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Boolean Space

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BOOLEAN SPACE

by

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INTRODUCTION

M. H. A. Stone showed in 1937 and subsequently that many interesting and important results of general topology involve latices and Boolean rings. This type of result forms the substance of this thesis.

Theorem 4, page 11, states that for any $r \neq 0$ in a Boolean ring, there exists a homomorphism h into I_2, (the field of integers modulo 2), such that $h(r) = 1$.

Theorem 3, page 6, states that any subring of a characteristic ring of a Boolean space X is the whole ring if it has the two points property (that is, given x, y in X and a, b in I_2, there exista a g such that $g(x) = a$ and $g(y) = b$)

From these two theorems follows the Stone Representation theorem which states that any Boolean ring is isomorphic to the characteristic ring of its Stone space.

Theorem 1, page 11, is independent of other theorems. It states that any compact Hausdorff space is the continuous image of some closed subset in a Cantor space.

Theorem 5, page 23, states that a topological space can be embedded in a Cantor space as a subspace if and only if it is Boolean.

This theorem uses the Dual Representation theorem as its sufficient part. It states that any Boolean space is homomorphic to the Stone space of its characteristic ring.

Theorem , page 25, is a problem suggested by Dr. Ju-kwei Wang. I am not sure whether it is original or not.

BOOLEAN RINGS

Maximal Ideals in Lattices

Definition: A Lattice is a non void set X with a partial ordering s.t. For every pair of $x \cdot y$ in X, there exists a (unique) smallest element $x \vee y$ which is greater than x and y, and a (unique) largest element $x \wedge y$ which is smaller than each. The lattice is distributive if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Definition: A subset A of a distributive lattice X is an ideal (a dual ideal) if whenever $y \geq x$ and $y \in A$ imply $x \in A$ and if y and z belong to A, so does $y \vee z$ (respectively, whenever $x \geq y$ and $y \in A$, then $x \in A$; and if $y \in A$, $z \in A$, then $y \wedge z \in A$).

Lemma 1: Let A be an ideal and B be a disjoint dual ideal in a distributive lattice X; then the family of all ideals which contain A and are disjoint from B contains a maximal ideal A.

Similarly, there is a dual ideal B which contains B, is disjoint from A, and is maximal with respect to the property.

Proof: We order the elements of this family S by

$M \geq N$ if $M \supseteq N$, for $M, N \in S$, then $\geq$ is a partial order in this family.

For each chain in S, consider the union of the members of it; then we get an ideal which is an upper bound in this chain, so applying Zorn's lemma to S, we can get a maximal ideal A with the property that it contains A and is disjoint from B.

Lemma 2: The smallest ideal which contains A and a member c of X is $E = \{x : x \leq c \text{ or } x \leq c \vee y \text{ for some } y \text{ in } A'\}$. If $c \notin A'$ and $c \notin B$, then $c \vee x \in B$ for some $x \in A'$.

Proof:

1. Any ideal which contains A and c contains

   $\{x : x \leq c\}$ and $\{x : x \leq c \vee y \text{ for some } y \in A'\}$

   Therefore, it contains E.

2. If $z \leq x$ for some $x \in E$

   Since $x \leq c$ or $x \leq c \vee y$ for some $y \in A'$

   Therefore, $z \leq c$ or $z \leq c \vee y$, $z \in E$

   If z and $x \in E$

   Suppose $z \leq c$, $x \leq y \vee c$, then $c \leq y \vee c$.

   $z \leq y \vee c$. Therefore, $z \vee c \leq y \vee c$, $z \vee c \in E$.

   Suppose $z, x \leq c$ or $z, x \leq y \vee c$.

   Then $z \vee x \leq c$ or $z \vee x \leq y \vee c$.

   Thus E is an ideal.

3. If $c \not\in A'$ and $c \not\in B$, then $E = X$, since A' is a maximal ideal which contains A. Thus, for any $y \in B$. $y \leq c \vee x$ for some $x \in A'$. Otherwise, $y \leq c$ will imply $c \in B$. Therefore, $c \vee x \in B$ for some $x \in A'$.

<u>Theorem 1:</u> $A' \cup B' = X$.

<u>Proof:</u> By lemme 2, if $c \in A'$ and $c \in B'$, then $c \in B$.

