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Integral Representation Theorems

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INTEGRAL REPRESENTATION THEOREMS

by

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ABSTRACT

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Since F. Riesz showed in 1909 that the dual of C[0,1] is BV[0,1] (the functions of bounded variation on [0,1] with $\|g\|_{BV} = V(g)$) via the Stieltjes integral, obtaining representations for linear operators in various settings has been a problem of interest. This paper shows the historical manner of representations, the road map type theorems and representations obtained via the v-integral.

(44 pages)

INTRODUCTION

Historically, integration has been studied in the sup-norm or weaker topology on the function space as Riemann and Lebesgue integrations in which integrable functions are approximated by step functions and simple functions, respectively. This presents complications in obtaining a representation for linear operators on the space of continuous functions, for step functions and simple functions are not continuous. One must work in the weak sequential extension of a space in order to extend the domain of linear operators in question. The main topic of this paper is to discuss historical methods for the integral representation and the characterization of the linear operators on the space of continuous functions on [0,1] with the BV norm (the norm given by the $\| f \| = V(f)$, the variation of f over [0,1].

Dual of C[0,1], Riesz Representation Theorem

In 1909, F. Riesz [17] characterized the dual of the space C[0,1], the space of continuous functions on the interval [0,1] with the sup-norm topology. The Riemann-Stieltjes integral is defined to be $\lim_{|\sigma| \to 0} \sum_{\sigma} f(t_i)[g(x_{i+1}) - g(x_i)]$ where $\sigma = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is a partition of [0,1], $x_{i-1} \leq t_i \leq x_i$, $i = 1,2, \cdots n$ and $|\sigma| = \max \{|x_i - x_{i-1}| : i = 1,2,\cdots,n\}$.

Theorem 2.1

For each $g \in BV[0,1]$, the space of the functions of bounded variation on [0,1], F defined by $F(f) = \int_0^1 f dg$ for each $f \in C[0,1]$ is a continuous linear functional on C[0,1], where the above integral is the Riemann-Stieltjes integral. Furthermore, $\|F\| \leq V(g)$.

Proof. Clearly F is linear since the integral depends linearly on $f \varepsilon C[0,1]$, and on $g \varepsilon BV[0,1]$. For arbitrary partition $0 = x_o < x_1 < \cdots < x_n = 1$, and $x_{i-1} \leq t_i \leq x_i$ for each i, we have

$$\left|\sum_{i=1}^{n} f(t_i)[g(x_i) - g(x_{i-1})]\right| \leq \sum_{i=1}^{n} |f(t_i)||g(x_i) - g(x_{i-1})| \leq$$

$$\| f \| \sum_{i=1}^{n} | g(x_i) - g(x_{i-1})|$$

from which it follows that $\left|\int_0^1 fdg\right| \leq \| f \| V(g)$ and hence $\|F\| \leq V(g)$.

Any two choices of g which differ by an additive constant define the same functional F. Thus, we may consider only the functions in BV[0,1] which are zero at zero. However, the following example shows that it is necessary to impose a further condition on g in order to obtain the uniqueness of g.

Example 2.2

Let $g_1(x) = 0$ if $x < 1/2$ and $g_1(x) = 1$ if $x \geq 1/2$, and let $g_2(x) = 0$ if $x \leq 1/2$ and $g_2(x) = 1$ if $x > 1/2$. For each $f \varepsilon C[0,1]$, $\int_0^1 fdg_1 = \int_0^1 fdg_2 = f(1/2)$. Although both g_1 and g_2 vanish at zero, they are distinct and generate the same functional.

The following lemma will eliminate this ambiguity.

Lemma 2.3

For each $g \varepsilon BV[0,1]$, there exists a unique $\bar{g}$ which is zero at zero and continuous from the right such that $\int_0^1 fdg = \int_0^1 fd\bar{g}$ for each $f \varepsilon C[0,1]$ and $V(\bar{g}) \leq V(g)$.

Proof. Let $g \varepsilon BV[0,1]$ and define $\bar{g}$ as follows.

$\bar{g}(0) = 0$, $\bar{g}(1) = g(1) - g(0)$ and $\bar{g}(t) = g(t^+) - g(0)$ if $0 < t < 1$. Clearly $\bar{g}$ is continuous from the right. The uniqueness follows from the definition of $\bar{g}$. For arbitrary partition $0 = t_o < t_1 < \cdots < t_n = 1$, choose $s_1, s_2, \cdots, s_{n-1}$ at which g is continuous with each s_k sufficiently close that $|g(t_k^+) - g(s_k)| < \varepsilon/_{2n}$. If $s_o = 0$ and $s_n = 1$, then $\sum_{k=1}^{n} |\bar{g}(t_k) - \bar{g}(t_{k-1})| \leq \sum_{k=1}^{n} |g(s_k) - g(s_{k-1})| + \varepsilon \leq V(g) + \varepsilon$ from which it follows that $V(\bar{g}) \leq V(g)$. For any $f \varepsilon C[0,1]$, we have

$|\sum_{k=1}^{n} f(x_i)[\bar{g}(t_k) - \bar{g}(t_{k-1})]| \leq |\sum_{k=1}^{n} f(x_i)[g(s_k) - g(s_{k-1})]| + \|f\| \varepsilon$, hence $\int_0^1 f d\bar{g} = \int_0^1 f dg$ for each $f \varepsilon C[0,1]$.

Let BVN[0,1] denote the subspace of functions in BV[0,1] which are zero at zero and continuous from the right. Therefore, each $g \varepsilon BVN[0,1]$ determines a unique functional F on C[0,1].

Theorem 2.4

For each F in the dual of C[0,1], there exists a unique $g \varepsilon BVN[0,1]$ such that $F(f) = \int_0^1 f dg$ for every $f \varepsilon C[0,1]$. Furthermore, $\|F\| = V(g)$.

Proof. Consider C[0,1] as a subspace of the space M[0,1] of bounded functions on [0,1] with the sup-norm topology. For each $f \varepsilon M[0,1]$, p is defined by $p(f) = \|F\| \|f\|$ has the properties that $p(f_1 + f_2) \leq p(f_1) + p(f_2)$ for any $f_1, f_2 \varepsilon M[0,1]$, $p(\alpha f) = \alpha p(f)$ for $\alpha \geq 0$ for each $f \varepsilon M[0,1]$ and finally $F(f) \leq p(f)$ on C[0,1]. It follows from the Hahn-Banach theorem [5] that F has a norm-preserving extension to all of M[0,1]. For each $x \varepsilon [0,1]$ let ϕ_x be the characteristic function on [0,x). Since F has been extended to all of M[0,1], we may define $g(x) = F(\phi_x)$. If $0 = x_o < x_1 < \cdots < x_n = 1$ is any partition of [0,1], then

$$\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| = \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})] \operatorname{sgn} [g(x_i) - g(x_{i-1})]$$

$$= F\left(\sum_{i=1}^{n} (\phi_{x_i} - \phi_{x_{i-1}}) \operatorname{sgn} [g(x_i) - g(x_{i-1})]\right)$$

$$\leq \| F \| \left\| \sum_{i=1}^{n} (\phi_{x_i} - \phi_{x_{i-1}}) \operatorname{sgn} [g(x_i) - g(x_{i-1})] \right\| = \| F \|.$$

Hence $g \varepsilon BVN[0,1]$ so that $V(\bar{g}) \leq V(g)$ and $\int_0^1 f d\bar{g} = \int_0^1 f dg$ for each $f \varepsilon C[0,1]$.

Let $f \varepsilon C[0,1]$ and define f_n by $f_n(t) = \sum_{k=1}^{n} f(\frac{k}{n})[\phi_{k/n}(t) - \phi_{(k-1)/n}(t)]$.

Each f_n is a step function having the value $f(\frac{k}{n})$ on the interval $[(k-1)/n, k/n]$ for $k = 1,2,\cdots,n$. Hence $\{f_n\}$ converges uniformly to f and by the continuity of F, $\{F(f_n)\}$ converges to $F(f)$. Now $F(f_n) = \sum_{k=1}^{n} f(\frac{k}{n})[F(\phi_{k/n}) - F(\phi_{(k-1)/n})] = \sum_{k=1}^{n} f(\frac{k}{n})[g(\frac{k}{n}) - g(\frac{(k-1)}{n})]$ from which it follows that $\{F(f_n)\}$ converges to $\int_0^1 f dg$. Therefore $F(f) = \int_0^1 f dg$ for each $f \varepsilon C[0,1]$. Now we can apply Theorem 2.4 to prove the following.

Theorem 2.5

If $\{f_n\}$ is a sequence of functions from C[0,1], then $\{f_n\}$ converges weakly ($\{F(f_n)\}$ converges for each $F \varepsilon C^*[0,1]$) to $f \varepsilon C^{**}[0,1]$ if and only if $\{\| f_n \|\}$ is uniformly bounded and $\{f_n\}$ converges pointwise to f.

Proof. For each $x \varepsilon [0,1]$, F_x defined by $F_x(f) = f(x)$ for each $f \varepsilon C[0,1]$ is a linear functional on C[0,1]. Therefore, if $\{f_n\}$ converges weakly to f, then $\{f_n\}$ converges pointwise to f and $\{\| f_n \|\}$ is uniformly bounded by the uniform boundedness principle [5].

Conversely, for each $F\varepsilon C^*[0,1]$ there is a unique $g\varepsilon BVN[0,1]$ such that $F(f) = \int_0^1 fdg$ for each $f\varepsilon C[0,1]$ by Theorem 2.4. Since each dg generates a unique regular bounded additive set functions $\mu(dg)$ defined on the field generated by the closed sets [18] so that $F(f) = \int_0^1 fdg = \int_0^1 fd\mu(dg)$. If $\{f_n\}$ converges pointwise to f, then f is Lebesgue integrable and f is bounded since $\{\|f_n\|\}$ is uniformly bounded. Therefore by the Lebesgue Dominated Convergence Theorem [18] we have

$\lim_n \int_0^1 f_n d\mu(dg) = \int_0^1 fd\mu(dg) = \lim_n F(f_n) = F(f)$ from which it follows that the sequence $\{f_n\}$ converges weakly to f.

