation and without missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = \sigma_e^2 + \frac{a \sum_j b_j^2}{b-1}$$

$$\text{EMS (Treatments)} = \sigma_e^2 + \frac{b \sum_i a_i^2}{a-1}$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 4.2.1.

Table 4.2.1

Due To	df	EMS
Blocks	b-1	$\sigma_e^2 + \frac{a \sum_j b_j^2}{b-1}$
Treatments	a-1	$\sigma_e^2 + \frac{b \sum_i a_i^2}{a-1}$
Error	(a-1)(b-1)	σ_e^2
Total	ab-1	

Proof:

$Y_{i.} = b\,\mu + ba_i + \sum_j e_{ij}$ (since $\sum_j b_j = o$)

$Y_{i.}^2 = b^2\mu^2 + b^2 a_i^2 + \sum_j e_{ij}^2$ + cross product terms

$$E\left(\frac{\sum_i Y_{i.}^2}{b}\right) = ab\,\mu^2 + b\sum_i a_i^2 + a\,\sigma_e^2$$

$Y_{..} = ab\,\mu^2 + \sum_i \sum_j e_{ij}$ $\left(\sum_i a_i = \sum_j b_j = o\right)$

$Y_{..}^2 = a^2 b^2 \mu^2 + \sum_i \sum_j e_{ij}^2$ + cross product terms

$$E\left(\frac{Y_{..}^2}{ab}\right) = ab\,\mu^2 + \sigma_e^2$$

$$E\left(\frac{\sum Y_{i.}^2}{b} - \frac{Y_{..}^2}{ab}\right) = (a-1)\sigma_e^2 + b\sum_i a_i^2$$

$$\text{EMS(Treatments)} = \sigma_e^2 + \frac{b\sum_i a_i^2}{a-1}$$

Likewise,

$$\text{EMS(Blocks)} = \sigma_e^2 + \frac{a\sum_j b_j^2}{b-1}$$

$$Y_{ij} = \mu + a_i + b_j + e_{ij}$$

$$Y_{ij}^2 = \mu^2 + a_i^2 + b_j^2 + e_{ij}^2 + \text{cross product terms}$$

$$E\ (\sum_i \sum_j Y_{ij}^2) = ab\,\mu^2 + b \sum_i a_i^2 + a \sum_j b_j^2 + ab\,\sigma_e^2$$

by subtraction,

$$\text{EMS (Error)} = \sigma_e^2$$

Notice that the derivation of the fixed model differs from that of the random model in only that the $E(a_i^2) = a_i^2 \neq \sigma_a^2$, and $E(b_j^2) = b_j^2 \neq \sigma_b^2$. Mathematically, the fixed model is just a linear transformation of random model. Therefore, from here on the proof of all of the fixed model will be omitted in order to save space.

Without correlation but with a missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{a \sum_j b_j^2}{b-1}$$

$$\text{EMS (Treatments)} = \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{b \sum_i a_i^2}{a-1}$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 4.2.2.

Table 4.2.2

Due To	df	EMS
Blocks	b-1	$\left[1+ \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{a \sum_j b_j^2}{b-1}$
Treatments	a-1	$\left[1+ \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{b \sum_i a_i^2}{a-1}$
Error	(a-1)(b-1)-1	σ_e^2
Total	ab-2	

Proof:

See random model.

With correlation but without missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = \left[\sigma_e^2 - C_{ab'}\right] + (a-1)\left[C_{a'b} - C_{a'b'}\right] + \frac{a \sum_j b_j^2}{(b-1)}$$

$$\text{EMS (Treatments)} = \left[\sigma_e^2 - C_{a'b}\right] + (b-1)\left[C_{ab'} - C_{a'b'}\right] + \frac{b \sum_i a_i^2}{a-1}$$

$$\text{EMS (Error)} = \sigma_e^2 - C_{a'b} - C_{ab'} + C_{a'b'}$$

The ANOVA is then as shown in Table 4.2.3.

Table 4.2.3

Due To	df	EMS
Blocks	b-1	$\left[\sigma_e^2 - C_{ab'}\right] + (a-1)\left[C_{a'b} - C_{a'b'}\right] + \frac{a \sum_j b_j^2}{(b-1)}$
Treatments	a-1	$\left[\sigma_e^2 - C_{a'b}\right] + (b-1)\left[C_{ab'} - C_{a'b'}\right] + \frac{b \sum_i a_i^2}{a-1}$
Error	(a-1)(b-1)	$\sigma_e^2 - C_{a'b} - C_{ab'} + C_{a'b'}$
Total	ab-1	

Proof:

See random model.

With correlation and with a missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 - \left[1 - \frac{1}{(a-1)(b-1)}\right] C_{ab'} + \frac{(a-1)^2(b-2)+a(a-2)}{(a-1)\ (b-1)}\ C_{a'b} - \left[(a-1) - \frac{1}{(a-1)(b-1)}\ C_{a'b'}\right] + \frac{a \sum_j b_j^2}{b-1}$$

$$\text{EMS (Treatments)} = \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{(b-1)^2(a-2)+b(b-2)}{(a-1)\ (b-1)}\ C_{ab'} - \left[1 - \frac{1}{(a-1)(b-1)}\right] C_{a'b} - \left[(b-1) - \frac{1}{(a-1)(b-1)}\right] C_{a'b'} + \frac{b \sum_i a_i^2}{a-1}$$

$$\text{EMS (Error)} = \sigma_e^2 - C_{a'b} - C_{ab'} + C_{a'b'}$$

The ANOVA is then as shown in Table 4.2.4.

Table 4.2.4

Due To	df	EMS
Blocks	b-1	$\left[1+ \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 - \left[1- \frac{1}{(a-1)(b-1)}\right] C_{ab'} + \frac{(a-1)^2(b-2)+a(a-2)}{(a-1)(b-1)}\ C_{a'b} - \left[(a-1) - \frac{1}{(a-1)(b-1)}\right] C_{a'b'} + \frac{a \sum_j b_j^2}{b-1}$
Treatments	a-1	$\left[1+ \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{(b-1)^2(a-2)+b(b-2)}{(a-1)\ (b-1)}\ C_{ab'} - \left[1- \frac{1}{(a-1)(b-1)}\right] C_{a'b} - \left[(b-1)- \frac{1}{(a-1)(b-1)}\right] C_{a'b'} + \frac{b \sum_i a_i^2}{a-1}$
Error	(a-1)(b-1)-1	$\sigma_e^2 - C_{a'b} - C_{ab'} + C_{a'b'}$
Total	ab-2	

(C) Mixed Model

Strictly speaking, all of the models are mixed models, i.e., μ is always fixed and the rest of components are random. For our purpose in the Randomized Block Design, only the mixed model in which treatments are selected random but blocks are fixed is considered.

Let the model be,

$$Y_{ij} = \mu + a_i + b_j + e_{ij} \qquad \text{where} \begin{cases} i = 1, 2, ---- a \\ j = 1, 2, ---- b \end{cases}$$

a_i (Treatment effect) $\backsim N(o, \sigma_a)$

b_j (Block effect) has the property that $\sum_j b_j = o$

e_{ij} (random element) $\backsim N(o, \sigma_e)$

Mathematically, the derivation of the mixed model is between those of the random model and the fixed model, and for this reason, a detailed proof will not be given.

Without correlation and without missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = \sigma_e^2 + \frac{a \sum_j b_j^2}{b-1}$$

$$\text{EMS (Treatments)} = \sigma_e^2 + b\, \sigma_a^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 4.3.1.

Table 4.3.1

Due To	df	EMS
Blocks	b-1	$\sigma_e^2 + \frac{a \sum_j b_j^2}{b-1}$
Treatments	a-1	$\sigma_e^2 + b\, \sigma_a^2$
Error	(a-1)(b-1)	σ_e^2
Total	ab-1	

Proof:

See random and fixed model.

