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A Report on the Statistical Properties of the Coefficient of Variation and Some Applications

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A REPORT ON THE STATISTICAL PROPERTIES OF THE COEFFICIENT OF VARIATION AND SOME APPLICATIONS

by

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ABSTRACT

A Report on the Statistical Properties of the Coefficient of Variation and Some Applications

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Examples from four disciplines were used to introduce the coefficient of variation which was considered to have considerable usage and application in solving Quality Control and Reliability problems.

The statistical properties were found in the statistical literature and are presented, namely, the mean and the variance of the coefficient of variation. The cumulative probability function was determined by two approximate methods and by using the noncentral t distribution. A graphical method to determine approximate confidence intervals and a method to determine if the coefficients of variation from two samples were significantly different from each other are also provided (with examples).

Applications of the coefficient of variation to solving some of the main problems encountered in industry that are included in this report are: (a) using the coefficient of variation to measure relative efficiency, (b) acceptance sampling, (c) stress versus strength reliability problem, and (d) estimating the shape parameter of the two parameter Weibull.

(84 pages)

INTRODUCTION

This presentation will include a history summary of the coefficient of variation and some of the uses to which it has been applied with examples and comparisons. Further, an attempt will be made to differentiate, if possible, between the coefficient of variation and what is normally referred to in engineering as a "safety factor." The coefficient of variation will be identified as C.V. Historically the coefficient of variation has been associated with economics, engineering, sociology, psychology, quality control, etc., in applications which provide a comparison of relative dispersion.

This study is addressed to Engineers who may have had two or three method courses in statistics and are concerned with applying statistical techniques to evaluating inspection and test measurements for quality control and reliability purposes.

Statistical techniques, as was indicated previously, are universal (in that general methods have been developed and applied to evaluating data resulting from experiments performed in each scientific discipline); the examples contained in this section are not restricted to quality control and reliability. This is considered to have merit since it broadens the base for comparison and provides more depth of application.

Definition of C.V.

The coefficient of variation (C.V.) is the ratio of $\frac{\sigma}{\mu}$ where μ and σ are the true population mean and standard deviation, respectively. However, the true parameters and are very seldom known and, therefore, must be estimated from sample data which provide the statistics $\bar{X}$ and s. For the normal distribution

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

and

$$s = \sqrt{\frac{\sum_{i=1}^{n} X_i - \frac{(\sum_{i=1}^{n} X_i)}{n}}{n-1}}$$

where

X_i is the value of the ith measurement from the sample containing n items.

General Examples

C.V. compared to the quartile comparator.[1] The first example is presented to compare the wages of group (a) with group (b) where:

(a) is a classification used for all males under 16 years of age in the central states working in foundaries and metal works;

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[1] W.L. Crum and A. Patton, Economic Statistics, (New York A. W. Shaw Company, 1925) pp. 194-195.

(b) is a classification used for all males over 16 years of age employed in the tanneries of the United States.

Q_C is often used as an index to compare the relationship of the middle 50 percent of one sample with the middle 50 percent of another sample. This index uses Q_1 (the value below which 25 percent of the sample lies) and Q_3 (the value above which 25 percent of the sample lies) to provide Q_C, the quartile comparator. Q_C does not provide a consistent measure of relative variation since it is sensitive to the values of Q_1 and Q_3. Table I is presented as an example to portray that the C.V. is a more consistent comparator of relative variation then Q_C.

TABLE I

Coefficient of Variation Versus Quartile Comparator

	A	B
n	·26	519
$\bar{X}$	$5 84	$10.73
s	1 21	2.24
Q_1 (first quartile)	3 67	10.01
Q_2 (median)	4 00	10.42
Q_3 (third quartile)	5 86	12.26
$Q_C = \dfrac{Q_3 - Q_1}{Q_3 + Q_1}$	23	.10
C.V. $= \dfrac{s}{\bar{X}}$	21	.21

C.V. used as a relative measure. The second consideration is one that was proposed by Fredrick C. Kent[2] and is as follows:

The significance of the value calculated for the standard deviation depends on the size of the measurements. Thus a variation of two feet in a measure of 100 feet has the same significance as a variation of 20 feet in a measurement of 1,000 feet. It is the custom to divide the standard deviation by the mean in order to bring out its proper relation to the measurements. The quotient thus obtained is called the Coefficient of Variability.

Quality variability and C.V. The third application was presented as a quality control technique by H. A Freeman.[3]

Producer and buyer risk using the coefficient of variation. Specification of average quality and variability in quality may be separately provided by other methods. However, it is sometimes desired to make use of a hybrid statistic to control both the average and variation of a quality characteristic One such statistic is the coefficient of variation which is given by

$$\frac{\text{Standard deviation}}{\text{Arithmetic mean}}$$

---

[2] F. C. Kent, Elements of Statistics, (New York; McGraw-Hill Book Co., Inc., 1924) p. 87.

[3] H. A. Freeman, Industrial Statistics, (New York; John Wiley & Sons, Inc.; 1942) p. 153.

High values of this statistic will result from high variability in quality and low mean quality, both of which we take in our examples to be unfavorable. Correspondingly, low values of the coefficient of variation are considered favorable.

C.V. over a time domain. Snedecor and Cochran[4] give an example which can be used as a model to present comparative statistics over a time domain. The mean stature in centimeters, the standard deviation, and the coefficient of variation are plotted in Figure 1 (p. 6) to show the growth pattern of girls from age 1 to 18 years. From 1 to 12 years the standard deviation increases at a greater rate relative to the mean stature growth. This difference in growth causes the C.V. to decrease the first year from 3.75 percent to approximately 3.25 percent from year one to year two. From the second to the twelfth year there is a somewhat steady increase of C.V. to its maximum of approximately 4.75 percent. During the time from the twelfth year to the fifteenth year, the C.V. drops off sharply from 4.75 percent to 3 percent and then returns to its original position of 3.75 percent by the seventeenth year, and it is expected to remain quite stable from then on.

Figure 1 provides the factors of the distribution in relation to comparing growth with respect to what the mean and the standard deviation are each doing with respect to time. However, the C.V. by itself may not be meaningful unless the experimenter has additional information to supplement that of the C.V.

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[4] G. W. Snedecor and W. G. Cochran, Statistical Methods (6th ed.; Ames, Iowa: The Iowa University Press, 1967), p. 63.

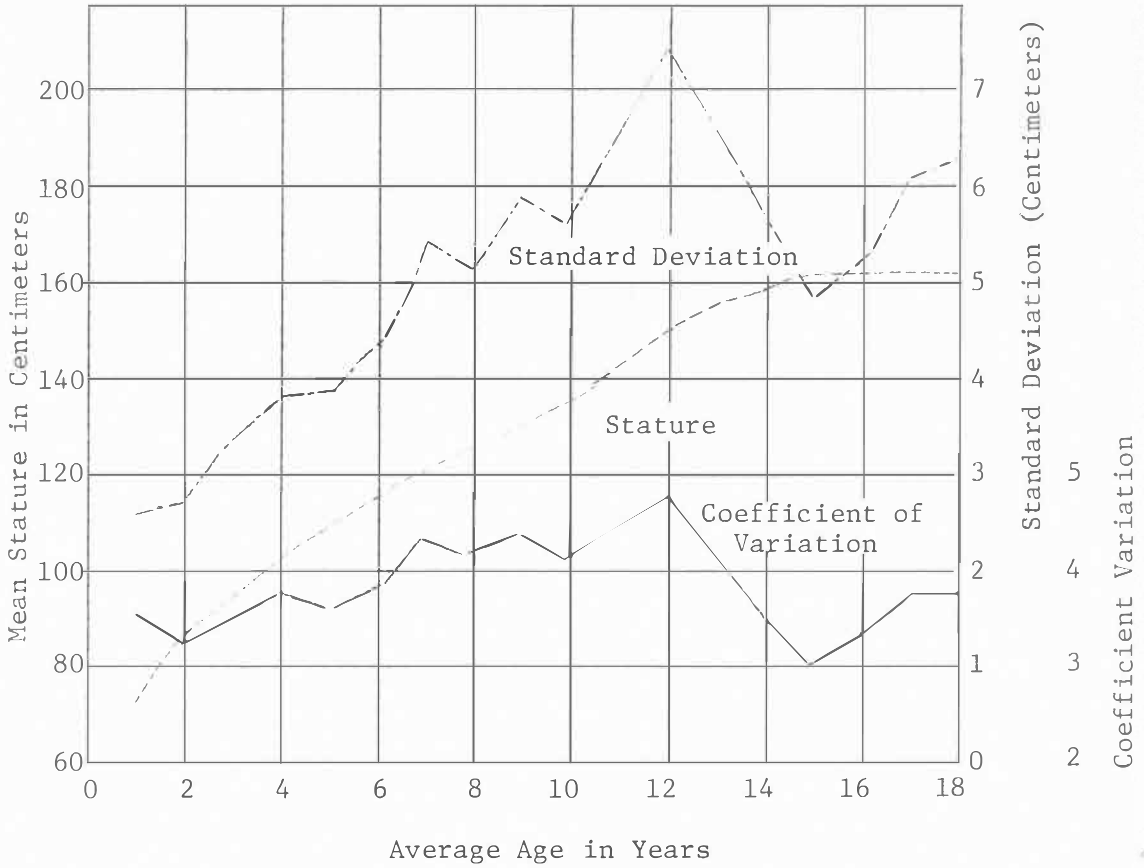

Figure 1. Charting of Three Time Series; Mean Stature, Standard Deviation of Stature, and the Coefficient of Variation for Girls from 1 to 18 Years of Age

C.V. used to compare test scores. The last example of this section was taken from Yamone.[5] Assume that a group of students took two tests. The first test has an average of 60 points and a standard deviation of 6 points with a maximum of 100 points. The second test has an average of 700 points and a standard deviation of 7 points with a maximum of 1,000 points. Which of the two tests has a larger scatter (dispersion)? Here we are comparing the dispersion of two frequency distributions.

One can readily see that from an absolute standpoint the 7 points is a larger scatter than the 6 points, but from a relative standpoint we can see that the students were much closer together in the second test. To bring this idea out explicitly, a measure of relative dispersion has been formulated. The coefficient of variation is used (Yamone[5]) to compare the results of the two tests as follows:

$$\text{First test,} \quad C.V. = \frac{6}{60} = \frac{1}{10}$$

$$\text{Second test,} \quad C.V. = \frac{7}{700} = \frac{1}{100} .$$

We observe that the relative dispersion of the second test is only 1/10 of the first. In such problems as this, by use of the coefficient of variation, the dispersion of different frequency distributions can be compared.

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[5] Taro, Yamone Statistics, An Introductory Analysis (1st ed. New York; Harper and Rowe Publishers, 1964) p. 75.

Estimation of the standard deviation of a new batch using C.V. In addition to providing a measure of relative variation, such as provided in the examples that have been presented, the C.V. may be used as a standard to compare two or more experimental results or as a means to rapidly estimate the standard deviation of a sample. In a number of cases $\overline{X}$ and s change together so that the C.V. is approximately constant. In such a situation, if there are several sets of experimental data that involve calculation of $\overline{X}$ and s, calculating the C.V.'s and comparing them with a given C.V. as well as with each other will serve as a check.

Also, if C.V. is available from previous data and $\overline{X}$ is known for a new batch of data, s may be estimated for this new sample by $s = \overline{X}(C.V.)$.

The following sections contain the results found in the literature and various reliability manuals which have been provided by industrial concerns. Although the literature search has been quite extensive, it has not been all inclusive. Additional literature search will probably provide greater theoretical depth which would provide additional uses to which the coefficient of variation can be applied.

STATISTICAL PROPERTIES OF SAMPLE C.V.

Note on population and sample distributions:

Before calculating the mean and the variance of C.V. from a random sample, it is necessary to comment on the sampling distribution and population distributions. If an item is selected, by a random process, from a population then the probability that the item selected will have a value no greater than x is the distribution function F(x). Similarly if the item selected has n variates (measurable characteristics) of concern, then the probability that the item will have a value of the first variate x no greater than x_1, a value of the second variate x no greater than x_2, . . . and a value of the nth variate no greater than x_n is the multivariate distributions function $G(x_1, x_2 \ldots x_n)$. Also, if the variates are independent the rth variate considered has the distribution function $(F_r(x_r) = F_1(x_1)F_2(x_2) \ldots F_r(x_r)$. Applying this concept to an unvariate population and selecting a sample of n items from the populations, each time the sample is taken there will be n values x_1 $x_2, \ldots x_n$. The nature of this multivariate distribution depends on the sampling process used as well as the population. If the distribution is $G(x_1, x_2 \ldots x_n)$ then this function represents the probability that a random sample will result in n values, the first not greater than x_1, the second not greater than $x_2, \ldots,$ and the nth not greater than x_n. The x's can be regarded as corresponding to n random variables $\xi_1, \xi_2, \ldots \xi_n$.

Since the C.V. is estimated from $\bar{x}$ and s, the sample statistic, and it is desired to estimate the true but unknown C.V., it is necessary to calculate the differences between the true C.V. of the population and the sample C.V.

