AN EVALUATION OF TRUNCATED SEQUENTIAL TEST

by

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Ryh-Thinn Chang
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ABSTRACT

An Evaluation of Truncated Sequential Test

by

Ryh-Thinn Chang, Master of Science

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Major Professor: Dr. David White
Department: Applied Statistics and Computer Science

The development of sequential analysis has led to the proposal of tests that are more economical in that the Average Sample Number (A. S. N.) of the sequential test is smaller than the sample size of the fixed sample test. Although these tests usually have a smaller A. S. N. than the equivalent fixed sample procedure, there still remains the possibility that an extremely large sample size will be necessary to make a decision. To remedy this, truncated sequential tests have been developed.

A method of truncation for testing a composite hypotheses is studied. This method is formed by mixing a fixed sample test and a sequential test and is applied to the exponential distribution and normal distribution to establish its usefulness.

It is proved that our truncation method can give a similar Operating Characteristic (O.C.) curve to that of corresponding fixed sample test if the test parameters are properly chosen. The average sample size required by our truncation method as compared with other
existing truncation methods gives us a satisfactory result. Though the truncation method we suggested in this study is not an optimum truncation, it is still worthwhile, especially, when we are interested in the testing of a composite hypotheses.
Sequential analysis is a method of statistical inference whose characteristic feature is that the number of observations required by the procedure is not determined in advance of the experiment. Usually, in the traditional statistical test, the sample space is divided into acceptance region and rejection region. In sequential analysis, the sample space is divided into acceptance region, rejection region, and continuation region. The sequential decision procedure to terminate the experiment depends, at each stage, on the results of the observations previously made. At the first step, an observation is taken from a distribution. On the basis of this observation, a decision is made of the accepting the hypothesis $H_0$ if the test statistic $W_1$ falls in the acceptance region, accepting $H_1$ if $W_1$ falls in the rejection region, or taking another sample if $W_1$ falls in the continuation region of the test. This process is repeated until the test statistic $W_m$ corresponding to observation $m$ falls in either the acceptance or rejection region. A merit of the sequential method, as applied to testing statistical hypotheses, is that test procedures can be constructed which require, on the average, a substantially smaller number of observations.
The Operating Characteristic (O.C.) is defined as the probability of accepting $H_0$. The Average Sample Number (A.S.N.) is defined as the average sample size required for terminating the sequential test. Both the O.C. and A.S.N. are functions of the parameters assumed in the density functions for the observations. The O.C. function describes how well the test procedure achieves its objectives of making a correct decision, and the A.S.N. function represents the price we have to pay in terms of observations required for the test. Both the O.C. and A.S.N. functions play a very important role in sequential analysis.

Thus, in judging the relative merits of two different test procedures, we shall compare the O.C. and A.S.N. functions of these two tests. An ideal O.C. curve will be as large as possible when $H_0$ is true, and as small as possible when it is false. Generally, the nearer the O.C. function to the ideal function and the smaller the expected number of observations required, the more desirable is the sequential test.

Wald (11) was the first to give a complete sequential procedure for testing statistical hypotheses, it is proved that the sequential test can give a similar power to that of the current fixed sample size test, if the test parameters are properly chosen. The average sample size required by a sequential test, in general, is smaller than that of the fixed sample size test. However, in some cases it requires relatively a larger sample to make a decision than the current fixed sample size test. For the purpose of saving time and costs, it seems necessary to set definite limits for the number of observations required, so that the
limits we set in a sequential test will yield a procedure with a smaller
A. S. N. than the fixed sample size test for any values of the parameters
and at the same time the power of the test will not be hurt. Truncation
of the sequential test was developed for this purpose.

In the literature, a few truncated sequential tests have been
developed. Wald (11) developed a rule for truncating the sequential
test but no formulas available for calculating O. C. and A. S. N. functions.
L. A. Aroian (4) applied Wald's truncation method to the normal dis­
tribution and developed a method to calculate exact O. C. and A. S. N.
functions. The Wald truncation procedure is given in Figure 1. Woodall
and Kurkjian (12) and L. A. Aroian (3) developed a different truncation
method for the exponential distribution. This method we call a "right
angle truncation" and is shown in Figure 2. More detailed discussion
of these truncation methods will be given in later chapters.

Generally, both of these two truncation methods are designed for
the testing of the hypotheses; \( H_0 : \theta = \theta_0 \) against \( H_1 : \theta = \theta_1 \), i.e. for a
testing simple hypothesis versus a simple alternative. However, their
objectives, to reduce the average sample size required and to maintain
the power of the test at the same level as a non-truncated sequential
test and also as a fixed sample size test are the same.

In the case of composite hypotheses; \( H_0 : \theta \leq \theta_0 \) against \( H_1 : \theta \leq \theta_0 \)
Tokko (10) did a right angle shaped truncation and developed a method
to determine the truncation values for the Poisson distribution. His
Figure 1. Wald's Truncation procedure.

Figure 2. Right angle shaped truncation.
truncation method is accomplished by superimposing an appropriate fixed sample size test on Wald's non-truncated sequential test. Tokko has compared his method with Wald's non-truncated sequential test and has found that his truncation will give a smaller A.S.N. than the non-truncated procedure and this truncation test has approximately the same power as the non-truncated procedure. Since the description and development of Tokko's truncation method is limited to a test of a Poisson parameter and the test he selected for comparison is Wald's non-truncated sequential test, it is difficult to ascertain how good his truncation method is. The main purpose of this study is to generalize Tokko's truncation method to other distributions and to evaluate its usefulness.

In order to establish the usefulness of the Tokko's truncation method and find is this method an optimum truncation, we will compare Tokko's truncation method with those tests which have exactly same truncation shape as the Tokko's truncation test but with different truncation values. That is, the determination of the truncation values for those tests will be chosen arbitrarily and not follow Tokko's truncation rule.

Woodall and Kurkjian's right angle truncation will also be selected as our comparison when we deal with the exponential distribution. Woodall and Kurkjian's right angle truncation is good only for the testing of simple hypotheses, and up to now, in the exponential case, no method of truncation for the testing of composite hypotheses has been
developed. When making comparison of the right angle truncation procedure and Tokko's method, we will reduce the original composite hypotheses to a simple hypotheses in Tokko's truncation test so that the comparison is possible. One benefit of adopting the right angle truncation as our comparison is that our truncation method also forms a right angle shaped truncation and the method for calculating exact the O.C. for the right angle shaped truncation is obtained by Woodall and Kurkjian.

In the fixed sample size test, if we are given the significance level and the sample size, certainly, we can determine a best critical region and an O.C. curve. Using the value corresponding to the critical region and the sample size as two truncation values in our test, then, the sequential parameters can be determined properly so that the O.C. curve of the corresponding fixed sample size test will be maintained in our truncated sequential test. The test parameters of the other truncation tests will also be determined properly so that their O.C. curves will as close as possible to the O.C. curve of our test. The A.S.N. is compared under the condition that the O.C. curve of our test and the o.c. curve of the other truncation tests are approximately the same. This is because when we compare the A.S.N. of any two sequential tests, both of the tests should have approximately same O.C. curve; otherwise the comparison is not reasonable.

The exact O.C. is obtained by using some formulas. The important thing we have to do is select the sequential parameters
appropriately so that the establishment of the two sequential boundaries of the test will produce the desired O.C. when truncating the sequential test. We will also use Monte Carlo methods to evaluate and select the test parameters. The A.S.N. is obtained by Monte Carlo methods.

In the light of the already mentioned points, this study may be divided into four parts:

1. Generalize Tokko’s truncation method to other distributions.
2. Reproduce the characteristics of the fixed sample size test with a sequential test.
3. Select sequential test parameters properly for our truncation method and the other truncation tests so that the O.C. curves will be as close as possible to that of fixed sample size test.
4. Compare the A.S.N. of these tests and establish the usefulness of our truncation method. Is our truncation method a better one? (i.e. A.S.N. is smaller?)

Some statistical and sequential concepts which are needed to define the characteristics of a test are reviewed in the second and third chapter.

In Chapter IV, the method of calculating the exact O.C. of a right angle shaped truncation test for simple hypotheses is studied. We consider the detailed characteristics of the exponential distribution. Some formulas which can be easily programmed by a computer language are given in this chapter.
Chapter V contains the procedures for constructing a truncated sequential test of a composite hypotheses which duplicates the characteristics of the corresponding fixed sample size test. A rough method to determine the upper and lower boundaries for our truncation method is suggested. This rough method will provide us a clue as to how to obtain the desired O.C. curve.

In Chapter VI, the exact O.C. curve for our test and for those tests which were selected for comparison are calculated. The A.S.N. is obtained by using a Monte Carlo method. The byproduct is the empirical O.C. curve which is supposed to coincide with that of the exact O.C. curve we got. Fortunately, the results we got by using Monte Carlo methods are very satisfactory in our study. Therefore, what we can say is that the use of Monte Carlo methods to simulate the A.S.N. will give us little bias. The comparison of the A.S.N. between our test and the other tests is contained in this chapter. The application of our truncation method to the normal distribution is also investigated in this chapter as another result of this study.

Chapter VII, the final chapter, is a conclusion of the study. Evaluation of our truncation method and areas of further study are suggested in this final chapter.
CHAPTER II

STATISTICAL BACKGROUNDS IN TESTING HYPOTHESES

A statistical hypothesis is an assertion about the distribution of one or more random variables. If the statistical hypothesis completely specifies the distribution, it is called a simple hypothesis; if it does not, it is called a composite hypothesis. Generally speaking, it is more difficult to handle composite hypotheses than simple hypotheses in sequential analysis. In this study we will deal with composite hypotheses.

There are two kinds of error inherent and unavoidable with statistical tests. If the hypothesis is true but the test rejects the hypothesis an error known as a Type I error is made. On the other hand, if the hypothesis is false but the test accepts the hypothesis, an error known as a Type II error is made. The relative importance of these two kinds of error depends upon what actions to be taken as a result of the test.

A good test is clearly one which makes the probabilities of both errors as small as possible. However, it is impossible to reduce both errors simultaneously. Reducing the Type I error will automatically increase the Type II error, and reducing the Type II error will automatically increase the Type I error. The common procedure is to fix the
Type I error arbitrarily and then choose the critical region so as to minimize the probability of the Type II error. The quantity

$$1 - \beta = 1 - \text{probability of Type II error}$$

is called the power of the test. The power functions play a very important role in the evaluation of statistical tests, particularly, in the comparison of several regions which might all be used to test a given null hypothesis $H_0$ against a given alternative $H_1$. In terms of this concept the principle for setting up a test is to fix the probability of Type I error and then choose a critical region so as to maximize the power of the test.

Certainly, a test specifies a critical region; it can also be said that a choice of a critical region defines a test. For instance, if one is given the critical region $C = (x_1, x_2, x_3); x_1 + x_2 + x_3 \leq 1)$, the test is determined. Three random variables $X_1, X_2, X_3$ are to be considered; if the observed values of $x_1, x_2, x_3$, accept $H_0$ if $x_1 + x_2 + x_3 > 1$; otherwise reject $H_0$. That is, the terms "test" and "critical region" can in this sense, be used interchangeably. Thus, if we define a best critical region, we have defined a best test. The critical region is said to be best or most powerful, if the Type I error is fixed and the corresponding Type II error is minimum. The procedures of deciding a best critical region have been developed by Neyman and Pearson. Suppose the test is $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$, where $\mu$ is the parameter of a population
given by \( f(x;\mu) \). Let \( L_0 \) and \( L_1 \) denote the likelihoods of a random sample of size \( n \) from the given population when its parameter \( \mu \) is respectively, \( \mu_0 \) and \( \mu_1 \) symbolically,

\[
L_0 = \prod_{i=1}^{n} f(x_i;\mu_0) \quad \text{and} \quad L_1 = \prod_{i=1}^{n} f(x_i;\mu_1) \quad (2.1)
\]

If there exists a critical region \( C \) of size \( \alpha \) and a constant \( K \) such that

1. \( \frac{L_0}{L_1} \leq K \) for each point \( (x_1, x_2, x_3, \ldots, x_n) \) inside \( C \) (2.2)

2. \( \frac{L_0}{L_1} \geq K \) for each point \( (x_1, x_2, x_3, \ldots, x_n) \) outside \( C \) (2.3)

3. \( \alpha = \Pr \{ (x_1, x_2, x_3, \ldots, x_n) \in C; H_0 \} \) (2.3)

then \( C \) is a most powerful critical region of size \( \alpha \) for testing \( \mu = \mu_0 \) against \( \mu = \mu_1 \).

