5-1969

A Concept of Buoyancy in Topological Spaces, with Applications to the Foundations of Real Variables

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A CONCEPT OF BUOYANCY IN TOPOLOGICAL SPACES, WITH APPLICATIONS TO THE FOUNDATIONS OF REAL VARIABLES

by

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A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

Approved:

UTAH STATE UNIVERSITY
Logan, Utah

1969

ACKNOWLEDGEMENT

The writer wishes to express sincere appreciation to Dr. John E. Kimber, Jr., whose constant advice made this work possible.

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ABSTRACT

A CONCEPT OF BUOYANCY IN TOPOLOGICAL SPACES, WITH APPLICATIONS TO THE FOUNDATIONS OF REAL VARIABLES

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Utah State University, 1969

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The Buoyancy Theorem states that a compact set is buoyant if every point of the compact set has a neighborhood whose intersection with the compact set is buoyant. In this paper, the Buoyancy Theorem is used to prove several standard results involving compact sets. The proof of such a result may be a direct application of the Buoyancy Theorem or the proof may rely on a certain compactness argument which follows from the Buoyancy Theorem. The last application in this paper is such an example.

The method used is to, first of all, define a buoyancy on the compact set; secondly, show that every point of the compact set has a neighborhood whose intersection with the compact set is buoyant; and finally, apply the Buoyancy Theorem to conclude that the compact set is buoyant.

(24 pages)

INTRODUCTION

S. T. Hu generalizes the concept of boundedness in his book Introduction to General Topology.[1] In his terminology, a boundedness in a topological space is a non-empty family of subsets satisfying the following two conditions; (1) every subset of a bounded set is bounded and, (2) the union of a finite number of bounded sets is bounded. A member of such a family of subsets is called a bounded set. He calls a topological space with a boundedness defined on it, a universe. A universe X is called locally bounded if every point in X has a bounded neighborhood in X.

Hu proves that every compact subset of a locally bounded universe is bounded. This will be referred to as Hu's Theorem.

We introduce the concept of buoyancy, which is somewhat less restrictive than Hu's boundedness, and develop some applications of Hu's Theorem in this paper. The applications in this paper could be accomplished using the boundedness defined by S. T. Hu. However, the buoyancy has the virtue that there are fewer conditions to verify.

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[1] S. T. Hu, Introduction to General Topology (San Francisco, California: Holden-Day, Inc., 1966), p. 184-192.

NOTATIONS AND CONVENTIONS

Let Y be a subset of a topological space S. A <u>buoyancy on Y</u> is a collection β of subsets of Y with the property that the union of two elements of β is an element of β. Any set belonging to β will be called <u>buoyant</u>. Subsets of Y that are not buoyant will be called <u>nonbuoyant</u>.

A buoyancy on Y induces a buoyancy on X by taking a subset of X to be buoyant if its intersection with Y is a buoyant subset of Y.

An example of a buoyancy which is not a boundedness follows. Let Y be a subset of a topological space X. Call a subset of Y buoyant if it is open in X. Let A and B be two open subsets of Y such that A is non-empty. $A \cup B$ is open and thus we have a buoyancy defined on Y. The set A being non-empty contains a point z and the set consisting of the single point z is not open. Therefore the property of being open is not a boundedness.

Hu's Theorem may be modified in terms of buoyancy as follows.

Buoyancy Theorem. If Y is a compact set and a buoyancy is defined on Y such that every point of Y has a neighborhood whose intersection with Y is buoyant, then Y is buoyant.

Proof. Employing mathematical induction, we see that a finite collection of buoyant sets is buoyant.

For each point y in Y, choose a neighborhood whose intersection with Y is buoyant. Each selected buoyant neighborhood N_y of y contains an open neighborhood M_y. Since Y is compact, the open covering $\{M_y | y \in Y\}$ contains a finite subcollection $\{M_{y_1}, \ldots, M_{y_k}\}$ which covers Y. Hence $\{N_{y_1}, \ldots, N_{y_k}\}$ is a finite covering of Y by buoyant sets.

