RELIABILITY AND MAINTAINABILITY SAMPLING PROCEDURES
FOR LIFE CYCLE COST EVALUATION

by

Doyle H. Harris

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in
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Doyle H. Harris
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ABSTRACT

Reliability and Maintainability Sampling Procedures
for Life Cycle Cost Evaluation

by

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Utah State University, 1968

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The intent of this thesis is to investigate, develop, and apply techniques to determine the reliability and maintainability of populations of items. These techniques are to be used in determining the total life-time operating costs of the populations so that those items with the lowest life-time costs can be bought. To do this, the author has explored current techniques for determining compliance to some minimum required Mean Time Between Failure (MTBF) in what is referred to as a Phase I testing. After the requirements of Phase I testing have been met, testing may be continued at the option of the contractor and confidence limits constructed about the Bid MTBF to determine compliance to it. Methods by which incentives or penalties may be rewarded or assessed the contractor as a result of the Phase II testing are included.

The author next investigated techniques which can be used to determine the maintainability parameters and the accuracy of these parameters. Finally, since the reliability techniques explored were all based on the exponential distribution, techniques were included to prove if the failure rate was exponential. If not, discussions were incorporated on how to handle this situation.

(85 pages)
INTRODUCTION

Up to this time, methods applied by the Air Force in purchasing items have primarily been on a low-bid initial cost basis. The contractor who offers the lowest price per item to the Air Force is rewarded the contract if the product meets the minimum specification—even though another product may exist which exceeds the minimum specification by a substantial amount. This type of procurement has often resulted in the Air Force accepting products with lower reliability and maintainability than necessary. Ultimately this leads to a total Air Force cost far in excess of that which would have been incurred had a superior product been bought since more money must be spent to keep the equipment operating. In addition, today's weapon systems are becoming so complex and must operate in such narrow performance ranges that an inferior product could extremely reduce their effectiveness. If this occurs, more weapon systems must be deployed to complete the same mission. This creates large impacts on all logistic requirements. When this impact is not properly managed, it could conceivably affect the national security.

It would, therefore, be beneficial to the Air Force if a program could be developed to determine the Life Cycle Cost of an item—that total cost necessary to procure, maintain, and operate that item during its intended lifetime. In assessing the Life Cycle Cost, consideration must be given not only to the initial cost, but to the cost of documentations and specifications, training, storage, transportation, and
repair (labor and material)--to mention a few. Many of these items are negotiable, and a fixed price can be assessed to them. The costs associated with items such as labor, material, and transportation are variable, however. They are largely dependent on the operational life of an item, how frequently an item fails, and if repairable, how long it takes to repair it, i.e., they are largely dependent on the reliability and maintainability of the equipment. Recent studies have indicated that annual maintenance costs of military electronics equipment range from 60 to 1000 per cent of its original procurement cost ([5, p. 1-1, 1-2]).

If a Life Cycle Cost Program is to be employed within the Air Force Procurement system, however, it will be mandatory that estimates of the reliability and maintainability parameters be achieved. These parameter estimates must be accurate enough to compute the associated costs in a manner acceptable to both the contractor and Air Force. They must indicate with some desired confidence that the parameters exceed their minimum limits which are required to adequately support the system in which they function. All reputable contractors whose product exceeds these minimum limits can then compete through competitive bidding to determine which item will be the most economical to the Air Force over its lifetime or planned inventory period.

It is the intent of this thesis to describe procedures by which these parameters can be determined--procedures that will provide the Air Force with a product that will meet the requirements of the system in which it operates and which will have the smallest Life Cycle Cost.

It is intended that these procedures will be fair to both the Air Force and the contractors and that they will promote competitive
enterprise by rewarding the competent contractors and penalizing the irresponsible ones.
The reliability and maintainability costs of an item are a function of how often the item fails and the frequency of any scheduled maintenance actions. Upon failure, costs are incurred to detect, isolate, remove, re-install, and to repair or replace the item. Calculation of the reliability and maintainability costs, therefore, requires knowledge of the item's Mean Time Between Failure (MTBF) and of the frequency of scheduled maintenance. Scheduled maintenance is normally accomplished at designated intervals and is therefore fixed for each item. The MTBF, on the other hand, is a parameter which must be estimated. When the item under consideration has previously been in the inventory, data may exist which will provide an accurate estimate. If the item has not been in the inventory, or if it is being modified, no data exists and the MTBF must be determined through statistical tests. Various tests which can be used to test the MTBF will be explained later, but first a definition of reliability and a discussion of the test criteria and assumptions are appropriate.

Reliability Statement

Reliability is defined as the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered (3, p. 1). It should be observed that the definition stresses four elements; namely, probability, adequate performance, time, and operating conditions.
Probability, the first element of the reliability definition, is a quantitative term and is expressed as a rational number between zero and one. It signifies the proportion of times an event is expected to occur. In reliability, probability is a function of time, i.e., the probability that a device will operate for 50 hours will be greater than the probability that the same device will operate for 100 hours.

Adequate performance criteria must clearly define what is considered to be satisfactory operation. As an example, an electronics tube might create a great deal of noise in a radio. In this case the radio's performance could be considered either adequate or inadequate, depending on the criteria. However, if the tube failed completely, there is no question that this condition would be considered inadequate. Details concerning failure criteria must be stated in the test procedures so that both the government and the contractor are knowledgeable, but in general, the following criteria should be included:

Equipment failure--The inability of an item to perform its required function within previously established limits.

Pattern failure--The occurrence of repetitive failures of the same part in identical or equivalent applications whose combined failure rate exceeds that predicted.

Primary failure--A failure which will independently cause equipment to perform outside specified limits.

Secondary failure--A failure of a part which is a direct result of a primary failure.

Operating conditions that insure the same factors are present during test which exists in the operational environment must be specified.
in detail. Some of the factors to be considered are: temperature, vibration, equipment on/off cycling, humidity, altitude, equipment loads, and sand and dust conditions. These factors are variable from item to item. They must be spelled out by the engineers and technicians that are responsible for the hardware under question and who are familiar with the conditions under which the equipment will be operated.

The most important element and the cornerstone of the reliability concept is time. Without knowledge of the probability of a device functioning or surviving for a given time, there is no way of assessing the probability of completing a mission or task which is scheduled to last for a given period.

Therefore, in reliability (time) sampling, precise statements as to what constitutes a failure and the conditions under which the item is subjected must be made. Operational tests are then conducted for a planned number of hours on a specified number of items selected at random from the production process. The number of failures are recorded and decisions pertaining to the item's conformance to a specified MTBF are reached. Pre-assigned risk levels indicate with what confidence the decisions are made.
Random Sampling

Time sampling is used to verify the reliability of parts, components or equipment. In time sampling, as in quality control sampling, random sampling must be accomplished if any inferences are to be made from the time sampling results.

Any appropriate sampling plan must specify a random sample of such a size that it will assure, with maximum confidence at reasonable cost, that the reliability of the lot has been properly assessed. Almost everyone dealing in statistics knows the procedures for selecting a random sample from a production lot. In reliability, however, a sampling plan consists of sampling some items for a specified number of hours. Therefore, a sample consists of so many hours of testing. Theoretically then, random sampling in reliability applications not only involves the selection of the test items in some random manner but also involves the sampling of time. To illustrate this, think of the continuum of time in which items could feasibly be operated. A sample from this continuum must be selected so that the results obtained from the time sample will be representative of the continuum of time and will give a good estimate of product reliability. In a strict sense, one might say the time cannot be selected in a random manner since it is not economical to test items at random times throughout their operating life. It has been observed, however, that items tend to fail in an unpredictable (random) manner throughout their operational life. Therefore, we can say that the failures through time are random. To insure randomness, however, care must be taken to test items during the period of constant failures and not during the infant mortality or wearout period. These failure periods are defined as follows:
Infant mortality -- The infant mortality period is the first portion of the practical reliability curve (See Figure 1) characterized by a decreasing failure rate. Failures during this period are due to assignable causes such as immaturity of design, lack of process control, etc.

![Theoretical reliability curve](image)

**Figure 1. Theoretical reliability curve**

Constant failure -- This is the period of the curve of longest duration and follows an exponential failure distribution (constant failure rate). Failures during this period are of random causes and are not repetitious in nature. It is often referred to as the useful operating life of an item.

Wearout -- The wearout portion of the curve occurs as an item approaches the end of its usable life. The failure rate increases rapidly as repetitive failures occur, and the failures are often normally distributed. During this period, failure occurs at such a rapid pace that it is almost impossible to maintain a high reliability level.
Exponential Distribution

In reliability testing, we are interested in the number of isolated events (failures) in specified time increments (exposures) and consequently the binomial distribution. In addition, most equipment is designed so that the operating time is large \((t \to \infty)\) and the failure rate is small \((r \to 0)\), such that \(rt = \lambda\) becomes and remains constant. When this occurs, the binomial distribution converges to a continuous distribution, namely the Poisson, i.e.,

\[
\binom{t}{x} r^x (1 - r)^{t - x} \to \frac{\lambda^x}{x!} e^{-\lambda} \quad \text{as } t \to \infty \text{ and } r \to 0.\]

Once the average number of occurrences per time increment of an event has been determined, the Poisson distribution can be used to predict the probability of 0, 1, 2, etc. occurrences. In reliability applications, the Poisson distribution is applied in calculating the probability of 0 failures. This probability is given by the first term of the Poisson distribution and is often called the exponential failure law or the probability of survival \((Ps)\) and is written:

\[
Ps = e^{-nt/\theta} \quad \text{or } e^{-rT} \quad \text{or } e^{-d}
\]

where

- \(n\) = number of items on test
- \(t\) = total time to test one item
- \(\theta = 1/r\) = mean time between failures in hours
- \(r\) = failure rate in failures per hour
- \(T\) = total time in hours of all items on test
- \(d\) = expected number of failures in time \(T\)

*See Appendix 1.*
Proof of Minimum Compliance-Acceptance Sampling

It is recommended that testing be accomplished in two phases for items under Life Cycle Cost procurement. The first phase of testing would be to assure that the minimum MTBF ($\theta_1$) as specified by the government is met. The second phase of testing would be to determine the degree of compliance to the contractor's bid MTBF ($\theta_1^1$).* This second phase will determine if the contractor can produce the product quality he claims, or if it is significantly different to justify application of incentives or penalties. Determination of the accuracy of the bid MTBF is extremely important since it is used in evaluating the total cost of an item and consequently awarding of the contract.

Compliance to the minimum acceptable MTBF can be accomplished by testing in accordance with standard reliability sampling procedures. Many documents exist which cover these techniques. Of these documents, the most pertinent to reliability are those based on sequential sampling techniques. The sequential sampling theory and the equations underlying these sampling techniques are not usually contained in these documents and will, therefore, be included in this thesis.

Sequential Sampling: Exponential Distribution

Reliability (time) sampling has been found to be expensive. Complex pieces of equipment must be bought, operated, and maintained (normally for long periods of time) in reliability testing. This could reduce the item's useful operating life. It therefore is very important

*See Section on Compliance to Bid MTBF
to meet the criterion of a sampling plan with the smallest amount of testing. A testing procedure which will meet the sampling plan criterion with less testing than any other type of plan is sequential sampling on an item-by-item basis. This type of sequential sampling plan was initially developed by Wald (15) and is based on what he terms the PR (Probability Ratio). The PR may be defined as

$$\text{PR} = \frac{\text{Probability of f failures where the sample is of } \theta_1 \text{ reliability}}{\text{Probability of f failures where the sample is of } \theta_0 \text{ reliability}}$$

To define the operating characteristics of this plan the following four values must be selected:

1. **Desired MTBF (\(\theta_0\))**
   
   The MTBF that, for the purposes of acceptance testing, can be considered acceptable as a process average. Items submitted of this quality will have a specified high probability of acceptance \((1 - \alpha)\).

