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Barypact Topological Spaces

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by

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TABLE OF CONTENTS

INTRODUCTION. 1
WEIGHTS AND BARYSUBSETS 2
THE STONE-WEIERSTRAUSS THEOREM 8
THE ASCOLI THEOREM. 16
THE DINI THEOREM 19
BIBLIOGRAPHY. 20

INTRODUCTION

Recently, Kimber [3] has discovered a general class of topological spaces, the members of which are termed barypact spaces, that includes the compact topological spaces. This class is distinct from the set of all compact topological spaces, but its members possess many of the useful properties associated with compactness. As a consequence, several standard compactness theorems become special cases of corresponding theorems in a more general setting and the techniques of proof applied to these extensions provide new, and sometimes remarkably simple, proofs of the very theorems they generalize. The purpose of this paper is to extend to this class three compactness theorems of topology: the Stone-Weierstrauss theorem, the Ascoli theorem, and the Dini theorem.

It is assumed throughout this paper that the reader is familiar with the standard set theoretic notation and with such concepts as topological space, compact topological space, metric space, continuity, convergence, uniform convergence, and so on. Sometimes theorems that are used in support of this paper, but are not directly part of it, will be stated without proof; however, sources for such material are included in the bibliography.

WEIGHTS AND BARYSUBSETS

DEFINITION. A <u>weighting</u> of a subset S of a topological space X is a classification of the subsets of S into two types, <u>light</u> <u>and</u> <u>heavy</u>, subject to the conditions that the empty set is light and the union of two subsets of S is light iff both subsets are light.

Obviously a subset of a light subset of S is light; or equivalently, a subset of S containing a heavy subset of S is heavy.

DEFINITION. Let S be a weighted subset of a topological space X. A point x of X is said to be a <u>barypoint</u> of S iff every neighborhood of x has a heavy intersection with S.

Since every neighborhood U of a barypoint of S has a heavy intersection with S, $U \cap S$ cannot be the empty set. Thus a barypoint of S is a point of closure of S.

DEFINITION. A subset S of a topological space X is said to be a <u>barysubset</u> of X iff whenever S is weighted and heavy, S has a barypoint.

As will be shown later in this paper, a subset of a compact subset of a space X is a barysubset of X. Since the closure of a bounded subset of R^n is compact, it follows that every bounded subset of R^n is a barysubset.

Some consequences of the above remarks that, in addition, illustrate methods of proof will now be given.

THEOREM 1. Let S be a finite subset of a topological space X. Then S is a barysubset of X.

Proof. Let S be weighted and heavy with no barypoint. Then for each x in S there exists a neighborhood U_x of x having a light intersection with S. But $S = \bigcup_{x \in S} (U_x \cap S)$ which is light. This contradiction establishes the theorem.

THEOREM 2. Let U be a barysubset of the space X with $V \subset U$. Then V is a barysubset of X.

Proof. Suppose V is weighted and heavy. Extend this weighting to U by taking a subset N of U heavy iff $N \cap V$ is heavy. This weights U and U is heavy. Let b be a barypoint of U. If W is any neighborhood of b, then $W \cap U$ is heavy. But $W \cap U$ is weighted as $(W \cap U) \cap V = W \cap V$. Thus $W \cap V$ is heavy and it follows that b is a barypoint of V.

THEOREM 3. The union of two barysubsets of a space X is a barysubset of X.

Proof. Let B_1, B_2 be barysubsets of X with $B = B_1 \cup B_2$ weighted and heavy. Assume without loss of generality that B_1 is heavy. Since each subset of B_1 is contained in B, B_1 is automatically weighted. Let b be a barypoint of B_1 and let W be any neighborhood of b. Then $W \cap B = (W \cap B_1) \cup (W \cap B_2)$ which is heavy. Hence b is a barypoint of B.

THEOREM 4. The continuous image of a barysubset is a barysubset.

