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Exact Analysis of Variance with Unequal Variances

Noriaki Yanagi
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EXACT ANALYSIS OF VARIANCE WITH UNEQUAL VARIANCES

by

Noriaki Yanagi

A thesis submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Applied Statistics

Approved:

UTAH STATE UNIVERSITY
Logan, Utah
1980
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Noriaki Yanagi
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ABSTRACT

Exact Analysis Variance with Unequal Variance

by

Noriaki Yanagi, Master of Science
Utah State University, 1980

Major Professor: Dr. David White
Department: Applied Statistics

The purpose of this paper was to present the exact analysis of variance with unequal variances. Bishop presented the new procedure for the r-way layout ANOVA. In this paper, one and two way layout ANOVA were explained and Bishop's method and Standard method were compared by using a Monte Carlo method.

(45 pages)
CHAPTER I
PREFACE AND GOALS

Test procedures in the r-way layout model in ANOVA are based on the assumptions of normality, independence, and equality of the variances of the errors. Studies of the robustness of the F-test, for example in the one-way layout, have shown that the violation of normality has little effect on inferences about the means. However, the violation of equality of variances can have a serious effect on inferences about the means, especially if the cell sample sizes are unequal.

To solve this problem, Thomas A. Bishop, [1] for example in the one-way layout, attempts to test the null hypothesis $H_0 : \mu_1 = \mu_2 \ldots = \mu_I$ in such a way that the level and power of the test are controllable, and not dependent upon the unknown variances. The plan of this thesis is to compare the standard ANOVA and Bishop's method.

In Chapter III, "Stein's method" is discussed because Bishop's method was based on it. In Chapter II, the theory needed to explain Stein's method is treated, namely likelihood ratio test.

In Chapter VI, Bishop's method is described and in Chapters V, and VII, the theories needed to explain Bishop's method are discussed.

Finally, experiments will be conducted for the one-way layout ANOVA by using Bishop's method and the results compared with standard ANOVA results. In Chapter IX, conclusions are given concerning these experiments.
Let us now present a general method for constructing critical regions, which in most cases have very satisfactory properties. It is called the likelihood ratio method. We shall discuss this method here with reference to tests concerning one parameter \( \theta \) and continuous populations, but all of our arguments can be extended to the multi-parameter case and to discrete populations.

To illustrate the likelihood ratio technique, let us suppose the \( x_1, x_2, \ldots \) and \( x_n \) is a random sample of size \( n \) from a population whose density is given by \( f(x, \theta) \), and that \( \Theta \) is the set of values which the parameter \( \theta \) can take on. The null hypothesis we shall want to test is:

\[ H_0 : \theta \text{ is an element of } \Theta \]

where \( \Theta \) is a subset of \( \Omega \), and the alternative hypothesis is \( H_1 : \theta \text{ is an element of } \Theta^c \). Where \( \Theta^c \) is the complement of \( \Theta \) with respect to \( \Omega \). Thus the set of values which the parameter \( \theta \) can take on is partitioned into the disjoint sets \( \Theta \) and \( \Theta^c \); according to the null hypothesis \( \theta \) is an element of the first set, and according to the alternative hypothesis it is an element of the second. In most problems \( \Omega \) is either the set of all real numbers, the set of all positive real numbers, some interval of real numbers, or a discrete set of real numbers. When \( H_0 \) and \( H_1 \) are both simple hypotheses, \( \Theta \) and \( \Theta^c \) each have only one element, and we construct tests by comparing the likelihoods \( L_0 \) and \( L_1 \). In the general case, where at least one of the two hypothesis is composite, we compare instead the two quantities \( \max L_0 \) and \( \max L_1 \).
where \( \max L \) is the maximum value of the likelihood function for all values of \( \theta \) in \( \omega \), and \( \max L \) is the maximum of the likelihood function for all values of \( \theta \) in \( \omega \).

