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Positive Controllability of Systems with Nearly-Non-Negative Matrices

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POSITIVE CONTROLLABILITY OF SYSTEMS WITH NEARLY-NON-NEGATIVE MATRICES

by

Theodore Sonne Perry

A thesis submitted in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Mathematics

Approved:

UTAH STATE UNIVERSITY
Logan, Utah

1976
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Theodore Sonne Perry
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ABSTRACT

Positive Controllability of Systems With Nearly-non-negative Matrices
by
Theodore Sonne Perry, Master of Science
Utah State University, 1976

Major Professor: Dr. Robert W. Gunderson
Department: Mathematics

This paper analyzes the controllability of constant coefficient linear differential equations and presents two proofs of a major theorem on controllability. Properties of nearly-non-negative matrices are discussed and in particular a theorem on the behavior of the exponential matrix of nearly-non-negative matrices is proven. These results are then used to prove that the reachable set for systems with nearly-non-negative matrices is limited to the positive hyperoctant.

(27 pages)
INTRODUCTION

Control theory had its beginnings about thirty years ago within the topic of differential equations. Since then it has rapidly become a discipline in its own right. This subject deals basically with the problem of changing a system from one state to another state by regulating a variable over which one has control. These types of problems range from maintaining the proper pH in a batch of chemicals to stopping a rocket.

This thesis is only concerned with the problem of determining whether or not a system can be controlled, that is given one state of the system can the system be changed to another given state. No attempt is made to study the problem of optimal control, that is finding the most efficient or most practical or most economical way to make such a change.

More specifically, the question this thesis attempts to answer is whether or not certain systems can be controlled by using only positive controllers, that is the variable over which one has control can only assume positive values or be zero. An example of this is the problem of stopping a swinging
pendulum by tapping it only from one side. In some applications, if a system can be controlled by only positive controls, perhaps half the cost of installing the controller could be eliminated by installing a controller that only assumes nonnegative values instead of one that assumes both positive and negative values.

In order to answer this question of controllability the relevant sections from standard control theory will be developed and then the theory will be extended to the areas of positive control.

The particular type of systems that these controls will be applied to will be systems of constant coefficient, differential equations of the form $x=Ax+bu$ where $A$ is an $n$ by $n$, nearly-non-negative matrix. Certain theorems concerning nearly-non-negative matrices will be developed and these will be used to prove a new theorem concerning the reachable set for systems with such matrices.
MATHEMATICAL FORMULATION

Linearization of the Problem

Many physical systems can be represented mathematically by a set of simultaneous differential equations of the form

\[ \dot{x}(t) = f(x(t), u(t), t) \]

where \( x(t) \) is an \( n \)-dimensional column vector which describes the state of the system and \( u(t) \) is a \( k \)-dimensional column vector representing input to the system.

To simplify analysis the system is linearized in the following manner. Suppose \( x_0(t) \) satisfies \( \dot{x}_0(t) = f(x_0(t), u_0(t), t) \), i.e. \( x_0(t) \) is the known state of the system from a given input \( u_0(t) \).

Let \( u(t) = u_0(t) + u^*(t) \)
and \( x(t) = x_0(t) + x^*(t) \).

Then

\[ \dot{x}_0(t) + x^*(t) = f(x_0(t), u_0(t), t) + J_x(x_0(t), u_0(t), t)x^*(t) + J_u(x_0(t), u_0(t), t)u^*(t) + h(t) \]

where \( J_x \) and \( J_u \) are the Jacobian matrices of \( f \) with respect to \( x \) and \( u \) respectively and \( h(t) \) is an expression that should be small with respect to \( x^*(t) \) and \( u^*(t) \).

Thus \( x^*(t) \) and \( u^*(t) \) approximately satisfy the linear equation

\[ \dot{x}^*(t) = A(t)x^*(t) + B(t)u^*(t) \]

where \( A = J_x \) and \( B = J_u \).
Solving the Differential Equations

To describe the physical system given mathematically in the preceding section it is necessary to solve the differential equations. Consider
\[ x(t) = A(t)x(t) + B(t)u(t). \]
If \( A(t) \) is continuous for all \( t \) then it is well-known \((1, pp. 37-48)\) that the homogeneous equation
\[ x(t) = A(t)x(t) \]
always has a solution which can be expressed as
\[ x(t) = F(t)x(0) \]
where \( F(t) \) has the properties \( F(0) = I \) where \( I \) is the \( n \) by \( n \) identity matrix and \( F(t) \) is non-singular for all \( t \).