Proof. Let f map the space X continuously into the space Y, and let B be a barysubset of X with f(B) weighted and heavy. Weight each subset N of B as f(N). This weights B and B is heavy. Let b be a barypoint of B. Then for each neighborhood U of b, $U \cap B$ is heavy. Let V be a neighborhood of f(b). Due to the continuity of f, $f^{-1}(V)$ is a neighborhood of b. Thus $f^{-1}(V) \cap B$ is heavy. But $f(f^{-1}(V) \cap B) \subset V \cap f(B)$ so that $V \cap f(B)$ is heavy. Thus f(b) is a barypoint of f(B).

THEOREM 5. The product $B_1 \times B_2$ of a barysubset B_1 of a space X_1 and a barysubset B_2 of a space X_2 is a barysubset of the product space $X_1 \times X_2$.

Proof. Let $B_1 \times B_2$ be weighted and heavy. Weight each subset U of B_1 as $U \times B_2$. This weights B_1, and B_1 is heavy. Let b_1 be a barypoint of B_1. Now define a weight on B_2 by taking a subset V of B_2 heavy iff $(U \times V) \cap (B_1 \times B_2)$ is heavy for each neighborhood U of b_1. This weights B_2, and B_2 is heavy. Let b_2 be a barypoint of B_2. Then (b_1, b_2) is a barypoint of $B_1 \times B_2$, for let W be any neighborhood of (b_1, b_2). Then W contains a set of the form $U \times V$ where U is a neighborhood of b_1 in X_1 and V is a neighborhood of b_2 in X_2. Consequently,

$W \cap (B_1 \times B_2) \supset (U \times V) \cap (B_1 \times B_2) = (U \times (V \cap B_2) \cap (B_1 \times B_2)$ which is heavy. Thus $W \cap (B_1 \times B_2)$ is heavy and the theorem is established.

Kimber [ibid.] has proved the following extension of the Tychonoff theorem.

THEOREM 6. If B_α is a barysubset of the space X_α for each α in an indexing set A, then $\prod_{\alpha \in A} B_\alpha$ is a barysubset of $\prod_{\alpha \in A} X_\alpha$.

The above theorems were obtained from basic compactness theorems by replacing the words "compact subset" by the word "barysubset".

The following theorems establish the relation between barysubsets and compactness.

THEOREM 7. A subset Y of a compact space X is a barysubset of X.

Proof. Let Y be weighted with no barypoint. Then for each x in X there exists an open neighborhood N_x of x having a light intersection with Y. Since X is compact, a finite number of the N_x cover X, say $N_{x_1}, N_{x_2}, \dots, N_{x_n}$, and consequently $Y = (Y \cap N_{x_1}) \cup (Y \cap N_{x_2}) \cdots \cup (Y \cap N_{x_n})$ which is light.

COROLLARY. A subset Y of a topological space X that is contained in a compact subset Z of X is a barysubset of X.

Proof. Y is a barysubset of Z by Theorem 7. If b is a barypoint of Y, the intersection of a neighborhood U of b in X with Y is heavy. This is true because $U \cap Z$ is a neighborhood of b in the relative topology for Z and $U \cap Y = (U \cap Z) \cap Y$ which is heavy.

THEOREM 8. If a topological space X is a barysubset of itself, then X is compact.

Proof. Let $\langle N_\alpha \rangle$ be an open cover of X. Weight X by taking a subset E of X light iff a finite number of the N_α cover E. Since every point x of X is in a light neighborhood N_α, X has no barypoint. Hence X is light.

Combining Theorems 7 and 8 we obtain

THEOREM 9. A topological space X is compact iff X is a barysubset of X.

The following example demonstrates that the converse of the corollary to Theorem 7 is false.

