A Confidence Interval Estimate of Percentile

How Coung Jou

Utah State University

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A CONFIDENCE INTERVAL ESTIMATE OF PERCENTILE

by

Jou, How Coung

A thesis submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Applied Statistics

UTAH STATE UNIVERSITY
Logan, Utah

1980
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Also, I would like to extend my thanks to Dr. David L. Turner and Dr. Gregory W. Jones for their kind help.

Jou, How-Coung
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ABSTRACT

A Confidence Interval Estimate of Percentile

by

Jou, How Cough, Master of Science

Utah State University, 1980

Major Professor: Dr. Ronald V. Canfield
Department: Applied Statistics

The confidence interval estimate of percentile and its applications were studied. The three methods of estimating a confidence interval were introduced. Some properties of order statistics were reviewed. The Monte Carlo Method--used to estimate the confidence interval was the most important one among the three methods. The generation of ordered random variables and the estimation of parameters were discussed clearly. The comparison of the three methods showed that the Monte Carlo method would always work, but the K-S and the simplified methods would not.

(45 pages)
CHAPTER I
INTRODUCTION

Statistical analysis has become a very important part of the preliminary work in dams and dikes design. It is very important to understand the flooding characteristics of the water system which the structure serves. Thus, many design criteria include capacity to contain the "N year flood" or simulate some measure of the flow. "N year flood" is the yearly maximum of daily stream flows which is exceeded with probability 1/N, where N is specified.

The usual method of determining the N year flood is to record the yearly maximum for Y years, select a representative distribution, then estimate the parameters. The N year flood is estimated as the \((N-1)/N\)th percentile of the estimated distribution. There are other methods which are also used to determine the design flood for dams and dikes.

No matter how the design flood is determined, the available information is the observed data. Therefore it is subject to the same inadequacies of any estimate of a random phenomenon. It is not precisely determined. The usual statistical characterization of this lack of the precision is the confidence interval. However for the case of design
flood, no attempt has been made to estimate its accuracy. It seems that such an evaluation should be a necessity when the consequences of inadequate design are considered.

The problem of deriving confidence limits for percentiles of a distribution are considered in this thesis. An existing method using the Kolomogorov-Smirnov statistic is shown to be inadequate for the high (or low) percentiles, and a new method based on Monte Carlo simulation is proposed.

A review of the Kolomogorov-Smirnov confidence interval and of the distribution of order statistics fundamental to later derivations is given in Chapter II. The new confidence interval using Monte Carlo simulation is derived in Chapter III, and applications of this method are given in Chapter IV. A simpler method which does not involve Monte Carlo simulation is evaluated in Chapter V. This method has some intuitive appeal but as noted in Chapter V is very biased for the higher percentiles. The conclusions and recommendations of this study are summarized in Chapter VI.
CHAPTER II
REVIEW

The Kolomogrov-Smirnov confidence interval on a distribution function could be used to derive a confidence interval on the percentiles of the distributions. This method uses the sample distribution function \( F_n(x) \) with sample size \( n \).

\[
F_n(x) = \frac{j}{n} \quad \text{for} \quad x_j < x < x_{j+1},
\]

\[
j = 0, 1, ..., n
\]

\[
(x_0 = -\infty, x_{n+1} = +\infty).
\]

This function will generally differ from the population distribution function. But if it differs from an assumed distribution \( F(x) \) by too much, we will reject the hypothesis that \( F(x) \) is the population distribution function. That is, the amount of the difference between the empirical and assumed distribution function should be a usual tool in determining whether or not to accept the assumed distribution as correct.

The least upper bound of \( |F_n(x) - F(x)| \) is the statistic used to test \( H_0 \): the population distribution function is \( F(x) \). That statistic is known as the Kolomogrov-Smirnov statistic:
\[ D_n = \sup_x \left| F_n(x) - F(x) \right| \]

This statistic has a known distribution under \( H_0 \). From Table 1 we can find the critical value for rejecting \( H_0 \) with a specified \( n \) and \( \alpha \). For large \( n \) the asymptotic values for certain \( \alpha \) level are given in Table 1.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0.2</th>
<th>0.15</th>
<th>0.1</th>
<th>0.05</th>
<th>0.01</th>
</tr>
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<td>Limitation</td>
<td>( \frac{1.07}{\sqrt{n}} )</td>
<td>( \frac{1.14}{\sqrt{n}} )</td>
<td>( \frac{1.22}{\sqrt{n}} )</td>
<td>( \frac{1.36}{\sqrt{n}} )</td>
<td>( \frac{1.03}{\sqrt{n}} )</td>
</tr>
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</table>