A common problem in testing hypotheses is that of testing a particular parameter values, say \( \mu_0 \), against a set of other values of \( \mu \) for a family of a distribution \( f(x;\mu) \). The basic idea may be illustrated by a particular example. This example will be used throughout this study.

Example 2.1.

Suppose the population is known to have an exponential distribution with the density function \( f(x;\Theta) = \frac{1}{\Theta} e^{-\frac{x}{\Theta}}, \Theta > 0 \), and suppose
it is further known that the mean $\mu$ (here $\mu = \frac{1}{3}$) is less than or equal to 3. A random sample $x_1, x_2, x_3, \ldots, x_9$ has been drawn, on the basis of these 9 observations, we shall test the null hypothesis $H_0 : \mu \leq 3$. The alternatives of this hypothesis are all of the values $\mu > 3$, and we shall accept $H_0$ (state that $\mu \leq 3$) or reject $H_0$ (state that $\mu > 3$). We shall require a test for which the probability of a Type I error is 0.0038. If a particular value $\theta$ of $\mu$ is considered then,

$$\frac{L_0}{L_1} = \frac{\sum_{i=1}^{9} f(x_i; 3)}{\sum_{i=1}^{9} f(x_i; \theta)} = \frac{(3 e^{-3 \sum x_i})}{(8 e^{-8 \sum x_i})}$$

$$= \left(\frac{3}{8}\right)^9 e^{5 \sum x_i}$$

Thus, we must find a constant $K$ and region $C$ the sample space such that

$$\frac{L_0}{L_1} = \left(\frac{3}{8}\right)^9 e^{5 \sum x_i} \leq K \quad \text{inside } C$$

$$\frac{L_0}{L_1} = \left(\frac{3}{8}\right)^9 e^{5 \sum x_i} \geq K \quad \text{outside } C$$

take logarithm on both sides

then
\[ \sum x_i \leq \frac{1}{5} (\log K - 9 (\log 3 - \log 8)) \]

\[ \sum x_i \geq \frac{1}{5} (\log K - 9 (\log 3 - \log 8)) \]

let

\[ V_0 = \frac{1}{5} (\log K - 9 (\log 3 - \log 8)) \]

then we have

\[ \sum x_i < V_0 \quad \text{reject the hypothesis that } H_0 : \mu \leq 3 \]

\[ \sum x_i \geq V_0 \quad \text{accept the hypothesis that } H_0 : \mu \leq 3 \]

The best critical region is therefore an interval \( \sum x_i < V_0 \), and \( V_0 \) is to be chosen so that

\[ \Pr (\sum x_i < V_0) = 0.0038 \]

\( V_0 = 1 \) approximately, in this example.

An important thing to observe here is that the critical region is independent of the selected value \( \theta \). Any value of \( \mu \) greater than 3 would have given rise to the same critical region, but this is not a general situation. It is not in general true that the inequality

\[ \frac{L_0}{L_1} \leq K \quad \text{inside } C \]
\[ \frac{L_0}{L_1} \geq K \quad \text{outside } C \]

will give rise to the same critical region for all possible values of \( \mu \) alternative to a value \( \mu_0 \) specified by a null hypothesis. When it is true that all alternatives give rise to the same critical region, the test is called uniformly most powerful test. Uniformly most powerful tests do exist for many important problems in statistics, while there are other equally important problems which do not have uniformly most powerful tests. No discussion for this problem will be given in this study.

Based on this critical region we have found and the given sample size, the probabilities of accepting \( H_0 \) (O.C.) for the various values of the parameter \( \mu \) and the operating characteristic curve can be obtained. The method of calculating O.C. for the exponential distribution has been programed and is given in Appendix A. Table 1 shows the O.C. for the various values of the parameter \( \mu \) and Figure 1 shows the O.C. curve for the test \( H_0 : \mu \leq 3 \) against \( H_1 : \mu > 3 \).

It is possible to reduce a composite hypotheses to a simple hypotheses (i.e. the composite hypotheses with composite alternative can be reduced to simple hypothesis with simple alternative) for some distributions, such as normal distribution, exponential distribution, Poisson distribution, ..., etc. Take example 2.1 again, the original composite hypotheses been considered is \( H_0 : \mu \leq 3 \) against \( H_1 : \mu > 3 \), this
composite hypotheses can be reduced to a simple hypotheses; \( H_0 : \mu = 3 \) against \( H_1 : \mu = 14 \), if we keep \( \alpha = 0.0038 \) and \( \beta = 0.06206 \). The reason that the mean to be set at 3 under the null hypothesis and the mean to be set at 14 under the alternative hypothesis can be seen clearly from Table 1. This is because we have specified the values of \( \alpha \) and \( \beta \) to be 0.0038, 0.06206 respectively.

Table 1. Probabilities of accepting \( H_0 \) (O.C.), with \( \alpha = 0.0038 \), \( n = 9 \)

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>O.C.</th>
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<tbody>
<tr>
<td>1</td>
<td>0.99999</td>
</tr>
<tr>
<td>2</td>
<td>0.99976</td>
</tr>
<tr>
<td>3</td>
<td>0.99620</td>
</tr>
<tr>
<td>4</td>
<td>0.97864</td>
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<tr>
<td>5</td>
<td>0.93191</td>
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<td>6</td>
<td>0.84724</td>
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</tr>
<tr>
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<td>0.00387</td>
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<tr>
<td>20</td>
<td>0.00209</td>
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</table>
Figure 3. O.C. curve for the exponential distribution with the test $H_0: \mu \leq 3$ against $H_1: \mu > 3$. 
Generally, reducing a composite hypotheses to a simple hypotheses, the likelihood ratio for this test should be a monotone function of the parameter. And the power of some part of the ranges of the parameter will have no protection. For example, reducing the composite hypotheses: $H_0 : \mu < 3$ against $H_1 : \mu > 3$ to the simple hypotheses; $H_0 : \mu = 3$ against $H_1 : \mu = 14$, then, the parameter $\mu$ within this range $3 < \mu < 14$ will have no protection against making a wrong decision.

In the later chapters, when we deal with composite hypotheses, we will try to reduce it to a simple hypotheses, because simple hypotheses is easier to handle.

Figure 2 is a graphical illustration of a fixed sample size test, which mixed with the concepts of sequential analyses. At the sample size $n = 9$, draw a vertical line, and at the point $\sum x_i = V_0 = 1$ (the critical value we found), draw a horizontal line. The two lines crosses at the point A. When the cumulative sum reaches the vertical line and falls above the point A, the hypothesis is accepted. When the cumulative sum reaches the vertical line and falls below the point A, the hypothesis is rejected. No decision can be made if the vertical line was not reached.

The purpose of drawing this kind of graph is that in the later chapters, this graph will be combined with other graphs and thus constitute our truncation method. A detailed discussion of sequential analysis will be given in the next chapter.
Figure 4. Fixed sample size test.
CHAPTER III

SEQUENTIAL ANALYSIS FOR SIMPLE HYPOTHESES

3.1 Sequential analysis

Sequential analysis is a procedure which leads to a statistical inference and in which the number of observations to be made is not determined before the experiment is begun. The procedure indicates when sufficient observations have been gathered to make our decisions with the risks we have chosen. On the average, fewer observations will be required by this procedure, and its use will not increase the risks \( \alpha \) and \( \beta \). For some problems only half the number of observations will be required on the average for the sequential procedure as compared with the number required if the sample size is fixed in advance.

3.2 Procedure of a sequential test

Let \( f(x; \mu) \) be a family of densities or (discrete) probability functions of a random variable \( X \) with \( \mu \) a parameter. Suppose that \( \mu \) is unknown, and that we are going to take observations on \( X \) to determine whether \( \mu \) is large or small. One way of formulating this problem is to say we are going to test the null hypothesis \( H_0 \) that \( \mu = \mu_0 \) against the alternative hypothesis \( H_1 \) that \( \mu = \mu_1 \), where \( \mu \) and \( \mu_1 \) are two suitably chosen numbers. The sequential procedure for testing such a
hypothesis can be characterized as follows: Observations are taken one at a time and after every observation we decide to do one of the following three things:

1. Accept the hypothesis
2. Reject the hypothesis
3. Make an additional observation

Let us denote the successive observations on X by \( x_1, x_2, x_3, \ldots \) etc.

In order to determine which one of these three possible actions to take we must determine critical sample size. To do this we compute \( P_{\text{om}} \) and \( P_{\text{lm}} \) where

\[
P_{\text{om}} = f(x_1; \mu_0) f(x_2; \mu_0) f(x_3; \mu_0) \cdots f(x_m; \mu_0)
\]

the probability that the \( m \) observations collected thus far would occur if our hypothesis \( H_0 \) were true.

\[
P_{\text{lm}} = f(x_1; \mu_1) f(x_2; \mu_1) f(x_3; \mu_1) \cdots f(x_m; \mu_1)
\]

the probability that the \( m \) observations would occur if the alternative statement \( H_1 \) were true.

To find \( P_{\text{om}} \), we assume that sampling is from the population stated in \( H_0 \) and compute the probability that such a result would occur.

Similarly, to find \( P_{\text{lm}} \), we assume that sampling is from the population
stated in $H_1$ and again compute the probability that such a result would occur.

The sequential probability ratio test for testing $H_0$ against $H_1$ is defined as follows: Two positive constants $A$ and $B$ ($B<A$) are chosen. At each stage of the experiment:

1. If $\frac{P_{\text{im}}}{P_{\text{om}}} > A$ the process is terminated with the rejection of $H_0$. 3.2.1

2. If $\frac{P_{\text{im}}}{P_{\text{om}}} < B$ the process is terminated with the acceptance of $H_0$. 3.2.2

3. If $B < \frac{P_{\text{im}}}{P_{\text{om}}} < A$ the process is continued by taking an additional observation. 3.2.3.

For the purpose of practical computation, it is much more convenient to compute the logarithm of the ratio $\frac{P_{\text{im}}}{P_{\text{om}}}$ than the ratio $\frac{P_{\text{im}}}{P_{\text{om}}}$ itself.

The reason for this is that $\log \frac{P_{\text{im}}}{P_{\text{om}}}$ can be written as the sum of $m$ terms, i.e.

$$
\log \frac{P_{\text{im}}}{P_{\text{om}}} = \log \frac{f(x_1; \mu_1)}{f(x_1; \mu_0)} + \log \frac{f(x_2; \mu_1)}{f(x_2; \mu_0)} + \ldots + \log \frac{f(x_m; \mu_1)}{f(x_m; \mu_0)}
$$
If we denote the $i^{th}$ term of this sum by $Z_i$, i.e.

$$Z_i = \log \frac{f(x_i; \mu)}{f(x_i; \mu_1)}$$

then the inequalities (3.2.1.), (3.2.2.) and (3.2.3.) will be respectively

$$Z_1 + Z_2 + Z_3 + \ldots + Z_m \geq \log^A$$  \hspace{1cm} 3.2.4.

$$Z_1 + Z_2 + Z_3 + \ldots + Z_m \leq \log^B$$  \hspace{1cm} 3.2.5.

$$\log^B < Z_1 + Z_2 + Z_3 + \ldots + Z_m < \log^A$$  \hspace{1cm} 3.2.6.