Therefore, there is $x \in A'$ s. t. $c \vee x \in B$, $c \vee x \in B'$

There is $y \in B'$ s. t. $y \quad c \in A$, $c \wedge y \in A'$, $(c \vee x) \wedge y =$ $(c \wedge y) \vee (x \wedge y)$

1. $c \wedge y \in A'$, $x \wedge y \leq x \in A'$

   Therefore, $x \wedge y \in A'$, $(c \vee x) \wedge y \in A'$.

2. $c \vee x \in B'$, $y \in B'$

   $(c \vee x) \wedge y \in B'$

   But $A' \cap B' = \phi$ for each $c \in X$; then $c \in A'$ or $c \in B'$.

   Therefore, $A' \cup B' = X$.

<u>Definition:</u> A <u>Boolean ring</u> is a ring $(R, +, \cdot)$, such that $r \cdot r = r$ for each $r \in R$.

<u>Theorem 2:</u> A Boolean ring is commutative and $r + r = 0$ for each $r \in R$.

<u>Proof:</u> $(r + r) \cdot (r + r) = r + r$. Therefore, $r + r = 0$

$(r + s)\ (r + s) = r + s$,

$r \cdot s + s \cdot r = 0$

Therefore, $r \cdot s + (s \cdot r + s \cdot r) = 0 + s \cdot r$

$r \cdot s = s \cdot r$

Use the notation $\underline{A \triangle B = (A \cap B^c) \cup (A^c \cap B)}$ for subsets A, B of a set X and $\mathfrak{O}$ <u>the family of all subsets of a set X.</u>

$(\mathfrak{O}, \triangle, \cap)$ is a Boolean ring with the unit. Here, only check the associative law and the unit with the operation $\triangle$.

Denoting $\phi = 0$, $A \triangle 0 = 0 \triangle A = A$

$(A \triangle B) \triangle C$

$= [[(A \cap B^c) \cup (A^c \cap B) \cap C^c] \cup [[(A \cap B^c) \cup (B \cap A^c)]^c \cap C]$

$= (A \cap B^c \cap C^c) \cup (B \;\; A^c \cap C^c) \cup [(A^c \cup B) \;\; (B^c \cap A) \cap C]$

$= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup [(A^c \cap B) \;\; (B^c \cap A) \cup (A \cap C)]$

$= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C)$

Since $\mathfrak{O}$ is commutative with $\triangle$, therefore,

$(A \triangle B) \triangle C = C \triangle (A \triangle B) = A \triangle (B \triangle C)$

Use the notation $\underline{I_2}$, being the field of integers modulo 2, and $\underline{I_2^X}$, the family of all functions on X to I_2, and we define

$(f + g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)\, g(x)$

for $f, g \in I_2^X$, $x \in X$.

<u>Theorem 3:</u> Let I_2^X be the family of all functions on a non-empty set X; then $(I_2^X, +, \cdot)$ is a Boolean ring and it is isomorphic to $(\mathfrak{O}, \triangle, \cap)$ where $\mathfrak{O}$ is the family of all subsets of X.

Proof: It is obvious that $f + g = g + f \in I_2^X$, $f g \in I_2^X$

and $f \cdot f = f$, $f + f = 0$

$f(g + h) = f g + f h$, $f(g h) = (f g) h$

$I f = f I = f$ $(I(x) = 1$, for $x \in X)$

Therefore, $(I_2^X, +, \cdot)$ is a Boolean ring with unit.

There is a one-to-one correspondence between

I_2^X and $\mathfrak{A}$ in such a way $f \leftrightarrow A$ iff $f^{-1}[1] = A$

Since $(f + g)^{-1}[1] = f^{-1}[1] \triangle g^{-1}[1]$

Therefore, $f + g \leftrightarrow f^{-1}[1] \triangle g^{-1}[1]$

$(f g)^{-1}[1] = f^{-1}[1] \triangle g^{-1}[1]$

Therefore, $f g \leftrightarrow f^{-1}[1] \cap g^{-1}[1]$

$[f(g + h)]^{-1}[1] = f^{-1}[1] \cap [g^{-1}[1] \triangle h^{-1}[1]]$

$= (f^{-1}[1] \cap g^{-1}[1]) \triangle (f^{-1}[1] \cap h^{-1}[1])$

Therefore, $f(g + h) = f g + f h \leftrightarrow (f^{-1}[1] \cap g^{-1}[1])$

$\triangle (f^{-1}[1] \cap h^{-1}[1])$

$(I_2^X, +, \cdot) \cong (\mathfrak{A}, \triangle, \cap)$

Boolean Rings

Theorem 1: Let $(R, +, \cdot)$ be a Boolean ring in which we define $r \geq s$ if $r \cdot s = s$ for $r, s \in R$. Then R is a lattice with the relation $\geq$.