The mean of C.V.

If the values of x_1, x_2, . . . x_n are a sample of n taken from a population that has a mean μ and standard deviation σ, then the true but unknown population C.V. is $\frac{\sigma}{\mu}$ which must be estimated by using the ratio $\frac{s}{\bar{x}}$, the sample statistics. The distribution of

$$\frac{x_1-\mu}{\sigma}, \frac{x_2-\mu}{\sigma}, \ldots \frac{x_n-\mu}{\sigma}$$

has a mean of zero; that is, consider the transform $Z_i = \frac{x_i-\mu}{\sigma}$.

For a finite sample of n, μ is estimated by

$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}$$

and the mean of Z for n values is

$$\bar{Z} = \frac{\sum_{i=1}^{n} Z_i}{n}$$ is zero. The proof is

$$\bar{Z} = \frac{Z_1 + Z_2 . . . + Z_n}{n}$$

$$= \frac{1}{n}\left[\frac{x_1-\hat{\mu}}{\sigma} + \frac{x_2-\hat{\mu}}{\sigma} + , \ldots + \frac{x_n-\hat{\mu}}{\sigma}\right]$$

$$= \frac{1}{n\sigma}\left[(x_1-\hat{\mu}) + (x_2-\hat{\mu}) +, \ldots + (x_n-\hat{\mu})\right]$$

$$= \frac{1}{n\sigma}(0) \quad \text{since} \quad \sum_{i=1}^{n}(x_i-\hat{\mu}) = 0 .$$

This shows that a transformation of the form $Z_i = \frac{x_i-\mu}{\sigma}$ made on the sample values and that the mean of these transform values is zero which is said to be expressed in standard measure or is standardized.[6] The mean of several C.V.'s that have been determined from samples that have been drawn repetitively from a continuous process would form a distribution of ratios. The magnitude of these ratios would depend upon the underlying distribution as well as μ and σ. In order to estimate the average of the C.V.'s from samples of n items it appears necessary to sample a random process of a known distribution by simulation or to use the relevant statistics $\overline{x}_i$ and s_i from a continuous process.

However, for the normal distribution and n large by applying the simple but important property of mean values which is "the mean value of a product of two functions is the product of their mean values if

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[6] M. G. Kendall and A. Stuart, The Advanced Theory of Statistics, Vol. I (6th ed.; New York: Hafner Publishing Company, 1952), pp. 48, 51.

each function depends on a set of varjiates indpendent of the set on which the other depends"[6] and the aritthmetic man of the sample C.V. $E(\frac{\bar{s}}{\bar{x}})$ is $E(\frac{s}{x})$. When $\bar{x}$ approaches zerro the vaue of using the C.V. appears questionable and other statisttics shoud be used.

Variance of C.V.

The variance of C.V. from a sampl1e of n iems is the ratio of two random variables x_1/x_2, and requirres that $_2>$ 0 for the discrete case and $x_2>0$ if it is continuous. 1The rth mment, m_r' (for a sample), is the expected value of the powers off the ranom variable, from which the variance of the ratio x_1/x_2, that is $V(\frac{x_1}{x_2})$ is found from

$$V\left(\frac{x_1}{x_2}\right) = \frac{V(x_1)}{m_1^2} + \frac{m_1^2 \quad V(x_2))}{m_2^4} - \frac{2m\mathrm{Cov}(x_1x_2)}{m_2^3} .$$

Since m_r' is defined as the rth moment statisti of a sample that corresponds to the rth moment, μ_r, of a populaion, that is $m_r' = \frac{1}{n} \sum_{i=1}^{n} x_i^r$ is the sample moment frcom n samles where the ith item of the sample has a measurement $x\varsigma_i$. If te sample moments are substituted then,

$$V\left(\frac{x_1}{x_2}\right) = \left[\frac{E(x_1)}{E(x_2)}\right]^2 \left[\frac{V(x_1)}{[E(x_1)]^2} +\!\!+ \frac{V(x_2)}{[E(x_2)^2} = \frac{2\ \mathrm{Cov}(x_1x_2)}{E(x_1)\ E(x_2)}\right] \qquad (1)$$

if the population C.V. is θ and $\theta = \frac{1\iota 00\ m_2}{m_1}$

where $\theta \neq 0$

and $\mu_1' \neq 0$

then

$$V(\theta) = \left(\frac{\theta}{100}\right)^2 \left[\frac{V(\sqrt{m_2})}{E(\sqrt{m_2})} + \frac{V(m_1')}{[E(m_1')]^2} - \frac{2\,Cov\,(\sqrt{m_2})\ m_1'}{E(\sqrt{m_2})\ E(m_1')}\right] \quad (2)$$

noting that $V(m_1) = \frac{\mu_2}{n}$; $V(\sqrt{m_2}) = \frac{V(m_2)}{4\mu_2} = \frac{\mu_4 - \mu_2^2}{4n\,\mu_2}$

and

$$Cov(\sqrt{m_2},\ m_1') = \frac{1}{2\sqrt{\mu_2}}\ Cov\ (m_2\ m_1') = \frac{\mu_3}{2n\sqrt{\mu_2}}$$

These equalities are substituted in equattion (2) which becomes

$$V(C.V.) = \frac{\theta^2}{n}\left[\frac{\mu_4 - \mu_2^2}{4\mu_2^2} + \frac{\mu_2}{(\mu_1')^2} - \frac{\mu_3}{\mu_2\ \mu_1'}\right] \quad (3)$$

Applying (3) to the normal distribution ($(\mu_3 = 0$ and $\mu_4 = 3\mu_2^2)$ the V(C.V.) is

$$V(C.V.) = \frac{\theta^2}{n}\left[\frac{\mu_2^2(3-1)}{4\mu_2^2} + \frac{\mu_2}{(\mu_1')^2} - \frac{0}{\mu_2\mu_1'}\right]$$

$$= \frac{\theta^2}{n}\left[\frac{1}{2} + \frac{\mu_2}{(\mu_1')^2}\right] = \frac{\theta^2}{2n}\left[1 + 2\left(\frac{\theta}{100}\right)^2\right] \qquad (4)$$

$$\cong \frac{\theta^2}{2n}$$

This relationship can be used to estimate the standard deviation of each of the C.V.s which could then be used in comparing the C.V. of one sample with the C.V. of another sample. For each sample the true but unknown standard deviation of the C.V. is estimated, using the sample results and (4) as

$$\theta' = \sqrt{\frac{n\cdot V(C.V)}{\left[\frac{1}{2} + \frac{\mu_2}{(\mu_1')^2}\right]}} = \sqrt{\frac{n\cdot V(C.V)}{\left[\frac{1}{2} + \frac{E(s^2)}{(\bar{X})^2}\right]}} \qquad (5)$$

Distribution of C.V.

To find an approximation to the distribution of C.V. which is a function of two random variables, $\bar{X}$ and s, and that $\bar{X}$ is normally distributed with parameters $(\xi, \frac{\sigma^2}{n})$; s is approximately normally distributed with parameters $(\sigma, \frac{\sigma^2}{2f})$ for large f; and $\bar{X}$ and s are stochastically independent.[7] The mean is $\overline{C.V.} \cong \frac{\sigma}{\xi} = \gamma$, the variance of C.V. is $V(C.V.) \cong \frac{\gamma^2}{2f}(1 + 2\gamma^2)$ which may be considered as approximately normally distributed with mean $\frac{\sigma}{\xi}$ and variance $\frac{\gamma^2}{2f}(1 + 2\gamma^2)$ for values of degrees of freedom, f, are large and small values of γ.

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[7] A. Hald, Statistical Theory with Engineering Applications (New York: John Wiley & Sons, Inc., 1962), pp. 301-303.

The P-fractile is C.V.$_P$, where $C.V._P \cong \gamma \left(1 + \frac{Z_P}{\sqrt{2f}} \sqrt{1 + 2\gamma^2}\right)$ (6)

and this fractile can be found by substituting $\frac{s}{\bar{X}} = \gamma'$ and inserting values of Z_P from the tabular values of the normal distribution. An example for this approximation is: let $\gamma' = .05$ and $f = 30$ then the calculation of C.V.$_P$ associated with the Pth fractile are tabulated in cumulative distribution function form as follows:

$P(\gamma < C.V._P)$	.01	.02	.025	.050	.10	.25	.50	.75	.90	.95	.975	.98	.99
Value of C.V.	.035	.037	.0375	.040	.042	.046	.050	.054	.058	.061	.0625	.063	.065

A better approximation to the distribution of C.V. is obtained by solving $P(C.V. < C.V._P)$ or Prob $(z < 0) = P$ where $Z = s - \bar{X}\, C.V._P$ and is considered to be approximately normally distributed with mean

$$\mu_Z \cong \sigma - \xi\, C.V._P \tag{7}$$

and variance

$$V(Z) \cong \sigma^2 \left(\frac{1}{2f} + \frac{(C.V._P)^2}{n}\right). \tag{8}$$

The distribution of the variable Z is a linear function of a normally distributed and an approximately normally distributed function which will usually deviate less from the normal distribution than C.V., the quotient between the same two random variables. Solving

$$P(S - \overline{X}\,C.V._P < 0) = \Phi\left(\frac{\xi\,C.V._P - \sigma}{\sigma\sqrt{\dfrac{1}{2f} + \dfrac{C.V._P{}^2}{n}}}\right) = \Phi\left(\frac{C.V._P/\gamma - 1}{\sqrt{\dfrac{1}{2f} + \dfrac{C.V._P{}^2}{n}}}\right) = P \quad , \qquad (9)$$

we have

$$\frac{C.V._P}{\gamma} \cong 1 + Z_P\sqrt{\frac{1}{2f} + \frac{C.V._P{}^2}{n}} \quad , \qquad (10)$$

and

$$C.V._P \cong \gamma\left[\frac{1 + Z_P\sqrt{\dfrac{1}{2f}\left(1 - \dfrac{\gamma^2 Z_P^2}{n}\right) + \dfrac{\gamma^2}{n}}}{1 - \dfrac{\gamma^2 Z_P^2}{n}}\right] . \qquad (11)$$

Comparison of this statistical procedure with the previous procedure used is accomplished by comparing the results of solving for P by this statistical procedure with the results obtained by using the previous approximation used to determine the Pth fractiles of C.V. when $\sigma = .05$ and $f = 30$. This was done and the two distributions calculated were identical to three decimal places.

The population coefficient of variation, θ, as previously defined is $\theta = \frac{\sigma}{\mu}$ where σ is the standard deviation and μ is the mean of the distribution. Let $\bar{X}$ be the mean of a sample (calculated from n observations) and let s be the standard deviation of the same sample based on f degrees of freedom where the sample observations are from a normal distribution. In some sampling situations, namely, the single-sample problem where one sample of n observations only is drawn, f = n-1, but the results need not be limited to just the single-sample case. The distribution of C.V. $= \frac{s}{\bar{X}}$, the sample coefficient of variation is the problem of interest.

For a positive constant, k, it is necessary to compute the probability that a noncentral t-distribution is greater than $t = \frac{\sqrt{n}}{k}$ with a noncentrality parameter $\delta = \frac{\mu\sqrt{n}}{\sigma}$; to add to this the probability that the noncentral t is zero. That is, to find the probability $\frac{s}{\bar{X}} < k$ we must form the probability statement to find the

$$\text{Prob}\left[\frac{s}{\bar{X}} < -c\right] = \text{Prob}\left[T_f \geq \frac{\sqrt{n}}{c} \middle| \delta = \frac{\mu\sqrt{n}}{\sigma}\right] + \text{Prob}\left[T_f < 0 \middle| \delta = \frac{\mu\sqrt{n}}{\sigma}\right]$$

where T_f is distributed as the noncentral t-distribution.

This general method of finding the distribution function of C.V was indicated by Johnson and Welch[8] to provide a precise characterization of the probability distribution functions of the sample C.V.

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[8] N. L. Johnson and B. L. Welch, "Applications of the Noncentral t-Distribution," Biometrika, XXXI (1939-40), pp. 362-389.