Without correlation but with a missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{a \sum_j b_j^2}{b-1}$$

$$\text{EMS (Treatments)} = \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + b\,\sigma_a^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 4.3.2.

Table 4.3.2

Due To	df	EMS
Blocks	b-1	$\left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{a \sum_j b_j^2}{b-1}$
Treatments	a-1	$\left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + b\,\sigma_a^2$
Error	(a-1)(b-1)-1	σ_e^2
Total	ab-2	

Proof:

See random and fixed model.

With correlation but without missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = \sigma_e^2 - C_{ab'} + (a-1)(C_{a'b} - C_{a'b'}) + \frac{a \sum_j b_j^2}{b-1}$$

$$\text{EMS (Treatments)} = b(\sigma_a^2 - C_a) + \sigma_e^2 - C_{a'b} + (b-1)(C_{ab'} - C_{a'b'})$$

EMS (Error) = $\sigma_e^2 - C_{a'b} - C_{ab'} + C_{a'b'}$

The ANOVA is then as shown in Table 4.3.3.

Table 4.3.3

Due To	df	EMS
Blocks	b-1	$\sigma_e^2 - C_{ab'} + (a-1)(C_{a'b} - C_{a'b'}) + \frac{a \sum_j b_j^2}{b-1}$
Treatments	a-1	$b(\sigma_a^2 - C_a) + \sigma_e^2 - C_{a'b} + (b-1)(C_{ab'} - C_{a'b'})$
Error	(a-1)(b-1)	$\sigma_e^2 - C_{a'b} - C_{ab'} + C_{a'b'}$
Total	ab-1	

Proof:

See random and fixed model.

With correlation and with a missing value. The EMS for each of the three components recognized in the model are:

$$\text{EMS (Blocks)} = \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 - \left[1 - \frac{1}{(a-1)(b-1)}\right] C_{ab'} + \frac{(a-1)^2(b-2)+a(a-2)}{(a-1)\ (b-1)} C_{a'b} - \left[(a-1) - \frac{1}{(a-1)(b-1)}\right] C_{a'b'} + \frac{a \sum_j b_j^2}{b-1}$$

$$\text{EMS (Treatments)} = \left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{(b-1)^2(a-2)+b(b-2)}{(a-1)\ (b-1)} C_{ab'} - \left[1 - \frac{1}{(a-1)(b-1)}\right] C_{a'b} - \left[(b-1) - \frac{1}{(a-1)(b-1)}\right] C_{a'b'} + b\left[\sigma_a^2 - C_a\right]$$

$$\text{EMS (Error)} = \sigma_e^2 - C_{a'b} - C_{ab'} + C_{a'b'}$$

The ANOVA is then as shown in Table 4.3.4.

Table 4.3.4

Due To	df	EMS
Blocks	b-1	$\left[1+\frac{1}{(a-1)(b-1)}\right]\sigma_e^2 - \left[1-\frac{1}{(a-1)(b-1)}\right]C_{ab'} + \frac{(a-1)^2(b-2)+a(a-2)}{(a-1)\ (b-1)}C_{a'b} - \left[(a-1) - \frac{1}{(a-1)(b-1)}\right]C_{a'b'} + \frac{a\sum_j b_j^2}{b-1}$
Treatments	a-1	$\left[1+\frac{1}{(a-1)(b-1)}\right]\sigma_e^2 + \frac{(b-1)^2(a-2)+b(b-2)}{(a-1)\ (b-1)}C_{ab'} - \left[1-\frac{1}{(a-1)(b-1)}\right]C_{a'b} - \left[(b-1) - \frac{1}{(a-1)(b-1)}\right]C_{a'b'} + b\left[\sigma_a^2 - C_a\right]$
Error	(a-1)(b-1)-1	$\sigma_e^2 - C_{a'b} - C_{ab'} + C_{a'b'}$
Total	ab-2	

Proof:

See random and fixed model.

(D) Exact Test

Table 4.1.2, 4.2.2, and 4.3.2 have given the analysis of variance of the Randomized Block Design without correlation but with a missing value. An approximate test of significance of the null hypothesis that the treatments have no differential effects may be obtained by analyzing those tables (augumented table) in the usual way, with the modification that the degrees of freedom for the error sum of squares is diminished by one. This test can be shown to be biased in that the expectation of the treatment mean square (EMS treatment) is greater than the expectation of the error mean square under the null hypothesis. Now take Table 4.1.2 as an illustration. Assume the missing value occurs in k^{th} treatment and 1^{th} block.

Due To	df	SS	EMS	F
Blocks	b-1	$\frac{\sum_{j\neq l} Y_{ij}^2}{a} + \frac{(Y_{.l}'+X)^2}{a} -$ $\frac{(Y_{..}'+X)^2}{ab}$		
Treatments	a-1	$\frac{\sum_{i\neq k} Y_{i.}^2}{b} + \frac{(Y_{k.}'+X)^2}{b} -$ $\frac{(Y_{..}'+X)^2}{ab}$	$b\sigma_a^2 + [1+$ $\frac{1}{(a-1)(b-1)}]\sigma_e^2$	$F=\frac{\cancel{b}\sigma_a^2+[1+}{1^2}$ $\frac{\frac{1}{(a-1)(b-1)}]\sigma_e^2}{\sigma_e^2}$
Error	(a-1)(b-1)-1	by subtraction	σ_e^2	
Total	ab-2	$\sum_{(i\neq k}\sum_{j\neq l)} Y_{ij}^2 + X^2 -$	$\frac{(Y_{..}'+X)^2}{ab}$	

From the above Table we can easily see that if the null hypothesis is true, i.e., $\sigma_a^2 = o$, then $F = \frac{\left[1 + \frac{1}{(a-1)(b-1)}\right]\sigma_e^2}{\sigma_e^2}$ which is greater than one.

However, if the approximate test of significance indicates that there are no significant treatment difference, there is no need to perform the accurate test of significance.

The accurate test of significance in the above case is made by the analysis of variance given in Table 4.4.1.

Table 4.4.1

Due To	df	SS	EMS	F
Blocks	b-1	$\frac{\sum_{j\neq 1} Y_{.j}^2}{a} + \frac{Y_{.1}'^2}{(a-1)} - \frac{Y_{..}'^2}{(ab-1)}$		
Treatments	a-1	by subtraction	$\frac{ab-b-1}{a-1}\sigma_a^2+\sigma_e^2$	$F=\frac{\frac{ab-b-1}{a-1}\sigma_a^2+\sigma_e^2}{\sigma_e^2}$
Error	(a-1)(b-1)-1	As in the ANOVA of the augumented table	σ_e^2	
Total	ab-2	$(\sum_{i\neq k} \sum_{j\neq 1} Y_{ij}^2 - \frac{Y_{..}'^2}{(ab-1)}$		

Proof:

For other blocks where missing value does not occur,

$$Y_{.j} = \sum_i Y_{ij} = \sum_i (\mu + a_i + b_j + e_{ij}) = a\,\mu + \sum_i a_i + ab_j + \sum_i e_{ij}$$

$$Y_{.j}^2 = a^2\mu^2 + \sum_i a_i^2 + a^2 b_j^2 + \sum_i e_{ij}^2 + \text{cross product terms}$$

$$\sum_{j\neq 1} Y_{.j}^2 = a^2(b-1)\,\mu^2 + (b-1)\sum_i a_i^2 + a^2 \sum_{j\neq 1} b_j^2 + \sum_i \sum_{j\neq 1} e_{ij}^2 + \text{cross product terms}$$

$$E\left(\frac{\sum_{j\neq 1} Y_{.j}^2}{a}\right) = a(b-1)\,\mu^2 + (b-1)\sigma_a^2 + a(b-1)\sigma_b^2 + (b-1)\sigma_e^2$$

