STATISTICAL ANALYSIS
FOR TOLERANCES OF NOXIOUS WEED SEEDS

by

Yadolah Dodge

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of
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in
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ABSTRACT

Statistical Analysis
for the Tolerances of Noxious Weed Seeds

by

Yadolah Dodge, Master of Science
Utah State University, 1971

Major Professor: Dr. Ronald V. Canfield
Department: Applied Statistics

An analysis of the previous method for testing tolerances of noxious weed seeds was performed. Problems of the current techniques were discussed, and the solution to these problems was given.

A new technique of testing through the sequential test ratio was developed, and results examined.

The sequential test was found to be useful enough to include the use of it in determining tolerances for noxious weed seeds.

This study did show that the use of sequential tests does have excellent potential and flexibility as a statistical tool for the tolerances of noxious weed seeds.

(75 pages)
CHAPTER I

INTRODUCTION

The technique of analysis of tolerances for noxious weed seeds was originated by G. N. Collins (1929). In his study, Collins used a normal approximation of the poisson distribution for large values of the mean. M. T. Mann and C. W. Leggatt (1932) brought out the fact that the present tolerances of the Association of Official Seed Analysts in Rules of Seed Testing appear to have no scientific basis. Leggatt (1939) used a Chi-square technique for the tolerances of noxious weed seeds. Then late in 1946, Leggatt very briefly mentioned the use of a sequential test and stated that it was something which should be kept in mind by seed analysts as a possible solution to the pressure of work, size of test, and the relation of these to the accuracy of the results obtained in seed analysis. (Unfortunately, since 1946, few efforts using this idea have been made to find a solution.)

In 1947 tables of tolerances for noxious weed seeds were presented at the thirty-seventh annual meeting of The Association of Official Seed Analysts. These tables were based on the poisson distribution (Rules of Seed Testing). S. R. Miles and L. C. Shenberger in 1954 used a special sampling technique to improve the problem of tolerances, and later Miles and Shenberger (1955) discussed the variation in seed analysis from bag to bag. Miles, Shenberger, and A. S. Carter (1958) gave some comments and discussed the tolerances
for chaffy and non-chaffy seeds in relation to sampling and other factors that may occur during operations. They also gave some useful tables and formulas not related to the tolerances of noxious weed seeds. L. C. Shenberger in 1962 discussed the variation in noxious weed seed number using the Chi-square method to find out whether the number of weed seeds found corresponded satisfactorily with the number expected from statistical theory. In 1965 the same table for tolerances of noxious weed seeds as was given in 1947 was again presented in Rules for Seed Testing, based on the poisson distribution.

Don Niffenegger, assistant director of the Biometrical Services Staff, wrote in a letter to Dr. Rex L. Hurst and dated February 22, 1971, that, "As you will see from the enclosed copy of my letter to M. Hanford Day, I've been wanting to locate someone with the ability and interest to carry on some work that already has a start." Enclosed was the letter and a copy of Seed Technology Research in North America, volume 41, November 3, 1967, which stated that there is no current research on the tolerances of noxious weed seeds.

In reply to this letter, Ronald V. Canfield, in determining the problems that may arise after finding the new approach to this problem, stated (May 27, 1971) that,

"It appears that this table is based on a two sided test of the hypothesis that a lot of seed is correctly labeled. . . . The main subject that I would like to consider concerns the philosophy of the testing programs. The philosophy I speak of is one which is implied by the statistical procedures used. In most cases when a statistical test is
developed, standard techniques are used and a 'philosophy' is seldom used."

In light of the already mentioned points, this study concentrates basically on three questions:

1. Can previous studies on the tolerances of noxious weed seeds be valid?

2. Can a new approach be found to the solution of the tolerances of noxious weed seeds?

3. What problems, if any, may be associated with the new approach?

The first part of the study will give the mathematical formulation of poisson distribution. This chapter will also indicate the relation of the poisson distribution to the distribution of noxious weed seeds in a bag.

Chapter III contains a discussion of the first question concerning the feasibility of using the poisson distribution to the tolerances of noxious weed seeds. To examine this problem a sample problem is analyzed using the method that has been given by the Rules Commission for testing seeds. A discussion of the shortcomings of the present technique of testing is also contained in Chapter III. These shortcomings are associated both with inaccuracies in the table and with problems involving producer and consumer risks. A new table is presented which corrects the inaccuracies of the old one.

Chapter IV contains the mathematical formulation of the new approach using sequential techniques in the solution of the problem.
Sequential analysis appears to substantially reduce the required sample size in seed testing. Chapter V contains the application of sequential analysis to the tolerances of noxious weed seeds.

Chapter VI, the final chapter, is a summary of the study. In this chapter areas of further research are suggested for the consideration of the reader.
CHAPTER II

MATHEMATICAL FORMULATION

2.1 The binomial distribution

The binomial distribution was first originated by Jacob Bernoulli in 1710 and by Abraham DeMoivre in 1718. Bernoulli's first paper on the subject appeared in 1713, after his death. He defines the binomial distribution as the number of successes in a sequence of success-failure experiments, or

\[ b(x, n, p) = \binom{n}{x} p^x (1-p)^{n-x} \]

where \( p \) is the probability of success.

Consider an experiment of the repetitive type in which we are only interested in the number of occurrences or nonoccurrences of an event. Suppose the probability that the event A occurs when the experiment performed is \( p \).

\[ P(A) = p \]

Now suppose that the experiment is performed \( n \) times, and these are "independent" repetitions. (That is, the result of any one trial does not influence, nor is it influenced by, the result of any other trial.) Each experiment produces success or failure. Let \( q = 1-p \) denote the probability that event A fails to occur. Then consider the random variable

\[ X = \text{number of times that event A occurs} \]

and assume that the experiment is performed only once. Then \( X \) may
have the value 0 or 1, depending on whether or not A occurs. The question is: What is the probability distribution of the number of successes? Another way to represent the problem is:

If $X_1, X_2, \ldots, X_n$ are $n$ independent indicator random variables with

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases} \quad (0 \leq p \leq 1)$$

Then the distribution of $X = X_1 + X_2 + \ldots + X_n$ is

$$p(X = x) = \binom{n}{x} p^x q^{n-x} \quad x = 0, 1, \ldots, n$$  \hspace{1cm} 2.1.2

To derive the formula consider the event $X = x$ which means that in $x$ of $n$ trials $A$ occurs and in $n-x$ trials $A$ does not occur. $n$ events are independent; therefore, if $P(A) = p$, and $P(A$ does not occur) equals $q = 1-p$. Then:

$$p^x q^{n-x} = p^x (1-p)^{n-x}$$  \hspace{1cm} 2.1.3

However, this is only one order of arranging $x$ $p$'s and $n-x$ $q$'s. The number of orders is the number of all possible outcomes (combinations) with $n$ letters of which $x$ are alike. The number of such combinations then is

$$\binom{n}{x}$$  \hspace{1cm} 2.1.4

Multiplying 2.1.4 by 2.1.3 results in

$$\binom{n}{x} p^x q^{n-x}$$
which is the probability of obtaining \( x \) successes (occurrences of event A) in \( n \) independent trials.

Or

\[
f(x) = \frac{n!}{x! \cdot (n-x)!} \cdot p^x \cdot q^{n-x} \quad 2.1.5
\]

The name binomial distribution comes from the fact that terms in the distribution function are in the binomial expansion.

\[
(p+q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x \cdot q^{n-x} = q^n + \binom{n}{1} p q^{n-1} + \ldots + p^n \quad 2.1.6
\]

The distribution function is \( F(k) = 0 \) if \( k < 0 \) and

\[
F(k) = \sum_{x=0}^{k} \binom{n}{x} p^x \cdot q^{n-x} \quad \text{when } x \geq 0 \quad 2.1.7
\]

The binomial distribution is a good mathematical model that can be used in many problems of real life, but in any application it is necessary to estimate the parameters \( x \) and \( P \).

Suppose a bag contains \( n \) balls, of which \( A \) are red and \( B \) are blue. Let the random variable \( X \) be the number of red balls drawn without replacement in a sample of size \( n \). Then

\[
P(X=x) = \frac{\binom{A}{x} \binom{B}{n-x}}{\binom{A+B}{n}} \quad 2.1.8
\]

where the numerator gives the number of arrangements of \( n \) balls with \( x \) red ones and the denominator shows the total number of arrangements of \( n \) balls from a population of \( N = A + B \) balls. This distribution is known as the hypergeometric distribution. If the size of the population is large compared to the sample size \( n \), then the
probabilities of success for sampling without replacement have negligible change from one sample to the succeeding sample. For example, suppose $n = 10^6$ and $a = 10^3$. Then the probability of a red ball on the second draw, given a red one on the first draw, is

$$\frac{10^3 - 1}{10^6 - 1}$$

which is close to $10^{-3}$. Thus, in this case, sampling without replacement differs insignificantly from sampling with replacement. The latter is seen to be the binomial case.