2. **Producers Risk (\(\alpha\))**
   
   The probability or risk of rejecting a lot which has a MTBF equal to \(\theta_0\). This is the probability \((\alpha)\) of rejecting items with \(\theta_0\) MTBF when they should have been accepted. Items with a MTBF greater than \(\theta_0\) will have a probability less than \(\alpha\) of being rejected.

3. **Minimum Acceptable MTBF (\(\theta_1\))**
   
   The MTBF that, for the purposes of acceptance testing, cannot be considered acceptable as a process average. Items that are submitted of this quality will have a specified low probability of acceptance \((\beta)\).
4. Consumers Risk ($\beta$)

The probability or risk of accepting a lot which has a MTBF equal to $\theta_1$. This is the probability ($\beta$) of accepting items with $\theta_1$ MTBF when they should have been rejected. Items with a MTBF less than $\theta_1$ will have a probability less than $\beta$ of being accepted.

These parameters can now be used to develop the following equations for the sequential sampling plan:

Rejection Line: $T_1 = sr - h_1$  

(2)

Acceptance Line: $T_2 = sr + h_0$  

(3)

where $r =$ number of failures

$T =$ total test time, and

$h_0 = \ln [(1 - \alpha)/\beta] / \left(\frac{\theta_0 - \theta_1}{\theta_0 \theta_1}\right) = \ln [(1 - \alpha)/\beta] / \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)$  

(4)

$h_1 = \ln [(1 - \beta)/\alpha] / \left(\frac{\theta_0 - \theta_1}{\theta_0 \theta_1}\right) = \ln [(1 - \beta)/\alpha] / \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)$  

(5)

$s = \ln [\theta_0/\theta_1] / \left(\frac{\theta_0 - \theta_1}{\theta_0 \theta_1}\right) = \ln [\theta_0/\theta_1] / \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)$  

(6)

$H_0$ and $h_1$ are the respective intercepts and $s$ is the slope of the decision lines.

The decision criteria for a sequential sampling plan is as follows:

After $T$ test hours are accumulated with $r$ observed failures;

reject if $T \leq (sr - h_1)$,

accept if $T \geq (sr + h_0)$, and

continue testing if $(sr - h_1) < T < (sr + h_0)$.

*See Appendix 2.
It is now becoming obvious that it would be quite cumbersome if a sampling plan had to be designed using the above equations for every case. An ideal situation would be to develop a family of plans that can be utilized for a variety of operating characteristics. This has been accomplished in Mil-Std-781A (9). To accomplish this, the ratio of $\theta_0/\theta_1$ was defined as DR (Discrimination Ratio). The horizontal scale of the sequential plans were constructed in multiples of $\theta_0$ with $\theta_1 = \theta_0/DR$. By doing this the equations become

$$h_0 = \left[ \ln \left( \frac{1 - \alpha}{\beta} \right) / (DR - 1) \right] \theta_0$$

(7)

$$h_1 = \left[ \ln \left( \frac{1 - \beta}{\alpha} \right) / (DR - 1) \right] \theta_0$$

(8)

$$s = \left[ \ln \left( \frac{DR}{DR - 1} \right) \right] \theta_0$$

(9)

Therefore, for any given $\alpha$, $\beta$ and DR, a sampling plan can be constructed.

It is interesting to note that if a plan is desired that is in terms of $\theta_1$ rather than $\theta_0$, it is necessary only to multiply the above equations by DR (multiply the values in Mil-Std-781A by DR and change the time scale to units of $\theta_1$). As systems become more complex, the consumer (Air Force) has become more conscious of the necessity of designing plans according to $\theta_1$ and $\beta$. This is necessary to protect against installing inferior components in a complex system and thus degrading the reliability of the entire system.

**Truncation.** Although the probability is one that sequential testing will eventually terminate, sampling utilizing a sequential sampling plan can continue for an indefinite length of time. To prevent
this from occurring, it is often desirable to truncate the sequential test. A plan can be truncated at any given time or number of defects. The place of truncation will alter the sampling risks $\alpha$ and $\beta$, however.

One method is to truncate at $r_0$ on the failure scale (where $r_0$ is the rejection number of the predetermined time sampling plan with the same operating characteristics (See Table 2, page 23) and at $sr_0$ on the time scale. Then with aid of a scientific computer the exact $\alpha$ and $\beta$ values are calculated by summing up all the probabilities associated with rejecting $\theta_0$ and accepting $\theta_1$ at each possible decision point on the reject and accept decision lines respectively. If the exact $\alpha$ and $\beta$ values are not close to the desired $\alpha$ and $\beta$, the decision lines are manually adjusted and the exact $\alpha$ and $\beta$ recomputed. The process is repeated until the exact $\alpha$ and $\beta$ are close to the desired $\alpha$ and $\beta$.

This method is applied to Mil-Std-781A and also by Aroian (1) in his tables of "Exact Sequential Plans."

If one does not have access to a computer and desires a plan that has not already been constructed, it has been shown that the $\alpha$ and $\beta$ risks do not change appreciably if the plan is truncated out far enough from the origin. The government has established that if the plans are truncated at $3r_0$ and at $3sr_0$, there will be negligible effect on the $\alpha$ and $\beta$ values, i.e., the probability is nearly one that truncation will terminate prior to $3r_0$ (where $r_0$ is the rejection number for the corresponding predetermined time sampling plan)(10, p. 2.58).

Operating Characteristic Curve.* The O. C. Curve for sequential sampling, i.e., $L(\theta)$ (the probability of accepting $H_0: \theta = \theta_0$ when $\theta$ is the mean life value) is given approximately by the following pair of

*See Appendix 3
The values of \( L(\theta) \) and \( \theta \) can be determined by assigning values of \( h (-\infty < h < \infty) \) and solving the above equations.

Five points on a sequential O. C. Curve can readily be obtained without solving the above equations. These five points are shown in Table 1. For most purposes, they should be sufficient to get a good estimation of the O. C. Curve.

### Table 1. Five points on the O. C. Curve

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \theta )</th>
<th>( L(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-( \infty )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>( \theta_1 )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>0</td>
<td>( s )</td>
<td>( h_1/(h_0+h_1) )</td>
</tr>
<tr>
<td>+1</td>
<td>( \theta_0 )</td>
<td>( 1-\alpha )</td>
</tr>
<tr>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>1</td>
</tr>
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Example 1. Construct a truncated sequential sampling plan; calculate \( E(r|\theta), E(T|\theta) \) and \( E(WT|\theta) \); and plot the O. C. Curve for the plan. The parameters of the plan are: \( \alpha = \beta = .10, \theta_0 = 500 \) hours and
\( \theta_1 = 100 \) hours. Five items are available for test.

Solution.

\[
DR = \frac{\theta_0}{\theta_1} = \frac{500}{100} = 5
\]

\[
h_0 = \ln \left( \frac{1 - \alpha}{\beta} \right) / (DR - 1) = \ln \frac{.90}{.10} / 4
\]

\[= \ln 9/4 = 2.1972/4 = 0.549\]

\[
h_1 = \ln \left( \frac{1 - \beta}{\alpha} \right) / (DR - 1) = \ln \frac{.90}{.10} / 4
\]

\[= \ln 9/4 = 2.1972/4 = 0.549\]

\[
s = \ln DR/(DR - 1) = 5/4 = 1.6094/4 = 0.402
\]

Rejection Line: \( T = [0.402 \, r - 0.549] \, \theta_0 \)

Acceptance Line: \( T = [0.402 \, r + 0.549] \, \theta_0 \)

From Table 2. The plan will be terminated at \( 3r_0 = 3(3) = 9 \) failures and \( [s \, (3r_0)] \theta_0 = 0.402 \, (9) \, \theta_0 = 3.618 \, \theta_0 \) hours.

\[
E(r|\theta_0) = \frac{(1 - \alpha) \ln \left( \frac{\beta}{(1 - \alpha)} \right) + \alpha \ln \left( \frac{(1 - \beta)}{\alpha} \right)}{\ln DR - (DR - 1)}
\]

\[= \frac{.90 \ln 1/9 + .10 \ln 9}{\ln 5 - 4} = \frac{.90 (-2.1974) + .10 (2.1974)}{1.6094 - 4}\]

\[= -1.758 + 0.220 = -1.978 + 0.220 = -2.391\]

\[= 0.735 \text{ failures}\]

\[
E(r|\theta_1) = \frac{\beta \ln \left[ \frac{\beta}{(1 - \alpha)} \right] + (1 - \beta) \ln \left[ \frac{(1 - \beta)}{\alpha} \right]}{\ln DR - DR - 1} \frac{1}{DR}
\]

\[= \frac{.10 \ln 1/9 + .90 \ln 9}{\ln 5 - 4/5} = \frac{.10 (-2.1974) + .90 (2.1974)}{1.609 - .800}\]

\[= -2.20 + 1.978 = 1.758 = 2.18 \text{ failures}\]
\[
E(T|\theta_0) = \theta_0[E(r|\theta_0)] = 0.735 \theta_0
\]
\[
E(T|\theta_1) = \theta_1[E(r|\theta_1)] = 2.18 \theta_1 = [2.18/DR]\theta_0 = .43 \theta_0
\]
\[
E(WT|\theta_0) = \frac{\theta_0}{n} E(r|\theta_0) = \frac{\theta_0}{5} (.735) = .147 \theta_0
\]
\[
E(WT|\theta_1) = \frac{\theta_1}{n} E(r|\theta_1) = \frac{2.18}{5} \theta_1 = .43 \theta_1 = .086 \theta_0
\]

**O. C. Curve.** Five values on the O. C. Curve are found to be:

<table>
<thead>
<tr>
<th>(h)</th>
<th>(\theta)</th>
<th>(L(\theta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>(\theta_1 = 1/5\theta_0 = 100)</td>
<td>.10</td>
</tr>
<tr>
<td>0</td>
<td>.402</td>
<td>.50</td>
</tr>
<tr>
<td>+1</td>
<td>(\theta_0 = 500)</td>
<td>.90</td>
</tr>
<tr>
<td>+(\infty)</td>
<td>+(\infty)</td>
<td>1</td>
</tr>
</tbody>
</table>

If other points are desired (say -.5 and +.5), then equations (10) and (11) are used, i.e.,

\[
\theta_{.5} = \frac{5^{.5} - 1}{5^{(4)}} = .62
\]

\[
L(\theta) = \frac{9^{-.5} - 1}{9^{-.5} - (1/9)^{+.5}} = \frac{3 - 1}{3 - .33} = \frac{2}{2.67} = .75
\]

and

\[
\theta_{-.5} = \frac{5^{-.5} - 1}{(-.5)(4)} = .28
\]

\[
L(\theta) = \frac{9^{-.5} - 1}{9^{-.5} - (1/9)^{-.5}} = .25
\]

The sequential plan is illustrated in Figure 2. Figure 3 shows the O. C. Curve for this plan.
Figure 2. Sequential sampling plan

\[
t = [0.402r - 0.549] \theta_o
\]

\[
t = [0.402r + 0.549] \theta_o
\]

<table>
<thead>
<tr>
<th>r</th>
<th>AC</th>
<th>RJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.547</td>
<td>N/A</td>
</tr>
<tr>
<td>1</td>
<td>0.951</td>
<td>N/A</td>
</tr>
<tr>
<td>5</td>
<td>2.559</td>
<td>1.461</td>
</tr>
<tr>
<td>10</td>
<td>4.569</td>
<td>3.471</td>
</tr>
</tbody>
</table>

\[\alpha = \beta = 0.10\]

DR = 5
Figure 3. Operating characteristic curve
Expected Number of Failures. The expected number of failures before a decision is reached is:

\[
E(r|\theta_0) = \frac{(1 - \alpha) \ln \left[ \beta/(1 - \alpha) \right] + \ln \left[ (1 - \beta)/\alpha \right]}{\ln DR - (DR - 1)}
\]

\[
E(r|\theta_1) = \frac{\beta \ln \left( 1 - \alpha \right) + (1 - \beta) \ln \left( 1 - \beta \alpha \right)}{\ln DR - (DR - 1) DR}
\]

Other values of interest are the expected total test time and the expected waiting time. They are given as:

\[
E(T|\theta) = \theta E(r|\theta)
\]

\[
E(WT|\theta) = \frac{\theta}{n} E(r|\theta) = E(T|\theta)/n
\]

where \( E(T|\theta) \) is the expected test time given mean life \( \theta \); and \( E(WT|\theta) \) is the expected waiting time until completion of the test.

