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A density for a Generalized Likelihood-Ratio Test When the Sample Size is a Random Variable

Raymond H. Neville
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A DENSITY FOR A GENERALIZED LIKELIHOOD-RATIO TEST
WHEN THE SAMPLE SIZE IS A RANDOM VARIABLE

by

Raymond H. Neville

A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Applied Statistics

Approved:

UTAH STATE UNIVERSITY
Logan, Utah

1966
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Raymond H. Neville
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INTRODUCTION AND REVIEW OF LITERATURE

Many articles, books, papers, and abstracts have been published, which describe the analysis and design of experiments. R. A. Fisher’s (1951) book, The Design of Experiments, is referred to by D. J. Finney (1960) as the classic for experimental designs. Some of the more outstanding books and publications in this area are those by Cochran and Cox (1957), Cox (1958), Davis et al., (1954), Federer (1955), Quenouille (1953), and Kempthorne (1952).

In most of these publications, the underlying model for the analysis, the assumptions necessary for the correct inferences, and detailed descriptions of the appropriate Analysis of Variance (A. N. O. V.) are clearly and explicitly presented. It is of interest to note, however, that the above information is based upon the assumption (although not explicitly mentioned) that the size of the samples in an experiment is a predetermined fixed quantity. That is, the experimenter, after he chooses an appropriate design, will determine the size of the samples before the experiment is actually performed. In so doing, he is also making the assumption that even though the experiment is repeated, the sample sizes will remain the same and will not vary.

The actual process of the experiment might result in lost or destroyed observations, however, making the sample sizes fluctuate
or vary considerably from experiment to experiment. This variation is an indication of the randomness of the sample sizes. If there could be assigned to each possible value of the sample sizes a probability of its occurrence, then the sample sizes could be interpreted as random variables. This concept leads one to investigate the effects it might have on the A. N. O. V. for experimental designs, or on the numerous tests of hypotheses that one commonly performs, or even on the analysis of missing observations.

Although much information can be found pertaining to the analysis and design of experiments when the sample size is fixed, as indicated earlier, little has been done on this subject when sample sizes are considered to be random variables. The Statistical Theory and Method Abstracts, which give a review for all of the major statistical journals from 1959 to 1965, have been reviewed as most of the statistical literature available in the Utah State University Library, and relatively little information was found which pertained to this problem as is outlined.

Because of the apparent lack of material on this aspect of the analysis of the designs of experiments, this work will be primarily a preliminary investigation or inquiry into the effects that the assumption of random sample size might have on the tests of hypotheses in experimental designs. Also, since this is a preliminary examination, this investigation will be restricted to the simplest of designs: one-way classification.
The main objective of this work will be to examine the hypothesis that all the treatment means are the same and equal to some unknown quantity, when we know that the variance is the same for each sample, and to determine if the conventional method for making this test (the F-test) is applicable when the sample sizes are assumed to be random variables. This hypothesis can be tested by using a likelihood-ratio test. To do this, a density function or distribution has to be found for this ratio, thus permitting us to make probability statements about the occurrence of this ratio under the null hypothesis.

Throughout this development, reference will be made to many concepts of which understanding will be essential to the comprehension of the methods that have been used. Thus, a brief introduction and review will be attempted in the next few pages to prepare the reader for the material to follow. It will be assumed that the reader will have a knowledge of basic statistical terms, such as: sample, random samples, population, experimental unit, treatment, statistic, and other common, general expressions.

**One-way Classification**

The one-way classification, or Completely Randomized Design, is the most elementary of the experimental designs. It is represented symbolically in Tables 1 and 2. In these tables, there are t-treatments allotted at random to N' experimental units, yielding N and \( n_i \).
observations to the $i^{th}$ treatment, respectively. $Y_{ij}$ represents the $j^{th}$ observation in the $i^{th}$ treatment for each table. 

Table 1. Symbolic representation of data in a one-way classification, $N$ observations in $i^{th}$ treatment

<table>
<thead>
<tr>
<th>Treatments</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$i$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_{11}$</td>
<td>$y_{11}$</td>
<td>$y_{21}$</td>
<td>$y_{31}$</td>
<td>$\ldots$</td>
<td>$y_{i1}$</td>
</tr>
<tr>
<td>$y_{12}$</td>
<td>$y_{12}$</td>
<td>$y_{22}$</td>
<td>$y_{32}$</td>
<td>$\ldots$</td>
<td>$y_{i2}$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>Observations</td>
<td>$y_{1j}$</td>
<td>$y_{2j}$</td>
<td>$y_{3j}$</td>
<td>$\ldots$</td>
<td>$y_{ij}$</td>
</tr>
<tr>
<td>$y_{1N}$</td>
<td>$y_{1N}$</td>
<td>$y_{2N}$</td>
<td>$y_{3N}$</td>
<td>$\ldots$</td>
<td>$y_{1N}$</td>
</tr>
</tbody>
</table>

Designs are usually represented by models. A model can be defined or thought of as a mathematical equation involving random variables, mathematical variables, and parameters. The distribution of the random variables, if it is known, is considered part of the model. A model for the one-way classification is given by

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij},$$

where $\mu$ is the overall mean of the experiment, $\tau_i$ is the deviation of the $i^{th}$ treatment mean ($\bar{Y}_i$) from the overall mean ($\mu$), and $\epsilon_{ij}$ is the deviation of the $j^{th}$ observation in the $i^{th}$ treatment ($Y_{ij}$) from
the \( i \text{th} \) treatment mean \( \bar{Y}_{i.} \) and \( \epsilon_{ij} \) is normally distributed with mean zero and variance, \( \sigma^2 \).

\[
\epsilon_{ij} \sim N(0, \sigma^2).
\]

Table 2. Symbolic representation of data in a one-way classification, \( n_i \) observations in \( i\text{th} \) treatment

<table>
<thead>
<tr>
<th>Treatments</th>
<th>i</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_{11} )</td>
<td>( Y_{12} )</td>
<td>( Y_{13} )</td>
</tr>
<tr>
<td>( y_{12} )</td>
<td>( y_{22} )</td>
<td>( y_{32} )</td>
</tr>
<tr>
<td>( y_{13} )</td>
<td>( y_{23} )</td>
<td>( y_{33} )</td>
</tr>
<tr>
<td>( \cdots ) &amp; ( \cdots ) &amp; ( \cdots ) &amp; ( \cdots )</td>
<td>( \cdots ) &amp; ( \cdots )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( y_{ij} ) &amp; ( y_{2j} ) &amp; ( y_{3j} ) &amp; ( \cdots )</td>
<td>( y_{ij} ) &amp; ( \cdots )</td>
<td>( y_{tj} )</td>
</tr>
<tr>
<td>( \cdots ) &amp; ( \cdots ) &amp; ( \cdots ) &amp; ( \cdots )</td>
<td>( \cdots ) &amp; ( \cdots )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( y_{ln1} ) &amp; ( y_{2n2} ) &amp; ( y_{3n3} ) &amp; ( \cdots )</td>
<td>( y_{lnj} ) &amp; ( \cdots )</td>
<td>( y_{tnl} )</td>
</tr>
</tbody>
</table>

The specification of this model is not complete because nothing has been said about the \( \tau_i \). There are two possibilities:

(a) the researcher can be concerned only with the \( t \) treatments in the experiment, in which case one interprets the \( \tau_i \) as being a fixed effect and that \( \sum_{i=1}^{t} \tau_i = 0 \), or

(b) he can be concerned with a population of treatments of which the \( t\text{th} \) treatment is a random sample. The latter
case implies that $\tau_i$ is a random effect and that $\tau_i \sim \text{NID}(0, \sigma^2)$. These two possibilities are often expressed by designating the models as Model I if $\tau_i$ is a fixed effect and as Model II if $\tau_i$ is a random effect.