Let X be the closed unit disk in R^2 and Y its interior. Retopologize X by taking a neighborhood of a point of Y to be any subset of X containing an open disk in Y containing this point, and a neighborhood of a point x of $X-Y$ to be any subset of X containing the union of $\{x\}$ with the intersection of Y with a neighborhood of x in R^2. Recall that a compact subset of a Hausdorff space X is a closed subset of X (see [2]). Now <u>the set Y is contained in no compact subset of X</u>, for suppose $Y \subset Z \subset X$ with Z compact. Then $Z \neq X$ because the open cover $\{\{x\} \cup Y : x \in X-Y\}$ of X has no finite subcover. But X is Hausdorff and $\overline{Z} = X$

so that Z is not closed in X, a contradiction. However, as will be shown next, Y is a barysubset of X. For Y being bounded as a subspace of R^2 is a barysubset of R^2. If a point b of R^2 is a barypoint of Y in R^2, b is also a barypoint of Y in X since every neighborhood of b in X contains the intersection of Y with a neighborhood of b in R^2.

This example suggests the existence of a general class of topological spaces X that contain barysubsets that are contained in no compact subsets of X. In the example given, Y is dense in X in the topology considered. This suggests the following definition.

DEFINITION. A topological space X is said to be barypact iff X contains a dense barysubset.

An immediate consequence of this definition is that every compact topological space is barypact. The converse, as the example also points out, is false.

The following theorem will be important in the sections to come.

THEOREM 10. Let f be a continuous real valued function defined on the barypact space K. Then f is bounded.

Proof. Let S be a dense barysubset of K and take a subset A of S to light iff f(A) is bounded. This weights S. Suppose S is heavy and let b be a barypoint of S. It follows that f is unbounded on every neighborhood of b, a contradiction. Thus S is light and f(S) is bounded. Due to the continuity of f, $f(\overline{S}) \subset \overline{f(S)}$, and since $\overline{S} = K$, f is bounded.

THE STONE-WEIERSTRAUSS THEOREM

DEFINITION. A family A of real valued functions defined on a set E is said to be an <u>algebra</u> iff $f+g \in A$, $fg \in A$, and $cf \in A$ for all $f \in A$, $g \in A$, and all real constants c.

THEOREM 11. Let C(K) denote the algebra of all continuous real valued functions defined on the barypact space K. Then C(K) becomes a complete metric space when distance is defined by the rule: $d(f,g) = \sup_{x \in K}(|f(x)-g(x)|)$ for $f,g \in C(K)$.

The proof of the above theorem under the hypothesis that K is compact is readily available (see for example [4]). The role compactness plays in the proof is only to insure the boundedness of the members of C(K). Consequently, due to Theorem 10, the proof under the hypothesis that K is barypact is no different.

A consequence of Theorem 11 is that a sequence of functions $\langle f_n \rangle$ of C(K) converges to f in C(K) iff $\langle f_n \rangle$ converges uniformly to f on K. It follows that $\langle f_n \rangle$ is a Cauchy sequence in C(K) iff $\langle f_n \rangle$ converges oniformly on K. Also, observe that a member f of C(K) belongs to the closure of a subset A of C(K) iff there is a sequence of members of A converging uniformly to f on K.

DEFINITION. Let A be a family of real valued functions defined on a set E. Then A is said to <u>separate points</u> on E iff to each pair x_1, x_2 of distinct points of E and each pair a_1, a_2 of real numbers there corresponds $f \in A$ such that $f(x_1) = a_1$ and $f(x_2) = a_2$.

An example of an algebra which separates points is the set of all polynomials in one variable on R^1, for if x_1, x_2 are distinct points of R^1 and a_1, a_2 are any real numbers, the polynomial $P(x) = a_2(x-x_1)(x_2-x_1)^{-1} + a_1(x-x_2)(x_1-x_2)^{-1}$ has the property that $P(x_1) = a_1$ and $P(x_2) = a_2$. An example of an algebra which does not separate points is the set of all even polynomials on $[-1,1]$, since $f(-x) = f(x)$ for every even function f.

