Probable Circular Error (CEP) of Ballistic Missiles

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PROBABLE CIRCULAR ERROR (CEP)
OF BALLISTIC MISSILES

by

James Edward Moran, Jr.

A thesis submitted in partial fulfillment
of the requirements for the degree
of
MASTER OF SCIENCE
in
Applied Statistics

Approved:

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Logan, Utah

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James Edward Moran, Jr.
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INTRODUCTION

The survival of our nation, during a nuclear exchange, depends upon an effective national defense structure. The prime weapon system in this defense structure is the ballistic missile. Although many factors enter into an evaluation of the effectiveness of a ballistic missile, one of the most important measures is accuracy. Without an accurate weapon system we have no weapon system.

The Department of Defense has placed emphasis on using a method of accuracy evaluation called "Probable Circular Error (CEP)." Probable Circular Error is defined as "The radius of a circle, centered at the intended target, within which 50% of the missiles would be expected to impact" or "The probability is 0.50 that an individual missile will impact within a circle whose radius is equal to the CEP." The statistical techniques and assumptions used in generating a CEP value will be investigated.
USE OF THE JOINT DENSITY FUNCTION

If a rifle were held stationary by clamping it in one position and it was fired at a target, the distribution of the shots might look like Figure 1.

Even though the rifle was properly aimed and held stationary, all of the shots would not hit the center of the target. This is due to minor variations such as differences in the weight of the bullets and shape of the bullets, the effect of humidity on the powder and many other incalculable factors.

The coordinates of the individual shot will be denoted by X and Y miss distances measured in relation to the intended impact point (0,0). If a large number of shots were fired, we could divide the target into squares and compile the total number of shots which landed in each square. The frequency of shots landing in a particular square would then be represented by the height of the respective

Figure 1. Measurement of miss distances through Cartesian coordinates.
column above that square. In other words, the volume of the column estimates the probability that a shot will occur in the square over which the column is placed. Let's assume that there is a function \( f(x, y) \) which can be superimposed over the xy plane, coinciding with the columns which are plotted perpendicular to the xy plane. The probability that a shot falls anywhere within a region \( A \) is equivalent to the volume under that region. See Figure 2.

\[
\text{Figure 2. Density function of the bivariate normal distribution}
\]

Referring to Figure 2:

\[
P(A) = P[(x, y) \in A] = \int_{A} \int f(x, y) \, dx \, dy \tag{2.1}
\]

\[
P(0 < x < a, 0 < y < b) = \int_{0}^{a} \int_{0}^{b} f(x, y) \, dx \, dy \tag{2.2}
\]
If, \( f(x,y) \geq 0 \), for \( -\infty < x < \infty \) \hspace{1cm} (2.3)

and, \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = 1 \) \hspace{1cm} (2.4)

Then \( f(x,y) \) is the joint density of variates \( x \) and \( y \). If \( f(x,y) \) has a bivariate normal distribution then it takes this form:

\[
f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho \frac{x-\mu_x}{\sigma_x} \frac{y-\mu_y}{\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right] \right\}
\]

(2.5)

Where:

- \( \infty < x < \infty \)
- \( \infty < y < \infty \)
- \( -1 < \rho < +1 \)
- \( \sigma_y > 0 \)
- \( \sigma_x > 0 \)
- \( -\infty < \mu_x < \infty \)
- \( -\infty < \mu_y < \infty \)

\( \rho \) = correlation coefficient between \( x \) and \( y \)

\( \sigma_y \) = the population standard deviation of the variable \( y \)

\( \sigma_x \) = the population standard deviation of the variable \( x \)

\( \mu_y \) = population mean of the variable \( y \)

\( \mu_x \) = population mean of the variable \( x \)

This is the form of the bivariate distribution which will be incorporated in this study. The use of this distribution will be justified in the next section.
The use of the bivariate normal distribution

The miss distance of a ballistic missile can be primarily attributed to two major sources: Guidance errors and nonguidance errors. Guidance errors originate in the inertial guidance system of the missile. Nonguidance errors are principally of three types: the re-entry errors, errors due to geophysical uncertainties and engine cut-off anomalies. Re-entry errors include errors caused by winds, by uncertainties in the density of the atmosphere, by attitude control system errors and by the effects of separation of the nose cone from the afterbody. Geophysical uncertainties occur because of geodetic differences between the missile launch site and the target site, and uncertainties in the gravitational field of the earth. Engine cut-off errors arise due to imperfect cut-off in that after the command is given some thrust remains, decaying for a few milliseconds.

It is quite obvious that a target miss due to an individual source may be treated as a random variable since the miss is expressed in terms of some quantity (e.g., wind variation) which varies in an unpredictable fashion from instrument to instrument, day to day, or from point to point on the earth. Since all of these sources are quite diversified in origin and effect, they can all be considered to be independent random variables. Consequently, we arrive at a conclusion which is of paramount importance in estimating missile accuracy and that is the right to use the central limit theorem.
In essence, the Central Limit Theorem states that the sum of a large number of independent random variables is approximately normally distributed regardless of the particular distributions of the individual components of the sum. The individual components which make up the total miss have been listed previously and their independency has been stated. These components are made up of numerous sources. They are so non-related that it is not feasible to even think of them as being dependent.

Let \( Y = X_1 + X_2 + \ldots + X_n \) represent the total downrange miss of a missile. The \( X \)'s are the innumerable effects which contributed to this over-all miss. Consequently, the total miss of a missile can be considered to consist of the sum of many individual independent misses. Therefore, the total miss tends to be normally distributed and we can now make probabilistic statements concerning missile accuracy with the use of the joint probability density function.

Sources of error in a ballistic missile system

The error sources which will be considered in this study are:

a. Guidance System
b. Sustainer Impulse
c. Translation System
d. Residual

The available instrumentation both in the missile and that associated with tracking system allows us to get accurate estimates of these sources. There are some other errors which are impossible to isolate and consequently then are considered in the residual error. For example, it is impossible to estimate the contribution to a miss
due to the re-entry vehicle because of the lack of instrumentation.

Each error source will be calculated from measured data and separated into downrange and crossrange miss distances or x and y miss distances (e.g. the re-entry vehicle impacted 0.8 nautical miles downrange or long and 0.4 nautical miles left of target or crossrange for an over-all radial miss of 0.895 nautical miles).