Representation in the Vector-Valued Setting

In this section X and Y denote linear normed spaces with the supnorm, C the set of X-valued continuous functions defined on [0,1] and B the space B[X,Y] of bounded linear operators from X into Y. If T is a linear operator from X into Y, then T is continuous if and only if T is bounded [18]. Thus we shall use "bounded linear operator" and "continuous linear operator" interchangeably. If X and Y are the spaces of real numbers a representation is given in the previous secion in terms of the Riemann-Stieltjes integral. In 1936, M. Gowurin [6] wrote a paper on the Stieltjes integral for vector valued functions as follows. If $K(t)\varepsilon B$ for each $t\varepsilon[0,1]$ then K is said to have the Gowurin w-property provided that there exists a constant $M > 0$ such that for each partition $0 = t_o < t_1 < \cdots < t_n = 1$ and each subset $\{x_i\}_{i=0}^{n-1}$ of X, $\left\|\sum_{i=0}^{n=1}[K(t_{i+1}) - K(t_i)]x_i\right\| \leq M \max_{(i)}\|x_i\|$. This is equivalent to bounded variation when K(t) is real and X is the space of real numbers.

The smallest such constant will be denoted by WK. The Gowurin w-property is called the semi-variation. The total variation is defined using $\sum_{i=0}^{n-1} \|[K(t_{i+1}) - K(t_i)]x_i\| \leq M \max_{(i)} \| x_i \|$. Therefore in the case of real-valued setting the total variation and semi-variation are equivalent.

The integral used throughout this section is defined as follows. For $f \varepsilon C$ and $K(t) \varepsilon B$ for each $t \varepsilon [0,1]$, $\int_0^1 dkf = \lim_{\sigma} \sum_{\sigma} [K(t_{i+1}) - K(t_i)] f(\xi_i)$ whenever this limit exists. The following theorem concerning conditions under which the integral exists was first shown by Gowurin [6].

Theorem 3.1

If Y is complete, then $\int_0^1 dKf$ exists for all $f \varepsilon C$ if K has the w-property and this integral defines a continuous linear operator from C into Y. Furthermore, for each $f \varepsilon CR$, the space of continuous real valued functions on [0,1] with the sup-norm, $\lim_{\sigma} \sum_{\sigma} [K(t_{i+1}) - K(t_i)] f(\xi_i)$ exists in $\bar{B}$ and hence is denoted by $\int_0^1 dKf$.

Proof. Since $f \varepsilon C$ is uniformly continuous on [0,1], given $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $\| f(t) - f(s) \|_X < \varepsilon$ for all t and s with $| t - s | < \delta$. Let σ and σ' be any two partitions with the mesh fineness less than δ. Then,

$$\| \sum_{\sigma} [K(t_{i+1}) - K(t_i)] f(\xi_i) - \sum_{\sigma'} [K(t_{j+1}) - K(t_j)] f(\eta_j) \|$$

$= \| \sum_{\sigma \cup \sigma'} [K(t_{k+1}) - K(t_k)] (f(\xi_\nu) - f(\eta_\mu) \| \leq \varepsilon WK$. It follows that $\int_0^1 dKf$ exists because Y is complete. Let $f \varepsilon CR$, then for each $x \varepsilon X$ $f(t)x \varepsilon C$. Existence of $\int_0^1 dKf$ is shown exactly as above. For any partition σ,

$$\| \sum_{\sigma} [K(t_{i+1}) - K(t_i)] f(\xi_i) x \| \leq WK \max_{(i)} \| f(\xi_i) x \|$$

$= WK \max_{(i)}|f(\xi_i)|\,\|x\|$. It follows that $\|\sum_\sigma[K(t_{i+1}) - K(t_i)]f(\xi_i)\|_B$ $\leq WK \max_{(i)}|f(\xi_i)|$. Hence the norm of $\int_0^1 dKf$ in $\bar{B}$ is less than or equal to WK.

The following example obtained by D. H. Tucker [20] shows that not all of B[C,Y] are represented by the integral $\int_0^1 dK(\cdot)$ where K has values in B[X,Y].

Example 3.2

Let X be the space of real numbers and I be the identity operator on C = Y, then B[X,Y] = B[R,Y] = B[R,C]. For each $x\varepsilon[0,1]$, let K(x) be $\chi_{(0,x]}$, the characteristic function on (0,x]. We show that K generates the transformation I.

Let $f\varepsilon C$, then $\int_0^1 dKf = \lim_\sigma \sum_\sigma[K(t_{i+1}) - K(t_i)]f(\xi_i) = \lim_\sigma \sum_\sigma \chi_{(t_i, t_{i+1}]} f(\xi_i)$

$= f = I(f)$. But $K(x) \notin B[X,Y]$ except x equals 0 or 1 and we see that $K(x)\varepsilon B[X,Y^+]$ by Theorem 2.5.

D. H. Tucker [20] represented the linear operators from C into Y. Although the development involved in this is far more complicated than that of the previous section, the historical methods of building an integral representation is clearly observable. Since continuous functions are approximated by step functions which are discontinuous, we first investigate the weak sequential extension of arbitrary linear normed space S.

Lemma 3.3

The weak sequential extension S^+ of S, the space of equivalence classes of weakly convergent sequences in S, can be viewed as a linear normed space and the inclusions $\bar{S} \subset S^+ \subset \bar{S}^+ \subset S^{**}$ hold ismetrically and isomorphically where $\bar{S}$ and $\bar{S}^+$ denote the closure of S and S^+, respectively.

<u>Proof.</u> Suppose $\{s_n\}$ is a sequence of points in S which converges weakly. Since S can be imbedded isometrically and isomorphically in S^{**}, $\{s_n\}$ may be considered as a sequence in S^{**}, where the identification $s \leftrightarrow s^{**}$ is given by $s^*(s) = s^{**}(s^*)$ for each $s^* \varepsilon S^*$. Thus for each $s^* \varepsilon S^*$, $\lim_n s_n(s^*) = s^{**}(s^*)$ exists and s^{**} is linear. By the uniform boundedness principle the sequence $\{s_n\}$ is bounded and $\| s^{**}(s^*) \| = \lim_n \| s_n(s^*) \| \leq \lim_n \| s_n \| \| s^* \| \leq \| s^* \| \sup_n \| s_n \|$ from which it follows that s^{**} is bounded and $\|s^{**}\| \leq \sup_n \| s_n \|$. Define the norm on S^+ by $\| \{s_n\} \| = \| s^{**} \|_{S^{**}} = \sup_{\|s^*\| \leq 1} \|s^{**}(s^*)\|$

$= \sup_{\| s^* \| \leq 1} |\lim_n s^*(s_n)|$. If $\{s_n\}$ converges weakly to $s \varepsilon S$, then $\| \{s_n\} \| = \| s \|_S$.

<u>Lemma 3.4</u>

If $K \varepsilon B^+$, then K represents a bounded linear operator $\bar{K}$ from X into Y^+.

<u>Proof.</u> Let $\{b_n\}$ be an element in K and $y^* \varepsilon Y^*$. Then $y^*(b_n) \varepsilon X^*$ and $\| y^*(b_n) \| \leq \| y^* \| \| b_n \|$. For a fixed $x \varepsilon X, y^*(\cdot)x \varepsilon B^*$ and since $|y^*(b)x| \leq \| y^* \| \| b(x) \| \leq \| y^* \| \| b \| \| x \|$, $\| y^*(\cdot)x \| \leq \| y^* \| \| x \|$ and $\{ y^*(b_n)x \}$ converges. Let $b_n(x) = x_n \varepsilon Y$, then $\{x_n\}$ converges weakly in Y and $\{x_n\} \varepsilon \bar{K}(x) \varepsilon Y^+$. $\bar{K}(x)$ may be considered an element of Y^{**} and hence a linear operator from X into Y^+. If $\| x \|_X = 1$, then $\| \bar{K}(x) \|_{Y^+} = \sup_{\| y^* \| \leq 1} |\lim_n y^*(b_n)x| \leq \sup_{\| y^* \| \leq 1} \lim_n \| y^* \| \|b_n(x)\|$

$= \lim_n \| b_n(x) \| \leq \sup_{\| b^* \| \leq 1} |\lim_n b^*(b_n)| = \| K \|_{B^+}$ and hence $\| \bar{K} \|_{B[X,Y^+]} \leq \| K \|_{B^+}$.

Lemma 3.5

If T is a continuous linear operator from C into Y, then T has a norm preserving extension T** from C** into Y** and hence from C^+ into Y^+.

Proof. Define a function T* from Y* into C* by taking (T*y*)(x) = y*(Tx) for each xεC. We show that T*εB[Y*,C*] and $\|T^*\| = \|T\|$. By the definition of T* the linearity of T* follows immediately. For any xεC and y*εY*, $|(T^*y^*)(x)| = |y^*(Tx)| \leq \|y^*\| \, \|Tx\|$ $\leq \|y^*\| \, \|T\| \, \|x\|$. Thus $\|T^*y^*\| \leq \|y^*\| \, \|T\|$ from which it follows that $\|T^*\| \leq \|T\|$ and T*εB[Y*,C*]. Let $\varepsilon > 0$ and xεC with $\|x\| \leq 1$ and $\|Tx\| > \|T\| - \varepsilon$. Choose y*$\varepsilon$Y* so that $\|y^*\| = 1$ and $|y^*(Tx)|$ $= \|Tx\|$ [5]. Then $|(T^*y^*)(x)| = |y^*(Tx)| = \|Tx\| > \|T\| - \varepsilon$, hence $\|T^*y^*\| \geq \|T\| - \varepsilon$ since $\|x\| \leq 1$. It follows that $\|T^*\| \geq \|T\| - \varepsilon$ since $\|y^*\| = 1$, and hence $\|T^*\| \geq \|T\|$. We now have the equality $\|T^*\| = \|T\|$. Similarly, we may define T** from C** into Y** so that T**εB[C**,Y**] and $\|T^{**}\| = \|T^*\| = \|T\|$.

Lemma 3.6

If CR is the space of real valued continuous functions on [0,1] with the sup-norm, then for fεCR and xεXf(t)xεC and T(f(t)x) = T(f•x) induces a continuous linear operator $\mathcal{T}$ from CR into B by taking $\mathcal{T}$(f)x = T(f•x) and $\|\mathcal{T}\| \leq \|T\|$ [19].

Proof. $\mathcal{T}(f)(a_1x_1 + a_2x_2) = T[f(t)(a_1x_1 + a_2x_2)] = a_1T(f(t)x_1)$ $+ a_2T(f(t)x_2) = a_1\mathcal{T}(f)x_1 + a_2\mathcal{T}(f)x_2$.