Note that the random variable, C.V., is largest when the non-central t random variable T_f is near zero, which indicates the extremes or tails of the distribution of C.V. are around zero for the random variable T_f. That is, consider c positive, then:

$$P_r\left(\frac{s}{\bar{X}} > c\right) = P_r\left(0 \le T_f \le \frac{\sqrt{n}}{c} \,\middle|\, \delta = \frac{\mu\sqrt{n}}{\sigma}\right) \tag{12}$$

$$P_r\left(\frac{s}{\bar{X}} < -c\right) = P_r\left(-\frac{\sqrt{n}}{c} \le T_f \le 0 \,\middle|\, \delta = \frac{\mu\sqrt{n}}{\sigma}\right) \tag{13}$$

and

$$P_r\left(\frac{s}{\bar{X}} > -c\right) = P_r\left(T_f \le -\frac{\sqrt{n}}{c} \,\middle|\, \delta = \frac{\mu\sqrt{n}}{\sigma}\right) + P_r\left(T_f < 0 \,\middle|\, \delta = \frac{\mu\sqrt{n}}{\sigma}\right). \tag{14}$$

The noncentral t-distribution, T_f, is the relationship $t = \frac{Z + \delta}{\sqrt{2f}}$ where Z is distributed about zero with unit standard deviation, δ is the noncentrality parameter and f is the degrees of freedom. In order to determine the probability that t exceeds some value of t_o it is necessary to calculate $y = (1+ \frac{t_o^2}{2f})^{-\frac{1}{2}}$ and $Y' = \left[\frac{t_o}{\sqrt{2f}} (1+ \frac{t_o^2}{2f})\right]^{\frac{1}{2}}$ which are estimates used to find the appropriate values of a constant, in the Table of X at the desired probability γ. Then value of λ, a constant that is associated with each γ must be used to establish δ for each γ by solving for the noncentrality parameter $\delta = t_o - \lambda(1+ \frac{t_o^2}{2f})^{\frac{1}{2}}$. For example, if the C.V. = 2.8 from a sample of n = 17 then f = 16. Then calculate $t_o = \pm \frac{\sqrt{17}}{2.8} = \pm 1.47254$; $y = (1+ \frac{t_o^2}{2f})^{-\frac{1}{2}} = .96775$; $Y' = \frac{t_o}{\sqrt{2f}} (1+\frac{t_o^2}{2f})^{\frac{1}{2}}$

that is $Y' = .2519$ and $-Y' = -.2519$. Tables of T_f at γ are entered to find the appropriate λ to use in order to determine δ with each γ, i.e., $\delta = t_o - \lambda\left(1 + \frac{t_o^2}{2f}\right)^{\frac{1}{2}}$.

Tables II and III provide listings of the parameters and calculations necessary to determine Table IV which provides the distribution of equation (2) above for this example

TABLE II

Values of λ for $+Y'$ and $-Y'$ for Specific γ

γ	λ For $+Y'$	λ For $-Y'$
.40	.276	.231
.30	.546	.504
.20	.863	.819
.10	1.303	1.259
.05	1.665	1.623
.025	1.979	1.938
.01	2.345	2.307
.005	2.594	2.556

TABLE III

Values of δ and $G(-\delta)$ for $+Y'$ and $-Y'$ for Specific γ

For $+Y'$				For $-Y'$		
$\frac{\theta}{\delta}$	γ	δ	$G(-\delta)$	δ	$G(-\delta)$	$\frac{\theta}{\delta}$
3.4727	.40	1.1873	.1175	-1.7112	.9565	2.4095
4.5394	.30	.9083	.1819	-1.9912	.9768	2.0707
7.1002	.20	.5807	.2808	-2.3188	.9893	1.7781
32.6971	.10	.1261	.4498	-2.7735	.9972	1.4866
16.6254	.05	-.2480	.5979	-3.1496	.9992	1.3091
7.1969	.025	-.5729	.7166	-3.4751	.99974	1.1865
4.3374	.01	-.9506	.8291	-3.8564	.99994	1.0692
3.4132	.005	-1.208	.8863	-4.1137	.99998	1.0023

TABLE IV

Cumulative Distribution of T_{16} for Values of δ

δ	$P_r(T_{16} \leq 1.47254 \mid \delta)$
4.1137	.005
3.8564	.010
3.4751	.025
3.1496	.050
2.7735	.100
2.3188	.200
1.9912	.300
1.7112	.400
1.1873	.600
.9083	.700
.5807	.800
.1261	.900
0	.915*
-.2480	.950
-.5729	.975
-.9506	.990
-1.2080	.995

* Obtained from a table of Student's t-distribution.

Since $P_r(T_f \leq 0 | \delta) = G(-\delta)$, then consider $\theta = \frac{\sqrt{n}}{\delta}$ and calculate the relationship $P_r(C.V. > 2.8) = P_r(T_{16} \leq 1.47254) - G(-\delta)$ which is presented in Table V.

TABLE V

Probability that C.V. is greater than 2.8 for θ

$\theta = \frac{\sqrt{n}}{\delta}$	$P_r(C.V. > 2.8)$
$-\infty$	.9154 - .5000 = .4154
-16.6254	.95 - .5979 = .3521
- 7.1969	.975 - .7166 = .2584
- 4.3374	.99 - .8291 = .1609
- 3.4132	.995 - .8863 = .1087
0	1.000 -1.000 = 0
1.0023	.005 - .00002= .00498
1.0692	.01 - .00006= .00994
1.1864	.025 - .00026= .02474
1.3091	.05 - .0008 = .0492
1.4866	.10 - .0028 = .0972
1.7781	.20 - .0107 = .1893
2.0707	.30 - .0232 = .2768
2.4095	.40 - .0435 = .3565
3.4727	.60 - .1175 = .4825
4.5394	.70 - .1819 = .5181
7.1002	.80 - .2808 = .5192
32.6971	.90 - .4498 = .4502
∞	.9154 - .5000 = .4154

Note that the above probability distribution has a maximum of .415 for $\theta = -\infty$ (i.e., $-\delta \to 0$), decreases to zero for $\theta = 0$ (i.e., $\delta \to \infty$), is a maximum somewhere in the interval of $\theta = 4.54$ to $\theta = 7.1$, and then decreases to .415 for $\theta = \infty$ (i.e., $\delta \to 0$ from the positive side). Specifically this probability distribution is not a monotone function of θ.

Now assume that $\theta \equiv 2.8$ and it is desired to know the probability distribution of C.V. for samples with f = 16 degrees of freedom and n = 17. In order to find this probability distribution first compute $\delta = \frac{\sqrt{17}}{2.8} = 1.47254$ and then calculate η where $\eta = \frac{\delta}{\sqrt{2f}} (1 + \frac{\delta^2}{2f})^{-\frac{1}{2}} = .25192$ which is needed to obtain values of λ for specific γ in order to calculate

$$t = \frac{\delta + \lambda \left(1 + \frac{\delta^2}{2f} - \frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1 - \frac{\lambda^2}{2f}\right)} . \qquad (15)$$

for each specific γ.

These are provided in the following Table

TABLE VI

Noncentral t Probability Distribution

γ	λ For $+\eta$	λ For $-\eta$	t For $+\eta$	t For $-\eta$
.75	.7071	.6620	2.2323	- .8039
.90	1.3181	1.2786	2.9601	- .1932
.95	1.6796	1.6476	3.4380	.1758
.975	1.9889	1.9675	3.8853	.5028
.99	2.3414	2.3391	4.4521	.8957
.995	2.5759	2.5914	4.8721	1.1741

Using the equality $P_r(T_f \leq t \mid \delta) = 1 - P_r(T_f \leq -t \mid -\delta)$ the non-central t probability distribution is obtained from the above tables using γ and t which are listed in Table VII.

TABLE VII

Noncentral t Probability Distribution with Noncentrality Parameter $\delta = \frac{\sqrt{17}}{2.8}$ for Specific Values of t

t	$P_r[T_f \leq t \mid \delta = \frac{\sqrt{17}}{2.8}]$
-1.1741	.005
- .8957	.010
- .5028	.025
- .1758	.050
.1932	.100
.8039	.250
2.2323	.750
2.9601	.900
3.4380	.950
3.8853	.975
4.4321	.990
4.8721	.995

Since $G(-\delta) = G[-\frac{\sqrt{17}}{2.8}] = .07044$, it is necessary to subtract .07044 from each of the above probabilities (i.e., one must be added to negative values of t) in order to find the probability distribution of $s/\overline{X}$ when $\theta = 2.8$. This provides the relationship given in Table VIII which is the probability distribution of C.V. being greater than $\frac{\sqrt{17}}{t}$ given that $\theta \equiv 2.8$.

TABLE VIII

Probability that a Coefficient of Variation of $\frac{\sqrt{17}}{t}$ Being Observed in a Sample of 17 when the Population Coefficient of Variation is 2.8.

$\frac{\sqrt{17}}{t}$	$P_r(C.V. > \frac{\sqrt{17}}{t} \mid \theta = 2.8)$
21.3365	.02956
5.1290	.17956
1.8470	.67956
1.3929	.82956
1.1993	.87956
1.0612	.90456
.9261	.91956
.8463	.92456
-3.5118	.93456
-4.6033	.93956
-8.2010	.95456
-23.4510	.97956

Difference between two sample C.V.'s

An approximation to compare the difference between two coefficients of variation were presented by Ferber[9] and Hald[10]. The general principle of hypothesis testing presented by Ferber is to consider the ratio, T, of

$$\frac{\text{A sample statistic - statistic from another sample}}{\text{Estimated standard error of the difference between the two statistics}}$$

in order to determine if the degree of random variation between the two statistics is due to chance or chance is ruled out. Considering large sample sizes (each item of each sample is drawn at random), if it is desired to perform a test of hypothesis on the C.V.'s of two samples, such a test would be as follows:

(a) H_o: $C.V._1 = C.V._2$ is the null hypothesis and H_A: $C.V._1 \neq C.V._2$ is the alternate hypothesis.

(b) Select the critical value of T.

(c) Subtract $C.V._2$ from $C.V._1$.

(d) Calculate the standard error of the difference between the two sample C.V.'s, which is:

$$\hat{\sigma}_{C.V._2 - C.V._1} = \sqrt{\frac{(C.V._1)^2}{n_1} + \frac{(C.V._2)^2}{n_2}} \tag{16}$$

[9] Robert Ferber, Statistical Techniques in Marketing Research, (1st ed.; New York: McGraw-Hill Book Company, Inc., 1949), p. 123.

[10] Hald, p. 302.

where $C.V._1$ and $C.V._2$ are the coefficient of variance of sample one and sample two, respectively,

and

n_1 and n_2 are the number of items in sample one and sample two, respectively.

(e) Compare Z' to the Z_α tabulated value of the normal curve

$$Z' = \frac{C.V._2 - C.V._1}{\hat{\sigma}_{C.V._1 - C.V._2}} . \tag{17}$$

(f) Action to be taken is (1), accept H_0, if Z_α tabulated is larger than Z' calculated, or (2) reject H_0, if Z' calculated is equal to or greater than Z_α. The preceding procedure provides a method to calculate $\hat{\sigma}_{C.V._2 - C.V._1}$ for large samples that are uncorrelated. When the samples are correlated, estimate

$$\hat{\sigma}_{C.V._1 - C.V._2} = \sqrt{\frac{(C.V._1)^2}{n_1} + \frac{(C.V._2)^2}{n_2} - \frac{r_{12}^2\, \hat{\sigma}_1\, \hat{\sigma}_2}{\sqrt{n_1 - n_2}}} \tag{18}$$

where

$$r_{12}^2 = \frac{\sum_{i=1}^{n} X_{1i}\, X_{2i}}{\sqrt{\sum_{i=1}^{n} X_{1i}^2 - \sum_{i=1}^{n} X_{2i}^2}} \qquad X_{1i} = X_{1i} - \overline{X}_1 \quad X_{2i} = X_{2i} - \overline{X}_2 \tag{19}$$

$$\hat{\sigma}_1^2 = \frac{\Sigma X_{1i}^2 - \dfrac{(\Sigma X_{2i})^2}{n_2}}{n_1 - 1} \tag{20}$$

$$\hat{\sigma}_2^2 = \frac{\Sigma X_{2i}^2 - \dfrac{(\Sigma X_{2i})^2}{n_2}}{n_2 - 1} \quad . \tag{21}$$

Confidence limits of C.V. (an approximate method)

The estimation of the population C.V. when the population is assumed normally distributed but σ is unknown, raises the problem of dependence between μ and σ^2. That is, we must consider using Student's t-distribution with n-1 degrees of freedom rather than the normal distribution. The distribution of means formed by drawing k samples of n from a population is normally distributed with mean $\dfrac{\sum_{i=1}^{k} \overline{X}_i}{k}$ and variance $\dfrac{\sigma^2}{n}$ but the distribution of the variance is χ^2 and the sample standard deviation, s, is distributed as $\sqrt{\chi^2}$. Placing $1-\alpha_i$ confidence limits about $\bar{X}$ and s of the sample (individually) results in a rectangle which will be too pessimistic except for small sample sizes and when very large confidence interval estimates are desired.

The joint boundary region in which μ and σ are expected to lie with $(1-\alpha)$ 100 percent confidence can be estimated by considering the independent distributions of X and $\sum_{i=1}^{n} (X_i - X)^2$ such that

$$P\left[-a < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < a\right] = \sqrt{1-\alpha} \quad \text{and} \quad P\left[a' < \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sigma^2} < b'\right] = \sqrt{1-\alpha} \tag{22}$$

from which the joint probability distribution of

$$P\left[-a < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < a, \quad a' < \left\{\frac{\sum_{i=1}^{n} (X_i - \bar{X})}{\sigma^2}\right\} < b\right] = 1-\alpha \tag{23}$$

due to the independence of the variables.[11] The boundaries of the joint probability distributions involving μ, $\bar{X}$, σ, and s are found by solving $(\mu - \bar{X}) = \pm \frac{a\,\sigma}{\sqrt{n}}$ where a, the constant of α probability of the normal distribution need not be used since $\mu - \bar{X}$ can be charted directly.