Proof:

1. $\geq$ is a partial order relation in R .

$r \geq r$ iff $r \cdot r = r$

$r \geq s$, $s \geq r$ $r \cdot s = s$, $s \cdot r = r$

$(r \cdot s)\, r = r$ $(r \cdot r)\, s = r$ $r = s$

$r \geq s$, $s \geq Z$ $r \cdot s = s$, $s \cdot Z = Z$

Therefore, $(r \cdot s)\, Z = Z$, $rZ = Z$ $r \geq Z$

2. Define $(r \vee s) = r + s + rs$

$(r \vee s)\, r = r$

$(r \vee s)\, s = s$ $\therefore$ $r \vee s \geq r$ and $r \vee s \geq s$

If $m \geq r$ and $m \geq s$, then,

$m\,(r \vee s) = mr + ms + mrs = r + s + rs = (r \vee s)$

Therefore, $m \geq (r \vee s)$

Define $r \wedge s = rs$, and then we can prove that $r \wedge s$ is the greatest element which precedes r and s.

$r \wedge (s \vee t) = rs + rt + rst$

$(r \wedge s) \vee (r \wedge t) = rs + rt + rst$

Therefore, $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$

It is also true that $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$

Definition: Let $(R, +, \cdot)$ be a Boolean ring. A subset T of R is called a dual ideal iff $r \wedge s \in T$ whenever $r, s \in T$, and $t \in T$ whenever t follows a member of T.

Theorem 2: There is a one-to-one correspondence between maximal ideals in $(R, +, \cdot)$ and homomorphisms in I_2 which are not identically zero.

Proof: Let h be a non-zero homomorphism between $(R, +, \cdot)$ and I_2, and let $K = h^{-1}(0)$

1. $a, b \in K$, $h(a-b) = h(a) - h(b) = 0$

   Thus, $a - b \in K$

   $r \in R$, $h(ra) = h(r)\, h(a) = h(r)\; 0 = 0$

   Thus, $ra \in K$

   Since h is not zero, therefore, $h^{-1}(0) = K \neq R$

   Therefore, K is a proper ideal of R.

2. Let K_1 be an ideal in R and $K_1 \not\leq K$,

   Choose an element $a \in K_1 - K$ for any $r \notin K$.

   $h(x - a) = h(x) - h(a) = 1 - 1 = 0$

   Thus, $x - a = r \in K$, $x = a + r \in K_1$

   $K_1 = R$; K is a maximal ideal.

   Conversely, if K is a maximal ideal in R, define a mopping g on R into I_2 by

   $g(x) = 0$ for $x \in K$, $g(x) = 1$ for $x \notin K$

Then $g(x + y) = g(x) + g(y)$ and $g(xy) = g(x)\, g(y)$

are true for x, y in the following cases:

a. If $x \in K$, $y \in K$, then $x + y \in K$

b. If $x \notin K$, $y \in K$, then $x + y \notin K$. Otherwise

$(x + y) + y = x \in K$.

c. Let $x \notin K$, $y \notin K$, and let M be the ideal generated

by x and K.

Since K is a maximal ideal, $M = R$.

Let $y = 1x + K$ for some $1x + K \in M$. Then,

$Lx \notin K$; otherwise $y \in K$.

If $xy \in K$, then $xy + xK = 1x + xK + xK = 1x \in K$

Thus, $xy \notin K$

d. Let $x \notin K$, $y \notin K$; then $x + y \in K$; otherwise $x + y \notin K$

and by case c, $xy \notin K$, $yx(x + y) \notin K$

But $yx(x + y) = xy + xy = 0 \in K$.

<u>Theorem 3:</u> If S is an ideal in a Boolean ring $(R, +, \cdot)$, T is

a disjoint dual ideal in R, and $S \cup T = R$. Then the

function which is zero on S and 1 on T is a homomorphism of R into I_2.