One is interested when working with more than one treatment in examining various hypotheses concerning the effects of the treatments or about the populations of the treatments. Before actually making the necessary tests, certain assumptions must be made about the models. These assumptions are outlined very well in Eisenhart's (1947) papers. For the one-way classification they are: (a) the observations $Y_{ij}$ are normally and independently distributed with mean $\mu_i$ and equal variances $\sigma^2 \left( Y_{ij} \sim \text{NID}(\mu_i, \sigma^2) \right)$; (b) homoscedasticity; (c) $\epsilon_{ij} \sim \text{NID}(0, \sigma^2)$; and (d) whether the model is fixed or random. The description of the tests that are made can be given in many ways. One method used is that of the likelihood-ratio test.

**Test of Hypotheses**

Testing hypotheses in general involves the setting up of a hypothesis denoted $H_0$ concerning a phenomenon in nature, and then through experimentation and sample evidence accepting or rejecting the hypothesis. It is important to note that a general hypothesis can never be proved, but can be disproved. When the experimenter takes observations and uses them as a basis for rejecting or accepting an
hypothesis, he is liable to two kinds of error - the Type I error and the Type II error. The Type I error is the rejection of a hypothesis when it is true. The Type II error is the acceptance of a hypothesis when it is false. Ideally, we would like to minimize the possibility of making either of these types of errors. One usually decides on the Type I error that is permissible and then minimizes the Type II error or maximizes the power of the test. The power of the test $B(\theta)$ is defined as $B(\theta) = 1 - P(II)$, where $P(II)$ is the probability of the Type II error. The power of the test $B(\theta)$ is the probability of rejecting the hypothesis when it is false. It is general practice to choose $P(I)$, the probability of making the Type I error, in advance, and then to maximize $B(\theta)$. A test which gives certain optimum properties is the likelihood-ratio test.

**Likelihood-ratio Test**

Define the parameter space $\Omega$ to be the set of all values that the parameters $\theta_1, \theta_2, \ldots, \theta_n$ can have and let $\omega$ denote a subspace of $\Omega$. If we have a frequency function $f(x, \theta_1, \theta_2, \ldots, \theta_n)$, then for a sample of size $n$, the likelihood function is $L = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2, \ldots, \theta_n)$. If we want to test the hypothesis $H_0 \left[ (\theta_1, \theta_2, \ldots, \theta_n) \in \Omega \right]$ against the alternative hypothesis $H_A \left[ (\theta_1, \theta_2, \ldots, \theta_n) \in \Omega - \omega \right]$, we form the ratio

$$\lambda = \frac{L(\hat{\theta})}{L(\tilde{\theta})}.$$
In the ratio above, \( L(\hat{\omega}) \) is the maximum of the likelihood function in the region \( \omega \) with respect to the parameters that are in \( \omega \), and \( L(\hat{\Omega}) \) is the maximum of the likelihood function in the region \( \Omega \) with respect to the parameters \( \theta_1, \theta_2, \ldots, \theta_n \) that are in \( \Omega \).

\( \lambda \) is such that \( 0 \leq \lambda \leq 1 \). We will reject the hypothesis if \( L(\hat{\omega}) \) is distant from \( L(\hat{\Omega}) \), and accept the hypothesis if \( L(\hat{\omega}) \) is close to \( L(\hat{\Omega}) \). We need to fix the Type I error \((\alpha)\) and find a constant \( A \) so that the rejection region is between 0 and \( A \). Thus, when the hypothesis \( H_0 \) is true, the Type I error, \( P(I) \), will be

\[
P(I) = \int_{0}^{A} g(\lambda/H_0) \, d\lambda = \alpha,
\]

where \( g(\lambda/H_0) \) is the distribution for the likelihood-ratio. Thus, if \( \lambda \) falls in the region 0 to \( A \), then the hypothesis \( H_0 \) is rejected. If \( \lambda \) falls in the region \( A \) to 1, then the hypothesis is accepted.

The rest of this work will be devoted to: (a) the creation of a joint density for the observations when the sample sizes are considered to be random variables, (b) the development of a likelihood function for the joint density, and the maximum likelihood estimators for the parameters in \( \omega \) and \( \Omega \), and (c) making the test \( H_0 (\mu_1 = \mu_2 = \mu_3 = \ldots = \mu_n = \mu) \) when the sample variances are assumed to be the same unknown quantity to determine the distribution of \( \lambda \). The important points will be summarized in the conclusion.
PROCEDURE AND RESULTS

Joint Density for the Observations

Consider the simplest of the experimental designs, a one-way classification, with \( t \) treatments and \( N \) experimental units per treatment, as is represented symbolically in Table 1. If we assume that there is one observation in each experimental unit, the performance of the actual experiment might result in some of the observations being lost or destroyed in some way, thus resulting in fewer experimental units per treatment and/or fewer treatments if all experimental units are lost in any one treatment. Now, it seems feasible that there could be associated with each observation a probability of its being present or absent after the experiment is performed. In other words, each experimental unit has the possibility of being lost or destroyed. This would result in the sample size associated with each treatment to vary. That is, it would become a random variable whose range of values would be from 0 to \( N \).

Let us represent this idea symbolically. Note that in Table 1, \( Y_{ij} \) represents the \( j^{th} \) experimental unit or observation in the \( i^{th} \) treatment, and that before the experiment is performed there are \( N \) observations in each treatment. Associate with each \( Y_{ij} \) a random variable \( X_{ij} \), which takes on the possible values 0 and 1, such that, if \( X_{ij} = 0 \), then \( Y_{ij} \) is absent or lost, and if \( X_{ij} = 1 \), then
$Y_{ij}$ is present in the experiment. Denote the probability that $X_{ij}$ is one, $P(X_{ij} = 1)$, by $p_{ij}$ and the probability that $X_{ij}$ is zero, $P(X_{ij} = 0)$, by $q_{ij}$ where $q_{ij} = 1 - p_{ij}$ and $p_{ij} + q_{ij} = 1$. Let the $X_{ij}$'s be independent of one another. By independent is meant the occurrence of any one of the $X_{ij}$'s in no way affects the probability of the occurrence of the other $X_{ij}$'s. It is clear that each $X_{ij}$ has two possible outcomes, zero and one. Therefore, in any trial, $X_{ij}$ will be zero or one, and the probability density function for $X_{ij}$ is

$$f(X_{ij}) = \begin{cases} p_{ij} & \text{if } X_{ij} = 1 \\ q_{ij} & \text{if } X_{ij} = 0 \end{cases}$$

If we have $N$ trials, then the probability that there will be $n_{ij}$ ones is

$$P(n_{ij}) = \binom{N}{n_{ij}} p_{ij}^{n_{ij}} q_{ij}^{N-n_{ij}}.$$

Now examine an experiment with a total of $N'$ observations, $N$ observations per treatment with $t$ groups or treatments. If observation $Y_{ij}$ is present, then $X_{ij}$ will be one. If observation $Y_{ij}$ is absent (destroyed), then $X_{ij}$ will be zero. Therefore, the probability that $Y_{ij}$ is present in the experiment is

$$P(Y_{ij} \text{ is present}) = P(X_{ij} = 1),$$
and the probability that \( Y_{ij} \) is absent (destroyed) is

\[
P(Y_{ij} \text{ is destroyed}) = P(X_{ij} = 0).
\]

Note also that the number of observations in the \( i^{th} \) treatment, after the experiment has been performed, is equal to the number of \( X_{ij} \)'s, \( j = 1, 2, 3, \ldots, N \) that are equal to one, i.e., \( n_i \) = the number of observations in \( k^{th} \) treatment = \( \sum_{j=1}^{N} X_{ij} \). Thus, the probability that the number of observations in the \( i^{th} \) treatment is \( n_i \) is given by

\[
P(n_i \text{ observations in } i^{th} \text{ treatment}) = \binom{N}{n_i} n_i^{n_i} (N-n_i)^{N-n_i}
\]

where \( 0 \leq n_i \leq N \).