$\|\mathcal{T}(f)x\| = \|T(f(t)x)\|_Y \leq \|T\| \int_0^1 \|f(t)x\|_X dt$

$= \|T\| \int_0^1 |f(t)| dt \, \|x\|_X \leq \|T\| \, \|f\|_{CR} \|x\|_X$, hence $\|\mathcal{T}\| \leq \|T\|$.

$[a\mathcal{T}(f) + b\mathcal{T}(g)]x = T(af(t)x) + T(bg(t)x) = T[af(t)x + bg(t)x]$

$= T[(af(t) + bg(t))x] = \mathcal{T}(af + bg)x$.

Theorem 2.5 shows that $(CR)^+$ contains the step functions. Therefore, if $f_1, f_2, \cdots, f_n$ are characteristic functions of subintervals of $[0,1]$ and $x_1, x_2, \cdots, x_n$ are points in X, then $\mathcal{T}(f_1)x_1 + \cdots + \mathcal{T}(f_n)x_n = T(f_1x_1 + \cdots + f_nx_n)$. We may identify $\mathcal{T}$ with T and make no notational distinction between them and their extensions to $(CR)^+$ and C^+, respectively.

Lemma 3.7

Let G_{ab} denote the characteristic function on $[a,b)$ and define $K(t) = \mathcal{T}(G_{ot}(s))$ if $0 < t < 1$, $K(0) = \mathcal{T}(0)$ and $K(1) = \mathcal{T}(1)$. Then K has the w-property and $WK \le \| T \|$.

Proof. If $0 = t_o < t_1 < \cdots < t_n = 1$ is a partition of $[0,1]$ and $x_o, x_1, \cdots, x_{n-1}$ are points in X, then

$$\left\| \sum_{i=o}^{n-1} [K(t_{i+1}) - K(t_i)]x_i \right\|_{Y^+} = \left\| \sum_{i=o}^{n-1} \mathcal{T}_1(G_{t_i t_{i+1}}(s))\, x_i \right\|_{Y^+}$$

$$= \left\| T[\sum_{i=o}^{n-1} G_{t_i t_{i+1}}(s)x_i] \right\|_{Y^+} \le \| T \| \max \| x_i \| .$$

Suppose $f \varepsilon C$, then the sequence of step functions $\{\phi_n\}$ converges uniformly to f where each $\phi_n(t) = \sum_{i=o}^{n-1} G_{i/n,\,(i+1)/n}(t) f(\xi_i)$ and $i/n \le \xi_i \le (i+1)/n$, and $T(\phi_n) = \sum_{i=1}^{n-1} \mathcal{T}(G_{i/n,\,(i+1)/n}) f(\xi_i)$
$= \sum_{i=1}^{n-1} [K((i+1)/n) - K(i/n)] f(\xi_i)$.

Theorem 3.8

$\{T(\phi_n)\}$ converges to $T(f)$ in the norm of Y^+.

Proof. Since each step function is a weak limit of continuous functions, we first construct a sequence $\{\theta_{n,m}\}_{m \ge 2n}$ of continuous functions so that $\{T(\theta_{n,m})\}$ converges to $T(\phi_n)$ in the norm of Y^+ as m tends to infinity. Let $\theta_{n,m}(t) = f(\xi_o)$ if $0 \leq t \leq {}^{i}/_{n} - {}^{i}/_{m}$; $\theta_{n,m}(t) = \lambda f(\xi_i) + (1 - \lambda)f(\xi_{i-1})$, $0 \leq \lambda \leq 1$, if ${}^{1}/_{n} - {}^{1}/_{m} \leq t \leq {}^{i}/_{n}$; $\theta_{n,m}(t) = f(\xi_{n-1})$ if ${}^{(n-1)}/_{n} \leq t \leq 1$ and $\theta_{n,m}(t) = f(\xi_i)$ if ${}^{i}/_{n} \leq t \leq {}^{(i+1)}/_{n} - {}^{1}/_{m}$. Clearly $\|\theta_{n,m}\| = \|\phi_n\| = \max \|f(\xi_i)\|$ for each m. Since $\|\theta_{n,m} - f\|_C = \max \|\theta_{n,m}(t) - f(t)\|_X$ and this maximum value is assumed in one of the four cases in the definition of $\theta_{n,m}$ it is dominated by twice the maximum oscillation of f on an interval of length ${}^{2}/_{n}$. From the continuity the oscillation of f on any interval tends to zero as the length of the interval tends to zero and hence $\{T(\theta_{n,m})\}$ converges weakly to $T(f)$ as n tends to infinity. Now T of the nth row of the triangular array $\{\theta_{n,m}\}_{m \geq 2n}, n = 1,2,\cdots,$ is an element of $T(\phi_n)$ and each sequence $\{\theta_{n_i,m_i}\}$ converges uniformly to f as n_i tends to infinity.

If we consider $T(f) \varepsilon Y$ as a point in Y^+, $T(f)$ is an equivalence class. We shall show that the sequence $\{T(\phi_n)\}$ of equivalence classes converges to $T(f)$ in the norm of Y^+. By construction $\{T(\theta_{n,m})\}_{m \geq 2n}$ is an element of the equivalence class $T(\phi_n)$ and will be used as a representative element of the class. For a representative element of the equivalence class $T(f)$, $\{T(f_m)\}$ where $f_m = f$ for each m will be used.

$$\| T(\phi_n) - T(f) \|_{Y^+} = \sup_{\|y^*\| \leq 1} \left| \lim_m y^*[T(\theta_{n,m}) - T(f_m)] \right|$$

$$= \sup_{\|y^*\| \leq 1} \left| \lim_m y^*[T(\theta_{n,m} - f_m)] \right| \leq \sup_m \|T\| \, \|\theta_{n,m} - f_m\|_C .$$ As shown before $\|\theta_{n,m} - f_m\|_C$ tends to zero as n tends to infinity from which it follows that $\{T(\phi_n)\}$ converges to $T(f)$ in the norm of Y^+. Therefore

$T(f) = \int_0^1 dKf$ makes sense where the convergence of the integral is in the norm of Y^+. Moreover, $\| T(f) \|_Y = \| T(f) \|_{Y^+} = \| \int dKf \|_{Y^+}$

$$= \lim_n \| \sum_{i=0}^{n-1} [K(^{i+1}/_n) - K(^i/_n)] f(\xi_i) \|_{Y^+}$$

$$\le \sup_i WK \max \| f(\xi_i) \| \le WK \| f \|_C.$$

Thus $\|T\| \le Wk$ and together with the reverse inequality established in Lemma 3.7, we have $\|T\| = WK$.

We now have the following theorem.

Theorem 3.9

If T is a bounded linear operator from C into Y, then there exists a function K on [0,1] with values in B^+, hence in $B[X,Y^+]$, such that K has the w-property and $T(f) = \int_0^1 dKf$ for each $f \varepsilon C$ where the convergence of the integral is in the norm of Y^+. Furthermore, $\| T \| = WK$.

As Example 3.2 shows, the converse of Theorem 3.9 is not true. Given a funtion K with the w-property and its values in B^+, K determines a continuous linear operator from C into $\bar{Y}^+$, the completion of Y^+. The question under what conditions K determine a continuous linear operator from C into Y has been answered in [15] for the case in which X and Y are both complete, and a complete characterization is given in [10].

Now we will investigate the uniqueness of the function K.

Lemma 3.10

If K has the w-property with values in $B[X,Y^+]$ for each $t \varepsilon [0,1]$, then for each $F \varepsilon B^*[X,Y^+]$, FK is of bounded variation on [0,1] and $V(FK) \le \| F \| WK$.

Proof. If $0 = t_o < t_1 < \cdots < t_n = 1$ is a partition of [0,1]; and each $s_i = \text{sgn}\, F[K(t_{i+1}) - K(t_i)]$, then

$$\sum_{i=o}^{n-1} |FK(t_{i+1}) - FK(t_i)| = \sum_{i=o}^{n-1} |F[K(t_{i+1}) - K(t_i)]|$$

$$= \sum_{i=o}^{n-1} s_i F[K(t_{i+1}) - K(t_i)]$$

$$= |F \sum_{i=o}^{n-1} s_i [K(t_{i+1}) - K(t_i)]| \leq \|F\| \left\| \sum_{i=o}^{n-1} s_i [K(t_{i+1}) - K(t_i)] \right\|_{B^+}$$

$$= \|F\| \sup_{\|x\| \leq 1} \left\| \sum_{i=o}^{n-1} s_i [K(t_{i+1}) - K(t_i)] x \right\|_{Y^+}$$

$$= \|F\| \sup_{\|x\| \leq 1} \left\| \sum_{i=o}^{n-1} [K(t_{i+1}) - K(t_i)](s_i x) \right\|_{Y^+} \leq \|F\| \ WK.$$

Theorem 3.11

Two functions K_1 and K_2 generate the same continuous linear operator T from C into Y if and only if (i) each of K_1 and K_2 generates such a T and (ii) there exists a point $d \varepsilon B^+$ such that $K_1(0) - K_2(0) = K_1(1) - K_2(1) = d$ and for each $F \varepsilon B^*[X, Y^+]$, $F[K_1(t) - K_2(t)] = F(d)$ except on a countable set E and that $\sum_E | F[K_1(t) - K_2(t)]|$ is finite.

Proof. Suppose K_1 and K_2 have the w-property and generate the same continuous linear operator T from C into Y, then $K = K_1 - K_2$ generates the zero operator on C. There is no loss of generality in assuming $K(0) = 0$, for if any two K differ by an additive constant then they generate the same operator. For each $F \varepsilon B^*[X, Y^+]$, $\int_0^1 dFK \cdot f = F \int_0^1 dKf$ for each $f \varepsilon CR$ since $\lim_\sigma \sum_\sigma [FK(t_{i+1}) - FK(t_i)] f(\xi_i) = \lim_\sigma F(\sum_\sigma [K(t_{i+1}) - K(t_i)] f(\xi_i))$ by Theorem 3.1. Since $\int_0^1 dKg = 0$ for each $g \varepsilon C$, $\int_0^1 dK(f \cdot x) = 0$ for each $f \varepsilon CR$ and each $x \varepsilon X$ and it follows that $\int_0^1 dKf = 0$ and $\int_0^1 dFK \cdot f = 0$ for each $f \varepsilon CR$. $K(0) = 0$ implies

$FK(0) = 0$ and $\int_0^1 dFK\cdot 1 = 0$ implies $FK(1) = 0$ for each F, hence $K(1) = 0$. Let $f_s(t) = t$ if $0 \le t \le s$ and $f_s(t) = s$ if $s \le t \le 1$ where $0 \le s \le 1$. An integration by parts yields $\int_0^1 dFK\cdot f_s = -\int_0^1 FKdf_s = -\int_o^s FK(t)dt$ from which it follows that $FK(t) = 0$ a.e. Since FK is of bounded variation, the set $E = \{t\varepsilon[0,1] | FK(t) \neq 0\}$ is countable and $\sum_E |FK(t)|$ is finite.