<u>Proof:</u>

1. $x \in T$, $y \in T$, then $xy \in T$ and $x + y \notin T$

Otherwise, $x + y \in T$, $xy(x + y) = xy + xy = 0 \in T$, but

$0 \in S$.

2. $x \in S$, $y \in S$, then $x + y \in S$, $xy \in S$

3. $x \in S$, $y \in S$, then $xy \in S$ and $x + y \in T$.

Otherwise, $x + y \in S$, $x + (x + y) = y \in S$ but $S \cap T = \phi$. By the proof of Theorem 2, there exists a homomorphism of R into I_2 , and S is a maximal ideal in R.

<u>Theorem 4:</u> If r is a non-zero element of a Boolean ring R , then there is a homomorphism h of the ring into I_2 such that $h(r) = 1$.

<u>Proof:</u> Let $S = \{0\}$, zero ideal in R .

$T = \{x : x \geq r\}$, a dual ideal in R .

$S \cap T = \phi$

By Theorem 1, Section 1, and Theorem 3 this theorem holds.

<u>Cantor Space</u>

<u>Definition:</u> <u>A product space 2^A</u> (that is, all functions on a set A to the discrete space whose only members are 0 and , with the product topology) is called a <u>Cantor space</u>.

<u>Theorem 1:</u> Each compact Hausdorff space is the continuous image of a closed subset of some Cantor space.

<u>Proof:</u> Let F be the family of all functions on 2 such that $f(0)$ and $f(1)$ are closed subsets of the compact Hausdorff

space X and $f(0) \cup f(1) = X$.

Let $a \in X$, and we choose $x \in 2^F$.

s.t. $x_f = \epsilon$ ($\epsilon = 0$, or 1) and $a \in f(\epsilon)$ for every $f \in F$.

Then $a \in \{\cap f(x_f) : f \in F\}$

If $a \neq b$ since X is Hausdorff, thus, there is two open neighborhoods N_a, N_b s.t. $a \in N_a$ $b \in N_b$ and

$N_a \quad N_b = \phi$, $a \in N_b^C$ $b \in N_a^C$ and

$N_a^C \cup N_b^C = (N_a \cap N_b)^C = \phi^C = X$

Define $g(0) = N_b^C$ $g(1) = N_a^C$

Then $g \in F$, $x_g = 0$

and $b \notin \{\cap f(x_f) : f \in F\}$

Therefore, $a = \{\cap f(x_f) : f \in F\}$

We define $\Phi(x) = \{\cap f(x_f) : f \in F\}$ when $\{\cap f(x_f) : f \in F\}$ is non void.

Let y be a limit point of the points in the domain of Φ.

Suppose $\{\cap f(y_f) : f \in F\} = \phi$

Since X is compact and $f(y_f)$ are closed subsets in X, therefore, there is finite $f_1(y_1)$, $f_2(y_2)$ . . . $f_n(y_n)$

s.t. $f_1(y_1) \cdot f_2(y_2) \quad \cdots\cdots \quad f_n(y_n) = \phi$

Let $y_{f_i} = \epsilon_i$ $i = 1, 2, \ldots, n$

Then $B = \{x : x_{f_i} = \epsilon_i \;\; i = 1, 2, \ldots, n \;\; x \in 2^F\}$

is a base element in 2^F.

For any $x \in B$, $\bigcap_{i=1}^{n} f_i(x_{f_i}) = \phi$

Therefore, $\Phi(x) = \{\bigcap f(x_f) : f \in F\} = \phi$

This is a contradiction to the fact that y is a limit point of $\Phi^{-1}(x)$.

Let y be a limit point of $\Phi^{-1}(x)$ where E is a closed subset of X.

If $\Phi(y) \not\subset E$ (that is, $y \not\in \Phi^{-1}(E)$), then

$\{\bigcap f(y_f) \quad E : f \in F\} = \phi$

Therefore, $(f(y_f) \cap E) = \phi$ for $f \in F$

Since X is compact and each $f(y_f) \quad E$ is closed in X, there exists finite $\bigcap_{i=1}^{m} (f_i(y_{fi}) \cap E) = \phi$

Let $y_{f_i} = \epsilon_i \quad i = 1, 2, \ldots, m$

Then $D = \{x : x_{f_i} = \epsilon_i \quad i = 1, 2, \ldots, m \quad x \in 2^F\}$ is a base element of 2^F for each $x \in D$.