As was mentioned in the introduction, one of the assumptions associated with the one-way classification is that the observations \( Y_{ij} \) are independent normally distributed random variables with mean \( \mu_i \) and variances \( \sigma^2 \), \( Y_{ij} \sim \text{NID}(\mu_i, \sigma^2) \). The density function for \( Y_{ij} \) is

\[
f(Y_{ij}) = \begin{cases} 
\left( \frac{1}{2\pi\sigma^2} \right)^{1/2} \exp \left( -\frac{(Y_{ij} - \mu_i)^2}{2\sigma^2} \right), & -\infty < Y_{ij} < \infty, \\
0, & \text{otherwise}
\end{cases}
\]
With the same assumption holding for each \( Y_{ij} \), and the added association of \( X_{ij} \) with \( Y_{ij} \), the proposed distribution for the observations will now be conditional on \( X_{ij} \) and will be given by

\[
f(Y_{ij} / X_{ij}) = \begin{cases} 
\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(Y_{ij} - \mu_1)^2}{2\sigma^2} \right] & \text{for } -\infty < Y_{ij} < \infty \text{ given that } X_{ij} = 1 \\
0 & \text{for } -\infty < Y_{ij} < \infty \text{ given that } X_{ij} = 0 \\
1 & \text{for } Y \text{ destroyed given that } X_{ij} = 0 \\
0 & \text{for } Y \text{ destroyed given that } X_{ij} = 1 
\end{cases}
\]

(2)

In order to investigate (2) to see if it is a density function, it will be necessary to examine the sample description space associated with (2). A sample description space \( S \) is defined as the set of all possible outcomes of an experiment. Therefore, the sample description space for (2) is

\[
S = \{ Y_{ij} \mid Y_{ij} \text{ is any real number or } Y_{ij} \text{ does not exist} \}
\]

That is, \( S \) is composed of the set of all positive or negative numbers, and those points where \( Y_{ij} \) does not exist. \( S \) can be divided into two events. The event that \( Y_{ij} \) exists and the event that \( Y_{ij} \) does not exist. In symbolic notation,
\[ S = \{E_1 \cup E_2\} , \text{ where} \]
\[ E_1 = \{Y_{ij} \mid Y_{ij} \text{ is any real number}\} \quad \text{and} \]
\[ E_2 = \{Y_{ij} \mid Y_{ij} \text{ does not exist}\} . \]

A density must satisfy these two rules:

i) \( f(Q) \geq 0 \)

ii) \( \int_{S} f(Q) \, dQ = 1 \).

It is obvious that rule i) is satisfied by our density, for \( f(Y_{ij}/X_{ij}) \) is always equal to or greater than zero when \( Y_{ij} \) is or is not present, because \( f(Y_{ij}/X_{ij}) \) is the normal distribution if \( Y_{ij} \) is present given that \( X_{ij} = 1 \) and \( f(Y_{ij}/X_{ij}) \) is equal to one when \( Y_{ij} \) is absent given that \( X_{ij} = 0 \), both of which are greater than zero.

Also, \( f(Y_{ij}/X_{ij}) = 0 \), when \( Y_{ij} \) is present given that \( X_{ij} = 0 \), and when \( Y_{ij} \) is absent given that \( X_{ij} = 1 \). (2) also satisfies rule ii), since

\[
\int_{S} f(Y_{ij}/X_{ij}) \, dY_{ij} = \int_{E_1} f(Y_{ij}/X_{ij}) \, dY_{ij} + \int_{E_2} f(Y_{ij}/X_{ij}) \, dY_{ij} .
\]

\[ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3) \]

When \( X_{ij} = 0 \) or \( X_{ij} = 1 \), (3) equals one. This is obvious since
\[
\int_{E_1} f(Y_{ij}/X_{ij} = 0) \, dY_{ij} + \int_{E_2} f(Y_{ij}/X_{ij} = 0) \, dY_{ij} = 1,
\]

and

\[
\int_{E_1} f(Y_{ij}/X_{ij} = 1) \, dY_{ij} + \int_{E_2} f(Y_{ij}/X_{ij} = 1) \, dY_{ij} = 1.
\]

for

\[
\int_{E_1} f(Y_{ij}/X_{ij} = 0) \, dY_{ij} = \int_{E_2} f(Y_{ij}/X_{ij} = 1) \, dY_{ij} = 0,
\]

and

\[
\int_{E_1} f(Y_{ij}/X_{ij} = 1) \, dY_{ij} = \int_{E_2} f(Y_{ij}/X_{ij} = 0) \, dY_{ij} = 1.
\]

Therefore, (3) is always one which implies that (2) is a density function. The joint density of \( Y_{ij} \) and \( X_{ij} \) is

\[
g(Y_{ij}, X_{ij}) = f(Y_{ij}/X_{ij}) \, P(X_{ij}).
\]

That is,
\[ g(Y_{ij}, X_{ij}) = \begin{cases} 
0 & \text{for } -\infty < Y_{ij} < \infty, X_{ij} = 0 \text{ and } 0 < p_i < 1 \\
\frac{X_{ij}}{2\pi\sigma^2} \exp \left[ -\frac{X_{ij}^2 (Y_{ij} - \mu_i)^2}{2\sigma^2} \right] q_i & \text{for } -\infty < Y_{ij} < \infty, X_{ij} = 1, \text{ and } 0 < q_i < 1 \\
0 & \text{for } Y \text{ destroyed, } X_{ij} = 1, 0 < p_i < 1 \\
p_i & \text{for } Y \text{ destroyed, } X_{ij} = 0, 0 < p_i < 1 
\end{cases} \] (4)

The Sample Description Space \( S \) for this joint distribution is:

\[ S = \{(Y_{ij}, X_{ij}) \mid -\infty < Y_{ij} < \infty, Y_{ij} \text{ is destroyed, } X_{ij} = 0, 1\}. \]