Now $N_{y_i} \cap Y$ is buoyant for each i and $\{N_{y_i} \cap Y\}$ is a finite collection of buoyant sets such that $\bigcup_i (N_{y_i} \cap Y) = Y$. Therefore Y is buoyant. Q.E.D.

The purpose of this paper is to give several applications of the Buoyancy Theorem. Hu's Theorem may be proved using the Buoyancy Theorem.

Hu's Theorem. Every compact subset Y of a locally bounded universe X is bounded.

Proof. Let S be the boundedness associated with the universe X. Call A, a subset of Y, buoyant if A is in S. The definitions of boundedness and buoyancy imply that every boundedness is a buoyancy. Thus S defines a buoyancy on Y.

Since the universe X is locally bounded, it follows that Y is also locally bounded in S. Let z be a point in Y. The point z has a neighborhood N in S and $N \cap Y$, being a subset of N, is also in S. Thus z has a neighborhood whose intersection with Y is buoyant.

As a result of the Buoyancy Theorem, Y is buoyant and therefore bounded. Q.E.D.

The Buoyancy Theorem can be proved using Hu's Theorem.

Buoyancy Theorem. If Y is compact and a buoyancy is defined on Y such that every point of Y has a neighborhood whose intersection with Y is buoyant, then Y is buoyant.

Proof. Since Hu defines a boundedness on the whole space X rather than on a subset Y, consider Y itself to be a topological space with the induced topology.

Let β be a given buoyancy on Y and let σ be the set of all subsets of Y which are subsets of elements of β. If M is in σ then M is a subset of an element of β and so is every subset of M. Thus every subset of M is in σ.

Consider $\{S_1, S_2, \ldots, S_k\}$, a finite number of sets in σ. S_1 and S_2 are subsets of elements of β, say B_1 and B_2 respectively. Since $S_1 \cup S_2 \subset B_1 \cup B_2$, an element of the buoyancy β, $S_1 \cup S_2$ is in σ. By induction on k, $\bigcup_k \{S_1, S_2, \ldots, S_k\}$ is in σ. Therefore, σ is a boundedness on Y.

If every point of Y has a buoyant neighborhood, then it also has a bounded neighborhood, in fact, the same neighborhood. Hence Y is locally bounded. Since Y is compact, Y is bounded by Hu's Theorem.

Being bounded, Y is a subset of a buoyant subset of Y, which can only be Y. Therefore, Y is buoyant. Q.E.D.

The proof of the following theorem is representative of the use of the buoyancy in routine compactness arguments.

Theorem. If X is a Hausdorff space, F is a compact subset of X and x is a point of X not in F, then x and F have disjoint neighborhoods.

Proof. Let x and F satisfy the hypothesis. Take $A \subset F$ to be buoyant if there exists disjoint neighborhoods W_A of x and V_A of A.

Let A and B be buoyant. Then there exist disjoint neighborhoods W_A of x and V_A of A and also W_B of x and V_B of B. Let $W_{A \cup B} = W_A \cap W_B$ which is a neighborhood of x, and $V_{A \cup B} = V_A \cup V_B$ which is a neighborhood of $A \cup B$. Since $W_{A \cup B} \cap V_{A \cup B} = (W_A \cap W_B) \cap (V_A \cup V_B) = \emptyset$, $A \cup B$ is buoyant. Thus the buoyancy is well-defined.

If c is a point in F, we wish to show that c has a neighborhood whose intersection with F is buoyant. Since X is a Hausdorff space, there exist disjoint neighborhoods, N_c of c and N_x of x. The neighborhood N_c

and thus $N_c \cap F$ is buoyant. By the Buoyancy Theorem, F is buoyant. Q.E.D.