For the 1^{th} block,

$$Y_{.1}' = \sum_{i\neq k} Y_{i1} = \sum_{i\neq k} (\mu + a_i + b_1 + e_{i1}) = (a-1)\,\mu + \sum_{i\neq k} a_i + (a-1)b_1 + \sum_{i\neq k} e_{ij}$$

$$Y_{.1}'^{2} = (a-1)^2\mu^2 + \sum_{i\neq k} a_i^2 + (a-1)^2 b_1^2 + \sum_{i\neq k} e_{ij}^2 + \text{cross product terms}$$

$$E\left(\frac{Y'_{.1}}{a-1}\right) = (a-1)\,\mu^2 + \sigma_a^2 + (a-1)\,\sigma_b^2 + \sigma_e^2$$

$$E\left(\frac{\sum_{j\neq 1} Y_{.j}^2}{a} + \frac{Y_{.1}'^{2}}{a-1}\right) = (ab-1)\,\mu^2 + b\,\sigma_a^2 + (ab-1)\,\sigma_b^2 + b\,\sigma_e^2$$

$$Y_{ij} = \mu + a_i + b_j + e_{ij}$$

$$Y_{ij}^2 = \mu^2 + a_i^2 + b_j^2 + e_{ij}^2 + \text{cross product terms}$$

$$E\left[\sum_{(i\neq k}\ \sum_{j\neq 1)} Y_{ij}^2\right] = (ab-1)\,(\mu^2 + \sigma_a^2 + \sigma_b^2 + \sigma_e^2)$$

$$\text{ESS (Treatments)} = E\left[\sum_{(i\neq k}\ \sum_{j\neq 1)} Y_{ij}^2 - \left(\frac{\sum_{j\neq 1} Y_{.j}^2}{a} + \frac{Y_{.1}'^{2}}{a-1}\right)\right] - \left[(a-1)(b-1)-1\right]\sigma_e^2$$

$$= (ab-b-1)\,\sigma_a^2 + (ab-1-b-ab+a+b)\,\sigma_e^2$$

$$= (ab-b-1)\,\sigma_a^2 + (a-1)\,\sigma_e^2$$

$$\text{EMS (Treatments)} = \frac{ab-b-1}{a-1}\,\sigma_a^2 + \sigma_e^2$$

This completes the proof. If the null hypothesis is true, i.e., $\sigma_a^2 = 0$, then F = 1. This test is accurate on the basis of normal law theory.

CHAPTER V

LATIN SQUARE DESIGN

Latin Square Design may be regarded as an arrangement in which treatment are randomized under two restrictions, namely, that each treatment occurs in each row and once and only once in each column. Such arrangement has fairly obvious properties, the important one being that any comparison of treatments is unaffected by average difference which exists between the rows or between columns. Such differences will not affect the errors of treatment comparison so that such arrangement is likely to lead to greatly increased precision. The Latin Square Design differs from the Randomized Block Design in not only that one more restriction is imposed, but also in that the number of treatments, rows, and columns are all equal. Therefore, one can easily see that the Latin Square Design is nothing but a k X k experiment with two restrictions.

The content of this chapter is constructed the same pattern as that of the last chapter. The ANOVA of each model for different cases will be examined. Also, many trival proofs are omitted owing to the fact that only linear transformation is involved.

Let the model be,

$$Y_{ijh} = u + a_i + b_j + c_h + e_{ijh} \qquad \text{where} \begin{cases} i = 1, 2, ---- k \\ j = 1, 2, ---- k \\ h = 1, 2, ---- k \end{cases}$$

a_i is treatment effect

b_j is row effect

c_h is column effect

e_{ijh} is random element associated with i^{th} treatment, j^{th} row, and h^{th} column.

The missing data formula which we can derive from Theorem I by considering row and column as two restrictions has also been given by almost all the standard textbooks of Statistics as follows:

$$X = \frac{k(Y_{i..}' + Y_{.j.}' + Y_{..h}') - 2Y_{...}'}{(k-1)(k-2)}$$

where assuming the value in i^{th} treatment, j^{th} row and h^{th} column is missing.

(A) Random Model

The model is $Y_{ijh} = \mu + a_i + b_j + c_h + e_{ijh}$ where $\begin{cases} i = 1, 2, ---- k \\ j = 1, 2, ---- k \\ h = 1, 2, ---- k \end{cases}$

and μ is fixed a_i's, b_j's, c_h's, and e_{ijh}'s are random variables with the following distributions:

$a_i \backsim N(o, \sigma_a)$

$b_j \backsim N(o, \sigma_b)$

$c_h \backsim N(o, \sigma_c)$

$e_{ijh} \backsim N(o, \sigma_e)$

Without correlation and without missing value. The EMS for each of the four components recognized in the model are:

EMS (Rows) $= k\sigma_b^2 + \sigma_e^2$

EMS (Columns) $= k\sigma_c^2 + \sigma_e^2$

EMS (Treatments) $= k\sigma_a^2 + \sigma_e^2$

EMS (Errors) $= \sigma_e^2$

The ANOVA is then as shown in Table 5.1.1.

Table 5.1.1

Due To	df	EMS
Rows	k-1	$k\sigma_b^2 + \sigma_e^2$
Columns	k-1	$k\sigma_c^2 + \sigma_e^2$
Treatments	k-1	$k\sigma_a^2 + \sigma_e^2$
Error	(k-1)(k-2)	σ_e^2
Total	k^2-1	

Proof:

See any standard textbook of Statistics.

Without correlation but with a missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_b^2 + \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2$$

$$\text{EMS (Columns)} = \sigma_c^2 + \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2$$

$$\text{EMS (Treatments)} = \sigma_a^2 + \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 5.1.2.

Table 5.1.2

Due To	df	EMS
Rows	k-1	$\sigma_b^2 + \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2$
Columns	k-1	$\sigma_c^2 + \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2$
Treatments	k-1	$\sigma_a^2 + \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2$
Error	(k-1)(k-2)-1	σ_e^2
Total	k^2-2	

Proof:

See Theorem II.

With correlation but without missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = k(\sigma_b^2 - C_b) + \sigma_e^2 + (k-1)\, C_{a'bc'} - (C_{ab'c'} + C_{a'b'c}) - (k-2)\, C_{a'b'c'}$$

$$\text{EMS (Columns)} = k(\sigma_c^2 - C_c) + \sigma_e^2 + (k-1)\, C_{a'b'c} - (C_{ab'c'} + C_{a'bc'}) - (k-2)\, C_{a'b'c'}$$

$$\text{EMS (Treatments)} = k(\sigma_a^2 - C_a) + \sigma_e^2 + (k-1)\, C_{ab'c'} - (C_{a'bc'} - C_{a'b'c}) - (k-2)\, C_{a'b'c'}$$

$$\text{EMS (Error)} = \sigma_e^2 - (C_{ab'c'} + C_{a'bc'} + C_{a'b'c}) + 2C_{a'b'c'}$$

The ANOVA is then as shown in Table 5.1.3.

Table 5.1.3

Due To	df	EMS
Rows	k-1	$k(\sigma_b^2-C_b)+\sigma_e^2+(k-1)C_{a'bc'}-(C_{ab'c'}+C_{a'b'c})+(k-2)C_{a'b'c'}$
Columns	k-1	$k(\sigma_c^2-C_c)+\sigma_e^2+(k-1)C_{a'b'c}-(C_{ab'c'}+C_{a'bc'})-(k-2)C_{a'b'c'}$
Treatments	k-1	$k(\sigma_a^2-C_a)+\sigma_e^2+(k-1)C_{ab'c'}-(C_{a'bc'}+C_{a'b'c})-(k-2)C_{a'b'c'}$
Error	(k-1)(k-2)	$\sigma_e^2-(C_{ab'c'}+C_{a'bc'}-C_{a'b'c})+2C_{a'b'c'}$
Total	k^2-1	

Proof:

See Theorem III.