The statistic \( D_n \) is two-sided, involving the "absolute" difference of \( F(x) \) and \( F_n(x) \). The critical region is \( D_n > \text{table value} \). Using this property, a confidence interval with significant level \( \alpha \) can be derived. Another method for calculating the asymptotic percentiles is from the limiting distribution:

\[
\lim_{n \to \infty} \mathbb{P}(D_n < \frac{Z}{\sqrt{n}}) = 1 - 2 \sum_{j=1}^{\infty} (-1)^{j-1} \exp(-2j^2 Z^2) \\
\approx 1 - 2 \exp(-2Z^2).
\]

Through upper limit and lower limit we can get a confidence interval \((x_1, x_2)\), as shown in Figure 1.
Order statistics play an important role in statistical inference partly because some of their properties do not depend on the distribution from which the random sample is obtained. Let $x_1, x_2, \ldots, x_n$ denote a random sample from a distribution of the continuous type having probability density function $f(x)$. Let $y_1$ be the smallest of these $x_i$, $y_2$ be the next $x_i$ in order of magnitude, ..., and $y_n$ the largest $x_i$, i.e., $y_1 < y_2, \ldots, < y_n$. Then $y_i$, $i = 1, 2, \ldots, n$, is called the $i$th order statistic of the random sample $x_1, x_2, \ldots, x_n$. The density of a continuous random variable may be defined as the derivatives of the
cumulative distribution function. Let \( f(x) \) and \( F(x) \) represent the density and cumulative distribution functions respectively of a random variable \( X \). Then by definition of derivative

\[
f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.
\] (1)

The numerator on the right of (1) can be interpreted as the probability of the event that the random variable lies in the interval \((x, x+h)\).

Consider now the \( k \)th order statistic \( Y_k \) a sample of size \( n \) of the random variable of \( X \). It will now be shown that the probability density function \( Y_k \) is

\[
q_k(Y_k) = \frac{n!}{(k-1)!(n-k)!} F(Y_k)^{k-1}(1-F(Y_k))^{n-k} f(Y_k)
\]

\[ k = 1, 2, \ldots, n. \] (2)

As in (2), the probability that \( Y_k \) lies in the interval \((y_k', y_k+h)\) will be used to derive the density function. This event requires that (a) \( k-1 \) observations lie in \((-\infty, y_k)\) (b) 1 observation lies in \((y_k, y_k+h)\) and (c) \( n-k \) observation lie in \((y_k+h, \infty)\). The probability that events (a), (b) and (c) occur simultaneously is

\[
Ph = \frac{n!}{(k-1)!(n-k)!} F(Y_k)^{k-1}(F(Y_k+h) - F(Y_k))
\]

\[ (1-F(Y_k+h))^{n-k} \] . (3)
Thus
\[ g_k(y_k) = \lim_{h \to 0} \frac{P_h}{h} \]
\[ = \frac{n!}{(k-1)! (n-k)!} F(y_k)^{k-1} (1-F(y_k))^{n-k} f(y_k). \quad (4) \]

Consider now the transformed random variable \( U = F(X) \) where \( F(.) \) is the cumulative density function of \( X \). The c.d.f. of \( U \) is
\[ P(U < u) = P(F(X) < u) = P(X < F^{-1}(u)) = u. \]

Therefore \( U \) is uniformly distributed on \([0, 1]\).

Let \( y_1, y_2, \ldots, y_n \) be the order statistics from a sample of size \( n \) of the random variable \( X \). Let \( U_1 = F(y_1), U_2 = F(y_2), \ldots, U_n = F(y_n) \). Because \( F(x) \) is monotone increasing, the smallest \( U \) is the transform of the smallest \( X \), and so forth, so that the ordered \( U \)'s are, respectively, the transforms of the ordered \( X \)'s:
\[ U_k = F(y_k). \]

The distribution of \((U_1, U_2, \ldots, U_n)\) is, therefore, the distribution of the order statistics of a random sample from the uniform population on \([0, 1]\).