In other words, if we have a random sample of size \( n \) from a population which is given by \( f(x, \theta) \), \( \hat{\theta} \) is the maximum likelihood estimate of \( \theta \) subject to the restriction that \( \theta \) must be an element of \( \omega \), and \( \hat{\theta} \) is the maximum likelihood estimate of \( \theta \) for all values of \( \theta \) in \( \omega \), then

\[
\max L_0 = \max_{\hat{\theta}} \prod_{i=1}^{n} f(x_i; \hat{\theta}) \quad \text{and} \quad \max L = \max_{\hat{\theta}} \prod_{i=1}^{n} f(x_i; \hat{\theta}) \quad \text{(2.1)}
\]

These quantities are both values of random variables, they depend on the observed sample values \( x_1, x_2, \ldots, x_n \) and their ratio

\[
\lambda = \frac{\max L_0}{\max L} \quad \text{(2.3)}
\]

is referred to as a value of the likelihood ratio statistic \( \lambda \).

Since \( \max L_0 \) is apt to be small compared to \( \max L \) when the null hypothesis is false, it stands to reason that the null hypothesis should be rejected when \( \lambda \) is small. Indeed, the critical region \( \lambda = K \) defines the likelihood ratio test of the null hypothesis that \( \theta \) is an element of \( \omega \) against the alternative hypothesis that \( \theta \) is an element of \( \omega \). If \( H_0 \) is a simple hypothesis, \( K \) is chosen so that the size of the critical regions equal \( \alpha \); if \( H_0 \) is composite, \( K \) is chosen so that the probability of a Type I error is less than or equal to \( \alpha \) for all \( \theta \) in \( \omega \), and equal to \( \alpha \) (if possible) for at least one value of \( \theta \) in \( \omega \). Thus, if \( H_0 \) is a simple hypothesis and \( g(\lambda) \) is the density of \( \lambda \) when \( H_0 \) is true, then \( K \)
must be such that

$$
\int_0^X g(\lambda) d\lambda = \alpha \quad (2.4)
$$

In the discrete case, the integral (2.4) is replaced by a sum and

K is taken to be the largest value for which the sum is less than or equal
to \( \alpha \). To illustrate the likelihood ratio technique, suppose we have a
random sample of size \( n \) from a normal population with the known variance
\( \sigma^2 \), and that we want to test the simple null hypothesis \( H_0: \mu = \mu_0 \)
against the composite alternative \( H_0: \mu \neq \mu_0 \) where \( \mu \) is, of course,
the mean of the population. Since \( \omega \) contains only \( \mu_0 \), it follows that
\( \hat{\mu} = \mu_0 \), and since \( \Omega \) is the set of all real numbers, it follows by the
method that \( \hat{\mu} = \bar{x} \).

Thus, \( \max L_0 = \left( \frac{1}{\sigma \sqrt{2\pi n}} \right)^n \cdot e^{-\frac{1}{2\sigma^2} \sum (x_i - \mu_0)^2} \)

and \( \max L = \left( \frac{1}{\sigma \sqrt{2\pi n}} \right)^n \cdot e^{-\frac{1}{2\sigma^2} \sum (x_i - \bar{x})^2} \)

where the summation extended from \( i = 1 \) to \( i = n \), and the value of the
likelihood ratio statistic becomes

$$
\lambda = e \cdot \frac{1}{2\sigma^2} \sum (x_i - \mu_0)^2 / e \cdot \frac{1}{2\sigma^2} \sum (x_i - \bar{x})^2
$$

$$
\lambda = e - \frac{n}{2\sigma^2} (\bar{x} - \mu_0)^2 \quad (2.5)
$$

The critical region of likelihood ratio test is

$$
e - \frac{n}{2\sigma^2} (\bar{x} - \mu_0)^2 \leq \kappa
$$

and, after taking logarithms and dividing by \( -\frac{n}{2\sigma^2} \)
it becomes \((\bar{x} - \mu_0)^2 \geq \frac{-2\sigma^2}{n}\). \(\forall \kappa \) or \(|\bar{x} - \mu_0| \geq \kappa^*\).

where \(\kappa\) will have to be determined so that the size of the critical region is \(\sigma\).