Predetermined Time Sampling: Exponential Distribution**

Although sequential sampling requires less testing than any other sampling plan for the same risks, it has certain disadvantages which may warrant selection of another type of plan. They are:

a. Sequential plans are more complicated to construct.

b. They are harder to implement and administer.

c. It is often necessary to pre-plan a sampling experiment.

You need to know how many items to buy or you have a certain time in which to complete the test.

When this is the case, then a predetermined time sampling plan

*See Appendix 4. **See Appendix 5.
should be used. A predetermined sampling plan is one in which the number of test hours and the acceptance and rejection numbers are fixed.

To develop tests which are terminated after a specified number of test hours or with a preassigned number of failures, whichever occurs first, the test time and the rejection number (critical number of failures) can be computed, given the test specifications—$\theta_0$, $\theta_1$, $\alpha$, and $\beta$.

If $T$ is the test termination time (the maximum number of hours all items will accumulate) and $r_0$ is the rejection number, the decision rule is as follows:

If $r_0$ failures occur before $T$ test hours reject the lot and discontinue testing.

If $T$ test hours are accumulated before $r_0$ failures occur accept the lot.

The rejection number ($r_0$) must be selected so that the test specifications are met. This is accomplished if $r_0$ is selected so that the following poisson equation is satisfied:

$$L (\theta) = \sum_{X = 0}^{r_0 - 1} \left( \frac{nt}{\theta} \right)^X \frac{e^{-nt/\theta}}{X!}$$

(16)

where $L (\theta_0) \geq 1 - \alpha$; $L (\theta_1) \leq \beta$.

$L (\theta)$ is the probability of acceptance, i.e., it must satisfy the following equations:

$$1 - \alpha \leq \sum_{X = 0}^{r_0 - 1} \left( \frac{nt}{\theta_0} \right)^X \frac{e^{-nt/\theta}}{X!}$$

and
\[
\alpha \leq \sum_{x=0}^{r_0-1} \frac{\left(\frac{nt}{\theta_1}\right)^x e^{-nt/\theta_1}}{x!}
\]

Since the expected number of failures is equal to \(nt/\theta\), for any \(\theta\) let \(d = nt/\theta_0\) and since \(DR = \theta_0/\theta_1\), then \(DR(d) = nt/\theta_1\). Substituting in equation (16) the following simultaneous equations can be solved for \(r_0\) and \(d\):

\[
1 - \alpha \leq \sum_{x=0}^{r_0-1} \frac{d^x e^{-d}}{x!}
\]

\[
\beta \geq \sum_{x=0}^{r_0-1} \frac{[DR(d)]^x e^{-DR(d)}}{x!}
\]

where the rejection number is \(r_0\) and if either \(n\), \(t\), or \(T\) is specified, the other two test parameters can be determined from the relationship

\[
\frac{nt}{\theta_0} = \frac{T}{\theta_0} = d
\]

Equations (17) and (18) can be solved for \(r_0\) and \(d\) by the use of a computer program for various values of \(DR\), \(\alpha\), and \(\beta\). Table 2 has been extracted from Arinc (14) for easy reference.

**Manual Calculations.** If a computer is not available, and a sampling plan is desired that has specifications different than in Table 1, the poisson tables may be utilized to construct the plan. The procedure is as follows:

1. Select an acceptance number \((c = r_0 - 1)\) that seems close to the correct number.
Table 2. Predetermined time sampling plans

<table>
<thead>
<tr>
<th>$DR = \frac{\theta_0}{\theta_1}$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$d$</th>
<th>$r_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.05</td>
<td>0.05</td>
<td>54.13</td>
<td>67</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.10</td>
<td>43.40</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.25</td>
<td>25.87</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.05</td>
<td>43.00</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.10</td>
<td>33.04</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.25</td>
<td>18.84</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.05</td>
<td>28.02</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.10</td>
<td>19.61</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.25</td>
<td>9.52</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
<td>15.72</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.10</td>
<td>12.44</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.25</td>
<td>7.69</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.05</td>
<td>12.82</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.10</td>
<td>10.30</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.25</td>
<td>5.43</td>
<td>9</td>
</tr>
<tr>
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<td>0.25</td>
<td>0.05</td>
<td>8.62</td>
<td>11</td>
</tr>
<tr>
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<td>5.96</td>
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</tr>
<tr>
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<td>0.25</td>
<td>0.25</td>
<td>3.37</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>0.05</td>
<td>5.43</td>
<td>10</td>
</tr>
<tr>
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<td>0.05</td>
<td>0.10</td>
<td>3.98</td>
<td>8</td>
</tr>
<tr>
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<td>0.05</td>
<td>0.25</td>
<td>2.61</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.05</td>
<td>4.66</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.10</td>
<td>3.15</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.25</td>
<td>1.74</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.05</td>
<td>3.37</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.10</td>
<td>2.54</td>
<td>4</td>
</tr>
<tr>
<td></td>
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<td>0.25</td>
<td>0.96</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
<td>0.05</td>
<td>1.97</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.10</td>
<td>1.37</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.25</td>
<td>0.82</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.05</td>
<td>1.74</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.10</td>
<td>1.10</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.25</td>
<td>1.10</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.05</td>
<td>0.96</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.10</td>
<td>0.96</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.25</td>
<td>0.29</td>
<td>1</td>
</tr>
</tbody>
</table>

2. Go down the table under $c$ until the probability of acceptance $(1 - \alpha)$ is located.
3. Go across this table and locate the value of \( d \) (np).

4. Multiply \( d \) by the discrimination ratio DR (\( d = d' \)).

5. Locate this new value for \( d = d' \) and cross over into the column under \( c \).

6. This is the \( \beta \) risk. If this value is too large, a larger value of \( c \) must be selected. If too small, a smaller value of \( c \) should be selected.

7. Repeat steps one through six until the value of \( c \) is selected, the one that most closely conforms to the \( \alpha \) and \( \beta \) risks. This is the acceptance number of the sampling plan \( (r_0 - 1) \). Equation (19) may then be used to determine the other parameters, using the value of \( d \) (np) corresponding to \( 1 - \alpha \) for the \( c \) value selected.

**Operating Characteristic Curve.** The operating curves for any fixed time sampling plan can be constructed by substituting various values of \( \theta \) into equation (16), once \( r_0 \) is known.

Other values that are of interest for this type of sampling plan are the expected number of failures, the expected total test time, and the expected waiting time when \( \theta = \theta_0 \) and \( \theta = \theta_1 \). The formula for these when the mean life is \( \theta \) has been shown by Epstein (4) to be:

\[
\begin{align*}
E(r|\theta) &= d \sum_{x=0}^{r_0-2} \frac{d^x e^{-d}}{x!} + r_0 \left[ 1 - \sum_{x=0}^{r_0-1} \frac{d^x e^{-d}}{x!} \right] \\
E(T|\theta) &= \theta E(r|\theta) \\
E(WT|\theta) &= \frac{\theta}{n} \left[ E(r|\theta) = \frac{E(T|\theta)}{n} \right]
\end{align*}
\]

*See Appendix 5.*
where $E(T|\theta)$ is the expected total test time and $E(WT|\theta)$ is the expected waiting time until completion of the test.

**Compliance to Bid MTBF**

If it were possible to buy items from all contractors who were interested in furnishing a product to the government and in testing those items to determine which contractor's product was best, there would be little need for incentive schemes. The contract would be awarded to the contractor who could provide the Air Force the most for its money. Unfortunately this is seldom the case. It is not normally economical to test more than one contractor's product. Many items are not manufactured until the contract is awarded. Therefore, if the Air Force is to apply the Life Cycle Cost concept, it must initially accept the contractor's word concerning the reliability of their product, evaluate and award the contract based on this stated reliability, then test to verify his word and apply incentives to keep the contractor honest.*

Incentive plans should be based upon the premise that the contractor will share the reward of reduced Life Cycle Costs due to product performance over and above that which he bid or to share the burden of increased Life Cycle Costs due to performance less than that bid. The Life Cycle Cost contracts should be established as incentive contracts and the incentive-penalty methodology defined prior to awarding the contract.

Relating the incentive to the accuracy of the contractor's bid MTBF could be a strong motivation for more accurate estimating by the contractor. If he attempts to guarantee an incentive by being conservative

*Incentives will be defined to include both incentives and penalties, i.e., negative and positive incentives.
in his performance estimates (ones which he is sure he can easily achieve), he runs the risk of having the contract awarded to another contractor who promises higher performance. If he is over-optimistic in his estimates, penalties could cut into the profit he can expect.

The estimates included in the contractors' bid packages will be used to compute the Life Cycle Cost of each contractor's bid. The contract will then be awarded on the basis of lowest bid Life Cycle Cost. After contract award, the items delivered by the successful contractor will be tested according to test conditions specified in the contract. The results of this test will be used to determine the item's Life Cycle Cost.

The incentive plan must share the benefits of a reduced Life Cycle Cost with the contractor, giving him an incentive when the performance is above that stated, and on the other hand penalizing the contractors whose performance is below that stated by making him forfeit a percentage of his profit. Each individual incentive plan must be designed upon the merits of its application so that the incentive-penalty plan is sensitive enough to motivate the contractor and yet not too sensitive as to discourage him.