Therefore, $\{\bigcap f(x_f) \cap E : f \in 2^F\} = \phi$

This is a contradiction to the fact that y is a limit point of $\Phi^{-1}(E)$.

BOOLEAN SPACE

Boolean Space and Characteristic Ring

Let $(R, +, \cdot)$ be a Boolean ring; let s' be the set of all ring homomorphisms of R into I_2, and let $S = S' - \{0\}$ where 0 is the homomorphism which is identically zero. Then S' is the subset of the product I_2^R.

Definition: The Stone space of the ring R is S with the relative product topology, where I_2 is assigned the discrete topology.

Definition: A Boolean space is a Hausdorff space such that the family of all sets which are both open and compact is the Base for the topology.

Theorem 1: The set of all continuous functions f on a Boolean space into I_2 such that $f^{-1}[1]$ is compact is a ring.

Proof: Let X be a Boolean space, and let F be the set of all continuous functions on X into I_2. s.t. $f^{-1}[1]$ is compact. 1 is open-closed in I_2; therefore, $f^{-1}[1]$ is open-closed in X

Let $f, g \in F$

$f^{-1}[1] \cap g^{-1}[1]$ is closed-open, since $f^{-1}[1]$ compact,

$f^{-1}[1] \cap g^{-1}[1]$ compact.

$(fg)^{-1}[1] = f^{-1}[1] \cap g^{-1}[1]$. Therefore, $f \cdot g \in f$

$f^{-1}[1] - f^{-1}[1] \cap g^{-1}[1]$ is open-closed and contained in the compact subset $f^{-1}[1]$.

Thus, it is compact.

The same $g^{-1}[1] - f^{-1}[1] \cap g^{-1}[1]$ is open-closed and compact.

$(f + g)^{-1}[1] = (g^{-1}[1] - f^{-1}[1] \cap g^{-1}[1]) \cup (f^{-1}[1] - f^{-1}[1] \cap g^{-1}[1])$ is compact .

Therefore, $f + g \in F$

It is clear $f(g + h) = fg + fh$ $(fg)h = f(gh)$

$$f + f = 0 \quad f \cdot f = f$$

Therefore F is a Boolean ring .

<u>Corollary:</u> $F \cong (\mathfrak{A}, \Delta, \cap)$ where F is the characteristic ring of a Boolean space X , and $\mathfrak{A}$ is the family of all compact subsets of the space X.

<u>Definition:</u> The ring of all continuous functions on a Boolean space into I_2 such that $f^{-1}[1]$ is compact is called the <u>characteristic ring</u> of the Boolean space.

<u>Theorem 2:</u> The Stone space of a Boolean ring $(R, +, \cdot)$ is a Boolean space and is compact whenever R has a unit.

Proof:

1. Since I_2 is Hausdorff and compact, then I_2^R is Hausdorff with the relative product topology.

2. Let x be any element of R. Then each of the sets

$\{f : f(x) = 0 , f \in I_2^R\}$, $\{f : f(x) = 1 , f \in I_2^R\}$

is open-closed.

This follows from the fact that each is the inverse image of an open-closed set in I_2 under the projection of I_2^R onto I_2.

The subbase consists of open-compact sets; so, too, does the base of I_2^R.

Therefore, S is a Boolean space with the relative topology.

3. If $1 \in R$, S equals the intersection of the following three subsets of I_2^R.

$$\bigcap_{x, y \in R} \{f : f(x + y) = f(x) + f(y)\} \qquad (1)$$

$$\bigcap_{x, y \in R} \{f : f(xy) = f(x)\, f(y)\} \qquad (2)$$

$$\{f : f(1) = 1\} \qquad (3)$$

From (2), we know that (3) is closed, given any two elements $x, y \in R$.

$\{ f : f(x) = 0 , \quad f(y) = 0, \quad f(x + y) = 0 \}$,

$\{ f : f(x) = 1 , \quad f(y) = 0, \quad f(x + y) = 1 \}$,

$\{ f : f(x) = 0 , \quad f(y) = 1, \quad f(x + y) = 1 \}$,

and

$\{ f : f(x) = 1 , \quad f(y) = 1, \quad f(x + y) = 0 \}$.