APPLICATIONS TO LOCALLY COMPACT SPACES

If F is a subset of a topological space, let $\overline{F}$ indicate the closure of F.

Theorem 1. If X is a locally compact space and K is a compact subset of X, then there exists an open set F containing K such that $\overline{F}$ is compact.

Proof. Take A a subset of K to be buoyant if there exists an open set F_A containing A such that $\overline{F}_A$ is compact.

If A and B are buoyant, there exists open sets F_A containing A and F_B containing B so that $\overline{F}_A$ and $\overline{F}_B$ are both compact. Let $F_{A \cup B} = F_A \cup F_B$; then $F_{A \cup B}$ is open and contains $A \cup B$. Moreover, $\overline{F}_{A \cup B} = \overline{F_A \cup F_B} = \overline{F}_A \cup \overline{F}_B$, which is compact.

Consider a point x in K. Since X is locally compact, there is an open set N_x containing x where $\overline{N}_x$ is compact. $(N_x \cap K) \subset N_x$ and is therefore buoyant. As a result of the Buoyancy Theorem, K is buoyant. Q.E.D.

Theorem 2. Let X be a locally compact Hausdorff space. If C is a compact set and M and N are open sets such that $C \subset M \cup N$, then there exists compact sets D and E such that $D \subset M$, $E \subset N$ and $C = D \cup E$.

Proof. Call $A \subset C$ buoyant if there exist compact sets A_1 and A_2 with $A_1 \subset M$, $A_2 \subset N$ and $\overline{A} = A_1 \cup A_2$.

If A and B are buoyant, then there are compact sets $A_1 \subset M$, $A_2 \subset N$, $B_1 \subset M$ and $B_2 \subset N$ with $\overline{A} = A_1 \cup A_2$ and $\overline{B} = B_1 \cup B_2$. Thus we have compact sets $A_1 \cup B_1 \subset M$ and $A_2 \cup B_2 \subset N$ with $\overline{A \cup B} = \overline{A} \cup \overline{B} = (A_1 \cup B_1) \cup (A_2 \cup B_2)$ causing $A \cup B$ to be buoyant. Thus the buoyancy is well-defined.

Let c be any point in C. We would like to show that c has a neighborhood whose intersection with C is buoyant. Since M and N form an open covering of C, c is in M or N; suppose c is in M. A locally compact Hausdorff space is regular so there exists a neighborhood of c, R_c, such that $\overline{R_c} \subset M$. The space X is locally compact and thus there is a neighborhood of c, call it K, such that $\overline{K}$ is compact.

The proof will be completed by establishing that $R_c \cap K$ is a buoyant neighborhood of c. Since $\overline{R_c \cap K} \subset \overline{K}$ and is thus compact, we have two compact sets, $\overline{R_c \cap K}$ and $\emptyset$ with $\overline{R_c \cap K} \subset M$ and $\emptyset \subset N$ such that $\overline{R_c \cap K} = \overline{R_c \cap K} \cup \emptyset$. This shows that $R_c \cap K$ is a buoyant neighborhood of the point c. By the Buoyancy Theorem, C is buoyant. Q.E.D.

For the next application define H to be the class of continuous functions which map a locally compact Hausdorff space X into the closed interval $[0,1]$.

Theorem 3. Let C be a compact subset of a locally compact Hausdorff space X. If F is a closed set such that $C \cap F = \emptyset$, then there exists a function F in H such that $f(x) = 0$ for x in C and $f(x) = 1$ for x in F.

Proof. Let $A \subset C$ be buoyant if there exists a continuous function g such that $g:X \to [0,1]$, $g(x) < \frac{1}{2}$ if x is in A, and $g(x) = 1$ if x is in F.

If A and B are both buoyant, then there exist functions g_A and g_B in H such that $g_A(x) < \frac{1}{2}$ for x in A, $g_B(x) < \frac{1}{2}$ for x in B and $g_A(x) = g_B(x) = 1$ for x in F. $A \cup B$ is seen to be buoyant by letting $g_{A \cup B} = g_A \cap g_B$, where $(g_A \cap g_B)(x) = \min(g_A(x), g_B(x))$.