The four major sources of error are:

a. Guidance errors

This source is usually the largest contributor to a miss since it is actually the "brains" of the missile and, consequently, has many functions to perform with very close tolerances required. Additional complications are also introduced by the fact that these errors are a function of the range and azimuth of the target. In other words, a different guidance error would be introduced if we were considering a target in Asia rather than in the Pacific where the United States tests their missiles.

There is a known bias introduced in measuring the guidance error contribution to a miss. This bias is long downrange and left crossrange. The bias originates in the ground tracking stations on the test range and is subtracted from the miss due to guidance so that a true guidance miss can be calculated.

\[ M_{DG} = \text{miss downrange in nautical miles due to the guidance system} \]

\[ M_{CG} = \text{miss crossrange in nautical miles due to the guidance system} \]

b. Sustainer Impulse

This error is due to the slow decay of thrust in the second stage engine (or sustainer engine) after the engine has been cut-off.
This error obviously imparts additional velocity to the re-entry vehicle and causes an error in the downrange direction. The cross-range error is very minute, if any, and therefore is disregarded. The error in measuring this miss contributor is negligible.

\[
M_{DS} = \text{miss downrange in nautical miles due to the slow decay of the second stage engine thrust after cut-off}
\]

c. Translation System

At the time the second stage separates from the re-entry vehicle, both are essentially at such an altitude that if they were allowed to continue on their own at this time they would have very similar trajectories. Since the second stage is quite large, enemy radar would have no trouble pinpointing its position and consequently pinpointing the position of the weapon which is in the re-entry vehicle. Therefore, a network of small rockets have been placed on the second stage and are activated after second stage cut-off, causing the burned-out second stage to tumble and thus place it in a different trajectory than the R/V (re-entry vehicle). The rockets of the translation system not only cause the second stage to tumble but also impart an increment of velocity to the R/V which causes a positive downrange error. The crossrange error is negligible.

\[
M_{DT} = \text{miss downrange in nautical miles due to the exhaust of the translation system rockets}
\]

d. Residual Error

This is the amount of miss remaining when all measured or calculated errors have been subtracted from the over-all miss. It has downrange and crossrange components and is composed of incalculable R/V errors, geodetic and geophysical errors and all other
errors that cannot be identified or isolated. Mathematically, residual error is calculated this way:

\[ M_{\text{DR}} = M_D - M_{\text{DG}} - M_{\text{DS}} - M_{\text{DT}} \]  
\[ M_{\text{CR}} = M_C - M_{\text{CG}} \]

where,

\( M_D \) = the over-all downrange miss in nautical miles
\( M_C \) = the over-all crossrange miss in nautical miles
\( M_{\text{DR}} \) = miss downrange in nautical miles due to the residual error
\( M_{\text{CR}} \) = miss crossrange in nautical miles due to the residual error

Therefore, the following equations form the basic model for calculation of probable circular error (CEP).

\[ M_D = M_{\text{DG}} + M_{\text{DS}} + M_{\text{DT}} + M_{\text{DR}} \]  
\[ M_C = M_{\text{CG}} + M_{\text{CR}} \]
SIMULATING A BIVARIATE NORMAL DISTRIBUTION

The basic model of the miss downrange and crossrange was calculated as:

\[ M_D = M_{DG} + M_{DT} + M_{DS} + M_{DR} \]
\[ M_C = M_{CG} + M_{CR} \]

The assumption that \( M_D \) and \( M_C \) are normally distributed is based upon the components that make up the two variables. These components are considered to be independent random variables because of their unpredictable, nonrelated occurrence.

By the use of a random number generator designed for a large scale computer and a programming language called JOVIAL (Jule's Own Version of the International Algebraic Language), a very large population of random numbers was simulated. The values produced by this generator follow a uniform distribution. Integers were generated between 0 and 1000 and then operated on to produce variables between +0.5 and -0.5. These numbers simulated values for \( M_{DG}, M_{DT}, M_{DS}, M_{DR}, M_{CG}, M_{CR} \), etc. In the case of \( M_D \), 900 sets of four random variables were randomly selected and summed to obtain 900 values of \( M_D \). The process was repeated for \( M_C \) except only two random variables were selected. Therefore, \( M_D \) and \( M_C \) are both sums of independent random variables and can be considered to be normally distributed because of the Central Limit Theorem. In other words, \( M_D \) and \( M_C \) are the sums of independent random variables, i.e. the numerous anomalies which contribute to a miss. Appendixes A and B
contain the flow chart and program which were designed by the author.

Therefore, a simulated firing of 900 missiles has been accomplished and the miss distance of a missile has been tabulated in terms of the miss downrange, \( M_D \), and the miss crossrange, \( M_C \). With this population of 900 missile firings, a random sample can be chosen and considered to be the number of missiles which have been actually fired at the test range. This sample of simulated firings will be used to calculate the CEP of the whole missile fleet (in this case, the population of 900 simulated firings) using different methods.

To make this simulation more meaningful and powerful, five different sample sizes will be taken: 5, 10, 15, 20 and 25. Each of these samples will be taken ten different times to obtain a good representation of samples from this population of 900.

Appendix C is a representation of what the computer print-out contained. The calculated values were: sum of \( M_D \), sum of \( M_D^2 \), mean of \( M_D \) and standard deviation of \( M_D \). These were computed for the whole population and each sample. A "t" statistic was also calculated to be used for a significance test. It is interesting to note that when a test of hypothesis (\( \alpha = 0.10 \)) was conducted on each sample to see if the population mean was equal to 0.01111, exactly five of fifty samples (ten per cent) rejected this hypothesis.

A print-out similar to Table 2 was obtained for values of \( M_D \) and \( M_C \) with \( \sigma_{MD} = \sigma_{MC} \), and for values of \( M_D \) and \( M_C \) with \( \sigma_{MD} \neq \sigma_{MC} \).
CEP CALCULATIONS FOR TWO MAIN CASES

The definition of CEP involves the solution of the integral in terms of the radius of the circle, equating the integral to a probability of 0.50. In other words, if \( f(x,y) \) is the form of the bivariate distribution, then the CEP is calculated from this equation:

\[
0.50 = \int_{C_R} \int f(x,y) \, dx\,dy
\]

(5.1)

where \( C_R \) designates a circle of radius \( R \) (CEP).