Each of these sets, being itself the intersection of three closed sets, is closed, so

$\{ f : f(x + y) = f(x) + f(y) \}$ is closed.

$\bigcap_{x, y \in R} \{ f : f(x + y) = f(x) + f(y) \}$ is the intersection of closed sets.

Therefore, (1) is closed. By the same argument, (2) is also closed.

Thus, S is closed in I_2^R . Since I_2^R is compact, S is a compact subspace of I^R .

Definition: Let F be the characteristic ring of a Boolean space X , and let G be a subring of F which has the two points property; that is, for distinct x and y of X and for a and b in I_2 , there is a g in G such that $g(x) = a$ and $g(y) = b$.

Theorem 3: (Stone-Weierstrass modulo 2)

Let F be the characteristic ring of a Boolean space X, and let G be a subring of F which has the two points property. Then $F = G$.

Proof:

Let Y be a compact subset of X, given an element $x \in X - Y$.

To each $y \in Y$, there exists a $g \in G$ such that $g(x) = 0$, and $g(y) = 1$. g is continuous.

Let N_y be a neighborhood of y s.t. $g(z) = 1$ for each $z \in N_y$.

Therefore, $\bigcup_{y \in Y} N_y \supset Y$, Since Y is compact, we can get a finite number of $g_1, g_2, \ldots g_n$ in G with the corresponding $N_{y_1}, N_{y_2}, \ldots N_{y_n}$

s. t.

$$\bigcup_{i=1}^{n} N_{y_i} \supset Y \qquad g_i(z) = 1 \quad \text{for } z \in N_{y_i}$$

$$g_i(x) = 0$$

G is a subring of F, so

$$\sum_{i=1}^{n} g_i - \sum_{i<j} g_i g_j + \sum_{i<j<k} g_i g_j g_k - \ldots + (-1)^{n+1} \bigcap_{i=1}^{n} g_i = g_x$$

is in G, and $g_x(x) = 0$; $g_x(z) = 1$ for each $z \in Y$

For each $x \in X - Y$, we can get a function g_x in G.

Let $h = \bigcap_{x \in X-Y} g_x$; then

$h \in G$ and $h(x) = 0$, $h(z) = 1$ for each $z \in Y$

Therefore, $G = F$.

Theorem 4: Each Boolean ring is isomorphic to the characteristic ring of its Stone space.

Proof:

Let X be a Boolean ring and let S be its Stone space. For each $x \in X$, we define the evaluation map e by

$$e(x)_f = f(x) = 1 \text{ or } 0 , \ f \in S$$

$$e(x)^{-1}[1] = \{f : f(x) = 1 \text{ for } f \in S\}$$

If $x = 0$, $e(x)^{-1}[1] = \phi$

If $x \neq 0$, by Theorem of Section II, there exists a

$h \in S$ s.t. $h(x) = 1$. Thus $e(x)^{-1}[1]$ is non-empty.

$e(x)^{-1}[1]$ is compact in S, since I_2^R is compact, and $\{f : f(x) = 1 \text{ for } f \in I_2^R\}$ is closed in I_2^R. Thus, it is compact.

$$e(x)^{-1}[1] = S \cap \{f : f(x) = 1 , \ f \in I_2^R\}$$

$$\{f : f(x + y) = 1 \ f \in S\}$$

$$= \{f : f(x) = 1 , \ f \in S\} \triangle \{f : f(y) = 1 , \ f \in S\}$$

Therefore, $e(x + y)^{-1}[1] = e(x)^{-1}[1] + e(y)^{-1}[1]$

$$\{f : f(xy) = f(x) f(y) = 1 \ f \in S\}$$

$$= \{f : f(x) = 1, \ f \in S\} \cap \{f : f(y) = 1 \ f \in S\}$$

Therefore, $e(xy)^{-1}[1] = e(x)^{-1}[1] \cap e(y)^{-1}[1]$

By the Corollary of Theorem 1

R is osomorphic to a subring of the characteristic ring of the Stone space S .

Given $f \neq g$ in S . There is $x \in X$ s.t. $f(x) \neq g(x)$

Let $g(x) = 0$, $f(x) = 1$, gince $g \neq 0$

Therefore, $y \in X$ s.t.