To show the converse suppose K_1 and K_2 generate T_1 and T_2, respectively and the conditions (i) and (ii) are satisfied. Let $K = K_1 - K_2$ and assume $K(0) = K(1) = 0$. For each $F\varepsilon B^*[X,Y^+]$, (ii) implies that FK is of bounded variation and $\int_0^1 FKdf = 0$ for each $f\varepsilon CR$. An integration by parts shows that $F\int_0^1 dKf = 0$ for each $f\varepsilon CR$ and each $F\varepsilon B^*[X,Y^+]$. Therefore $\int_0^1 dKf = 0$ for each $f\varepsilon CR$. Since K has the w-property, $T(g) = \int_0^1 dKg$ exists for each $g\varepsilon C$ and $||T|| \le WK$. Let $g_n(t) = \sum_{k=o}^{n} \binom{n}{k} t^k(1-t)^{n-k} g(^k/_n)$ for each n, then $\{g_n\}$ converges uniformly to g. By the continuity of T $\{T(g_n)\}$ converges to $T(g)$. $T(g_n) = \sum_{k=o}^{n} \binom{n}{k} [\int_0^1 dKt^k(1-t)^{n-k}] g(^k/_n) = 0$, for $\int_0^1 dKt^k(1-t)^{n-k} = 0$ for each k, from which it follows that $T(g) = 0$ for each $g\varepsilon C$.

A Unifying Representation Theorem

The technique used in the previous section motivated the work in this section [7]. The main result in this section generarizes many representation theorems [3,4,14,17,20,21, and 22] and gives a road map for obtaining a representation.

Let H be an arbitrary point set, Σ be a field of sets in p(H), the power class consisting of all subsets of H, X and Y be linear normed spaces and B[X,Y] be the space of bounded linear operators from X into Y. Let F denote a linear normed space of functions from H into X with a norm not stronger than the sup-norm. The space of X valued simple functions over Σ is denoted by $S(\Sigma,X)$. The elements in $S(\Sigma,X)$ are of the form $\sum_{i=1}^{n} \chi_{E_i} x_i$ where $E_i \varepsilon \Sigma, E_i \cap E_j = \phi$ if $i \neq j$, and $x_i \varepsilon X$. The set function K from Σ into B[X,Y] is said to be quasi-Gowurin relative to the norm $\| \cdot \|_{S(\Sigma,X)}$ on $S(\Sigma,X)$ provided there exists a positive constant M such that for each partition $p = \{E_i\}$ of H into elements of Σ and elements $\{x_i\}$ in X,

$\| \sum_p [K(E_i)]x_i \|_Y \leq M \| \sum_p \chi_{E_i} x_i \|_{S(\Sigma,X)}$. The smallest such constant M will be denoted by WK. In the case the norm on $S(\Sigma,X)$ is the sup-norm, this definition agrees the Gowurin w-property [3, 19, 20, and 21] and in the case H = [0,1], then WK is the Gowurin w constant as defined by Gowurin [6].

The following lemma is straightforward and hence is stated without proof.

Lemma 4.1

Suppose $f \varepsilon cl\{S(\Sigma,X)\}$ under the sup-norm. Then for each net of partitions of H, $\{\{E_i^p\}_{i=1}^{n(p)}\}p$, which is cofinal with the net of partitions of H (there always exists a partition in the net of partitions of H which is finer than a given one), $\lim_p \Sigma\chi_{E_i} x_i^p = f$ where convergence is in the sup-norm and for each pair p and i, $x_i^p \varepsilon f(E_i)$.

From Lemma 4.1 it follows that if $\| \cdot \|'$ is a norm on $S(\Sigma,X)$ which is not stronger than the sup-norm and if K is quasi-Gowurin with respect to $\| \cdot \|'$, then f is K - integrable, i.e., $\int dKf$ exists. The following lemma is also immediate.

Lemma 4.2

If each f_n and f are K integrable and $\lim_n f_n = f$ in the sup-norm, then $\lim_n \int dKf_n = \int dKf$.

Proof. $\| \int dKf_n - \int dKf \|_Y \leq WK \| f_n - f \|$.

Theorem 4.3

Suppose F is a linear normed space with norm $\| \cdot \|'$ which is not stronger than the sup-norm, and suppose that there is a field of sets Σ such that the norm $\| \cdot \|'$ can be extended to $S(\Sigma,X)$ and such that there is a subspace $S'(\Sigma,X) \subset S(\Sigma,X)$ satisfying (i) $F \subset cl\{S'(\Sigma,X)\}$ under the sup-norm and (ii) there is a linear operator $\theta: S(\Sigma,X) + F \to F^+$ which is the identity on F and is continuous when restricted to each of $S(\Sigma,X)$ and $S'(\Sigma,X) + F$, both under the norm $\| \cdot \|'$. Then if T is a continuous linear operator from F into Y, there is a finitely additive set function K on Σ with values in $B[X,Y^+]$ which is quasi-Gowurin with respect to $\| \cdot \|'$ such that $T(f) = \int dKf$ for each $f \varepsilon F$. Furthermore, ${}^{WK}/_{\| \theta \|_1} \leq \| T \| \leq WK$ where $\| \theta \|_1$ is the norm of θ restricted to $S(\Sigma,X)$.

Proof. By Lemma 3.5, T has a norm preserving extension T^+ from F^+ into Y^+. Define the set function K from Σ into $B[X,Y^+]$ by taking $K(E)x = T^+(\theta(\chi_E x))$. Clearly K is finitely additive.

Suppose $\{E_i\}$ is a partition of H over Σ and that $\{x_i\}$ is a corresponding subset of X. Then

$\| \Sigma[K(E_i)]x_i \| = \| T^+(\theta(\Sigma \chi_{E_i} x_i)) \| \leq \|T^+\| \, \| \theta \|_1 \, \| \Sigma \chi_{E_i} x_i \|'$ from which it follows that K is quasi-Gowurin and $WK \leq \| T \| \, \| \theta \|_1$ or equivalently $\| T \| \geq {}^{WK}/_{\| \theta \|_1}$.

For any $f \varepsilon F \subset cl\{S'(\Sigma,X)\}$, $\int dKf$ exists in $\bar{Y}$, the completion of Y, by Lemma 4.1 and there is a sequence $\{s_n\} = \{\Sigma \chi_{E_i^n} x_i^n\}$ which converges to f in the sup-norm. By Lemma 4.2 $\lim_n \Sigma[K(E_i^n)]x_i^n = \lim_n \int dKs_n = \int dKf$. Since $\|\cdot\|'$ is a norm not stronger than the sup-norm, $\{s_n\}$ converges to f in the norm $\|\cdot\|'$. Hence the continuity of θ on $S'(\Sigma,X) + F$ implies that $\{\theta(s_n)\}$ converges to $\theta(f) = f$ in the norm of F^+ and the continuity of T^+ implies that $\{T^+(\theta(s_n))\}$ converges to $T^+(f) = T(f)$. But for each n, $T^+(\theta(s_n)) = \Sigma[K(E_i^n)]x_i^n$ and it follows that $T(f)$ $= \lim_n T^+(\theta(s_n)) = \lim_n \Sigma[K(E_i^n)]x_i^n = \lim_n \int dKs_n = \int dKf$. Since K is quasi-Gowurin,

$\|T(f)\| = \|\int dKf\| \lim_p \|\Sigma K(E_i^p)f(t_i^p)\| \leq \overline{\lim_p} WK \|\Sigma \chi_{E_i^p} f(t_i^p)\|$, where $t_i^p \varepsilon E_i^p$. It follows from Lemma 4.1 that $\Sigma \chi_{E_i^p} f(t_i^p)$ converges to f in the sup-norm and hence in the norm $\|\cdot\|'$. Since $\overline{\lim_p} \|\Sigma \chi_{E_i^p} f(t_i^p)\|'$ $= \|f\|'$, we have the inequality $\|T(f)\| \leq WK \|f\|'$ from which it follows that $\|T\| \leq WK$.

Remark 4.4

If θ in Theorem 4.3 maps into F instead of into F^+, then the definition of K given in the proof of Theorem 4.3 becomes $K(E)x$ $= T(\theta(\chi_E x))$. Therefore K takes its values in $B[X,Y]$ instead of in $B[X,Y^+]$.

Remark 4.5

In general, conditions for the uniqueness of K are not known. For example, if F consists of only the identically zero function, then it is clear that K need not be unique.

Representations for Functionals Continuous in the BV Norm

In this section we are back to the space of real valued continuous functions on the interval [0,1]. This section [8] shows that it is possible to get away from the sup-norm topology on a space of functions and the norm will be changed to the stronger norm. In the BV norm neither the class of step functions nor that of simple functions can get close to the continuous functions. It will be shown that the set of polygonal functions is dense in the subspace AC of absolutely continuous functions in the BV norm.

If any two functions f and g differ by an additive constant, then $\| f - g \|_{BV} = 0$. Hence we shall always choose the representative element from each class in BV[0,1] so that $f(0) = 0$.

Theorem 5.1

The space AC is complete in the BV norm and equals the closure of the set of polygonal functions in the BV norm.

Proof. $f \varepsilon AC$ implies that $f' \varepsilon L^1$ and hence there is a sequence $\{s_i\}$ of step functions converging to f' in the L^1 norm. Let $p_i(x) = \int_0^x s_i d\mu$, then p_i is polygonal and $\| f - p_i \|_{BV} = L \int |(f - p_i)'| \, d\mu = \| f' - s_i \|_{L^1}$. Therefore $\{p_i\}$ converges to f in the BV norm.