The constant $A$ and $B$ need to be determined so that the test will have the preassigned strength $(\alpha, \beta)$. How to determine the two constants of $A$ and $B$ will be studied in the next section.

The above procedure can be described in the plan $m$ and $Z$.

There are two lines $Z = a$ and $Z = b$, sampling is stopped as soon as the sequence $Z_1, Z_2, \ldots$ leaves the strip between two lines and the decision depends on which line is crossed. The following graph illustrates this decision procedure. Figure 5 is shown, the size we need for this test is 12, and the result is accept the hypothesis.
Figure 5. Sequential decision procedure.
3.3 Determination of two boundary constants $A$ and $B$

Let

$$R_n = \{x_1, x_2, x_3, \ldots, x_n\}; \quad B \leq \frac{P_{lm}}{P_{om}} = \frac{f(x_{1l}; \mu_1) \cdots f(x_{ml}; \mu_l)}{f(x_{1o}; \mu_0) \cdots f(x_{m0}; \mu_0)} < A$$

for $m = 1, 2, 3, \ldots, n-1$ and $\frac{P_{ln}}{P_{on}} < B$)

$$S_n = \{x_1, x_2, x_3, \ldots, x_n\}; \quad B \leq \frac{P_{lm}}{P_{om}} = \frac{f(x_{1l}; \mu_1) \cdots f(x_{ml}; \mu_l)}{f(x_{1o}; \mu_0) \cdots f(x_{m0}; \mu_0)} < A$$

for $m = 1, 2, 3, \ldots, n-1$ and $\frac{P_{ln}}{P_{on}} \geq A$)

We denote the power of the test when $H_0$ is true by the symbol $\alpha$ and the power of the test when $H_1$ is true by the symbol $1-\beta$. Thus, $\alpha$ is the probability of committing a Type I error, and $\beta$ is the probability of committing a Type II error. With the sets $R_n$ and $S_n$ as previously defined, and with random variables of continuous type, we then have

$$\alpha = \sum_{l=1}^{n} \int_{S_n} f(x_i; \mu_0) \, dx_i$$

$$\beta = \sum_{l=1}^{n} \int_{R_n} f(x_i; \mu_1) \, dx_i$$
If
\[(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n,\]
we have
\[\prod_{i=1}^{n} f(x_i; \mu_1) \leq B(\prod_{i=1}^{n} f(x_i; \mu_0)),\]
hence it is clear that
\[\beta = \sum_{i=1}^{n} \int_{\mathbb{R}^{1\ldots n}} f(x_i; \mu_1) \, dx_i < \sum_{i=1}^{n} \int_{\mathbb{R}^{1\ldots n}} B(\prod_{i=1}^{n} f(x_i; \mu_0)) \, dx_i = B(1-\alpha)\]

Because
\[\prod_{i=1}^{n} f(x_i; \mu_1) \geq A(\prod_{i=1}^{n} f(x_i; \mu_0))\]
at each point of the set \(S_n\), we have
\[1 - \beta = \sum_{i=1}^{n} \int_{S_{i\ldots n}} f(x_i; \mu_1) \, dx_i \geq \sum_{i=1}^{n} \int_{S_{i\ldots n}} A(\prod_{i=1}^{n} f(x_i; \mu_0)) \, dx_i = A \alpha\]
Accordingly, it follows that
\[B \geq \beta/(1-\alpha)\]
\[A \leq (1-\beta)/\alpha\]
provided that $\alpha$ is not equal to zero or 1. Thus $(1 - \beta)/\alpha$ is an upper limit for $A$, and $\beta/(1 - \alpha)$ is a lower limit for $B$.

These inequalities are of considerable value in practical applications, since they furnish upper limits for $\alpha$ and $\beta$ for given values of $A$ and $B$. Suppose we wish to have a test procedure of strength $(\alpha, \beta)$, then, our problem is to determine the constants $A$ and $B$ such that resulting test will have the desired strength $(\alpha, \beta)$. The two constants are two appropriate values.

The determinations of these two exact values of $A$ and $B$ is usually very laborious, but Wald (11) has proved that the use of approximate values of $A$ and $B$ instead of true values $A$ and $B$ respectively, can not result in any appreciable increase in the value of either $\alpha$ or $\beta$. In other words, for all practical purposes, if we set $B = \beta/(1 - \alpha)$, $A = (1 - \beta)/\alpha$ the test provides at least the same protection against wrong decisions as the test using the true values of $A$ and $B$.

As we studied in this section, the constants $A$ and $B$ can be determined by the two types of error $\alpha$ and $\beta$. These probabilities will give two important points on the O.C. curve of a sequential test and will allow us to draw an appropriate O.C. curve of a test. These knowledges are very important and useful to define the test parameters for the sequential test.
CHAPTER IV
THE EXACT OPERATING CHARACTERISTIC FOR TRUNCATED
SEQUENTIAL TESTS IN THE EXPONENTIAL CASE

In this chapter, detailed explanation of the procedures of calculating exact O.C. function for the exponential distribution is made, only the truncated case is considered. At first, we assume $V_0$ and $I_0$ are two truncation values, draw two lines from $V_0$ and $I_0$ respectively, then the two lines cross at a point, say, $G$, and formed a truncation shape like Figure 6 with the other two sequential boundaries. A decision, either accept or reject the hypothesis $H_0$, must be made when either of the truncation lines is reached.

4.1 Mathematical formulation

Let a random variable be exponentially distributed with the density

$$f(x; \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad \text{for } x > 0, \theta > 0$$ 4.1.1.

Where $\theta$ is unknown parameter, let it be required to test

$$H_0: \theta = \theta_0$$
against

\[ H_1: \theta = \theta_1 \]

where

\[ \theta_1 < \theta_0 \]

Application of Wald's probability ratio test, the probability of obtaining a sequence of sample \((x_1, x_2, x_3, \ldots, x_n)\) is

\[ P_{lm} = \prod_{j=1}^{i} f(x_j; \theta_1) = \left( \frac{1}{\theta_1} \right)^i e^{\frac{1}{\theta_1} \sum_{j=1}^{i} x_j} \]

When \( H_1: \theta = \theta_1 \) is true

\[ P_{om} = \prod_{j=1}^{i} f(x_j; \theta_0) = \left( \frac{1}{\theta_0} \right)^i e^{\frac{1}{\theta_0} \sum_{j=1}^{i} x_j} \]

When \( H_0: \theta = \theta_0 \) is true

Consequently, the sequential probability ratio becomes

\[
\frac{P_{lm}}{P_{om}} = \frac{f(x; \theta_1)}{f(x; \theta_0)} = \frac{\left( \frac{1}{\theta_1} \right)^i e^{\frac{1}{\theta_1} \sum_{j=1}^{i} x_j}}{\left( \frac{1}{\theta_0} \right)^i e^{\frac{1}{\theta_0} \sum_{j=1}^{i} x_j}}
\]
By 3.2.3.

\[ B < \left( \frac{\Theta_0}{\Theta_1} \right)^i e^i \sum_{j=1}^{i} x_j \left( \frac{1}{\Theta_o} - \frac{1}{\Theta_1} \right) < A \]  \hspace{1cm} 4.1.5.

For practical purposes taking the logarithms of inequalities (4.1.5.)

\[
\log B < \log \left( \left( \frac{\Theta_0}{\Theta_1} \right)^i e^i \sum_{j=1}^{i} x_j \left( \frac{1}{\Theta_o} - \frac{1}{\Theta_1} \right) \right) < \log A
\]

\[
\log B < i \log \left( \frac{\Theta_0}{\Theta_1} \right) + i \sum_{j=1}^{i} x_j \left( \frac{1}{\Theta_o} - \frac{1}{\Theta_1} \right) < \log A
\]

\[
\log B - i \log \left( \frac{\Theta_0}{\Theta_1} \right) < i \sum_{j=1}^{i} x_j \left( \frac{1}{\Theta_o} - \frac{1}{\Theta_1} \right) < \log A - \log \left( \frac{\Theta_0}{\Theta_1} \right)
\]

when

\[
\Theta_0 > \Theta_1
\]

\[
(1/(\Theta_1^{-1} - \Theta_o^{-1}))(\log B + i \log \left( \frac{\Theta_0}{\Theta_1} \right)) \sum_{j=1}^{i} x_j > (1/(\Theta_1^{-1} - \Theta_o^{-1}))(\log A + i \log \left( \frac{\Theta_0}{\Theta_1} \right))
\]

\[
\Theta_0
\]

\[
4.1.6.
\]
These inequalities can be used for any \( \theta_0 \) and \( \theta_1 \) in practice.

Where

\[ B = \beta / (1 - \alpha), \quad A = (1 - \beta) / \alpha \]

Let

\[ h_0 = -\log B / (\theta_1^{-1} - \theta_0^{-1}) \quad \text{(4.1.7)} \]

\[ h_1 = \log A / (\theta_1^{-1} - \theta_0^{-1}) \quad \text{(4.1.8)} \]

\[ S = \log \left( \frac{\theta_0^{-1}}{\theta_1^{-1}} \right) / (\theta_1^{-1} - \theta_0^{-1}) \quad \text{(4.1.9)} \]

we define the acceptance line by

\[ a_i = h_0 + i S \quad \text{(4.2.0)} \]

and the rejection line by

\[ r_i = h_1 + i S \quad \text{(4.2.1)} \]

Let

\[ \sum_{j=1}^{i} x_j = V(t) = \text{accumulated time of all items tested up to time } t \]

\[ i = \text{the observed number of failures up to time } t \]

then
If \( V(t) \geq a_i \) 

accept \( H_0 : \theta = \theta_0 \)

If \( V(t) \leq r_i \) 

accept \( H_1 : \theta = \theta_1 \)

If \( a_i < V(t) < r_i \) 

continue the test

4.2 The construction of grid points

The method of calculating an exact O.C. curve for the exponential distribution depends on identifying truncated sequential tests with a random walk governed by a Poisson stochastic process which itself is a Markov Chain. The method is perfectly general and may be applied to other density functions.

We will illustrate the computational procedures by the aid of some graphs. In all the Figures of this section, \( V(t) \), on the y axis, denotes the accumulated time of all items tested up to time \( t \). \( N \), on the x axis, denotes the observed number of failures up to time \( t \).

In Figure 6, the cross points, labeled \( x \), are all obtained from the rejection line \( r_i = -h_1 + iS, i = 1, 2, 3, \ldots, Io \), such that \( r_i \geq 0 \). The first cross point occurs on the rejection line at \( (i_o, r_i) \), where \( i_o \) is the first integer \( i \), such that \( r_i > 0 \). Figure 6, \( i_o \) happens to be 6. This means that, between time zero and time \( r_i \), the hypothesis \( \theta = \theta_0 \) is rejected if \( i \) or more failures occur. In Figure 6, the first rejection interval occurs at 6 or more failures in the time interval 0 to \( r_6 \). \( i \) cannot be zero, but may be one or more. Since the test is truncated at \( Io \) failures, \( i \), the first opportunity to reject the item being tested must be
less than \( I_0 \) failures. We have graphed time on \( y \) axis, and failures on \( x \) axis, thus, \((i, r_i)\) means that \( N = i \) and \( V(t) = r_i \). The first row of crosses comes from \( N = i \), intersecting the rejection line \( r_i = -h_i + i S \). Any cross on the rejection line is circled and never used thereafter.

The first row of cross is given by \((0, r_i),(1, r_i),(2, r_i),\ldots,(6, r_i)\). In Figure 6, the first row of crosses occurs at \((0, r_6),(1, r_6),\ldots,(6, r_6)\).