\( S \) can be partitioned into four subsets or events:

\[ S = (E_1 \cup E_2 \cup E_3 \cup E_4) \text{ where} \]

\[ E_1 = \{(Y_{ij}, X_{ij}) \mid (Y_{ij} \text{ is any real number and } X_{ij} = 0)\}, \]

\[ E_2 = \{(Y_{ij}, X_{ij}) \mid (Y_{ij} \text{ is any real number and } X_{ij} = 1)\}, \]

\[ E_3 = \{(Y_{ij}, X_{ij}) \mid (Y_{ij} \text{ does not exist and } X_{ij} = 0)\}, \text{ and} \]

\[ E_4 = \{(Y_{ij}, X_{ij}) \mid (Y_{ij} \text{ does not exist and } X_{ij} = 1)\}. \]
It is obvious from (4) that \( f(Y_{ij}, X_{ij}) \geq 0 \) and therefore, rule i) is satisfied. Rule ii) will also be satisfied, since,

\[
\int_1^{\infty} \sum_{X_{ij} = 0}^1 q(Y_{ij}, X_{ij})dY_{ij} = \int_1^{\infty} \sum_{X_{ij} = 0}^1 f(Y_{ij} / X_{ij}) \cdot P(X_{ij})dY_{ij}
\]

\[
= \int_1^{\infty} \left( f(Y_{ij} / X_{ij} = 0) \cdot q_1 + f(Y_{ij} / X_{ij} = 1) \cdot p_1 \right) dY_{ij}
\]

\[
= q_1 \left( \int_{E_1} f(Y_{ij} / X_{ij} = 0) dY_{ij} + \int_{E_2} f(Y_{ij} / X_{ij} = 0) dY_{ij} \right)
\]

\[
+ p_1 \left( \int_{E_1} f(Y_{ij} / X_{ij} = 1) dY_{ij} + \int_{E_2} f(Y_{ij} / X_{ij} = 1) dY_{ij} \right)
\]

\[
= q_1 + p_1 = 1.
\]

Therefore, (4) is a density function.

It can be shown, Parzen (1960), and Feller (1957), that if \( A \) and \( B \) are independent random variables, then their joint density can be obtained by the product of their respective density functions. That is,

\[
f(A, B) = f(A)f(B)
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5)
\]
Since the $Y_{ij}$'s, $j = 1, 2, \ldots, N$, are independent normally distributed random variables, their joint density given the corresponding $X_{ij}$'s, $j = 1, 2, \ldots, N$, is

$$f(Y_{i1}', Y_{i2}', \ldots, Y_{iN}', X_{i1}, X_{i2}, \ldots, X_{iN})$$

$$= \prod_{j=1}^{N} f(Y_{ij}'/X_{ij}) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp \left(\sum_{j=1}^{N} X_{ij}^2 \frac{(Y_{ij}' - \mu_j)^2}{2\sigma^2}\right)$$

for $-\infty < Y_{ij}' < \infty$ given $X_{ij} = 1$,

$Y_{ij}$ destroyed given $X_{ij} = 0$;

$$= 0$$

otherwise.

The joint density for the $Y_{ij}$ and $X_{ij}$, $j = 1, \ldots, N$, is

$$f(Y_{i1}, \ldots, Y_{iN}, X_{i1}, \ldots, X_{iN}) = f(Y_{i1}', \ldots, Y_{iN}/X_{i1}', \ldots, X_{iN})$$

or,

$$f(Y_{i1}', \ldots, Y_{iN}', X_{i1}', \ldots, X_{iN}') = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp \left(\sum_{j=1}^{N} X_{ij}^2 \frac{(Y_{ij}' - \mu_j)^2}{2\sigma^2}\right) \cdot \binom{N}{n_i} p_i^{n_i} q_i^{N-n_i}$$
for $-\infty < Y_{ij} < \infty$, and $X_{ij} = 1$, $Y_{ij}$ destroyed and $X_{ij} = 0$, $0 \leq p_{ij}, q_{ij} \leq 1$, $0 \leq n_{ij} \leq N$, and 0 otherwise.

$$
\text{for } \ldots \ldots \ldots \ldots \ldots.
$$

This is a density function because each density $f(Y_{ij}, X_{ij})$ is greater than or equal to zero for all values of $Y_{ij}$ and $X_{ij}$, thus, the product of positive quantities will be positive, and

$$
\sum_{S} \sum_{X_{ij} = 0} f(Y_{ij}, X_{ij}) dY_{ij} = 1.
$$

Therefore,

$$
\sum_{S_{1}} \sum_{S_{2}} \ldots \sum_{S_{N}} \sum_{X_{11} = 0} \sum_{X_{i2} = 0} \ldots \sum_{X_{iN} = 0} \prod_{j=1}^{N} f(Y_{ij}, X_{ij}) dY_{ij}
$$

$$
= \prod_{j=1}^{N} \sum_{S_{j}} \sum_{X_{ij} = 0} f(Y_{ij}, X_{ij}) dY_{ij} = 1,
$$

and rule ii) is satisfied.

The joint distribution of the $Y_{ij}$'s and $X_{ij}$'s for all of the $t$ treatments is given by

$$
f(Y_{1l'}, Y_{12'}, \ldots, Y_{1N'}, Y_{2l'}, \ldots, Y_{2N'}, \ldots, Y_{tN'}, X_{1l'}, \ldots, X_{1N'}, \ldots, X_{tN'}) = \prod_{i=1}^{t} f(Y_{ii'}, \ldots, Y_{iN'/X_{il'}}, \ldots, X_{iN'})
$$
\[ P(X_{11}, \ldots, X_{1N}) = \prod_{i=1}^{t} \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{n_i}{2}} \exp \left[ -\sum_{j=1}^{N} \frac{X_{ij}(Y_{ij} - \mu_1)^2}{2\sigma^2} \right] \left( \begin{array}{c} N \\ n_i \end{array} \right)^{n_i} \left( \begin{array}{c} N-n_i \\ \mu_i \end{array} \right) \] 

for \(-\infty < Y_{ij} < \infty\) and \(X_{ij} = 1, Y_{ij} = 0\), \(0 \leq n_i \leq N\), \(1 \geq p_i, q_i \geq 0\), \(p_i + q_i = 1\), and 0 otherwise. By the same reasoning as that used previously, it can be shown that (7) is also a density function.

Now that we have the joint density function, we are in a position to test the hypothesis, \(H_0: (\mu_1 = \mu_2 \ldots = \mu_t = \mu_0)\). As was mentioned in the introduction, we can make the test by the likelihood-ratio criterion; however, to do this we need the maximum likelihood estimators for the unknown parameters. This, then, will be the next topic discussed.

**Maximum Likelihood Estimators**

As was indicated in the introduction, to find the maximum likelihood estimators of the unknown parameters, we maximize the likelihood functions with respect to the unknown parameters over the regions in which these parameters are defined. We, in essence, have two regions, \(\omega\) and \(\Omega\).

The Likelihood Function for the Region $\omega$

The subspace $\omega$ is defined as follows:

$$\omega = \{\mu, \sigma^2, p_1 | \mu = \mu_0, \sigma^2 > 0, 0 \leq p_1 \leq 1 \}.$$

Therefore, the likelihood function defined on the region $\omega$ is

$$L(\mu, \sigma^2, p_1) = \prod_{i=1}^{t} \left( \frac{1}{2\pi\sigma^2} \right)^{n_i/2} \exp \left( \sum_{i=1}^{t} \sum_{j=1}^{N} \frac{X_{ij}(Y_{ij} - \mu_0)^2}{2\sigma^2} \right).$$

or

$$L(\mu, \sigma^2, p_1) = \left( \frac{1}{2\pi\sigma^2} \right)^{\sum_{i=1}^{t} n_i/2} \exp \left[ \sum_{i=1}^{t} \sum_{j=1}^{N} \frac{X_{ij}(Y_{ij} - \mu_0)^2}{2\sigma^2} \right].$$

The natural log of this function gives

$$L^*(\mu, \sigma^2, p_1) = -\sum_{i=1}^{t} \ln(2\pi\sigma^2) \sum_{i=1}^{t} \sum_{j=1}^{N} \frac{X_{ij}(Y_{ij} - \mu_0)^2}{2\sigma^2}. $$
\begin{align*}
  &+ \sum_{i=1}^{t} \left[ \ln \left( \frac{N}{n_i} \right) + n_i \ln(p_i) + (N - n_i) \ln(q_i) \right].
\end{align*}