The correct value for the CEP is now found by performing the indicated integration and solving for \( R \).

Since the basis for using the bivariate normal density function has already been established, this function will be used. So,

\[ f(x,y) \sim \mathcal{N}(\mu_x, \mu_y, \sigma_x, \sigma_y) \]

For this study, the following designations will be used:

- \( x = M_C \) = the miss crossrange
- \( y = M_D \) = the miss downrange
- \( \mu_x = \mu_{MC} \) = the population mean of the crossrange miss distance
- \( \mu_y = \mu_{MD} \) = the population mean of the downrange miss distance
- \( \sigma_x = \sigma_{MC} \) = the population standard deviation of the crossrange miss distance
- \( \sigma_y = \sigma_{MD} \) = the population standard deviation of the downrange miss distance
\[ \rho = \text{the correlation coefficient between the crossrange miss} \ (M_C) \text{ and downrange miss} \ (M_D) \]

\[ M_C = \text{the sample mean of the crossrange miss distance} \]

\[ M_D = \text{the sample mean of the downrange miss distance} \]

\[ S_{MC} = \text{the sample standard deviation of the crossrange miss distance} \]

\[ S_{MD} = \text{the sample standard deviation of the downrange miss distance} \]

\[ N = \text{number of missiles in the entire fleet (population)} \]

\[ n = \text{number of missiles in the sample tested} \]

Now the CEP or radius of the circle in question is obtained by solving the following expression:

\[
0.50 = \frac{1}{2\pi \sigma_{MC} \sigma_{MD} \sqrt{1-\rho^2}} \int_{C_R} \int (\frac{M_C-\mu_{MC}}{\sigma_{MC}})^2 - 2\rho \frac{(M_C-\mu_{MC})(M_D-\mu_{MD})}{\sigma_{MC} \sigma_{MD}} + (\frac{M_D-\mu_{MD}}{\sigma_{MD}})^2 \ dM_DdM_C \quad (5.2)
\]

The relative difficulty in solving this integral is determined by the values of the variables: \( \mu_{MC}, \mu_{MD}, \sigma_{MC}, \sigma_{MD}, \) and \( \rho. \)

From the numerous tests that have been run and after analyzing gross amounts of data, the ballistic missile engineer has proved conclusively that it is physically impossible for the miss crossrange to have any effect on the miss downrange and vice versa. Consequently, it is safe to assume that \( \rho = 0. \)

The possible combinations that arise concerning the other four variables will be considered by discussing the two main cases that occur in actual tests.
Case I

\[ \rho = 0 \]
\[ \mu_{MD} = 0 \]
\[ \mu_{MC} = 0 \]
\[ \sigma_{MC} = \sigma_{MD} = \sigma \]

This is the case where, of course, \( \rho = 0 \), but the two means are also zero. In other words, all of the bias has been removed from the system and the over-all means of the downrange and crossrange misses are located at the point \((0,0)\) or right on the target. Since the standard deviations are equal, the result is a circular bivariate normal distribution. Figure 3 displays what the target might look like if this were the case.

![Circular bivariate normal distribution](image)

Figure 3. Circular bivariate normal distribution
The complicated double integral now reduces down to:

\[ P[(M_C, M_D) \in S] = 0.50 = \frac{1}{2\pi \sigma^2} \int_S \int r^{-1/2} \frac{1}{2\sigma^2} [M_C^2 + M_D^2] \, dM_C \, dM_D \]  
(5.3)

where \( S \) defines the region within which the shots have impacted. \( S \) is a circle in this case.

Transforming to polar coordinates:

\[ M_C = r \cos \theta \]

\[ M_D = r \sin \theta \]

where \( 0 \leq r \leq R \)

\[ 0 \leq \theta \leq 2\pi \]

\[ R = \sqrt{M_D^2 + M_C^2} \]

\[ 0.50 = \frac{1}{2\pi \sigma^2} \int_0^R \int_0^{2\pi} r^{-1/2} \frac{1}{2\sigma^2} [r^2 \cos^2 \theta + r^2 \sin^2 \theta] \, r \, dr \, d\theta \]  
(5.4)

\[ = \frac{1}{2\pi \sigma^2} \int_0^R \int_0^{2\pi} r^{-r/2} \frac{1}{2\sigma^2} \, r \, dr \, d\theta \]  
(5.5)

\[ = \frac{1}{2\pi \sigma^2} \left[ \sigma^2 (1 - r^{-R^2/2\sigma^2}) \right] \int_0^{2\pi} d\theta \]  
(5.6)

\[ = 1 - r^{-R^2/2\sigma^2} \]  
(5.7)

\[ r^{-R^2/2\sigma^2} = 0.50 \]  
(5.8)

\[ -R^2/2\sigma^2 \ln r = 1 \ln 0.50 \]  
(5.9)

\[ -R^2/2\sigma^2 = -0.69315 \]  
(5.10)

\[ R = \sigma \sqrt{2(0.69315)} \]  
(5.11)

\[ R = 1.1774 \sigma \]  
(5.12)

\[ CEP = 1.1774 \sigma \]  
(5.13)
Therefore, the population CEP is obtained by multiplying the common population standard deviation by a constant, 1.1774.

Example:

If the population standard deviations were known the calculation would be quite easy. But the real world situation is that only $S_{MC}$ and $S_{MD}$ are known and the probability of these two statistics being equal is very low even though the parameters $\sigma_{MC}$ and $\sigma_{MD}$ are equal. Therefore, a pooled estimate of the variance must be computed.

$$s^2_p = \text{the common variance}$$

$$s^2_p = \frac{\sum (M_D - \overline{M_D})^2 + \sum (M_C - \overline{M_C})^2}{2(n-1)} = \frac{s^2_{MD} + s^2_{MC}}{2} \quad (5.14)$$

\(\widehat{CEP}\) = the estimated or sample CEP obtained through the sample statistics.