The second row of crosses starts on the rejection line at \((i_0 + 1, r_{i_0} + 1)\), and continues down to \((i_0, r_{i_0} + 1),\ldots,(0, r_{i_0} + 1)\). The second rejection interval occurs between \( r_{i_o} \) and \( r_{i_0} + 1 \) with \( i + 1 \) or more failures.

In Figure 6, the second row of crosses are at \((0, r_7),(1, r_7),\ldots,(7, r_7)\), provided none of these last points lands either on the acceptance line or goes into the acceptance region. The second rejection interval occurs between \( r_6 \) and \( r_7 \). No crosses are ever on the acceptance line, unless it happens that the line of crosses coincide with a line of dots.

In such cases, we continue with rows of one kind, either cross or dot.

The dots\ldots are generated by the acceptance line. The first acceptance point occurs at \( N = 0 \) and \( V(t) = a_0 \). This value of \( V(t) \) we designate by \( a_0 \) which is the first value of \( V(t) \) on the acceptance line, \( a_i = h_0 + i S \). The last value of \( V(t) \) is at \( V_0 \), one of the two truncation values which are determined by the truncation rule we use.

In Figure 6, we can see the test is truncated at the points \( I_0 \) and \( V_0 \). Since the two truncation lines cross at the point \( G \) and form a right angle, that is why we call it a right angle truncation. How to decide the
Figure 6. Grid points.
points $I_0$ and $V_0$ for the case of testing a composite hypotheses will be discussed in the next chapter.

4.3 Graphical illustration of random walk and the Poisson Stochastic process

We start a random walk at the origin. If no failures occur we proceed steadily in time until we reach the acceptance line at $(0, a_0)$. If this happens we accept the hypothesis that $\Theta = \Theta_0$. Suppose, however, we have three failures before time $a_0$ and we continue with no failures until we reach time $a_1$ and then have one failure at this time. This puts us at $(4, a_1)$, we continue the test with no failures until we reach time $a_3$ and then have one failure at this time. This puts us at $(5, a_3)$. We continue the test and assuming no more failures, we would reach $(5, a_5)$ and accept $\Theta = \Theta_0$.

If a random walk ends in the rejection region, or on the rejection line, $N = I_0$, we reject $\Theta = \Theta_0$. If the random walk ends in the acceptance region or on the horizontal truncation line $V(t) = V_0$, we accept $\Theta = \Theta_0$. Some walks illustrate these situations in Figure 7.

The random walk is governed by a Poisson process.

$$P(x; t) = e^{-ut} \frac{(ut)^x}{x!} \quad x = 0, 1, 2, 3, \ldots$$

Where $(x; t)$ is the probability of obtaining $x$ failures in time $t$, if the failure rate is $u$ per unit time. Let $t_0 = 0, t_1, t_2, t_3, \ldots$ be the epochs at which the events occur in Poisson process.
Figure 7. Random walk.
Let $z_n = t_n - t_{n-1}$ ($n = 1, 2, 3, \ldots$), see above Figure; clearly, $z_1, z_2, z_3, \ldots$ are the random variables representing the time intervals between two successive occurrences (interoccurrence time) of Poisson process. Then, the interoccurrence times of Poisson events have a probability density function $u e^{-ux}$. The proof can be seen in (5).

Now the probability of reaching any rows of either dots or crosses depends only on the previous rows of either dots or crosses. This is the Markov Chain property which must be satisfied.

4.4 Calculation of probabilities at crosses and at the dots

To find the probability that the process will arrive at any point, we define at each dot or cross a probability. To find probabilities for any row of dots or crosses we look at the row just preceding it, whether the previous row consists of dots or crosses. The first row of crosses for Figure 6 will have the following probabilities.

$$P(0, r_6) = e^{-u(r_6 - 0)}$$

$$P(1, r_6) = u (r_6 - 0) e^{-u(r_6 - 0)}$$

$$P(2, r_6) = \frac{(u(r_6 - 0))^2}{2!} e^{-u(r_6 - 0)}$$
\[ P(3, r_6) = \frac{3}{3!} e^{-u(r_6 - 0)} \]
\[ P(4, r_6) = \frac{4}{4!} e^{-u(r_6 - 0)} \]
\[ P(5, r_6) = \frac{5}{5!} e^{-u(r_6 - 0)} \]

However

\[ P(6, r_6) = 1 - \sum_{i=0}^{5} P(i, r_6) \]

Since \((6, r_6)\) is rejection point; and, therefore, the probability represented by it is the probability of obtaining six or more failures. \((6, r_6)\) is circled since it is a prohibited point and will never be used again to calculate probabilities.

The probabilities for the second row of crosses are found from the first row as a sum of either a jump (a point goes from present row to the next row, on the same column) or a shift (a point goes from present column to the next column, on the same row) of 1, 2, 3, … etc. The probability

\[ P(0, r_7) = e^{-u(r_7 - r_6)} P(0, r_6) \]

This is so since we make a jump on the same line. To get \(P(1, r_7)\) we can make a shift from \((0, r_6)\) and a jump from \((1, r_6)\) hence

\[ P(1, r_7) = e^{-u(r_7 - r_6)} (P(1, r_6) + u(r_7 - r_6) P(0, r_6)) \]
Similarly, $P(5, r_7)$ comes from a jump of $(5, r_6)$ plus a shift from $(4, r_6)$ plus a double shift from $(3, r_6)$ plus a triple shift from $(2, r_6)$ plus a four shift from $(1, r_6)$ plus a five shift from $(0, r_6)$. Noted that when calculate $P(6, r_7)$, $(6, r_6)$ makes no contribution to it, since $(6, r_6)$ is prohibited point. $P(7, r_7)$ can be calculated by substraction. i.e.

$$P(7, r_7) = \sum_{i=0}^{5} P(i, r_6) - \sum_{i=0}^{6} P(i, r_7)$$

As soon as we come to a row of dots we have a slightly different situation. The first dot of Figure 6 is at $(0, a_o)$; its probability is

$$P(0, a_o) = e^{-u(a_o - r_{11})} P(0, r_{11})$$

The probability for the dot at $(1, a_o)$ is the sum of a jump from the cross in the previous row on the same column, $(1, r_{11})$, and a shift from the cross in the previous row on the row left, $(0, r_{11})$. A similar calculation method extends to $(2, a_o), (3, a_o), \ldots$. The probability at the rejection dot $(12, a_o)$ is the difference between

$$\sum_{i=0}^{10} P(i, r_{11}) - \sum_{i=0}^{11} P(i, a_o)$$

The circled point $(0, a_o)$ is used once in the preceeding calculation but never again.
The next row of probabilities at the crosses is found from the row of probabilities at the dot.

4.5 The exact O.C. function

The operating characteristic function or curve is obtained by addition of the probabilities at all the acceptance points. This will be the sum of the probabilities of all points on the acceptance line at which acceptance occurs at integral values of the failures, and all points on the truncation line in the acceptance region.

In Figure 6, the exact O.C. function is given by the sum

\[ \sum_{i=0}^{8} P(i, a_i) + \sum_{i=0}^{14} P(i, V_0) \]

Where \( V_0 \) is the truncation time.

As we have seen in the previous sections, the calculation of the exact O.C. function is really laborious. To simplify it, we introduce a method which was developed by R. C. Woodall and B. M. Kurkjian (12), with this method it is not only easy to compute the probability of acceptance but also easy to program. The method is characterized as follows:

Let \( P_a(u) \) denote the O.C.; and let \( J \) be the largest integer less than \( (V_0 - h_0)/S \), then,
\[ P_a (u) = \sum_{i=0}^{J} S_i \exp(-a_i * 1/\theta) + \sum_{i=J+1}^{I_0-1} S_i \exp(-V_0 * 1/\theta) \text{ for } V_0 > h_o \] 4.5.1.

\[ = \sum_{i=0}^{I_0-1} S_i \exp(-V_0 * 1/\theta) \text{ for } V_0 < h_o \] 4.5.2.

Where

\[ S_i = 1 \text{ for } i = 0 \] 4.5.3.

\[ = \sum_{j=1}^{i} (-1)^{j+1} / j! \left(a_i - j \right)^j S_{i-j} \text{ for } i = 1, 2, \ldots, n(0) \] 4.5.4.

\[ = \sum_{j=1}^{i} (-1)^{j+1} / j! \left(a_i - r_i \right)^j S_{i-j} \text{ for } i = n(0)+1, \ldots m \] 4.5.5.

\[ = \sum_{j=1}^{m} (-1)^{j+1} / j! \left(a_i - r_i \right)^j S_{i-j} \text{ for } i = m+1, \ldots \] 4.5.6.

\[ n(0) \text{ is the integer denotes the largest index } i \text{ for which } r_i = 0, \text{ satisfying the relation} \]

\[ (h_1 / S) - 1 \leq n(0) \leq (h_1 / S) \] 4.5.7.

\[ m \text{ is the integer designated by the relation} \]

\[ r_m \leq h \leq r_{m+1} \] 4.5.8.
$S^*$ is the same as $S_i$ (as defined in 4.5.3. through 4.5.6) except $a_i$ is replaced by $V_0$ whenever $a_i > V_0$.

The proofs of these formulas can be seen in (12). Figure 8 shows the places of $n(0)$, $m$, $J$, and $(V_0 - h_0)/S$.

The theory behind these formulas (4.5.1. through 4.5.6.) are the same as we mentioned in the previous sections. The most important advantage of using these formulas is that the exact O.C. curve is easy to get. These formulas have been programed and are given in Appendix B.
Figure 8. Right angle truncation.
CHAPTER V
APPLICATION OF A FIXED SAMPLE SIZE TEST TO A SEQUENTIAL TEST

The development of sequential analysis has led to the proposal of tests that are more economical in A.S. N. than are their fixed sample counterparts. Even though we can be sure that a decision is reached in a finite number of trials, another difficulty exists with the sequential probability ratio test. For most cases the number of observations is a random variable which is unbounded and has a positive probability of requiring an unacceptably large number of test stages to terminate, and this is the main criticism of the sequential test, therefore, any practical sequential test procedure must include a truncation rule.

In a field or greenhouse experiment involving lengthy and costly observations, such as complex chemical analyses of the plants, the experimenter may adopt a sequential test procedure in order to give himself a chance of reaching a decision after as small a number of observations as possible. However, it is also of interest to consider cases in which the truncation of a sequential procedure, rather than being a necessary technical restriction is desirable in order to improve the performance of the procedure. In particular, truncation of a test may reduce the expected sample size for some values of the parameters.
involved. If a priori information is available concerning the values (or distributions) of these parameters, it may be practicable to truncate the procedure for this reason alone.

The literature on truncated sequential analysis is quite extensive, especially, in the discussion of normal distribution. Wald developed a method of truncation and studied the changes of $\alpha$ and $\beta$. T. W. Anderson (1) got the approximate O.C. and A.S.N. functions for the normal distribution by using a Weiner process. M. A. Schneiderman and P. Armitage (2) developed a "wedge" shaped boundary to truncate the sequential test, in order to get a smaller A.S.N. Clearly, a lot of methods can be used to truncate the test there is no particular need to use straight lines as boundaries or two parallel lines for boundaries. An important item that should be noticed is that when reducing the A.S.N. of a test, the desired O.C. should be maintained. A test can't be said to be a good test, if it reduces the power of the test and the sample size required simultaneously.

In this chapter, we will introduce a method of truncation for the case of the exponential distribution when composite hypotheses are being considered. Reducing the average sample size required and maintaining the desired O.C. curve will be regarded as two equally important things in our method of truncation. How good is our truncation method will be examined in the next chapter.
5.1 Method of truncation

The method we introduced here will be called the "mixed method." The reason we name it a mixed method is that this truncation method is accomplished by superimposing an appropriate fixed sample size test on a non-truncated sequential test. Or, we can say, this method, a mixture of a fixed sample size test procedure and a sequential test procedure is the same as the sequential test with truncation, and removes the main criticism of the sequential test.