With correlation and with a missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = k(\sigma_b^2-C_b) + \left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'bc'} - \left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'b'c}) + \left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}$$

$$\text{EMS (Columns)} = k(\sigma_c^2-C_c) + \left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'b'c} - \left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'b'c}) + \left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}$$

EMS (Treatments) = $k(\sigma_a^2-C_a)+\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]$

$C_{ab'c'}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{a'bc'}+C_{a'b'c})+\left[(k-2)-\right.$

$\left.\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}$

EMS (Error) = $\sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$

The ANOVA is then as shown in Table 5.1.4.

Table 5.1.4

Due To	df	EMS
Rows	$k-1$	$k(\sigma_b^2-C_b)+\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'bc'}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'b'c}+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}$
Columns	$k-1$	$k(\sigma_c^2-C_c)+\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'b'c}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'bc'})+\left[(k-2)-\frac{1}{(k-1)(k-2)}\right]C_{a'b'c'}$
Treatments	$k-1$	$k(\sigma_a^2-C_a)+\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{ab'c'}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{a'bc'}+C_{a'b'c})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}$
Error	$(k-1)(k-2)-1$	$\sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$
Total	k^2-2	

Proof:

See Theorem IV.

(B) Fixed Model

Let the model be,

$$Y_{ijh} = \mu + a_i + b_j + c_h + e_{ijh} \quad \text{where} \left\{ \begin{array}{ll} i = 1, 2, ---- k & \\ j = 1, 2, ---- k & \text{with properties} \\ h = 1, 2, ---- k & \text{that} \end{array} \right.$$

$\sum_i a_i = \sum_j b_j = \sum_h c_h = o$ and $e_{ijh} \backsim N(o, \sigma_e)$

Without correlation and without missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns} = \sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$$

$$\text{EMS (Treatments)} = \sigma_e^2 + \frac{k \sum_i a_i^2}{k-1}$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 5.2.1.

Table 5.2.1

Due To	df	EMS
Rows	k-1	$\sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$
Columns	k-1	$\sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$
Treatments	k-1	$\sigma_e^2 + \frac{k \sum_i a_i^2}{k-1}$
Error	(k-1)(k-2)	σ_e^2
Total	k^2-1	

Proof:

See random model.

Without correlation but with a missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k\Sigma_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k\Sigma_h c_h^2}{k-1}$$

$$\text{EMS (Treatments)} = \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k\Sigma_i a_i^2}{k-1}$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 5.2.2.

Table 5.2.2

Due To	df	EMS
Rows	k-1	$\left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k\Sigma_j b_j^2}{k-1}$
Columns	k-1	$\left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k\Sigma_h c_h^2}{k-1}$
Treatments	k-1	$\left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k\Sigma_i a_i^2}{k-1}$
Error	(k-1)(k-2)-1	σ_e^2
Total	k^2-2	

Proof:

See random model.

With correlation but without missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_e^2+(k-1)C_{a'bc'}-(C_{ab'c'}+C_{a'b'c})-(k-2)C_{a'b'c'}+\frac{k\Sigma_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \sigma_e^2+(k-1)C_{a'b'c}-(C_{ab'c'}+C_{a'bc'})-(k-2)C_{a'b'c'}+\frac{k\Sigma_h c_h^2}{k-1}$$

$$\text{EMS (Treatments)} = \sigma_e^2+(k-1)C_{ab'c'}-(C_{a'bc'}+C_{a'b'c})-(k-2)C_{a'b'c'}+\frac{k\Sigma_i a_i^2}{k-1}$$

$$\text{EMS (Error)} = \sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$$

The ANOVA is then as shown in Table 5.2.3.

Table 5.2.3

Due To	df	EMS
Rows	$k-1$	$\sigma_e^2+(k-1)C_{a'bc'}-(C_{ab'c'}+C_{a'b'c})-(k-2)C_{a'b'c'}+\frac{k\Sigma_j b_j^2}{k-1}$
Columns	$k-1$	$\sigma_e^2+(k-1)C_{a'b'c}-(C_{ab'c'}+C_{a'bc'})-(k-2)C_{a'b'c'}+\frac{k\Sigma_h c_h^2}{k-1}$
Treatments	$k-1$	$\sigma_e^2+(k-1)C_{ab'c'}-(C_{a'bc'}+C_{a'b'c})-(k-2)C_{a'b'c'}+\frac{k\Sigma_i a_i^2}{k-1}$
Error	$(k-1)(k-2)$	$\sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$
Total	k^2-1	

Proof:

See random model.

With correlation and with a missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \left[1+ \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \left[(k-1) - \frac{1}{(k-1)(k-2)}\right] C_{a'bc'} - \left[1 - \frac{1}{(k-1)(k-2)}\right] (C_{ab'c'} + C_{a'b'c}) + \left[(k-2) - \frac{2}{(k-1)(k-2)}\right] C_{a'b'c'} + \frac{k \sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \left[1+ \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \left[(k-1) - \frac{1}{(k-1)(k-2)}\right] C_{a'b'c} - \left[1 - \frac{1}{(k-1)(k-2)}\right] (C_{ab'c'} + C_{a'bc'}) + \left[(k-2) - \frac{2}{(k-1)(k-2)}\right] C_{a'b'c'} + \frac{k \sum_h c_h^2}{k-1}$$

$$\text{EMS (Treatments)} = \left[1+ \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \left[(k-1) - \frac{1}{(k-1)(k-2)}\right] C_{ab'c'} \left[1 - \frac{1}{(k-1)(k-2)}\right] (C_{a'bc'} + C_{a'b'c}) + \left[(k-2) - \frac{2}{(k-1)(k-2)}\right] C_{a'b'c'} + \frac{k \sum_i a_i^2}{k-1}$$

$$\text{EMS (Error)} = \sigma_e^2 - (C_{ab'c'} + C_{a'bc'} + C_{a'b'c}) + 2C_{a'b'c'}$$

The ANOVA is then as shown in Table 5.2.4.

Proof:

See random model.

(C) Mixed Model

Let the model be,

$$Y_{ijh} = \mu + a_i + b_j + c_h + e_{ijh} \quad \text{where} \begin{cases} i = 1, 2, ---- k \\ j = 1, 2, ---- k \\ h = 1, 2, ---- k \end{cases}$$ with the following properties:

a_i (Treatment effect) $\backsim N(o, \sigma_a)$

b_j (Row effect) $\sum_j b_j = o$

Table 5.2.4

Due To	df	EMS
Rows	$k-1$	$\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'bc'}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'b'c})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}+\frac{k\sum_j b_j^2}{k-1}$
Columns	$k-1$	$\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'b'c}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'bc'})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}+\frac{k\sum_h c_h^2}{k-1}$
Treatments	$k-1$	$\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{ab'c'}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{a'bc'}+C_{a'b'c})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}+\frac{k\sum_i a_i^2}{k-1}$
Error	$(k-1)(k-2)-1$	$\sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$
Total	k^2-2	

c_h (Column effect) $\sum_h c_h = o$

e_{ijh} (Random effect) $\sim N(o, \sigma_e)$

Without correlation and without missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$$

$$\text{EMS (Treatments)} = \sigma_e^2 + k\,\sigma_a^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 5.3.1.

Table 5.3.1

Due To	df	EMS
Rows	k-1	$\sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$
Columns	k-1	$\sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$
Treatments	k-1	$\sigma_e^2 + k\,\sigma_a^2$
Error	(k-1)(k-2)	σ_e^2
Total	k^2-1	

Proof:

See random and fixed model.

Without correlation but with a missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$$

$$\text{EMS (Treatments)} = \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + k\sigma_a^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 5.3.2.

Table 5.3.2

Due To	df	EMS
Rows	k-1	$\left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$
Columns	k-1	$\left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$
Treatments	k-1	$\left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2 + k\,\sigma_a^2$
Error	(k-1)(k-2)-1	σ_e^2
Total	k^2-2	

Proof:

See random and fixed model.