Theorem 2.1

If \(\bar{x}\) is the mean of a random sample of size \(n\) from a normal population with the mean \(\mu\) and the variance \(\sigma^2\), then the sampling distribution of \(\bar{x}\) is a normal distribution with the mean \(\mu\) and the variance \(\sigma^2/n\).

Making use of Theorem 2.1, we find that the critical region of this likelihood ratio test is \(|\bar{x} - \mu_0| \geq Z_{\alpha/2}\sigma/\sqrt{n}\).

In other words, the null hypothesis is to be rejected if \(\bar{x}\) takes on a value greater than or equal to \(\mu_0 + Z_{\alpha/2}\sigma/\sqrt{n}\)
or a value less than or equal to \(\mu_0 - Z_{\alpha/2}\sigma/\sqrt{n}\).

In the preceding example it was easy to find the constant which made the size of the critical region equal to \(\sigma\), because we were able to refer to the known distribution of \(\bar{x}\) instead of the distribution of the likelihood ratio statistic \(\lambda\), itself. Since the distribution of \(\lambda\) is generally rather complicated, which makes it difficult to evaluate \(\kappa\) with the use of (2.4), it is often convenient to use an approximation based on the following theorem.

Theorem 2.2

For large \(n\), the distribution of \(-2 \ln L\) approaches, under very general conditions, the chi-square distribution with 1 degree of freedom.
We should add that this theorem applies only to the one-parameter case; if the population involves more than one unknown parameter, upon which the null hypothesis imposes r restrictions, the number of degrees of freedom in the chi-square approximation of the distribution of \(-2 \cdot \ln \lambda\) is equal to r. Thus, if we wanted to test the null hypothesis that the unknown mean and variance of a normal population are, respectively, \(\mu = \mu_0\) and \(\sigma^2 = \sigma_0^2\) against the alternative hypothesis that \(\mu \neq \mu_0\) and \(\sigma^2 \neq \sigma_0^2\) the number of degrees of freedom in the chi-square approximation of the distribution of \(-2 \ln \lambda\) would be 2; the two restrictions are \(\mu = \mu_0\) and \(\sigma^2 = \sigma_0^2\).

Using theorem 2.2, we can write the critical region of this approximate likelihood ratio test as
\[
-2 \cdot \ln \lambda \geq \chi^2_{\alpha,1}
\]
where \(\chi^2_{\alpha,1}\) is as defined \(\int_{\chi_{\alpha,1}}^{\infty} f(\chi) d\chi = 2\).

**Theorem 2.3**

If \(x\) has the standard normal distribution, then \(x^2\) has the chi-square distribution with 1 degree of freedom. We find that according to theorem 2.1 and 2.3 and (2.5)

\[
-2 \cdot \ln \lambda = \frac{n}{\sigma^2} (\bar{x} - \mu_0)^2 = (\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}})^2
\]

actually is a value of a random variable having the chi-square distribution with 1 degree of freedom.
CHAPTER III
TWO-SAMPLE TEST FOR A LINEAR HYPOTHESIS WHOSE
POWER IS INDEPENDENT OF THE VARIANCE [6]

Suppose $x_i, i=1, 2, \ldots$ are independently normally distributed with mean $\mu$ and variance $\sigma^2$. We wish to test the hypothesis $\mu = \mu_0$, the power of the test to depend only upon $\mu - \mu_0$, not upon $\sigma^2$. For this purpose we define a statistic $t'$ as follows. A sample of $n$ observations, $x_1, x_2, \ldots x_n$ is taken (an initial observation), and the sample estimate, $S^2$, of the variance computed by

$$S^2 = \frac{1}{n-1} \left\{ \frac{n}{n} \sum x^2_i - \frac{1}{n} \left( \sum x_i \right)^2 \right\}$$

Then $N$ (an actual observation) is defined by

$$N = \max \left\{ \left\lfloor \frac{S^2}{Z} \right\rfloor + 1, n + 1 \right\}$$

where $Z$ is a previously specified positive constant.