The goal of this section is the following extension of the Stone-Weierstrauss theorem.

THEOREM 12. Let C(K) be as in Theorem 11. Then a subalgebra A of C(K) which separates points on K is dense in C(K).

This theorem states that every member of C(K) is the uniform limit of a sequence of functions of A; or equivalently that every member of C(K) can be uniformly approximated arbitrarily closely by a member of A. Before proceeding with the proof, the following lemma (see [1]) will be needed.

LEMMA 1. Let $e>0$ and $\alpha \leq \varphi \leq \beta$ be any real interval. Then there exists a polynomial $P(\varphi)$ in the real variable φ with $P(0) = 0$ and such that $\left| |\varphi| - P(\varphi) \right| < e$ for $\alpha \leq \varphi \leq \beta$.

Proof. If the point $\varphi=0$ does not belong to the interval $[\alpha,\beta]$ it will suffice to take $P(\varphi) = \pm\varphi$ according as $\alpha>0,\beta>0$, or $\alpha<0,\beta<0$. Thus there is no loss of generality in considering only intervals of the form $[-\tau,\tau]$, since $[\alpha,\beta]$ can be included in an interval of this form. Furthermore, it is sufficient to confine our attention to the interval $[-1,1]$ since if $Q(\eta)$, $Q(0)=0$, is a polynomial such that $\left| |\eta| - Q(\eta) \right| < e/\tau$ for $-1\leq\eta\leq 1$ and $\tau>0$, then $P(\varphi)=\tau Q(\varphi/\tau)$ is a polynomial such that $P(0)=0$ and $\left| |\varphi| - P(\varphi) \right| < e$ for $-\tau\leq\varphi\leq\tau$. Having made this observation, define a sequence of constants recursively as follows: Let $\alpha_1=1/2$, $\alpha_k=1/2\sum_{m+n=k}\alpha_m\alpha_n= 1/2(\alpha_1\alpha_{k-1}+\alpha_2\alpha_{k-2}+\cdots+\alpha_{k-1}\alpha_1)$.

Then $\alpha_k>0$ for all k because $a_1>0$ and assuming $\alpha_n>0$ for all $n<M$, $\alpha_M=\sum_{i+j=M}\alpha_i\alpha_j= 1/2(\alpha_1\alpha_{M-1}+\cdots+\alpha_{M-1}\alpha_1) > 0$. Set $\pi_n =\sum_{k=1}^{n}\alpha_k$. Then $\pi_n<1$ for all n because $\pi_1<1$ and $\pi_n<1$ implies $\pi_{n+1} =\sum_{r=1}^{n+1}\alpha_r = \alpha_1 +\sum_{r=2}^{n+1}\alpha_r = 1/2 + 1/2\sum_{r=2}^{n+1}\sum_{i+j=r}\alpha_i\alpha_j$

$\leq 1/2 + 1/2\sum_{i,j=1}^{n}\alpha_i\alpha_j = 1/2 + 1/2(\pi_n^2) = 1/2(1 + \pi_n^2) <1/2(2)$

$= 1$. Thus the positive term series $\sum_{n=1}^{\infty}\alpha_n$ converges to a sum δ satisfying the inequality $\delta\leq 1$, and it follows that the series $\sum_{n=1}^{\infty}\alpha_n\varphi^n$ converges uniformly for $-1\leq\varphi\leq 1$ to a continuous function $\delta(\varphi)$. Consider the following idenity:

$$\left[\sum_{i=1}^{n} \alpha_i \omega^i\right]\left[2 - \sum_{j=1}^{n} \alpha_j \omega^j\right]$$

$$= 2\sum_{i=1}^{n} \alpha_i \omega^i - \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \omega^{i+j}$$