If we can determine the test parameters of a sequential test which can reproduce a similar O.C. curve to that of corresponding fixed sample size test it may reduce the sample size of the test with the same power against the wrong decision. The mixed truncation method is formed for this purpose.

As we mentioned in Chapter II, in the fixed sample size test, the sample size required for the test, Io, is fixed. If we preassigned a value as the Type I error, for the test $H_0: \mu \leq 3$ against $H_1: \mu > 3$, a best critical region can be found. And this kind of test is known as a uniformly most powerful test. Our truncation method is formulated in this way; let the x axis denotes the number of observations, and we select the sample size required, Io, for the fixed sample size test as a vertical truncation value in our mixed truncation. That is, if no decision has been made by the non-truncated sequential procedure with Io-1 trials, when the number of observations reaches Io, a decision must be made at
this time. All the cumulative totals fall on this vertical line, and the hypothesis must be rejected or accepted. Let the $y$ axis denote the cumulative total; we select the critical value we found in the fixed sample size test as a horizontal truncation value in our mixed truncation method. If the cumulative total crosses this horizontal line the hypothesis is accepted. Briefly speaking, the theory for fixed sample tests was applied to the sequential test. Thus, we mix them together in our new method of truncation. If we can maintain the O.C. curve of the corresponding fixed sample test in our truncation test, then, our mixed truncation test may be an uniformly most powerful truncation test. This is because all the characteristics of fixed sample size tests are duplicated with our method of truncation. The next thing we have to do is to determine how to set two parallel boundaries, i.e. the upper boundary and the lower boundary in our truncation test, so that the desired O.C. can be obtained. The upper boundary should cross the horizontal truncation line and the lower boundary should cross the vertical truncation line.

The determination of the two parallel boundaries is discussed in the following section.

5.2. Rough method of determining two parallel boundaries for the mixed truncation method

As we mentioned in Chapter II, it is more difficult to handle composite hypotheses than simple hypotheses in sequential analysis. In the case of simple hypotheses, both the parameters under the null and
alternative hypothesis are completely specified. In the non-truncated sequential test, Wald (11) has given a method to construct the test regions for simple hypotheses. For instance, in the exponential case, consider the test $H_0 : \Theta = \Theta_0$ against $H_1 : \Theta = \Theta_1$, and with preassigned two values $\alpha$ and $\beta$. In terms of these four values $\Theta_0$, $\Theta_1$, $\alpha$, $\beta$, and using the inequalities 4.1.6. through 4.1.9. the two boundaries are completely determined. The O.C. curve of the test with these two boundaries is very similar to the O.C. curve of the fixed sample size test. However, in the case of composite hypotheses, both parameters under the null and alternative hypotheses are not specified. This increases the difficulty in setting the upper and lower boundaries for the sequential test, since without specified parameters Wald's formulas to construct the two boundaries can't be applied. The same difficulty occurs with truncated sequential test.

In this section, we will reduce original composite hypotheses to a simple hypotheses, so that Wald's formulas for setting up test regions can be applied. Usually, to reduce a composite hypotheses to a simple hypotheses, we mean, to give specified values for the parameters under the null hypothesis and the alternative hypothesis. Owing to the measurement error and the inaccuracy of drawing an O.C. curve, a small deviation from the exact value is unavoidable.

Generally, determining the upper and lower boundaries for our mixed truncation test by reducing the original composite hypothesis to a
simple hypotheses, the desired O.C. for the test can not be maintained sometimes. To remedy this shortcoming, we introduce a "rough method." The method can be characterized as follows:

1. Reduce the original composite hypotheses to a simple hypotheses
2. Apply Wald's method to build two boundaries for the test
3. Use the boundaries we previously found as two sequential boundaries in our mixed truncation test at first. If the O.C. curve we got with these two boundaries are close to that of corresponding fixed sample size test, then these two boundaries become the boundaries of our mixed truncation test. If not, then,
4. Adjust the two boundaries with the aid of computer, the process of adjustment is repeated several times until the desired O.C. is obtained.

We emphasize the importance of the determination of the two boundaries, because with our experience we found it affects the O.C. curve profoundly.

Here, we use example 2.1. again to illustrate our rough method. The test $H_0: \mu < 3$ against $H_1: \mu > 3$, with $\alpha = 0.0038$, $n = 9$. The O.C. curve for this test is given in Figure III. Suppose that we want to duplicate this curve by our mixed truncation test, it is easy to find $\alpha$ and $\beta$ when $\alpha = 0.0038$ and $\beta = 0.0038$. If we follow the dashed lines in
Figure 3, with O.C. = 0.99620, and O.C. = 0.0038 respectively, then the parameters $\mu_0$ and $\mu_1$ should be decided at 3 and around 19 respectively.

Actually, the Type I and Type II errors we got in our mixed truncation test are not exactly 0.0038, but will not be far from 0.0038. If we substitute the $\mu_0$ and $\mu_1$ we found and with the values $\alpha = 0.0038$, $\beta = 0.0038$, into the inequalities 4.1.6. through 4.1.9. then, $h_0 = 35$, $h_1 = -0.35$, $S = 0.115$. These values will not give a very close O.C. curve to that of corresponding fixed sample test. However, if we adjust the values $h_0$, $h_1$, $S$ to be 0.44, $-0.44$, 0.118 respectively, then, the O.C. curve we got is very very close to that of corresponding fixed sample test.

Wald (11) in his non-truncated sequential test used two parallel straight lines as the upper boundary and lower boundary, it is proved that these two parallel straight boundaries produced a very satisfactory result. The adoption of two parallel straight lines as two boundaries in our mixed truncation test is not only for the purpose of easying to handle but also for the reason we mentioned on the above.

5.3 Clues for adjusting the two sequential boundaries

In the above section we mentioned that in order to get the desired O.C. some adjustments of the two sequential boundaries in our truncation test are necessary. In this section, we give some clues on how to
adjust the two boundaries. We will use some graphs to illustrate this problem.

We already know that as soon as we defined the values, $h_0$, $h_1$, and $S$, we have defined the two boundaries. In Figure 9 we denote the solid line as the O.C. curve of the fixed sample size test, and this O.C. curve is the O.C. curve we want to match. Let the dashed line denote the O.C. curve we got by using Wald's boundaries. From Figure 9, we can see the O.C. curve (dashed line) is uniformly lower than the O.C. curve (solid line) of the fixed sample size test. That is, by taking Wald's boundaries as two boundaries in our truncation test will cause the probabilities of acceptance uniformly lower than that of fixed sample size test. At this time, we have to increase the probabilities of acceptance a little bit higher so that these two O.C. curves can be as close as possible. In this case, we may lower $h_0$ and keep $h_1$ and $S$ unchanged, or, we may keep $h_0$, $h_1$ unchanged and lower $S$ a little bit, then the desired O.C. can be obtained. However, according to our experience, changing the slope $S$, is easier to get the desired result. Figure 10 is a graph illustration, dashed lines are the new boundaries of our truncation test which may give us desired O.C. curve and the solid lines are the Wald's boundaries. Clearly, from Figure 10, the two new boundaries will give us more acceptance and less rejections so that the probabilities of acceptance are increased if the O.C. curve which we obtained by using Wald's boundaries is uniformly higher than
the O.C. curve of the fixed sample size test, we could use the same procedure as indicated above, but in an opposite way.

When the O.C. curve we get is like the curve (denoted by dots) shown in Figure 9, we have to increase the probabilities of acceptance as $\mu \leq 4$ and decrease the probabilities of acceptance as $\mu > 4$. In this case, we may lower $h_o$, $h_1$, and raise the slope so that we will get more acceptances when the parameter $\mu$ is small and get more rejections when the parameter $\mu$ is large. The dotted line denotes the new boundaries for our truncation test.

Certainly, by using Wald's boundaries in our truncation test the O.C. curve we got might not be like the curves we showed in Figure 9; however, we can see what the shape of the curve looks like and then decide to lower (or raise) $h_o$, $h_1$, and $S$, or keep some of the values unchanged and changing the others.

Generally, the process of adjustment should be repeated several times. However, the procedures we just mentioned will help us to get the desired result with less effort.

5.4 Graphical illustration of the new truncation method

By overlapping Figures 3 through 5, we finally have one like Figure 9. This is our mixed truncation test. We truncate the test at $I_0 = 9$, the sample size required for the fixed sample test, and $V_0 = 1$, the critical value of fixed sample test with $\alpha = 0.0038$, $n = 9$. The
Figure 9. Effects on O.C. curve with different boundaries.
Figure 10. Effects of adjusting two sequential boundaries.
first sample path, labeled 1, at $Io = 3$ crosses the upper boundary, the hypothesis $H_o: \mu \leq 3$ is accepted. The second sample path, labeled 2, at $Io = 6$ crosses the horizontal truncation line and the hypothesis $H_o: \mu \leq 3$ is accepted. If the test is Wald's nontruncated sequential test no decision could be made for the second sample path at this time. Because the final dot on the second sample path still remains in the continuation region, therefore, our truncation test in this case as compared with non-truncated sequential test will make the decision earlier. The third sample path, labeled 3, crosses at the vertical truncation line, the hypothesis $H_o: \mu \leq 3$ is rejected. If this case happened in the non-truncated sequential test, there is still no decision that can be made. In our truncation method, all the dots fall on this vertical truncation line decision must be made and the experiment is terminated with the rejection of the hypothesis. The fourth sample path, labeled 4, crosses the lower boundary at $Io = 7$, and the hypothesis $H_o: \mu \leq 3$ is rejected.

The upper and lower boundaries are determined by using the rough method we mentioned in previous section. With these two truncation lines and the two boundaries, the power of the test against making wrong decision for our truncation test is almost the same as the corresponding fixed sample test. The average sample size required for our truncation test, however, in general, will be less than 9, if the worst situation happened the sample size required for our truncation test will be equal to 9 but never greater than 9.
Figure 11. Mixed truncation procedure.
CHAPTER VI
EXAMINATION OF EFFECTS OF THE MIXED TRUNCATION METHOD

As we mentioned in the introductory chapter, we will compare two different sequential test procedures in terms of the O.C. and A.S.N. functions. Generally, the smaller the average sample size required and the more powerful the test, the better the test procedure. Therefore, a test procedure which needs a much smaller A.S.N. to terminate testing but with poorer power for the test cannot be said a good test.

In this study, in order to find whether our truncation rule is an optimum truncation rule or not, we will do the following things:

1. Both the truncation values Vo and Io are determined by following our truncation rules. The original composite hypotheses is reduced to a simple hypotheses. Using the rough method we suggested, the two boundaries for our test are determined so that the O.C. curve for our mixed truncation test will be as close as possible to that of corresponding fixed sample test.

2. It may be that this procedure is not optimum; if so, changing either Io or Vo should result in a greater A.S.N. At first, Io will be kept the same as the Io we used above, but Vo will be selected arbitrarily. The two boundaries used in this test will be the same as the boundaries
used above at first. However, in general, these two boundaries will not
give us the desired O.C. curve; therefore, several time's adjustment
for these two boundaries is needed until we got the O.C. curve close to
the O.C. curve of the fixed sample size test. In the case of keeping Vo
unchanged and changing the value of Io, the test procedure is exactly
the same.

3. Under the condition that these test procedures have approxi­
mately the same O.C. curve, the A.S.N. for these test procedures are
compared.

The O.C. curves for these test procedures are calculated by
using the formulas given in Chapter IV. Unfortunately, no easy formulas
are available for calculating exact A.S.N. However, Monte Carlo
methods will be used to obtain the A.S.N. for these tests.