Conversely if $\{p_i\}$ is a Cauchy sequence of polygonal functions in the BV norm, then $\{p_i'\}$ is a sequence of step functions and $\| p_i \|_{BV} = \int_0^1 |p_i'| d\mu$ implies that $\{p_i'\}$ is Cauchy in the L^1 norm. Since L^1 is complete, there is a function $f' \varepsilon L^1$ such that $\{p_i'\}$ converges to f' in the L^1 norm, i.e., $\{p_i\}$ converges to f in the BV norm where $f(x) = \int_0^x f' d\mu$.

It is possible to show that a Cauchy sequence $\{p_i\}$ of polygonal functions converges to an AC function with a straightforward calculus type argument which does not depend on any knowledge of L^1. Such a proof gives insight and shows that the sequence $\{p_i\}$ is equi-absolutely-continuous. Since the results in the vector valued setting depend on this proof, we outline the proof as follows.

Since $\{p_i\}$ is Cauchy in the BV norm, it is Cauchy in the sup-norm and the pointwise limit function f is continuous. If σ is any partition of $[0,1]$, then

$$(1)\ \sum_\sigma |f(x_{i+1}) - f(x_i)| \leq \sum_\sigma |f(x_{i+1}) - p_n(x_{i+1})| + \sum_\sigma |p_n(x_{i+1}) - p_n(x_i)| + \sum_\sigma |p_n(x_i) - f(x_i)|.$$

The first and third terms on the right of (1) can be made small for sufficiently larger n and the middle term is less than $\| p_n \|_{BV}$. Since $\sup_n \| p_n \|_{BV}$ is finite, the left side of (1) is bounded independent of σ and is an increasing function of σ. It follows that $f \varepsilon BV$.

Choose M so that $n,m > M$ implies $\| p_n - p_m \|_{BV} < \varepsilon$ and let $\Delta_i f = f(x_{i+1}) - f(x_i)$.

$$\begin{aligned}\sum_\sigma |\Delta_i (f - p_m)| &\leq \sum_\sigma |\Delta_i (f - p_n)| + \sum_\sigma |\Delta_i (p_n - p_m)| \\ &\leq \overline{\lim_n} \sum_\sigma |\Delta_i (f - p_n)| + \overline{\lim_n} |\Delta_i (p_n - p_m)| \\ &\leq 0 + \lim_n \| p_n - p_m \|_{BV} \leq \varepsilon .\end{aligned}$$

It follows $\{p_n\}$ converges to f in the BV norm.

Let $\varepsilon > 0$ and choose M so that $n,m \geq M$ implies $\| p_n - p_m \|_{BV} < \varepsilon/3$. For p_M there exists a $\delta > 0$ such that if $\Sigma |y_i - x_i| < \delta$ then $\Sigma |p_M(y_i) - p_M(x_i)| < \varepsilon/3$. For any finite collection $\{[x_i, y_i]\}$ of intervals with $\Sigma |y_i - x_i| < \delta$, we have

(2) $\Sigma|f(y_i) - f(x_i)| = \Sigma|f(y_i) - p_n(y_i)| + \Sigma|(p_n(y_i) - p_M(y_i)) - (p_n(x_i) - p_M(x_i))| + \Sigma|p_M(y_i) - p_M(x_i)| + \Sigma|p_n(x_i) - f(x_i)|$.

The first and fourth terms can be made less than $\varepsilon/_3$ for sufficiently large n, the second term is dominated by $\| p_n - p_M\|_{BV}$ which is less than $\varepsilon/_3$ and the third term is less than $\varepsilon/_3$ by the absolute continuity of p_M. Therefore $f\varepsilon AC$.

Theorem 5.2

If pf_σ is the polygonal function with corners at exactly the points $(x_i, f(x_i))$ for $x_i \varepsilon \sigma$, then $f\varepsilon AC$ implies $\lim_\sigma pf_\sigma = f$ in the BV norm.

Proof. Let $\{q_n\}$ be a sequence of polygonal functions converging to f in the BV norm and σ be the values of x at which q_n has corners. Since $\sum_\sigma|\Delta_i(pf_\sigma - q_n)| = \| pf_\sigma - q_n \|_{BV}$ and is also an approximating sum to the value of $\| f - q_n \|_{BV}$ which is obtained as the limit with respect to partitions of a nondecreasing function of partitions, $\| pf_\sigma - q_n \|_{BV} \leq \| f - q_n \|_{BV}$. Hence if $\| f - q_n \|_{BV} < \varepsilon$ then for σ' finer than σ $\| f - pf_{\sigma'} \|_{BV} \leq \| f - q_n\|_{BV} + \| q_n - pf_{\sigma'} \|_{BV} \leq \| f - q_n \|_{BV} + \| q_n - f\|_{BV} < 2\varepsilon$.

Theorem 5.3

Let $|\sigma|$ denote the max $\{|x_{i+1} - x_i|\}$ for $x_{i+1}, x_i \varepsilon \sigma$. Then $f\varepsilon AC$ implies $\lim_{|\sigma| \to 0} pf_\sigma = f$ in the BV norm.

Proof. Let σ_1 be a partition such that σ_2 finer than σ_1 implies $\|pf_{\sigma_2} - pf_{\sigma_1}\|_{BV} < \varepsilon/_3$. Let $\delta_1 > 0$ such that $\Sigma|y_i - x_i| < \delta_1$ implies $\Sigma|f(y_i) - f(x_i)| < \varepsilon/_3$ and N be the number of points in σ_1. Choose σ' to be any partition with $|\sigma'| = \delta < \delta_1/_{2N}$ and let $\sigma = \sigma' \cup \sigma_1$.

Since σ is finer than σ_1, $\| pf_\sigma - f \| < \varepsilon/3$. Let $A = \{[x_i, x_{i+1}] | x_i, x_{i+1} \varepsilon \sigma$ and either $x_i \varepsilon \sigma_1$ or $x_{i+1} \varepsilon \sigma_1\}$. Then $[x_i, x_{i+1}] \varepsilon A$ implies $|x_{i+1} - x_i| < \delta < \delta_1/2N$ and hence $\sum_A |x_{i+1} - x_i| < \delta_1$. Since $\| pf_\sigma - pf_{\sigma'} \|_{BV} = \sum_A \| pf_\sigma - pf_{\sigma'} \|_{BV} \leq \sum_A \| pf_{\sigma'} \|_{BV} < \varepsilon/3 + \varepsilon/3$, $\| f - pf_{\sigma'} \|_{BV} \leq \| f - pf_\sigma \|_{BV} + \| pf_\sigma - pf_{\sigma'} \|_{BV} < \varepsilon$ for any σ' with $|\sigma'| < \delta$.

A half-open interval $(a,b] \subset (0,1]$ will be called a fundamental set. The set function K is said to be convex with respect to length provided if the fundamental set H is the finite union of disjoint fundamental sets $\{H_i\}_{i=1}^n$ then $K(H) = \sum_{i=1}^n \lambda_i K(H_i)$ where λ is the ratio of the length of H_i to the length of H. K is said to be fundamentally bounded if for some constant M, $|K(H)| \leq M$ for each fundamental set H and the smallest such constant, denoted by WK, is called the fundamental bound for K. The integral involved in the representation theorems is defined as follows. If K is a set function defined on the fundamental sets, then the v-integral of K with respect to f, denoted by $v \int K df$, is defined to be $v \int K df = \lim_\sigma \sum_\sigma K((x_i, x_{i+1}])[f(x_{i+1}) - f(x_i)]$ whenever this limit exists.

Example 5.4

If f is a continuous function satisfying $f(0) = 0$, then the Riemann integral $R \int_0^1 f dx = v \int_0^1 K_1 df$ where $K_1((a,b]) = 1 - b$.
$v \int_0^1 K_1 df = \lim_\sigma \sum_\sigma (1 - x_{i+1}) [f(x_{i+1}) - f(x_i)] = s \int_0^1 (1 - x) df$ where the last integral is the Stieltjes integral. An integration by parts yields $s \int_0^1 (1 - x) df = f(1) \cdot 0 - f(0) \cdot 1 + R \int_0^1 f dx = R \int_0^1 f dx$. Note that the set function K_1 is fundamentally bounded but not convex with respect to length. If $0 < t < 1$, $s \int_0^t (1 - x) df = f(t)(1 - t)$

$+ R \int_o^t fdx$ and it follows that $R \int_o^t fdx \neq v \int_o^t K_1 df$. The definition of $v \int_o^t K_1 df$ is in the beginning of the next section.

Example 5.5

If f is a continuous function satisfying $f(0) = 0$, then $R \int_0^1 fdx = v\int_0^1 K_2 df$ where $K_2((a,b]) = 1 - {}^{(a + b)}/_2$. Since

$$v \int_0^1 K_2 df = \lim_{\sigma} \sum_{\sigma}(1 - {}^{(x_i + x_{i+1})}/_2)[f(x_{i+1}) - f(x_i)] = s \int_0^1 (1 - x)df,$$

$R \int_0^1 fdx = v \int_0^1 K_2 df$. K_2 is fundamentally bounded and is convex with respect to length.

Example 5.6

Let $K((a,b]) = 0$ if $b \leq 1/2$ or if $a > 1/2$, $K((1/2,b]) = {}^1/_{(b - 1/2)}$, and $K((a,b]) = [(b - 1/2)/(b - a)]K((1/2,b]) = 1/(b - a)$ if $a < 1/2 < b$. Then K is fundamentally bounded and convex with respect to length. Furthermore, $v \int_0^1 Kdf$ equals the right-hand derivative of f at 1/2. Thus $v \int_0^1 Kdf$ generates a continuous linear functional on C_1, the space of continuously differentiable functions on [0,1] with the C_1 norm given by $\| f \|_{C_1} = \| f \|_\infty + \| f' \|_{ess - sup}$. The v-integral is extended to represent the continuous linear operators on spaces of continuously differentiable vector-valued functions in [9].

If $H = (a,b]$, then the function ψ_H is called the fundamental function determined by H where $\psi_H(t) = 0$ if $t \leq a$, $\psi_H(t) = (t - a)/(b - a)$ if $a < t < b$ and $\psi_H(t) = 1$ if $t \geq b$.

Suppose p is a polygonal function anchored at zero which has corners at each point of $\sigma = \{0 = x_o < x_i < \cdots < x_n = 1\}$, and let α_i denote $f(x_{i+1}) - f(x_i)$ for each i. Then $p = \sum_{i=1}^{n-1} \alpha_i \psi_{H_i}$ where $H_i = (x_i, x_{i+1}]$. This shows that the set of fundamental functions forms a basis for the space of polygonal functions which are zero at zero.