$$= 2\sum_{i=1}^{n} \alpha_i \omega^i - \left[\sum_{\substack{i+j\le n\\ 1\le i,j<n}} \alpha_i \alpha_j \omega^{i+j} + \sum_{\substack{i+j\ge n+1\\ 1\le i,j\le n}} \alpha_i \alpha_j \omega^{i+j}\right]$$

$$= 2\sum_{i=1}^{n} \alpha_i \omega^i - \left[\sum_{k=2}^{n} \omega^k \sum_{\substack{i+j=k\\ 1\le i,j<k}} \alpha_i \alpha_j + \sum_{\substack{i+j\ge n+1\\ 1\le i,j\le n}} \alpha_i \alpha_j \omega^{i+j}\right]$$

$$= 2\sum_{i=1}^{n} \alpha_i \omega^i - \left[2\sum_{k=2}^{n} \alpha_k \omega^k + \sum_{\substack{i+j\ge n+1\\ 1\le i,j\le n}} \alpha_i \alpha_j \omega^{i+j}\right]$$

$$= \omega - \sum_{\substack{i+j\ge n+1\\ 1\le i,j\le n}} \alpha_i \alpha_j \omega^{i+j}.$$

The final term is estimated as follows:

$$\left|\sum_{\substack{1\le i,j\le n\\ i+j\ge n+1}} \alpha_i\alpha_j\varphi^{i+j}\right| \le \sum_{\substack{1\le i,j\le n\\ i+j\ge n+1}} \alpha_i\alpha_j|\varphi|^{i+j} \le \sum_{\substack{1\le i,j\le n\\ i+j\ge n+1}} \alpha_i\alpha_j$$

$$\le \sum_{i+j\ge n+1} \alpha_i\alpha_j = \sum_{k=n+1}^{\infty}\sum_{i+j=k}\alpha_i\alpha_j$$

$$= 2\sum_{k=n+1}^{\infty}\alpha_k$$

As $n\longrightarrow\infty$, this last term approaches zero and passage to the limit in the above idenity yields $\delta(\varphi)(2-\delta(\varphi))=\varphi$. For each φ such that $-1\le\varphi\le 1$ we have $\delta(\varphi) = 1 \pm \sqrt{1-\varphi}$. Now $\delta(1)=1$ independently of the choice of sign and hence $\sum_{k=1}^{\infty}\alpha_k = \delta(1) = 1$. Since α_k is positive, it follows that

$\delta(\varphi) \le \delta(|\varphi|) \le \delta(1) = 1$ for $|\varphi| \le 1$ and hence the lower sign is the proper choice. Thus the power series for $\sqrt{1-\varphi}$ is given by $\sqrt{1-\varphi} = 1 - \delta(\varphi) = 1 - \sum_{k=1}^{\infty}\alpha_k\varphi^k = \sum_{k=1}^{\infty}\alpha_k(1-\varphi^k)$

the series being uniformly convergent for $-1\le\varphi\le 1$. If $-1\le\varphi\le 1$, then $0\le 1-\varphi^2\le 1$ and hence $\varphi = \sqrt{\varphi^2} = \sqrt{1-(1-\varphi^2)} = \sum_{k=1}^{\infty}\alpha_k(1-(1-\varphi^2)^k)$. As noted above, the series is uniformly

convergent and its general term is a polynomial which vanishes for $\varphi=0$. Hence a suitable one of its partial sums will serve as the required polynomial $P(\varphi)$.

The proof of Theorem 12 will now proceed with the following lemmas.

LEMMA 2. If $f\in A$, then $|f|\in \bar{A}$.

Proof. Since K is barypact, f is bounded. Assuming that $\alpha\leq f(x)\leq\beta$ for each x in K, by the preceeding lemma there exists for each n a polynomial $P_n(\varphi)$ such that $\left| |\varphi| - P_n(\varphi)\right| < 1/n$ for $\alpha\leq\varphi\leq\beta$ and $P_n(0) = 0$. It is clear that $P_n(f)\in A$ and that $\left| |f(x)| - P_n(f(x))\right| < 1/n$ for each x in K. Thus $|f|$ is the uniform limit of a sequence of functions of A and hence belongs to $\bar{A}$.