2.2 Mean and variance of the binomial distribution

The mean of the binomial or any discrete random variable $X$ is defined by

$$E(X) = \sum_{i=1}^{k} x_i f(x_i)$$

Apply 2.1.2 to 2.2.1.

$$E(X) = \sum_{x=0}^{n} x \frac{n!}{x! (n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x! (n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)! (n-x)!} p^x q^{n-x}$$

If $n$ and $p$ are factored out, this becomes

$$E(X) = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)! (n-x)!} p^x q^{n-x}$$

Let $y = x - 1$. Then
\[ E(X) = np \sum_{y=0}^{n-1} \frac{(n-1)!}{(n-1-y)!} p^y q^{n-1-y} \]

By 2.1.2 the quantity being summed is the binomial probability of success in \( n-1 \) trials. Since the sum is over all possible values of \( y \), the sum must be equal to one, then

\[ \text{mean} = E(X) = \mu = np \]

To derive the variance, consider

\[ (E(X) - \mu)^2 = E(X^2 - 2\mu X + \mu^2) \]

\[ = E(X^2) - 2\mu EX + \mu^2 \]

\[ = E(X^2) - \mu^2 \]

Now

\[ E(X)^2 = \sum_{x=0}^{n} x^2 \frac{n!}{x! (n-x)!} p^x q^{n-x} \]

\[ = \sum_{x=0}^{n} [x(x-1) + x] \frac{n!}{x! (n-x)!} \]

\[ = \sum_{x=0}^{n} x(x-1) \frac{n!}{x! (n-x)!} p^x q^{n-x} + \mu \]

Since the terms for \( x = 0 \) and \( x = 1 \) are equal to 0 because of the factor \( x(x-1) \), the summation can begin with \( x = 2 \); hence

\[ \mu_2 = \sum_{x=2}^{n} x(x-1) \frac{n!}{x! (n-x)!} p^x q^{n-x} + \mu \]

\[ = \sum_{x=2}^{n} \frac{n!}{(x-2)! (n-x)!} p^{x-2} q^{n-x} + \mu \]

If \( n(n-1)p^2 \) is factored out, this becomes
\[ \mu_2' = n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)! (n-x)!} p^x q^{n-2-x} + \mu \]

Letting \( z = x-2 \), the right side can be written as

\[ \mu_2' = n(n-1)p^2 \sum_{z=0}^{n-2} \frac{(n-2)!}{z! (n-2-z)!} p^z q^{n-2-z} + \mu \]

The quantity being summed is the probability of \( z \) successes in \( n-2 \) trials. Since the sum is over all possible values of \( z \), its value must be one. Using this result and the earlier result that \( \mu = np \), 2.2.4 reduces to

\[ \mu_2' = n(n-1)p^2 + np \]

If formula 2.2.3 is applied to the results just obtained for the binomial distribution,

\[ \mu_2 = n(n-1)p^2 + np - n^2 p^2 \]

\[ = -np^2 + np \]

\[ = npq \]

2.3 Poisson distribution

The distribution with probability function

\[ f(x) = \frac{\mu^x}{x!} e^{-\mu} \quad (x = 0, 1, 2, \ldots) \]

is called poisson distribution, named after S. D. Poisson who introduced it in 1837. The parameter \( \mu \) is the mean of the distribution. From 2.3.1 we see that the distribution function of the poisson distribution is

\[ F(k) = e^{-\mu} \sum_{x=k}^{\infty} \frac{\mu^x}{x!} \quad \text{when } x \geq 0 \text{ and } F(x) = 0 \text{ if } x < 0. \]
The poisson distribution has important applications. In fact, this distribution is a convenient approximation of the binomial distribution in cases of a large number \( n \) of trials and a small probability \( p \) of success in a single trial.

2.4 **Mean and variance of poisson distribution**

Consider the poisson distribution with parameter \( \mu \).

\[
f(x) = \frac{e^{-\mu} \mu^x}{x!} \quad x = 0, 1, \ldots
\]

Apply this to 2.2.1. Then

\[
E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\mu} \mu^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\mu} \mu^x}{(x-1)!}
\]

If \( \mu \) factors out,

\[
\sum_{x=1}^{n} \frac{e^{-\mu} \mu^x}{(x-1)!}
\]

Let \( (x-1) = y \). Then

\[
E(X) = \mu \left( \sum_{y=0}^{\infty} \frac{e^{-\mu} \mu^y}{y!} \right) = \mu
\]

Since the result is the same over all \( y \), the quantity in parenthesis must be equal to one. Therefore,

\[
\text{mean} = E(X) = \mu
\]

The variance of the poisson distribution is calculated in a similar manner and is equal to \( \mu \).

Consider what happens to the binomial density function when \( n \) becomes infinite and \( p \) approaches zero in such a manner that the mean \( \mu = np \) remains fixed.
Rewrite 2.1.1

\[ f(x) = \frac{n(n-1)\ldots(n-x+1)}{x!} \cdot p^x (1-p)^{n-x} \]

If the numerator and denominator are multiplied by \( n^x \) and the indicated algebraic manipulations are performed,

\[ f(x) = \frac{n(n-1)\ldots(n-x+1)}{n^x x!} \cdot \frac{\mu^x}{x!} \cdot (1-p)^{n-x} \]

\[ = (1 - \frac{1}{n}) (1 - \frac{2}{n}) \ldots (1 - \frac{x-1}{n}) \cdot \frac{\mu^x}{x!} \cdot (1-p)^{n-x} \]

\[ = \frac{(1 - \frac{1}{n}) (1 - \frac{2}{n}) \ldots (1 - \frac{x-1}{n})}{(1-p)^x} \cdot \frac{\mu^x}{x!} \cdot (1-p)^n \]

Next, express \((1-p)^n\) in the form

\[ (1-p)^n = [(1-p) - \frac{1}{p}]^{-np} = [(1-p) - \frac{1}{p}]^{-\mu} \]

Now, from the definition of e,

\[ \lim_{z \to 0} (1+z)^{\frac{1}{z}} = e \]

Hence, letting \( z = -p \),

\[ \lim_{p \to 0} [(1-p) - \frac{1}{p}]^{-\mu} = e^{-\mu} \]

Furthermore,

\[ \lim_{n \to \infty} \frac{(1 - \frac{1}{n}) (1 - \frac{2}{n}) \ldots (1 - \frac{x-1}{n})}{(1-p)^x} = 1 \]

because \( p \to 0 \) as \( n \to \infty \) when \( np = \mu \) is fixed. By applying these two results to the right side of 2.1.1, it will be seen that
The mathematical formulation in this study follows the procedures suggested by Paul G. Hoel (1971). (In fact, the proof can be found in most mathematical probability books.)

If, in the binomial distribution, the probability of success is very small, such that the assurance of success is rare but the number of trials $n$ is large, the computations, involving the use of the binomial formula, becomes laborious. Therefore, a good approximation of the binomial distribution is the poisson distribution. This is why the poisson distribution is often used to approximate the binomial distribution for $n$ large and $p$ small.

Figures 1 and 2 are given to indicate how rapidly the binomial distribution approaches the poisson distribution.

![Figure 1. Binomial (—) and poisson(---) distribution for $\mu = 4$, $p = 1/24$.](image-url)
Figure 2. Binomial (—) and poisson (---) distribution for \( \mu = 4 \) and \( p = 1/3 \).

The broken lines represent the fixed poisson distribution for \( \mu \) chosen equal to 4, and the solid lines represent the binomial distribution for \( p = 1/3 \) and \( p = 1/24 \), in Figures 1 and 2 respectively. It would appear from inspecting these graphs that the poisson approximation should be sufficient for most applications if \( n > 100 \) and \( p < .05 \).

Poisson distribution has been introduced by means of an approximating property, for the binomial distribution is a very useful model for treating certain types of problems related to binomial distribution.

The distribution of the number of noxious weed seeds in a sample of \( n \) seeds from a population of \( N \) seeds is seed to follow the hypergeometric distribution since sampling is without replacement.
However, since the population size \( N \) is very large compared to \( n \), the sample size in binomial distribution provides a very good approximation to this distribution. Again, since the number of seeds sampled (\( n \)) is also a large quantity in general and the probability of finding a noxious seed is small, the poisson approximation to the binomial is valid.

**EXAMPLE:** Consider the problem of finding 5 or less noxious weed seeds in a sample which contains 200 seeds. The bag is labeled to say that 2\% of the seeds are noxious weed seeds. Then

\[
\mu = np = 200 \times 0.02 = 4
\]

by using

\[
p(X \leq 5) = \sum_{x=0}^{5} \frac{e^{-4} \cdot 4^x}{x!}
\]

\[
= e^{-4} \left(1 + 4 + \frac{4^2}{2} + \frac{4^3}{6} + \frac{4^4}{24} + \frac{4^5}{120}\right) = 0.785
\]

Thus the probability of finding 5 or fewer noxious seeds is 0.785.
CHAPTER III

CURRENT TECHNIQUES IN RETROSPECT

In 1947 the Association of Official Seed Analysts published a table giving the tolerances for the number of noxious weed seeds in seed tests. This table is based upon the poisson distribution and is calculated using the formula

\[ y = x + \sqrt{\frac{3.841}{x}} + 1 \]  \hspace{1cm} 3.1.1

where \( y \) is the maximum number within tolerance in a test. This formula was presented without further description or information. In 1954, 1960, and 1965 the same table was presented again based upon poisson distribution and calculated from the formula

\[ y = x + 1 + 1.96 \sqrt{x} \]  \hspace{1cm} 3.1.2

A tolerance based on a degree of certainty of 5 percent (\( p = .05 \)) will be recognized. Because of the importance of this table in latter work, it is given as Table 1. This table is used to determine whether or not the number of noxious weed seeds found in the weight of seed examined exceeds the maximum number within tolerance.