With correlation but without missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_e^2 + (k-1)C_{a'bc'} - (C_{ab'c'} + C_{a'b'c}) - (k-2)C_{a'b'c'} + \frac{k \sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \sigma_e^2+(k-1)C_{a'b'c}-(C_{ab'c'}+C_{a'bc'})-(k-2)C_{a'b'c'} + \frac{k\sum_h c_h^2}{k-1}$$

$$\text{EMS (Treatments)} = k(\sigma_a^2-C_a)+ \sigma_e^2+(k-1)C_{ab'c'}-(C_{a'bc'}+C_{a'b'c})-(k-2)C_{a'b'c'}$$

$$\text{EMS (Error)} = \sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$$

The ANOVA is then as shown in Table 5.3.3.

Table 5.3.3

Due To	df	EMS
Rows	$k-1$	$\sigma_e^2 +(k-1)C_{a'bc'}-(C_{ab'c'}+C_{a'b'c})-(k-2)C_{a'b'c'}$ $+ \frac{k\sum_j b_j^2}{k-1}$
Columns	$k-1$	$\sigma_e^2+(k-1)C_{a'b'c}-(C_{ab'c'}+C_{a'bc'})-(k-2)C_{a'b'c'}$ $+ \frac{k\sum_h c_h^2}{k-1}$
Treatments	$k-1$	$k(\sigma_a^2-C_a)+\sigma_e^2+(k-1)C_{ab'c'}-(C_{a'bc'}+C_{a'b'c'})$ $-(k-2)C_{a'b'c'}$
Error	$(k-1)(k-2)$	$\sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$
Total	k^2-2	

Proof:

See random and fixed model.

With correlation and with a missing value. The EMS for each of the four components recognized in the model are:

$$\text{EMS (Rows)} = \left[1 + \frac{1}{(k-1)(k-2)}\right]\sigma_e^2+ \left[(k-1)- \frac{1}{(k-1)(k-2)}\right] C_{a'bc'}-\left[1- \frac{1}{(k-1)(k-2)}\right]$$
$$(C_{ab'c'}+C_{a'b'c})+ \left[(k-2)- \frac{2}{(k-1)(k-2)}\right] C_{a'b'c'}+ \frac{k\sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'b'c}-$$

$$\left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'bc'})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]$$

$$C_{a'b'c'}+\frac{k\sum_h c_h^2}{k-1}$$

$$\text{EMS (Treatments)} = \left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{ab'c'}-$$

$$\left[1-\frac{1}{(k-1)(k-2)}\right](C_{a'bc'}+C_{a'b'c})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]$$

$$C_{a'b'c'}+k(\sigma_a^2-C_a)$$

$$\text{EMS (Error)} = \sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$$

The ANOVA is then as shown in Table 5.3.4.

Proof:

See random and fixed model.

(D) Exact Test

We have seen in the Randomized Block Design that, with a missing value, an exact test can be performed by constructing an augumented table. For the Latin Square Design, the approximate test of significance may be performed by the usual way, but the F value will have an upward bias. Thus, if non-sigificance is found, we can stop. If F value is significant, however, we cannot be sure that it is due to treatments and not to this bias.

Exact tests of significance in terms of infinite model theory, although easy to describe in terms of general theory, are somewhat difficult to obtain. It is necessary to evaluate the sum of squares attributable to rows and columns ignoring treatments, and to rows, columns, and treatments.

Table 5.3.4

Due To	df	EMS
Rows	k-1	$\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'bc'}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'b'c})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}+\frac{k\Sigma_j b_j^2}{k-1}$
Columns	k-1	$\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{a'b'c}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{ab'c'}+C_{a'bc'})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}+\frac{k\Sigma_h c_h^2}{k-1}$
Treatments	k-1	$\left[1+\frac{1}{(k-1)(k-2)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-2)}\right]C_{ab'c'}-\left[1-\frac{1}{(k-1)(k-2)}\right](C_{a'bc'}+C_{a'b'c})+\left[(k-2)-\frac{2}{(k-1)(k-2)}\right]C_{a'b'c'}+K(\sigma_a^2-C_a)$
Error	(k-1)(k-2)-1	$\sigma_e^2-(C_{ab'c'}+C_{a'bc'}+C_{a'b'c})+2C_{a'b'c'}$
Total	k^2-2	

The difference of these two sum of squares may then be tested against the error with the reduced sum of squares. To compute the necessary quantities, the experiment is first regarded as an experiment in rows and columns with one observation missing and the minimum sum of squares for error obtained. This sum of squares, W, say, will have (k^2-2k) degrees of freedom. The minimum sum of squares for error, when treatments are taken into account, is obtained by analyzing the augumented table, E, say, with (k^2-3k+1) degrees of freedom. The quantity $(W-E)/(k-1)$ is the mean square for treatments, which is tested against $E/(k^2-3k+1)$ by the F test with $(k-1)$ and (k^2-3k+1) degrees of freedom.

Assuming, the missing value occurs in l^{th} treatment, m^{th} row and n^{th} column, then the accurate test of significance of a Randomized Block Design with a missing value is made by the ANOVA given in Table 5.4.1.

Proof:

$$E\left(\frac{\sum_{j\neq m} Y_{.j.}^2}{k}\right) = k(k-1)\,\mu^2+(k-1)\sigma_a^2+k(k-1)\,\sigma_b^2+(k-1)\,\sigma_c^2+(k-1)\,\sigma_e^2$$

$$E\left(\frac{Y'^2_{.m.}}{k-1}\right) = (k-1)\,\mu^2+\sigma_a^2+(k-1)\,\sigma_b^2+\sigma_c^2+\sigma_e^2$$

$$E\left(\frac{\sum_{j\neq m} Y_{.j.}^2}{k}+\frac{Y'^2_{.m.}}{k-1}\right) = (k^2-1)\,\mu^2+k\,\sigma_a^2+(k^2-1)\sigma_b^2+k\,\sigma_c^2+k\sigma_e^2$$

likewise,

$$E\left(\frac{\sum_{h\neq n} Y_{..h}^2}{k}+\frac{Y'^2_{..n}}{k-1}\right) = (k^2-1)\,\mu^2+k\,\sigma_a^2+k\sigma_b^2+(k^2-1)\,\sigma_c^2+k\sigma_e^2$$

$$Y'_{...} = k(k-1)\,\mu+k\sum_{i\neq l} a_i+(k-1)\sum_j b_j+(k-1)\sum_h c_h+\sum_{i\neq l}\sum_{(jh)} e_{ijh}+(k-1)\,\mu+(k-1)a_l$$

$$+\sum_{j\neq m} b_j+\sum_{h\neq n} c_h+\sum_{(jh)} e_{ljh}$$

Table 5.4.1

Due To	df	SS	EMS	F
Rows	k-1	$\frac{\sum_{j \neq m} Y_{.j.}^{2}}{k} + \frac{Y'^{2}_{.m.}}{k-1} - \frac{Y'^{2}_{...}}{k^2-1}$		
Columns	k-1	$\frac{\sum_{i \neq n} Y_{..h}^{2}}{k} + \frac{Y'^{2}_{..n}}{k-1} - \frac{Y'^{2}_{...}}{k^2-1}$		
Treatments	k-1	by subtraction	$\frac{k^3-2k-2}{k^2-1}\sigma_a^2 - \frac{1}{k^2-1}(\sigma_b^2+\sigma_c^2)$ $+ \sigma_e^2$	$F= \frac{\frac{k - 2k-2}{k^2-1}\sigma_a^2 - \frac{1}{k^2-1}}{\sigma_e^2}$ $\frac{(\sigma_b^2+\sigma_c^2,+ e^2}{\sigma_e^2}$
Error	(k-1)(k-2)-1	As in the ANOVA of the augumented table	σ_e^2	
Total	k^2-2			

$$E\left(\frac{Y'^{\,2}_{\cdots}}{k^2-1}\right) = (k^2-1)\,\mu^2 + \frac{(k-1)(k^2+k-1)}{k^2-1}\,(\sigma_a^2+\sigma_b^2+\sigma_c^2) + \sigma_e^2$$

$$E\left(\sum_{(i\neq l}\ \sum_{j\neq m}\ \sum_{h\neq n)} Y_{ijh}^{\;2}\right) = (k^2-1)\left[\mu^2 + \sigma_a^2 + \sigma_b^2 + \sigma_c^2 + \sigma_e^2\right]$$

$$W=E\left[\sum_{(i\neq l}\ \sum_{j\neq m}\ \sum_{h\neq n)} Y_{ijh}^{\;2} - \frac{\sum_{i\neq m} Y_{\cdot j \cdot}^{\;2}}{k} + \frac{Y'^{\,2}_{\cdot m \cdot}}{k-1} - \frac{\sum_{h\neq n} Y_{\cdot\cdot h}^{\;2}}{k} + \frac{Y'^{\,2}_{\cdot\cdot n}}{k-1} + \frac{Y'^{\,2}_{\cdots}}{k^2-1}\right]$$

$$= \frac{k^3-2k+2}{k+1}\,\sigma_a^2 - \frac{1}{k+1}\,(\sigma_b^2+\sigma_c^2) + (k^2-2k)\,\sigma_e^2$$