Now, we want to find the estimates of the parameters which will maximize \( L(\mu_0, \sigma^2, p_i) \). To do this, we need to take the partial derivatives of \( L(\mu_0, \sigma^2, p_i) \) with respect to the parameters \( \mu_0, \sigma^2 \), and \( p_i \), and by setting these first order derivatives to zero solve for \( \mu_0, \sigma^2 \), and \( p_i \), if the values exist.

\begin{align*}
\frac{\partial L^* (\mu_0, \sigma^2, p_i)}{\partial \mu_0} &= -2 \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \mu_0)^2}{2\sigma^2} (-1)
\end{align*}

Setting this quantity to zero implies that

\begin{align*}
\hat{\mu}_0 &= \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} Y_{ij}}{\sum_{i=1}^{t} n_i} = \frac{Y_{..}}{t}.
\end{align*}

\begin{align*}
\frac{\partial L^* (\mu_0, \sigma^2, p_i)}{\partial \sigma^2} &= -\sum_{i=1}^{t} \frac{n_i}{\sigma^2} + \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \frac{Y_{..}}{t})^2}{2(\sigma^2)^2}.
\end{align*}

Setting this quantity to zero implies that

\begin{align*}
-\sum_{i=1}^{t} \frac{n_i}{\sigma} \frac{\sigma^2}{4} + \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \frac{Y_{..}}{t})^2}{\sigma^4} &= 0.
\end{align*}
This implies that

\[ \hat{\omega}^2 = \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} \chi_{ij} (Y_{ij} - \bar{Y})^2}{\sum_{i=1}^{t} n_i} \]

Setting this quantity to zero implies that

\[ \frac{n_i}{p_i} - \frac{(N-n_i)}{(1-p_i)} = \frac{n_i - n_i p_i - p_i N + n_i p_i}{p_i (1 - p_i)} = 0 . \]

This implies that

\[ \hat{p}_i = \frac{n_i}{N} \]

The parameter space \( \Omega \) is defined as the set of all values of \( \mu_i, \mu_2, \ldots, \mu_t, \sigma^2 \), and \( p_i, i=1, 111, t \), such that \( \mu_i \neq \mu_o, \sigma^2 > 0 \), and \( 1 \geq p_i \geq 0 \). Symbolically, \( \Omega = \{ \mu_i, p_i, i=1, \ldots, t, \sigma^2 | \mu_i \neq \mu_o, \sigma^2 > 0, 0 \leq p_i \leq 1 \} \). The likelihood function defined on the parameter space \( \Omega \) is

\[ L(\mu_1, \mu_2, ..., \mu_t, \sigma^2, p_i) = \prod_{i=1}^{t} \left( \frac{1}{2\pi \sigma^2} \right)^{n_i/2} \]
\[
\exp \left\{ \frac{-\sum_{j=1}^{N} X_{ij} (Y_{ij} - \mu_i)^2}{2\sigma^2} \right\} \cdot \left( \frac{N}{n_i} \right)^{n_i} \frac{(N-n_i)}{q_i} \cdot P(n_1, \ldots, n_t).
\]

The natural log of this function gives
\[
\log L(\mu_1, \ldots, \mu_t, \sigma^2, p_i) = \frac{-\sum_{i=1}^{t} n_i}{2 \cdot 2\sigma^2} \ln(2\pi\sigma^2)
\]
\[
- \sum_{i=1}^{t} \sum_{j=1}^{N} \frac{X_{ij} (Y_{ij} - \mu_i)^2}{2\sigma^2} + \ln \left( P(n_1, n_2, \ldots, n_t) \right).
\]

The maximum likelihood estimators are obtained as follows:
\[
\frac{\partial L^*(\mu_1, \ldots, \mu_t, \sigma^2, p_i)}{\partial \mu_i} = -2 \sum_{j=1}^{N} \frac{X_{ij} (Y_{ij} - \mu_i)^2}{2\sigma^2} (-1)
\]
\[
\frac{N}{\sigma^2} \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \mu_i)^2}{2\sigma^2} \frac{(-1)}{2}\]
Setting this quantity to zero implies that

\[
\hat{\mu}_i = \frac{\sum_{j=1}^{N} X_{ij} Y_{ij}}{\sum_{j=1}^{N} X_{ij}} = \overline{Y}_i, \quad i = 1, 2, \ldots, t,
\]

and

\[
\frac{\partial L^*}{\partial \sigma^2} = \frac{t}{\sum_{i=1}^{t} n_i} \frac{1}{\sigma^2}
\]

\[
= \frac{t}{\sum_{i=1}^{t} n_i} \frac{1}{\sigma^2} \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \overline{Y}_{ij} - \overline{Y}_i)^2
\]

Setting this quantity to zero implies that

\[
\frac{t}{\sum_{i=1}^{t} n_i} \sigma^2 + \frac{t}{\sum_{i=1}^{t} n_i} \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \overline{Y}_{ij} - \overline{Y}_i)^2 \]

\[- \frac{t}{\sum_{i=1}^{t} n_i} \sigma^2 + \frac{t}{\sum_{i=1}^{t} n_i} \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \overline{Y}_{ij} - \overline{Y}_i)^2 = 0
\]

\[
\hat{\sigma}^2 = \frac{t}{\sum_{i=1}^{t} n_i} \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \overline{Y}_i)^2
\]

\[
\hat{\Omega} = \frac{t}{\sum_{i=1}^{t} n_i}
\]

and
\[ \hat{p}_\Omega = \frac{n_i}{N} \]

**Likelihood-ratio**

It was indicated in the introduction that to test the hypothesis \( H_0(\theta \in \omega) \) against the hypothesis \( H_A(\theta \in \Omega - \omega) \) we need to calculate the following ratio

\[
\lambda = \frac{L(\hat{\theta})}{L(\hat{\theta})} = \frac{L(\hat{\mu}_o, \hat{\sigma}^2_\omega)}{L(\hat{\mu}_1, \ldots, \hat{\mu}_t, \hat{\sigma}^2_\Omega)}
\]

For our problem, the likelihood-ratio is

\[
\lambda = \frac{\sum_{i=1}^{t} \frac{n_i}{2} \left( \frac{1}{2\pi\hat{\sigma}^2_\omega} \right) \exp \left( -\frac{1}{2\hat{\sigma}^2_\omega} \left( \sum_{i=1}^{t} \sum_{j=1}^{N} x_{ij}(y_{ij} - \bar{y})^2 \right) \right) \ P(n_1, n_2, \ldots, n_t)}{\sum_{i=1}^{t} \frac{n_i}{2} \left( \frac{1}{2\pi\hat{\sigma}^2_\Omega} \right) \exp \left( -\frac{1}{2\hat{\sigma}^2_\Omega} \left( \sum_{i=1}^{t} \sum_{j=1}^{N} x_{ij}(y_{ij} - \bar{y})^2 \right) \right) \ P(n_1, \ldots, n_t)}
\]

but in \( \omega \)

\[
\hat{\sigma}^2_\omega = \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} x_{ij}(y_{ij} - \bar{y})^2}{\sum_{i=1}^{t} n_i}
\]
and in $\Omega$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \bar{Y}_i)^2}{\sum_{i=1}^{t} n_i}.$$ 