To show C is buoyant it is sufficient to show that every point of C has a buoyant neighborhood and thus a neighborhood whose intersection with C is buoyant.

Since X is completely regular, there exists a continuous function h such that $h:X \to [0,1]$, $h(c) = 0$, and $h(x) = 1$ if x is in F. Since h is continuous and $[0,\frac{1}{2})$ is open in $[0,1]$, $h^{-1}([0,\frac{1}{2}))$ is a neighborhood of c. Since $h(x) < \frac{1}{2}$ for x in $h^{-1}([0,\frac{1}{2}))$ and $h(x) = 1$ for x in F, $h^{-1}([0,\frac{1}{2}))$ is a buoyant neighborhood of c.

Thus C is buoyant and we have a function g in H with $g(x) < \frac{1}{2}$ for x in C and $g(x) = 1$ for x in F.

Let $f = (2g-1) \cup 0$ where $((2g-1) \cup 0)(x) = \max(2g(x) - 1, 0)$. Then f is in H, $f(x) = 0$ for x in C and $f(x) = 1$ for x in F. Q.E.D.

Theorem 4. Let A be a Hausdorff space, B a topological space and let $f:A \to B$ be a function. Let $G = \{(a,f(a)):a \in A\}$ be its graph in $A \times B$. If G is a compact set, then f is continuous.

Proof. If $(a,f(a))$ is a given point in G and we wish to show f is continuous at a, it is sufficient to prove that if N is a neighborhood of $f(a)$, then there exists a neighborhood M of a so that $f(x)$ is in N if x is in M.

Let a neighborhood N of $f(a)$ be given. Call $S \subset G$ buoyant if there exists a neighborhood M of a so that $f(x)$ is in N if x is in M and $(x,f(x))$ is in S.

If R and T, each a subset of G, are both buoyant, then there are neighborhoods, M_R and M_T, of a so that $f(x)$ is in N whenever x is in M_R or M_T and $(x,f(x))$ is in R or T, respectively. Then $M_{R \cup T} = M_R \cap M_T$ is a neighborhood of a such that $f(x)$ is in N whenever x is in $M_{R \cup T}$ and $(x,f(x))$ is in $R \cup T$. Thus $R \cup T$ is buoyant.

To prove that f is continuous on A, it suffices to prove that G is buoyant, which will follow if it is extablished that every point of G has a neighborhood whose intersection with G is buoyant.

Let $(z,f(z))$ be a point of G. If $z \neq a$, take a neighborhood W of z so that $a \notin \overline{W}$. Let $M = A - \overline{W}$ so that M is a neighborhood of a. Then $L = W \times B$ is a neighborhood of $(z,f(z))$ whose intersection with G is buoyant for there is no x such that x is in M and $(x,f(x))$ is in $L \cap G$.

If $z = a$, take $L = A \times N$ and let $M = A$. We have $f(x)$ in N if x is in M and $(x,f(x))$ is in $L \cap G$. So again L is a neighborhood of $(z,f(z)) = (a,f(a))$ whose intersection with G is buoyant. Q.E.D.

AN APPLICATION TO TOPOLOGICAL GROUPS

In a topological group, let e denote the identity element and Z the class of neighborhoods of the identity.

Theorem. If C is a compact subset of an open set U in a topological group X, then there exists a neighborhood V of e such that $VCV \subset U$ where $VCV = \{vcv' \mid v \in V, c \in C \text{ and } v' \in V\}$.

Proof. Let $A \subset C$ be buoyant if there exists a neighborhood V of the identity e such that $VAV \subset U$.