$g(y) = 1$, $f(y) = 1$ or $f(y) = 0$

If $f(y) = 1$, then $f(x + y) = f(x) + f(y) = 0$

Therefore, there are two points x, y (or x + y) in X s.t.

$f(x) = 1$, $g(x) = 0$

$f(y) = 0$, $g(y) = 1$ (or $f(y) = 1$, $g(y) = 1$, then, $f(x + y) = 0$, $g(x + y) = 1$)

Therefore, $e(x)_f = 1$, $e(x)_g = 0$

$e(y)_f = 0$, $e(y)_g = 1$ (or $e(x + y)_f = 0$, $e(x + y)_g = 1$)

By Theorem 3 , $R^r \cong$ the characteristic ring of S.

Lemma 1: Let F be the characteristic ring of a Boolean space X . T is a maximal proper ideal in F iff for some $x \in X$, $T = \{f : f(x) = 0,\ f \in F\}$

Proof: Let T be a maximal proper ideal in F . Suppose there does not exist a x in X s.t. $T = \{f : f(x) = 0,\ f \in F\}$, then to each $x \in X$ there is a $f \in T$ and $f(x) = 1$

Let g be an element in F, $g^{-1}[1] = B$

For each $x \in B$, there is a $f \in T$ s.t. $f(x) = 1$, there is an open neighborhood Nx of x, $f(y) = 1$ for $y \in N_x$

Since B is compact, therefore, there is a finite number of functions $f_1, f_2, \ldots f_n$ in T with their corresponding Nx covering B.

Let

$$h = \sum_{i=1}^{n} f_i - \sum_{i<j} f_i f_j + \ldots + (-1)^{n+1} \bigcap_{i=1}^{n} f_i \text{, then}$$

$h \in T$, and $h(z) = 1$ for $z \in B$

$h \cdot g = g$, $h \cdot g \in T$. Therefore, $g \in T$.

Therefore, $T = F$. This is a contradiction to T being proper;maximal in F.

Let $T = \{ f{:}f(x) = 0 \; f \in F \}$ for some $x \in X$. Then T is a proper ideal in F. Suppose it is not a maximal ideal. Let E be an ideal. $E \supsetneq T$ and $E \neq F$, then there is a $h \in E - T$, $h(x) = 1$ for any $s \in F - T$, $(s + h)(x) = s(x) + h(x) = 0$

Therefore, $s + h = r \in T$, $s + (h + h) = r + h \in E$

$$s = r + h \in E, \; E = F$$

This contradicts $E \neq T$.

<u>Definition:</u> Let F be a family of functions such that each f F is on a topology sapce x to a space Y_f . The <u>evaluation map</u> is a map e on X into $\prod_{f \in F} Y_f$ s.t. $e(x)_f = f(x)$

<u>Definition:</u> A family F of functions on X <u>distinguishes points</u> iff for each pair of distinct points x and y, there is f in F s.t. $f(x) \neq f(y)$. The family <u>distinguishes points and closed sets</u> iff for each <u>closed</u> subset A of X and each $x \in X - A$, there is $f \in F$ s.t. $f(x) \notin f\,[A]$.

<u>Lemma 2:</u> Let F be a family of functions, each member f of F being on a topological space X to a topological space Y_f. Then

(a) The evaluation map e is continuous on X to the product space $X \{Y_f : f \in F\}$

(b) The function e is an open map of X onto e[X] if F distinguishes points and closed sets.

(c) The function e is one to one iff F distinguishes points.

<u>Proof:</u> (Kelley)

The map e followed by projection P_f into the f^{th} coordinate space is continuous because

$$P_f \cdot e(x) = f(x)$$

Choose a member f of F such that f(x) does not belong to the closure of $f(X - U)$. The set of all y in the product such that $y_f \notin \overline{f(X - U)}$ is open, and evidently its intersection with $e[x]$ is a subset of $e[U]$. Hence, e is an open map of X into $e[x]$.

Statement (c) is clear.

Theorem 5: A topological space can be embedded in a Cantor space if it is Boolean.

Proof:

Let X be a Boolean space, F its characteristic ring, and S the Stone space of F.

We consider the evaluation map e on X to 2^F.

$e(x)_f = f(x)$ for $x \in X$, $f \in 2^F$

Given $x \in X$

$e(x)^{-1}[0] = \{f:f(x) = 0 \ f \in 2^F\}$ by Lemma 1

is a proper maximal ideal in 2^F.