As dealing with composite hypotheses, our mixed truncation
method reduces it to a simple hypotheses, therefore, we will also com­
pare our truncation test with those existing truncated sequential tests
which are designed for the testing of a simple hypothesis versus a
simple alternative. The following section is a brief description about
these truncated sequential tests.

6.1 Brief description of existing
truncated sequential tests

In the exponential case, few truncated sequential tests have been
developed in the literature. Woodall and Kurkjian (12) and Aroian (3)
developed what we call "right angle truncation" simultaneously; their approach to this problem is the same. The main difference is that easier formulas are given in Woodall and Kurkjian's paper to calculate the exact O.C. function.

The truncation shape of the right angle truncation is exactly the same as the shape shown in Figure 11. The rules of accepting, rejecting or continuing the test are also the same as we described in section 5.4, except that the determination of the horizontal and vertical truncation values is different from the mixed truncation method.

This right angle truncation is simply designed for testing a simple hypotheses, such as \( H_0: \mu = \mu_1 \). The derivation of the truncation formulas of this right angle truncation are simple, but a little bit tricky. Briefly, let \( h_0 = h_1 = 0 \) for the acceptance and rejection lines, so that they coincide, and we got a single line, next, consider the intersection of this line \( a_1 = iS \), and \( N = Io \) (see Figure 8). This gives *Io, IoS*) for this point. Draw a parallel line to x axis and cross y axis at point \( V_0 = IoS \). That is, select the horizontal truncation value \( V_0 \) (or Io, the vertical truncation value) arbitrarily and determine \( Io \) (or \( V_0 \)) from the relation \( V_0 = IoS \). As pointed out by Woodall and Kurkjian (12) this is a reasonable truncation procedure. After reducing a composite hypotheses to a simple hypotheses for our mixed truncation test we will compare our truncation method with Woodall and Kurkjian's right angle truncation method which is based on this relation.
Wald (11) developed a method of truncation, which is good only for the testing of a simple hypotheses. By giving a new rule for the acceptance or rejection of $H_0$ at the $I_0^{th}$ stage if the sequential process did not lead to a final decision for $i < I_0$, the method of truncation is studied. If no decision is reached by $I_0^{th}$ stage, sampling is stopped anyway with $H_0$ accepted if

\[
\log \frac{P_{I_0}}{P_{i_0}} < 0 \quad \text{6.1.1.}
\]

and reject if

\[
\log \frac{P_{I_0}}{P_{i_0}} > 0 \quad \text{6.1.2.}
\]

Figure 1 is a graphical illustration of the stopping rule.

L. A. Aroian (4) applied Wald's truncation method to the testing of the mean of a normal distribution with known variance and got exact O.C. and A.S.N. functions. As dealing with normal distribution, Wald's truncation method will be selected as a comparison with the mixed truncation method. Some results already done by Aroian will be picked up to compare with our Monte Carlo results.

6.2 Empirical A.S.N. curve by Monte Carlo Methods

Quite often in scientific problems where probabilities are difficult to determine mathematically, the frequency theory is used to obtain an
empirical approximation. The actual experiment or simulation thereof is repeated a large number of times. The observed proportion of the outcomes of the event of interest is determined and is used as an approximation to the theoretical probability. Such procedures are widely used and are called Monte Carlo techniques. Computing the exact A.S.N. on all possible alternative values of $\mu$ to obtain a theoretical A.S.N. curve of a truncated sequential test is very tiresome. And we have enough information to apply a Monte Carlo method to find the approximations to draw an A.S.N. curve of a truncated sequential test.

First of all, to apply this method, an exponential random variable was generated. Appendix C gives the program to generate exponential random variables and to calculate empirical A.S.N. curve.

Following are brief descriptions of the method used to obtain an A.S.N. curve for our truncation test and for those tests whose truncation values $V_o$ and $I_o$ are deliberately changed to determine whether the mixed method is optimum. The rules of accepting, rejecting, or continuing the test are exactly the same as we described in section 5.4, except that the determination of $V_o$ and $I_o$ depends on what runcation rules we use.

1. We obtained the upper boundary and lower boundary which we found by using Wald's formulas and then adjusting to get the correct O.C. curve. We used these as the two boundaries in applying the Monte Carlo methods.
2. We generated an exponential random variable with the parameter \( \mu \) which is the true mean.

3. If an observation generated by the above step did not fall into one of the decision regions, another observation was required. We kept sampling until we reached a decision, or until we reached the horizontal truncation line or the vertical truncation line. Figure 8 is a graphical illustration of this situation.

4. One hundred sets of trials were made for each alternative mean. Generally, the more sets of trials, the more accurate the results. The number of trials is also dependent on the computer budget, because it is expensive to get results by using Monte Carlo methods. One hundred sets of trials seems not large enough but it gives us a quite satisfactory result.

5. The proportion of times that \( H_0 \) was accepted was also computed. This proportion of acceptance shows the O.C. of the test on that alternative mean which is used to generate the exponential random variables for that one hundred sets of trials. The O.C. we got here should coincide with the exact O.C. we got by using some formulas. Thus, this provides a method to check the accuracy of Monte Carlo methods we adopted. If the deviations between exact O.C. and Monte Carlo O.C. are so small that we can ignore them, then we can say the simulation of A.S.N. by Monte Carlo methods will have little bias.

6. The number of observations for every trial, at the point when the decision was reached (either accepting or rejecting the
hypothesis), was recorded. After one hundred set of trials were finished, the accumulated total was divided by the total number of trials and the average sample size required to terminate the test for each alternative mean was obtained.

6.3 Numerical results and comparison

The exact O.C. for our mixed truncation test and for those tests with adjusted Vo and Io is obtained by using formulas 4.5.1. through 4.5.6. Monte Carlo methods were conducted on these tests to get the empirical A.S.N. curve according to the steps we just mentioned in the above section.

Example 2.1 was being used again, with the method of truncation we introduced in section 5.1. Our mixed truncation test will set the horizontal truncation value Vo (critical value which corresponds to the critical region we found in the fixed sample size test) at 1 and the vertical truncation value Io (sample size required for a fixed sample test) at 9. The determination of upper and lower boundaries for our mixed truncation test is obtained by using rough method we suggested. Here, the mean \( \mu_0 \) should be set at 3 and the mean \( \mu_1 \) should be set around 19, with \( \alpha = 0.0038 \), \( \beta = 0.0038 \). That is, the original composite hypotheses \( H_0: \mu < 3 \) against \( H_1: \mu > 3 \) has been reduced to a simple hypotheses \( H_0: \mu = 3 \) against \( H_1: \mu = 19 \). Using the equalities 4.1.7. through 4.2.1., \( h_0, h_1, S \) are determined to be 0.35, -0.35, 0.115 respectively. As soon as
h₀, h₁ and S are obtained we can say we have defined two boundaries for the test already. With these values Vo, Io, h₀, h₁, and S, we use the program already written (given in Appendix B), and the O.C. curve for this test is obtained.

However, with these values Vo = 1, Io = 9, h₀ = 0.35, h₁ = 0.35, and S = 0.115, the O.C. curve we got will not be close enough to that of corresponding fixed sample size test, a little bit adjustment is needed. If we have Vo, Io, h₀, h₁, and S to be 1, 9, 0.44, −0.44, and 0.118 respectively, then, the result obtained is quite satisfactory. Comparing the results on Table 1 and Table 2, we can see the exact O.C. curve of our mixed truncation test is pretty close to that of the corresponding fixed sample size test. The Monte Carlo method also gives us an answer very close to this result.

Next, in order to establish the usefulness of our mixed truncation method, we select VO=1.2 instead of VO = 1 in our mixed truncation test and keep Io = 9 unchanged. The upper and lower boundaries we use in the above test were used again in the present test, i.e. h₀, h₁, and S are still to be 0.44, −0.44, 0.118 respectively. With these two boundaries the O.C. curve we got for the present test is far far away from the O.C. curve of the fixed sample test, therefore, an adjustment is needed. The process of adjustment is repeated several times until the desired O.C. is obtained. In the present test the final values of h₀, h₁, and S should be determined at 0.52, −0.52, and 0.092
respectively, then, the desired O.C. can be obtained. The results for this test are given in Table 2.

From Table 2, we can see the O.C. curves of our mixed truncation test and the test with different Vo are very close. Obviously, we can't get exactly the same O.C. curves for both tests; but small deviations can be ignored.

Figure 12 shows the O.C. curve of the fixed sample size test, denoted by the solid line, of the mixed truncation test, denoted by cross (x), and of the mixed truncation test with different Vo, denoted by the dashed line. These three curves can be regarded as one curve, because they are close enough.

The O.C. curves of our mixed truncation test and the truncation test with different Vo are so close that we can compare the A.S.N. of these two test procedures. Table 3 shows the values of A.S.N. function by using Monte Carlo methods. Figure 13 is the comparison of A.S.N. curves for these two test procedures. Either from Table 3 or from Figure 13, we can see that with \( \mu \leq 9 \) the A.S.N. values of our mixed truncation test are somewhat greater than that of the test with different Vo. However, when \( \mu > 9 \) the A.S.N. of our mixed truncation test is smaller than that of the test with different Vo. As \( \mu \) goes larger and larger the deviation of these two A.S.N. curves is getting bigger and bigger.

Now suppose that we keep the horizontal truncation value unchanged, i.e. \( Vo = 1 \) and change the vertical truncation value from \( Io = 9 \)
Table 2. Values of O.C. function

<table>
<thead>
<tr>
<th>Procedure</th>
<th>( v_o = 1, I_o = 9, h_o = 0.44, h_1 = -0.44, S = 0.118 )</th>
<th>( v_o = 1.2, I_o = 9, h_o = 0.52, h_1 = -0.52, S = 0.092 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(( \mu ))</td>
<td>Exact results</td>
<td>Monte Carlo results</td>
</tr>
<tr>
<td>1</td>
<td>0.99999</td>
<td>0.99999</td>
</tr>
<tr>
<td>2</td>
<td>0.99973</td>
<td>0.99976</td>
</tr>
<tr>
<td>3</td>
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<td>0.99614</td>
</tr>
<tr>
<td>4</td>
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</tr>
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<td>5</td>
<td>0.93126</td>
<td>0.93162</td>
</tr>
<tr>
<td>6</td>
<td>0.84673</td>
<td>0.84689</td>
</tr>
<tr>
<td>7</td>
<td>0.72908</td>
<td>0.72880</td>
</tr>
<tr>
<td>8</td>
<td>0.59325</td>
<td>0.59239</td>
</tr>
<tr>
<td>9</td>
<td>0.45704</td>
<td>0.45565</td>
</tr>
<tr>
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<td>0.33469</td>
<td>0.33294</td>
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</tr>
<tr>
<td>19</td>
<td>0.00421</td>
<td>0.00390</td>
</tr>
<tr>
<td>20</td>
<td>0.00232</td>
<td>0.00210</td>
</tr>
</tbody>
</table>
Figure 12. Effect of truncation on O.C. curve.
to $I_0 = 10$ in the mixed truncation test. Then, the final values of $h_0, h_1,$ and $S$ are 0.48, $-0.48$, and 0.1475 respectively. Unfortunately, with these values the O.C. curve of this test still can get close to the O.C. curve of the fixed sample size test. We tried several combinations of $h_0, h_1,$ and $S$ but to no avail; the answer we got was not better than the answer we got by using the values $h_0 = 0.48, h_1 = -0.48,$ and $S = 0.1475$.

As we said before, to compare the A.S.N. of two different test procedures, the O.C. curves of these two tests should be approximately the same, otherwise; the comparison is not reasonable. Table 4 gives the O.C. functions for our mixed truncation test and for the test with different $I_0$. No comparison of A.S.N. for these two tests is possible.