LEMMA 3. If $g\in\bar{A}$, then $|g|\in \bar{A}$.

Proof. For each n, select $g_n\in A$ such that $d(g_n,g) < 1/n$ (here d is the metric constructed for C(K) in Theorem 11). The lemma now follows from the observation that $d(|g_n|,|g|) \leq d(g_n,g)$, the fact that $\bar{\bar{A}} = \bar{A}$, and Lemma 2.

LEMMA 4. If $f,g\in A$, then $\max(f,g)\in\bar{A}$ and $\min(f,g)\in\bar{A}$.

Proof. This lemma follows from the relations

$\max(f,g) = 1/2(f+g + |f-g|)$

$\min(f,g) = 1/2(f+g - |f-g|)$.

LEMMA 5. If $f,g\in\bar{A}$, then $\max(f,g)\in\bar{A}$ and $\min(f,g)\in\bar{A}$.

Proof. This follows from Lemmas 3 and 4 and the easily verified observation that $\bar{A}$ is itself an algebra.

LEMMA 6. Given a function f continuous on K, a point $x\in K$, and $e>0$, there exists a function $g_x \in \bar{A}$ such that $g_x(x) = f(x)$ and $g_x(t) \geq f(t) - e$ for each $t \in K$.

Proof. Let S be a dense barysubset of K and call a subset B of S heavy iff there exists no function $g \in \bar{A}$ such that $g(x)=f(x)$ and $g(t) \geq f(t)-e$ for all $t \in B$. This weights S because the empty set is light and B,C both light subsets of S implies there exist functions g_B, g_C belonging to $\bar{A}$ such that $g_B(x)=f(x)$, $g_C(x)=f(x)$, $g_B(t) \geq f(t)-e$ for each $t \in B$, and $g_C(t) \geq f(t)-e$ for each $t \in C$. Define $g_{B\cup C}=\max(g_B, g_C)$. Then $g_{B\cup C} \in \bar{A}$ by Lemma 5, and $g_{B\cup C}(x) =f(x)$, $g_{B\cup C}(t) \geq f(t)-e$ for each $t \in B\cup C$. Thus $B\cup C$ is light. The remaining portion of the demonstration that S is weighted is trivial. Proceeding with the proof, suppose that S is heavy and let b be a barypoint of S. Since $A \subset \bar{A}$ and A separates points, $\bar{A}$ separates points. Hence there exists a function $h \in \bar{A}$ such that $h(x)=f(x)$ and $h(b)=f(b)$. Since h is continuous (because the uniform limit of a sequence of functions of A is continuous), there exists a neighborhood U of b such that $h(t)>f(t)-e$ for all $t \in U$. This contradicts the fact that $U \cap S$ is heavy. Hence the existence of a function $g_x \in \bar{A}$ such that $g_x(x)=f(x)$ and $g_x(t) \geq f(t)-e$ for each $t \in S$ is established. Since $S \subset \{t: g_x(t) \geq f(t)-e\}$ and the latter is closed in K, $K= \bar{S} \subset \{t: g_x(x) \geq f(t)-e\}$ and the theorem is established.

LEMMA 7. Given a function f continuous on K and $e>0$, there exists a function $h \in \bar{A}$ such that $|h(x)-f(x)| \leq e$ for each $x \in K$.