EXAMPLE: A lot of red clover is labeled to contain 18 dodder per pound. Five dodder were found in 50 grams. The table number is at rate 2 per 50 grams. In Table 1 on the line which has 2 in column \( X \), shows 6 in column \( Y \) as the maximum number within tolerance. The label is considered satisfactory as far as dodder are concerned,
Table 1. Previous tolerances for noxious weed seeds.

<table>
<thead>
<tr>
<th>Number labeled or represented X</th>
<th>Maximum number within tolerance Y</th>
<th>Number labeled or represented X</th>
<th>Maximum number within tolerance Y</th>
<th>Number labeled or represented X</th>
<th>Maximum number within tolerance Y</th>
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</thead>
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<tr>
<td>0</td>
<td>2</td>
<td>43</td>
<td>57</td>
<td>86</td>
<td>105</td>
</tr>
<tr>
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<td>44</td>
<td>58</td>
<td>87</td>
<td>106</td>
</tr>
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<td>2</td>
<td>6</td>
<td>45</td>
<td>59</td>
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because the number found does not exceed the maximum number within tolerances.

This example is taken from the proceedings of the Association of Official Seed Analysts.

Before going into further detail of this example and Table 1 which is related to this example, type I and type II errors will be defined.

Statistical decisions in which there are just two possible actions constitute an important class called hypothesis testing problems. The possible states of nature are called hypotheses about nature. A simple hypothesis, then, is a complete specification of a probability distribution—the distribution of the population on which observations are obtained for inference.

Let two actions be open to the decision marker and be denoted $A$ and $B$. The analysis will be given in terms of the regret function, which has the property that for each state of nature, one action (at least) has zero regret. This action is correct for that state of nature. The set of all states for which $A$ is the correct action is called the acceptance region.

Therefore, if action $A$ is taken as a decision, the hypothesis is said to be accepted (reject hypothesis). The region of the sample space containing the values for which we reject the hypothesis is called the critical region or rejection area.

Now consider a random variable $X$ with unknown parameter $\theta$. If the hypothesis is $\theta = \theta_0$, then related to this hypothesis there exist three types of alternatives:

3.1.3

θ < θ₀

θ > θ₀

θ ≠ θ₀

3.1.5 is a two-sided alternative while 3.1.3 and 3.1.4 are one-sided alternatives.

Now assume that θ is used to index the states of nature and A and B denote the two available actions. Let the losses associated for actions A and B be defined by:

\[
L(θ, A) = \begin{cases} 
0 & \text{if } θ \text{ is in } H₀ \\
β & \text{if } θ \text{ is in } H₁ 
\end{cases}
\]

\[
L(θ, B) = \begin{cases} 
α & \text{if } θ \text{ is in } H₀ \\
0 & \text{if } θ \text{ is in } H₁ 
\end{cases}
\]

Choosing action A when θ is actually in the set \( H₁ \) is an error, and choosing action B when θ is actually in \( H₀ \) is also an error. Taking the first action is called type II error and the second one is called a type I error. In other words, the rejection of a true null hypothesis is called a type I error, and the acceptance of a false null hypothesis is called a type II error. Or:

In type I errors (table 2) the hypothesis is true, but is rejected. α is defined such that

\[ P(\text{accept alternate} \mid \text{Hypothesis is true}) = α \]

or

\[ P(H₁ \mid H) = α \]

In type II errors (table 2) the hypothesis is false, but not rejected. β is defined such that

\[ P(\text{accept hypothesis} \mid \text{Alternate is true}) = β \]

or

\[ P(H \mid H₁) = β \]
The quantity
\[ n = 1 - \beta \]
is called the power of the test, or the probability of avoiding type II errors, and usually depends on \( \theta \), \( \eta(\theta) = 1 - \beta \).

Table 2. Relation of type I and type II errors.

<table>
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<tr>
<th>unknown truth</th>
<th>( \theta = \theta_0 )</th>
<th>( \theta = \theta_1 )</th>
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<td>true decision</td>
<td>type II error</td>
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<td>( \theta = \theta_0 )</td>
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<td>( P = \beta )</td>
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<tr>
<td>type I error</td>
<td>true decision</td>
<td></td>
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<tr>
<td>( \theta = \theta_1 )</td>
<td>( P = \alpha )</td>
<td>( P = 1 - \beta )</td>
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The probability \( \alpha \) is called type I risk or producer's risk, and the corresponding \( \beta \) is called type II risk, or consumer's risk.

Consider the type II error (consumer's risk) when a bag of seeds labeled \( \mu = 2 \) weed seeds per 50 grams has been mislabeled by 300%. That is, the bag came from a seed population which has an average of 6 weed seeds per 50 grams. The probability of a type II error is shown in Table 3 for various sample sizes. This table is computed from Table 1 which was taken from Rules for Testing Seeds.

The consumer risk may be very high, depending on sample size. This risk is even larger if alternatives closer to the published
Consider the relation of type I and type II errors on the same sample problem where the mean number of noxious weed seeds is actually 5/50 grams, and let the probability of type I error (producer's risk) be equal to 0.05. Suppose the hypothesis to be tested is:

\[ H_0: \mu = 2 \text{ noxious weed seeds in 50 grams} \]
\[ H_1: \mu > 2 \text{ noxious weed seeds in 50 grams} \]

where the actual value of \( \mu \) is 250% of the original mean. According to Table 1, the maximum number within the tolerances is equal to 6. Assume that a sample of 50 grams is analyzed for weed seeds. The type II error for this hypothesis and sample size is:

\[ p(\gamma \leq 6 | \mu = 5) = .77 \]

or the consumer risk in accepting the hypothesis while the alternate is true is .77 which is 15.4 times as great as the producer's risk.
In simple words, even if the bag is mislabeled by 250% of its labeled value, the consumer has little chance of detecting the erroneous labeling.

Suppose now the sample size is doubled, i.e., a 100 gram sample is analyzed. Then the appropriate hypothesis is

\[ H_0: \mu = 4 \]
\[ H_1: \mu \neq 4 \]

and the actual value for the population is \( \mu = 10 \). According to Table 1, the maximum number within the tolerances is equal to 9. In this case, the type II error is:

\[ p(y < 9 | \mu = 10) = 0.468 \]

In general, an increase in sample size produces a smaller probability of a type II error.

The consumer has vital interest in the type II error. Thus, he is interested in the sample required to provide him with an acceptable probability of a type II error.

However, the problem is not completely determined by selecting an acceptable probability of type II error because this error depends upon the mean number of noxious seeds in a sample of a fixed size, which is unknown. The consumer may have a continuum of acceptable risk levels. For example, if the bag is mislabeled by only a small amount, he is not seriously affected and thus is willing to take a large risk. However, if the bag is grossly mislabeled, he desires a small risk of a type II error. This inverse relationship between the amount of mislabeling and acceptable risk is a property
provided by the test as given. By selecting a sample size, a risk curve for type II errors is determined which gives small risk for gross mislabeling and vice versa. Thus a family of curves, one for each possible sample size, results. The problem now becomes one of choosing an acceptable curve which then determines the appropriate sample size for the test.

A curve may be determined by choosing the acceptable consumer's risk at a specified value of \( \mu \) within the alternate hypothesis, the choice of significance level \( \alpha = .05 \). It seems that any procedure which considers the type II error should have some guidelines, or better still, some accepted standard, such as the standard of \( \alpha = .05 \) for type I error.

The choice of a standard is beyond the scope of this work. However, the following considerations will be useful in establishing guidelines or a standard.

A meaningful reference level for a test with a fixed sample size is the value of \( \mu \) in the alternate which requires for \( \alpha = \beta = 0.05 \). This may be compared with the value of \( \mu \) hypothesized.

If the producer claims that there are on the average 1 noxious weed seeds per 50 grams, consider the graph of \( \beta \) plotted against \( \mu \) for a fixed sample of 50 grams as shown in Figure 3.

It can be seen from Figure 3 that the number of noxious weed seeds has to be 900% more than the labeled value to have \( \beta = .05 \). In other words, if in a test, a label of 1 noxious weed seed per 50 grams is assumed, and if the sample size of 50 grams (fixed) for a test is used, the true \( \alpha \) has to be approximately 9 times larger than
the labeled value in order to be detected with the same probability as given by the producer's risk.

3.1 Corrections on the previous method

Up to this point, the problems of the previous method have been discussed. To correct these problems, two major things are to be developed: (1) Corrected values of $X$, and "maximum number within tolerance" in Table 1, (2) A general method of selecting the appropriate sample size.