\(\widehat{CEP} = 1.1774s_p = 1.1774 \sqrt{\frac{s^2_{MD} + s^2_{MC}}{2}} \quad (5.15)\)

If, $S_{MD} = 0.5188$ nautical miles

$S_{MC} = 0.4024$ nautical miles

CEP* = 0.5466 nm

Case II

$\rho = 0$

$\mu_{MC} = 0$

$\mu_{MD} = 0$

$\sigma_{MC} \neq \sigma_{MD}$

Again, there is no correlation between $M_D$ and $M_C$ and all of the bias has been removed from the system making the point $(\overline{M_C}, \overline{M_D})$
coincide with the target \((0,0)\). The difference from Case I is due to the inequality of the two standard deviations. Now the distribution is a bivariate normal which takes on the form of an ellipse with the major and minor axes being equal to \(\sigma_{MD}\) and \(\sigma_{MC}\) respectively. This actually is the case which predominates in missile firings due to the fact that there are many more sources of error causing downrange misses. The crossrange or azimuth error sources are very few. Figure 4 depicts the probable distribution of the impact points in this case.

\[
P[(M_D, M_C) \in S] = 0.50 = \frac{1}{2} \int_{S} \int_{S} e^{-1/2[(M_C^2/\sigma_{MC}^2 + M_D^2/\sigma_{MD}^2)} \, dM_D \, dM_C \quad (5.16)
\]

where \(S\) defines the region within which the shots have impacted. \(S\) is an ellipse in this case.
The solution of this integral will be carried out as far as is necessary for our purposes. Since a complete, detailed solution would involve the use of numerical integration and Bessel functions, the appropriate sources will be referenced.

Transforming to polar coordinates again:

\[ M_C = r \cos \theta \]

\[ M_D = r \sin \theta \]

where, \( 0 \leq r \leq R \)

\[ 0 \leq \theta \leq 2\pi \]

\[ R = \sqrt{M_D^2 + M_C^2} \]

\[
0.50 = \frac{1}{2\pi \sigma_{MD} \sigma_{MC}} \int_0^{2\pi} \int_0^R \frac{r^2}{2} [r^2 \cos^2 \theta \sigma_{MC}^2 + r^2 \sin^2 \theta \sigma_{MD}^2] \, r \, dr \, d\theta \quad (5.17)
\]

\[
0.50 = \frac{1}{2\pi \sigma_{MD} \sigma_{MC}} \int_0^R \int_0^{2\pi} -r^2/2 [\cos^2 \theta \sigma_{MC}^2 + \sin^2 \theta \sigma_{MD}^2] \, r \, dr \, d\theta \quad (5.18)
\]

Since,

\[ \cos^2 \theta = 1/2(1 + \cos 2\theta); \quad \text{and,} \quad \sin^2 \theta = 1/2(1 - \cos 2\theta) \quad (5.19) \]

\[
0.50 = \frac{1}{2\pi \sigma_{MD} \sigma_{MC}} \int_0^R \int_0^{2\pi} -r^2/4 [1 + \cos 2\theta \sigma_{MC}^2 + 1 - \cos 2\theta \sigma_{MD}^2] \, r \, dr \, d\theta \quad (5.20)
\]

\[ = \frac{1}{2\pi \sigma_{MD} \sigma_{MC}} \int_0^R \int_0^{2\pi} -r^2/4 [1/\sigma_{MC}^2 + 1/\sigma_{MD}^2 - r^2/4 \frac{\cos 2\theta}{\sigma_{MC}^2} - \frac{\cos 2\theta}{\sigma_{MD}^2}] \, r \, dr \, d\theta \quad (5.21)\]

\[ = \frac{1}{2\pi \sigma_{MD} \sigma_{MC}} \int_0^R r \left[ -r^2/4 [1/\sigma_{MC}^2 + 1/\sigma_{MD}^2] \right] \quad (5.22)\]
Let,
\[ 2\theta = \phi, \quad \theta = \phi/2 \]  
\[ d\phi = 2d\theta, \quad d\theta = d\phi/2 \]  
then
\[ \theta = \pi/2 \]
so, \[ \phi = \pi \]  
\[ 0.50 = \int_{0}^{R} \frac{-r^2/4\sigma_{MD}^2}{1 + \sigma_{MD}^2/\sigma_{MC}^2} \]  
\[ \left[\frac{1}{\pi} \int_{0}^{\pi} \frac{-r^2/4\sigma_{MD}^2}{\sigma_{MD}^2/\sigma_{MC}^2 - 1}\cos \phi \, d\phi\right] \, dr \] (5.26)

The expression in square brackets is a zero order Bessel function of the second kind (1, p. 376). In other words:
\[ I_0[Z] = \frac{1}{\pi} \int_{0}^{\pi} \frac{-r^2/4\sigma_{MD}^2}{1} \cos \phi \, d\phi \] (5.27)

where, in this case:
\[ Z = \frac{r^2}{4\sigma_{MD}^2} \left(\frac{\sigma_{MD}^2}{\sigma_{MC}^2} - 1\right) \] (5.28)

Therefore:
\[ \frac{1}{\pi} \int_{0}^{\pi} \frac{-r^2/4\sigma_{MD}^2}{1 + \sigma_{MD}^2/\sigma_{MC}^2} \cos \phi \, d\phi = I_0\left[\frac{r^2}{4\sigma_{MD}^2}\left(\frac{\sigma_{MD}^2}{\sigma_{MC}^2} - 1\right)\right] \] (5.29)

and,
\[ 0.50 = \frac{1}{\sigma_{MD}\sigma_{MC}} \int_{0}^{R} \frac{-r^2/4\sigma_{MD}^2}{1 + \sigma_{MD}^2/\sigma_{MC}^2} \left[ I_0\left[\frac{r^2}{4\sigma_{MD}^2}\left(\frac{\sigma_{MD}^2}{\sigma_{MC}^2} - 1\right)\right]\right] \, dr \] (5.30)
Substituting:

\[
v = \frac{r^2}{4\sigma_M^2} \left( 1 + \frac{\sigma^2}{\sigma_{MC}^2} \right), \quad \text{and} \quad a = \frac{\sigma_{MC}}{\sigma_{MD}}
\] (5.31)

then,

\[
0.50 = \frac{2a}{1 + a^2} \int_{0}^{R^2/4\sigma^2} \left( 1 + \frac{1}{1 + \alpha^2/\alpha^2} \right) e^{-\nu[I_0(v(\frac{1-a^2}{1+a^2}))]} \, dv
\] (5.32)