Proof. Let S be a dense barysubset of K. There exist closed neighborhoods V_x of x such that $g_x(t) \leq f(t)+e$ for each $t \in V_x$, where g_x is constructed for each $x \in K$ as in Lemma 6. Weight S by taking $B \subset S$ light iff a finite number of the V_x cover $\bar{B}$. Suppose S is heavy and let b be a barypoint of S. Then $V_p \cap S$ is heavy which is impossible since $\overline{V_p \cap S} \subset V_p$. Thus S is light so that $\bar{S} = K = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_n}$. Let $h=\min(g_{x_1}, \ldots, g_{x_n})$. Then $h \in \bar{A}$ and $h(t) \leq f(t)+e$ for each $t \in K$. But from Lemma 6, $h(t) \geq f(t)-e$ for each $t \in K$. Thus $|h(t)-f(t)| \leq e$ for each $t \in K$, as required.

The theorem now follows. For let $f \in C(K)$. Then for each n there exists a function $f_n \in \bar{A}$ such that $|f_n(x)-f(x)| < 1/n$ for all $x \in K$. Thus the sequence $\langle f_n \rangle$ of functions of $\bar{A}$ converges uniformly to f on K, whence $f \in \bar{\bar{A}} = \bar{A}$. From this it follows that $\bar{A} = C(K)$ and the proof is complete.

THE ASCOLI THEOREM

DEFINITION. Let $\langle f_n \rangle$ be a sequence of real valued functions defined on a set E. Then $\langle f_n \rangle$ is said to be <u>pointwise bounded</u> on E iff there exists a finite real valued function φ defined on E such that $|f_n(x)| \leq \varphi(x)$ for each $x \in E$ and all n.

An example of a pointwise bounded sequence is the sequence $\langle (-1)^n (nx)^{-1} \rangle$ defined on the half open interval $]0,1]$. Here φ is defined by $\varphi(x)=(x)^{-1}$ for each $x \in]0,1]$.

DEFINITION. Let F be a family of functions from a topological space X to a metric space $[Y,d]$. Then F is said <u>to be equicontinuous at the point $x \in X$</u> iff for each $e>0$ there exists a neighborhood U of x such that $d(f(x),f(y)) < e$ whenever $y \in U$ and $f \in F$. The family F is said to be <u>equicontinuous on X</u> iff F is equicontinuous at each point of X.

It is clear that each member of an equicontinuous family is a continuous function.

DEFINITION. Let $e>0$. A sequence $\langle a_n \rangle$ of real numbers is said to be an <u>e-sequence</u> iff $|a_n - a_m| < e$ for all m,n. A sequence $\langle g_n \rangle$ of real valued functions defined on a set Y is called an <u>e-sequence of functions</u> iff $\langle g_n(x) \rangle$ is an e-sequence for all $x \in Y$.

The purpose of this section is the following extension of the Ascoli theorem.

THEOREM 13. Given an equicontinuous pointwise bounded sequence $\langle f_n\rangle$ of real valued functions defined on the barypact space X, there exists a uniformly convergent subsequence.

Proof. Let S be a dense barysubset of X and $e>0$. Take $A\subset S$ to be light iff every subsequence of $\langle f_n|\bar{A}\rangle$ contains an e-subsequence. This weights S, for suppose A,B are light subsets of S. Let $\langle h_n\rangle$ be a subsequence of $\langle f_n|\overline{A\cup B}\rangle$. Since A is light, there exists a subsequence $\langle g_n\rangle$ of $\langle h_n\rangle$ such that $\langle g_n|\bar{A}\rangle$ is an e-subsequence. Since B is light, there exists a subsequence $\langle k_n\rangle$ of $\langle h_n\rangle$ such that $\langle k_n|\bar{B}\rangle$ is an e-subsequence. Then $\langle k_n|\overline{A\cup B}\rangle$ is an e-subsequence of $\langle h_n\rangle$, and $A\cup B$ is light. Proceeding with the proof, suppose that S is heavy with b a barypoint of S. Let W be a closed neighborhood of b such that for all n and all x in W, $|f_n(x)-f_n(b)|\leq e/3$. Since $W\cap S$ is heavy, some subsequence $\langle g_n\rangle$ of $\langle f_n|\overline{W\cap S}\rangle = \langle f_n|W\rangle$ contains no e-subsequence. Since $\langle g_n(b)\rangle$ is bounded, there exists a subsequence $\langle h_n\rangle$ of $\langle g_n\rangle$ such that $\langle h_n(b)\rangle$ is an e/3 sequence (extract a convergent subsequence $\langle k_n(b)\rangle$ of $\langle g_n(b)\rangle$. Then $\langle k_n(b)\rangle$ is a Cauchy sequence so there exists an index N such that $m,n>N$ imply $|k_m(b)-k_n(b)|<e/3$. Define $h_n(b) = k_{N+n}(b)$). if $x\in W$ we have $|h_n(x)-h_m(x)|\leq|h_n(x)-h_n(b)|+|h_m(b)-h_n(b)|+|h_m(b)-h_m(x)| < 3(e/3) = e$, so that $\langle h_n\rangle$ is an e-subsequence of $\langle g_n\rangle$.