(1) Correct values of $X$, and "maximum number within tolerance" in Table 1. As a result of Chapter II and beginning in this chapter, the distribution of noxious weed seeds is based on Poisson's binomial limit. The probability of finding $X$ defective units in a random sample of $N$ units drawn from an infinite universe (general output of uniform product) in which the fraction defective is $p$ given by

Figure 3. Relation of type II error with mean.
As mentioned before, when \( p \) is less than 0.10 and \( n \) is large, an approximation to this expression that is satisfactory for most practical purposes is given by the poisson probability distribution

\[
p(x) = \frac{(pn)^x e^{-pn}}{x!}
\]

where \( np = \text{mean, of the poisson distribution or the expected number of defective units in a sample.} \) Or, if \( N \) units in which the number of defective units is \( M = PN \) is given exactly by the hypergeometric distribution

\[
\frac{\binom{N-M}{n-x} \binom{M}{x}}{\binom{N}{x}}
\]

When \( p \) is small and \( \frac{n}{N} \) is small, the poisson approximation can be used as described before.

Table 4, which is the correction of Table 1, is given here based on

\[
\sum_{x=c}^{\infty} \frac{\mu^x e^{-\mu}}{x!} = 0.95 \quad c = 0, 1, 2, ...
\]

where \( \mu \) is equal to mean of poisson distribution and, in this problem, \( \mu = 1, 2, ..., 180. \)

The problem which was used to compute the entries in this table is included in Appendix A. (See the American Mathematical Monthly, June, 1913 for values of \( \mu \), 0.0001 to 928, for further information.)

(2) A general method of selecting the appropriate sample size. Table 4 alone is not as useful as it might be because it
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<td>154</td>
<td>173</td>
</tr>
<tr>
<td>155</td>
<td>174</td>
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</tbody>
</table>
gives no information which may be used in finding an acceptable level for the consumer's risk. The question may be asked relative to the consequences of the average number of noxious weed seeds in a lot: What is a reasonable deviation from the published mean to be detected with $\beta = .05$? If, for example, it is determined that double the published amount is not a serious deviation but triple the amount should be detected. A deviation of 300% should be set as the alternative which should be detected with probability of .95. The sample size may then be determined. This determination may be greatly facilitated by using a graph. To find a unique graph that could be used for any particular deviation, consider the functional relation between deviation $\gamma$ and sample size $n$.

**Definition:** Basic sample unit (BSU). A (BSU) is the size of sample for which the average number of noxious weed seeds (as labeled) is one.

For example, suppose a bag is labeled 5 noxious seeds per 50 grams. Then the BSU is 10 grams. Or suppose it is labeled "less than .01% weed seeds." Equivalently, the label could then be written as 1 weed seed per 10,000 seeds. The BSU is then 10,000 seeds.

Figure 4 gives the graph of percent deviation ($\gamma$) from the labeled value plotted against sample size for $\beta = 0.05$, 0.10, and 0.30. The sample size is expressed in BSU. The following example will serve to illustrate the use of this graph.

Suppose it is determined that a lot of seeds is labeled "less than .01% noxious seeds." The BSU is then 10,000 seeds. It has
Figure 4. Relation of BSU to deviation from hypothesized mean.
been determined previously that 0.1% contamination by the type of weed seed normally found with the particular type of seed being considered results in significant degradation of the drop. Thus $\gamma = 900\%$ and $\alpha = \beta = .05$ should be the protection levels of the type I and type II errors. From the graph it may be seen that a sample of 1 BSU or 10,000 seeds provides more than the required protection. If the producer is willing to lower the label, a smaller sample may be used, since in the first case more than the required protection was provided. Suppose the label 0.05% noxious seeds is used (i.e., 5/10,000 or 1/2,000). Then the BSU is 2,000 seeds, $\gamma = 400\%$.

From Figure 4, the sample size is 2 BSU or 4,000 seeds.

As another example, suppose the lot is labeled "not more than 2 noxious seeds per 50 grams." The BSU is then 25 grams. Suppose also that a critical deviation from the labeled value is 200%. From Figure 4, the required sample size for $\alpha = \beta = .05$ at $\gamma = 200\%$ is 5 BSU or 125 grams. One might ask: What are the characteristics of a test with this sample size for values of $\beta$ other than .05? Two other curves are plotted in Figure 4 which will aid in evaluating the values of $\beta$ corresponding to other deviations. Thus, in this case a deviation of 184% would be detected with probability .30.
4.1 Sequential analysis

The first idea of a sequential test procedure, i.e., a test for which the number of observations is not determined in advance but is dependent on the outcome of the observations as they are made, was constructed by H. F. Dodge and H. G. Roming in 1929.

This technique was further developed by Abraham Wald (1947). Wald's first paper appeared in 1944 and a second paper appeared in 1945. The complete book was published in 1947. Wald (1947) described sequential test as follows: A rule is given for making one of the following three decisions at any stage of the experiment (at the mth trial for each integral value of m): (1) to accept the hypothesis $H$, (2) to reject the hypothesis $H$, (3) to continue the experiment by making an additional observation. Thus, such a test procedure is carried out sequentially. On the basis of the first observation, one of the aforementioned three decisions is made. If the first or second decision is made, the process is terminated. If the third decision is made, the second trial is performed. The process is continued until either the first or the second decision is made. The number $n$ of observations required by such a test procedure is a random variable since the value of $n$ depends on the outcome of the observations.
4.2 The sequential probability ratio test for testing a simple hypothesis $H_0$ against a single alternate $H_1$.

Wald in 1947 defined the sequential probability ratio as follows:

Let $f(x, \theta)$ denote the distribution of the random variable $x$ under consideration. Let $H_0$ be the hypothesis that $\theta = \theta_0$, and $H_1$ the hypothesis that $\theta = \theta_1$. Thus, the distribution of $x$ is given by $f(x, \theta_0)$ when $H_0$ is true, and by $f(x, \theta_1)$ when $H_1$ is true. We shall denote the successive observations on $x$ by $x_1, x_2, \ldots$, etc.

For any positive integral value $m$, the probability that a sample $x_1, \ldots, x_m$ will be obtained is given by

$$p_1^m = f(x_1, \theta_1) \cdot f(x_2, \theta_1) \cdots f(x_m, \theta_1) \tag{4.2.1}$$
when $H_1$ is true, and by

$$p_0^m = f(x_1, \theta_0) \cdot f(x_2, \theta_0) \cdots f(x_m, \theta_0) \tag{4.2.2}$$
when $H_0$ is true.

The sequential probability ratio for testing $H_0$ against $H_1$ is defined as follows: Two positive constants $A$ and $B$ ($B < A$) are chosen. At each stage of the experiment (at the $m$th trial for any integral value $m$), the probability ratio

$$\frac{p_1^m}{p_0^m} \quad \text{is computed if } B < \frac{p_1^m}{p_0^m} < A \tag{4.2.3}$$

The experiment is continued by taking an additional observation. If

$$\frac{p_1^m}{p_0^m} \geq A \tag{4.2.4}$$
The process is terminated with the rejection of $H_0$ (acceptance of $H_1$).

If

$$\frac{p_1^m}{p_0^m} \leq B \quad 4.2.5$$

The process is terminated with the acceptance of $H_0$. The constants A and B are determined so that the test will have the prescribed strength $(\alpha, \beta)$.

For the purposes of practical computation, it is often more convenient to compute the logarithm of the ratio $p_1^m/p_0^m$ than the ratio $p_1^m/p_0^m$ itself. The reason for this is that $\log(p_1^m/p_0^m)$ can be written as the sum of m items,

$$Z_i = \log \left( \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} \right) \quad 4.2.6$$

The test procedure is carried out as follows: The quantities $Z_i$ ($i = 1, 2, \ldots$) are used. At each stage of the experiment (at the $m$th trial for each integral value of $m$), the cumulative sum $Z_1 + Z_2 + \ldots + Z_m$ is computed.

If

$$\log B < Z_1 + Z_2 + \ldots + Z_m < \log A \quad 4.2.7$$

The process is continued by taking an additional observation.

If

$$Z_1 + \ldots + Z_m > \log A \quad 4.2.8$$

The experiment is terminated with the rejection of $H_0$. If

$$Z_1 + \ldots + Z_m < \log B \quad 4.2.9$$

The process is terminated with the acceptance of $H_0$. 
4.3 Fundamental relations among the quantities \( \alpha, \beta, A, \) and \( B \)

Wald (1947) described this relationship as: Sample 

\( (x_1, \ldots, x_m) \) is of type 0 if

\[
B < \frac{p_1^m}{p_0^m} = \frac{f(x_1, \theta_1) \cdots f(x_m, \theta_1)}{f(x_1, \theta_0) \cdots f(x_m, \theta_0)} < A \tag{4.3.1}
\]

for \( m = 1, 2, \ldots, m-1 \), and \( \frac{p_1^m}{p_0^m} < B \).

Similarly, a sample \( (x_1, \ldots, x_m) \) is of type 1 if

\[
B < \frac{p_1^m}{p_0^m} = \frac{f(x_1, \theta_1) \cdots f(x_m, \theta_1)}{f(x_1, \theta_0) \cdots f(x_m, \theta_0)} < A \tag{4.3.2}
\]

for \( m = 1, \ldots, m-1 \), and \( \frac{p_1^m}{p_0^m} > A \).

Thus, a sample of type 0 leads to the acceptance of \( H_0 \) and a sample of type 1 leads to the acceptance of \( H_1 \) (rejection of \( H_0 \)).