Substituting equals for equals and cancelling like terms, the likelihood-ratio becomes

$$\lambda = \frac{\left[ \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \bar{Y}_i)^2 \right]}{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \bar{Y}_i)^2} = \frac{\left[ \sum_{i=1}^{t} n_i \right]}{2}.$$ 

(8)

It can be shown quite easily that

$$\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \bar{Y}_i)^2 = \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \bar{Y}_i)^2 + \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (\bar{Y}_i - \bar{Y})^2;$$

hence, by substituting this relationship into the expression above, (8) becomes
\[ \chi = \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \bar{Y}_{1i}).^2}{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \bar{Y}_{1i}). + \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (\bar{Y}_{1i} - \bar{Y}_{..}).^2} \] t \sum_{i=1}^{n_i} \frac{n_i}{2} (9) 

\[ \chi = \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \bar{Y}_{1i}).^2}{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (\bar{Y}_{1i} - \bar{Y}_{..}).^2} \] t \sum_{i=1}^{n_i} \frac{n_i}{2} (10) 

Note 1

\[ V = \frac{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (\bar{Y}_{1i} - \bar{Y}_{..}).^2}{\sigma^2} \] 

is distributed as a Chi-square variate with \( t - 1 \) degrees of freedom, where

\[ \ell = \sum_{i=1}^{t} \left[ 1 - \sum_{k=0}^{n_i} \binom{n_i}{k} (-1)^k \right] . \]
The relationship for \(l\) is obtained by noting that if all the observations in any one treatment are missing, then likewise so is the treatment. This is represented symbolically in this manner:

\[
X = \begin{cases} 
0 & \text{if } \text{ij}^{th} \text{ observation is missing} \\
1 & \text{if } \text{ij}^{th} \text{ observation is present}
\end{cases}
\]

\[
\frac{N}{\prod_{j=1}^{l} (1 - X_{ij})} = \begin{cases} 
0 & \text{if at least one observation in } i^{th} \text{ treatment is present} \\
1 & \text{if all observations in } i^{th} \text{ treatment are missing}
\end{cases}
\]

\[
\frac{N}{1 - \prod_{j=1}^{l} (1 - X_{ij})} = \begin{cases} 
0 & \text{if all observations in } i^{th} \text{ treatment are missing} \\
1 & \text{if at least one observation is present in } i^{th} \text{ treatment}
\end{cases}
\]

\[
N \frac{1}{\prod_{j=1}^{l} (1 - X_{ij})} \text{ can be represented by } \sum_{k=0}^{n_1} (-1)^k \binom{n_1}{k}, \text{ by noting that}
\]

\[
(X - Y)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X^k Y^{n-k}
\]

Let \(X = Y = 1\), and define \(0^0 = 1\), then

\[
\frac{N}{\prod_{j=1}^{l} (1 - X_{ij})} = (1 - 1)^{n_1} = 0^{n_1} = \begin{cases} 
0 & \text{if } n_1 > 0 \\
1 & \text{if } n_1 = 0
\end{cases}
\]

This implies that if \(X = Y = 1\), then
\[(X - Y)^{n_i} = \sum_{k=0}^{n_i} (-1)^k \binom{n_i}{k} \]

hence,

\[\ell = \sum_{i=1}^{t} \sum_{j=1}^{N} (1 - \Pi_j (1 - X_{ij})) = \sum_{i=1}^{t} \sum_{k=0}^{n_i} (-1)^k \binom{n_i}{k}.\]

The range for \(\ell\) is 0 to \(t\).

Note 2

\[U = \sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - Y_i)^2\]

is distributed as a Chi-square variate with \(\sum_{i=1}^{t} (n_i - l)\) degrees of freedom.

Note 3

\[\frac{V}{\frac{m}{U}}, \text{ where } m = l - 1 \text{ and } n = \sum_{i=1}^{t} (n_i - l)\]

are the degrees of freedom associated with the Chi-square variates \(V\) and \(U\), respectively, is distributed as an F variate with \(m\) and \(n\) degrees of freedom. This relationship is conditional on holding the \(n_i\)'s constant.
Note 4

The distribution for $F$, Mood and Graybill (1963), is given by

$$h(F) = \frac{\left(\frac{m+n-2}{2}\right)!}{\left(\frac{m-2}{2}\right)! \left(\frac{n-2}{2}\right)!} \frac{m^2}{F} \frac{m-2}{\left(1 + \frac{m}{n} F\right)^{\frac{m+n}{2}}}$$

for $F > 0$.

Using the information in the above notes, we see that

$$\frac{t \sum_{i=1}^{t} \left( \sum_{j=1}^{N} X_{ij} (Y_{ij} - \overline{Y_i})^2 \right)}{\sigma^2 ( \ell - 1 )} \frac{t}{\sum_{i=1}^{t} \sum_{j=1}^{N} X_{ij} (Y_{ij} - \overline{Y_i})^2} \frac{t}{\sum_{i=1}^{t} (n_i - 1)}$$

$$\sigma^2 \left( \sum_{i=1}^{t} (n_i - 1) \right)$$

is distributed as an $F$ variate with $m = \ell - 1$ and $n = \sum_{i=1}^{t} (n_i - 1)$ degrees of freedom, and therefore (10) becomes

$$\lambda = \left[ \frac{1}{1 + \frac{1}{\sum_{i=1}^{t} (n_i - 1) \frac{F}{\overline{F}(\ell - 1)}}} \right]^{2}$$

\[
\begin{bmatrix}
\frac{t}{\sum_{i=1}^{t} n_i} \\
\frac{1}{\sum_{i=1}^{t} (n_i - 1) \frac{F}{\overline{F}(\ell - 1)}}
\end{bmatrix}
\]

(13)
By a simple transformation, Parzen (1960), and Mood and Graybill (1963), of variables and letting $m = t - 1$, $Z = \Sigma n_i$, and $n = \sum (n_i - 1)$, we can find the distribution of $\lambda$ as follows:

$$\lambda = \left( \frac{1}{1 + \frac{m}{n} F} \right)^{Z/2}$$

Solving for $F$, gives

$$\lambda^{2/Z} = \left( \frac{1}{1 + \frac{m}{n} F} \right)$$

$$\lambda^{2/Z} \left( 1 + \frac{m}{n} F \right) = 1$$

$$\frac{m}{n} F = \left( \lambda^{-2/Z} - 1 \right)$$

$$F = \left( \lambda^{-2/Z} - 1 \right) \left( \frac{n}{m} \right)$$

$$\left| \frac{dF}{d\lambda} \right| = \left[ (2/Z) \lambda^{-2/Z} - 1 \right] \left( \frac{n}{m} \right)$$
\[ h(\lambda) = \frac{\binom{m+n-2}{2}}{(\binom{m-2}{2})! (\binom{n-2}{2})!} \left( \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right)^{m/2} \left( \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right)^{m/2} \left[ 1 + \frac{m}{n} \cdot \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right]^{m+n} \]

\[
= C \cdot \binom{m}{n}^{m/2} \cdot \binom{n}{m}^{m/2} \cdot \binom{2}{Z} \left[ \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right]^{m/2} \left[ \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right]^{m/2} \frac{m+n}{\lambda^{2/Z} + 1 - \lambda^{2/Z}} \frac{m+n}{\lambda^{2/Z}} \frac{1}{\lambda^{(2/Z+1)}}
\]

\[
= C \cdot \frac{2/Z}{\lambda^{2/Z}} \left[ \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right]^{m/2} \left[ \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right]^{m/2} \frac{m+n}{\lambda^{2/Z} + 1 - \lambda^{2/Z}} \cdot \frac{m+n}{\lambda^{2/Z}} \frac{1}{\lambda^{(2/Z+1)}}
\]