We already know

$$E = (k^2-3k+1)\ \sigma_e^2$$

$$W-E = \frac{k^3-2k-2}{k+1}\,\sigma_a^2 - \frac{1}{(k+1)}\,(\sigma_b^2+\sigma_c^2) + (k-1)\,\sigma_e^2$$

The proof is complete. Notice that this is only an accurate test not an exact test because σ_b^2 and σ_c^2 do not completely vanish. However, as k becomes larger, the F value approaches to one if the null hypothesis $\sigma_a^2 = o$ is true. Therefore, this is an accurate test instead of an exact test.

CHAPTER VI

GRAECO-LATIN SQUARE DESIGN

The Graeco-Latin Square Design may be used to impose three restrictions: rows, columns, and Greek letters on a k X k experiment. It differs from the Latin Square Design in that one more restriction is imposed. Hence, its error structure is changed only by adding one more source of variation into the ANOVA.

Let the model be,

$$Y_{ijhl} = \mu + a_i + b_j + c_h + d_l + e_{ijhl} \quad \text{where} \begin{cases} i = 1, 2, .---- k \\ j = 1, 2, ---- k \\ h = 1, 2, ---- k \\ l = 1, 2, ---- k \end{cases}$$

a_i is treatment effect

b_j is row effect

c_h is column effect

d_l is Greek letter effect

e_{ijhl} is random element association with i^{th} treatment, j^{th} row, h^{th} column, and l^{th} Greek letter.

The missing data formula which we can derive from Theorem I by considering rows, columns, and Greek letters as three restrictions is,

$$X = \frac{k(Y'_{i...} + Y'_{.j..} + Y'_{..h.} + Y'_{...l}) - 3Y'_{....}}{(k-1)\ (k-3)}$$

where assuming the value in i^{th} treatments, j^{th} row, h^{th} column, and l^{th} Greek letter is missing.

(A) Random Model

The model is $Y_{ijhl} = \mu + a_i + b_j + c_h + d_l + e_{ijhl}$ where $\begin{cases} i = 1, 2, ---- k \\ j = 1, 2, ---- k \\ h = 1, 2, ---- k \\ l = 1, 2, ---- k \end{cases}$

and μ is fixed, a_i's, b_j's, c_h's, d_l's, and e_{ijhl}'s are random variables with the following distributions:

$a_i \backsim N(o, \sigma_a)$

$b_j \backsim N(o, \sigma_b)$

$c_h \backsim N(o, \sigma_c)$

$d_l \backsim N(o, \sigma_d)$

$e_{ijhl} \backsim N(o, \sigma_e)$

Without correlation and without missing value. The EMS for each of the five components recognized in the model are:

EMS (Rows) = $k \sigma_b^2 + \sigma_e^2$

EMS (Columns) = $k \sigma_c^2 + \sigma_e^2$

EMS (Greek letters) = $k \sigma_d^2 + \sigma_e^2$

EMS (Treatments) = $k \sigma_a^2 + \sigma_e^2$

EMS (Error) = σ_e^2

The ANOVA is then as shown in Table 6.1.1.

Table 6.1.1

Due To	df	EMS
Rows	k-1	$k\sigma_b^2 + \sigma_e^2$
Columns	k-1	$k\sigma_c^2 + \sigma_e^2$
Greek letters	k-1	$k\sigma_d^2 + \sigma_e^2$
Treatments	k-1	$k\sigma_a^2 + \sigma_e^2$
Error	(k-1)(k-3)	σ_e^2
Total	k^2-1	

Proof:

See any standard textbook.

Without correlation but with a missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_b^2 + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2$$

$$\text{EMS (Columns)} = \sigma_c^2 + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2$$

$$\text{EMS (Greek letters)} = \sigma_d^2 + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2$$

$$\text{EMS (Treatments)} = \sigma_a^2 + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 6.1.2.

Table 6.1.2

Due To	df	EMS
Rows	k-1	$\sigma_b^2 + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2$
Columns	k-1	$\sigma_c^2 + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2$
Greek letters	k-1	$\sigma_d^2 + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2$
Treatments	k-1	$\sigma_a^2 + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2$
Errors	(k-1)(k-3)-1	σ_e^2
Total	k^2-1	

Proof:

See Theorem II.

With correlation but without missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = k(\sigma_b^2-C_b)+\sigma_e^2+(k-1)C_{a'bc'd'}-(C_{ab'c'd'}+C_{a'b'cd'}-C_{a'b'c'd})$$
$$-(k-3)C_{a'b'c'd'}$$

$$\text{EMS (Columns)} = k(\sigma_c^2-C_c)+\sigma_e^2+(k-1)C_{a'b'cd'}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})$$
$$-(k-3)C_{a'b'c'd'}$$

$$\text{EMS (Greek letters)} = k(\sigma_d^2-C_d)+\sigma_e^2+(k-1)C_{a'b'c'd}-(C_{ab'c'd'}+C_{a'bc'd'}$$
$$+C_{a'b'c'd}) - (k-3)C_{a'b'c'd'}$$

$$\text{EMS (Treatments)} = k(\sigma_a^2-C_a)+\sigma_e^2+(k-1)C_{ab'c'd'}-(C_{a'bc'd'}+C_{a'b'cd'}$$
$$+C_{a'b'c'd}) - (k-3)C_{a'b'c'd'}$$

$$\text{EMS (Error)} = \sigma_e^2-(C_{ab'c'd'}-C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$$

The ANOVA is then shown in Table 6.1.3.

Table 6.1.3

Due To	df	EMS
Rows	k-1	$k(\sigma_b^2-C_b)+\sigma_e^2+(k-1)C_{a'bc'd'}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}$
Columns	k-1	$k(\sigma_c^2-C_c)+\sigma_e^2+(k-1)C_{a'b'cd'}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}$
Greek letters	k-1	$k(\sigma_d^2-C_d)+\sigma_e^2+(k-1)C_{a'b'c'd}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})-(k-3)C_{a'b'c'd'}$
Treatments	k-1	$k(\sigma_a^2-C_a)+\sigma_e^2+(k-1)C_{ab'c'd'}-(C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}$
Error	(k-1)(k-3)	$\sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$
Total	k^2-1	

Proof:

See Theorem III.