Clearly, for any given sample \( (x_1, \ldots, x_m) \) of type 1 the probability of obtaining such a sample is at least \( A \) times as large under hypothesis \( H_1 \) as under \( H_0 \). Thus, the probability measure of the totality of all samples of type 1 is the same as the probability that the sequential process will terminate with the acceptance of \( H_1 \) (rejection of \( H_0 \)). But the latter probability is equal to \( \alpha \) when \( H_0 \) is true and to \( 1-\beta \) when \( H_1 \) is true. Thus, we obtain the inequality

\[
1 - \beta \geq A\alpha \tag{4.3.3}
\]

or

\[
A \geq \frac{1-\beta}{\alpha}.
\]
Thus, \((1-\beta)/\alpha\) is an upper limit for \(A\). A lower limit for \(B\) can be derived in a similar way. In fact, for any given sample \((x_1, \ldots, x_m)\) of type \(O\) the probability of obtaining such a sample under \(H_1\) is at most \(B\) times as large when \(H_1\) is true as when \(H_0\) is true.

Since the probability of accepting \(H_0\) is \(1 = \alpha\) when \(H_0\) is true and \(\beta\) when \(H_1\) is true, we obtain the inequality

\[
\beta \leq (1 - \alpha)B
\]

or

\[
B > \frac{\beta}{1-\alpha}
\]

Thus, \(\beta/(1-\alpha)\) is a lower limit for \(B\). The two inequalities can also be written as

\[
\frac{1}{A} \geq \frac{\beta}{1-\alpha}
\]

and

\[
B > \frac{\beta}{1-\alpha}
\]

These two inequalities prove in this situation that the success observations \(x_1, x_2, \ldots\) are independent observations on \(x\). The two inequalities also remain valid even if the success observations are dependent.
CHAPTER V
APPLICATION OF SEQUENTIAL ANALYSIS
TO THE TOLERANCES OF NOXIOUS WEED SEEDS

5.1 Methodology

Since the distribution of noxious weed seed in a lot is approximately poisson distributed with the mean number of noxious weed seeds $\mu$ the sequential probability ratios with

$$H_0: \mu = \mu_0$$

$$H_1: \mu = \mu_1$$

becomes

$$\frac{p_{1m}}{p_{0m}} = \frac{n!}{\mu_0^{-\mu_0} \mu_1^{\sum x_i}} \frac{\mu_0^{\sum x_i}}{N^{\mu_0} \mu_1^{\sum x_i}}$$

5.1.1

By 4.3.1 and 5.1.2,

$$\log(\frac{1-\alpha}{1-\beta}) < \log [e^{-N(\mu_0-\mu_1)(\mu_0/\mu_1)\sum x_i}] < \log (\frac{1-\beta}{\alpha})$$

These inequalities become
\[
\log \left( \frac{\beta}{1-\alpha} \right) < -N(\mu_0 - \mu_1) + \Sigma x \log(\mu_0 / \mu_1) < \log \left( \frac{1-\beta}{\alpha} \right)
\]

5.1.3

\[
\log \left( \frac{\beta}{1-\alpha} \right) + N(\mu_0 - \mu_1) < \Sigma x \log(\mu_0 / \mu_1) < \log \left( \frac{1-\beta}{\alpha} \right) + N(\mu_0 - \mu_1)
\]

\[
\log \left( \frac{\beta}{1-\alpha} \right) + N(\mu_0 - \mu_1) < \Sigma x \log(\mu_0 / \mu_1) < \log \left( \frac{1-\beta}{\alpha} \right) + N(\mu_0 - \mu_1)
\]

For \( \mu_1 > \mu_0 \) these inequalities would be reversed which makes \( \log(\mu_0 / \mu_1) \) negative.

Assume that an equal size of type I and type II errors is required for a fixed alternative value of \( \mu \) or \( \alpha = \beta = 0.05 \). Hence, we are willing to risk rejection of lots 5% of the time with mean equal to \( \mu_0 \), and acceptance of a lot with mean equal to \( \mu_1 \), 5% of the time. Then 5.1.3 becomes

\[
\log \left( \frac{0.05}{1-0.05} \right) + N(\mu_0 - \mu_1) < \Sigma x \log(\mu_0 / \mu_1) < \log \left( \frac{1-0.05}{0.05} \right) + N(\mu_0 - \mu_1)
\]

5.1.4

\[
\log \left( \frac{0.05}{1-0.05} \right) + N(\mu_0 - \mu_1) < \Sigma x \log(\mu_0 / \mu_1) < \log \left( \frac{1-0.05}{0.05} \right) + N(\mu_0 - \mu_1)
\]

These inequalities can be used for any \( \mu_0 \) and \( \mu_1 \). For our purpose, a particular hypothesis with different deviations from the hypothesized value as alternative must be selected, such that it could be used as general. The quantity \( \log(\mu_0 / \mu_1) \) for the alternate hypothesis specified according to this deviation from \( \mu_0 \) is given in Table 5.
Table 5. Relation of percent deviation to log(µ₀/µ₁).

<table>
<thead>
<tr>
<th>log(µ₀/µ₁)</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>75</th>
<th>10</th>
<th>125</th>
<th>150</th>
<th>175</th>
<th>200</th>
<th>300</th>
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<tbody>
<tr>
<td></td>
<td>0.18232</td>
<td>0.26236</td>
<td>0.33647</td>
<td>0.40547</td>
<td>0.55962</td>
<td>0.69314</td>
<td>0.81093</td>
<td>0.91629</td>
<td>1.01160</td>
<td>1.09861</td>
<td>1.38629</td>
</tr>
</tbody>
</table>

The value of log(19) is 2.94444. The derivation (µ₀-µ₁) is a deviation from the true mean (labeled) and for a specific test is constant. N is the only variable in the sequential test and will take on the values 1, 2, ... up to the time that the test is terminated. Xᵢ is a number of poisson in a unit weight of sample, and Exᵢ is the sum of poisson after the ith sample is taken from the lot.

EXAMPLE: Consider a lot which is labeled 1 noxious weed seed per 50 grams of lot. The consumer would like to test:

H₀: µ₀ = 1
H₁: µ = 100% deviation from hypothesis or select alternate + µ₀ = 2 or twice as much as it is labeled.

In this example, log(µ₀/µ₁) = -0.69314 and µ₀-µ₁ = -1.

Applied to 5.1.4,

-2.94444 - 1.0 N < Exᵢ(-.69314) < +2.94444 - 1.0 N
\[
\frac{2.94444 + 1.0 N}{0.69314} > \xi > \frac{-2.94444 + 1.0 N}{0.69314}
\]

These inequalities are inversed now. The consumer will accept the hypothesis (reject alternate) if

\[
\xi < \frac{-2.94444 + 1.0 N}{0.69314}
\]

and will reject the hypothesis (accept the alternate) if

\[
\xi > \frac{2.94444 + 1.0 N}{0.69314}
\]

If neither inequality is satisfied, another observation will be taken.

This problem was simulated on the computer. Poisson random variables were simulated and \(\xi\) recorded at each stage. The method of generating poisson random variables is found in Appendix B.

Compare the \(\xi\) with upper and lower bounds, if \(\xi\) is bigger than the upper limit, the hypothesis will be rejected, if \(\xi\) is smaller than the lower limit, the inspector will accept the hypothesis. If neither of the two cases are true, another sample will be selected. This illustration is shown in Figure 5.

5.2 Illustration of the problem in practice

The method given in this chapter is suggested to be used instead of the previous method. The following are the main reasons for the change:

1. This method is designed to avoid unnecessary observations. In most cases, the sample size will be much smaller than required in the usual test.

2. The probabilities of type I (producer's risk) and type II
Figure 5. Sequential test for the problem done by the computer.
(consumer's risk) error can be decided in advance and easily incorporated into the test.

3. The test is inherently simpler than the usual procedure.

To illustrate the problem, a lot which is labeled 1 noxious weed seed per 50 grams of sample will be discussed. At the same time, the following four major problems need to be covered:

(1) Choosing alternate or deviation from the true mean; (2) Upper and lower limits related to deviation from the true mean (labeled); (3) Relation of the other means ($\mu_0$) (labeled) to the sample size in base $\mu_0 = 1$ noxious weed seed per unit weight; (4) Expected sample size needed to reach to an acceptance or rejection of the hypothesis.

(1) Choosing alternatives (deviation from the true means, or labeled). Since it is necessary to have an assumption before the test starts, different possibilities that may occur during the test related to previous assumptions need to be determined in advance.

Let the value labeled be 1 noxious weed seed per 50 grams of sample from the lot. Table 6 shows the value of $\mu_1$ for various deviations from $\mu_0$ expressed in percentage.

EXAMPLE: A bag is labeled 1/50 grams. The inspector would like to test under what alternate the bag should be tested. If it is legal to have the deviation of 50%, then the alternate will be $\mu_1 = 1.5$. Then the test will be:

$$H_0: \mu_0 = 1$$
$$H_1: \mu_1 = 1.5$$
and sampling after setting upper and lower limits will be started until one of the decisions is made.