Now, if \( B(t) \) and \( u(t) \) are piecewise continuous for all \( t \), the solution to (2) is
\[ x(t) = F(t)x(0) + F(t) \int_0^t F^{-1}(s)B(s)u(s) \, ds \]
(See Appendix).

The problem now remains of finding \( F(t) \). If \( A(t) = A \) where \( A \) is a constant matrix, then the homogenous equation
\[ x(t) = Ax(t) \]
has solution
\[ F(t) = e^{At} \]
where \( e^M = I + M + M^2/2! + M^3/3! + \ldots \) which can be shown to converge for all \( M \) \((1, p. 56)\).

One method of evaluating \( e^{At} \) is by the use of
Laplace transforms. Consider the time-invariant, linear homogeneous differential equation (6). Laplace transformation yields
\[ sX(s) - X(0) = AX(s) \] or
\[ X(s) = (sI - A)^{-1}X(0). \]
Now from (4) and (7)
\[ x(t) = e^{At}x(0). \]
Taking the Laplace transform yields
\[ X(s) = L(e^{At})X(0) \] where \( L(e^{At}) \) denotes the Laplace transform of \( e^{At} \). Thus
\[ L(e^{At}) = (sI - A)^{-1} \] and \( (sI - A)^{-1} \) can be evaluated by numerical methods (3, p. 34). \( x(t) \) is then found by taking an inverse Laplace transform.

As a simple example consider
\[ x(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t). \]

Then \( (sI - A)^{-1} = \frac{1}{s^2} \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} \).

Inverse Laplace transformation yields
\[ e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]

For the inhomogeneous equation
\[ x(t) = Ax(t) + Bu(t) \]
Laplace transformation yields
\[
sX(x) - X(0) = AX(s) + BU(s) \text{ or } X(s) = (sI - A)^{-1}X(0) + (sI - A)^{-1}BU(s).
\]
Again numerical evaluation and inverse Laplace transformation give \( x(t) \).

Another method of evaluating \( e^{At} \) is the eigenvalue and eigenvector method. If \( A \) has \( n \) linearly independent eigenvectors \( v_1, \ldots, v_n \) corresponding to eigenvalues \( \lambda_1, \ldots, \lambda_n \), then \( e^{At} \) equals a non-singular constant matrix multiplied by the matrix whose columns are the vectors \( \exp(\lambda_1 t)v_1, \ldots, \exp(\lambda_n t)v_n \). Even if the eigenvalues are not distinct the method can be modified to determine \( e^{At} \) for any matrix \( A \) (1, pp. 62-72).

Thus when \( A \) is constant, equation (5) becomes
\[
(8) \quad x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}B(s)u(s) \, ds.
\]
CONTROLLABILITY

Definitions

In general system (1) is said to be completely controllable at $t_0$ if for each pair of points, $x_0$ and $x_1$, there exists a bounded, measurable, vector-valued function $u(t)$ on $t_0 \leq t \leq t_1$, where $t_1$ is less than infinity, such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

In the introduction it was shown that locally system (1) can be approximated in many cases by a constant coefficient, linear system. Thus, to simplify analysis from this point on, consider only the linear differential system in real $n$-dimensional Euclidean space $\mathbb{R}^n$.

\begin{equation}
\dot{x}(t) = Ax(t) + bu(t) \quad x(0) = 0
\end{equation}

where $A$ is a constant $n$ by $n$ matrix and $x$ and $b$ are $n$-dimensional column vectors and $u$ is a scalar function of time.

Define the restraint set $S$ to be a closed interval in $\mathbb{R}^1$. If $S$ equals the closed interval $[0, p]$ where $p$ is a positive real number, then $S$ is called positive and (9) is said to have positive controls.

Define the attainable set $K(t^*)$ for $t^*$ greater than or equal to zero by $K(t^*) = \bigcup_{u \in U_S} x_u(t^*)$ where $U_S$ is the set of all measurable functions $u(t)$ from
the non-negative real numbers to $S$, and where $x_u(t)$ is the unique solution to system (9) given by formula (8). Define the reachable set $K$ by $K = \bigcup_{t^* \geq 0} K(t^*)$. Then system (9) is defined to be controllable if for some $t^*$, $K(t^*)$ contains a neighborhood of the origin. Controllability is thus slightly less restrictive than complete controllability but in many cases it is equivalent.