This is a contradiction. Hence S is light. Thus we have that every subsequence of $\langle f_n \rangle$ contains an e-subsequence for all $e>0$. The theorem now follows, for let $\langle f_{1,n} \rangle$ be a 1-subsequence of $\langle f_n \rangle$, let $f_{2,n}\rangle$ be a 1/2-subsequence of $\langle f_{1,n} \rangle$, let $\langle f_{3,n} \rangle$ be a 1/3-subsequence of $\langle f_{2,n} \rangle$, and so on. The subsequence $\langle g_n \rangle = \langle f_{n,n} \rangle$ of $\langle f_n \rangle$ is a Cauchy sequence in $C(X)$, for let $e>0$ and select $k>1/e$. Then for $m,n>k$ we have $d(g_n,g_m) \leq \max(1/m,1/n) < e$. Thus $\langle g_n \rangle$ converges uniformly on X.

THE DINI THEOREM

THEOREM 14 (Dini). Let $\langle f_n \rangle$ be a monotonically increasing sequence of continuous real valued functions defined on the barypact space X which converges pointwise to a continuous function f. Then $\langle f_n \rangle$ converges uniformly to f on X.

Proof. Let S be a dense barysubset of X, let $e>0$, and define $G_n = \{x: f_n(x)-f(x) \leq e\}$. Call a subset A of S light iff there exists an index N such that $\overline{A} \subset G_N$. This defines a weight on S, for suppose A,B are both light subsets of S. Then there exists an index M and an index N such that $\overline{A} \subset G_M$ and $\overline{B} \subset G_N$. Let $K=\max(M,N)$. Since the sequence $\langle f_n - f \rangle$ is monotonically decreasing, $G_M \subset G_K$ and $G_N \subset G_K$. Thus $\overline{A \cup B} \subset G_K$ and $A \cup B$ is light.
Now suppose that S is heavy and let b be a barypoint of S. Then each closed neighborhood of b cannot be contained in any G_n. Since $\langle f_n \rangle$ converges pointwise to f, there exists an index $N=N(b)$ such that $|f_N(b)-f(b)| < e/3$. Due to the continuity of f_N and f, there exists a neighborhood U of b such that for all $x \in U$, $|f_N(x)-f(x)| < e/3$ and $|f(x)-f(b)| < e/3$. But $|f_N(x)-f(x)| \leq |f_N(x)-f_N(b)| + |f_N(b)-f(b)| + |f(x)-f(b)| < 3(e/3) = e$ for all $x \in U$, and consequently $U \subset G_N$. Since G_N is closed, $\overline{U} \subset G_N$ and this is a contradiction. Hence S is light. Let $\overline{S}=X=G_K$. Then $|f_K(x)-f(x)| \leq e$ for all $x \in X$. If $L \geq K$, then $0 \leq |f_L(x)-f(x)| \leq |f_K(x)-f(x)| \leq e$ for all $x \in X$. Thus $f_n \longrightarrow f$ uniformly on X.

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