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11 A. M. Mood and F. A. Graybill, Introduction to the Theory of Statistics, (New York: McGraw-Hill Book Company, Inc., 1963), p. 255.

(a) Let $\mu = \bar{X}$ then $\sigma = 0$.

(b) Solve $\mu = \bar{X} \pm \frac{\hat{\sigma}}{\sqrt{n}} t_{\frac{\alpha}{2}; n-1}$ Note: $a = t_{\frac{\alpha}{2}; n-1}$.

(c) Solve upper limit of $\hat{\sigma} = \dfrac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{a'}$; $a' = \chi^2_{1-\frac{\alpha}{2}; n-1}$.

(d) Solve lower limit of $\hat{\sigma} = \dfrac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{b'}$; $b' = \chi^2_{\frac{\alpha}{2}; n-1}$.

(e) Plot the values calculated in a, b, c, and d above.

(f) Select μ', σ', μ'' and σ'' at the intersection of μ's calculated and the appropriate limits of σ from the chart (d above).

(g) Use the values selected to provide the confidence limits for C.V. in the following probability statement,

$$P \left[\frac{\sigma''}{\bar{X} + \frac{a\,\sigma}{\sqrt{n}}} \leq \frac{\sigma}{\mu} \leq \frac{\sigma'}{\bar{X} - \frac{a\,\sigma}{\sqrt{n}}} \right] = 1 - \frac{\alpha}{2}$$ where σ' is upper limit

on σ' and σ'' is the lower limit on σ (selected at the appropriate points of Figure 2).

Example: A sample composed of 25 measurements was taken and

$$\sum_{i=1}^{25} (X_i - \bar{X})^2$$

was found to be 384 and X = 50; what is the 95.0 percent confidence interval estimate for the population C.V. This is solved by considering $(\mu - \bar{X}) = \pm \frac{a\,\sigma}{\sqrt{n}}$ at three points.

If $\mu - \overline{X}$ is zero then σ is zero. Substitute $t_{.025;24}$ for a and $\hat{\sigma}$ for σ to solve $\mu = \overline{X} \pm \frac{a\ s}{\sqrt{n}}$. That is, solve for two points of μ using the standard deviation of the sample and $\pm$ t of Student's t-distribution at $\alpha = .025$ and n-1 degrees of freedom since σ is unknown. This provides the points $\mu = 51.6$ versus $\sigma = 4$ and $\mu = 48.36$ for $\sigma = 4$ which are used to construct the two straight lines identified as $\overline{X} + \frac{a\ \hat{\sigma}}{\sqrt{n}}$ and $\overline{X} - \frac{a\ \hat{\sigma}}{\sqrt{n}}$ in Figure 2. These two lines intersect at $\mu = \overline{X}$ and $\sigma = 0$.

Next, calculate the upper and lower confidence limits of $\hat{\sigma}$ by

$$\sigma' \text{ is upper confidence limit of } \hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})}{a'}} = \sqrt{\frac{384}{12.4}} = 5.7,$$

$$\sigma'' \text{ is lower confidence limit of } \hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})}{b'}} = \sqrt{\frac{384}{39.4}} = 3.1.$$

Note: $a' = \chi^2_{.975;24} = 12.4$, and $b' = \chi^2_{.025;24} = 39.4$.

The values to determine the confidence limits on the C.V. can be taken from Figure 2 as lower limit of C.V. $= \frac{\sigma''}{\mu''}$ and upper limit of C.V. $= \frac{\sigma'}{\mu'}$, or calculated directly. The direct calculation is to calculate the upper confidence limit on $\hat{\sigma}$ as σ'' as before, then use σ'' to determine the lower limit on $\overline{X}$ (i.e., $\mu' = \overline{X} - \frac{\sigma'}{\sqrt{n}}\ t_{.025;n-1}$). Calculate the lower confidence limit on $\hat{\sigma}$ as σ'' and then use this σ'' to determine the upper limit on X (i.e., $\mu'' = \overline{X} + \frac{\sigma''}{\sqrt{n}}\ t_{.025;n-1}$). These calculated values are: $\sigma' = 5.7$, $\sigma'' = 3.1$, $\mu' = 50 - \frac{(2.064)(5.7)}{5} \cong 47.4$; $\mu'' = 50 + \frac{(2.064)(3.1)}{5} \cong 51.3$, and the lower limit on C.V. is $\frac{3.1}{51.3} \cong .06$, and the upper limit is $\frac{5.7}{47.4} \cong .12$. Therefore, the approximate 95 percent confidence interval in which the true but unknown population coefficient of variation, θ, is expected to lie is between .06 and .12.

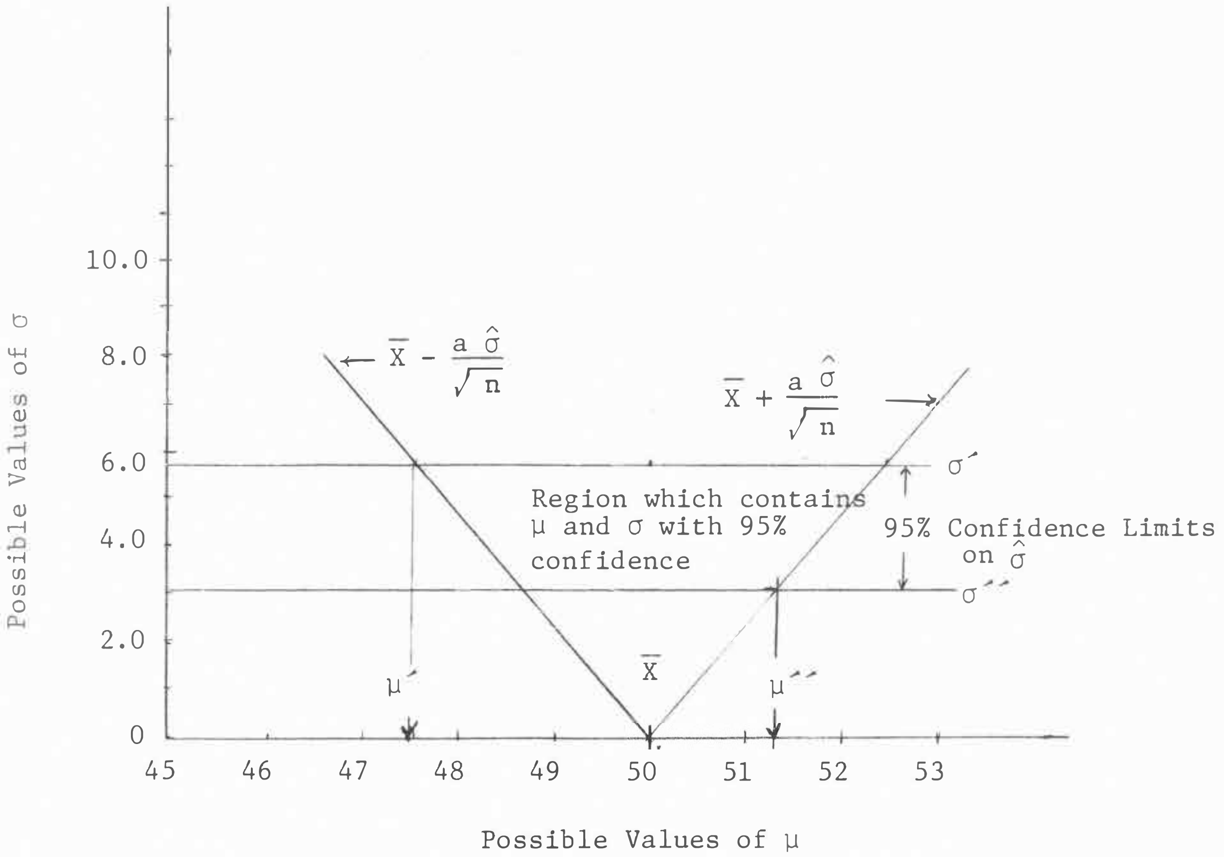

Figure 2. Chart of the Region which Contains Both μ and σ with 95 Percent Confidence

This approximation is considered to be conservative. However, for most practical purposes it should give satisfactory results.[12]

Upper and lower bounds on C.V.'s

Sigeiti Moriguti considered the upper and lower bounds for the expectation, the coefficient of variation, and the variance of the extreme member of the sample from a symetrically distributed population with a finite variance.[13] Specific discussion was concentrated on the largest member and considered the mean of the population equal to zero. These conventions do not imply any essential restriction. This is included since at times, the experimenter wants to know the maximum or the minimum C.V. that could occur.

The cumulative density function (cdf) is denoted by $F(x)$, then the cdf of the largest member x_n from a sample of size n is $[F(x)]^n$, and the expectation of the largest member can be expressed by $E(X_n)$ $= \int_{-\infty}^{-\infty} x_n \, [F(x)]^{n-1} \, \partial F(x)$. $F(x)$'s inverse function of $x(F)$ must be considered along with an additional definition for points of discontinuity, if any exist, for $F(x)$. Then $E(x_n)$ can be written as

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[12] J. Earl Faulkner, Associate Professor of Statistics, Brigham Young University, "The Comparison of Coefficients of Variation for Normal Random Variables" (paper presented at the 10th Western Regional Meeting of American Statistical Association, Salt Lake City, Utah; May 16, 1969).

[13] Sigeiti Moriguti, "Annals of Mathematical Statistics" Journal, XXII, No. 4 (December, 1951); pp. 523-528.

$\int_0^1 x(F)_n F^{n-1} \partial F$ and because of symmetry, $x(F) = -X(1-F)$ holds almost everywhere. Then $E(x_n) = \int_{\frac{1}{2}}^1 x(F)_n [F^{n-1}-(1-F)^{n-1}] \partial F.$ (24)

Also, the sample variance is

$$V(x_n) = \int_{\frac{1}{2}}^1 [x(F)]^2 n [F^{n-1} + (1-F)^{n-1}] \partial F - [E(x_n)]^2 \qquad (25)$$

and the population variance is given by

$$\sigma^2 = 2 \int_{\frac{1}{2}}^1 [x(F)]^2 \, dF. \qquad (26)$$

The bounds for the largest member is determined by (Swartz's) inequality which is used as follows:

$$\left(\int_a^b f(F)\, g(F) dF\right)^2 \leq \int_a^b [f(F)]^2 \, dF \int_a^b [g(F)]^2 \, dF \qquad (27)$$

setting

$$a = \frac{1}{2}, \; b = 1, \; f(F) \equiv x(F), \; g(F) \equiv n [F^{n-1} - (1-F)^{n-1}]$$

results in a formula which means in view of $E(x_n)$ and σ^2 given above that

$$E(x_n) \leq \frac{\sigma}{\sqrt{2}} \; n \left(\int_{\frac{1}{2}}^1 [F^{n-1} - (1-F)^{n-1}]^2 \, \partial F\right)^{\frac{1}{2}} \qquad (28)$$

where equality is satisfied if and only if $f(F) =$ (a constant)

$g(F)$, that is, $x(F) = (\text{constant})\ [F^{n-1}-(1-F)^{n-1}]$. Therefore, the expectation of the largest member is the right-hand side of (28)as an upper bound, which is actually achieved for a type of distribution described by $x(F)$ above.

The integral in ( ) is evaluated as follows:

$$\int_{\frac{1}{2}}^{1}[F^{n-1}-(1-F)^{n-1}]dF = \frac{1}{2}\int_{0}^{1}[F^{n-2}+(1-F)^{n-2}-2F^{n-1}(1-F)^{n-1}]dF$$

$$= \frac{1}{2}\left[\frac{1}{2n-1}+\frac{1}{2n-1}-2\beta(n,n)\right]$$

$$= \frac{1}{2}\left[\frac{2}{2n-1}-2\beta(n,n)\right]$$

$$= \frac{1}{2n-1}-\beta(n,n)\ . \tag{29}$$

Now applying the equal integral arguments for the Beta functions which is expressed as $\beta(n,n) = \dfrac{1}{(2n-1)\binom{2n-2}{n-1}}$;

then the extreme bound for $E(x_n)$ is given by:

$$E(x_n) \leq \frac{n}{\sqrt{2(2n-1)}}\left[1-\frac{1}{\binom{2n-2}{n-1}}\right]^{\frac{1}{2}}\sigma \tag{30}$$

The value of $E(x_n)$ is calculated for various sample sizes and compared with the values of $E(x_n)/\sigma$ for normal and rectangular populations in Table IX.