Confidence Interval Calculations.* The upper or lower control limits on \( \theta \) are determined by:

\[
\hat{\theta}_{LCL} = \frac{2T}{X^2_{\alpha, 2R}} = \frac{2R\theta}{X^2_{\alpha, 2R}} \tag{23}
\]

If terminated at a fixed number of failures \( R \) or

\[
\frac{2T}{X^2_{\alpha, (2R+2)}} = \frac{2R\theta}{X^2_{\alpha, (2R+2)}} \tag{24}
\]

*See Appendix 6.
if terminated at a fixed time \((t)\) and

\[
\hat{\theta} \text{ UCL} = \frac{2T}{X (1 - \alpha), 2R} = \frac{2R\hat{\theta}}{X (1 - \alpha), 2R} \tag{14}
\]

\[
\hat{\theta} \text{ LCL} = \frac{T}{X (1 - \alpha), 2R} = \frac{R\hat{\theta}}{X (1 - \alpha), 2R} \tag{25}
\]

**Testing.** When testing has been accomplished to determine compliance to the minimum acceptable MTBF \((\theta_1)\), and a decision has been made to accept the product under test, a determination will be made as to whether the point estimate of the MTBF \((\hat{\theta})\) is in a penalty, no incentive, or incentive region.

This is done by determining the upper and lower confidence limits of the bid MTBF. This must be done for each conceivable failure. The contractor should have the option of terminating the test upon proving the minimum specified MTBF has been met or to continue at his own expense for an indefinite amount of time. Continuation of the test will award a contractor whose product is better than that bid since it will qualify him for incentives. Continuation of the test will most likely penalize a contractor whose product is worse than that specified since it will narrow the confidence interval and increase the amount of the penalty. The government will be rewarded by a continuation of the test since more accurate information is available concerning the product. This will enable better management of the item being bought.

When the test is terminated, the decision criteria will be as follows:

1. If the intervals about the bid MTBF brackets the time actually tested then it will be determined that no significant difference exists.

2. If the actual test time is greater than the computed test time for the upper limit then the decision will be that the bid MTBF is
significantly better than specified. In this case, the Air Force will award the contractor incentives. The amount of incentives will be determined for each Life Cycle Cost procurement prior to the invitation for bid.

3. If the actual test time is less than the computed test time for the lower limit then the decision that the bid MTBF is significantly worse than that specified is reached. In this case, the contractor will be required to return a certain amount of the bid price of the contract to the Air Force. The amount to be returned will be specified in the incentive plans.

Once a contractor has selected to do testing in addition to that required for initial acceptance he must coordinate his decision to terminate testing with the Air Force representatives. The representatives will have the privilege to inspect all test items to insure that testing was not terminated to preclude failures. If this was the case the failures must be counted or testing resumed until such a time that termination can be agreed upon. This is essential to insure the test is not terminated at a point which could bias the test results.

It is recommended that confidence intervals be constructed at the 90% level. This will maintain uniformity in procurement actions.

Test Time Required to Demonstrate Bid MTBF. The formula for establishing confidence intervals about the MTBF of an exponential distribution can be used to determine whether the point estimate of the MTBF is in the penalty, no-incentive, or incentive region. This is done by calculating the test time required to be out of the penalty area or into incentive area for all conceivable number of failures. The calculations are as follows:
Since

\[ LCL = \frac{2T}{\chi^2_{2R + 2, \alpha}} \]  

the time required to demonstrate a lower limit is

\[ T = LCL \left[ \frac{\chi^2}{2(2R + 2, 1 - \alpha)} \right] \frac{1}{2}. \]

By setting the LCL equal to the bid MTBF we can find the test time required before it can be demonstrated with confidence that the actual MTBF is equal to or better than the bid MTBF. Similarly, by setting the bid MTBF equal to the upper confidence limit, we can compute the test time required to demonstrate with confidence that the MTBF is greater than that bid, i.e., we calculate the test time necessary to demonstrate at the prescribed confidence level that the actual MTBF is greater than the bid. Thus the test time to get out of the penalty area is calculated by:

\[ T = \text{Bid MTBF} \left[ \frac{\chi^2}{2R, \alpha} \right] \frac{1}{2} \]  

and the time required to get into the incentive region is calculated by:

\[ T = \text{Bid MTBF} \left[ \frac{\chi^2}{2R + 2, 1 - \alpha} \right] \frac{1}{2} \]  

These limits can be constructed on the same graph as the sequential plan. Table 3 tabulates the above \( \chi^2 \) values for the various number of failures; the approximate time required for test can be readily computed

*The confidence interval for time terminated tests will be used since termination is seldom made at the time of a failure.
and plotted from this table. In addition Figures 4 and 5 can be used to
give an approximation of the test time to get out of the penalty region or
into the incentive region for any bid MTBF and corresponding number of
failures. Figure 6 is an example of the confidence intervals applied to
Example 1. The bid MTBF was set equal to $\theta_0$.

Table 3. Factors for determining test time required to
verify 90% upper (lower) limits on the bid MTBF*

<table>
<thead>
<tr>
<th>Number of Failures</th>
<th>Multiply Bid MTBF by this value to determine time required to get out of penalty region (upper limit)**</th>
<th>Multiply Bid MTBF by this value to determine time required to get into incentive region (lower limit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>--</td>
<td>$(\chi^2_{2R, .10})/2$</td>
</tr>
<tr>
<td>1</td>
<td>2.303</td>
<td>2.303</td>
</tr>
<tr>
<td>2</td>
<td>2.390</td>
<td>3.532</td>
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<td>15</td>
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</tbody>
</table>

*For values of $\nu > 30$ ($\nu = 2R$ when the Bid MTBF is set equal to the upper confidence limit and $\nu = 2R + 2$ when the Bid MTBF is set equal to the lower confidence limit), solve the equation $\sqrt{2x^2} = \sqrt{2\nu - 1} + Z$ where $Z$ is the standard normal deviate (1.282).

**Bid MTBF must be in terms of $\theta_0$ to plot the two on the same graph.
Figure 4. Total test time required to enter incentive region
Figure 5. Total test time required to leave penalty region
Figure 6. Combined sequential sampling and incentive plan
Comment. Upon acceptance of the hypothesis that the minimum
specified MTBF ($\bar{\theta}_1$) has been met, a contractor may be in any of three
incentive regions, depending on what he bid in comparison to $\bar{\theta}_1$. * When
the bid MTBF is much larger than the specified MTBF, more testing will
normally be required of the contractor to insure he is as good as or
better than what he specified. Although it is recommended that the
contractor pay for this extra testing, he is still awarded since:

a. His high bid enabled him to get the contract.

b. Awarding the contract on a Life Cycle Cost basis allows
the contractor to charge more for a good quality product.

Incentive Schemes

When the bid MTBF ($\bar{\theta}_1$) falls outside the computed test times
for the upper or lower limits (Region-of-No-Incentives (See Figure 7)),
incentives will be rewarded or assessed on the contractor. Since the
amount of incentives is highly dependent on the magnitude of the contract,
they must be determined separately for each contract. In essence, they
must be of sufficient size to be effective, but at the same time they
must be realistic to both parties involved. Penalties assessed on a
contractor should not usually exceed their expected profit. Rewards
to a contractor should not exceed a certain percent of the Air Force's
savings.

The first requirement of incentive awards is to determine the
difference in the expected cost for the government to operate an item for
its operational life. If a significant difference exists between $\bar{\theta}$ and

*Except for the case where acceptance was made with zero failures.
Then the contractor can be in only one of two regions--no incentive or
incentive.
so that $\theta^1$ is no longer bracketed by the 90% confidence intervals on $\hat{\theta}$, there becomes a region of significant difference; i.e.,

$$+ \Delta \hat{\theta} = \frac{2(\text{Actual test time} - \text{Required test time})}{X^2 2R + 2, \alpha} \quad (29)$$

and

$$- \Delta \hat{\theta} = \frac{2(\text{Required test time} - \text{Actual test time})}{X^2 2R, (1 - \alpha)} \quad (30)$$

Figure 7 illustrates this difference ($\pm \Delta \theta$). The expected cost can then be calculated as:

$$E(C) = \left[ N \ast \text{Hrs Usage/Mo} \right] \ast \text{Mo. Oper Life} * \text{Repair Cost/Item} \pm \Delta \hat{\theta} \quad (31)$$

where $N$ is the number of items procured.

For each procurement, a certain percent on this expected cost will be established as the percent of adjustment to be made. This will normally range between 40 and 60 percent.

**Conclusion**

Testing required for Life Cycle Cost Procurements should be
accomplished in two phases. The first phase will determine compliance to some minimum MTBF. The second phase will determine compliance to a Bid MTBF and incentives will be applied if compliance is not demonstrated. Testing costs will be negotiable and will be fixed with the bid price.

To satisfactorily complete testing of an item, the following information must be included within the test directive:

1. Failure modes must be defined and their parameters specified. This must be accomplished by the hardware engineers.

2. Testing should not continue into the wearout failure period. This will penalize the contractor unjustly.

3. Environmental conditions under which the equipment must operate are to be determined by the hardware engineers and incorporated into the test directive. These conditions must be adhered to, through simulation when possible and in the operational environment when simulation cannot be realistically accomplished.

4. The production phase must be the same as the pre-production phase from which the test samples were selected.

5. Repetitive failures must be corrected. If not, they are cause for rejections.

6. Test items should be repaired whenever possible and placed back on test.

7. Test samples must be randomly selected. No pre-selection or censorship of the items will be allowed.

8. If the contractor selects to continue testing after completion of the Phase I testing, his decision to terminate the test must be coordinated with the Air Force representatives.
MAINTAINABILITY

The previous sections of this thesis have explained methods by which the expected number of equipment failures can be accurately identified. To complete the information necessary to assess the Life Cycle Cost of an item, consideration must now be given to procedures which will estimate the second variable parameter, Maintainability.

In order to determine the maintainability of an item, an estimate of the mean time of maintenance action (\( \hat{\phi} \)) must be made. Mean time of maintenance action will be defined as the average repair time necessary to perform any type of maintenance action. This includes all time expended to keep the item in an operational condition either through preventative or corrective actions. The point estimate of \( \hat{\phi} \) is calculated as:

\[
\hat{\phi} = \frac{\text{total maintenance action time (in hours)}}{\text{number of maintenance actions}} \quad (32)
\]

Accuracy of Point Estimate

In order for the point estimate of \( \hat{\phi} \) to be valid for Life Cycle Cost calculations, a measure of its accuracy must be known. In industrial engineering, they have classified the estimates of \( \hat{\phi} \) into different types of labor standards. AFLC has defined these labor standards according to their accuracy in AFLCM 66-4 (6). To classify a standard 1A, it must be shown with 95% confidence that \( \hat{\phi} \) is within ±10% of \( \phi \) (the population mean time of maintenance actions). For a
standard to be Type 1B, it must be shown with 95% confidence that \( \hat{\theta} \) is within \( \pm 25\% \) of \( \theta \). In addition to classifying labor standards 1A through time studies, standards can also be classified 1A by using standard data. The most important standard data system is known as Methods-Time-Measurement.

Although it would be desirable in Life Cycle Cost to stick with a 10% risk (90% confidence), it is felt that the advantages of conforming to accepted industrial engineering practices outweigh the desire to standardize the confidence level at 90%. This is true since AFLC has already established labor standards for most items and the expense of maintainability testing will probably prohibit the contractors from testing. It is therefore recommended that the criteria for a Type 1A standard as described in AFLCM 66-4 be accepted as the criteria to which \( \hat{\theta} \) must conform in order to be acceptable for Life Cycle Cost calculations.

**Criteria for Making Maintainability a Biddable Item**

The time required to perform maintenance actions can be divided into two parts: (1) That time that will be constant for like items and which cannot be affected by altering the design of the item. This time is fixed by the system in which the item operates. (2) That time that is variable by changing the design of the item. This is the biddable portion of the time.