Lemma 5.7

Let K be a convex with respect to length set function, p be a polygonal function and $\sigma = \{x_i\}$ be the partition determined by the corners of p. If $\sigma' = \{x_{ij}\}$, $x_i < x_{ij} < x_{i+1}$, is a refinement of σ, then $\sum_\sigma [p(x_{i+1}) - p(x_i)]K(H_i) = \sum_{\sigma'} [p(x_{i(j+1)}) - p(x_{ij})]K(H_{ij})$. Hence it follows that $v \int K dp = \sum_\sigma K(H_i)\Delta_i p$.

Proof. Since p has a constant slope on $[x_i, x_{i+1}]$, for each j $[p(x_{i(j+1)}) - p(x_{ij})]/[p(x_{i+1}) - p(x_i)] = (x_{i(j+1)} - x_{ij})/(x_{i+1} - x_i)$. By the convexity of K, $K(H_i) = \sum_j [(x_{i(j+1)} - x_{ij})/(x_{i+1} - x_i)]K(H_{ij}) = \sum_j \{[p(x_{i(j+1)}) - p(x_{ij})]/[p(x_{i+1}) - p(x_i)]\} K(H_{ij})$ from which the lemma follows.

We now state and prove the main result of this section, a characterization of AC* (the dual of AC with the BV norm).

Theorem 5.8

$T \in AC^*$ if and only if there exists a unique fundamentally bounded set function K which is convex with respect to length such that $T(f) = v \int_0^1 K df$ for each $f \in AC$. Furthermore, $\| T \| = WK$.

Proof. For each fundamental set H let ψ_H denote the corresponding fundamental function and define the set function K by taking $K(H) = T(\psi_H)$. If $f \in AC$, then it follows from Theorem 5.2 that pf_σ converges to f in the BV norm. Thus $T(f) = \lim_\sigma T(pf_\sigma) = \lim_\sigma T(\sum_\sigma \Delta_i pf_\sigma \psi_{H_i}) = \lim_\sigma \sum_\sigma K(H_i)\Delta_i f = v \int K df$. Furthermore,

$$\| T \| = \sup_{\| f \|_{BV} = 1} |T(f)| \geq \sup_{\psi_H} | T(\psi_H)| = \sup_H | K(H)| = WK, \text{ and}$$

$$\| T \| = \sup_{\| f \|_{BV} = 1} |T(f)| = \sup_{\| f \|_{BV} = 1} |v \int Kdf| = \sup_{\| f \|_{BV} = 1} |\lim_{\sigma} \sum_{\sigma} K(H_i)\Delta_i f|$$

$$\leq \sup_{\| f \|_{BV} = 1} \lim_{\sigma} \sum_{\sigma} |K(H_i)| \; |\Delta_i f| \leq \sup_{\| f \|_{BV} = 1} \lim_{\sigma} \sum_{\sigma} WK|\Delta_i f|$$

$$= \sup_{\| f \|_{BV} = 1} WK \; \| f \|_{BV} = WK.$$

Note that the inequality $|v \int Kdf| \leq WK \; \| f \|_{BV}$ always holds. Since $T(\psi_H) = \int Kd\psi_H = K(H)$, K is unique.

Conversely, if K is a fundamentally bounded convex set function, then we will show that $\lim_{\sigma} \sum_{\sigma} K(H_i)\Delta_i f$ exists. From Theorem 5.1 $\sum_{\sigma} K(H_i)\Delta_i f = \Sigma K(H_i)\Delta_i pf_{\sigma} = v \int Kdpf_{\sigma}$. Since $|v \int Kdpf_{\sigma} - v \int Kdp_{\sigma'}|$ $= |\sum_{\sigma \cup \sigma'} K(H_i)\Delta_i(pf_{\sigma} - pf_{\sigma'})| \leq WK \; \| pf_{\sigma} - pf_{\sigma'} \|_{BV}$, $\{v \int Kdpf_{\sigma}\}$ is Cauchy and it follows that $v \int Kdf$ exists. Furthermore if $\{f_n\}$ converges to f in the BV norm then $\lim_n v \int Kdf_n = v \int Kdf$, from which it follows that $T(f) = v \int Kdf$ is a continuous operator and the linearity is immediate from the definition of the v-integral.

Remark 5.9

As shown in Example 5.4, the convexity of K is not necessary to generate a linear functional via $v \int Kdf$. In the case of Lebesgue integral additivity of a measure μ is not required to generate a linear functional $L \int fd\mu$. Once a linear functional is generated from a bounded set function then there is a unique additive (in the case of Lebesgue) or a unique convex (in Theorem 5.8) set function which generates the same transformation. This is illustrated by Example 5.4 and Example 5.5.

Remark 5.10

Theorem 5.3 allows the definition of the v-integral to be restated in terms of $\lim_{|\sigma| \to 0}$ rather than in terms of $\lim_{\sigma}$ which is a limit with respect to the net of partitions. Likewise, Lemma 5.7 and Theorem 5.8 can also be restated. Hence the v-integral is as computable as the Riemann integral.

Some Consequences of the Calculus of the v-Integral

In this section L denotes the linear normed space of Lipschitz functions on [0,1] which are zero at zero with the norm $\|f\|_L$ given by the Lipschitz constant. We will show L is isometric and isomorphic to the linear normed space Ω of bounded convex set functions defined on fundamental sets of (0,1] with the norm given by the fundamental bound.

If $K\varepsilon\Omega$ and $v\int_0^1 Kdf$ exists, then we define $v\int_a^b Kdf = v\int_0^1 Kdf_{ab}$ where $f_{ab}(x) = f(a)$ for $x \leq a$, $f_{ab}(x) = f(x)$ for $a \leq x \leq b$ and $f_{ab}(x) = f(b)$ for $x \geq b$.

Theorem 6.1

Let $f\varepsilon AC$ and $K\varepsilon\Omega$. Then $v\int_a^b Kdf$ exists for $a,b\varepsilon[0,1]$ and $v\int_a^c Kdf = v\int_a^b Kdf + v\int_b^c Kdf$ for $a < b < c$.

Proof. $f\varepsilon AC$ implies $f_{ab}\varepsilon AC$. It follows from Theorem 5.2 that $v\int_a^b Kdf$ exists for $a,b\varepsilon[0,1]$. Suppose σ is a partition which is finer than $\{0,a,b,c,1\}$. Then

$$\sum_{a\sigma}^{c} K((x_i,x_{i+1}])\Delta_i f = \sum_{a\sigma}^{b} K((x_i,x_{i+1}])\Delta_i f + \sum_{b\sigma}^{c} K((x_i,x_{i+1}])\Delta_i f.$$ By taking the limit over σ, we obtain the desired equality.

Theorem 6.2

If $F(x) = v\int_0^x K\,df$ for $f\varepsilon AC$ and $K\varepsilon\Omega$, then $F\varepsilon AC$.

Proof. Given $\varepsilon > 0$, there is a $\delta > 0$ such that if $\{[x_i, x_{i+1}]\}$ is a collection of intervals with $\Sigma\Delta_i x < \delta$ then $\Sigma|\Delta_i f| < \varepsilon/_{WK}$.

$$\Sigma|\Delta_i F| = \Sigma\left|\int_0^{x_{i+1}} K\,df - \int_0^{x_i} K\,df\right| = \Sigma\left|\int_{x_i}^{x_{i+1}} K\,df\right| = \Sigma\left|\int_0^1 K\,df_{x_i x_{i+1}}\right|$$

$$\leq WK\,\Sigma\,\|f_{x_i x_{i+1}}\|_{BV} < WK\,\varepsilon/_{WK} = \varepsilon.$$ It follows that $F\varepsilon AC$.

In the next two lemmas the isometry and isomorphism between Ω and L are established.

Lemma 6.3

If $K\varepsilon\Omega$ then there is a unique Lipschitz function $G_K\varepsilon L$ defined by $G_K(x) = XK((0,x])$. Furthermore $\|G_K\|_L = WK$.

Proof. The proof to the above Lemma is straightforward and omitted.

Lemma 6.4

If $G\varepsilon L$ with $\|G\|_L = L$, then the set function $K_G\varepsilon\Omega$ where K_G is defined by $K_G((a,b]) = [G(b) - G(a)]/(b - a)$. Furthermore $WK_G = L$.

Proof. Since G is a Lipschitz function it follows that K_G is fundamentally bounded and that $WK_G = L$. We next show K_G is convex with respect to length . If $(a,b]\subset(0,1]$ and $\{x_i\}$ is a partition of $(a,b]$, then

$$\Sigma[\Delta_i x/_{(b-a)}]K_G((x_i, x_{i+1}]) = \Sigma[\Delta_i x/_{(b-a)}][\Delta_i G/_{\Delta_i x}] = (b-a)^{-1}\Sigma\Delta_i G$$

$$= [G(b) - G(a)]/(b - a) = K_G((a,b]).$$

We now have established the following theorem.

Theorem 6.5

The space Ω is isometric and isomorphic to the space L with the mapping given by $K \longleftrightarrow G_K$. It follows that AC* is isometric and isomorphic to L.

The well-known classical characterization given in [2] for AC* states that $T \in AC^*$ if and only if there exists $g \in L^\infty$ such that $T(f) = L \int f' g d\mu$ for each $f \in AC$. We now relate this to Theorem 5.2 and show that for some elements $T \in AC^*$ Theorem 5.2 gives a more explicit representation of the norm preserving extension of T on BV[0,1] than that of [2]. Since the set of polygonal functions is dense in AC in the BV norm, it is not difficult to see the following lemma which is stated without proof.

Lemma 6.6

The closure of C_1, the continuously differentiable functions on [0,1] which are zero at zero, is AC in the BV norm.

Theorem 6.7

If a sequence $\{g_n\}$ of L^∞ functions converges to g in the L^1 norm and if $G_n(x) = L \int_0^x g_n$ (Lebesgue integral) for each n, then $\lim_n v \int K_{G_n} df = v \int K_G df$ for $f \in \mathcal{L}$.