Since \( F(x) = x \) for the uniform distribution, it follows from (4) that the probability density function of \( u_k \) is
\[ g_{uk}(y) = \frac{n!}{(k-1)!(n-k)!} \, y^{F-1} (1-y)^{n-k}, \quad 0 \leq y \leq 1. \quad (5) \]

This is a Beta distribution with parameters $v = k$, $w = n-k+1$. 
CHAPTER III
MONTE CARLO CONFIDENCE INTERVAL

In this chapter a confidence interval estimate of the $p^{th}$ percentile of a distribution is developed. The technique is based upon a method of estimation developed for the Weibull distribution (see Bain and Antls, 1968) but which can be adapted for many other distributions. This method of estimation is best explained by example. The Weibull and Normal distribution are illustrated here.

Let $X$ be a Weibull random variable. Then

$$F(x) = 1 - e^{-(x/\theta)^r}, \quad x \geq 0.$$  

Let $y_1, y_2, \ldots, y_n$ and $u_1, u_2, \ldots, u_n$ be the order statistics of a sample of size $n$ of the Weibull and an independent uniform random variable respectively. The parameters $r$ and $\theta$ are estimated by choosing values which provide the "best fit" of $F(x)$ through the points $(y_k, u_k)$, $k = 1, 2, \ldots, n$. The "best fit" criterion may be least squares from $\sum(u_i - F(y_i))^2$. However, in practice it is convenient to transform the values $u_k = F(y_k)$ and $y_k$ so that a simple linear relationship holds. For the Weibull case

$$\ln(-\ln(1-F(x))) = r \ln x - r \ln \theta.$$  

Thus $\ln (-\ln(1-u_k))$ has a linear relationship to $\ln(y_k)$ and the least square fit is
the regression line through the points \((\ln(-\ln(1-u_k)), \ln(y_k)), k = 1, 2, \ldots, n\). The slope of this line estimates \(r\) and the intercept estimates \(-r\ln\theta\). Let \(\eta_k = \ln(-\ln(1-u_k))\) and \(r_k = \ln(y_k)\) for \(k = 1, 2, \ldots, n\). Then it follows that

\[
\hat{\gamma} = \frac{\Sigma \eta_k r_k - \Sigma \eta_k \Sigma r_k / n}{\Sigma r_k^2 - (\Sigma r_k)^2 / n}
\]

and

\[
\hat{\theta} = \exp\left(-\frac{\Sigma \eta_k}{n} - \frac{\Sigma \eta_k}{r_k n}\right).
\]

Consider the \(y_k, k = 1, 2, \ldots, n\) values fixed. For each Monte Carlo sample \(u_k, k = 1, \ldots, n\), there results estimates of \(\theta\) and \(r\) in turn the estimated \(p\)-th percentile \(x_p\). The process of sampling \(u_1, u_2, \ldots, u_n\) and estimating \(x_p\) is repeated for a large number of times, say 500. Then the confidence interval is interpolated from the empirical distribution function of \(x_p\). For example, let \(x_{p1}', x_{p2}', \ldots, x_{p500}'\) be the ordered values of 500 \(x_p\)'s computed as described previously. Then a 0.95 level confidence interval is

\[(x_{p12.5}', x_{p487.5}')\), where \(x_{p12.5} = (x_{p12} + x_{p13}) / 2,\]

\(x_{p487.5} = (x_{p487} + x_{p488}) / 2\). Since any interval containing 475 \(x_{pi}\) values is a 95% confidence interval it is reasonable to explore the distribution to find the narrowest interval which contains 475 points. This procedure is time consuming,
however, and usually differs very little from the equal tails interval given previously.

The normal distribution illustrates another method of deriving the confidence interval on $x_p$. Let $y_1, y_2, \ldots, y_n$ and $u_1, u_2, \ldots, u_n$ be the order statistics from random samples of size $n$ from the normal and uniform populations. As noted previously, the $u_k$ represent possible values of $F(y_k)$. Each sample $u_1, u_2, \ldots, u_n$ is apriori equally likely. Let $\phi(z)$ be the standard normal distribution function (CDF). Then $\phi^{-1}(u_k) = z_k, k = 1, \ldots, n$ are likely values of $(y_k - \mu)/\sigma$. Therefore there is a linear relationship between $z_k$ and $y_k$ where the slope is $1/\sigma$ and the intercept is $-\mu/\sigma$. Using least squares, 

$$\hat{\sigma} = \left( \frac{\sum y_k z_k - \sum y_k \sum z_k/k}{\sum y_k^2 - (\sum y_k)^2/n} \right)^{-1}$$ \hspace{1cm} (8)$$

and 

$$\hat{\mu} = \hat{\sigma} \sum z_k/n + \sum y_k/n.$$ \hspace{1cm} (9)

As before the $x_p$ is sampled by obtaining Monte Carlo samples $u_1, u_2, \ldots, u_n$. Then $x_p$ is computed using as parameters the estimates $\mu$ and $\sigma$ obtained from each Monte Carlo sample $u_1, u_2, \ldots, u_n$ with the fixed sample $y_1, y_2, \ldots, y_n$. The confidence interval is interpolated from the empirical distribution of the $x_p$'s.
CHAPTER IV
APPLICATIONS

In this chapter the method developed in Chapter III is applied to the Weibull distribution. Generation of Weibull and uniform random variables is considered first. In the final section, the Monte Carlo confidence interval is illustrated.