With correlation and with a missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = k(\sigma_b^2 - C_b) + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \left[(k-1) - \frac{1}{(k-1)(k-3)}\right] C_{a'bc'd'}$$

$$- \left[1 + \frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'} - C_{a'b'cd'} + C_{a'b'c'd}) + \left[\frac{3}{(k-1)(k-3)} - (k-3)\right] C_{a'b'c'd'}$$

$$\text{EMS (Columns)} = k(\sigma_c^2 - C_c) + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \left[(k-1) - \frac{1}{(k-1)(k-3)}\right] C_{a'b'cd'}$$

$$- \left[1 + \frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'} + C_{a'bc'd'} + C_{a'b'c'd}) + \left[\frac{3}{(k-1)(k-3)} - (k-3)\right] C_{a'b'c'd'}$$

$$\text{EMS (Greek letters)} = k(\sigma_d^2 - C_d) + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \left[(k-1) - \frac{1}{(k-1)(k-3)}\right]$$

$$C_{a'b'c'd} - \left[1 + \frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'} + C_{a'bc'd'} + C_{a'b'cd'}) + \left[\frac{3}{(k-1)(k-3)} - (k-3)\right]$$

$$C_{a'b'c'd'}$$

$$\text{EMS (Treatments)} = k(\sigma_a^2 - C_a) + \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \left[(k-1) - \frac{1}{(k-1)(k-3)}\right]$$

$$C_{ab'c'd'} - \left[1 + \frac{1}{(k-1)(k-3)}\right](C_{a'bc'd'} + C_{a'b'cd'} + C_{a'b'c'd}) + \left[\frac{3}{(k-1)(k-3)} - (k-3)\right]$$

$$C_{a'b'c'd'}$$

$$\text{EMS (Error)} = \sigma_e^2 - (C_{ab'c'd'} - C_{a'bc'd'} + C_{a'b'cd'} + C_{a'b'c'd}) + 3C_{a'b'c'd'}$$

The ANOVA is then as shown in Table 6.1.4.

Proof:

See Theorem IV.

Table 6.1.4

Due To	df	EMS
Rows	k-1	$k(\sigma_b^2-C_b)+\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'bc'd'}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}$
Columns	k-1	$k(\sigma_c^2-C_c)+\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'cd'}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}$
Greek letters	k-1	$k(\sigma_d^2-C_d)+\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'c'd}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}$
Treatments	k-1	$k(\sigma_a^2-C_a)+\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{ab'c'd'}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}$
Error	(k-1)(k-3)-1	$\sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$
Total	k^2-1	

(B) Fixed Model

Let the model be,

$$Y_{ijhl} = \mu + a_i + b_j + c_h + d_l + e_{ijhl} \qquad \text{where} \begin{cases} i = 1, 2, \text{----} k \\ j = 1, 2, \text{----} k \\ h = 1, 2, \text{----} k \\ l = 1, 2, \text{----} k \end{cases}$$

with properties that

$$\sum_i a_i = \sum_j b_j = \sum_h c_h = \sum_l d_l = o, \text{ and } e_{ijhl} \backsim N(o, \sigma_e)$$

Without correlation and without missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$$

$$\text{EMS (Greek letters)} = \sigma_e^2 + \frac{k \sum_l d_l^2}{k-1}$$

$$\text{EMS (Treatments)} = \sigma_e^2 + \frac{k \sum_i a_i^2}{k-1}$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 6.2.1.

Proof:

See random model.

Without correlation but with a missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$$

Table 6.2.1

Due To	df	EMS
Rows	k-1	$\sigma_e^2 + \frac{k \Sigma_j b_j^2}{k-1}$
Columns	k-1	$\sigma_e^2 + \frac{k \Sigma_h c_h^2}{k-1}$
Greek letters	k-1	$\sigma_e^2 + \frac{k \Sigma_l d_l^2}{k-1}$
Treatments	k-1	$\sigma_e^2 + \frac{k \Sigma_i a_i^2}{k-1}$
Error	(k-1)(k-3)	σ_e^2
Total	k^2-1	

$$\text{EMS (Greek letters)} = \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k \sum\limits_l d_l^2}{k-1}$$

$$\text{EMS (Treatments)} = \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k \sum\limits_i a_i^2}{k-1}$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 6.2.2.

Proof:

See random model.

With correlation but without missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_e^2+(k-1)C_{a'bc'd'}-(C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'} + \frac{k \sum\limits_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \sigma_e^2+(k-1)C_{a'b'cd'}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'} + \frac{k \sum\limits_h c_h^2}{k-1}$$

$$\text{EMS (Greek letters)} = \sigma_e^2+(k-1)C_{a'b'c'd}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})-(k-3) C_{a'b'c'd'} + \frac{k \sum\limits_l d_l^2}{k-1}$$

$$\text{EMS (Treatments)} = \sigma_e^2+(k-1)C_{ab'c'd'}-(C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3) C_{a'b'c'd'} + \frac{k \sum\limits_i a_i^2}{k-1}$$

$$\text{EMS (Error)} = \sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$$

The ANOVA is then as shown in Table 6.2.3.

Proof:

See random model.

With correlation and with a missing value. The EMS for each of the five components recognized in the model are:

Table 6.2.2

Due To	df	EMS
Rows	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\frac{k\sum_j b_j^2}{k-1}$
Columns	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\frac{k\sum_h c_h^2}{k-1}$
Greek letters	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\frac{k\sum_l d_l^2}{k-1}$
Treatments	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\frac{k\sum_i a_i^2}{k-1}$
Error	(k-1)(k-3)-1	σ_e^2
Total	k^2-2	

Table 6.2.3

Due To	df	EMS
Rows	k-1	$\sigma_e^2+(k-1)C_{a'bc'd'}-(C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}+\frac{k\sum_j b_j^2}{k-1}$
Columns	k-1	$\sigma_e^2+(k-1)C_{a'b'cd'}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})-(k-3)C_{a'b'c'd'}+\frac{k\sum_h c_h^2}{k-1}$
Greek letters	k-1	$\sigma_e^2+(k-1)C_{a'b'c'd}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})-(k-3)C_{a'b'c'd'}+\frac{k\sum_l d_l^2}{k-1}$
Treatments	k-1	$\sigma_e^2+(k-1)C_{ab'c'd'}-(C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}+\frac{k\sum_i a_i^2}{k-1}$
Error	(k-1)(k-3)	$\sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$
Total	k^2-1	

$$\text{EMS (Rows)} = \left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'bc'd'}$$

$$-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]$$

$$C_{a'b'c'd'}+\frac{k\sum_j b_j^2}{k-1}$$

$$\text{EMS(Columns)} = \left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'cd'}$$

$$-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]$$

$$C_{a'b'c'd'}+\frac{k\sum_h c_h^2}{k-1}$$

$$\text{EMS (Greek letters)} = \left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'c'd}$$

$$-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]$$

$$C_{a'b'c'd'}+\frac{k\sum_l d_l^2}{k-1}$$

$$\text{EMS (Treatments)} = \left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{ab'c'd'}$$

$$-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]$$

$$C_{a'b'c'd'}+\frac{k\sum_i a_i^2}{k-1}$$

$$\text{EMS (Error)} = \sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$$

The ANOVA is then shown in Table 6.2.4.

Proof:

See random model.

Table 6.2.4

Due To	df	EMS
Rows	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'bc'd'}$ $-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})$ $+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}+\frac{k\sum_j b_j^2}{k-1}$
Columns	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'cd'}$ $-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})$ $+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}+\frac{k\sum_h c_h^2}{k-1}$
Greek letters	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'c'd}$ $-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})$ $+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}+\frac{k\sum_l d_l^2}{k-1}$
Treatments	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{ab'c'd'}$ $-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})$ $+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}+\frac{k\sum_i a_i^2}{k-1}$
Error	(k-1)(k-3)-1	$\sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})$ $+3C_{a'b'c'd'}$
Total	k^2-2	

(C) Mixed Model

Let the model be,

$$Y_{ijhl} = \mu + a_i + b_j + c_h + d_l + e_{ijhl} \quad \text{where} \begin{cases} i = 1, 2, \text{----}\ k \\ j = 1, 2, \text{----}\ k \\ h = 1, 2, \text{----}\ k \\ l = 1, 2, \text{----}\ k \end{cases}$$

with the following properties:

a_i (Treatment effect) $\backsim N(o, \sigma_a)$

b_j (Row effect) $\sum_j b_j = o$

c_h (Column effect) $\sum_h c_h = o$

d_l (Greek letter effect) $\sum_l d_l = o$

e_{ijhl} (Random effect) $\backsim N(o\ \sigma_e)$

Without correlation and without missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_e^2 + \frac{k \sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \sigma_e^2 + \frac{k \sum_h c_h^2}{k-1}$$

$$\text{EMS (Greek letter)} = \sigma_e^2 + \frac{k \sum_l d_l^2}{k-1}$$

$$\text{EMS (Treatments)} = \sigma_e^2 + k\,\sigma_a^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then as shown in Table 6.3.1.