(2) Upper and lower limits related to deviations from the true mean (labeled). The second step of the procedure is to determine the limits of the test. All of these assumptions are under consideration of equal chance of producer's and consumer's risks \((\alpha = \beta = .05)\). According to 5.1.4, upper and lower limits are

\[
\frac{- \log 19 + \frac{N(\mu_0 - \mu_1)}{\log(\mu_0/\mu_1)}}{\log(\mu_0/\mu_1)} \quad \text{as upper limit and}
\]

\[
\frac{\log 19 + \frac{N(\mu_0 - \mu_1)}{\log(\mu_0/\mu_1)}}{\log(\mu_0/\mu_1)} \quad \text{as lower limit}
\]

where \(\mu_0 - \mu_1\) is a deviation of the labeled mean \(\mu_0\) from alternate \(\mu_1\).

Table 7 shows upper and lower limits in relation to deviation from the true mean \((\mu_0)\).

This table can be changed very easily if other deviations or other type I or II errors are to be considered.

EXAMPLE: Assume that a lot is labeled 1 noxious weed seed per 50 grams. The inspector (or consumer) would like to test with 100% deviation from the mean \((\mu = 2)\). What will be upper and lower control limits for the test? By Table 7, under 100% deviation,

<table>
<thead>
<tr>
<th>% deviation</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>alternate</td>
<td>1.1</td>
<td>1.2</td>
<td>1.3</td>
<td>1.4</td>
<td>1.5</td>
<td>1.6</td>
<td>1.7</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
<td>3.5</td>
<td>4</td>
</tr>
</tbody>
</table>
Table 7. Relation of upper and lower limits with respect to deviation from the labeled value (BSU = 1/50).

<table>
<thead>
<tr>
<th>% deviation from tolerance</th>
<th>upper limit</th>
<th>lower limit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.94444 + .2N</td>
<td>-2.94444 + .1N</td>
</tr>
<tr>
<td></td>
<td>0.09531</td>
<td>0.09531</td>
</tr>
<tr>
<td>20</td>
<td>2.94444 + .2N</td>
<td>-2.94444 + .2N</td>
</tr>
<tr>
<td></td>
<td>0.18232</td>
<td>0.18232</td>
</tr>
<tr>
<td>30</td>
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<td>-2.94444 + .3N</td>
</tr>
<tr>
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<td>0.26236</td>
</tr>
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<td>-2.94444 + .4N</td>
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<td>0.33647</td>
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<td>-2.94444 + .5N</td>
</tr>
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</tr>
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<td>-2.94444 + .6N</td>
</tr>
<tr>
<td></td>
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<td>0.47000</td>
</tr>
<tr>
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<td>-2.94444 + .7N</td>
</tr>
<tr>
<td></td>
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<td>0.53063</td>
</tr>
<tr>
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<td>-2.94444 + 1.0N</td>
</tr>
<tr>
<td></td>
<td>0.69314</td>
<td>0.69314</td>
</tr>
<tr>
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<td>-2.94444 + 1.5N</td>
</tr>
<tr>
<td></td>
<td>0.91629</td>
<td>0.91629</td>
</tr>
<tr>
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<td>-2.94444 + 2.0N</td>
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<tr>
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<td>1.09861</td>
</tr>
<tr>
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<td>-2.94444 + 2.5N</td>
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<tr>
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<td>1.26276</td>
<td>1.26276</td>
</tr>
<tr>
<td>300</td>
<td>2.94444 + 3.0N</td>
<td>-2.94444 + 3.0N</td>
</tr>
<tr>
<td></td>
<td>1.38629</td>
<td>1.38629</td>
</tr>
</tbody>
</table>
upper and lower control limits are:

\[-2.94444 + 1.0N \quad \frac{0.69314}{0.69314}\]

as lower control limit or accepting region

and

\[2.94444 + 1.0N \quad \frac{0.69314}{0.69314}\]

as upper limit or rejection line

and

\[H_0: \mu_2 = 1\]

\[H_1: \mu_1 = 2\]

(3) Relation of the other mean (\(\mu\)) or labeled value to sample size in basic sample units (BSU). The following procedure is used to determine the appropriate sample size. Express \(\mu\) in terms of the BSU; thus, 10 noxious weed seeds per 50 grams would be converted as:

\[\frac{10}{50} = \frac{1}{5}\]

In other words, the BSU is 5 grams, or 5 grams of seed need to be tested.

(4) Expected sample size required to reach a decision. Wald(1947) developed the expected number of samples required. B. W. Lindgren (1960) proved the same idea with more explanation.

The number of observations required to reach a decision, in a given sequential likelihood ratio test, is a random variable and is given by

\[E(N) = E(\log \Delta_N)/E(Z)\]
where

\[ Z_i = \log \left[ \frac{F_0(X_i)}{F_1(X_i)} \right] \]

and \( \log \Delta_n \) for a random variable \( \Delta N \) is approximately a Bernoulli variable, with the value \( \log A \) if the decision reached is to reject \( H_0 \) and the value \( \log B \) if the decision reached is to accept \( H_0 \), then

\[ E_0(\log \Delta N) = (\log A)\pi(\theta) + (\log B)[1-\pi(\theta)] \]

where \( \pi(\theta) \) is the power function or probability that the given test rejects \( H_0 \), when the state is actually \( \theta \). Then

\[ E(N|H_0) = \frac{1}{E(Z|H_0)} \left[ a \log A + (1-a) \log B \right] \]

when the hypothesis is true and

\[ E(N|H_1) = \frac{1}{E(Z|H_1)} \left[ (1-\beta) \log A + \beta \log B \right] \]

when the alternate is true.

Table 8 shows the relation of sample size needed to be tested to deviation from the labeled mean when the hypothesis is true, and Table 9 shows the same relationship when the alternate is true.

For noxious weed seeds, \( E(Z|H_0) \) becomes:

\[ E(Z) = E\left(\log \frac{e^{-\mu_0} \frac{x}{\mu_0}}{e^{-\mu_1} \frac{x}{\mu_1}}\right) \]

\[ = E\left(\log [e^{-(\mu_0-\mu_1)}(\mu_0/\mu_1)^x]\right) \]
= -(μ₀-μ₁) + E[x log(μ₀/μ₁)]
= -(μ₀-μ₁) + log(μ₀/μ₁) E(x)

Instead of E(x), the value hypothesis μ₀ should be replaced if
the hypothesis is true and μ₀ if the alternate is true, then
E(log Δᵢ) becomes

E(log Δᵢ) = [α log A + (1-α) log B]

where α = β = 0.05. The result becomes

E(log Δᵢ) = 0.05 log 1/19 + (1-0.05) log 19
= 0.05(-log 19) + 0.95 (log 19)
= 0.05(-2.94444) + 0.95(2.94444) = 2.65

where the hypothesis is true, and

E(log Δᵢ) = (1-0.5) log A + .05 log B
= 0.95(-2.94444) + 0.05(2.94444) = -2.65

where the alternate is true.

Different values of E(Z|H₀) and E(Z|H₁) and the computer
program are in Appendix C.

EXAMPLE: A lot is labeled 1 noxious weed seed per unit of
weight (sample). If the state of the hypothesized are:

H₀: μ₀ = 1
H₁: μ₁ = 2 (100% deviation from labeled)

E(N|H₀) = \frac{2.65}{-(1-2) + 1 \log(1/2)}
Table 8. Relation of % deviation (labeled) with expected number of sample required when hypothesis is true.

<table>
<thead>
<tr>
<th>µ</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>70</th>
<th>100</th>
<th>200</th>
<th>300</th>
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<tbody>
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<td>1</td>
<td>150</td>
<td>70</td>
<td>42</td>
<td>28</td>
<td>16</td>
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<td>2</td>
<td>75</td>
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<tr>
<td>6</td>
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<td>11</td>
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<td>5</td>
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</tr>
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</table>
Table 9. Relation of % deviation (labeled) with expected number of sample required when alternate is true.

<table>
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<tr>
<th>μ (labeled)</th>
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<th>40</th>
<th>50</th>
<th>70</th>
<th>100</th>
<th>200</th>
<th>300</th>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
or 9 samples are required on the average to reach a decision. Of course the bigger the \( \mu_0 \) or \( \mu_1 \), the smaller the sample size required. Also, the sample size decreases as the percentage of the alternate increases. This relation can be seen in Tables 8 and 9.

COMPLETE EXAMPLE: Suppose a lot of seeds is labeled 10 noxious weed seeds per 50 grams. Let the test be conducted with \( \alpha = \beta = 0.05 \) and a deviation of 100% from the labeled. Suppose also that it is convenient to use a 10 gram sample. The test is set up as follows:

1. Convert 10 noxious weed seeds per 50 grams to a sample of 10 grams.
   \[
   \frac{10}{50} = \frac{1}{5}
   \]
   or, each sample must have 1 noxious weed seed per 5 grams of sample.

2. State the nature of the hypothesized value as follows:
   \[
   H_0: \mu_0 = 1
   \]
   \[
   H_1: \mu_1 = 2 \text{ (100\% deviation from labeled)}
   \]

3. Upper and lower limits from Table 7 will be
   \[
   2.94444 + 1.0\text{ON} \over 0.69314 \text{ as upper limit}
   \]
   and
   \[
   -2.94444 + 1.0\text{ON} \over 0.69314 \text{ as lower limit}
   \]
4. Expected number of samples required according to Tables 8 and 9 will be 9 if the hypothesis is true and 7 if the alternate is true.