It is interesting to note that if \( \sigma_{MC} = \sigma_{MC} \), we have the circular bivariate normal distribution and \( a = 1 \). Then this equation reduces to:

\[
0.50 = \int_{0}^{R^2/2\sigma^2} e^{-\nu} \, dv \quad ; \quad \text{because} \quad I_0(0) = 1
\] (5.33)

which eventually reduces to:

\[
R = 1.1774 \sigma = \text{CEP}
\] (5.34)

which verifies our original proof. Continuing,

\[
0.50 = \frac{2a}{1 + a^2} \int_{0}^{R^2/4\sigma^2} \left( 1 + \frac{1}{1 + \alpha^2/\alpha^2} \right) e^{-\nu[I_0(v(\frac{1-a^2}{1+a^2}))]} \, dv
\] (5.35)

Grad and Solomon (4) have tabulated different values of this integral and a linear approximation to the curve which estimates the solution will be used in this study. This linear approximation is:

\[
\frac{\text{CEP}}{\sigma_{MD}} \approx 0.614 a + 0.563
\] (5.36)

when \( a > 1/3 \) (5.37)
The condition placed on $\alpha$ is not a restrictive one since the actual data does place $\alpha$ in this region due to the elliptical distribution of the impact points. The relationship used to calculate the CEP is:

$$\text{CEP} = 0.614 \sigma_{MC} + 0.563 \sigma_{MD}$$

(5.38)

This is the equation which will be used when the two standard deviations are not equivalent.

Example:

If,

\begin{align*}
    S_{MD} &= 0.4525 \text{ n.m.} \\
    S_{MC} &= 0.3569 \text{ n.m.}
\end{align*}

\[ \hat{\text{CEP}} = 0.614 S_{MC} + 0.563 S_{MD} \]

\[ \hat{\text{CEP}} = 0.4739 \text{ n.m.} \]
CEP CONFIDENCE INTERVALS

Since the cost of a missile test is astronomical, it is imperative that a good, reliable estimate of the actual CEP be obtained through the smallest sample possible. This is accomplished through the use of confidence intervals. Just like the previous section, the discussion about confidence intervals will be separated into two main cases.

According to Bowker and Lieberman:

If \( X_1, X_2, \ldots, X \) are independent normally distributed random variables each having mean \( \mu \) and variance \( \sigma^2 \), the random variable \( (n-1) \frac{S^2}{\sigma^2} \) has a chi-square distribution with \( n-1 \) degrees of freedom. (2, p. 76)

The independent normally distributed random variables in this case are the values of \( M_D \) and \( M_C \). The sample variances would be \( S^2_{MC} \) and \( S^2_{MD} \) of the group of missile tests that have been made and \( \sigma^2_{MC} \) and \( \sigma^2_{MD} \) would be the actual population variances of the entire missile fleet.

Case I

\( \rho = 0 \)

\( \mu_{MC} = 0 \)

\( \mu_{MD} = 0 \)

\( \sigma_{MC} = \sigma_{MD} = \sigma \)

Therefore, CEP = 1.1774 \( \sigma \)  

(6.1)

Using the sample statistics:

\[
\hat{\text{CEP}} = 1.1774 \sqrt{\frac{S^2_{MC} + S^2_{MD}}{2}} = 1.1774 \sqrt{s^2_B}
\]

(6.2)
Since $M_C$ and $M_D$ are independent normally distributed random variables, then the following variates are chi-square variates.

\[ \chi^2_{MC} = \frac{(n-1)S_{MC}^2}{\sigma^2_{MC}} \quad \text{d.f.} = n - 1 \quad (6.3) \]

\[ \chi^2_{MD} = \frac{(n-1)S_{MD}^2}{\sigma^2_{MD}} \quad \text{d.f.} = n - 1 \quad (6.4) \]

According to Bowker and Lieberman (2), the sum of two chi-square variates is a chi-square variate itself.

\[ \chi^2 = \chi^2_{MC} + \chi^2_{MD} = \frac{(n-1)S_{MC}^2}{\sigma^2_{MC}} + \frac{(n-1)S_{MD}^2}{\sigma^2_{MD}} \quad (6.5) \]

\[ \chi^2 = \frac{(n-1)(S_{MC}^2 + S_{MD}^2)}{\sigma^2} \quad \text{d.f.} = n - 1 + n - 1 = 2n - 2 \quad (6.7) \]

Restating the total chi-square variate in terms of the CEP requires some manipulation.

\[ \chi^2 = \frac{(n-1)[(S_{MC}^2 + S_{MD}^2)/2]}{\sigma^2/2} \quad (6.8) \]

\[ (CEP)^2 = (1.1774^2)\left(\frac{S_{MC}^2 + S_{MD}^2}{2}\right), \quad \text{and,} \quad CEP^2 = (1.1774^2)\sigma^2 \quad (6.9) \]

So,

\[ \chi^2 = \frac{(n-1)[(CEP)^2/(1.1774)^2]}{CEP^2/2(1.1774)^2} = \frac{2(n-1)(CEP)^2}{CEP^2} \quad (6.10) \]

Therefore, the total chi-square variate becomes,

\[ \chi^2 = \frac{2(n-1)(CEP)^2}{CEP^2}, \quad \text{d.f.} = 2n - 2 \quad (6.11) \]
In Case I the probability statement which eventually arrives at a confidence interval is:

\[
P\left[\chi^2_{1-a/2;2n-2} \leq \frac{2(n-1)(\widehat{\text{CEP}})^2}{\text{CEP}^2} \leq \chi^2_{a/2;2n-2}\right] = 1 - \alpha \quad (6.12)
\]

where:

\[\alpha = \text{Type I error or critical area}\]

The interpretation of the equation is:

The probability that the variate, \(2(n-1)\left(\frac{\text{CEP}^2}{(\text{CEP})^2}\right)\) is bracketed by \(\chi^2_{1-a/2;2n-2}\) and \(\chi^2_{a/2;2n-2}\) is \(1 - \alpha\).

This equation may be rewritten:

\[
P\left[\widehat{\text{CEP}} \leq \text{CEP} \leq \widehat{\text{CEP}} \frac{\sqrt{2(n-1)/\chi^2_{1-a/2;2n-2}}}{\chi^2_{a/2;2n-2}}\right] = 1 - \alpha \quad (6.13)
\]

This confidence interval brackets the true CEP which is the statement that is of primary concern.