Proof:

See random and fixed model.

Without correlation and with a missing value. The EMS for each of the five components recognized in the model are:

Table 6.3.1

Due To	df	EMS
Rows	k-1	$\sigma_e^2 + \frac{k \Sigma_j b_j^2}{k-1}$
Columns	k-1	$\sigma_e^2 + \frac{k \Sigma_h c_h^2}{k-1}$
Greek letters	k-1	$\sigma_e^2 + \frac{k \Sigma_l d_l^2}{k-1}$
Treatments	k-1	$\sigma_e^2 + k\, \sigma_a^2$
Error	(k-1)(k-3)	σ_e^2
Total	k^2-1	

$$\text{EMS (Rows)} = \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k\sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k\sum_h c_h^2}{k-1}$$

$$\text{EMS (Greek letters)} = \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k\sum_l d_l^2}{k-1}$$

$$\text{EMS (Treatments)} = \left[1 + \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + k\quad{}_a^2$$

$$\text{EMS (Error)} = \sigma_e^2$$

The ANOVA is then shown in Table 6.3.2.

Table 6.3.2

Due To	df	EMS
Rows	k-1	$\left[1+ \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k\sum_j b_j^2}{k-1}$
Columns	k-1	$\left[1+ \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k\sum_h c_h^2}{k-1}$
Greek letters	k-1	$\left[1+ \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + \frac{k\sum_l d_l^2}{k-1}$
Treatments	k-1	$\left[1+ \frac{1}{(k-1)(k-3)}\right]\sigma_e^2 + k\quad{}_a^2$
Error	(k-1)(k-3)-1	σ_e^2
Total	k^2-2	

Proof:

See random and fixed model.

With correlation but without missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = \sigma_e^2+(k-1)C_{a'bc'd'}-(C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}+\frac{k\sum_j b_j^2}{k-1}$$

$$\text{EMS (Columns)} = \sigma_e^2+(k-1)C_{a'b'cd'}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}+\frac{k\sum_h c_h^2}{k-1}$$

$$\text{EMS (Greek letters)} = \sigma_e^2+(k-1)C_{a'b'c'd}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})-(k-3)C_{a'b'c'd'}+\frac{k\sum_l d_l^2}{k-1}$$

$$\text{EMS (Treatments)} = k(\sigma_a^2-C_a)+(k-1)C_{ab'c'd'}-(C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}$$

$$\text{EMS (Error)} = \sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$$

The ANOVA is then as shown in Table 6.3.3.

Proof:

See random and fixed model.

With correlation and with a missing value. The EMS for each of the five components recognized in the model are:

$$\text{EMS (Rows)} = \left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'bc'd'}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}+\frac{k\sum_j b_j^2}{k-1}$$

Table 6.3.3

Due To	df	EMS
Rows	$k-1$	$\sigma_e^2+(k-1)C_{a'bc'd'}-(C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}+\frac{k\sum_j b_j^2}{k-1}$
Columns	$k-1$	$\sigma_e^2+(k-1)C_{a'b'cd'}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}+\frac{k\sum_h c_h^2}{k-1}$
Greek letters	$k-1$	$\sigma_e^2+(k-1)C_{a'b'c'd}-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})-(k-3)C_{a'b'c'd'}+\frac{k\sum_l d_l^2}{k-1}$
Treatments	$k-1$	$k(\sigma_a^2-C_a)+\sigma_e^2+(k-1)C_{ab'c'd'}-(C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})-(k-3)C_{a'b'c'd'}$
Errors	$(k-1)(k-3)$	$\sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$
Total	k^2-1	

$$\text{EMS (Columns)} = \left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'cd'}$$

$$-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}$$

$$+\frac{k\sum_h c_h^2}{k-1}$$

$$\text{EMS (Greek letters)} = \left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'c'd}$$

$$-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}$$

$$+\frac{k\sum_l d_l^2}{k-1}$$

$$\text{EMS (Treatments)} = k(\sigma_a^2-C_a)+\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]$$

$$C_{ab'c'd'}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]$$

$$C_{a'b'c'd'}$$

$$\text{EMS (Error)} = \sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$$

The ANOVA is then as shown in Table 6.3.4.

Proof:

See random and fixed model.

(D) Exact Test

Similarly, in order to perform the exact test, it is necessary to evaluate the sum of squares attributable to rows, columns and Greek letters ignoring treatments, and to rows, columns, Greek letters, and treatments. The difference of these two sum of squares may then be tested against the error with the reduced sum of squares. To compute the necessary

Table 6.3.4

Due To	df	EMS
Rows	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'bc'd'}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'b'cd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}+\frac{k\Sigma_j b_j^2}{k-1}$
Columns	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'cd'}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}+\frac{k\Sigma_h c_h^2}{k-1}$
Greek letters	k-1	$\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{a'b'c'd}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}+\frac{k\Sigma_l d_l^2}{k-1}$
Treatments	k-1	$K(\sigma_a^2-C_a)+\left[1+\frac{1}{(k-1)(k-3)}\right]\sigma_e^2+\left[(k-1)-\frac{1}{(k-1)(k-3)}\right]C_{ab'c'd'}-\left[1+\frac{1}{(k-1)(k-3)}\right](C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+\left[\frac{3}{(k-1)(k-3)}-(k-3)\right]C_{a'b'c'd'}$
Error	(k-1)(k-3)-1	$\sigma_e^2-(C_{ab'c'd'}+C_{a'bc'd'}+C_{a'b'cd'}+C_{a'b'c'd})+3C_{a'b'c'd'}$
Total	k^2-2	

quantities, the experiment is first regarded as an experiment in rows, columns, and Greek letters with one observation missing and the minimum sum of squares for error obtained. This sum of squares, W, say, will have (k^2-3k+1) degrees of freedom. The minimum sum of squares for error, when treatments are taken into account, is obtained by analyzing the augumented table, E, say, with (k^2-4k+2) degrees of freedom. The quantity $(W-E)/(k-1)$ is the mean square for treatments, which is tested against $E/(k^2-4k+2)$ by the F test with $(k-1)$ and (k^2-4k+2) degrees of freedom.

The proof of this follows the similar way to those in the last two chapters; thus is omitted here. However, it can be shown that it is also an accurate test instead of an exact test.

CHAPTER VII

SUMMARY

Both correlation and missing value often occurs in an experiment. The existence of correlation will impair to some extent the standard properties on which the widespread utility of the analysis of variance depends. Also, even one missing value would jeopardize the accuracy of the experiment.

For the correlation, the experimenter may not always meet the ideal condition that the errors are uncorrelated due to the physical relationship of the experimental units. By the method of transformation and rotation to obtain the canonical form, and the use of randomization techniques, some of the correlation may be removed. It is still of interest to analyze the contribution of the remaining covariance to the expected mean squares of the analysis of variance.

For the missing value, though an estimate can be obtained by the use of least square method and an approximate analysis of variance can be computed, substitution of estimates for the missing data does not in any way recover the information that is lost through loss of data, as some experimenters have suggested. The only complete solution of the missing data problem is not to have them.

When both correlation and a missing value occurs together in an experiment, the error structure becomes more complicated. Not only the experiment loses accuracy, but also the analysis of variance loss symmetry and balance. Therefore, special methods are available for exact F-test.

One of the objects of this paper is to indicate the theoretical basis for some of the methods which have been used extensively by experimenters in dealing with either missing data or correlation or both.

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