Generations of Ordered Random Variables

The usual method of generating order statistics of a random variable with distribution $F(x)$ is to generate independent uniform values, $U_i$, $i = 1, ..., n$. Then using the inverse of $F(x)$, transform the $U_i$ to $X_i = F^{-1}(U_i)$.

The method is more efficient if the uniform random variables are generated as order statistics. Thus avoiding the operation of ordering the sample. This is accomplished using Fortran subroutine ORDER, the method is given by Hartley and Lurie (1972) in the following subroutine.

```
SUBROUTINE ORDER (X,N,M)
DIMENSION X(N)
TEMP=0.0
SEED=TIME(11)
DO 10 I=1,M
   V=RANDOM(SEED)
10 CONTINUE
```
Estimation of Parameters

Let $y_i, i = 1, \ldots, n$ represent the order statistics of a random variable with Weibull Density Function (CDF)

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^r}, \quad x > 0.$$ 

It was shown in the previous chapter that $x_i, i = 1, \ldots, n$ represent a random sample of observations of a random variable with CDF $F(x)$. These values must then be avoided thereby greatly decreasing the cost of Monte Carlo experiments which require order statistics. In here, I do not specify the method to get those $x_i, i = 1, \ldots, n$. Replacing $X(I) = U$ with $\Theta(-\log(1.0 - X))^{\frac{1}{R}}$ the inverse function of CDF of Weibull distribution. In Chapter III for the Weibull case we get $\ln(-\ln(1.0 - F(x))) = r\ln x - r\ln \theta$. Thus $\ln(-\ln(1.0 - U_k))$ was a linear relationship to $\ln(y_k)$ and the least square fit (where $y_k$ is independent uniform values, $k = 1, \ldots, n$) is the regression line through the points $(\ln(y_k), \ln(-\ln(1.0 - u_k))), k = 1, \ldots, n$. The slope of this line estimates $r$ and intercept estimates $-r\ln \theta$.

So we can get estimates of $r$ and $\theta$. The Fortran program listed in Appendix A generates the $y_i, i = 1, \ldots, n$ and
repeatedly generates uniform order statistics, with each new set of $u_i$, $i = 1, \ldots, n$ the parameters $r$ and $\theta$ are estimated. These estimates are then used in the program to compute the $p$th percentile for several values of $p$. In Appendix D is the data when $p = 0.9$, $r = 2.0$, $\theta = 10.0$ and $n = 40$. The first column is ordered Weibull random values $y_k$, $k = 1, \ldots, 40$, the second column is $\ln(y_k)$, $k = 1, \ldots, 40$, the third column is ordered uniform random values $u_k$, $k = 1, \ldots, n$, the fourth column is $\ln(-\ln(1.0-u_k))$, $k = 1, \ldots, n$. The plot $u_k$ vs $y_k$ ($k = 1, \ldots, 40$) and plot $\ln(-\ln(1.0-u_k))$ vs $\ln(y_k)$ ($k = 1, \ldots, 40$) are listed in Figures 2 and 3 respectively.

The estimated parameters $\hat{r} = 1.60999$, $\hat{\theta} = 11.5163$ and $x_{0.9} = 19.332724$. The 95% confidence interval on $x_p$ (in the thesis I try to do 500 times) has been obtained from the Monte Carlo distribution of $x_p$. The results are shown in Table 2.