It is felt that most of the time spent for maintenance actions will be fixed by the system in which an item operates. The time to isolate, get to, remove, package, and transport an item is normally far larger than the time to repair it. It is therefore felt that the
majority of Life Cycle Cost procurements will not make maintainability a biddable item. The Air Force will rely on its own work measurement system in acquiring the necessary labor standards to support Life Cycle Cost calculations.

Even when an item has maintenance time which can be affected by design, the only portion of the time that can save the Air Force money is the difference between the two designs. Furthermore, items from the various manufacturers who want to compete on a Life Cycle Cost basis may already be in stock. If this is the case, labor standards will already be established for the items. If there were any significant difference from one manufacturer's design to another one, separate labor standards would have been established in order to achieve the accuracy necessary for type IA standard.

In cases where it is desirable to make maintainability a biddable item and no labor standards exist that will do the job, one of the following courses of action must be taken:

1. Let each manufacturer bidding on the contract also bid on maintainability testing.
2. Have a third party establish the labor standards.
3. Have Air Force work measurement personnel establish the standards.

Establishing a Labor Standard

As previously mentioned, it is recommended that estimates of the mean time of maintenance actions meet the requirements of a type IA standard. For a standard to be type IA, it must be demonstrated within 95% confidence that the point estimate be within ± 10% of the population
mean. This is primarily a confidence interval problem and is illustrated in Figure 8.

\[ \hat{\mu} - .10 \hat{\mu} \quad \hat{\mu} \quad \hat{\mu} + .10 \hat{\mu} \]

Type 1A

(95% Confidence Interval)

Figure 8. Type 1A standard

Therefore the probability that the interval \((\hat{\mu} - .10 \hat{\mu}; \hat{\mu} + .10 \hat{\mu})\) brackets \(\mu\) must equal .95.

The sample size \((n)\) required to meet the specified criteria is determined by

\[
\sqrt{n} / 10 t_{n-1}, \frac{\alpha}{2} \geq S/\hat{\mu} \]

(33)

The values of \(\sqrt{n} / 10 t_{n-1}, \alpha/2\) have been computed and are tabulated for \(n\) ranging from 1 to 120 in Table 4. This has been accomplished for the two sided 95% confidence intervals. For values above 120, \(t_{n-1}, \alpha/2\) is approximately identical to the normal distribution and therefore 1.96 can be used.

The procedure for determining if \(\hat{\mu}\) meets the criteria of a type 1A standard is as follows:

1. Randomly select and measure from 10 to 20 maintenance actions performed on the item.

2. Calculate \(S = \sqrt{\left[\sum X^2 - (\sum X)^2/n\right] / n-1}\) and \(\hat{\mu} = \Sigma X/n\)

*See Appendix 7.
Table 4. Values for statistical determination of standard classification

<table>
<thead>
<tr>
<th>Number of Observations</th>
<th>Type 1A $S/\hat{\theta}$ ≤</th>
<th>Number of Observations</th>
<th>Type 1A $S/\hat{\theta}$ ≤</th>
<th>Number of Observations</th>
<th>Type 1A $S/\hat{\theta}$ ≤</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.0111</td>
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<td>.0402</td>
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<td>.0805</td>
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<td>.1952</td>
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<td>.1805</td>
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<td>.1877</td>
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<td>53</td>
<td>.3168</td>
<td>54</td>
<td>.3168</td>
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</tbody>
</table>

For $n > 121$ $S/\hat{\theta}$ ≤ $\sqrt{n}/19.6$
3. Calculate $S/\hat{\theta}$ and locate the next larger value than it in Table 4.

4. Note $n$ opposite this value.

5. If $n$ is less than the initial sample accept $\hat{\theta}$ as a type I A standard. If $n$ is greater than the initial sample more observations are necessary. Determine the difference between $n$ and the original sample and randomly measure that many more readings.

6. Repeat steps 1 through 5 until $\hat{\theta}$ can be classified a type I A standard. If the requirements for a type I A standard cannot be met the variability in the maintenance actions is too great. This can be controlled by either breaking units of measurement into smaller segments or by redesigning the item to achieve better maintainability. If a system is broken into smaller segments, the total maintenance time of a system must be established through the use of occurrence factors.

**Occurrence Factors**

Occurrence factors are established by measuring the frequency at which maintenance actions on any specific unit are being performed compared to the total number of maintenance actions on the system. This is expressed as:

$$\overline{p}_i = \frac{\text{Number maintenance action on } i^{th} \text{ unit}}{\text{Number system maintenance actions}} \quad (34)$$

This frequency must also be accurate enough to insure with 95% confidence that it is within $\pm 5\%$ absolute accuracy of the true frequency. This is determined by: (See Appendix 8)
\[ n > \frac{Z^2 \overline{P}(1-P)}{\delta^2} \]  

(35)

for 95% confidence limits \( Z = 1.96 \) and \( \delta \) equal the absolute accuracy (.05).

\[ n \geq 1.96^2 \frac{\overline{P}(1-P)}{.05^2} \geq 1536.7 \overline{P}(1-\overline{P}) \]

Table 5 lists the number of samples required for difference occurrence factors (\( \overline{P} \)) for an absolute accuracy (\( \delta \)) equal to .05.

**Example 2:** A labor standard was being established on how long it takes to replace a bearing in a motor generator. The bearing had to be replaced in 15 motors out of the 500 received. Determine if the standard is a type 1A and if enough observations have been made to establish an accurate occurrence factor. The 15 measurements are as follows: 27, 30, 35, 35, 37, 38, 41, 42, 34, 38, 40, 32, 34, 36, 42.

The calculated statistics are: \( S = 4.35, \hat{\theta} = 36.07, S/\hat{\theta} = .12 \). Therefore since \( S/\hat{\theta} \leq .1805 \) (from Table 4) the standard is classified 1A.

The occurrence factor is 15/500 = 3%. Table 5 shows that only 45 motors would have had to be received to determine the accuracy of this factor within ± 5%.

**System Mean Time of Maintenance Actions**

Once the various standards are set for the \( i \) units and their occurrence factors known, the overall system mean time of maintenance actions is calculated as:

\[ \hat{\theta}_s = \hat{\theta}_1 \overline{P}_1 + \hat{\theta}_2 \overline{P}_2 + \ldots + \hat{\theta}_n \overline{P}_n = \sum_{i=1}^{n} \hat{\theta}_i \overline{P}_i \]  

(36)
Table 5. Sample size requirements for a degree of accuracy

<table>
<thead>
<tr>
<th>Occurrence n =</th>
<th>n =</th>
<th>Occurrence n =</th>
<th>n =</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor ((\hat{p})) ((\delta = .05))</td>
<td>((\delta = .10))</td>
<td>Factor ((\hat{p})) ((\delta = .05))</td>
<td>((\delta = .10))</td>
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<tr>
<td>.25</td>
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<td>.50</td>
<td>384</td>
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</table>

*For values of \(\hat{p}\) greater than .50, the sample size is the same as for 1 - \(\hat{p}\).*

This estimate of the system mean time of maintenance actions is classified a type 1A standard if 95% of the element times making up the system standard are classified 1A.

**Importance of Maintainability**

Today's complex equipment and integrated systems have made us realize the importance of maintainability because the cost of maintenance has become prohibitively high. Moreover, the military must
make plans based on the immediate availability of weapons, even if the cost was tolerable.

Maintainability consists of two types: (1) preventive and (2) corrective. Preventive maintenance is to forestall the occurrence of a failure or malfunction. Corrective maintenance is performed only when a malfunction or failure occurs.

**Maintainability Factors**

Factors which affect maintainability may be divided into two groups—(1) design and (2) installation. Those factors which are related to design are: reliability, maintainability, complexity, interchangeability and replaceability, compatibility, visibility, and configuration.

Installation factors are chiefly related to human beings and the associated environment. They include: experience, training, skill, supervision, environment, technical publications, and test techniques.

All these factors must be considered in assessing the maintainability of an item. They are effectively incorporated into a design by applying modern methods to locate and correct failures rapidly and in providing for the deferment of immediate repair. These methods include such things as:

1. Fault-location and isolation devices such as built-in test equipment and marginal checking devices for the rapid detection of the need of either preventative or corrective maintenance.

2. Making parts accessible so that parts can be readily removed without removing adjacent parts.
3. Equipment is becoming "unitized" by being composed of a series of pluggable parts. In this way, a unit which malfunctions can be readily removed and replaced with a good unit and then repaired in the spare time.

4. Equipment is being developed with many redundant components so that when one fails the other can still perform the required mission.

5. Alternate systems are being developed so that when one fails, the other takes over.

All of these methods are of little use unless maintenance personnel are highly trained and skilled. Untrained personnel not only cost time, but create other problems by their clumsiness and unfamiliarity with the equipment.

Conclusion

Although it will probably be the exception when circumstances warrant making maintainability a biddable item, accurate estimates of the mean time of maintenance actions must be acquired in order to accurately estimate the Life Cycle Cost of an item. These estimates should normally be provided by the work measurement technicians. When it is beneficial to make maintainability biddable, it will be accomplished by:

1. Letting the manufacturers building the item bid on maintainability testing.

2. Contracting a third party to determine maintainability.

3. Doing it in-house.

No matter who is involved in establishing the estimate of $\emptyset$, these estimates must be accurate enough to conform to the criteria.
set forth for establishing a type IA standard in AFLCM 66-4.

While methods have been presented which measure the effect of the various maintainability factors discussed, a separate engineering review should be conducted of all candidate items where maintainability will be a biddable item to insure that the design is such that the effects of these factors are optimized.

Once the estimates of reliability and maintainability are achieved, the maintenance man-hours required to support the system during its lifetime can be computed. This figure can be converted to cost, and when combined with other costs such as material, packaging, and shipping, the expected cost of maintainability during the item's lifetime can be assessed.
IDENTIFICATION OF FAILURE RATE CHARACTERISTICS

As previously stated, it is very important to both the Air Force and the contractors that testing be accomplished during the constant failure period.

Since reliability testing is very expensive, verification testing is normally accomplished during the earlier part of the item's life. If an item is tested during an infant mortality period under the assumption of constant failure rate, it will result in quoting a higher failure rate than actually exists. This could ultimately lead to the awarding of a contract to a poorer manufacturer. When this is done, both the government and the superior contractors suffer—the government has an inferior product and the contractor loses a contract. On the other hand, the government must still repair items which fail during the infant mortality period. Since this costs money, adjustments must be incorporated in the Life Cycle Cost calculation, or a contractor must be willing to "burn-in" the items prior to delivery.

Trend Detection

To test the hypothesis that the failure rate is constant, various nonparametric methods may be applied as the test progresses. A graphical approach is to plot: (1) the log of the reliability function \( \log R(t_i) \) versus time* when testing is without replacement and

*Semi-log graph paper may be used to plot \( R(t_i) \) versus time directly.
(2) the number of failures versus time when testing is with replacement. If the data plots as a straight line then a constant failure rate exists. A concave line with respect to the time axis means a decreasing failure rate; a convex line means an increasing failure rate. The fact that the line is concave (convex) under the assumption of an increasing (decreasing) failure rate may be demonstrated by taking the second differential of the log of the reliability function for the Weibull distribution.

\[ R(t) = e^{-t^\beta/\alpha} \]

\[ \log_e [R(t)] = (-1/\alpha) t^\beta \]

\[ f'[\ln R(t)] = -\frac{\beta}{\alpha} t^{\beta-1} \]

\[ f''[\ln R(t)] = -\frac{\beta(\beta-1)}{\alpha} t^{\beta-2} \]

If \( R(t) \) is a decreasing function (\( \beta < 1 \)) then the second derivative must be positive, which it is. This plots as a concave line on semi-log paper. When \( \beta = 1 \), the second derivative is 0 and, therefore, the failure rate is constant. This plots as a straight line. When \( \beta > 1 \), the second derivative is negative, indicating an increasing function, which plots as a convex line.