Proof. $|v \int K_{G_n} df - v \int K_G df| = |\lim_\sigma \sum_\sigma [K_{G_n}((x_i, x_{i+1}]) - K_G((x_i, x_{i+1}])]\Delta_i f| = \lim_\sigma | \sum_\sigma [(\int_{x_i}^{x_{i+1}} g_n - g)/\Delta_i x]\Delta_i f|$

$$\leq \overline{\lim_\sigma} \sum_\sigma \left(\int_{x_i}^{x_{i+1}} |g_n - g| \right) |\Delta_i f / \Delta_i x|$$

$$\leq \left(L \int_0^1 |g_n - g|\right) \| f \|_{\mathcal{L}}.$$

Since $\{g_n\}$ converges to g in the L^1 norm, it follows that $v \int K_{G_n} df$ converges to $v \int K_G df$.

Theorem 6.8

If $f \varepsilon AC$, $g \varepsilon L^{\infty}$ and $G \varepsilon \mathcal{L}$ is given by $G(x) = L \int_o^x g$, then $v \int K_G df = L \int f'g$.

Proof. Let C denote the set of differentiable functions.

Case(1), $f \varepsilon C$ and $g \varepsilon C$.

$$v \int K_G df = \lim_{\sigma} \Sigma K_G((x_i, x_{i+1}])\Delta_i f = \lim_{\sigma} \Sigma[\Delta_i G/\Delta_i x]\Delta_i f$$

$$= \lim_{\sigma} \Sigma[\Delta_i G/\Delta_i x][\Delta_i f/\Delta_i x]\Delta_i x = \lim_{\sigma} \Sigma\, f'(\xi_i)g(\eta_i)\Delta_i x$$

by the Mean Value Theorem, where $x_i < \xi_i, \eta_i < x_{i+1}$. By Bliss' Theorem, the above limit is the Riemann integral $R \int f'g$ from which the special case follows.

Case(2) $f \varepsilon C_1$ and $g \varepsilon L^{\infty}$.

$g \varepsilon L^1$ implies that there is a bounded sequence $\{g_n\}$ in C which converges to g in the L^1 norm. By Theorem 6.7 $\lim_n v \int K_{G_n} df = v \int K_G df$, where $G_n(x) = L \int_o^x g$ for each n. For each n, $v \int K_{G_n} df = L \int f'g_n$ and by the Bounded Convergence Theorem $\lim_n L \int f'g_n = L \int f'g$ and hence $v \int K_G df = L \int f'g$.

Case(3) $f \varepsilon AC$ and $g \varepsilon L^{\infty}$.

Lemma 6.6 implies that there is a sequence $\{f_n\}$ in C_1 which converges to f in the BV norm from which it follows that $\lim v \int K_G df_n = v \int K_G df$. From Case(2) $v \int K_G df_n = L \int f'_n g$ for each n. Since $\{f_n\}$ converges to f in the BV norm, $\{f'_n\}$ converges to f' in the L^1 norm. Hölders inequality yields $L \int |(f_n - f)'g| \leq \| f_n - f \|_{L^1} \| g \|_{L^{\infty}}$ from which it follows that $\lim_n L \int f'_n g = L \int fg$ and hence $v \int K_G df = L \int f'g$.

Corollary 6.9

If $f\varepsilon AC$, $K\varepsilon\Omega$, and $F(x) = v\int_0^x K df$, then $F' = f'G_K'$ a.e.

Theorem 6.8 gives the most natural relations between the representation given by the v-integral and that in [2] by the isomorphism $g \longleftrightarrow T \longleftrightarrow K_G \longleftrightarrow G = L\int g$.

The following development gives a sufficient condition on the convex set function K so that the v-integral of K with respect to every BV function will exist (Corollary 6.12).

Theorem 6.10

Let $f\varepsilon BV$ and $\{K_n\} \subset \Omega$ such that the v-integral of each K_n with respect to f exists and suppose $\{K_n\}$ converges to $K\varepsilon\Omega$. Then the v-integral of K with respect to f exists and $\lim_n v\int K_n df = v\int K df$.

Proof. For each n let $g_n \varepsilon L^\infty$ for which $G_{K_n}(x) = L\int_0^x g_n$. $v\int K_{G_n} df$ is Cauchy, because

$$\begin{aligned}
\left|v\int K_{G_n} df - v\int K_{G_m} df\right| &= \lim_\sigma \left|\sum_\sigma [K_{G_n}((x_i,x_{i+1}]) - K_{G_m}((x_i,x_{i+1}])]\Delta_i f\right| \\
&= \lim_\sigma \left|\sum_\sigma [(L\int_{x_i}^{x_{i+1}} g_n - g_m)/\Delta_i x]\Delta_i f\right| \\
&\leq \overline{\lim_\sigma} \sum_\sigma (\| g_n - g_m \|_{L^\infty} \Delta_i x/\Delta_i x) |\Delta_i f| \\
&\leq \| g_n - g_m \|_{L^\infty} \| f \|_{BV}.
\end{aligned}$$

Since $\{K_n\}$ converges to $K\varepsilon\Omega$, $\{G_{K_n}\}$ converges to $G_K \varepsilon \mathcal{L}$ and hence $\{g_n\}$ converges to g in the L^∞ norm. Therefore $\lim_n v\int K_{G_n} df$ exists. Let $V = \lim_n v\int K_{G_n} df$ and we show that $V = v\int K df$. Given $\varepsilon > 0$, there is a positive integer N such that $n > N$ implies $\|g_n - g\| < \varepsilon/3\| f \|_{BV}$ and that $|V - v\int K_{G_n} df| < \varepsilon/3$. Let σ' be a partition so that if σ is finer than σ' then $|\sum_\sigma K_{G_n}((x_i,x_{i+1}])\Delta_i f - v\int K_{G_n} df| < \varepsilon/3$. For σ finer than σ'

$$|V - \sum_\sigma K_G((s_i,x_{i+1}])\Delta_i f| \le |V - v\int K_{G_n} df| + |v\int K_{G_n} df - \sum_\sigma K_{G_n}((x_i,x_{i+1}])\Delta_i f|$$

$$+ |\sum_\sigma K_{G_n}((x_i,x_{i+1}])\Delta_i f - \sum_\sigma K_G((x_i,x_{i+1}])\Delta_i f|$$

$$< \varepsilon/3 + \varepsilon/3 + \Sigma \| g_n - g \|_{L^\infty} |\Delta_i f|$$

$$= 2\varepsilon/3 + \|g_n - g\|_{L^\infty} \|f\|_{BV} < 2\varepsilon/3 + \varepsilon/3 = \varepsilon.$$

This completes the proof.

Theorem 6.11

If G is polygonal, then $v\int K_G df$ exists for each $f \varepsilon BV[0,1]$.

Proof. Let $\sigma_G = \{a_i\}$ be the partition of [0,1] determined by the corners of G. For $f\varepsilon BV$ and σ finer than σ_G, $\sum_\sigma K_G((x_i,x_{i+1}])\Delta_i f$ $= \sum_\sigma [\Delta_i G/\Delta_i x]\Delta_i f = \sum_\sigma \alpha_j \Delta_i f$ where α_j is the slope of G on the interval $[a_j, a_{j+1}]$ and $[x_i, x_{i+1}] \subset [a_j, a_{j+1}]$. Therefore,

$$\sum_\sigma K_G((x_i,x_{i+1}])\Delta_i f = \sum_j \alpha_j [f(a_{j+1}) - f(a_j)] = v\int K_G df.$$

If $K\varepsilon\Omega$ and $f\varepsilon BV[0,1]$, then $|v\int K df| \le WK \|f\|_{BV}$. Hence if the v-integral of K with respect to each $f\varepsilon BV[0,1]$ exists, then K defines a continuous linear functional on BV[0,1]. This observation together with Theorem 6.10 and Theorem 6.11 establish the following corollary.

Corollary 6.12

If g is in the sup-norm closure of the step functions and if G(x) $= L\int_0^x g$, then $T(f) = v\int K_G df$ exists for each $f\varepsilon BV[0,1]$ and $T\varepsilon BV^*[0,1]$.

The v-Integral in the Vector-Valued Setting

As one would expect it is a very natural thing to extend the results of the v-integral to the vector-valued setting [8], as D. H. Tucker [20] extended the result of the Riesz Representation Theorem. In this section X denotes a Banach space, Y denotes a linear normed space, and B[X,Y] is the linear normed space of bounded linear operators from X into Y. The variation of an X-valued function f is defined to be $V(f) = \sup_{\sigma} \Sigma \|\Delta_i f\|$ and BV(X) denotes the linear normed space of X-valued functions of bounded variation which are θ_X (the additive identity of X) at 0 with $\|f\|_{BV(X)} = V(f)$. The space of X-valued absolutely continuous function, AC(X), is defined in the analogous manner. A function of the form $p = \Sigma\psi_{H_i} x_i$ is called an X-valued polygonal function where each $x_i \varepsilon X$. Let Γ denote the closure of the set of X-valued polygonal functions in the BV norm. Although $\Gamma \ne AC$ [11], $\Gamma \subset AC$ as in the following theorem.

Theorem 7.1

If $\{p_n\}$ is a Cauchy sequence of polygonal functions in the BV norm, then there exists $f \varepsilon AC$ such that $\{p_n\}$ converges to f in the BV norm.

The proof follows as in the second proof of Theorem 5.1.

The following theorem is a generalization of two theorems from Section 5 and is stated without proof since the proof follows as earlier.

Theorem 7.2

(i) If $f \varepsilon \Gamma$, then $\lim_{\sigma} pf_{\sigma} = f$ where the convergence is in the BV norm. (ii) If K is a set function with values in B[X,Y] which is convex with respect to length, p is a polygonal function and σ is the partition of [0,1] determined by the corners of p, then σ' finer than σ $\sum_{\sigma'} K((x_i, x_{i+1}])\Delta_i p = \sum_{\sigma} K((x_i, x_{i+1}])\Delta_i p$ from which it follows that K is v-integrable with respect to p and $v\int K dp = \sum_{\sigma} K((x_i, x_{i+1}])\Delta_i p$.

Theorem 7.3

Suppose K is a fundamentally bounded convex with respect to length set function and takes values in B[X,Y]. Then the transformation given by $T(f) = v\int K df$ is a bounded linear operator from Γ into $\bar{Y}$, the completion of Y. Furthermore $\|T\| = WK$.

The proof follows as in Theorem 5.2.

Theorem 7.4

If T is a bounded linear operator from Γ into Y, then there is a unique fundamentally bounded convex with respect to length set function with values in B[X,Y] such that $T(f) = v\int K df$ for each $f \in \Gamma$. Furthermore, $\|T\| = WK$.