Conversely, given an element in S, then we can determine a proper maximal ideal in 2^F by Lemma 1. This proper maximal ideal is $T = \{f:f(y) = 0, f \in F\}$ for some $y \in X$

Therefore, $T = e(y)^{-1}[0]$

(1) e is continuous since $f \in F$ is continuous.

(2) Given a closed subset A in X, and a point x of X, $x \notin A$.

Let Nx be a compact-open neighborhood of x in $X - A$; then define a function of f.

$f(y) = 1$ for $y \in N_x$

$f(y) = 0$ for $y \notin N_x$

Thus, $f \in F$

$N_x \subset X - A$, $X - N_x \supset A$

$f(X - N_x) = 0$. $f(A) = 0$

Thus, $\overline{f(A)} = 0$, $f(x) \notin f(A)$

Therefore, e is an open mapping.

(3) Given $x \neq y$ in X, a, $b \in I_2$

there is $f \in F$ s.t. $f(x) = a$, $f(y) = b$

Thus e is 1 - 1

Therefore, R is homomorphic to S.

Let X be a topology space which can be embedded in a Cantor space C. Then

a. X is Hausdorff since C is Hausdorff.

b. C is compact and the subbase of C consists of open-closed sets. Therefore, the set of all open-compact sets in C is a base.

Thus the set of all open-compact sets in X is a base.

Therefore, X is Boolean.

Complete Lattice

Definition: A lattice is complete if every subset of it has at least upper bound.

Definition: A C ring of a Boolean space is the ring of all continuous functions f into I_2 such that $f^{-1}[1]$ is closed-open. (If a Boolean space is compace, the C ring is a characteristic ring.)

Theorem : A Boolean space has the property that the closure of every open set is open if and only if its C ring is complete.

Proof: Let X be a Boolean ring,

Then C is a lattice under the operations.

$(f \vee g)(x) = \text{maximum } [f(x), g(x)]$

$(f \wedge g)(x) = \text{minimum } [f(x), g(x)]$

And we write $f \geq g$ if $f(x) \geq g(x)$, $x \in X$

1. Suppose C is a complete lattice and E is an open set in X. Then $E = \bigcup_{\alpha} E_\alpha$ where E_α is open and closed. (In the bad case, we can choose E_α being the base elements of topology X.)

$K_E(x) = \sup_{\alpha}. f_\alpha(x)$, where f_α is the continuous function. s.t. $f_\alpha^{-1}[1] = E_\alpha$

Let $f_o = \vee f_\alpha$ be the least upper bound of the family $\{f_\alpha\}$

Then $K_E(x) \leq f_o(x)$ for $x \in X$

For each $x_o \notin \overline{E}$, we can have an open-closed neighborhood N_{x_o} of x_o

$$N_{x_o} \subset \overline{E}^c$$

Then there is a function g in C s.t.

$$g(z) = 1 \quad \text{for} \quad z \in N_{x_o}^c$$

$$g(z) = 0 \quad \text{for} \quad z \in N_{x_o}$$

Thus $g(x_o) = 0$ and $g(\overline{E}) = 1$

g is a bound for the family $\{f_\alpha\}$

Thus $K_E(x) \leq f_o(x) \leq g(x)$ for $x \in X$

Therefore $f_o(\overline{E}) = 1$, $f_o(\overline{E}^c) = 0$

As f_o is continuous, E is open and closed.

2. Suppose the closure of an open set in X is open.

Let $\{f_\alpha\}$ be a family of functions in C.

Then $\bigcup_\alpha f_\alpha^{-1}[1]$ is an open set in X.

Therefore $\overline{\bigcup_\alpha f_\alpha^{-1}[1]}$ is closed and open and any open-closed set N $\bigcup_\alpha f_\alpha^{-1}[1]$ contains $\overline{\bigcup f^{-1}[1]}$, since $\overline{N} = N$

Therefore, the function h defined on X s.t.

$$h(x) = 0 \qquad \text{for} \quad x \notin \bigcup_\alpha f_\alpha^{-1}[1]$$

$$h(x) = 1 \qquad \text{for} \quad x \in \bigcup_\alpha f_\alpha^{-1}[1]$$

$$h = \bigvee_\alpha f_\alpha$$

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