The O.C. curves for fixed (solid line), mixed (cross, x) and the test with different $I_0$ (dashed line) are given in Figure 14. From Figure 14 we can see the O.C. curve of our mixed truncation test is steeper than that of the test with different $I_0$. This indicates that our truncation test is a more powerful test.

As we reduced the original composite hypotheses $H_0: \mu \leq 3$ against $H_1: \mu > 3$ to a simple hypotheses $H_0: \mu = 3$ against $H_1: \mu = 19$ and compared our mixed truncation test with Woodall and Kurkjian's right angle truncation test, we found that the right angle truncation rule based on the relation $V_0 = I_0 S$ seems to coincide with our mixed truncation rule. Because we already know that in our mixed truncation test the value of the slope $S$, is equal to 0.118, in the right angle truncation test its
Table 3. Values of A.S.N. functions (Monte Carlo results)

<table>
<thead>
<tr>
<th>Procedure</th>
<th>( V_0=1, I_0=9, h_0 = 0.44, h_1 = -0.44, S=0.118 )</th>
<th>Monte Carlo results</th>
<th>( V_0=1.2, I_0=9, h_0 = 0.52, h_1 = -0.52, S=0.092 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(( \mu ))</td>
<td>( V_0=1, I_0=9, h_0 = 0.44, h_1 = -0.44, S=0.118 )</td>
<td>( V_0=1.2, I_0=9, h_0 = 0.52, h_1 = -0.52, S=0.092 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.64977</td>
<td>1.57647</td>
<td></td>
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<tr>
<td>2</td>
<td>2.46805</td>
<td>2.25706</td>
<td></td>
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<tr>
<td>3</td>
<td>3.42754</td>
<td>3.04933</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4.45574</td>
<td>3.93410</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5.45681</td>
<td>4.86280</td>
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</tr>
<tr>
<td>6</td>
<td>6.34405</td>
<td>5.58430</td>
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</tr>
<tr>
<td>7</td>
<td>7.06131</td>
<td>7.26956</td>
<td></td>
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<td>8</td>
<td>7.58850</td>
<td>7.46956</td>
<td></td>
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<tr>
<td>9</td>
<td>7.93460</td>
<td>7.80833</td>
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<tr>
<td>10</td>
<td>8.12574</td>
<td>8.20819</td>
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<tr>
<td>11</td>
<td>8.19409</td>
<td>8.49026</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>8.17068</td>
<td>8.68044</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>8.08197</td>
<td>8.80446</td>
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<td>14</td>
<td>7.94904</td>
<td>8.89839</td>
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</tr>
<tr>
<td>15</td>
<td>7.78809</td>
<td>8.92508</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>7.61135</td>
<td>8.95010</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>7.42796</td>
<td>8.96215</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>7.24459</td>
<td>8.96585</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>7.06599</td>
<td>8.95826</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>6.89540</td>
<td>8.95826</td>
<td></td>
</tr>
</tbody>
</table>
Figure 13. Comparison of two A.S.N. curves.
Table 4. Values of O.C. function.

<table>
<thead>
<tr>
<th>Procedure</th>
<th>$V_0=1, I_0=9, h_0=0.44, h_1=-0.44, S=0.118$</th>
<th>$V_0=1, I_0=10, h_0=0.48, h_1=-0.48, S=0.1475$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(mixed Truncation)</td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>Exact results</td>
<td>Exact results</td>
</tr>
<tr>
<td>1</td>
<td>0.99999</td>
<td>0.99998</td>
</tr>
<tr>
<td>2</td>
<td>0.99973</td>
<td>0.99945</td>
</tr>
<tr>
<td>3</td>
<td>0.99601</td>
<td>0.99531</td>
</tr>
<tr>
<td>4</td>
<td>0.97817</td>
<td>0.97943</td>
</tr>
<tr>
<td>5</td>
<td>0.93126</td>
<td>0.94035</td>
</tr>
<tr>
<td>6</td>
<td>0.84773</td>
<td>0.86980</td>
</tr>
<tr>
<td>7</td>
<td>0.72908</td>
<td>0.76834</td>
</tr>
<tr>
<td>8</td>
<td>0.59325</td>
<td>0.64560</td>
</tr>
<tr>
<td>9</td>
<td>0.45704</td>
<td>0.51586</td>
</tr>
<tr>
<td>10</td>
<td>0.33469</td>
<td>0.39278</td>
</tr>
<tr>
<td>11</td>
<td>0.23406</td>
<td>0.28590</td>
</tr>
<tr>
<td>12</td>
<td>0.15707</td>
<td>0.19969</td>
</tr>
<tr>
<td>13</td>
<td>0.10161</td>
<td>0.13431</td>
</tr>
<tr>
<td>14</td>
<td>0.06360</td>
<td>0.08731</td>
</tr>
<tr>
<td>15</td>
<td>0.03868</td>
<td>0.05502</td>
</tr>
<tr>
<td>16</td>
<td>0.02292</td>
<td>0.03372</td>
</tr>
<tr>
<td>17</td>
<td>0.01382</td>
<td>0.02014</td>
</tr>
<tr>
<td>18</td>
<td>0.00754</td>
<td>0.01176</td>
</tr>
<tr>
<td>19</td>
<td>0.00421</td>
<td>0.00762</td>
</tr>
<tr>
<td>20</td>
<td>0.00232</td>
<td>0.00377</td>
</tr>
</tbody>
</table>
Figure 14. Effect of Truncation on O.C. curve.
horizontal truncation value $V_o$ is obtained by $V_o = 9 \times 0.118 = 1.062$. Thus, comparing the $V_o = 1.062$ with the $V_o = 1$ in our mixed truncation test, we found that the difference is only 0.062. With this small deviation, it is obvious that the results obtained by using the right angle truncation rule will be almost the same as the results obtained by using our mixed truncation rule.

6.4 Application of mixed truncation method to a normal distribution with known variance

In order to be more sure of the usefulness of our mixed truncation method, we applied it again to the testing of the mean of a normal distribution with known variance. The procedures, such as the construction of test regions, the rules of truncation, the determination of the two boundaries, are exactly the same as for the case when we deal with the exponential distribution. The only difference is that the distribution we now deal with is a normal distribution. Therefore, in the following as we deal with this problem no detailed explanation is given.

Suppose in a normal distribution we want to test a composite hypotheses $H_0: \mu \geq 5$ against $H_1: \mu \leq 5$, with known $\sigma = 4$, and $\alpha = \beta = 0.025$, $n = 16$. Then, when we apply our truncation method to this test the following procedures are necessary:

1. Find a critical region and draw an O.C. curve for this test
2. Construct sequential test regions for this test
3. Reduce the original composite hypotheses to a simple hypotheses
4. Using the rough method suggested earlier, find the values \( h_0, h_1, \) and \( S \), so that the desired O.C. can be obtained.

5. Build a Monte Carlo method.

According to the above steps, we have found \( V_0=64.32, I_0=16 \) for this test, the mean \( \mu_0 \) and \( \mu_1 \) should be set around 5 and 3.05 respectively. With the values \( h_0=7.465, h_1=-7.465, \) and \( S = 4.025 \) and O.C. curve for this test is close to that of corresponding fixed sample size test. These did not have to be adjusted much, from the values obtained, using Wald's procedure.

Monte Carlo methods were used to get the empirical A.S.N. and O.C. functions. One thousand sets of trials were made for each alternative mean. Before we did one thousand sets of trials, we had tried one hundred, five hundred sets of trials; the variation is too much that we should increase the number of trials. After the number of trials reached one thousand, the variation became steadily. The program for generating normal random variables and calculating empirical A.S.N. and O.C. functions is given in Appendix D.

For the test; \( H_0: \mu = 5 \) against \( H_1: \mu = 3, \) with \( \sigma^2 = 4, \alpha = \beta = 0.025 \) and \( n = 16. \) L. A Aroian (4) had obtained exact O.C. and A.S.N. functions, as he applied Wald's truncation method to a normal distribution. In this study, we just picked up the results he already had done and compared it with the Monte Carlo results we got by using our mixed truncation method. The comparison is shown in Table 5.
Table 5. Values of A.S.N. and O.C. functions (normal distribution)

<table>
<thead>
<tr>
<th>Procedure</th>
<th>$V_0 = 64.32$, $h_0 = 7.465$, $h_1 = -7.465$, $S = 4.025$, $I_0 = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(mixed Truncation)</td>
</tr>
<tr>
<td></td>
<td>Monte Carlo results</td>
</tr>
<tr>
<td></td>
<td>(Wald Truncation)</td>
</tr>
<tr>
<td></td>
<td>Monte Carlo results</td>
</tr>
<tr>
<td>$(\mu)$</td>
<td>O.C.</td>
</tr>
<tr>
<td>5.5</td>
<td>0.998</td>
</tr>
<tr>
<td>5.0</td>
<td>0.977</td>
</tr>
<tr>
<td>4.5</td>
<td>0.849</td>
</tr>
<tr>
<td>4.0</td>
<td>0.500</td>
</tr>
<tr>
<td>3.5</td>
<td>0.172</td>
</tr>
<tr>
<td>3.0</td>
<td>0.027</td>
</tr>
<tr>
<td>2.5</td>
<td>0.000</td>
</tr>
</tbody>
</table>
From Table 5, we can see our truncation method is not an optimum truncation. The A.S.N. of our truncation test is better than the Wald's truncation only for part of the ranges of $\mu$, but not all over the ranges of $\mu$. Therefore, it is difficult to ascertain which truncation method is better.
CHAPTER VII
CONCLUSION

In actual practice, it is relatively rare that simple hypotheses are tested against simple alternative; usually, one or the other, or both, are composite. For the purpose of being practical, we deal with composite hypotheses in this study.

Generally, in truncated sequential analysis, when we deal with composite hypotheses, the problem of determining the truncation rule, the construction of test regions, or the determination of O.C. and A.S.N. functions becomes much more difficult. To solve this problem we have provided a method of truncation. For simplicity, the way we did is to reduce the original composite hypotheses to a simple hypotheses so that the whole test procedures will not be so difficult to handle. One thing that should be noted here is that not all composite hypotheses can be reduced to simple hypotheses; only those tests for which the likelihood ratio is a monotone function of the parameter have this property.

The theory behind our mixed truncation method is simple. Briefly with the combination of a fixed sample size test and a sequential test our truncation method is formed. A more difficult thing is that in order to maintain the desired O.C. curve in our mixed truncation method, the test parameters should be chosen properly. However, to
accomplish this purpose is not easy; although we can get a rough method to help us determine the two sequential boundaries, problems still exist. For example, in dealing with the exponential distribution in this study, we followed the rough method we suggested so that the two boundaries for the test can be determined. However, to obtain a desired O.C., the process of adjusting the two boundaries, should be repeated several times. When the truncation values in the mixed truncation test are not determined by our truncation rule; the determination of the two boundaries for the test becomes very difficult. The process of adjustment must be repeated many times. From the point of view of time and costs, to decide the two boundaries in such a way is really very tiresome and time-consuming. However, in dealing with normal distribution, we even don't need any adjustments, the O.C. curve we got is quite satisfactory. Anyway, to find a good method to determine the upper and lower boundaries for testing a composite hypotheses in truncated sequential test further investigation is needed.

Before doing this study, we thought the truncation method we suggested was an optimum truncation. Because the fixed sample size test we dealt with in this study was known as a uniformly most powerful test, all the characteristics of a fixed sample test were duplicated in our truncation test. We thought our truncation method might therefore be a uniformly most powerful truncation. Results shown in Chapter VI showed otherwise. Though our truncation method is not an optimum
truncation, the O.C. and A.S. N. functions obtained by using our truncation method are quite satisfactory as compared with other truncation methods.