Table 2. 95% confidence interval on the $p$th percentile (Monte Carlo Method)

<table>
<thead>
<tr>
<th>$p$</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>12.77745</td>
<td>20.26675</td>
</tr>
<tr>
<td>0.95</td>
<td>13.68255</td>
<td>24.84900</td>
</tr>
<tr>
<td>0.975</td>
<td>15.60380</td>
<td>29.59265</td>
</tr>
<tr>
<td>0.99</td>
<td>17.02290</td>
<td>31.09065</td>
</tr>
<tr>
<td>0.995</td>
<td>19.63265</td>
<td>35.89530</td>
</tr>
</tbody>
</table>
Figure 2. Plot $u_k$ vs $y_k$. 
Figure 3. Plot \( \ln(-\ln(1-u_k)) \) vs \( \ln(y_k) \).
Table 3 shows the corresponding confidence interval computed by the method of Kolomogorov-Smirnov using the same initial sample from the Weibull distribution. The Fortran program 2 listed in Appendix B is to calculate pth percentile. Note that in every case the upper bound for $U_k$, $k = 1, \ldots, n$ is 1.

Table 3. 95% confidence interval on the pth percentile (K-S Method)

<table>
<thead>
<tr>
<th>P</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>12.6619730</td>
<td>**</td>
</tr>
<tr>
<td>0.95</td>
<td>13.6153340</td>
<td>**</td>
</tr>
<tr>
<td>0.975</td>
<td>12.4740910</td>
<td>**</td>
</tr>
<tr>
<td>0.99</td>
<td>13.3881968</td>
<td>**</td>
</tr>
</tbody>
</table>

** = no meanings. For example, $p = 0.99$, the upper bound 24.02172 has $p$ value 0.75. It is much less than 0.99. So I say the upper bound received from extrapolating is meaningless. The data for $r = 2.0, \theta = 10.0$ and $n = 40, p = 0.95$ are shown in Appendix B.
CHAPTER V
SIMPLIFIED METHOD

The method described in Chapter III provides an approximation on the pth percentile whose accuracy is determined by the Monte Carlo sample size. Since this can be expensive if a very accurate interval is needed, a simpler method is evaluated in this section. Let \( U_i = F(y_i) \), \( i = 1, \ldots, n \) where \( y_i \), \( i = 1, \ldots, n \) are the order statistics of a random sample of size \( n \) from a population with CDF \( F(x) \).

It was shown in Chapter II that the distribution of \( U_i \) is Beta with parameters \( v = k \), and \( w = n-k+1 \). It seems intuitively reasonable to construct a confidence envelope for the CDF in the following manner. At each \( y_i \), \( i = 1, \ldots, n \), construct an \( (1 - \alpha) \) level confidence on \( U_i \), \( i = 1, \ldots, n \). Denote the upper and lower bounds \( U_i \) and \( L_i \) respectively. Then using the same estimation techniques used in Chapter IV, the Weibull parameters are estimated with the set of upper bound values \( (U_i, i = 1, \ldots, n) \) substituted for the \( U_i \) in the equation and then repeating the estimation with the set of lower bounds \( (L_i, i = 1, \ldots, n) \). The two resulting estimated CDF's constitute an envelope of possible CDF's based upon a \( 1 - \alpha \) level confidence interval.
The proposed confidence interval on the $p$th percentile is found by determining the percentile estimates from each of the estimated CDF's. Graphically the procedure is illustrated in Figure 4.

Figure 4. Confidence interval by simplified method.
The $L_i$ and $U_i$, $i = 1, \ldots, n$ are found as follows for a $1 - \alpha$ interval

$$\frac{1}{B(r, \theta)} \int_0^{L_i} t^{r-1}(1-t)^{\theta-1} \, dt = \alpha/2$$

$$\frac{1}{B(r, \theta)} \int_0^{U_i} t^{r-1}(1-t)^{\theta-1} \, dt = 1-\alpha/2.$$  \hfill (10)

An approximate solution in $L_i$ and $U_i$ is given in the Handbook of Mathematical Functions edited by M. Abramowitz and I. A. Stegun (1970) where

$$\frac{1}{B(r, \theta)} \int_0^{x_p} t^{r-1}(1-t)^{\theta-1} \, dt = p \quad 0 \leq p \leq 1$$

$$x_p = \frac{r}{(r+\theta e^{2w})}, \quad w = \frac{y_p (h+\theta)^{1/2}}{h} - \left(1/(2\theta-1)\right)$$

$$- 1/(2r-1))(\lambda+5/6 - 2/(3h))$$

$$h = 2(1/(2r-1) + 1/(2\theta-1))^{-1}, \quad \lambda = \sqrt{\ln(1/p^2)}$$

$$y_p = t - \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} + \varepsilon(p)$$

$c_0 = 2.515517$, $c_1 = 0.802853$, $c_2 = 0.010328$

$d_1 = 1.432788$, $d_2 = 0.189269$, $d_3 = 0.001308.$  \hfill (11)
The absolute value of the error in \( x_p \) for this approximation is given as less than \( 4.5 \times 10^{-4} \) (see program in Appendix B).