TABLE IX

Expectation of the largest member in the unit of:

Sample Size n	Upper Bound	For Normal Distribution	For Rectangular Distribution
2	.5774	.5642	.5774
3	.8660	.8463	.8660
4	1.0420	1.0294	1.0392
5	1.1701	1.1630	1.1547
6	1.2767	1.2672	1.2372
7	1.3721	1.3522	1.2990
8	1.4604	1.4236	1.3472
9	1.5434	1.4850	1.3856
10	1.6222	1.5388	1.4171
11	1.6974	1.5864	1.4434
12	1.7693	1.6292	1.4656
13	1.8385	1.6680	1.4846
14	1.9052	1.7034	1.5011
15	1.9696	1.7359	1.5155
16	2.0320	1.7660	1.5283
17	2.0926	1.7939	1.5396
18	2.1514	1.8200	1.5497
19	2.2087	1.8450	1.5588
20	2.2645	1.8673	1.5671

Note that the value for a normal distribution is quite close to the values of the upper bound when n is less than eight; close agreement of the upper bound and the rectangular distribution is when n is less than six.

Bounds for C.V. of the largest member of a sample are found by using (27). Let $a = \frac{1}{2}$, $b = 1$, $f(F) \equiv x(F)\sqrt{n}\,[F^{n-1}+(1-F)^{n-1}]^{\frac{1}{2}}$ and

$$g(F) \equiv \frac{\sqrt{n}\,[F^{n-1}-(1-F)^{n-1}]}{[F^{n-1}+(1-F)^{n-1}]^{\frac{1}{2}}}$$

then with respect to $E(x_n)$ and $V(x_n)$ there is the relationship

$$\frac{V(x_n)}{E(x_n)^2} \geq \frac{1}{M_n} - 1 \tag{31}$$

where

$$M_n = \int_{\frac{1}{2}}^{1} \frac{n[F^{n-1}-(1-F)^{n-1}]^2}{F^{n-1}+(1-F)^{n-1}}\, dF \quad .$$

Equality in (31) is satisfied if, and only if, f = (constant). g which is more precisely stated as $x(F) = (\text{constant})\, \frac{F^{n-1}-(1-F)^{n-1}}{F^{n-1}+(1-F)^{n-1}}$.

Therefore, the C.V. of the largest member has $\sqrt{\frac{1}{M_n} - 1}$ (32)

as a lower bound which is achieved for a particular type of population distribution given by x(F).

M_n is determined by evaluating the integral of (32) by a method of quadrature.

Results for small samples are shown as follows:

$$M_2 = .33333$$

$$M_3 = .64381$$

$$M_4 = .81677$$

$$M_5 = .90695$$

$$M_6 = .95300$$

As the sample size increases, the calculations of M_n become more laborious and numerical integration would be preferable for large values of n. M_n is then used in (32) to determine the lower bound. The C.V. of the largest members are given in Table X for a normal population, a rectangular population, and the lower bound.

TABLE X

Coefficient of variation of the largest member of the Lower Bound, Normal Population, and the Rectangular Population.

Sample Size n	Lower Bound	For Normal Population	For Rectangular Population
2	1.4142	1.4634	1.4142
3	.7438	.8838	.7746
4	.4737	.6812	.5443
5	.3203	.5752	.4226

APPLICATIONS OF C.V.

Utilizing a known C.V. to reduce the mean squared error

Estimation problems are solved for Bayesian approaches using "a priori" information. In a sense, this approach can be applied to reducing the variability exhibited by the means from one sample to the next. That is, a more accurate estimate of the interval in which, μ, the unknown mean is expected to lie can be made using prior information that is available to the experimenter. This prior information may be in the form of sample means, sample standard deviations, identifiable capacity of each unit/sample, environmental exposure on each test, etc. The statistical results of these prior experiment can be used to provide a weight, w, that is associated with each condition of a planned experiment and the subsequent evaluation of the n observations of the experiment.

Associating w_i to the appropriate ith condition may take other forms based on the specific scientific discipline and the statistical rationale but in this instance the C.V. is the statistic of application. This is due to relating gross effects that are exhibited by isolated factors on the sample $\bar{X}$ and s individually or both $\bar{X}$ and s may change appreciably.

Let us consider a random sample of n observations $y_1, y_2 \cdots y_n$ from which it is desired to estimate, μ, the true but unknown population average in such a manner that $E(\bar{y} - \mu)^2$ is a minimum. This is achieved by considering construction of an estimator, say

$\bar{y}' = w \sum_{j=1}^{n} y_j$ is to be compared with $\bar{y}$. Now using the Mean Squared Error of $\bar{y}$, $MSE(\bar{y})$, to the $MSE(\bar{y}')$ the relative efficiency gained by using the weighted sample average versus using the unweighted sample average can be determined. The $MSE(\bar{y})$ is simply $\frac{\hat{\sigma}^2}{n}$ (34)

and since $\bar{y}' = w \sum_{j=1}^{n} y_j$ the $MSE(\bar{y}') = n\, w^2 \hat{\sigma}^2 + \hat{\mu}^2 (1-nw)^2$. (35)

Now if $MSE(\bar{y}')$ is differentiated with respect to the weight, w, this will give

$$\frac{\partial'}{\partial w} [MSE(\bar{y}')] = 2n[w\, \hat{\sigma}^2 - \hat{\mu}^2\ (1-nw)] \tag{36}$$

and taking the second partial derivative provides

$$\frac{\partial''}{\partial w} [MSE(\bar{y}')] = 2n\ (\hat{\sigma}^2 + n\hat{\mu}^2\ )\ . \tag{37}$$

Now (37) is always positive, so the value of w can be found by setting (36) equal to zero and solving. The solution for w is:

$$2n[w\,\hat{\sigma}^2 - \hat{\mu}^2\ (1-nw)] = 0$$

$$\frac{w(\hat{\sigma}^2 + n\hat{\mu}^2)}{n^2} = \frac{\hat{\mu}^2}{\hat{\mu}^2} = 1$$

$$w(\frac{\hat{\sigma}^2}{\hat{\mu}^2} + n) = 1$$

$$w = \frac{1}{(C.V.)^2 + n} \tag{38}$$

and

$$\bar{y}' = [\frac{1}{(C.V)^2 + n}] \sum_{j=1}^{n} y_i \tag{39}$$

$$\text{the MSE}(\bar{y}') = \frac{\sigma^2}{(C.V)^2 + n} \tag{40}$$

Comparing the MSE$(\bar{y})$ to the MSE$(\bar{y}')$ provides R.E. $= \frac{MSE(\bar{y}\ )}{MSE(\bar{y}')}$

which is found by substituting equations (34) and (40) which reduces to $[1 + \frac{(C.V)^2}{n}]$ 100 percent.

That is, R.E. $= \dfrac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n + (C.V)^2}}$ we find the R.E. $= \frac{n + (C.V)^2}{n}$ or

$$100\ [\ 1 + \frac{(C.V)^2}{n}]\ \text{percent.}$$

Relative efficiencies for specific values of C.V. and sample sizes of n are listed in Table XI which follows. This table indicates the R.E. of small samples is largest and should be used when it is expensive to obtain additional observation for inspection or testing.

TABLE XI

R. E. in Percent of $\bar{y}'$ for Various Sample Sizes from a Distribution with a Given C.V.

C.V.	Sample Size								
	5	10	15	20	30	50	70	90	110
.25	101.25	100.62	100.42	100.32	100.21	100.13	100.09	100.07	100.06
.50	105.00	102.50	101.67	101.25	100.83	100.50	100.36	100.28	100.23
.75	111.25	105.63	103.75	102.81	101.88	101.12	100.80	100.63	100.51
1.00	120.00	110.00	106.67	105.00	103.33	102.00	101.43	101.11	100.91
1.25	131.25	115.62	110.42	107.81	105.21	103.12	102.23	101.74	101.42
1.50	145.00	122.50	115.00	111.25	107.50	104.50	103.21	102.50	102.05
1.75	161.25	130.62	120.42	115.31	110.21	106.12	104.38	103.40	102.78
2.00	180.00	140.00	126.67	120.00	113.33	108.00	105.71	104.44	103.64

Acceptance sampling using the C.V.

Controlling the variability of products sufficiently so that not too many orders, or in some instances lot sized shipments will be returned by the buyers, has been a major area for applying statistical techniques. This section is restricted to sampling incoming lots from a continuous production process. In this continuous process, it is usually assumed that failures (defects) are random events and that if a trend develops, this trend is due to an assignable cause such as tool wear which can be compensated for by taking some appropriate corrective action (i.e., adjustment, tool sharpening, etc.).

The final products of this continuous process are put into various quantities of size n to fill customer purchase orders. Some customers perform incoming inspections on each lot that is purchased while others may accept the lot and perform inspection as part of their assembly operation. This section is for application to incoming inspection where measurement of specific critical characteristics is performed and the lot is either accepted or rejected based on the statistical evaluation of the measurement data of each lot.

H. A. Freeman,[14] provided an evaluation of a large sample (n = 188) of the crushing strength, in tons, of bricks wherein the C.V. from the sample is .146. Further it is necessary to determine how many bricks should be tested, and what is the sample C.V. that detects the acceptable lots from the unacceptable lots? The essential parts of evaluating this typical quality control problem is to specify quantitative values which reflect the consumer and producer's interests, respectively. That is, assume a buyer is willing to accept bricks of lower average strength and higher variability in strength restricted to $C.V._B = 0.3$, say five percent of the time (i.e., let this be the buyer's risk, identified as B = .05). Also, assume the producer does not want to have more than one percent of his lots rejected (i.e., let this be the producer's risk, identified as P = .01). Then since the producer's statistically controlled output is characterized by $C.V._P = 0.146$ and using the following functions

$$\frac{n\ (C.V.)^2}{n + (C.V.)^2} \left(\frac{1}{(.146)^2} + 1 \right) = \chi^2_P \tag{41}$$

$$\frac{n(C.V.)^2}{1 + (C.V.)^2} \left(\frac{1}{(.3)^2} + 1 \right) = \chi^2_B \tag{42}$$

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[14] Freeman, p.63

which are approximations, the producer's and buyer's interests are provided for by the ratio

$$\frac{\chi^2_P}{\chi^2_B} = \frac{48}{12} = 4.$$

Entering the Chi-Square tables the value corresponding to the ratio 4 is $\chi^2_{.99}/\chi^2_{.05} = \frac{32.0}{7.96}$ for 16 degrees of freedom, hence n is 17. Using the appropriate χ value and n = 17 in either equation (1) or (2), it is found that C.V. = .202. This means that a sample of 17 should be drawn and if the sample coefficient of variation, C.V., is greater than .20, the lot should be rejected. Establishing when there is no longer any possibility of accepting a lot was investigated in an article by Robert D. Summers. The procedure is to order the samples from the lowest measured value to the highest and then to discontinue inspection when the number above a certain limit (as attributes) is exceeded. That is, consider that a bound on the C.V. exists and this can be expressed as

$$C.V._b \geq \sqrt{\frac{r}{n-r}} \qquad (43)$$

under the conditions:

x_i is the ith ordered sample value $1 \leq i \leq n$,

r is the number of negative sample values,

$\bar{x}$ is the sample mean (assumed positive),

$s = \left[\Sigma \frac{(x_i - \bar{x})}{n} \right]^{\frac{1}{2}}$ is the sample standard deviation.

The application proposed is in the sampling of variables where disposition of a lot or a group of items from a process (sublot) is decided on the basis that the reject criteria is:

If $\bar{y} + Ks' > U$, reject the lot or process.

$\bar{y}$ is the mean of n values of y_i

$$s' = \left[\Sigma \frac{(y_i - \bar{y})^2}{n - 1} \right]^{\frac{1}{2}} \qquad (44)$$

k is a constant associated with the sampling plan,

U is a limit such that $y_i > U$ identifies the ith largest sample item as a defective.

The sample values are expressed with relation to the deviations limit, that is $x_i = U - y_i$. The criterion now becomes reject if $\bar{x} - k_s' < 0$ which is equivalent to $\bar{x}/s' < k$ or $\frac{\bar{x}}{s} \left[\frac{n-1}{n}\right]^{\frac{1}{2}} <$

$k(C.V.)^{-1} \left[\frac{n-1}{n}\right]^{\frac{1}{2}} < k$ then a sufficient condition for rejection

is $\left[\frac{n-r}{n}\right]^{\frac{1}{2}} \left[\frac{n-1}{n}\right]^{\frac{1}{2}} < k$ or $r > \frac{n(n-1)}{nk^2 + n-1} = \frac{n^2 - n}{nk^2 + n - 1}$. (45)

If the number of defectives in the lot exceed $\frac{n(n-1)}{nk^2 + n-1}$, there is no chance to accept the lot. It is necessary to specify n and k prior to sampling and determine the maximum number of defectives that would be acceptable or to predetermine the number of defectives required to terminate sampling inspection.

Stress versus strength reliability problem

The Standard Handbook for Mechanical Engineers as revised by a staff of specialists[15] gives definitions of stress, strength, and safety factors in a context that is usually referred to by engineers. However, the definitions that are used in reliability assessment and evaluations have a somewhat different implication and it is necessary to indicate graphically the probability density functions that represent applied stress and material strength. Applied stress is, hopefully, less than the strength of the item to which the stress is being or is to be applied at some future time. Figure 3 provides a representation of the applied stress function and the end item strength function that has necessarily been of great concern in liquid propellant rocket design as well as in the design of solid propellant rocket motors.