It is hard to say which truncation method should be adopted; it depends on the time and costs we have to pay, and depends on which truncation method the experimenter is most interested in.
LITERATURE CITED


APPENDIXES
Appendix A

Computer Program for O.C. Curve of a Fixed Sample Size Test-Exponential

Distribution

This program computes the O.C. of each alternative mean from 1 to 20 with increment of 1 each time. The density function is assumed to be $ue^{-ux}$ which is governed by a Poisson stochastic process. This program is useful computing the O.C. for a large mean whose cumulative probabilities are not available in any Poisson table. The increment may be enlarged for convenience when the mean is large.

Program

C N DENOTES THE NUMBER OF OBSERVATIONS REQUIRED FOR THE TEST
READ(5,1) N
1 FORMAT (I5)
MEAN=1
3 POS=EXP(-MEAN)
EOE=POS
DO 2 I=1,N
XI=I
POS=MEAN*POS/XI
EOE=EOE+POS
2 CONTINUE
MEAN=MEAN+1
IF(MEAN, GE. 21) GO TO 4
GO TO 3
4 STOP
END
Appendix B

Computer Program for Calculating the Exact O.C. Curve

of a Right Angle Shaped Truncation Test—

Exponential Distribution

This program computes the exact O.C. for each alternative mean from 1 to 20 with increment of 1 each time. So long as the shape of the truncation is right angle shaped (see Figure 2), this program is useful computing the exact O.C. for any kind of truncation rules. In this program the values VO, IO, HO, Hl, and SLOPE are determined by what truncation rule we use. Different truncation rule with different values of VO, IO, HO, Hl, and SLOPE.

Program

DEMENSION A(12), R(12), S(12)
36 READ(5, 78, END=30) HO, Hl, SLOPE
78 FORMAT(=F6, 4)
WRITE(6, 45) HO, Hl, SLOPE
45 FORMAT(/////, 15X, 'HO='F6. 4, 5X, 'Hl='F6. 4, 5X, 'SLOPE='F6.4)
MEAN=1
C VO AND IO ARE TRUNCATION VALUES
35 VO=1
IO=9
C CALCULATE EACH VALUES OF Ai AND Ri FOR i FROM 1 TO Io
R(I)=Hl+SLOPE
A(I)=HO+SLOPE
DO 2 I=2, Io
K=I-1
A(I)=A(K)+SLOPE
2 R(I)=R(K)+SLOPE
C FIND NO AND M WHICH SATISFY THE RELATION (H1/S)-1<
   NO ≤ (H1/S) AND R_m<HO< R_{m+1} RESPECTIVELY
   DO 3 I=1, Io
   IF(R(I), GT. 0) GO TO 4
3 CONTINUE
4 NO=I-1
   DO 5 I=1, Io
   IF (R(I), GT. HO) GO TO 6
5 CONTINUE
6 M=I-1
   L=Io-1
C CALCULATE S_i VALUES FOR DIFFERENT RANGES OF i
   DO 7 I=1, Io
   IF(I, LE. NO) GO TO 8
   IF(I, LE. M) GO TO 9
   IF(I, GE. (M+1)) GO TO 10
C S_i VALUES FOR i FROM 1 TO NO
8 KK=I
   F=1
   DO 11 J=1, KK
   F=F*J
   IF(I, EQ. J) GO TO 150
   IF(A(I-J), GT. VO) GO TO 12
   S(I)=S(I)+((-1)**(J+1))/F)*((A(I-J)*MEAN)**J)*S(I-J)
   GO TO 11
12 \( S(I) = S(I) + \frac{((-1)^*(J+1))}{F} * ((V*MEAN)^*J) * S(I-J) \)
GO TO 11

150 \( S(I) = S(I) + \frac{((-1)^*(J+1))}{F} * ((H*MEAN)^*J) \)

11 CONTINUE
GO TO 7

C Si VALUES FOR i FROM NO+1 TO M

9 MM=I
F=1
DO 13 J=1, MM
F=F*J
IF(I,EQ,J) GO TO 151
IF(A(I-J), GT, VO) GO TO 14
S(I)=S(I)+\(\frac{((-1)^*(J+1))}{F} * ((A(I-J)-R(I))^*MEAN)^*J) * S(I-J) \)
GO TO 13
S(I)=S(I)+\(\frac{((-1)^*(J+1))}{F} * ((V-R(I))^*MEAN)^*J) * S(I-J) \)
GO TO 13

151 \( S(I) = S(I) + \frac{((-1)^*(J+1))}{F} * ((H-R(I))^*MEAN)^*J) \)

13 CONTINUE
GO TO 7

C Si VALUES FOR i FROM M+1 TO Io

10 LL=M
F=1
DO 15 J=1, LL
F=F*J
IF(A(I-J), GT, VO) GO TO 16
S(I)=S(I)+\(\frac{((-1)^*(J+1))}{F} * ((A(I-J)-R(I))^*MEAN)^*J) * S(I-J) \)
GO TO 15

16 AB=V-R(I)
IF(AB, LT, 0) AB=0
S(I)=S(I)+\(\frac{((-1)^*(J+1))}{F} * ((AB^*MEAN)^*J) * S(I-J) \)

15 CONTINUE
CONTINUE
C FIND THE LARGEST INTEGER J WHICH LESS THAN \((V_O - H_O)/SLOPE\)
D=\(((V_O - H_O)/SLOPE)\)
DO 19 I=1, I0
IF(I. GT. D) GO TO 40
19 CONTINUE
40 J=I-1
C CALCULATION OF EXACT O.C. WHEN \(i=0\)
G=\(\exp(-H_O*MEAN)\)
C CALCULATION OF EXACT O.C. WHEN \(i\) FROM 1 TO J
DO 20 I=1, J
20 P=P+S(I)*\(\exp(-A(I)*MEAN)\)
C CALCULATION OF EXACT O.C. WHEN \(i\) FROM J+1 TO I0-1
JJ=J+1
DO 21 I=JJ, L
21 PP=PP+S(I)*\(\exp(-V_O*MEAN)\)
C THE FINAL EXACT O.C. IS OBTAINED BY ADDITING PREVIOUS
RESULTS TOGETHER
OC=P+PP+G
WRITE(6, 101) MEAN, OC
101 FORMAT(10X, 'MEAN= 'F5.1, 5X, 'OC='F12.8)
DO 18 I=1, I0
A(I)=0
S(I)=0
18 R(I)=0
P=0
PP=0
MEAN=MEAN+1
IF(MEAN, GE. 21) GO TO 36
GO TO 35

30 STOP

END
Appendix C

Program for the Empirical A.S.N. Curve of a
Truncated Sequential Test-Exponential Case

This program was provided for an empirical A.S.N. curve of a truncated sequential test. Figure 11 is a graphical illustration of the method used in this program. The values HO, H1, SLOPE, VO, IO are determined by what truncation rule we use. However, the values of HO, H1, SLOPE, VO, and IO used here should be the same as those values of HO, H1, SLOPE, VO, IO used in previous program.

Program

DIMENSION ACC(100), REJ(100), SS(100)
LAR=999886
HO= 0.44
H1= -0.44
SLOPE=0.118
IO=9
VO=1

C NN DENOTES THE TOTAL NUMBER OF TRIALS
NN=100
XNN=NN
JJ=50

C GENERATE EXPONENTIAL RANDOM VARIABLES
DO 1 J=1, JJ
DO 2 K=1, NN
XA=H1
XB=Ho
XT=0
DO 13 I=1, N
XI=J-I+1
IF(XI. GE. 1) GO TO 10
XT=VO
GO TO 11
10 U=RANDOM(LAR)
X=(VO-XT)*(1-U**(1/XI))
XT=XT+X
XA=XA+SLOPE
IF(XT. LE. XA) GO TO 3
11 XB=XB+SLOPE
IF (XB. GT. VO) XB=VO
IF(XT. GE. XB) GO TO 4
13 CONTINUE
I=I-1
C RECORD AND ACCUMULATE THE NUMBER OF TIMES THE
TEST BEEN REJECTED
3 REJ(J)+REJ(J)+1
GO TO 5
C RECORD AND ACCUMULATE THE NUMBER OF TIMES THE
TEST BEEN ACCEPTED
4 ACC(J)=ACC(J)+1
C RECORD AND ACCUMULATE THE NUMBER OF TIMES THE
$ TEST EITHER BEEN REJECTED OR ACCEPTED
5 SS(J)=SS(J)+1
2 CONTINUE
ACC(J)=ACC(J)/XNN
REJ(J)=REJ(J)/XNN
SS(J)=SS(J)/XNN
1 CONTINUE

DO 21 I=1,100

XI=I

P=EXP(-XI)

AP=P

AV1=P

DO 22 J=1, JJ

XJ=J

P=XI*P/XJ

PA=PA+P*ACC(J)

AV1=AV1+P*SS(J)

22 CONTINUE

IF (PA .LT. 0.001) GO TO 24

WRITE(6, 7) I, PA, AV1

7 FORMAT(1x, I4, 2F12.8)

21 CONTINUE

24 STOP

END
Appendix D

Program for the Empirical A.S.N. and O.C. Curve

of a Truncated Sequential Test-

Normal Distribution

This program was provided for an empirical A.S.N. curve and
for an empirical O.C. curve of a truncated sequential test of a normal
distribution. This program computes the A.S.N. and O.C. of each
alternative from 6 to the first point whose O.C. is less than 0.001 with
decrement of 0.5 each time. Figure 11 can be regarded as a graphical
illustration of the method used in this program. The values HO, H1,
SLOPE, VO, IO are determined by what truncation rule we use.

Program

LAR=78982
HO=7.465
H1=-7.465
SLOPE=4.025
CV DENOTES MEAN
C VO AND IO ARE TRUNCATION VALUES
C NN DENOTES THE TOTAL NUMBER OF TRIALS
V=6
VO=64.32
I0=16
NN=1000
XNN=NN
1 ASN=0
AN=0
RN=0
DO 2 J=1, NN
XA=HO
XB=H1
XT=0
C GENERATE NORMAL RANDOM VARIABLES
DO 3 M=1, IO
U=RANDOM(LAR)
Z = (((1. - U) ** (-0.16239) - 1) ** (0.20517) - (U ** (-0.16239) - 1) ** (0.20517)) / 0.323968
X=2*Z+V
XT=XT+X
XA=XA+SLOPE
XB=XB+SLOPE
IF (XT. GE. XA. OR. XT. GE. VO) GO TO 4
IF(XT. LE. XB) GO TO 5
IF(M. EQ. IO. AND. XT. LT. VO) GO TO 5
3 CONTINUE
C RECORD AND ACCUMULATE THE NUMBER OF TIMES THE TEST BEEN ACCEPTED
4 AN=AN+1
GO TO 6
C RECORD AND ACCUMULATE THE NUMBER OF TIMES THE TEST BEEN REJECTED
5 RN=RN+1
C RECORD AND ACCUMULATE THE NUMBER OF TIMES THE TEST BEEN EITHER ACCEPTED OR REJECTED
6 ASN=ASN+M
2 CONTINUE
C CALCULATE THE PROBABILITY OF ACCEPTANCE AND THE
$\text{AVERAGE SAMPLE SIZE REQUIRED FOR TERMINATING THE}
\text{TEST}
\text{OC}=\frac{AN}{XNN}
\text{ASN}=\frac{ASN}{XNN}
\text{WRITE}(6,100)\ V,\ OC,\ ASN
100 \text{FORMAT}(10X, 'V='F4.1, 5X, 'OC='F8.5, 5X, 'ASN='F6.2)
\text{IF}(\text{OC}.<.0.001) \text{GO TO 101}
V=V-0.5
\text{GO TO 1}
101 \text{STOP}
\text{END}
VITA

Ryh-Thinn Chang

Candidate for the Degree of

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CREDITS

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