**Case II**

\[\mu = 0\]

\[\mu_{MC} = 0\]

\[\mu_{MD} = 0\]

\[\sigma_{MC} \neq \sigma_{MD}\]

Therefore, CEP = 0.614 \(\sigma_{MC} + 0.563 \sigma_{MD}\) \(6.14\)

or, using the sample statistics,

\[\widehat{\text{CEP}} = 0.614 \widehat{s}_{MC} + 0.563 \widehat{s}_{MD}\] \(6.15\)
In Case II the two standard deviations are unequal and consequently, the CEP is a function of the sum of two standard deviations.

But both of these standard deviations (or variances) are considered to have a chi-square distribution with \( n-1 \) degrees of freedom. Then the CEP which is a function of the sum of these two variates can also be assumed to be a chi-square variate with \( 2n-2 \) degrees of freedom (since \( n_C = n_D \)). But \( \sigma^2_{MC} \neq \sigma^2_{MD} \), so a pooled estimate of the variances cannot be used. Therefore,

\[
\chi^2 = \frac{(n-1)(\text{CEP})^2}{\text{CEP}^2} \sim \chi^2_{1-\alpha;2n-2} \tag{6.16}
\]

This is similar to the chi-square variate that was obtained for Case I and the confidence interval statement is obtained in the same manner.

\[
P[\sqrt{\frac{n-1}{\chi^2_{1-\alpha/2;2n-2}}} \leq \text{CEP} \leq \sqrt{\frac{n-1}{\chi^2_{1-\alpha/2;2n-2}}}] = 1 - \alpha \tag{6.17}
\]
EFFECT OF SAMPLE SIZE

As the sample size is increased, the estimate, \( \hat{\text{CEP}} \), should be getting closer to the population CEP. A test of hypothesis using a chi-square variate was incorporated to test the validity of this assumption. The chi-square variate was used since the test was concerned with CEP and CEP is a function of the standard deviation.

\[
H_0: \quad \text{CEP}^2 = \text{the actual population CEP}^2
\]

\[
H_A: \quad \text{CEP}^2 \neq \text{the actual population CEP}^2
\]

\[\alpha = 0.05 \quad \chi^2 = \frac{(n-1)(\hat{\text{CEP}}^2 - \text{CEP}^2)^2}{\text{CEP}^2} \quad (7.1)\]

Acceptance criteria:
\[
\chi^2_{1-\alpha/2;n-1} \leq \chi^2 \leq \chi^2_{\alpha/2;n-1} \quad (7.2)
\]

Example:

\[H_0: \quad \text{CEP}^2 = (0.4807)^2\]

\[H_A: \quad \text{CEP}^2 \neq (0.4807)^2\]

\[\alpha = 0.05 \quad \chi^2 = \frac{(4)(0.3274)^2}{(0.4807)^2}\]

\[\hat{\text{CEP}} = 0.3274 \quad \chi^2 = 1.855\]

Acceptance criteria: \(0.484 \leq \chi^2 \leq 11.143\) \quad (7.5)

Accept \(H_0: \quad \text{CEP} = 0.4807 \text{ n.m.}\)

Of the 50 samples, only one sample rejected the hypothesis and this was a sample of size five.
Calculations of a more exacting nature are necessary to get a good idea of the effect of sample size. The necessity of being able to estimate the true CEP with a minimum number of samples was mentioned earlier in this study. It is simply a matter of economics. The question is, how many samples should be taken? A satisfactory method of accomplishing this will now be presented.

What is desired is an estimation of the CEP which will be within a certain per cent (ε) of the true CEP.

\[ 1 - \varepsilon \leq \frac{\text{CEP}}{\text{CEP}^2} \leq 1 + \varepsilon \] (7.6)

\[ (1 - \varepsilon)^2 \leq \frac{\text{CEP}^2}{\text{CEP}^2} \leq (1 + \varepsilon)^2 \] (7.7)

\[ (2n-2)(1-\varepsilon)^2 \leq \frac{(2n-2)(\text{CEP}^2)}{\text{CEP}^2} \leq (2n-2)(1+\varepsilon)^2 \] (7.8)

\[ \chi^2 = \frac{(2n-2)(\text{CEP}^2)}{\text{CEP}^2} \] (7.9)

\[ P[(2n-2)(1-\varepsilon)^2 \leq \chi^2_{2n-2} \leq (2n-2)(1+\varepsilon)^2] = 1 - \alpha \] (7.10)

From this probability statement, estimates of the number of samples (n) necessary to estimate the CEP to within ε per cent of its true value may be obtained with a given confidence of 1 - α. Table 1 contains some of these values.

As an example for a true CEP of two nautical miles and a sample of ten, the estimate would be expected to be within 2/3 of a mile of the true value with a confidence of 95 per cent. Above a sample size of twenty, the increase in the accuracy of the estimate starts to become negligible.
Table 1. Number of missiles (n) necessary to estimate CEP to within ε per cent of its true value with 95 per cent confidence

<table>
<thead>
<tr>
<th>n</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>50%</td>
</tr>
<tr>
<td>10</td>
<td>35</td>
</tr>
<tr>
<td>15</td>
<td>25</td>
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<td>20</td>
<td>22</td>
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<tr>
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<td>35</td>
<td>17</td>
</tr>
<tr>
<td>40</td>
<td>16</td>
</tr>
<tr>
<td>45</td>
<td>15</td>
</tr>
</tbody>
</table>

The advantage of such calculations is tremendous, especially in the initial phases of development of the weapon system when the fleet size is determined. The total number of missiles initially contracted for is determined, in part, by the number of missiles needed for testing. Therefore, a concrete idea of how many total missiles needed can be achieved. This provides an advantage for the defense department when the initial contract is released for bids. It also insures sufficient samples for an adequate test program.
RECOMMENDATIONS AND SUMMARY

Sample size

Samples below five observations should not be seriously relied on, if at all. A crash program to build the sample size above twenty missiles should be scrutinized thoroughly. An examination of the trade-off between the cost of additional firings and the probable resultant increase in accuracy should be the main point in question. The reliability of the CEP estimates in the simulated population of 900 should be kept in mind. If the bias of the system has been removed and the majority of the errors are due to random, unpredictable sources, then sample sizes of five to twenty-five will allow concrete conclusions.