The method was applied to the same Weibull sample that was in Chapter IV, so that comparisons with the exact method could be made. The results were disappointing in that a large bias occurs in the high percentile region. The intervals are shown in Table 4 with the corresponding intervals from the exact method. The large bias in the higher percentiles is evident. The extent of the bias is seen graphically in Figure 5.

Table 4. 95% confidence interval on the pth percentile (Simplified method)

<table>
<thead>
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<th>P</th>
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<th>Upper bound</th>
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<tr>
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<td>14.132521</td>
<td>17.453690</td>
</tr>
<tr>
<td>97.5%</td>
<td>15.162933</td>
<td>18.135656</td>
</tr>
<tr>
<td>99%</td>
<td>16.423828</td>
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</tr>
<tr>
<td>99.5%</td>
<td>16.961182</td>
<td>18.592846</td>
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Figure 5. The plot of confidence interval vs different percentiles.
CHAPTER VI

CONCLUSIONS

Two methods for computing a 1-\( \alpha \) level confidence interval were developed and illustrated. The method in Chapter IV is limited in precision only by the Monte Carlo sample size. Therefore it is possible to specify any reasonable level of precision before computing the interval. The method works for any distribution which can be inverted, i.e., there exists a solution for \( F(x) = U \) which can be computed. The primary disadvantage is the expense of Monte Carlo simulation.

The second method was developed on intuitive grounds and is not based on firm theoretical principles. It is evident from Figure 5 that the method does not give reasonable intervals for the high percentiles. It is instructive to compare the confidence interval with the true value of the population percentile. The Weibull data used in the computation of the intervals in Tables 2 and 4 were generated using the Weibull distribution function

\[
F(x) = 1 - e^{-\left(\frac{x}{10}\right)^2}, \quad x \geq 0.
\]

The true percentiles for this distribution are tabulated in Table 5.
Table 5. Percentiles of the Weibull distribution

\[ F(x) = 1 - \exp\left(-\frac{x}{10}\right)^2 \]

<table>
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<tr>
<th>Percentile</th>
<th>90%</th>
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<th>97.5%</th>
<th>99%</th>
<th>99.5%</th>
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<td>Value</td>
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<td>17.308*</td>
<td>19.2064</td>
<td>21.4596</td>
<td>23.0180</td>
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</table>

*17.308 = 10.0*(-ln(1-0.95))^{1/2.0}

A rough check on the method of computing the confidence interval is to see if the intervals cover the true value at each percentile. Note from Table 4 that the Monte Carlo method does provide intervals which cover the true value at every point. However the short method indicates a severe bias at the higher percentiles (i.e. > 90%). Thus it is clear that this method is not good.
REFERENCES