Proof. Define a linear map $\mathcal{T}$ from AC into B[X,Y] by taking $\mathcal{T}(\psi_H)x = T(\psi_H x)$ for each fundamental function H. It follows from Lemma 3.6 that $\|\mathcal{T}\| \leq \|T\|$. Let K be the set function on fundamental sets defined by $K(H) = \mathcal{T}(\psi_H)$. The remainder of the proof follows as in Theorem 5.2.

If Y is complete, then the following corollary follows from Theorem 7.3 and Theorem 7.4.

Corollary 7.5

Suppose Y is complete, then a transformation T from Γ into Y is a bounded linear operator if and only if there is a unique fundamentally bounded set function K with values in B[X,Y] which is convex with respect to length such that $T(f) = v\int K\,df$ for each $f\in\Gamma$. Furthermore, $\|T\| = WK$.

The variation of an X-valued function f may be defined to be $SV(f) = \sup_{\sigma}\{\sup\{\|\sum_{\sigma} \alpha_i\Delta_i f\| : |\alpha_i| \leq 1\}\}$ which is called the semi-variation of f. Let BSV denote the space of functions of bounded semi-variation which are θ_X at zero and with the norm given by the semi-variation. The notion of semi-absolutely continuity is defined analogously and SAC denotes the corresponding function space. Let $S\Gamma$ be the closure of the set of X-valued polygonal functions in the BSV norm. A similar proof to that of Theorem 7.1 shows that $S\Gamma \subset SAC$.

Lemma 7.6

If p_1 and p_2 are X-valued polygonal functions, and σ_1 and σ_2 are the partition of [0,1] determined by the corners of p_1 and p_2, respectively, then

$$\|p_1 - p_2\|_{BSV} = \sup\{\|\sum_{\sigma_1\cup\sigma_2}\alpha_i\Delta_i(p_1 - p_2)\| : |\alpha_i| \leq 1\}.$$

Proof. Let $\{x_i\} = \sigma_1\cup\sigma_2$.

$$\|p_1 - p_2\|_{BSV} = \sup_{\sigma}\{\sup\{\|\sum_{\sigma} B_j\Delta_j(p_1 - p_2)\| : |B_j| \leq 1\}\}$$

$$= \sup_{\pi}\{\sup\{\|\sum_{\pi} B_j[(x_{i(j+1)} - x_{ij})/(x_{i+1} - x_i)]\Delta_i(p_1 - p_2)\| : |B_j| \leq 1\}\}$$

where π is any partition finer than $\sigma_1\cup\sigma_2$, and $[x_{ij}, x_{i(j+1)}] \subset [x_i, x_{i+1}]$. Since $|\sum_j B_j[(x_{i(j+1)} - x_{ij})/(x_{i+1} - x_i]| \leq$ for $|B_j| \leq 1$, the desired equality holds.

Due to Lemma 7.6 similar proofs to Theorem 7.2, 7.3, and 7.4 show the analogies of these theorems can be established in the semi-variation setting. The following is the analogous theorem to Corollary 7.5.

Theorem 7.7

Suppose Y is complete, then a transformation T from $S\Gamma$ into Y is a bounded linear operator if and only if there is a unique fundamentally bounded set function K with values in B[X,Y] which is convex with respect to length such that $T(f) = v\int K df$ for each $f \varepsilon S\Gamma$. Furthermore $||T|| = WK$.

In the special case Y = R, Theorem 7.5 and Theorem 7.7 imply $\Gamma^* = (S\Gamma)^*$.

Let $\Omega(B[X,Y])$, be the space of bounded convex with respect to length set functions K with the norm $||K|| = WK$. Let $L(B[X,Y])$ denote the space of Lipschitz functions on [0,1] with values in B[X,Y] which are $\theta_{B[X,Y]}$ at zero with the norm given by the Lipschitz constant L. The results obtained in Theorem 6.2 - 6.6 and 6.10 - 6.12 can be all carried over to the vector setting and hence will be stated without proof.

Theorem 7.8

The space $\Omega(B[X,Y])$ is isometric and isomorphic to the space $L(B[X,Y])$ by the map given by $G \leftrightarrow K_G$. Therefore Γ^* and $(S\Gamma)^*$ are isometric and isomorphic to $L(B[X,Y])$.

Theorem 7.9

If $f \varepsilon BV[0,1]$ and $\{K_n\}$ is a sequence in $\Omega(B[X,Y])$ which converges to $K \varepsilon \Omega(B[X,Y])$, and if each K_n is v-integrable with respect to f, then K is v-integrable with respect to f and $\lim_n v\int K_n df = v\int K df$.

Theorem 7.10

If G is polygonal, then $v \int K_G df$ exists for each $f \varepsilon BSV$.

Corollary 7.11

A sufficient condition that $T(f) = v \int K_G df$ be a bounded linear operator from BV[0,1] into $\bar{Y}$ is that G be in the closure of the polygonal functions in $\mathcal{L}(B[X,Y])$.

A v-Integral Representation for Linear Operators on a Space of Continuous Vector-Valued Functions

Section 3 shows a very elaborate and difficult construction by D. H. Tucker [20] to obtain a Stieltjes integral type representation for continuous linear operators from C into a linear normed space Y where C is the space of continuous functions on [0,1] with values in a linear normed space X. This last section of the paper shows a representation by the v-integral which is rather straightforward [10].

Let K be a set function defined on fundamental sets (half-open intervals) with values in B[X,Y]. If there is a constant M such that for any disjoint collection $\{H_i\}$ of fundamental sets and any corresponding subset $\{x_i\}$ of X $\| \Sigma K(H_i)x_i \| \leq M \max_j \| \sum_{i=1}^{j} x_i \|$, then K is said to be convex-Gowurin and the smallest such constant M is denoted by WK. For any continuous linear operator T from C into Y and any $f \varepsilon C$, $T(f) = T(f - \chi_{[0,1]} f(0)) + T(\chi_{[0,1]} f(0))$ and hence we only consider the functions in C which are θ_x at zero.

Theorem 8.1

If K is a set function with values in B[X,Y] which is convex-Gowurin and convex with respect to length, then $T(f) = v \int K df$ is a bounded linear operator from C into $\bar{Y}$. Furthermore $\| T \| = WK$.

Proof. Let σ and σ' be partitions of (0,1] and for $f\varepsilon C$ let pf_σ and $pf_{\sigma'}$ be the polygonal functions with corners determined by f, σ, and σ', respectively. Then $\|\sum_\sigma K((t_i,t_{i+1}])\Delta_i f - \sum_{\sigma'} K((t_j,t_{j+1}])\Delta_j f\|$

$\leq \|\sum_{\sigma\cup\sigma'} K((t_k,t_{k+1}])\Delta_k(pf_\sigma - pf_{\sigma'})\|$

$\leq WK \max_n \|\sum_{k=1}^{n} \Delta_k(pf_\sigma - pf_{\sigma'})\|$. Since pf_σ and $pf_{\sigma'}$ both converge to f as the mesh fineness of σ and ' tend to zero, $\max_n \|\sum_{k=1}^{n} \Delta_k(pf_\sigma - pf_{\sigma'})$ tends to zero, and it follows that $T(f) = v\int Kdf$ exists. $\|T\| = \sup_{\|f\|\leq 1} \|T(f)\|$

$= \sup_{\|f\|=1} \|v\int Kdf\| = \sup_{\|f\|=1} \|\lim_\sigma \sum_\sigma K((t_i,t_{i+1}])\Delta_i f\|$

$\leq WK \max_j \|\sum_{i=1}^{j} \Delta_i f\| = WK.$

Let ψ_H denote the fundamental function determined by a fundamental set H (see Section 5), then $v\int Kd(\psi_H x) = K(H)x$ for $x\varepsilon X$. If $\{H_i\}$ is a disjoint collection of fundamental sets and $\{x_i\}$ is a corresponding subset of X, then

$\|\Sigma K(H_i)x_i\| = \|v\int Kd(\Sigma\psi_{H_i}x_i)\| = \|T(\Sigma\psi_{H_i}x_i)\| \leq \|T\|\,\|\Sigma\psi_{H_i}x_i\|.$

But $\|\Sigma\psi_{H_i}x_i\| \leq \max_j \|\sum_{i=1}^{j} x_i\|$ from which it follows that $WK \leq \|T\|$, and we now have $\|T\| = WK$.

Theorem 8.2

If T is a bounded linear operator from C into Y, then there is a unique set function K with values in B[X,Y] which is convex-Gowurin and convex with respect to length such that $T(f) = v\int Kdf$ for each $f\varepsilon C$. Furthermore, $\|T\| = WK$.

Proof. Define $\mathcal{T}$ from CR, the space of continuous real valued functions on [0,1], into B[X,Y] by taking $\mathcal{T}(f)x = T(f\cdot x)$, then it follows from Lemma 3.6 that $\|\mathcal{T}\| \le \|T\|$. For each fundamental set H let $K(H) = \mathcal{T}(\psi_H)$. If H is a union of disjoint fundamental sets $\{H_i\}$, then $\psi_H = \Sigma\lambda_i\psi_{H_i}$ where λ_i is the ratio of the length of H_i to the length of H. Since $\mathcal{T}$ is linear, it follows that K is convex with respect to length. If $\{x_i\}$ is a subset of X, then

$$\|\Sigma K(H_i)x_i\| = \|\Sigma\mathcal{T}(\psi_{H_i})x_i\| = \|\Sigma T(\psi_{H_i}x_i)\| = \|T(\Sigma\psi_{H_i}x_i)\| \le \|T\|\,\|\Sigma\psi_{H_i}x_i\|$$

and $\|\Sigma\psi_{H_i}x_i\| \le \max_j \|\sum_{i=1}^{j} x_i\|$, hence K is convex-Gowurin. For any $f\varepsilon C$, $T(f) = \lim_\sigma T(pf_\sigma) = \lim_\sigma T(\Sigma_\sigma\psi_{H_i}\Delta_i f) = \lim_\sigma \Sigma_\sigma K(H_i)\Delta_i f = v\int K df$.
Since K determines T uniquely on polygonal functions which are dense in C, K is unique.

Therefore, in the case Y is complete we have a characterization, as opposed to a representation, of the linear operators which is not immediate in Section 3.

The v-integral development [8] is extended [1] to a more general setting [16] by parallel development. A v-derivative can be defined in the real valued setting appropriately [12,13] in order to establish the Fundamental Theorem of Calculus type theorem.

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