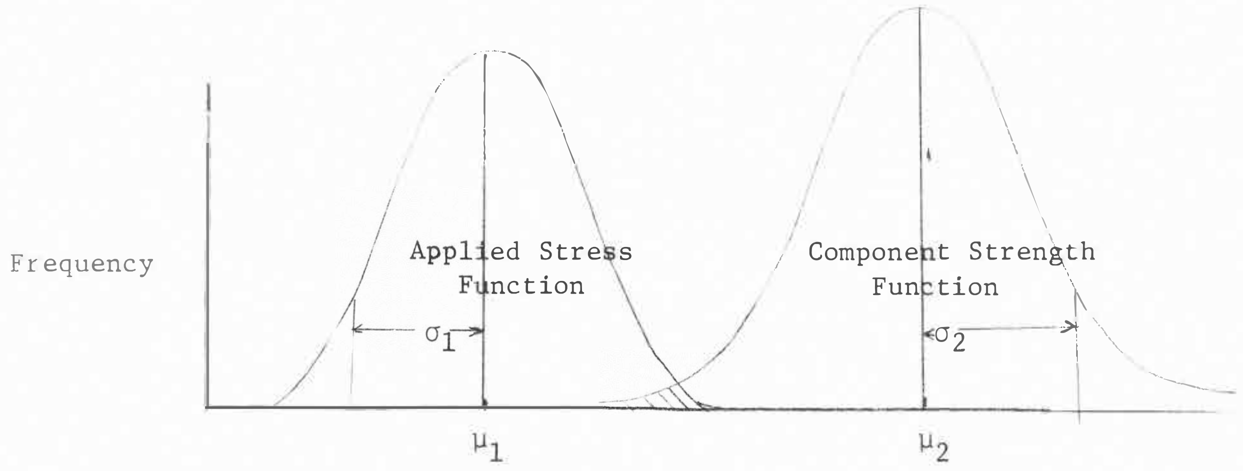

Figure 3. Relationship of Applied Stress to Strength

[15] Theodore Baumeister, Editor, Standard Handbook for Mechanical Engineers (New York: McGraw-Hill Book Company, Inc., 1967).

Applied stress is usually a combination of several environmental factors working on an item during its operational life whereas strength is the ability of the item to withstand the applied stress. Therefore, a failure will occur when the applied stress, as represented by, say, the maximum chamber pressure of a solid propellant rocket during its action time, exceeds the case-closure strength (i.e., the case-closure's ability to contain the stresses that are applied during action time without deformation). This is represented by the cross-hatches area of Figure 3.

Solution of this typical problem has been extensively explored in the reliability literature involving the use of liquid propellant and solid propellant rocket motors. Four methods are presented in this section; the first method is a straightforward evaluation based upon the difference between two random variables such as, X_i, strength of a case-closure,[16] minus Y_i, the stress applied on the same case-closure. As the distance between $\overline{X}$ and $\overline{Y}$ increases for σ_1 and σ_2 fixed, the probability of failure decreases and the probability of successful operation increases. This is usually solved by considering $\xi = X_i - Y_i$ to be normally distributed with mean $\mu_1 - \mu_2$ and a standard deviation of $(\sigma_1^2 + \sigma_2^2)^{\frac{1}{2}}$. That is,

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[16] D. K. Lloyd and M. Lipow , Reliability: Management, Methods, and Mathematics (Prentice Hall, Space Technology Series, 1962), pp. 237-238.

let X and Y be the random variables representing the burst pressure from a sample item and the maximum chamber pressure of a sample item, respectively. Then an estimator of the reliability, $\hat{R}$, and the lower confidence limit, $\hat{R}_1$, is found by substituting the sample means and standard deviation estimates in

$$\hat{R} = \phi\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) \quad \text{and } V(\hat{R}) \cong \frac{\phi^2}{n}\left[\frac{1}{1+(\frac{\sigma_1}{\sigma_2})^2} + \frac{(\frac{\mu_2 - \mu_1}{\sigma^2})^2}{2(1+(\frac{\sigma_1}{\sigma_2})^2)^3}\right] \tag{46}$$

where ϕ is the standard normal P.D.F. using

$$\phi\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) \tag{47}$$

i.e., N(0,1).

For example if μ_2 = 800 psia and σ_2 = 100 are known from design and test verification that has been performed over a long period of time and a sample of 20 rocket motors are tested which provides $\hat{\mu}_1$ = 450 psia, $\hat{\sigma}_1$ = 25 and the 90 percent confidence coefficient, γ, is desired for $\hat{R}$, then

$$\hat{R} = \phi(3.3955) \cong .999666$$

$$V(\hat{R}) = \frac{(1 - .99875)^2}{20} \left\{ \frac{1}{1 + (\frac{100}{25})^2} + \frac{(\frac{800 - 450}{25})^2}{2[\,1 + (\frac{100}{25})^2\,]^3} \right\} \cong 6.144 \times 10^{-8}$$

$$V(\hat{R})^{\frac{1}{2}} = .000248$$

and

$$\hat{R}_{.90} = .999666 - (1.282)(.000248) = .99935.$$

The second method is to consider the relationship of reliability to a safety margin. Reliability is the probability that an item will successfully perform its intended function for a specific required period of time in the environment specified. Safety margin usually is considered as the ratio of an equipment's average strength prior to the point of breakdown (maximum design load) and the average load that will be applied to equipment in its normal use conditions. Both of these are random variables with true but unknown parameters that are calculated and verified by testing a sample of n items.

Figure 3 (page 46) indicates that μ_1, μ_2, σ_1, and σ_2 are known parameters. In practice, this is seldom a true statement and in reliability evaluations there is a tendency to be overly pessimistic and require a large number of items to be tested. Due to cost constraints the number of items provided for destructive testing and the evaluation of design performance may be held to a minimum.

Predicting the reliability of very large solid propellant rocket motors is such a problem since each rocket motor test is very costly. Also, new test equipment may have to be purchased before the first rocket can be tested. However, the testing of critical components on a proposed system may be used to estimate the probability and safety factor, K, where K = μ_2/μ_1. When it is desired to establish a numerical value to determine if the safety factor is sufficient, using the C.V.'s as indicated in the next procedure will provide a solution to this (third) problem.

Let X_1 be the applied stress resulting from the level of environment and let X_2 be the strength level of the component material. If x = is the difference between X_2 and X_1 then $\bar{x} \equiv \bar{X}_2 - \bar{X}_1 > 0$; $\bar{x} = \bar{X}_2 - \bar{X}_1$ and $\sigma_x^2 = \sigma_2^2 + \sigma_1^2$. Substituting $K = \bar{X}_2/\bar{X}_1$ for μ_2/μ_1 we can find the one tail probability of $\frac{\bar{x}}{\sigma_x}$ for $\bar{X} - X_i$ thereby solving

$$\frac{\bar{x}}{\sigma_{\bar{x}}} = \frac{\bar{X}_2/\bar{X}_1 - 1}{\sqrt{(C.V._1)^2 + (C.V._2)^2\ (\bar{X}_2/\bar{X}_1)}} \tag{48}$$

and the probability of $\Phi\left(\frac{\bar{x}}{\sigma_{\bar{x}}}\right)$ can be found in a table of areas for the normal probability density function. For example, if $\bar{X}_1 = 350$, $\hat{\sigma}_1 = 35$, $\bar{X}_2 = 500$, and $\hat{\sigma}_2 = 25$, then the safety factor is 1.4285 and the probability of $\frac{\bar{x}}{\sigma_{\bar{x}}} = 3.44$ is .9997.

The fourth method used to solve for reliability of applied stress versus strength was presented in the Martin Company (Denver) Handbook of Reliability Problems. The reliability coefficient of variation, $C.V._R$, and the ratio of average strength to average applied stress are used to find the numerical value of reliability.

The procedure to estimate reliability from sample data is:

(a) Calculate average strength from a sample

$$\bar{X}_2 = \frac{\Sigma X_{2i}}{n_2} \tag{49}$$

(b) Calculate average applied stress from a sample

$$\bar{X}_1 = \frac{\Sigma X_{1i}}{n_1}$$

(c) Determine the ratio $\bar{F} = \frac{\bar{X}_2}{\bar{X}_1}$ (50)

(d) Calculate the sample variance of strength

$$\hat{\sigma}^2_2 = \frac{\sum_{i=1}^{n_2} X^2_{2i} \quad \frac{\left(\sum_{i=1}^{n_2} X_{2i}\right)^2}{n_2}}{n_2 - 1} \quad . \tag{51}$$

(e) Calculate the sample variance of applied stress

$$\hat{\sigma}^2_{X_1} = \frac{\sum_{i=1}^{n_1} X^2_{1i} \quad \frac{\left(\sum_{i=1}^{n_1} X_{1i}\right)^2}{n_1}}{n_1 - 1} \quad . \tag{52}$$

(f) Calculate the coefficient of variation of reliability

$$C.V._{\hat{R}} = \frac{\sqrt{\hat{\sigma}^2_2 + \hat{\sigma}^2_1}}{\overline{X}_2} \quad . \tag{53}$$

Note: $\overline{X}_1$ does not appear in the formula of $C.V._{\hat{R}}$ since the variance of $X_{2i} - X_{1i}$ is being compared to the average strength in order to determine the decimal ratio of $C.V._R = \frac{\hat{\sigma}_R}{\hat{R}}$ which can be used with the ratio of $\frac{\overline{X}_2}{\overline{X}_1}$ to provide the probability that $X_{2i} \geq X_{1i}$.

(g) Refer to Figure 5 ; find the intersection of $\overline{F}$ with the appropriate $C.V._{\hat{R}}$ line and read the reliability directly from the chart (c.f. page 66 for Figure 5).

Example

A pressure vessel is to be installed in a line that will have an average pressure of 1100 PSIG with a variance of 425. Destructive tests are performed on a number of these items, the data result in an average burst pressure of 1175 PSIG and a variance of 800; what is the reliability of the pressure vessel in this application?

1. Average strength; $\overline{X}_1$ = 1175 PSIG

2. Average load; $\overline{X}_2$ = 1100 PSIG

3. $\overline{F}$ the ratio of average strength to average load is

$$\frac{1175 \text{ PSIG}}{1100 \text{ PSIG}} = 1.068$$

4. Variance of strength $S^2_{\overline{X}_1}$ = 800

5. Variance of load $S^2_{\overline{X}_2}$ = 425

6. $C.V._{\hat{R}} = \sqrt{\frac{1225}{1175}} = \frac{35}{1175} = .03$

7. The reliability of the .03 and 1.07 point on the chart is .98 percent.

John Lupo[17] presented a method that provided an estimate of the safety margin as follows:

Let

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

be the average of strength resulting from testing n sample items where X_i is the strength of the ith sample item tested. Also, let the maximum of applied stress be R_b, the reliability boundary, and

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n - 1}},$$

the standard deviation of strength, calculated from the sample items tested. Then, the safety margin, S_m, is calculated by the relationship of the distance between $\hat{R}_b$ and $\bar{X}$ divided by the estimate of $\hat{\sigma}$ that resulted from testing the n sample items. That is,

$$\hat{S}_m = \frac{(\bar{X} - R_b)}{\hat{\sigma}} \quad . \qquad (54)$$

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[17] John E. Lupo, "Safety Margin Confidence Limits - the Non-central 't' Distribution," Evaluation Engineering, Chicago (January/February, 1966) p. 51.

It was then necessary to determine if $t = \sqrt{n\ S_m}$ was a noncentral t-distribution with n-1 degrees of freedom and had a noncentral parameter, $\delta = \sqrt{n\ S_m}$. The proof was completed and the equations from the proof were used to calculate confidence limits, γ_i, on estimated S_m for various sample sizes (Table XII-XV).

Example:

If the required safety margin is 3 and the sample size is 5, what must the estimated safety margin be to demonstrate the required safety margin at a confidence level of 80 percent?

This is solved by referring to the table for 80 percent confidence (Table XIII) and finding the number 3.0 in the S_m row and 5 in the sample column; the value in the 80 percent table where $S_m = 3.0$ and $n = 5$ intersect is 4.74 which is the safety margin that must be measured in order to demonstrate that the true safety margin is equal to or greater than 3.0.