APPENDICES
Appendix A

Monte Carlo Method
*FILE (B08115) PROGRAM ON PACK
100 DIMENSION X(50), UNIF(50), A(10, 10), SUM(10), ARRAY(1001)
200 DOUBLE PRECISION A, TEST, DET
300 COMMON R, THETA
400 READ(5,/) N, R, THETA, P
500 READ(5,/) ITIME
600 PP = P
700 TEMP = THETA
800 WRITE(6,1)
900 1 FORMAT(4X, "I", 10X, "ESTIMATED R", 10X, "ESTIMATED THETA"/)
1000 CALL ORDER(X, 50, N, 1)
1100 DO 5 I = 1, N
1200 X(I) = ALOG(X(I))
1300 DO 160 III = 1, ITIME
1400 C* GENERATE ORDER STATISTICS OF UNIFORM DIST. U(0,1).
1500 CALL ORDER(UNIF, 50, N, 2)
1600 C* USE LEAST SQUARE METHOD TO FIND THE ORDER STATISTIC VALUE
1700 C* Y=P
1800 DO 50 I = 1, 10
1900 SUM(I) = 0.0
2000 DO 50 J = 1, 10
2100 A(I, J) = 0.0
2200 DO 60 J = 1, N
2300 SUM(I) = X(J) + SUM(I)
2400 DO 60 I = 1, N
2500 UNIF(I) = ALOG((-1.) * ALOG(1. - UNIF(I)))
2600 SUM(2) = UNIF(I) + SUM(2)
2700 DO 50 I = 1, N
2800 A(I, 1) = X(I) * X(I) + A(I, 1)
2900 A(2, 2) = UNIF(I) * UNIF(I) + A(2, 2)
3000 DO 50 J = 1, 2
3100 A(I, J) = A(I, J) - SUM(I) * SUM(J) / FLOAT(N)
3200 DO 100 J = 1, 2
3300 100 A(J + 1, I) = A(I, J)
3400 TEST = 0.5E-10
3500 CALL DMATIV(A, 1, 1, 2, 2, DET, TEST, 10)
3600 BO = SUM(2) / FLOAT(N)
3700 BO = BO - A(1, 2) * SUM(1) / FLOAT(N)
3800 C* COMPUTE THE INTERSECTION POINT
3900 P = ALOG((-1.) * ALOG(1.0 - P))
4000 T = (P - BO) / A(1, 2)
4100 ARRAY(III) = EXP(T)
4200 THETA = DEXP((-1.) * BO / A(1, 2))
4300 WRITE(6, 122) III, A(1, 2), THETA
4400 122 FORMAT(I5, E21.6, E25.6)
4500 P = PP
4600 THETA = TEMP
4700 150 CONTINUE
4800 DO 200 I = 1, ITIME - 1
4900 200 J = ITIME - I
5000   IFLAG=0
5100   DO 250 K=1,J
5200   A1=ARRAY(K)
5300   A2=ARRAY(K+1)
5400   IF(A1.LE.A2) GO TO 250
5500   E=A1
5600   ARRAY(K)=A2
5700   ARRAY(K+1)=E
5800   IFLAG=1
5900   250 CONTINUE
6000   IF(IFLAG.EQ.0) GO TO 300
6100   200 CONTINUE
6200   WRITE(6,303)
6300   303 FORMAT(///4X,"I",10X,"INTERACTION POINTS"///)
6400   WRITE(6,305) (I,ARRAY(I),I=1,TIME)
6500   305 FORMAT(I5,E27.6)
6600   STOP
6700   END
6800   SUBROUTINE ORDER(X,N,M,II)
6900   DIMENSION X(N)
7000   COMMON R,THETA
7100   TEMP=0.0
7200   SEED=TIME(I1)
7300   DO 10 I=1,M
7400   U=1.0-(1.0-TEMP)*V**(1.0/(FLOAT(M-I)+1.0))
7500   IF(II.EQ.1) GO TO 1
7600   1   X(I)=U
7700   TEMP=U
7800   GO TO 10
7900   10 X(I)=THETA*(-ALOG(1.-U))**(1./R)
8000   RETURN
8100   END
Appendix B

Simplified Method
!FILE (808115) PROGRAM2 ON PACK
100 DIMENSION Y1(40), Y2(40), X(40), SUM(5), A(5, 5)
200 DOUBLE PRECISION A, DET, TEST
300 COMMON R, THETA
400 READ(5,/) DN, N
500 READ(5,/) R, THETA
600 DO 10 I = 1, N
700 TEMP = FLOAT(I)/FLOAT(N)
800 Y1(I) = TEMP + DN
900 Y2(I) = TEMP - DN
100 CALL ORDER(X, N)
110 DO 200 I = 1, N
120 FORMAT(6, 250) Y1(I), Y2(I), X(I)
130 CALL ORDER(X, N)
140 FORMAT(6, 250) Y1(I), Y2(I), X(I)
150 IFLAG = 1
160 DO 15 I = 1, 2
170 SUM(I) = 0.0
180 DO 15 J = 1, 2
190 A(I, J) = 0.0
200 SUM(I) = SUM(I) + X(I)
210 DO 20 I = 1, N
220 A(I, 1) = A(I, 1) + X(I)*X(I)
230 A(I, 2) = A(I, 2) + X(I)*Y1(I)
240 DO 30 T = 1, 2
250 A(T, 1) = X(I)*Y1(I) + A(T, 1)
260 DO 30 T = 1, 2
270 A(T, 2) = X(I)*Y1(I) + A(T, 2)
280 SUM(I) = SUM(I) + X(I)
290 SUM(I) = SUM(I) + X(I)
300 SUM(I) = SUM(I) + X(I)
310 SUM(I) = SUM(I) + X(I)
320 SUM(I) = SUM(I) + X(I)
330 SUM(I) = SUM(I) + X(I)
340 WRITE(5, 100) ANS
350 100 FORMAT("THE INTERACTION PT IS", F18.7/)
360 IF (IFLAG.EQ.2) GO TO 150
370 DO 50 I = 1, N
380 50 Y1(I) = Y2(I)
390 IFLAG = 2
400 GO TO 5
410 150 STOP
420 END
430 SUBROUTINE ORDER(X, N)
440 DIMENSION X(N)
450 TEMP = 0.0
460 SEED = TIME(11)
470 DO 10 I = 1, N
480 V = RANDOM(SEED)
490 U = 1.0 - (1.0 - TEMP) * V * (1.0/(FLOAT(N-I)+1.0))
500 X(I) = PINV(U)
5100   10 TEMP = U
5200      RETURN
5300      END
5400      REAL FUNCTION PINV(U)
5500      COMMON R, THETA
5600      A = (-ALOG(1. - U))**(1./R)
5700      PINV = THETA*A
5800      RETURN
5900      END
Appendix C