For instance consider a different system

$$x'(t) = Ax(t) + bu(t) \quad x'(0) = y \neq 0.$$  

At a given time $t^*$

$$x'(t^*) = e^{At^*}y + \int_0^{t^*} e^{A(t^*-s)}bu(s) \, ds$$

but the last term on the right is simply $x(t^*)$. Hence

$$x'(t^*) = e^{At^*}y + x(t^*).$$

Thus $K'(t^*)$ is just a translation of $K(t^*)$ by the constant vector $e^{At^*}y$.

Also it can be shown that if all eigenvalues of $A$ have non-positive real parts and $S$ contains zero as an interior point then system (9) is completely controllable if and only if it is controllable ($4$, p. 92).

**Standard Theorem on Controllability**

Now the standard result on the controllability of system (9) will be proven by the method of Lee
and Markus (4, pp. 81-82).

Theorem: The linear process \( x(t) = Ax(t) + bu(t) \) with restraint set \( S \) containing zero in its interior is controllable if and only if the \( n \times n \) matrix \( C \) having as columns the vectors \( b, Ab, A^2b, \ldots, A^{n-1}b \) has rank \( n \).

Proof: Assume the process is controllable and suppose the rank of \( C \) is less than \( n \). The rows of \( C \) are then linearly dependent and there exists a row vector \( v \) such that \( vC = 0 \) or \( vb = vAb = vA^2b = \ldots = vA^{n-1}b = 0 \). By the Cayley-Hamilton theorem (See Appendix) \( A \) satisfies its characteristic equation

\[
A^n + c_1A^{n-1} + c_2A^{n-2} + \ldots + c_nI = 0
\]

for certain real numbers \( c_1, \ldots, c_n \). Multiplying on the right by \( v \) and on the left by \( b \)

\[
vA^n b + c_1vA^{n-1}b + \ldots + c_nv^b = 0.
\]

All terms but the first are known to be zero from which it can be concluded that \( vA^n b = 0 \). Multiplying (10) by \( A \) yields

\[
A^{n+1} + c_1A^n + \ldots + c_nA = 0.
\]

Again multiplying appropriately by \( v \) and \( b \)

\[
vA^{n+1} b + c_1vA^n b + \ldots + c_nv^b = 0
\]

which implies that \( vA^{n+1} b = 0 \). Hence by induction \( vA^{n+k}b = 0 \) for all non-negative integers \( k \).
Thus
\[ v e^{At} b = v (I + At + A^2 t^2 / 2! + \ldots) b = 0 \]
for all \( t \). Now noting that the solution to the linear system (9) is
\[ x(t) = \int_0^t e^{A(t-s)} b u(s) \, ds \]
one obtains
\[ v \cdot x(t) = \int_0^t v e^{A(t-s)} b u(s) \, ds = 0 \]
and hence all solutions lie in the hyperplane orthogonal to \( v \), which contradicts the fact that the process was assumed to be controllable. Thus, if the process is controllable, the matrix \( C \) has rank \( n \).

Now assume that \( C \) has rank \( n \). Let \( K_1 \) be the set of points that the origin can be steered to in time \( t \) where \( t \) ranges from zero to one, by controllers satisfying \( |u(t)| \leq e \) where \( e \) is greater than zero and both \( +e \) and \( -e \) lie in \( S \). It can be shown that \( K_1 \) is compact and convex (4, pp. 69-71). Suppose that the space equal to the span of \( K_1 \) has dimension less than \( n \). Then there exists unit vector \( v \) such that \( x \cdot v = 0 \) where
\[ x = \int_0^1 e^{A(1-s)} b u(s) \, ds \]
where \( u \) is arbitrary. Thus
\[ v e^{A(1-s)} b = 0 \]
for all \( s \) ranging from zero to one.

If \( s \) is set equal to one, then \( v \cdot b = 0 \). Differentiation gives \( -v A e^{A(1-s)} b = 0 \) or \( v A b = 0 \) when \( s \) is set equal to one. Repeated differentiation yields
\[ v \cdot b = v A b = \ldots = v A^{n-1} b = 0, \]
which contradicts the assumption that matrix $C$ has rank $n$. Thus $K_1$ spans $\mathbb{R}^n$. Since $u(t)$ can be replaced by $-u(t)$, $K_1$ is symmetric about the origin. Since $K_1$ is compact and convex, the origin must be contained in its interior. Thus if $C$ has rank $n$ the system is controllable.