For the nonreplacement case the reliability function is determined as follows:

At the time of each failure, the percentage of survivals is known. This leads to an estimate of \( R(t) \) which is given by (14, p. 144-5)

\[ \hat{R}(t_i) = 1 - F(t_i) = 1 - \frac{i}{N+1} = \frac{N - i + 1}{N + 1} \]  \hspace{1cm} (37)
where $i$ represents the $i$th ordered incident of failure and $t_i$ represents the time to the $i$th failure. The function $\log R(t_i)$ may be plotted by the points $\log \frac{N - i + 1}{N + 1}$, $t_i$ or $\frac{N - i + 1}{N + 1}$, $t_i$ if semi-log paper is used.

If terminated (censored) observations occur and if the time of occurrence and the number of terminations are known, an estimate of $R(t)$ at the time of the $k$th failure is given by:

$$\hat{R}(t_k) = \prod_{i=1}^{k} \frac{N_i - r_i + 1}{N_i + 1}$$

(38)

Where $N_i$ is the number of survivors beginning the interval which precedes the $i$th failure and $r_i$ is the number of failures occurring at the time of the $i$th failure.

Example 3: Nineteen items were put on test, the time to failure of each was recorded, and the reliability function was calculated as follows:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t(i)$</th>
<th>$R(t_i)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>.95</td>
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<td>.45</td>
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</tbody>
</table>
Figure 9 shows the plot of \([t_{(i)}, R_{(i)}]\) on semi-log paper. As can be seen, the curve is concave in the early portions of testing, representing an infant mortality period. The line then becomes straight indicating the constant failure period and then becomes convex as the items begin to wear out.

The procedure for replacement items would be essentially the same only time versus the number of failures would be plotted on regular graph paper.

*Weibull Distribution*

Testing during the constant failure period is normally required since most reliability sampling procedures are based on the exponential distribution. Although not presently recommended for Life Cycle Cost procurement, a note should be made concerning the Weibull distribution. Sampling plans have been developed based on this distribution that do not require a constant failure rate \((11, 12, 13)\). They do, however, require that the failure rate or shape parameter be known in order to select a sampling plan. A decision to assume the exponential distribution in Life Cycle Cost testing was made for this reason, combined with the understanding that:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(t_{(i)})</th>
<th>(R_{(i)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>150</td>
<td>.40</td>
</tr>
<tr>
<td>13</td>
<td>185</td>
<td>.35</td>
</tr>
<tr>
<td>14</td>
<td>280</td>
<td>.30</td>
</tr>
<tr>
<td>15</td>
<td>340</td>
<td>.25</td>
</tr>
<tr>
<td>16</td>
<td>390</td>
<td>.20</td>
</tr>
<tr>
<td>17</td>
<td>420</td>
<td>.15</td>
</tr>
<tr>
<td>18</td>
<td>450</td>
<td>.10</td>
</tr>
<tr>
<td>19</td>
<td>460</td>
<td>.05</td>
</tr>
</tbody>
</table>
Figure 9. Reliability function
1. Extensive research and past experience indicate the failure rate of most equipment conforms to the theoretical reliability curve, which depicts a constant failure rate through the operational life of the item.

2. The failure rate prior to test is normally unknown. Since the Weibull distribution requires that the failure rate be specified, the best estimate of an unknown failure rate would have to be based on theory and would therefore be constant.

3. The Weibull distribution is less powerful than the exponential distribution since it is necessary to estimate one more parameter.

4. The failure rate of the Weibull must be monotonic, a condition which does not exist if testing occurs on overlapping portions of the theoretical reliability curve.

If previous knowledge of an item's failure rate does exist, however, and it is not constant, then consideration should be given to Weibull sampling plans. Since I feel that the above situation will be rare, no further discussion will be given to the Weibull distribution in this report.

**Hypothesis Test**

Barlon and Proschan (2, p. 232-5) have developed a nonparametric method of testing the null hypothesis, \( H_0: r \) is constant against the alternate hypothesis, \( H_1: r \) is increasing but not constant.

The test is conducted as follows: Let \( t_1, t_2, \ldots, t_n \) be the ordered observations; \( D_1 = t_1 \), \( D_2 = t_2 - t_1 \), \ldots, \( D_n = t_n - t_{n-1} \) the
spacings; and \( \bar{D}_1 = nD_1, \bar{D}_2 = (n - 1)D_2, \ldots, \bar{D}_n = D_n \) the normalized spacings. For \( i, j = 1, 2, \ldots, n \) let \( V_{ij} = 1 \) if \( \bar{D}_i > \bar{D}_j \); 0 otherwise.

The test statistic is

\[
V_n = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} V_{ij} \quad (39)
\]

We reject the null hypothesis at the \( \alpha \) level of significance if \( V_n > v_n, \alpha \) where \( v_n, \alpha \) is determined such that \( P[V_n > v_n, \alpha | H_0] = \alpha \).

Since the normalized spacings are independently distributed, having density \( \lambda e^{-\lambda t} \), it can be shown that \( U_n \) and \( \sigma_n^2 \), the mean and variance of \( V_n \) and given by

\[
U_n = \frac{n(n - 1)}{4} \quad (40)
\]

\[
\sigma_n^2 = \frac{(2n + 5)(n - 1)n}{72} \quad (41)
\]

and that \( V_n \) is asymptotically normal (7, p. 245-9).

Therefore the null hypothesis cannot be rejected if \( V_n \) is contained in the interval \((U_n - \frac{Z_{\alpha/2}}{n}, U_n + \frac{Z_{\alpha/2}}{n})\) and rejected otherwise.

Example 4: Using the data presented in example 3, test the hypothesis \( H_0: r \) is constant versus the alternate hypothesis \( H_A: r \) is increasing.

<table>
<thead>
<tr>
<th>( N(i) )</th>
<th>( t(i) )</th>
<th>( D(i) )</th>
<th>( \bar{D}(i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>10</td>
<td>10</td>
<td>190</td>
</tr>
<tr>
<td>18</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>11</td>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td>16</td>
<td>13</td>
<td>2</td>
<td>32</td>
</tr>
<tr>
<td>15</td>
<td>16</td>
<td>3</td>
<td>45</td>
</tr>
<tr>
<td>14</td>
<td>20</td>
<td>4</td>
<td>56</td>
</tr>
<tr>
<td>13</td>
<td>28</td>
<td>8</td>
<td>104</td>
</tr>
<tr>
<td>12</td>
<td>40</td>
<td>12</td>
<td>144</td>
</tr>
<tr>
<td>11</td>
<td>51</td>
<td>11</td>
<td>121</td>
</tr>
<tr>
<td>10</td>
<td>75</td>
<td>24</td>
<td>240</td>
</tr>
</tbody>
</table>
\[ V_n = 11 + 1 + 1 + 1 + 3 + 4 + 3 + 5 + 4 + 6 + 4 + 5 + \\
+ 4 + 3 + 2 + 1 = 59 \]

\[ \bar{D}(i) \]

\[ \sigma_n = 14.3 \]

\[ L_{CL, 90} = 85.5 - 1.645(14.3) = 62.0 \]

\[ U_{CL, 90} = 85.5 + 1.645(14.3) = 109.0 \]

Therefore, since \( V_n \) is not contained in the interval (62.0, 109.0), reject \( H_0 \) and conclude that the failure rate is not constant at the 90% significance level.

**Coefficient of Variation**

A look at the coefficient of variation is a quick way to get a rough feel for the failure rate behavior. If the failure rate is constant, the reliability function is exponential and the mean equals the standard deviation and \( CV = S/\bar{x} = 1.0 \). If the failure rate is
increasing (as in a normal distribution) C.V. < 1. With a decreasing failure rate the coefficient of variation would be > 1 (8, p. 12).

Adjustment of Failure Rate

If the failure rate is noticeably increasing or decreasing, it is highly improbable that the change is due to only random failure effects. Some out-of-control condition exists; either the manufacturing process is to blame or the design is not mature. The process should be halted, if possible, and an investigation conducted to determine and correct the assignable cause of variation, and then the test continued. (If a change in the design or process is necessary, the test must be started over again.) When an infant mortality period is observed and cannot be corrected through design or process changes, the contractor must either "burn-in" the items prior to government acceptance or the expected number of failures during the items' service life must be adjusted accordingly. If they select to burn the items in, the "burn-in" period must be sufficient to put the items in the constant failure period.

If the number of failures is to be adjusted, the following procedure should be followed:

1. Divide the service life of the item into subintervals so that the failure rate is fairly constant throughout each interval, i.e., so that the interval is small enough that the change in the failure rate is negligible.

2. Calculate the average failure rate of each interval:

\[ \bar{r}_i = \frac{F_i}{T_i} \]  

(42)
where \( F_i \) is the total number of failures in that interval and \( T_i \) is the total time accumulated during that interval for all items on test.

3. The expected number of failures during an item's service life is then calculated as:

\[
E(F) = \sum_{i=1}^{n} \bar{r}_i T_i
\]

This expected number of failures will then be used to calculate the Life Cycle Cost to the government.
SUMMARY AND CONCLUSIONS

To determine the reliability and maintainability parameters necessary to calculate the total life-time costs of populations of items, this thesis was divided into two main sections with a third section which dealt on determining if the underlying distributions of the populations were exponential.

The first section was to investigate, develop, and apply reliability sampling techniques for the above purpose. This was accomplished by first exploring techniques which can be used to determine compliance to some minimum required MTBF in what is referred to as Phase I testing. Once the requirements of Phase I testing are met, testing may be continued at the option of the contractor and confidence limits constructed about the Bid MTBF to determine compliance to it. Methods by which incentives or penalties can be rewarded or assessed the contractor was also investigated.

The second section involved researching techniques to determine the necessary maintainability parameters. Confidence interval calculations were applied to determine the accuracy of these parameters.