5. Start sampling with 5 grams in each sample. Suppose that in the first sample 3 noxious weed seeds were found, in the second sample 0 noxious weed seeds were found, in the third sample 4 noxious weed seeds were found, and in the fourth sample 4 were found. By Table 10, the sequential test will than be terminated at the fourth sample and the hypothesis that the label is correct will be rejected, because at the fourth trial $ ꞌ X \dagger$ lower limit.

Table 10. Illustration of the sample problem.

<table>
<thead>
<tr>
<th>Upper limit</th>
<th>5.69</th>
<th>7.13</th>
<th>8.58</th>
<th>10.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ ꞌ X \dagger$</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>11</td>
</tr>
</tbody>
</table>

| Lower limit | -2.81 | -1.36 | 0.08  | 1.52  |

5.3 Proof that the probability is 1 that the sequential probability ratio test will eventually terminate.

The question may be asked: What if sampling continues? Wald (1947) proved that the probability is 1 that the sequential probability ratio test will eventually terminate.

The sequential probability ratio test terminates at the nth trial where $ n $ is the smallest integer for which either
\[ z_1 + \ldots + z_n \geq \log A \]

or

\[ z_1 + \ldots + z_n \leq \log B \]

Let \( c = \log B + \log A \). Subdivide the infinite sequence \( z_1, z_2, z_3, \ldots \) into segments of length \( r \) where \( r \) is some positive integer. Thus, the first segment \( S_1 \) will consist of the elements \( z_1, \ldots, z_r \), the second segment \( S_2 \) will contain the elements \( z_{r+1}, \ldots, z_{2r} \), etc. In general, the \( k \)th segments \( S_k \) will consist of the elements \( z_{(k-1)r+1}, \ldots, z_{kr} \). Let \( \zeta_k \) denote the sum of the elements in the \( k \)th segment. It can be seen that if the infinite sequence \( z_1, z_2, \ldots \) etc. is such that the sequential process never terminates, then

\[ |\zeta_k| < c \quad \text{for } k = 1, 2, \ldots \quad 5.3.1 \]

Inequality 5.3.1 can also be written

\[ (\zeta_k)^2 < c^2 \quad \text{for } k = 1, 2, \ldots \quad 5.3.2 \]

Thus, in order to show that the probability is 1 that the sequential process will eventually terminate, it is sufficient to prove that the probability is 0 that 5.3.2 holds for all integral values \( k \). For any given positive integer \( i \) denote by \( P_i \) the probability that \( \zeta_i^2 < c^2 \). Since \( z_1, z_2, \ldots \), etc. are independently distributed, each having the same distribution, the distribution of \( \zeta_i \) must be the same for all values \( i \). Hence, \( P_i \) is also independent of \( i \) and is denoted by \( P \). Since \( \zeta_1, \zeta_2, \ldots \), etc. are independently distributed, the probability of the joint event that 5.3.2 holds
for all values \( k \), it is sufficient to show that \( P < 1 \). Clearly, if the expected value of \( \zeta_1^2 \) is greater than \( c^2 \), then \( P \) must be less than 1. Since the variance of \( z_1 \) is assumed to be positive, the expected value of \( \zeta_1^2 \) can be made arbitrarily large by choosing \( r \), which is the number of elements in a segment sufficiently large. Thus, \( P < 1 \), and the proposition is proved: The probability is 1 that the sequential probability ratio test procedure will eventually terminate.

5.4 Termination of the sequential test procedure

In 5.3 it was proved that the probability is 1 that the sequential probability test procedure will eventually terminate, but it is usually desirable to set a definite upper limit, say \( n_0 \), for the number of observations. Wald (1947) suggested a method of truncation of the sequential test by giving a near rule for the acceptance or rejection of \( H_0 \) at the \( n_{0\text{th}} \) trial if the sequential process did not lead to a final decision for \( n \leq n_0 \) as follows: If the sequential probability ratio test does not lead to a final decision for \( n \leq n_0 \), accept \( H_0 \) at the \( n_{0\text{th}} \) trial when

\[
\log B < \sum_{a=1}^{n_0} z_a \leq 0 \quad 5.4.1
\]

and reject \( H_0 \) when

\[
0 < \sum_{a=1}^{n_0} z_a \leq \log A \quad 5.4.2
\]

By truncating the sequential process at the \( n_{0\text{th}} \) trial, the probabilities of type I and type II errors must be changed.
Let \( \alpha \) and \( \beta \) be the probabilities of type I and type II errors respectively if the sequential test is not truncated. The effect of the truncation on \( \alpha \) and \( \beta \) will, of course, depend on the value of \( n_0 \). The larger the \( n_0 \), the smaller will be the effect of truncation on \( \alpha \) and \( \beta \). Denote the resulting probabilities of type I and type II errors by \( \alpha(n_0) \) and \( \beta(n_0) \), respectively, if the sequential process is truncated at \( n = n_0 \).

To obtain an upper limit for \( \alpha(n_0) \), consider the cases in which the truncated process leads to the rejection of \( H_0 \), while the non-truncated process leads to the acceptance of \( H_0 \). Denote by \( P_0(n_0) \) the probability under \( H_0 \) of obtaining a sample such that the truncated process leads to the rejection of \( H_0 \), while the non-truncated process leads to the acceptance of \( H_0 \). Then

\[
\alpha(n_0) \leq \alpha + P_0(n_0)
\]

5.4.3

The reason that in 5.4.3 the inequality sign holds instead of the equality sign is that there may be samples for which the truncated process leads to the acceptance of \( H_0 \), while the non-truncated process leads to the rejection of \( H_0 \). To obtain an upper bound for \( \alpha(n_0) \), we merely need to derive an upper bound for \( P_0(n_0) \). By definition, \( P_0(n_0) \) is the probability under \( H_0 \) that for the successive observations \( z_1, z_2, \ldots \), etc., the following three conditions are simultaneously fulfilled:

\[
\log B < \sum_{\alpha=1}^{n} z_\alpha < \log A \quad \text{for } n = 1, n_0 - 1
\]

5.4.4
When the sequential process is continued beyond \( n_0 \), it terminates with the acceptance of \( H_0 \).

Denote by \( P_0(n_0) \) the probability under \( H_0 \) that condition 5.4.2 will be fulfilled, i.e.,

\[
0 < \sum_{\alpha=1}^{n_0} z_\alpha < \log A
\]

Since the probability that condition 5.4.2 is fulfilled cannot be smaller than the probability that all three conditions are fulfilled simultaneously, then

\[
P_0(n_0) \geq P_0(n_0)
\]

and, therefore,

\[
\alpha(n_0) \leq \alpha + P_0(n_0)
\]

Thus, \( \alpha + P_0(n_0) \) is an upper bound for \( \alpha(n_0) \), which can easily be computed, as will be shown later. To obtain an upper bound for \( \beta(n_0) \) denote by \( P_1(n_0) \) the probability (under \( H_1 \)) that the successive observations will be such that the truncated process leads to the acceptance of \( H_0 \), while the non-truncated process leads to the rejection of \( H_0 \). In other words, \( P_1(n_0) \) is the probability under \( H_1 \) that the successive observations will satisfy the following three conditions simultaneously:

\[
\log B < \sum_{\alpha=1}^{n} z_\alpha < \log A \quad \text{for } n = 1, \ldots, n_0 - 1
\]
\[ \log B < \sum_{\alpha=1}^{n_0} z_\alpha \leq 0 \]

If the process is continued beyond the \( n_0 \)th trial, it terminates with the acceptance of \( H_1 \).

Clearly,

\[ \beta(n_0) \leq \beta + P_1(n_0) \quad 5.4.7 \]

Since it is difficult to determine the value of \( P_1(n_0) \), we shall derive a simple upper bound for it. Let \( \tilde{P}_1(n_0) \) be the probability under \( H_1 \) that condition 5.4.1 is fulfilled, i.e.,

\[ \tilde{P}_1(n_0) = P_1(\log B < \sum_{\alpha=1}^{n_0} z_\alpha \leq 0) \quad 5.4.8 \]

Then \( \tilde{P}_1(n_0) \geq P_1(n_0) \) and then

\[ \beta(n_0) \leq \beta + \tilde{P}_1(n_0) \quad 5.4.9 \]

5.5 **Values of \( P_0(n_0) \) and \( P_1(n_0) \)**

Assume that \( n_0 \) is sufficiently large so that \( z_1 + \ldots + z_{n_0} \) may be regarded as a normally distributed variable. When \( H_i \) is true (\( i = 0, 1 \)), the expected value of \( z_1 + \ldots + z_{n_0} \) is equal to \( n_0 E_i(z) \) and the standard deviation of \( z_1 + \ldots + z_{n_0} \) is equal to \( \sqrt{n_0} \sigma_i(z) \) where \( \sigma_i(z) \) denotes the standard deviation of \( z \) when \( H_i \) is true. To compute \( \tilde{P}_0(n_0) \), write the inequality

\[ 0 < \sum_{\alpha=1}^{n_0} z_\alpha < \log A \]

in the following form:
5.5.1

\[
\frac{-n_0E_0(z)}{\sqrt{n_0\delta_0(z)}} < \frac{z_1 + \ldots + z_n - n_0E_0(z)}{\sqrt{n_0\delta_0(z)}} < \frac{\log A - n_0E_0(z)}{\sqrt{n_0\delta_0(z)}}
\]