If A and B are both buoyant, then there exists neighborhoods of e, V_A and V_B, such that $V_A A V_A \subset U$ and $V_B B V_B \subset U$. If $W = V_A \cap V_B$, then W is a neighborhood of e and we have $(WAW) \cup (WBW) \subset U$. Thus $W(A \cup B)W \subset U$ and $A \cup B$ is buoyant.

Let x be a point in C. It will be sufficient to show that x has a buoyant neighborhood. For then the intersection of this neighborhood and C will also be buoyant. If x is in U, then e is in $x^{-1}U$ and there exists a set W in Z so that $WW \subset x^{-1}U$. Since W is in Z, there exists V in Z so that $VV \subset W$ and we have $VVW \subset x^{-1}U$ giving $xVVW \subset U$ or $(xVx^{-1})(xV)W \subset U$. Since xVx^{-1} is in Z, we may take $V_1 = xVx^{-1}$.

Let $V_2 = V_1 \cap W$. Then V_2 is a neighborhood of e such that $V_2(xV)V_2 \subset U$ so that xV is a neighborhood of x which is buoyant.

AN APPLICATION IN MEASURE THEORY

If E is any bounded set and F is any set with a non-empty interior, then define the ratio $E:F$ as the least non-negative integer n with the property that E may be covered by n left translations of F; i.e., that there exists a set $\{x_1, \ldots, x_n\}$ of n elements of x such that $E \subset \bigcup_{i=1}^{n} x_i F$.

Let A be a fixed compact set with a non-empty interior and Z the class of all neighborhoods of the identity. For each U in Z, we construct the set function λ_u, defined for all compact sets C by $\lambda_u(C) = \frac{C:U}{A:U}$.

To each set C in K, the set of compact sets, associate the closed interval $[0, C:A]$. Let P be the Cartesian product of all these intervals. Then the points of P are real-valued functions $\emptyset$ defined on K such that for each C in K, $0 \leq \emptyset(C) \leq C:A$.

For each U in Z the function λ_u is a point in this space. For each U in Z, let $\Delta(U) = \{\lambda_u : U \supset V \in Z\}$.

The proof of the theorem, "In every locally compact topological group X there exists at least one regular Haar measure", rests heavily on the following proposition which can be proved using the Buoyancy Theorem.

Proposition. For U in Z, there is a point λ in the intersection of the closure of all $\Delta(U)$; i.e., $\lambda \in \cap \{\Delta(U) | U \in Z\}$.

Proof. Let the space $P = \prod_{C \in k} [0,C:A]$. A subset $\prod_C Y_C$ of P will be called non-buoyant if for every U in Z there exists $V \subset U$ such that the point λ_V is in $\prod_C Y_C$; i.e., $\lambda_V(C)$ is in Y_C for each C in K.

Let A and B be subsets of P. If $A \cup B$ is non-buoyant, then for every U in Z, there exists a subset V of U such that the point λ_V is in $A \cup B$ indicating that λ_V is in A or B. Therefore A or B is non-buoyant and the buoyancy on P is well-defined.

$P = \prod_C [0,C:A]$ is non-buoyant since it contains every point λ_V. The set $[0,C:A]$ is compact for each C and thus the product space $\prod_C [0,C:A]$ is also compact. By the contrapositive of the Buoyancy Theorem, there is a point λ such that each of its neighborhoods has a non-buoyant intersection with P.

Suppose for some U in Z, $\lambda \notin \overline{\Delta(U)}$, then there is a neighborhood M of λ which does not intersect $\Delta(U)$; i.e., $\Delta(U) \cap M = \emptyset$. For the neighborhood U of e, there is no subset V of U such that λ_V is in M. Thus M is buoyant and this contradicts the assertion that every neighborhood of λ is non-buoyant. Therefore λ is in $\overline{\Delta(U)}$ for all U in Z and we have $\lambda \in \cap \{\overline{\Delta(U)} \mid U \in Z\}$.