Finally, a section was included to test if the underlying distributions of the populations were exponential. This was included since the technique used in the first section was based on the exponential distribution. A discussion was included which indicated what action could be taken if a distribution was not exponential.
LITERATURE CITED


Appendix I

Proof of Binomial Distribution Converging to Poisson Distribution as $t \to \infty$, $r \to 0$ such that $rt = \lambda$ Becomes and Remains Constant

$$\binom{t}{x} r^{x(1-r)} t^{-x} = \frac{t(t-1) \ldots (t-x+1)}{x!} r^{x-1} t^{-x}$$

but $rt = \lambda$ or $r = \lambda/t$

$$\therefore \binom{t}{x} r^{x(1-r)} t^{-x} = \frac{t(t-1) \ldots (t-x+1)}{x!} \frac{\lambda^x}{t^x} (1 - \lambda/t)^{t-x}$$

$$= \frac{t(t-1) \ldots (t-x+1)}{t^x} \frac{\lambda^x}{x!} (1 - \lambda/t)^{-x} (1 - \lambda/t)^t$$

$$= \left[ (1 - 1/t)(1 - 2/t) \ldots (1 - \frac{x-1}{t}) \right] (1 - \lambda/t)^{-x}$$

$$\frac{\lambda^x}{x!} (1 - \lambda/t)^t$$

as $t \to \infty$ and $r \to 0$, the bracketed terms approach 1 and $(1 - \lambda/t)^t$ approaches $e^{-\lambda}$

hence $$\binom{t}{x} r^{x(1-r)} t^{-x} \to \frac{\lambda^x}{x!} e^{-\lambda}$$
Appendix 2

Sequential Sampling Plan

Let:

a. \( f(x, \theta) \) denote the distribution of a random variable \( x \).

b. \( H_0 \) be the hypothesis that \( \theta = \theta_0 \), and
c. \( H_1 \) be the hypothesis that \( \theta = \theta_1 \).

For any positive integral value \( m \) the probability that a sample \( x_1, \ldots, x_m \) is obtained is given by

\[
P_{1m} = f(x_1, \theta_1) \ldots f(x_m, \theta_1)
\]

when \( H_1 \) is true, and by

\[
P_{0m} = f(x_1, \theta_0) \ldots f(x_m, \theta_0)
\]

when \( H_0 \) is true.

Select two positive constants \( A \) and \( B \) (\( B < A \)). The sequential probability ratio test for testing \( H_0 \) against \( H_1 \) is calculated by the probability ratio \( P_{1m}/P_{0m} \) at any trial \( m \). If

\[
B < P_{1m}/P_{0m} < A \text{ continue testing,}
\]

if

\[
P_{1m}/P_{0m} \geq A \text{ terminate the test and reject } H_0 \text{ and if}
\]

\[
P_{1m}/P_{0m} \leq B \text{ terminate the test and accept } H_0.
\]

Let us call a sample \( (X_1, \ldots, X_n) \) type I if

\( B < P_{1m}/P_{0m} < A \) for \( m = 1, \ldots, n-1 \)

and \( P_{1n}/P_{0n} \leq B \).

Similarly, call the sample type II if

\( B < P_{1m}/P_{0m} < A \) for \( m = 1, \ldots, n-1 \) and \( P_{1n}/P_{0n} \geq A \).
Thus a sample of type I leads to acceptance of $H_0$ and a sample of type II leads to the rejection of $H_0$ (acceptance of $H_1$). For any given sample $(X_1, \ldots, X_n)$ of type II the probability of obtaining such a sample is at least $A$ times as large under hypothesis $H_1$ and under hypothesis $H_0$, since $P_{1m}/P_{0m} \geq A$. Thus, the probability of all samples of type II is at least $A$ times as large under $H_1$ as under $H_0$. This is the same as the probability of terminating with the rejection of $H_0$ (acceptance of $H_1$). But this probability is equal to $\alpha$ when $H_0$ is true and to $1-\beta$ when $H_1$ is true. Therefore, we obtain the inequality

$$1-\beta \geq A\alpha$$

or

$$A \leq (1-\beta)/\alpha$$

Thus, $(1-\beta)/\alpha$ is the upper limit for $A$.

A lower limit for $B$ can be derived in a similar way. For a given sample $(X_1, \ldots, X_n)$ of type I the probability of obtaining such a sample under $H_1$ is at most $B$ times as large as the probability of obtaining such a sample when $H_0$ is true. Since the probability of accepting $H_0$ is $1-\alpha$ when $H_0$ is true and $\beta$ when $H_1$ is true, we obtain the inequality

$$B \leq (1-\alpha)B$$

or

$$B \geq \beta/(1-\alpha)$$

Thus, $\beta/(1-\alpha)$ is a lower limit for $B$. Sampling is continued as long as $\beta/(1-\alpha) < P_{1m}/P_{0m} < (1-\beta)/\alpha$, and the lot is rejected or accepted otherwise.
Sequential sampling: Exponential distribution

We can now derive the sequential sampling plan for the exponential distribution.

Since

\[ P_{1m} = \left( \frac{T}{\theta_1} \right)^r e^{-T/\theta_1} / r! \]

and

\[ P_{om} = \left( \frac{T}{\theta_0} \right)^r e^{-T/\theta_0} / r! \]

the probability ratio test for rejection becomes

\[
\frac{\left( \frac{T}{\theta_1} \right)^r e^{-T/\theta_1}}{\left( \frac{T}{\theta_0} \right)^r e^{-T/\theta_0}} \geq \left[ \frac{(1-\beta)/\alpha}{1 - S} \right]
\]

For simplicity in calculations take the natural log of the above equations, i.e.,

\[
\ln \left( \frac{T/\theta_1}{T/\theta_0} \right) + \ln \left( \frac{-T/\theta_1 + T/\theta_0}{e} \right) \geq \ln \left( \frac{(1-\beta)/\alpha}{1 - S} \right)
\]

or

\[
\ln \frac{\theta_0 - T \left( \frac{1}{\theta_0} - \frac{1}{\theta_1} \right)}{\theta_1} \geq \ln \left( \frac{(1-\beta)/\alpha}{1 - S} \right)
\]

Solving for \( T \)

\[
-T \geq \frac{\ln[(1-\beta)/\alpha] - \ln \theta_0/\theta_1}{1/\theta_1 - 1/\theta_0}
\]

or

\[
T \geq \frac{\ln \theta_0/\theta_1}{1/\theta_1 - 1/\theta_0} - \frac{\ln[(1-\beta)/\alpha]}{1/\theta_1 - 1/\theta_0}
\]
Let
\[ S = \frac{\ln \frac{\theta_0}{\theta_1}}{\frac{1}{\theta_1} - \frac{1}{\theta_0}}; \quad \text{and} \quad h_1 = \frac{\ln[(1-\beta)/\alpha]}{\frac{1}{\theta_1} - \frac{1}{\theta_0}}, \]
then
\[ T \geq Sr - h_1 \]

The acceptance line \((T = Sr + h_0)\) can be derived in a similar manner.
Appendix 3

Operative Characteristic (O.C.) Curve

Let $L(\theta)$ be defined as the probability that the sequential process will terminate with the acceptance of the null hypothesis $(H_0)$ when $\theta$ is the true value of the parameter. The straightforward way to compute $L(\theta)$ is to add the probabilities that $H_0$ will be accepted at each observation. Thus

$$L(\theta) = P \left[ \frac{f(x_1, \theta_1)}{f(x_1, \theta_0)} \right] \leq B + P \left[ B < \frac{f(x_1, \theta_1)}{f(x_1, \theta_0)} \frac{f(x_2, \theta_1)}{f(x_2, \theta_0)} \right]$$

where the second probabilities are

$$P = \int_A^B \int_0^B f(x_1, \theta) f(x_2, \theta) \, dx_1 \, dx_2$$

This procedure for determining the O.C. Curve is tedious and is usually so troublesome as to be completely out of the question in practice. Therefore, the following is an approximate method for calculating $L(\theta)$, which ignores the excess of $P_{1m}/P_{0m}$ over the boundaries $A$ and $B$ at the termination of the process.

Consider

$$\left[ \frac{f(x, \theta_1)}{f(x, \theta_0)} \right]^h$$

For each value of $\theta$, there exists exactly one non-zero value of $h$ such that
\begin{align*}
g(x, \theta) &= \left( \frac{f(x, \theta_1)}{f(x, \theta_0)} \right)^h f(x, \theta) \\
\text{is a density, i.e., } h \text{ is determined so that} \\
\int_{-\infty}^{\infty} g(x, \theta) \, dx &= 1.
\end{align*}

Let's consider the case where \( h > 0 \). Let \( H_0 \) denote the hypothesis that \( f(x, \theta) \) is the true distribution of \( x \) and \( H_1 \) the hypothesis that \( g(x, \theta) \) is the true distribution of \( x \). Consider the sequential probability ratio test for testing \( H_0 \) against \( H_1 \) to be defined as follows:

If \( B^h < \frac{g(x_i, \theta)}{f(x_i, \theta_0)} < A^h \) continue testing and cease to test when the ratio equals or falls outside these limits. Since \( h > 0 \), the above inequalities are the same as

\[
B < \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} < A, \quad \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} > A, \quad \text{and} \quad \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} < B.
\]

But, these inequalities are identical with those for defining the sequential test \( H_0 \) against \( H_1 \), when the constants \( A \) and \( B \) are used.

Thus the acceptance of \( H_0 \) implies the acceptance of \( H_1 \). It follows that \( L(\theta) \), which is the probability of accepting \( H_0 \) when \( \theta \) is true, is the same as the acceptance of \( H_0 \) when \( f(x, \theta) \) is the true distribution of \( x \).

This latter is easily calculated since \( H_0 \) will be accepted when it is true with probability \((1 - \alpha)\), where

\[
A^h = \frac{1 - \beta_1}{\alpha}, \quad \text{and} \quad B^h = \frac{\beta_1}{1 - \alpha}.
\]
On solving this pair of equations we find

\[ 1 - \alpha_l = L(\theta) = \frac{A^h - 1}{A^h - B^h} \]

Similarly, it can be proven that when \( h < 0 \) the above equation is valid.

**Exponential distribution:** \( h \) must be determined so that the expected value of \( g(x, \theta) \) equals 1, i.e.

\[
\sum_{x=0}^{\infty} \left[ \frac{f(x, \theta)}{f(x, \theta_0)} \right]^h f(x, \theta) = 1
\]

Substituting the poisson equation this becomes:

\[
\sum_{r=0}^{\infty} \left[ \frac{(T/\theta_1)^r e^{-T/\theta_1}}{(T/\theta_0)^r e^{-T/\theta_0}} \right]^h (T/\theta)^r e^{-T/\theta} / r! = 1
\]

To plot the O.C. Curve it is not necessary to solve the above equation for \( h \). We can treat it as a parameter and solve for \( \theta \). Thus we obtain

\[
\sum_{r=0}^{\infty} \frac{\theta_0}{\theta_1} e^{-hT(1/\theta_1 - 1/\theta_0)} (T/\theta)^r e^{-T/\theta} / r! = 1
\]

\[
e^{-hT(1/\theta_1 - 1/\theta_0)} \sum_{r=0}^{\infty} \frac{\theta_0}{\theta_1} (T/\theta)^r e^{-T/\theta} / r! = 1
\]

\[
e^{-T/\theta} \sum_{r=0}^{\infty} \left( \frac{\theta_0}{\theta_1} \right)^r \left( \frac{T}{\theta} \right)^r / r! = e^{hT(1/\theta_1 - 1/\theta_0)}
\]
The O. C. Curve is then plotted by substituting various values of \( h \) \((h \neq 0)\) in the following two equations and plotting the results.

\[
L(\Theta) = \frac{A^h - 1}{A^h - A^h} \quad ; \quad \Theta = \frac{(\Theta_0/\Theta_1)^h - 1}{h(1/\Theta_1 - 1/\Theta_0)}
\]
or

\[
L(\theta) = \frac{\left(\frac{1-\beta}{\alpha}\right)^h - 1}{\left(\frac{1-\beta}{\alpha}\right)^h - \left[\frac{\beta}{1-\alpha}\right]^h}
\]
Appendix 4

Average Sample Size

The sample size $T$ in sequential testing is a random variable with a density $f(n)$, which may be determined in terms of the true density $f(x; \theta)$. Thus

$$f(1) = P(\lambda_1 \geq B) + P(\lambda_1 \geq A)$$

$$f(2) = P(B < \lambda_1 < A, \lambda_2 \leq B) + P(B < \lambda_1 < A, \lambda_2 \geq A)$$

and so forth, where the probabilities are determined by integrals like that of Appendix 3. Since solving these integrals is very difficult, an approximate expression for the expected sample size $E(T)$ can be determined as follows.