Kolomogrov-Smirnov Method
PROGRAM 3 ON PACK

DIMENSION X(40), Y(40), A(10,10), SUM(10)
COMMON R, THETA
DOUBLE PRECISION A, DET, TEST
READ(5,/) R, THETA
READ(5,/) N, P
CALL ORDER(X, N)
DO 5 I=1, N
5 X(I) = ALOG(X(I))
IFLAG = 0
PP = P
SIGLEV = 0.025
100 DO 8 I = 1, 3
SUM(I) = 0.0
DO 8 J = 1, 3
8 A(I,J) = 0.0
DO 50 I = 1, N
SUM(1) = SUM(1) + X(I)
50 A(1,I) = A(1,I) + X(I) * X(I)
IF(IFLAG .GE. 2) GO TO 200
CALL CINTVL(Y, N, SIGLEV)
DO 10 I = 1, N
10 Y(I) = ALOG(-ALOG(1.0 - Y(I)))
DO 12 I = 1, N
SUM(2) = SUM(2) + Y(I)
A(1,2) = A(1,2) + X(I) * Y(I)
12 A(2,2) = A(2,2) + Y(I) * Y(I)
DO 15 I = 1, 2
DO 15 J = 1, 2
A(I,J) = A(I,J) - SUM(I) * SUM(J) / FLOAT(N)
15 A(J+1,I) = A(I,J)
TEST = 0.5E-10
CALL DMATIV(A, 1, 1, 2, 1, DET, TEST, 10)
BO = SUM(2) / FLOAT(N)
BO = BO - A(1,2) * SUM(1) / FLOAT(N)
WRITE(6,20) BO, A(1,2)
20 FORMAT(" THE REGRESSION COEFFICIENTS BO = ", F15.6, " B1 = ", F16.6/)
P = ALOG(-ALOG(1.0 - P))
T = (P - BO) / A(1,2)
T = EXP(T)
WRITE(6, 25) PP, T
25 FORMAT(" THE INTERVAL BOUND FOR P = ", F7.3, "X", " IS ", F15.6/)
SIGLEV = 0.975
IFLAG = IFLAG + 1
P = PP
GO TO 100
200 STOP
END
SUBROUTINE CINTVL(Y, N, P)
DIMENSION Y(N)
T = SQRT(ALOG(1.0/(P*P)))
T=T-((2.515517+.802853*T+.010329*T**2.)/(1.+1.432788*T+\n\-2.+.001308*T**3.))
TEMP=(T**2.3)/S.0
DO 10 I=1,N
TA=1./(2.*FLOAT(I)-1.)
TB=1./(2.*FLOAT(N-I+1)-1.)
TH=2.*1./(TA+TB)
W=T*SQRT(TH+TEMP)/TH
W=W-(TEMP+S./S.-2./(3.*TH))*TB-TA)
Y(I)=FLOAT(I)/(FLOAT(I)+FLOAT(N-I+1)*EXP(2.*W))
10 CONTINUE
RETURN
END
SUBROUTINE ORDER(X,N)
COMMON R,THETA
DIMENSION X(N)
TEMP=0.0
SEED=TIME(11)
DO 10 I=1,N
V=RANDOM(SEED)
U=1.0-(1.0-TEMP)*V**(1.0/(FLOAT(N-I)+1.0))
X(I)=((-ALOG(1.-U))**(1./R)))*THETA
10 TEMP=U
RETURN
END
Appendix D

Part of Data for Monte Carlo Method
<table>
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<tr>
<th>ROW</th>
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Appendix E

Part of Data for K-S Method
The first column is for upper bound.  
The second column is for lower bound.  
The third column is for Weibull order random numbers when $r = 2.0$, $\theta = 10.0$, $p = 0.95$, $n = 40$. 

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