**Alternative Proof of Standard Theorem**

Recently an alternative proof of this result has been given by Saperstone and Yorke (5).

**Theorem:** The linear process (9) $\dot{x} = Ax + bu \quad x(0) = 0$ with restraint set $S$ containing zero in its interior is controllable if and only if the $n$ by $n$ matrix $C$ having as columns the vectors $b$, $Ab$, $A^2b$, ..., $A^{n-1}b$ has rank $n$.

**Proof:** Let $u(t) = e$ where $e$ is chosen so that both $+e$ and $-e$ lie in $S$. Let $x(t)$ be the solution to the linear system (9) for this choice of $u$. Choose $T$ greater than zero. Let $L$ be the subspace spanned by $x(t)$ for $t$ ranging from zero to $T$. Now let $L^r$ be the space spanned by the vectors $b$, $Ab$, ..., $A^{r-1}b$. Since the dimension of $L^r$ is less than $n+1$ for all $r$, let $q$ be the smallest integer such that $L^q = L^{q+1}$, i.e. $A^q b$ is an element of $L^q$. Now it will be shown that $L = L^q$. Pick $y$ in $L^q$. There exist real numbers
$a_0, \ldots, a_{q-1}$ such that $y = a_0 b + a_1 A b + \cdots + a_{q-1} A^{q-1} b$.

Multiplying by $A$ one obtains $Ay = a_0 Ab + \cdots + a_{q-1} A^{q-1} b$.

Since all elements on the right are elements of $L^q$, $Ay$ is an element of $L^q$ for all $y$ in $L^q$. Also for any scalar function $u$, $Ay + bu$ is an element of $L^q$ if $y$ is in $L^q$. Since $x(0) = 0$ is in $L^q$, system (9) can be regarded as a differential equation on $L^q$. Hence all solutions are in $L^q$. In particular, $K(T)$ is a subset of $L^q$.

Now pick $t$ in the interval from zero to $T$.
Let $u^*(s) = 0$ for $0 \leq s \leq T-t$ and $u^*(s) = e$ for $T-t < s \leq T$.

Let $x^*$ be the solution of (9) corresponding to $u^*$.
Then note that $x^*(T)$ is an element of $K(T)$ and then note that $x(t) = x^*(T)$. Hence $x(t)$ is in $K(T)$. Since $K(T)$ is a subset of the subspace $L^q$, and $L$ is the subspace created by the span of $x(t)$ for $t$ ranging from zero to $T$, $L$ is a subset of $L^q$.

Now for $t$ and $t+s$ in the interval from zero to $T$.
$(x(t+s) - x(t))/s$ is in $L$ and thus $x(t)$ is in $L$. Again $(x(t+s) - x(t))/s$ is in $L$ and thus $x(t)$ is in $L$. By induction all derivatives of $x(t)$ are in $L$. Now evaluating the differential equation (9) at zero one obtains $x(0) = be$ and thus $b$ is in $L$. Evaluating the second derivative of the differential equation (9) at zero one obtains $x(0) = Abe$ and thus $Ab$ is in $L$. 

Taking higher derivatives shows that \( A^k b \) is in \( L \) for all non-negative integers \( k \) and thus \( L^q \) is a subset of \( L \). Hence it follows that \( L = L^q \).

Choose \( 0 \leq t_1 < t_2 < \ldots < t_q \leq T \) such that the vectors \( x(t_1), x(t_2), \ldots, x(t_q) \) are linearly independent. Such a choice is possible because \( L^q = L \) has dimension \( q \) and \( L \) equals the span of \( x(t) \) for \( t \) ranging from zero to \( T \). Since these vectors form a basis for \( L \), given any \( y \) in \( L \) there exist numbers \( c_1, \ldots, c_q \) such that \( y = c_1 x(t_1) + \ldots + c_q x(t_q) \). Furthermore a norm on \( L \) can be defined by norm of \( y = \left| c_1 \right| + \ldots + \left| c_q \right| \).