A COMPARISON OF HU'S BOUNDEDNESS AND THE BUOYANCY

A boundedness is defined to be a family of sets in a topological space whereas a buoyancy is a family of subsets of a specific set Y in a topological space. Also, every subset of a bounded set must be bounded, but a subset of a buoyant set need not be buoyant.

In general, a buoyancy is not a boundedness. However, each buoyancy in this paper is also a boundedness since, in each case, a subset of a given buoyant set can be easily shown to be buoyant. For example, consider the buoyancy defined in the first theorem on page five. $A \subset F$ is buoyant if there exist disjoint neighborhoods W_A of χ and V_A of A. If R is a subset of A, R can be shown to be buoyant by choosing the disjoint neighborhoods W_A of χ and V_A of A and thus of R.

Since the strength of the two conditions of the boundedness was not needed for the results of this paper, the weaker buoyancy was used. It is also evident that the buoyancy, being less restrictive than the boundedness, cannot be expected to produce all of Hu's results.

Some of Hu's results and terminology follow with a comparison of the buoyancy in each case.

The intersection of a non-empty family of bounded sets is bounded.[2]

It does not follow that the intersection of a family of buoyant sets is buoyant since the condition of buoyancy does not require a subset of a buoyant set to be buoyant.

If a universe X with boundedness β is not bounded, then the family $\mathfrak{C} = \{X-B | B \in \beta\}$ of the complements $X-B$ has the following properties:

(C1) None of the sets $X-B$ is bounded.

(C2) Every subset of X which contains a member of $\mathfrak{C}$ is itself a member of $\mathfrak{C}$.

(C3) The intersection of a finite number of members of $\mathfrak{C}$ is a member of $\mathfrak{C}$.[3]

In terms of a buoyancy β:

(C1) None of the sets $X-B$ is buoyant. $X = (X-B) \cup B$ and $X \notin \beta$. Since $B \in \beta, X-B \notin \beta$.

(C2) Every subset of X which contains a member of $\mathfrak{C}$ is itself a member of $\mathfrak{C}$. Suppose R contains a member of $\mathfrak{C}$, say $X-B$; then we need to show R is in $\mathfrak{C}$, that is to say $R = X-M$, where M is in β. But

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[2]S. T. Hu, Introduction to General Topology (San Francisco, California: Holden-day, Inc., 1966), p. 184.

[3]Ibid., p. 185.

$(X-R) \subset B \in \beta$. If β were a boundedness, we could conclude that $(X-R) \in \beta$. However, this need not hold for a buoyancy in which a subset of a buoyant set need not be buoyant.

(C3) The intersection of a finite number of members of $\mathfrak{C}$ is a member of $\mathfrak{C}$. Suppose $X-A$ and $X-B$ are in $\mathfrak{C}$ with A and B in β. $(X-A) \cap (X-B) = X-(A \cup B)$ where $A \cup B$ is in β. Thus the intersection of two members of $\mathfrak{C}$ is a member of $\mathfrak{C}$ and the result is obtained by induction on the number of sets.

Any family of subsets of X satisfying conditions (C2) and (C3) is called a filter in X. Hence the family $\mathfrak{C}$ will be referred to as the filter at infinity of the universe X.[4]

Since (C2) does not necessarily hold for a buoyancy, such a definition could not be made in terms of buoyancy.

Hu gives four different examples of a boundedness.[5] In each case, the given boundedness is also a buoyancy since the only condition of a buoyancy is one of the conditions of a boundedness.

A point χ of a universe X is said to be a finite point if and only if it has a bounded neighborhood in X;

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[4] <u>Ibid</u>.

[5] <u>Ibid</u>.

otherwise, χ is called a point at infinity. A universe X is said to be locally bounded if and only if every point in X is finite.[6]

These same definitions could be made using the buoyancy rather than the boundedness.

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[6] *Ibid.*, p. 186.

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Hu, S. T., *Introduction to General Topology*. San Francisco: Holden-Day, Inc., 1966.

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