Equality of the variances

It is absolutely essential that a test of hypothesis be conducted concerning the equality of $\sigma^2_{MD}$ and $\sigma^2_{MC}$. Since this will determine what CEP equation to use, it is of the utmost importance.

Example: $H_0$: $\sigma^2_{MC} = \sigma^2_{MD}$
$H_A$: $\sigma^2_{MC} \neq \sigma^2_{MD}$

\[
\begin{align*}
S^2_{MC} &= 0.0406 \\
S^2_{MD} &= 0.297 \\
\alpha &= 0.05
\end{align*}
\]

Acceptance criteria: $F_{0.975;9.9} \leq F \leq F_{0.025;9.9}$

\[
\begin{align*}
F &= \frac{S^2_{MD}}{S^2_{MC}} \\
F &= \frac{0.0406}{0.297} = 0.136 \quad (8.1) \\
F &= \frac{S^2_{MC}}{S^2_{MD}} \\
F &= \frac{0.297}{0.0406} = 7.337 \quad (8.2) \\
\alpha &= 0.05 \\
n_C &= 10 \\
n_D &= 10
\end{align*}
\]

\[0.248 \leq F \leq 4.03 \quad (8.4)\]
F = \frac{0.297}{0.0406} = 7.31 \quad F > 4.03 \quad (8.5)

\text{Reject } H_0: \quad \sigma^2_{MC} = \sigma^2_{MD} \quad (8.6)

And use, \( \text{CEP} = 0.614 \frac{S_{MC}}{MC} + 0.563 \frac{S_{MD}}{MD} \quad (8.7) \)

In the preceding example, it is interesting to note that the population from which the sample was derived had the following variances:

\( \sigma^2_{MC} = 0.0401 \quad (8.8) \)

\( \sigma^2_{MD} = 0.318 \quad (8.9) \)

In this same population of 900 missiles, a test of hypothesis was conducted on all fifty samples (i.e., ten samples of size five, ten samples of size ten, etc.). The results were:

\( H_0: \quad \sigma^2_{MC} = \sigma^2_{MD}; \quad \alpha = 0.05 \quad (8.10) \)

n = 5, \quad 8 \text{ samples accepted } H_0
n = 10, \quad 3 \text{ samples accepted } H_0
n = 15, \quad 0 \text{ samples accepted } H_0
n = 20, \quad 0 \text{ samples accepted } H_0
n = 25, \quad 0 \text{ samples accepted } H_0

Even allowing for an \( \alpha = 0.05 \) (rejecting the \( H_0 \) five per cent of the time when it is actually true), these results point out that the optimum sample size is around fifteen as far as checking for equality of variances. This point should receive prime consideration when testing for equal variances with a sample below ten observations.

**Confidence intervals**

Confidence intervals are a very useful tool in the management
of our defense industry. But the extent of this usefulness depends upon the statistician to properly utilize this tool and explain its ramifications to the manager. Although an individual may not understand the theory behind confidence statements, he tends to accept them for it gives him useful information about the weapon system which is not available otherwise.

Both the upper and lower CEP confidence limits are of interest to the manager. The upper limit is the more important of the two because it indicates how effective the missile is and what size nuclear weapon is needed to inflict the minimum amount of damage allowable. The lower limit will allow investigation into the possibility of redundantly destroying a target. This might be desirable if the target was hardened but would be a waste of weapons if it was a soft target.

Example:

\[
S_{MC} = 0.3569 \quad (8.11)
\]

\[
S_{MD} = 0.4525 \quad (8.12)
\]

\[n = 10 \quad (8.13)\]

\[\alpha = 0.05 \quad (8.14)\]

First, check for equality of variances:

\[H_0: \sigma^2_{MC} = \sigma^2_{MD} \quad (8.15)\]

\[H_A: \sigma^2_{MC} \neq \sigma^2_{MD} \quad (8.16)\]

Acceptance criteria:

\[F_{0.975;9.9} \leq F \leq F_{0.025;9.9} \quad (8.17)\]

\[0.248 \leq F \leq 4.03 \quad (8.18)\]

\[F = \frac{S^2_{MD}}{S^2_{MC}} = \frac{0.204}{0.127} = 1.18 \quad , \quad 0.248 < 1.18 < 4.03 \quad (8.19)\]
Accept: $H_0: \sigma^2_{MC} = \sigma^2_{MD}$ \hspace{1cm} (8.20)

And use: $\hat{\text{CEP}} = 1.1774 \frac{\sqrt{S^2_{MC} + S^2_{MD}}}{2} = 0.478 \text{ n.m.}$ \hspace{1cm} (8.21)

$$P[\text{CEP} \leq \sqrt{2(n-1)/x^2_{0.025;18}} \leq \text{CEP} \leq \sqrt{2(n-1)/x^2_{0.975;18}}] = 1 - \alpha$$ \hspace{1cm} (8.22)

$$P[0.478 \leq \text{CEP} \leq 0.478 \sqrt{2(9)/8.231}] = 0.95$$ \hspace{1cm} (8.23)

$$P[0.361 \leq \text{CEP} \leq 0.706] = 0.95$$ \hspace{1cm} (8.24)

The probability is 0.95 that the population CEP is bracketed by 0.361 n.m. and 0.706 n.m. In this population of 900 missiles, the CEP is actually 0.481.

Summary

The bivariate normal distribution and the theory of probable circular error are very economical and useful tools, especially in this age of cost reduction in the defense department. The proper use of these instruments of statistical inference allows the manager to make many far-reaching decisions concerning his weapon system without committing millions of defense dollars for additional tests. Every bit of data available must be reduced from each test to make the most economical use of these expensive tests. The statistical conclusions will only be as good as the authenticity of the collected data.