\[
= C \cdot \frac{2/Z}{\lambda^{2/Z}} \left[ \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right]^{m/2} \left[ \frac{1-\lambda^{2/Z}}{\lambda^{2/Z}} \right]^{m/2} \cdot \frac{m+n}{\lambda^{2/Z} + 1 - \lambda^{2/Z}} \cdot \lambda^{-(2/Z+1)}
\]

\[
= C \cdot \frac{2/Z}{\lambda^{2/Z}} \left[ 1 - \lambda^{2/Z} \right]^{m/2} \left[ \lambda^{2/Z} \right]^{m/2} \cdot \frac{m+n}{\lambda^{2/Z} + 1 - \lambda^{2/Z}} \cdot \lambda^{-(2/Z+1)}
\]

\[
= C \cdot \frac{2/Z}{\lambda^{2/Z}} \left[ 1 - \lambda^{2/Z} \right]^{m/2} \left[ \lambda^{2/Z} \right]^{m/2} \cdot \left[ \lambda^{2/Z} \right]^{-1} \cdot \lambda^{-2Z/2Z}
\]
\[
\begin{align*}
\frac{m-2}{2} \left( \frac{n+2-2}{2} \right) \cdot \left( \frac{2}{Z} \right)^2 \\
\frac{m-2}{2} \left( \frac{n-Z}{2} \right)
\end{align*}
\]

where \( m = \ell - 1, \quad n = \sum_{i=1}^{N} \left( \sum_{j=1}^{t} X_{ij} - 1 \right), \quad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}, \quad \text{and} \quad C = \frac{\binom{m+n-2}{2}}{\binom{m-2}{2} \binom{n-2}{2}}.
\]

Making the substitutions mentioned above, \( h(\lambda) \) becomes

\[
h(\lambda) = \frac{\left( \sum_{i=1}^{t} X_{ij} - 3 \right)!}{(\ell-3)!} \cdot \frac{2}{\sum_{i=1}^{m} X_{ij} - \ell - 2} \\
\left( \frac{2}{\sum_{i=1}^{m} X_{ij} - \ell - 2} \right)! \\
\left[ \left( \frac{2}{\sum_{i=1}^{m} X_{ij} - \ell - 2} \right)^2 \right] \\
\left[ \left( \frac{2}{\sum_{i=1}^{m} X_{ij} - \ell - 2} \right)^2 \right]
\]

and in terms of \( n_i = \sum_{j=1}^{N} X_{ij} \)

\[
h(\lambda) = \frac{\left( \sum_{i=1}^{n_i} n_i - 3 \right)!}{(\ell-3)!} \cdot \frac{2}{\sum_{i=1}^{n_i} n_i - \ell - 2} \\
\left( \frac{2}{\sum_{i=1}^{n_i} n_i - \ell - 2} \right)! \\
\left[ \left( \frac{2}{\sum_{i=1}^{n_i} n_i - \ell - 2} \right)^2 \right] \\
\left[ \left( \frac{2}{\sum_{i=1}^{n_i} n_i - \ell - 2} \right)^2 \right]
\]

where \( 0 \leq \lambda \leq 1 \).
A preliminary investigation of the distribution for \( \lambda \) that has just been obtained, reveals that this distribution depends upon the values of \( n_i, i = 1, 2, \ldots, t \), and is therefore conditional on \( n_i, i = 1, 2, \ldots, t \).

This can be represented as

\[
h(\lambda/n_1, n_2, \ldots, n_t) = \frac{\left(\frac{\Sigma n_i - \ell - 2}{\Sigma n_i}\right)!}{\left(\frac{\ell - 3}{2}\right)! \left[\Sigma n_i - \ell - 2\right]!} \left[\frac{2}{\Sigma n_i}\right]
\]

\[
\quad \left[\frac{\ell - 3}{2} \lambda^{\ell - 2} \right] \quad 0 < \lambda < 1.
\]

The joint density for \( \lambda \) and \( n_1, n_2, \ldots, n_t \) is

\[
h(\lambda, n_1, n_2, \ldots, n_t) = h(\lambda/n_1, n_2, \ldots, n_t) \cdot P(n_1, n_2, \ldots, n_t)
\]

where

\[
P(n_1, n_2, \ldots, n_t) = \prod_{i=1}^{t} \left(\frac{N}{n_i}\right)^{n_i} q_i^{N-n_i}.
\]
The distribution for $\lambda$ is found by summing (17) over all possible values for the $n_i$'s, $i=1, 2, \ldots, t$. That is

$$h(\lambda) = \sum_{n_i=0}^{N} \sum_{n_t=0}^{N} h(\lambda/n_1, \ldots, n_t) \cdot P(n_1, \ldots, n_t).$$

(18)

To simplify the form of this distribution, make the substitution

$$W = \frac{2}{\Sigma n_i}.$$

This implies that

$$\lambda = W \frac{\Sigma n_i}{2} \quad \text{and} \quad \left| \frac{d\lambda}{dW} \right| = \frac{t}{W} \left( \frac{\Sigma n_i/2 - 1}{2} \right).$$

Making the appropriate substitutions,

$$h(W/n_1, n_2, \ldots, n_t) = \frac{\left( \frac{\Sigma n_i - 3}{2} \right)!}{(\ell-3)! \left( \frac{\Sigma n_i - \ell - 2}{2} \right)!} \left( \frac{2}{\Sigma n_i} \right)^{\ell-3} \left( \frac{\Sigma n_i}{2} \right)^{\ell}\left(1-W\right)^{\ell-3} \cdot W^{-\ell/2} \cdot W^{-(\Sigma n_i/2 - 1)}$$

$$= \frac{\left( \frac{\Sigma n_i - 3}{2} \right)!}{(\ell-3)! \left( \frac{\Sigma n_i - \ell - 2}{2} \right)!} \left( \frac{\Sigma n_i - \ell - 2}{2} \right)^{\ell-3} \left(1-W\right)^{\ell-3} \left( \frac{\Sigma n_i}{2} \right)^{\ell}\left(1-W\right)^{\ell-3} \cdot W^{-\ell/2} \cdot W^{-(\Sigma n_i/2 - 1)}.$$

(19)
The joint distribution for $W$ and $n_i$, $i = 1, \ldots, t$ is

$$h(W, n_1, \ldots, n_t) = h(W/n_1, \ldots, n_t) \cdot P(n_1, \ldots, n_t),$$
or

$$h(W, n_1, \ldots, n_t) = \frac{\left(\frac{\sum n_i - 3}{2}\right)!}{\left(\frac{t-3}{2}\right)! \left(\frac{N-n_1-l-2}{2}\right)!} \left(1 - W\right)^{\frac{t-3}{2}}$$

Summing over all possible values of $n_1, \ldots, n_t$ will give the density for $W$, i.e.,

$$g(W) = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \ldots \sum_{n_t=0}^{N} h(W, n_1, \ldots, n_t)$$

This, then gives us a density function for $W$. To complete the test of hypothesis, one needs to find the critical region, $0 - A$, such that $P(0 < W < A) = \alpha$, where $\alpha$ is the Type I error probability and is usually chosen beforehand.