Let

\[
\nu_1 = \frac{-n_0E_0(z)}{\sqrt{n_0\delta_0(z)}} \quad \text{and} \quad \nu_2 = \frac{\log A - n_0E_0(z)}{\sqrt{n_0\delta_0(z)}}
\]

Since the middle term in 5.5.1 is normally distributed with zero mean and unit variance when \(H_0\) is true, the probability that 5.5.1 is fulfilled when \(H_0\) is true is equal to \(G(\nu_2) - G(\nu_1)\) where \(G(\nu)\) denotes the probability that a normally distributed variable with mean 0 and variance unity will take a value less than \(\nu\). Thus,

\[
P_0(n_0) = G(\nu_2) - G(\nu_1)
\]

5.5.3

To compute \(P_1(n_0)\), write the inequality \(\log B < \sum_{a=1}^{n_0} z_a \leq 0\) in the following form:

\[
\frac{\log B - n_0E_1(z)}{\sqrt{n_0\delta_1(z)}} < \frac{z_1 + \ldots + z_n - n_0E_1(z)}{\sqrt{n_0\delta_1(z)}} \leq \frac{-n_0E_1(z)}{\sqrt{n_0\delta_1(z)}}
\]

5.5.4

Let

\[
\nu_3 = \frac{\log B - n_0E_1(z)}{\sqrt{n_0\delta_1(z)}} \quad \text{and} \quad \nu_4 = \frac{-n_0E_1(z)}{\sqrt{n_0\delta_1(z)}}
\]

Since the middle term in 5.5.4 is normally distributed with mean 0 and variance unity when \(H_1\) is true, the probability (under \(H_1\)) that 5.5.4 holds is equal to \(G(\nu_4) - G(\nu_3)\). Hence

\[
P_1(n_0) = G(\nu_4) - G(\nu_3)
\]

5.5.6
These results can thus be summarized as follows:

\[ a(n_0) \leq a + G(v_2) - G(v_1) \]  \hspace{1cm} 5.5.7

and

\[ b(n_0) \leq b + G(v_4) - G(v_3) \]  \hspace{1cm} 5.5.8

where \( v_1, v_2, v_3, \) and \( v_4 \) are given in 5.5.2 and 5.5.4. These upper bounds may considerably exceed \( a(n_0) \) and \( b(n_0) \), respectively. It would be desirable to find closer limits.
CHAPTER VI

SUMMARY AND CONCLUSION

Statistical analysis of seed certification has been a subject of discussion for many years. Much effort is still needed to make this field of study complete, but in the philosophy of statistics, these efforts will be valid if they become practically useful.

The tolerances of noxious weed seeds, the subject of this study, has been studied since 1929, but unfortunately, little effort has been applied in this area.

Because of a lack of understanding of probability, especially of the type II error, the present testing program lacks the meaning and usefulness it might otherwise have. It has been shown that with no attention to sample size the risk to the consumer is very high. That is, the testing program encourages labeled values of seed contamination much higher than they should be.

The table used for tolerance values, as it exists, does not give equal consideration to both the producer and the consumer according to the philosophy of statistics.

Corrections of the previous methods have been made in this study by the unique graph. Of course, the graph can be extended according to the needs of the testing program. The graph illustrates the relation of type I and type II errors with respect to sample size and deviation (alternate) from the true mean or labeled.

A new approach to the solution of this problem was the intended
purpose of this study. Sequential analysis with its wide application was discussed as a solution to statistical analysis tolerances of noxious weed seeds.

In sequential analysis, decisions of risk can be made in advance; upper and lower control limits can be changed easily; and the most important advantage is that the test can be easily understood and practically used. By taking continuous sampling and making a decision at the end of each sampling, the inspector's duties are considerably eased.
LITERATURE CITED


Collins, G.N. 1929. The application of statistical methods to seed testing. United States Department of Agriculture Circular 79.


Appendix A

Computer Program for Cumulative Poisson Distribution

This program provides the solution to the following equation for given values of $\mu$:

\[
\sum_{x=0}^{\infty} \frac{\mu^x e^{-\mu}}{x!} = 0.95
\]

Program:

```plaintext
IMPLICIT REAL*8 (A-H,O-Z)
X=1
1 K=0
  FX=DEXP(-X)
  SUM=FX
2 IF(SUM.GE..95)GO TO 3
  IF(K.GT.350)GO TO 3
  K=K+1
  FX=FX*X/DFLOAT(K)
  SUM=SUM+FX
  GO TO 2
3 WRITE(6,100)X,SUM,K
100 FORMAT(1X,3G15.7)
IF(X.GE.180.)STOP
X=X+1.
GO TO 1
END
```
Appendix B

Method of Generating Poisson Random Variables

Consider the following process: A point $U_1$ is picked at random in $[0, 1]$, and a second point $U_2$ is picked at random in $[0, U_1]$. Then a third point $U_3$ is picked up at random in $[0, U_2]$, and so forth. An appropriate model for the distribution of $U_1, U_2, \ldots, U_n$ is $U_1 = X_1, U_2 = X_1X_2, \ldots, U_n = X_1X_2\ldots X_n$ where $X_1, \ldots, X_n$ are mutually independent, each uniformly distributed on $[0, 1]$. Let $z$ be the number of points $U_1, U_2, \ldots$, etc. that fall in $[C, 1]$, $(0 < C < 1)$. Then $z$ is poisson distributed with parameter $\mu = -\log C$ (Dwass, 1970).

The procedure for generating a single poisson observed random variable with $\mu = 2$, $c = 0.135335$ is as follows: Select a uniform distribution $(X_1)$ on $[0, 1]$. If that number is less than $0.135335$, the poisson observation is 0. If, however, the number $X_1$ is greater than $0.135335$, another uniform random variable $X$ must be selected from $U_2 = X_1X_2$. If $U_2$ is less than $C$, the poisson observation is 1. If $U_2$ is not less than $C$, select another uniform random variable $X_3$ from $U_3 = X_1X_2X_3$ and test it. If $U_3$ is less than $C$, the poisson observation is 2; otherwise, continue in the same manner until $U_k$ is less than $C$ for some $k$. Clearly, $U_k$ is less than $C$ for some $k$ since $U_k = \prod_{i=1}^{k} X_i$ is decreasing to 0.

The following program illustrates the problem with $\mu = 2$
as the hypothesis and \( \mu = 4 \) as the alternate in sequential testing.

The method of generating Poisson random variables is also given in the program. This program can be generalized for any Poisson random variable by changing the value of \( C \) and upper and lower control limits.

**Program:**

```
REAL N
100 FORMAT(T10,'SX=',G15.7,' P=',G15.7,' N=',G15.7)
101 FORMAT(' REJECT',T10,'SX=',G15.7,' P=',G15.7,' N=',G15.7,' C
*1=',G15.7)
102 FORMAT(' ACCEPT',T10,'SX=',G15.7,' P=',G15.7,' N=',G15.7,' C
*2=',G15.7)
103 FORMAT(1H,SF15.4)
C=0.135335
ICOUNT=1
6 N=1.
IF(ICOUNT.GT.100)STOP
SX=0.
3 SY=1.
P=0.
C1=(2.94444+2.0*N)/0.69314
C2=(-2.94444+2.0*N)/0.69314
2 SY=SY*RN(55591)
IF(SY.LT.C)GO TO 1
P=P+1.
WRITE(6,103)SY,P,C1,C2,N
GO TO 2
1 SX=SX+P
IF(SX.LE.C2)GO TO 4
IF(SX.GE.C1)GO TO 5
WRITE(6,100)SX,P,N
N=N+1.
IF(N.GT.25.)GO TO 6
GO TO 3
5 WRITE(6,101)SX,P,N,C1
ICOUNT=ICOUNT+1
GO TO 6
4 WRITE(6,102)SX,P,N,C2
ICOUNT=ICOUNT+1
GO TO 6
END
```
Appendix C

Computer Program for Expected Sample Size Required

A program for (ESR) is the same for both alternate and hypothesis, except for the values of the denominators which change as the percent deviation from the true mean (hypothesis) changes.

Program:

```fortran
DO 2 I=1,8
READ(5,100) X
100 FORMAT(F10.5)
DO 1 LAMDA=1,200
    EN=2.65/(X*FLOAT(LAMDA))
1 WRITE(6,101) LAMDA,EN
2 CONTINUE
STOP
END
```
VITA

Yadolah Dodge

Candidate for the Degree of

Master of Science

Theses: Statistical Analysis for the Tolerances of Noxious Weed Seeds

Major Field: Applied Statistics

Biographical Information:

Personal Data: Born at Abadan, Iran, March 30, 1944, son of Houssain and Fatemeh Dodge; married Kadijah Khatemi September 2, 1963; one child--Ali.

Education: Attended elementary school in Abadan, Iran, graduated from Farokhi High School in 1961; received the Bachelor of Science degree from Jundi-Shapur University in Agriculture in 1966; completed requirements for the Master of Science degree, in Applied Statistics, at Utah State University in 1971.

Professional Experience: June 1966 to 1968, Military duty; served as teachers' assistant at Jundi-Shapur University from 1968 to 1970. January 1970 to present, statistical consultant for research workers on campus of Utah State University.