The statistical theories that are incorporated in CEP should not be static. A dynamic program to research the latest techniques should be maintained. This should include investigation into new
sources of data. Advanced techniques of instrumentation should open new horizons in data collection. The statistician is the one person who can provide the impetus to such a program. Consequently, he must stay abreast of all developments to provide our country with an effective accuracy evaluation effort.
LITERATURE CITED


GENERATE RANDOM NUMBER \( G_i \)

GENERATE RANDOM NUMBER \( S_i \)

GENERATE RANDOM NUMBER \( T_i \)

GENERATE RANDOM NUMBER \( R_i \)

\[ X_i = G_i + S_i + T_i + R_i \]

\[ X_i^2 \]

SUMX = SUMX + R_i

\[ X_i^2 \]

\( I = I + 1 \)

\( I = 900 \)

YES

\( I \)
MEANX = SUMX/I
STDEVX = \sqrt{(SUMX^2 - (SUMX)^2/I)} / (I-1)

K = I

SUMXS = 0
SUMX2S = 0
MEANXS = 0
STDEVXS = 0
TVALUE = 0

L = 1

GENERATE RANDOM NUMBER QL

PRINT SUMX SUMX2 MEANX STDEVX

SIZE = (K) 5

M = 1

L = 1
\[ K = K + 1 \]

\[ \text{SUMXS} = \text{SUMXS} + X_i \]
\[ \text{SUMX2S} = \text{SUMX2S} + X_i^2 \]

\[ I_s \]

\[ L = L + 1 \]

\[ \text{MEANXS} = \frac{\text{SUMXS}}{L} \]
\[ \text{STDEVXS} = \frac{\left( \text{SUMX2S} - \left( \frac{\text{SUMXS}^2}{L} \right) \right)}{(L-1)} \]

\[ M = M + 1 \]

\[ K = K + 1 \]

\[ \text{TVALUE} \]

PRINT
SUMXS
SUMX2S
MEANXS
STDEVXS
TVALUE
SIZE

TERMINATE
Appendix B
Program Lines

* 1.00 FLOAT; NR=900;
* 2.00 ITEM SUB I; ITEM SIZE I;
* 3.00 SUMX=0; SUMX2=0;
* 4.00 SUMXS=0; SUMX2S=0;
* 5.00 SUMSTD=0; SUMSTD2=0;
* 6.00 ARRAY RAN 950 F;
* 7.00 FOR I=0,1,NR-1;
* 8.00 BEGIN
* 9.00 RN=RANDOM(1000);
* 10.00 RD=(RN*0.001)-0.500;
* 11.00 RN1=RANDOM(1000);
* 12.00 RD1=(RN1*0.001)-0.500;
* 13.00 RN2=RANDOM(1000);
* 14.00 RD2=(RN2*0.001)-0.500;
* 15.00 RN3=RANDOM(1000);
* 16.00 RD3=(RN3*0.001)-0.500;
* 17.00 RAN(I)=RD+RD1+RD2+RD3;
* 18.00 SUMX=SUMX+RAN(I);
* 19.00 SUMX2=SUMX2+(RAN(I))**2;
* 20.00 END
* 21.00 MEANX=SUMX/NR;
* 22.00 STDEVX=SQRT((SUMX2-((SUMX)**2)/NR)/(NR-1));
* 23.00 FOR I=0,1,NR-1;
* 24.00 BEGIN
* 25.00 SUMSTD=SUMSTD+(RAN(I)-(NR/4)-MEANX)/(STDEVX**2);
* 26.00 SQRT(NR/4));
* 27.00 SUMSTD2=SUMSTD2+((RAN(I)-(NR/4)-MEANX)/(STDEVX**2);
* 28.00 END
* 29.00 FOR K=1,1,5;
* 30.00 FORMAT NUMBR,S27,C*POPULATION*;
* 31.00 FORMAT VALUES,S30,13/;
* 32.00 PRINT NUMBR;
* 33.00 PRINT VALUES, NR;
* 34.00 FORMAT HEAD,S6,C*SUMX*,S10,C*SUMX2*,S9,C*MEANX*,;
* 35.00 S11,C*STDEVX*;
* 36.00 FORMAT CLEAR,4F14.9/;
* 37.00 PRINT HEAD;
* 38.00 PRINT CLEAR, SUMX,SUMX2,MEANX,STDEVX;
* 39.00 FOR K=1,1,5;
* 40.00 BEGIN
* 41.00 SIZE=NR+K*5;
* 42.00 FORMAT MAT,//S19,C*SAMPLE SIZE*;
* 43.00 FORMAT MATS,S25,12;
* 44.00 FORMAT HEADS,/S2,C*SUMXS*,S5,C*SUMX2S*,S5,C*
* 45.00 MEANXS*, S4, C*STDEVXS*, S5, C*TVALUE*, S4, C*
* 46.00 CHIVALEU*;
* 47.00 PRINT MAT;
* 48.00 PRINT MATS, SIZE-NR;
* 49.00 PRINT HEADS;
* 50.00 FOR M = 0, 1, 9;
* 51.00 BEGIN
* 52.00 SUMXS = 0; SUMX2S = 0; MEANXS = 0; STDEVXS = 0;
* 53.00 TVALUE = 0; CHIVALEU = 0;
* 54.00 FOR I = NR, 1, SIZE-1;
* 55.00 BEGIN
* 56.00 SUMXS = SUMXS + RAN(I);
* 57.00 J = SUB;
* 58.00 RAN(I) = RAN(J);
* 59.00 SUMXS = SUMXS + RAN(I);
* 60.00 SUMX2S = SUMX2S + RAN(I);
* 61.00 RAN(J) = RAN(I);
* 62.00 END
* 63.00 MEANXS = SUMXS / (SIZE-NR);
* 64.00 STDEVXS = SQRT((SUMX2S - ((SUMXS)**2) / (SIZE-NR)) / (SIZE
* 65.00 - NR-1));
* 66.00 TVALUE = ((SQRT(SIZE-NR) * (MEANXS - MEANX)) / STDEVXS;
* 67.00 CHIVALEU = ((SIZE-NR-1) * (STDEVXS)**2) / (STDEVX)**2;
* 68.00 FORMAT CLEARS, F8.3, F11.6, F11.6, F11.6, F11.6, F11.6;
* 69.00 PRINT CLEARS, SUMXS, SUMX2S, MEANXS, STDEVXS, TVALUE,
* 70.00 CHIVALEU;
* 71.00 END
* 72.00 END
* 73.00 MEANSTD = SUMSTD / NR;
* 74.00 STDEVSTD = SQRT((SUMSTD2 - ((SUMSTD)**2) / NR) / (NR-1));
* 75.00 FORMAT STAND, / S6, C*MEANSTD*, S8, C*STDEVSTD*;
* 76.00 FORMAT VALSTD, 2F14.9;
* 77.00 PRINT STAND;
* 78.00 PRINT VALSTD, MEANSTD, STDEVSTD;
## Appendix C

Random Number Statistics

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