Let $Z_i = \ln \frac{f(x_i; \theta_1)}{f(x_i; \theta_0)}$ and let $n$ be the smallest integer for which $Z_1 + Z_2 + \ldots + Z_n$ does not satisfy $\ln B < Z_n < \ln A$. Let $N$ be some very large but fixed value of $T$ so that the distribution of $T$ to the right of $N$ can be disregarded because the error is arbitrarily small. Since $N$ is fixed

$$E(Z_N^*) = NE(Z).$$

But since the value $Z_N^*$ can be written $Z_N^* = (Z_1 + Z_2 + \ldots + Z_T) + (Z_{T+1} + \ldots + Z_N) = Z_T^* + Z_T^*$; $E(Z_N^*)$ may be written $E(Z_T^* + Z_T^*) = NE(Z)$.

For $i > T$, the random variable $Z_i$ is distributed independently of $T$, and the expected value of $(Z_T + 1) + \ldots + Z_N$ is equal to the expected value of $(N - T)$ times the expected value of a single $Z$, i.e.,

$$E(Z_T + 1 + \ldots + Z_N) = E(N - T)E(Z) = NE(Z) - E(T)E(Z).$$
Therefore, from the above two equations it follows that
\[ E(Z_1 + \ldots + Z_T) - E(T)E(Z) = 0 \]
or
\[ E(T) = \frac{E(Z_1 + \ldots + Z_T)}{E(Z)} \]

**Exponential distribution.** If \( \theta \) is the true value of the parameter, then \( E(T) = E(T|\theta) \) by definition. If the excess of the probability ratio \( P_{1m}/P_{0m} \) over the boundaries A and B at the termination point is neglected, the random variable of \( (Z_1 + \ldots + Z_T) \) can only take on the values \( \ln A \) and \( \ln B \) with the probabilities \( [1 - L(\theta)] \) and \( L(\theta) \), respectively. Hence
\[ E(T|\theta) = L(\theta) \ln B + [1 - L(\theta)] \ln A \]
\[ \frac{E(Z|\theta)}{E(Z|\theta) \ln [\beta/(1-\alpha)] + [1 - L(\theta)] \ln [(1-\beta)/\alpha]} \]

To compute \( E(Z|\theta) \), we have
\[ E(Z|\theta) = E(\theta) \left[ \ln \frac{f(x,\theta_1)}{f(x,\theta_0)} \right] \]
where
\[ \ln \frac{f(x,\theta_1)}{f(x,\theta_0)} = \ln \left[ \frac{(1/\theta_1)^r e^{-1/\theta_1}}{r!} \right] \]
\[ = r \ln \theta_0/\theta_1 - (1/\theta_1 - 1/\theta_0) \]

Therefore
\[ E(Z|\theta) = \sum_{r=0}^{\infty} \left[ r \ln \frac{\theta_0}{\theta_1} - (1/\theta_1 - 1/\theta_0) \right] \frac{(1/\theta)^r e^{-1/\theta}}{r!} \]
\[
\begin{align*}
&= \sum_{r=0}^{\infty} \left[ r(1/\theta_0/\theta_1) (1/\theta)^r e^{-(1/\theta)} \right. \\
&\quad \left. - (1/\theta_1 - 1/\theta_0) (1/\theta)^r e^{-(1/\theta)} \right]^{74}
\end{align*}
\]

But
\[
\sum_{r=0}^{\infty} \frac{(1/\theta)^r e^{-(1/\theta)}}{r!} = 1
\]

and
\[
\sum_{r=0}^{\infty} \frac{r(1/\theta)^r e^{-(1/\theta)}}{r!} = (1/\theta) \sum_{r=0}^{\infty} \frac{(1/\theta)^{r-1} e^{-(1/\theta)}}{(r-1)!} = 1/\theta
\]

Hence \( E(Z|\theta) = \frac{1}{\theta} \ln \theta_0/\theta_1 - (1/\theta_1 - 1/\theta_0) \)

and \( E(T|\theta) = \frac{L(\theta) \ln [\beta/(1-\alpha)] + [1-L(\theta)] \ln [(1-\beta)/\alpha]}{1/\theta \ln \theta_0/\theta_1 - (1/\theta_1 - 1/\theta_0)} \)

or \( E(T|\theta_0) = \frac{(1-\alpha) \ln [\beta/(1-\alpha)] + \alpha \ln [(1-\beta)/\alpha]}{1/\theta_0 \ln \theta_0/\theta_1 - (1/\theta_1 - 1/\theta_0)} \)

and \( E(T|\theta_1) = \frac{\beta \ln [\beta/(1-\alpha)] + (1-\beta) \ln [(1-\beta)/\alpha]}{1/\theta_1 \ln \theta_0/\theta_1 - (1/\theta_1 - 1/\theta_0)} \)

Expected waiting time. The expected waiting time is calculated as:
\[
E(WT|\theta) = \frac{E(T|\theta)}{n}
\]

Expected number of failures. The expected number of failures is determined as follows:
\[
E(r|\theta) = \frac{E(T|\theta)}{\theta} = \frac{L(\theta) \ln [\beta/(1-\alpha)] + [1-L(\theta)] \ln [(1-\beta)/\alpha]}{\ln \theta_0/\theta_1 - \theta(1/\theta_1 - 1/\theta_0)}
\]
\[
E(r|\theta_0) = \frac{(1-\alpha) \ln [\beta/(1-\alpha)] + \alpha \ln [(1-\beta)/\alpha]}{\ln DR - (DR - 1)}
\]
\[
E(r|\theta_1) = \frac{\beta \ln [\beta/(1-\alpha)] + (1-\beta) \ln [(1-\beta)/\alpha]}{\ln DR - (DR - 1)/DR}
\]
Appendix 5

Predetermined Time Sample

The expected number of failures for a predetermined time sample is calculated as follows:

\[ E(R|\theta) = \sum_{x=0}^{r_0} \frac{xd^x e^{-d}}{x!} = 0 + \frac{d e^{-d}}{1!} + \frac{2d^2 e^{-d}}{2!} + \frac{3d^3 e^{-d}}{3!} + \ldots + \frac{(r_0 - 1)d(r_0 - 1) e^{-d}}{(r_0 - 1)!} + \frac{r_0 d^r_0 e^{-d}}{r_0!} \]

\[ = d \left[ e^{-d} + \frac{d e^{-d}}{1!} + \frac{d^2 e^{-d}}{2!} + \ldots + \frac{d(r_0 - 2) e^{-d}}{(r_0 - 2)!} \right] + r_0 \frac{d^r_0 e^{-d}}{r_0!} \]

But

\[ \sum_{x=0}^{r_0} \frac{d^x e^{-d}}{x!} = \sum_{r=0}^{r_0 - 1} \frac{d^r e^{-d}}{r!} + \frac{d^{r_0} e^{-d}}{r_0!} = 1 \]

or

\[ \frac{r_0 d^r_0 e^{-d}}{r_0!} = 1 - \sum_{x=0}^{r_0 - 1} \frac{d^x e^{-d}}{x!} \]

Since by definition the test is terminated at a maximum of \( r_0 \) failures, i.e., the probability of getting 1, 2, \ldots, \( r \) failures equals 1. Therefore

\[ E(r|\theta) = d \sum_{x=0}^{r_0 - 2} \frac{d^x e^{-d}}{x!} + r_0 \left[ 1 - \sum_{x=0}^{r_0 - 1} \frac{d^x e^{-d}}{x!} \right] \]
Appendix 6

Confidence Interval Calculations on $\hat{\theta}$

These limits can be readily explained if we examine the confidence limits for the variance of a population. The $\chi^2$ distribution is used to set these limits on the variance as follows:

$$\frac{n\sigma^2}{\chi^2_{\alpha/2}} \leq \hat{\sigma}^2 \leq \frac{n\sigma^2}{\chi^2_{1 - \alpha/2}}$$

An analysis of the above equation reveals that it is practically the same as the confidence limits for $\hat{\theta}$. This inequality is obvious when we realize that the exponential distribution has a standard deviation and mean equal to $\theta$. The degrees of freedom are set at $2R$ since it has been observed that a relationship exists between the $\chi^2$ and Poisson distributions. The relationship is expressed as

$$\chi^2_{2R, \alpha} = 2rT = 2R$$

Now if the total test time is divided by a lower (upper) limit on the number of failures ($R$), the lower (upper) limit of the estimator of the mean ($\theta$) is achieved. That is

$$\frac{T}{R(\text{upper limit})} \leq \hat{\theta} \leq \frac{T}{R(\text{lower limit})}$$

but since

$$2R = \chi^2_{2R, \alpha}$$

$$\frac{2T}{\chi^2_{2R, (\alpha/2)}} \leq \hat{\theta} \leq \frac{2T}{\chi^2_{2R, (1 - \alpha/2)}}$$

In the manner specified, confidence limits are constructed about the best estimate of the mean ($\theta$).
Appendix 7

Sample Size Required to Establish a Labor Standard

The sample size (n) required to meet the specified criteria is determined from the following confidence interval statement:

\[ \hat{\theta} \text{ UCL } \begin{cases} \hat{\theta} = \hat{\theta} \pm t_{n-1}, \alpha/2 \cdot \frac{S}{\sqrt{n}} \end{cases} \text{ LCL} \]

This is accomplished as follows:

\[ \hat{\theta} \text{ UCL } = \hat{\theta} + t_{n-1}, \alpha/2 \cdot \frac{S}{\sqrt{n}} \]

but if the criteria for a type Ia standard is to be met

\[ \text{UCL } \hat{\theta} \leq \hat{\theta} + .10 \cdot \hat{\theta} \]

therefore

\[ \hat{\theta} + .10 \cdot \hat{\theta} \geq \hat{\theta} + t_{n-1}, \alpha/2 \cdot \frac{S}{\sqrt{n}} \]

\[ \hat{\theta} - \hat{\theta} + .10 \cdot \hat{\theta} \geq t_{n-1}, \alpha/2 \cdot \frac{S}{\sqrt{n}} \]

\[ .10 \cdot \hat{\theta} \geq t_{n-1}, \alpha/2 \cdot \frac{S}{\sqrt{n}} \]

or

\[ \frac{.10 \cdot \sqrt{n}}{t_{n-1}, \alpha/2} \geq \frac{S}{\hat{\theta}} \]

and

\[ \sqrt{n} / 10 \cdot t_{n-1}, \alpha/2 \geq \frac{S}{\hat{\theta}} \]
Appendix 8

Sample Size Required to Establish Occurrence Factors

The sample size \( n \) required to meet the specified criteria is determined from the following confidence interval statement:

\[
\hat{p} + z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq \bar{p} + \varepsilon
\]

or

\[
\hat{p} - z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq \bar{p} - \varepsilon
\]

and

\[
n \geq \frac{z^2 \hat{p}(1-\hat{p})}{\varepsilon^2}
\]