It should be pointed out that
\[ P(0 < W < A) = \int_0^A g(W) dW = \int_0^A \sum_{n_t=0}^N \ldots \sum_{n_1=0}^N h(W, n_1, \ldots, n_t) dW \]

and

\[ P(0 < \lambda < A) = \int_0^A h(\lambda) d\lambda = \int_0^A \sum_{n_t=0}^N \ldots \sum_{n_1=0}^N h(\lambda, n_1, \ldots, n_t) d\lambda \]

are not immediately obtainable, and that a test for \( \lambda \) or \( W \) will not be obtained from these relationships until a closed expression can be found for \( g(W) \) and \( h(\lambda) \) or until an approximation can be obtained for them which will permit the calculation of the probabilities of (18) and (21).

Even if we were able to obtain (18) and (21) a test still might be difficult to obtain because both (18) and (21) involve the unknown parameters \( p_1 \) and \( q_1 \). The Type I error would vary for the different values of the parameters for a fixed critical region 0 to A.

Mood and Graybill (1963) indicate a method, however, which will permit us to construct a test. They point out that if a test criterion \( \lambda \) has a distribution \( f(\lambda; \theta_1, \theta_2, \ldots, \theta_n) \) which involves a set of unknown parameters \( \theta_1, \theta_2, \ldots, \theta_n \), and these parameters have a set of sufficient statistics \( \hat{\theta}_1, \ldots, \hat{\theta}_n \), then the joint density for \( \lambda \) and \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n \) may be expressed as
A sufficient statistic implies that the estimator contains all the information about the true parameter that the sample can give. From (22) one notes that the conditional density of \( \lambda \), given the sufficient statistics, will not involve the parameters. Using this conditional distribution, a number \( A(\hat{\theta}_1, \ldots, \hat{\theta}_n) \) may be found which for every \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n \)

\[
A(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n) \quad \int_0^\infty v_1(\lambda/\hat{\theta}_1, \ldots, \hat{\theta}_n) \, d\lambda = \alpha
\]

is true. Hence, one may test a hypothesis by using \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) and \( \lambda \). The test is actually a conditional test. We can observe \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) and test \( \lambda \) by the critical region \( 0 < \lambda < A(\theta_1, \theta_n) \), using the conditional distribution of \( \lambda \) given \( \hat{\theta}_1, \ldots, \hat{\theta}_n \).

The important point to be gleaned from the above discussion is that if a set of sufficient statistics exist, then

\[
A(\hat{\theta}_1, \ldots, \hat{\theta}_n) \quad \int_0^\infty v_1(\lambda/\hat{\theta}_1, \ldots, \hat{\theta}_n) \, d\lambda = \alpha
\]

will give a test for the likelihood criterion \( \lambda \) for arbitrary values of \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n \).
There is a theorem in Mood and Graybill (1963) that gives a
criterion for examining a set of statistics for sufficiency. This theorem
states that if a joint density of a random sample can be factored as
\[ g(X_1, \ldots, X_n; \theta_1, \ldots, \theta_n) = h(\hat{\theta}_1, \ldots, \hat{\theta}_n; \hat{\theta}_1, \ldots, \hat{\theta}_n) \cdot g(X_1, \ldots, X_n) \]
where \( g(X_1, \ldots, X_n) \) does not involve the \( \theta_i \), then
\( \hat{\theta}_1, \ldots, \hat{\theta}_n \) is a set of \( n \) sufficient statistics.

Now, the joint densities for \( \lambda \) or \( W \), which were just obtained
in (17) and (20), respectively, involve the unknown parameters \( p_1, p_2, \ldots, p_t \). Note that (17) and (20), represented symbolically below,

\[
\begin{align*}
    h(\lambda, n_1, \ldots, n_t; p_1, \ldots, p_t) &= g(\lambda/n_1, \ldots, n_t) f(n_1, \ldots, n_t; p_1, p_2, p_t) \\
    h(W, n_1, \ldots, n_t; p_1, \ldots, p_t) &= g(W/n_1, \ldots, n_t) f(n_1, \ldots, n_t; p_1, \ldots, p_t)
\end{align*}
\]

are the product of two densities where the first density does not involve
the parameters \( p_1, \ldots, p_t \), and the second density involves \( n_i \) and
\( p_i, \ i=1, \ldots, t \).

Using the criterion for sufficiency outlined previously, one sees
that the \( n_i \)'s, \( i=1, \ldots, t \) are sufficient statistics for the parameters
\( p_i, \ i=1, \ldots, t \). Therefore, a test can be performed on \( \lambda \) or \( W \) by
using
\[
\int_0 A(n_1, \ldots, n_t) g(\lambda/n_1, \ldots, n_t) \, d\lambda = \alpha
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots (23)
\]

or

\[
\int_0 A(n_1, \ldots, n_t) g(W/n_1, \ldots, n_t) \, dW = \alpha
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots (24)
\]

It is possible to demonstrate this by noting that to test the hypothesis \( H_0 (\mu_1 = \mu_2 = \ldots = \mu_t = \mu_0) \) we need to find a critical region for \( \lambda \) or \( W \), such that if any value of \( \lambda \) or \( W \) obtained by the likelihood-ratio falls in this region, then \( H_0 \) will be rejected. If the value for \( \lambda \) or \( W \) is not in this region then \( H_0 \) is accepted. This is expressed symbolically as

\[
\sum_{n_1} \int_{0}^{A(n_1)} g(s^*, n_1; p_1) \, ds^*
\]

\[
= \sum_{n_1} \int_{0}^{A(n_1)} g(s^*/n_1) f(n_1; p_1) \, ds^*
\]

\[
= \sum f(n_1; p_1) \int_{0}^{A(n_1)} g(s^*/n_1) \, ds^*
\]
\[ \sum_{n_i} f(n_i; p_i) \cdot \alpha = \alpha \cdot \sum_{n_i} f(n_i; p_i) = \alpha \]

where \( s^* \) represents \( \lambda \) or \( W \) and \( \sum_{n_i} f(n_i; p_i) = 1 \). Hence, to make a test one needs only to find an \( A(n_1, \ldots, n_t) \), such that

\[ A(n_1, \ldots, n_t) \int_0 g(s*/n_1, \ldots, n_t) \, ds^* = \alpha \]

But, \( g(s*/n_1, \ldots, n_t) \) is just a beta distribution in terms of \( W \) or a function of an \( F \) variate in terms of \( \lambda \). This, then, implies that to test the hypothesis \( H_0 (\mu_1 = \mu_2 = \ldots = \mu_t = \mu_0) \) when the sample variances are the same and when the sample sizes are assumed to be random variables, one uses the conventional \( F \)-test method. In other words the likelihood-ratio will have an \( F \) distribution.

To fully examine this test, however, one should examine the power function \( g(\theta) \). But, in order to be able to do this, it is necessary to actually be able to evaluate (18). At the moment, this is not possible, since the expression is too complex, or seems to be. Until an approximate expression can be found or this expression simplified to a point that (18) can be evaluated, the power function will not be able to be evaluated. This could develop into a thesis in and of itself.
CONCLUSIONS

In summary, a density has been found which describes the experiment when the sample sizes are assumed to be random variables. The likelihood-ratio was used to test the hypothesis $H_0 (\mu_1 = \mu_2 = \ldots = \mu_t = \mu_0)$.

A mathematical relationship was obtained for the density of the likelihood-ratio criterion, $\lambda$, which was too complex to obtain the necessary probabilities for testing $\lambda$. It was shown, however, that the conventional $F$ test could be used to make a test for $\lambda$, even though it is assumed that the sample sizes are random variables.

The power function, $\beta$, was not compared with the known power functions for fixed sample sizes due to the complexity of the aforementioned density function.
LITERATURE CITED