TABLE XII - 70 Percent Confidence Level of Safety Margin

	Sample Size n						
Smr	5	10	15	20	25	30	40
0.0	0.2545	0.1719	0.1386	0.1193	0.1063	0.0968	0.0836
1.0	1.4510	1.2651	1.2033	1.1703	1.1491	1.1340	1.1137
2.0	2.7532	2.4181	2.3134	2.2592	2.2250	2.2010	2.1692
3.0	4.0843	3.5903	3.4389	3.3612	3.3125	3.2786	3.2337
4.0	5.4254	4.7694	4.5700	4.4681	4.4044	4.3602	4.3017
5.0	6.7700	5.9513	5.7034	5.5770	5.4982	5.4435	5.3712

TABLE XIII - 80 Percent Confidence Level of Safety Margin

	Sample Size n						
Smr	5	10	15	20	25	30	40
0.0	0.4204	0.2793	0.2242	0.1926	0.1714	0.1560	0.1346
1.0	1.7382	1.4268	1.3261	1.2728	1.2387	1.2145	1.1820
2.0	3.2155	2.6654	2.4978	2.4115	2.3573	2.3194	2.2689
3.0	4.7405	3.9345	3.6938	3.5711	3.4944	3.4411	3.3703
4.0	6.2808	5.2149	4.8990	4.7385	4.6385	4.5691	4.4772
5.0	7.8276	6.5002	6.1082	5.9095	5.7859	5.7001	5.5867

TABLE XIV - 90 Percent Confidence Level of Safety Margin

Smr	Sample Size n 5	10	15	20	25	30	40
0.0	0.6857	0.4373	0.3473	0.2970	0.2637	0.2395	0.2061
1.0	2.2476	1.6844	1.5150	1.4276	1.3724	1.3337	1.2820
2.0	4.0580	3.0678	2.7868	2.6456	2.5580	2.4973	2.4170
3.0	5.9432	4.4983	4.0957	3.8952	3.7715	3.6862	3.5739
4.0	7.8526	5.9464	5.4187	5.1570	4.9959	4.8850	4.7393
5.0	9.7734	7.4026	6.7484	6.4245	6.2254	6.0884	5.9086

TABLE XV - 95 Percent Confidence Level of Safety Margin

Smr	Sample Size n 5	10	15	20	25	30	40
0.0	0.9539	0.5797	0.4548	0.3867	0.3423	0.3103	0.2664
1.0	2.8130	1.9343	1.6906	1.5685	1.4925	1.4398	1.3700
2.0	5.0068	3.4655	3.0601	2.8620	2.7407	2.6577	2.5488
3.0	7.2994	5.0584	4.4777	4.1963	4.0251	3.9082	3.7558
4.0	9.6280	6.6746	5.9139	5.5465	5.3235	5.1716	4.9737
5.0	11.9730	8.3017	7.3588	6.9042	6.6286	6.4410	6.1968

Proof:

Let Z be a random variable distributed normally about zero with unit standard deviation, and let W be a random variable distributed independently of Z as χ^2/f with f degrees of freedom. If t is defined by

$$t = \frac{(Z + \delta)}{\sqrt{W}}$$ where δ is some constant, then t

is said to have the noncentral t distribution with f degrees of freedom and noncentrality parameter δ.

The estimate of the safety margin can be related to t as follows:

$$\hat{S}_m = \frac{\overline{X} - R_b}{\sigma} \bigg/ \left(\frac{\hat{\sigma}}{\sigma}\right) \tag{55}$$

where

$\overline{X}$ = estimated mean of strengths = $\Sigma X_i/n$,

where X_i is observed strengths,

R_b = reliability boundary = maximum stress (known),

$\hat{\sigma}$ = estimated standard deviation of the strengths

= $[\Sigma(X_i - \overline{X})^2/(n-1)]^{\frac{1}{2}}$.

$$\hat{S}_m = \left[\frac{\bar{X} - \mu}{\sigma} + \frac{\mu - R_b}{\sigma}\right] \Bigg/ \left(\frac{\hat{\sigma}}{\sigma}\right) \tag{56}$$

where

μ is the true but unknown mean of the population of strengths,

σ is the true but unknown standard deviation of the population of strengths.

Multiply both sides by $\sqrt{N}$

$$\sqrt{N}\,\hat{S}_m = \left[\sqrt{N}\,\frac{\bar{X} - \mu}{\sigma} + \sqrt{N}\,\frac{\mu - R_b}{\sigma}\right] \Bigg/ \left(\frac{\hat{\sigma}}{\sigma}\right) . \tag{57}$$

Substitute $S_{mr} = (\mu - R_b)/\sigma$, where S_{mr} is the required safety margin.

$$\sqrt{N}\,\hat{S}_m = \left[\sqrt{N}\,\frac{\bar{X} - \mu}{\sigma} + \sqrt{N}\,S_{mr}\right] \Bigg/ \left(\frac{\hat{\sigma}}{\sigma}\right) . \tag{58}$$

The quantity $\sqrt{N}\,(\bar{X} - \mu/\sigma)$ has a normal distribution with mean zero and unit standard deviation, and $(\hat{\sigma}/\sigma)^2$ has a X^2/f distribution with N - 1 degrees of freedom. Therefore $\sqrt{N}\,\hat{S}_m = t$ and has a noncentral distribution with N - 1 degrees of freedom and a noncentrality parameter $\delta = \sqrt{N}\,S_m$.

Two parameter Weibull (estimate the shape parameter)

The two parameter Weibull distribution has a density function

$$f(x) = (\gamma/\theta)\, X^{\gamma-1} \exp\left(-X^{\gamma}/\theta\right) \qquad (59)$$

$x \geq 0$

$\gamma \geq 0$

$\theta \geq 0$

which becomes the one parameter exponential distribution when $\gamma = 1$. Other equivalent forms of the two parameter Weibull have been presented in the literature. Cohen[18] selected this form for the purpose of simplifying the deviations of the maximum likelihood estimating equations.

Censored or complete samples. In a typical life test N items are placed on test, and the behavior of each item is observed. The time that each failure occurs is recorded along with any comments pertinent to the item response during testing.

The testing is stopped at: (a) some predetermined

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[18] A. Clifford Cohen, "Maximum Likelihood Estimation in the Weibull Distribution based on Complete and on Censored Samples" Technometrics, VII, No. 4; November, 1965; p. 579.

time x_o or (b) when a predetermined number of failures x_n have occurred. Data consist of failures $x_1, x_2 \ldots x_n$ after $t_1, t_2 \ldots t_n$ test time exposure, respectively, on each item that failed. Also, there are N-n items that survived the test time of termination, x_o or x_n.

Testing programs that are terminated at a fixed x_o (time) censoring are referred to as type I, and n is a random variable. For testing programs terminated at a predetermined nth failure, then the time of termination x_n is a random variable and censoring is referred to as type II.

For the complete sample consisting of n observations, the likelihood function of (1) is $L(x_1, x_2 \ldots x_n; \gamma,\theta) = \prod_{i=1}^{n} (\gamma/\theta)\, X_i^{\gamma-1}$, $\exp(-X_i^{\gamma}/\theta)$.

The estimating equations are found by taking the log, lnL, differentiating with respect to γ and θ in-turn and equating the results to zero. This gives

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \ln x_i - \frac{1}{\theta} \sum_{1}^{n} X_i^{\gamma} \ln x_i = 0 \qquad (60)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} X_i^{\gamma} \qquad (61)$$

from which θ can be eliminated and upon simplifying becomes

$$\frac{\sum_{i=1}^{n} X_i^{\gamma} \ln X_i}{\sum_{i=1}^{n} X_i^{\gamma}} - \frac{1}{\gamma} = \frac{1}{n} \sum_{i=1}^{n} \ln X_i \tag{62}$$

or

$$\frac{\Sigma X_i^{\gamma} \ln X_i}{\Sigma X_i^{\gamma}} - \frac{1}{n} \sum_{i=1}^{n} \ln X_i = \frac{1}{\gamma} \tag{63}$$

then solve for the MLE of γ; (MLE(γ)). Standard iterative procedures may be used but in most cases a trial and error approach can be used to find the required value of γ.

Once γ is estimated (a $\hat{}$ or a ^{1} such as $\hat{\gamma}$ or γ indicates an estimate of the parameters γ) by using this estimate, $\hat{\gamma}$, in equation (61) and solving for $\hat{\theta}$, that is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} X_i^{\hat{\gamma}}}{n} \tag{64}$$

the likelihood function for simply censored and progressively censored samples.

Estimating the shape parameter γ by using the C.V.

Numerous articles and texts have been addressed to this problem of solving the parameters of the Weibull distribution (shape β, Scale γ, and location α). The shape parameter is the most difficult to estimate and various approximate methods have been proposed. Several references which contain additional information about the Weibull distribution are provided in the technical journals.

Cohen, in his article, suggested using a suitable graph or a table to establish a first approximation to his shape parameter $\hat{\gamma}$.[19] The C.V. of the Weibull [(C.V.)w] is a function of the shape parameter alone.

The kth noncentral moment is determined to be

$$\mu_k' = \theta^{k/\gamma}\Gamma[(k/\gamma) + 1] \tag{65}$$

where

$$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx \quad m > 0$$

$$\mu_1 = \Gamma[(1/\gamma) + 1] \tag{66}$$

$$V(X) = \Gamma[(2/\gamma) + 1] - \Gamma^2 (1/\gamma) + 1] \tag{67}$$

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19 Cohen, p. 579

that is,

$$V(X) = \mu_2' - (\mu_1')^2$$

and

$$\sqrt{\frac{[\Gamma\ (4\gamma) + 1] - \Gamma^2[(^1/\gamma) + 1]}{\Gamma[(^1/\gamma) + 1\ ]}} \; . \tag{68}$$

Tables of the Γ distribution can be found in most engineering handbooks or tables for mathematicians. Table XVI provides values of $C.V._w$ for specific values of γ (shape).

TABLE XVI

The Weibull C.V. as a Function of the Shape Parameter γ

γ	$C.V._w$	γ	$C.V._w$
.25	8.3066	2.5	.4279
1/3	4.3589	3.0	.3633
.50	2.2361	3.5	.3189
.75	1.3612	4.0	.2806
1.00	1.0000	4.5	.2498
1.25	.8050	5.0	.2290
1.5	.6754		
1.75	.5884		
2.0			

Figure 4 is a plotting of these values.

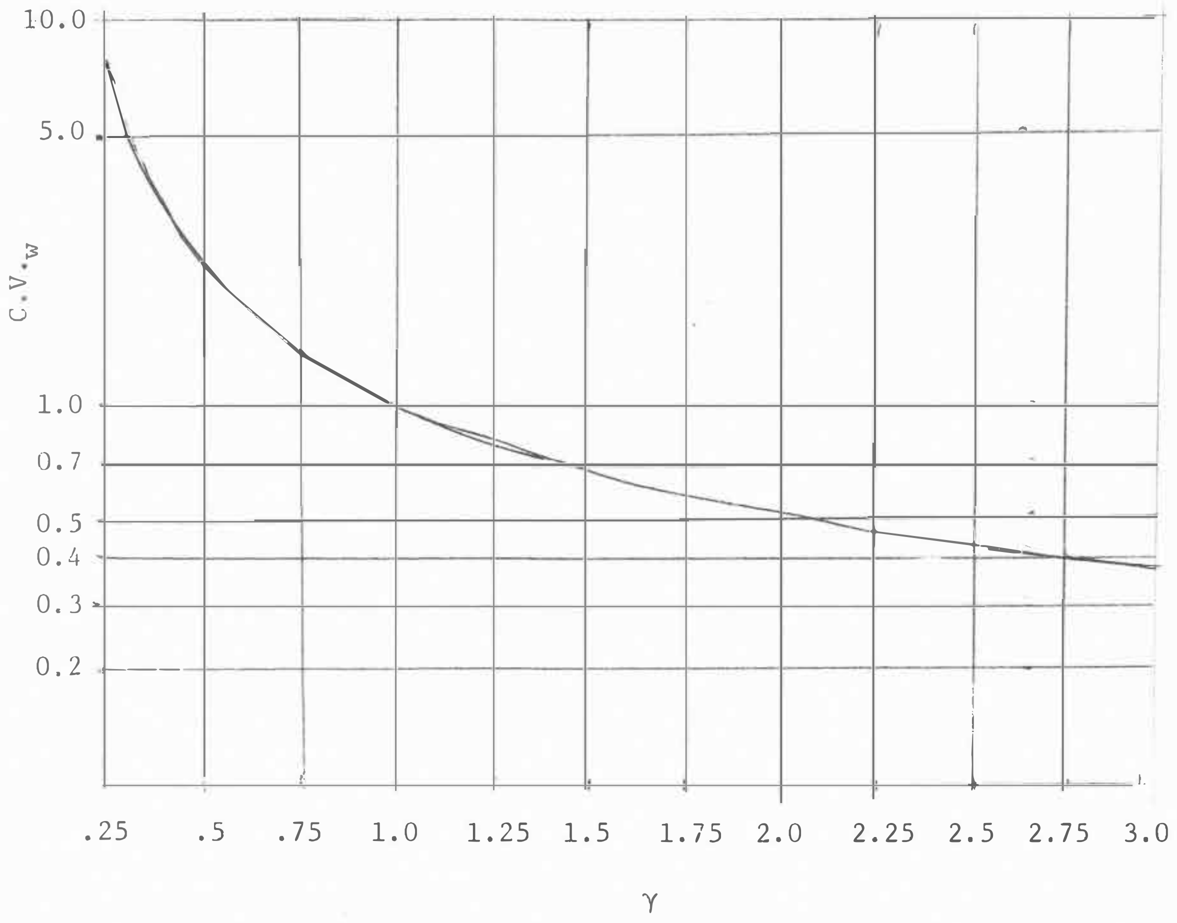

Figure 4. Coefficient of Variation for the Weibull Distribution for Values of the Shape Parameter γ