Since \( S \) is convex, then as mentioned earlier \( K(T) \) is convex. Since \( u \) can be replaced by \(-u\), \( K(T) \) is symmetric about the origin. Now \( x(t_i) \) is in \( K(T) \) for all integers \( i \) ranging from one to \( q \). Thus \(-x(t_i)\) is in \( K(T) \). Thus if \( W \) is the set of all vectors that have norm less than or equal to one, then \( W \) is a subset of \( K(T) \). Thus \( K(T) \) contains a neighborhood of zero in the topology defined on \( L \). Thus \( K(T) \) contains a neighborhood of the origin in \( \mathbb{R}^n \) if and only if the rank of \( L \) equals \( n \). Hence system (9) is controllable if and only if the rank of \( C \) equals \( n \).
NEARLY-NON-NEGATIVE MATRICES

Definitions

Having derived the standard result on controllability, attention will now be turned to expanding on these results for systems with nearly-non-negative matrices with positive controls.

A nearly-non-negative matrix is one in which all elements off the main diagonal are non-negative but no restriction is placed on the elements on the main diagonal. A non-negative matrix is one in which all elements are non-negative and is denoted by \( E \geq 0 \) where \( E \) is a non-negative matrix.

Existence of a Real Eigenvalue

A result that will be important in determining the controllability of systems with nearly-non-negative matrices will now be derived.

Theorem: If matrix \( A \) is nearly-non-negative, then it has a real eigenvalue.

Proof: There exists a real number \( s \) such that \( A + sI \geq 0 \) where \( I \) is the identity matrix. By a corollary to the Perron-Froebinius Theorem shown in Gantmacher (2, Vol. II, p. 66) \( A + sI \) has a real eigenvalue \( w \) with associated eigenvector \( v \). Thus \((A + sI)v = wv\) or
Av = (w-s)v and thus w-s is a real eigenvalue of the matrix A.

Behavior of Exponential Matrix

Another important result for determining the controllability of nearly-non-negative systems will now be derived. The proof here is similar to that used for a similar result by Varga (6, p. 257).

Theorem: If A is a nearly-non-negative matrix, then $e^{At}$ is non-negative for all $t$ greater than or equal to zero.

Proof: Again there exists real numbers such that $A + sI \geq 0$. Thus $(A + sI)^n \geq 0$ for all integers $n \geq 0$. Thus if $t$ is greater than or equal to zero, then $(A + sI)^n t^n / n! \geq 0$ for all non-negative integers $n$. Thus $I + (A + sI) t + (A + sI)^2 t^2 / 2! + \ldots = e^{(A + sI)t} \geq 0$. Now since the real number $e^{-st}$ is greater than zero, $e^{-st} e^{(A + sI)t}$ is non-negative. Hence by the familiar rules for working with exponential matrices $e^{At}$ is non-negative for all non-negative $t$. 
POSITIVE CONTROLLABILITY OF SYSTEMS WITH NEARLY-NON-NEGATIVE MATRICES

In order to show that systems with nearly-non-negative matrices are not controllable by positive controls according to the earlier definition of controllability it is necessary first to derive one more result on controllability. This result deals only with systems having positive controls and is proved along the lines given by Saperstone and Yorke (5).

Theorem: If $A$ has a real eigenvalue $\lambda$, then $x=0$ belongs to the boundary of $K$ for the system (9) $x=Ax+bu \quad x(0)=0$ with positive restraint set.

Proof: Assume $w$ is greater than or equal to zero. $w$ is also an eigenvalue of the transpose of $A$ designated $A^T$. Thus there exists $v$ in $\mathbb{R}^n$ such that $A^Tv=wv$ and so that $v \cdot b$ is non-negative. (Given eigenvector $v$ then $-v$ is also an eigenvector. Thus $v \cdot b$ or $-v \cdot b$ is non-negative.) Let $x(t)$ be a solution of $x=Ax+bu$ satisfying $x(0)=0$. Let $p(t)=v \cdot x(t)$. Then

$$\frac{dp(t)}{dt} = v \cdot x(t)$$

$$= v \cdot Ax(t) + v \cdot bu(t)$$

$$= A^Tv \cdot x(t) + v \cdot bu(t)$$
\[
\frac{d}{dt} p(t) = w v \cdot x(t) + v \cdot b u(t) \\
= w p(t) + v \cdot b u(t)
\]