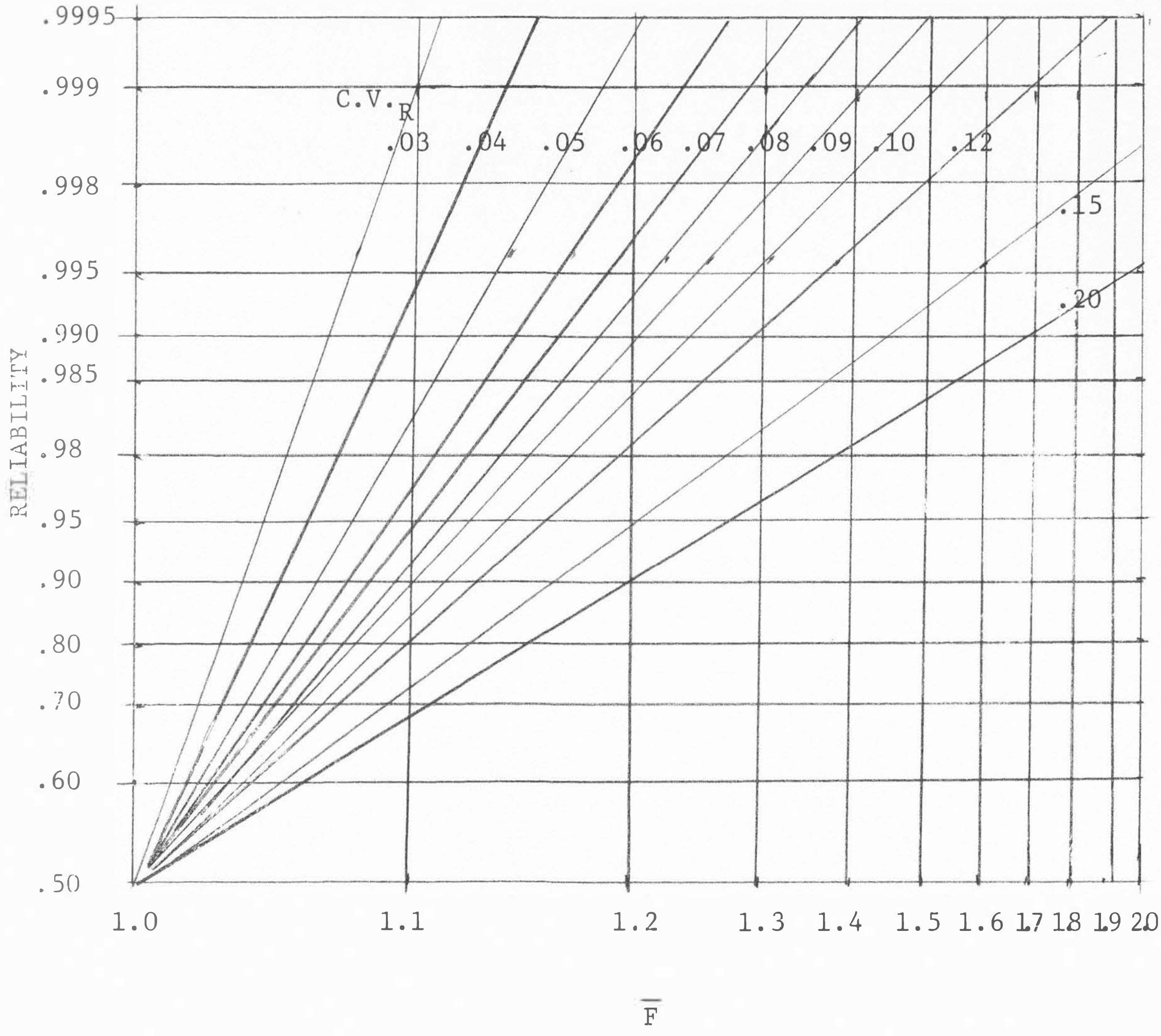

Figure 5. Reliability as a Function of $\overline{F}$ and $C.V._R$

SUMMARY AND CONCLUSIONS

The intent of this thesis was to determine the statistical properties of the coefficient of variation, C.V., and to make use of these properties in statistical procedures for solving Quality Control and Reliability problems. The mean and the standard deviation of the C.V. were found, in the statistical literature, and these can be used in the same manner as the properties of the normal distribution when the sample size is large.

The C.V. statistical properties covered in this report are approximate methods (two) to determine the cumulative probability distribution, using the noncentral t-distribution to determine the cumulative probability distribution; comparison of C.V.'s from two samples, approximate confidence limits for the C.V., the C.V.'s upper and lower bounds. These properties are applied to compare experimental results, acceptance sampling, solve stress versus strength problems in reliability and to estimate the shape parameter of the two parameter Weibull.

Using the noncentral t-distribution is the best of the three methods to determine the probability function of the sample C.V. However, it is much more complicated and requires more calculations than either of the other two methods. These methods are used more often due to the reduced number of calculations required.

Evaluation procedures using the C.V. have been developed to provide the experiementer with more precise estimates. Also, the experiementer can determine if the difference between the C.V. of his testing is due to chance variation or if the difference observed is nonrandom. In addition, the C.V. is useful for solving specific problems encountered in Reliability and Quality Control.

LITERATURE CITED

Baumeister, Theodore. Standard Handbook for Mechanical Engineers 7th ed. New York: McGraw-Hill Book Company, Inc., 1967.

Cohen, A. Clifford. "Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples." Technometrics, VII, No. 4 (November, 1965), 579.

Crum, W. L. and A. C. Patton. Economic Statistics. 1st Ed. New York: A. W. Shaw Company, 1925.

Faulkner, J. Earl, Associate Professor of Statistics, Brigham Young University. "The Comparison of Coefficients of Variation for Normal Random Variables." (Paper presented at the 10th Western Regional Meeting of American Statistical Association, Salt Lake City, Utah) May 16, 1969.

Ferber, Robert. Statistical Techniques in Marketing Research. 1st ed. New York: McGraw-Hill Book Company, Inc., 1949

Freeman, H. A. Industrial Statistics. 1st ed. New York: John Wiley & Sons, Inc., 1942.

Hald, A. Statistical Theory with Engineering Applications. New York: John Wiley & Sons, Inc., 1962.

Johnson, N. L. and B. L. Welch. "Applications of the Noncentral t-Distribution." Biometrika, XXXI, (September 1940).

Kendall, M. G. and A. Stuart. The Advanced Theory of Statistics. 6th ed. Vol. I. New York: Hafner Publishing Company, 1952.

Kent, Fredrick C. Elements of Statistics. 1st ed. New York: McGraw-Hill Book Company, Inc., 1924.

Lloyd, D. K. and M. Lipow. Reliability: Management, Methods, and Mathematics. 1st ed. New Jersey: Prentice-Hall, Inc., 1962.

Lupo, John. "Safety Margin Confidence Limits-the-Noncentral 't' Distribution." Evaluation Engineering, Chicago (January/February, 1966).

Mood, A. M. and F. A. Graybill. Introduction to the Theory of Statistics. 2nd ed. New York: McGraw-Hill Book Company, Inc., 1963.

Moriguti, Sigeiti. "Annals of Mathematical Statistics," _Journal,_ XXII, No. 4 (December, 1951)pp. 523-528.

Snedecor, George W. and William G. Cochran. _Statistical Methods._ 6th ed. Ames, Iowa: The Iowa University Press, 1967.

Yamone, Taro. _Statistics, An Introductory Analysis._ 1st ed. New York: Harper and Rowe Publishers, 1964.

A P P E N D I X

The probability density functions, the mean, variance, and coefficient of variation of the discrete probability functions that are very often used in Reliability and Quality Control are given in the first section of this Appendix. The second section provides the same information for the continuous probability functions that are also very often used in Reliability and Quality Control.

I. Discrete Probability Density Functions

Hypergeometric Distribution

If the total set contains N items and D of these items possess a given property, then the probability that a random sample of size n, without replacement, will contain exactly x items that possess the given property is

$$f(x:N,D,n) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$

$$= \frac{D!}{x!\,(D-x)!} \cdot \frac{(N-D)!}{(n-x)!\,(N-D-x+n)!} \cdot \frac{n!\,(N-n)!}{N!}$$

$$x = 0,1,2 \ldots n$$

$$N \geq n > 0$$

$$N - D \geq 0$$

$$= 0 \quad \text{elswhere} \ .$$

Note: $\binom{A}{B}$ is the combination of A items taken B at a time.

Three properties of the Hypergeometric distribution are:

Mean . . . . . . $\mu = \frac{nD}{N}$

Variance . . . . $\sigma^2 = \frac{nD}{N} \left(1 - \frac{D}{N}\right) \left(\frac{N-n}{N-1}\right)$

Coefficient of Variation $= \sqrt{\frac{N-D}{nD} \left(\frac{N-n}{N-1}\right)}$

Binomial Distribution

If θ denotes the probability of an event occurring at each of n observations, then the probability that the event will occur exactly x times is

$$f(x;n,\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

Three properties of the Binomial distribution are:

Mean . . . . . . $\mu = n\theta$

Variance . . . . $\sigma^2 = n\theta (1-\theta)$

Coefficient of Variation $= \sqrt{(1-\theta)/_{n\theta}}$

Geometric Distribution

Let θ denote the probability of the event occurring at each trial. Consider repeating each observation until the event occurs for the first time. The probability that x trials must be made is given by the geometric distribution

$$f(x,\theta) = \theta(1-\theta)^{x-1} \qquad x = 1, 2, \ldots$$

$$0 \leq \theta \leq 1$$

$$= 0 \qquad \text{elsewhere}$$

$$F(x) = \sum_{i=1}^{x} \theta(1-\theta)^{i-1} = 1 - (1-\theta)^{x}$$

Some properties of the Geometric distribution are:

Mean . . . . . . $\mu = 1/\theta$

Variance . . . . $\sigma^2 = (1-\theta)/\theta^2$

Coefficient of Variation $= \sqrt{1-\theta}$

Poisson Distribution

The Poisson distribution is a useful approximation to the binomial and hypergeometric distributions and also one in which arises when the number of possible events is large but the probability of occurrence over a given area or interval is small, e.g., defects, waiting lines.

$$f(x,\lambda) = \lambda^x e^{-\lambda}/x! \qquad x = 0, 1, 2, \ldots$$
$$= 0 \qquad \lambda > 0$$
$$\text{elsewhere}$$

$$F(x) = \sum_{i=0}^{x} \lambda^i e^{-\lambda}/i!\ .$$

Some properties of the Poisson distribution are:

Mean . . . . $\mu = \lambda$

Variance . . . . $\sigma^2 = \lambda$

Coefficient of Variation $= 1/\sqrt{\lambda}$

<u>Continuous Probability Density Functions</u>

Uniform Distribution. Uniform distribution is defined by the function

$$f(x) = 1/(b-a) \qquad a < x < b$$
$$= 0 \qquad \text{elsewhere}$$

$$F(x) = 0 \qquad x \leq a$$
$$= (x-a)/(b-a) \qquad a < x < b$$
$$= 1 \qquad x \geq b$$

Properties of the Uniform distribution are:

Mean . . . . . . $\mu = (b+a)/2$

Variance . . . . $\sigma^2 = (b-a)^2/12$

Coefficient of Variation $= (1/\sqrt{3})\ [(b-a)/(b+a)]$

Gamma Distribution

The gamma distribution is defined by the two-parameter function

$$f(x:\alpha,\beta) = (1/\alpha!\beta^{\alpha+1})x^{\alpha}e^{-x/\beta} \qquad x > 0$$
$$= 0 \qquad x \leq 0$$

where the scale parameter $\beta > 0$ and the shape parameter $\alpha > -1$.

$$F(x) = \int_0^x (1/\alpha!\beta^{\alpha+1})x^{\alpha}e^{-x/\beta}dx$$
$$= (1/\alpha!)\ \Gamma_{x/\beta}(\alpha+1)$$

where $\Gamma_{x/\beta}(\alpha+1)$ is the incomplete gamma function tabulated in Karl Pearson, Tables of the Incomplete Gamma Function, Cambridge University Press, London, 1922.

Some properties of the Gamma distribution are:

Mean . . . . . . $\mu = \beta(\alpha+1)$

Variance . . . . $\sigma^2 = \beta^2(\alpha+1)$

Coefficient of Variation $= 1/\sqrt{\alpha+1}$

Exponential Distribution

One of the most widely used distributions in the field of reliability is the one-parameter exponential function defined by

$$f(x:\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad x \geq 0$$

$$= 0 \qquad \text{elsewhere}$$

$$F(x) = \frac{1}{\theta} \int_0^2 e^{-x/\theta} dx = 1-e^{-x/\theta} \qquad x > 0$$

$$= 1 \qquad x \leq 0$$

Some properties of the Exponential distribution are:

Mean . . . . . . $\mu = \theta$

Variance . . . . $\sigma^2 = \theta^2$

Coefficient of Variation $= 1$