Now since \( w, u(t), \) and \( v \cdot b \) are non-negative and, since \( p(0) = 0, \) \( p(t) \) is non-negative for all non-negative \( t. \) Thus, for all \( x \) in the reachable set \( K, \) \( v \cdot x \) is non-negative. Thus if \( H \) is the half-space determined by all \( x \) such that \( v \cdot x \) is non-negative, then \( K \) is a subset of \( H. \)

Now using the trivial control function \( u(t) = 0 \) system (9) has solution \( x(t) = 0 \) and thus zero is in \( K. \) Since \( K \) is a subset of \( H, \) zero belongs to the boundary of \( K. \)

If \( w \) is less than zero, the vector \( v \) must be chosen so that \( v \cdot b \) is non-positive. Again all solutions will lie in a half-space. Thus \( x = 0 \) belongs to the boundary of \( K. \)

Now if \( A \) of system (9) is nearly-non-negative, then as shown in a previous theorem \( A \) has a real eigenvalue. Thus by the theorem just proved the reachable set \( K \) has zero on its boundary and thus the system is not controllable.

**Geometry of Reachable Set**

Even when a system is not controllable there are in general points to which the origin can be
steered. Thus the following theorem further delineates the positive controllability of systems with nearly-non-negative matrices.

**Theorem:** If the restraint set $S$ is positive and $A$ is nearly-non-negative and $b$ is non-negative, then the reachable set $K$ of system (9) $\dot{x} = Ax + bu$ $x(0)=0$ is a subset of the positive hyperoctant.

**Proof:** The solution to system (9) is $x(t)=\int_0^t e^{A(t-s)}bu(s) \, ds$.

Now since $A$ is nearly-non-negative, $e^{A(t-s)}$ is non-negative. Thus since $b$ and $u(t)$ are non-negative $x(t)$ is non-negative for all $t$ greater than or equal to zero. Thus $K$ is a subset of the first hyperoctant.

**Remarks**

This theorem completely characterizes the controllability of systems with nearly-non-negative matrices. Examples can easily be constructed where the reachable set $K$ consists of the entire positive hyperoctant or where $K$ consists of just the origin. Thus the original goals of this thesis topic have been entirely met.
APPENDIXES
Appendix A

Variation of Parameters Formula

Theorem: If the system $\dot{x}(t) = A(t)x(t)$ has solution $x(t) = F(t)x(0)$, then the system $\dot{x}(t) = A(t)x(t) + B(t)$ has solution $x(t) = F(t)x(0) + F(t) \int_0^t F^{-1}(s)B(s) \, ds$.

Proof: The solution given will be shown to be a solution of the differential equation. Differentiation of $x(t)$ yields

$$\dot{x}(t) = \dot{F}(t)x(0) + F(t) \int_0^t F^{-1}(s)B(s) \, ds + F(t)F^{-1}(t)B(t)$$

or

$$\dot{x}(t) = A(t)F(t)x(0) + A(t)F(t) \int_0^t F^{-1}(s)B(s) \, ds + B(t)$$

or

$$\dot{x}(t) = A(t)x(t) + B(t).$$

Thus $x(t)$ is a solution to the differential equation.
Appendix B

Cayley-Hamilton Theorem

The proof given here is taken from Gantmacher (2, pp. 82-83).

Theorem: Every square matrix $A$ satisfies its characteristic equation, i.e. $|wI-A|=0$.

Proof: Consider an arbitrary matrix polynomial $F(w)=F_0 w^m + F_1 w^{m-1} + \ldots + F_m$ where $F_0 \neq 0$.

Each $F_i$ is a square matrix of the same order as $A$.

Divide $F(w)$ on the right by $wI-A$ in the following manner

$$F(w)=F_0 w^{m-1}(wI-A)+(F_0 A + F_1) w^{m-2} + F_2 w^{m-2} + \ldots + F_m$$

Thus $F(w)$ is divisable on the right by $wI-A$ if and only if

$$F(A)=F_0 A^{m-1} + \ldots + F_m = 0.$$ 

Let $p(w)=|wI-A|$ and $F(w)=p(w)I$. Let $B$ be the adjoint matrix of $wI-A$. Then

$$B(wI-A)/p(w)=I$$

Thus $F(w)$ is divisable on the right by $wI-A$ and thus

$$F(A)=0$$ and thus $A$ satisfies its characteristic equation.
LITERATURE CITED


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