In an actual test statistic, the experimenter has to choose beforehand the constant $Z$. And, he compares his sample variance with $Z$. If $[S^2/Z] + 1$ is greater than $n + 1$, then he needs additional observations $x_{n+1}, x_{n+2}, \ldots x_N$, where $N = [S^2/Z] + 1$.

In accordance with an initially specified rule depending only upon $S^2$, real numbers $a_i, i = 1, 2, \ldots n$ are chosen in such a way that
\[ \sum_{i=1}^{n} a_i = 1, \ a_1 = a_2 = \ldots = a_n, \text{ and } S^2 = \sum_{i=1}^{n} a_i^2 = Z. \]

To show that there exists a set \([a_1, i = 1, 2, \ldots, n]\) with these proportions, we consider the minimum of \(Ea_i^2\) as follows: (using the method of lagrangian multipliers)

\[
\frac{2}{2a_s} \left( \sum_{i=1}^{N} a_i^2 - \lambda \left( \sum_{i=1}^{N} a_i - 1 \right) \right) = 0
\]

\[2a_s - \lambda = 0 \quad \lambda = 2a_s \text{ for all } S.\]

From the property of the \(a_i = a_2 = \ldots = a_n = C\)

\[
\sum_{i=1}^{N} a_i = 1 = NC = 1 \quad NC = 1/N \min \sum_{i=1}^{N} a_i^2 = (1/N)^2 = 1/N
\]

So, \(Ea_i^2 = Z/s^2 \rightarrow \min Ea_i^2 = 1/N \leq Z/s^2\)

We can attempt the test statistic in the following conditions

\[\sum_{i=1}^{N} a_i = 1, \ a_1 = a_2 = \ldots = a_n \]

Then \(t'\) is defined by \(t' = \frac{\sum_{i=1}^{N} a_i x_i - \mu_0}{\sqrt{Z}}\)

**Theorem 3.1**

If \(x_1, x_2, \ldots, x_n\) are random variables, \(a_1, a_2, \ldots, a_n\) are constants, and \(y = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n\),
then $E(y) = \sum_{i=1}^{n} a_i E(x_i)$ and $\text{Var}(y) = \sum_{i=1}^{n} a_i^2 \cdot \text{Var}(x_i)$

$$+ 2\sum_{i<j} a_{ij} \cdot \text{Cov}(x_i, x_j)$$

Where $\sum_{i<j}$ means that the summation extends over all values of $i$ and $j$ from 1 to $n$, for which $i < j$.

We defined that if $x_1, x_2, \ldots, x_n$ are random variable, $a_1, a_2, \ldots, a_n$ are constants and $y = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$, then

$$E(y) = \sum_{i=1}^{n} a_i E(x_i), \quad \text{Var}(y) = \sum_{i=1}^{n} a_i^2 \cdot \text{Var}(x_i)$$

where $Z = \sum_{i=1}^{n} a_i^2$ and $\sum_{i=1}^{n} a_i x_i$ correspond the theorem 3.1. Therefore, the method $(t' = \frac{\sum a_i x_i - \mu_0}{\sqrt{Z}})$ corresponds to the standard testing of a hypothesis about $\mu$, namely

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}, \text{ where } H_0 : \mu = \mu_0$$

But in our case, $\sigma/\sqrt{n}$ is replaced with $\sqrt{Z}$, which is beforehand established by experimenter and $x$ is replaced with $\sum a_i x_i$ which is adjusted by constant $z$.

Also, $t' = \frac{\sum_{i=1}^{n} a_i x_i - \mu_0}{\sqrt{Z}} = \frac{\sum_{i=1}^{n} a_i (x_i - \mu)}{\sqrt{Z}} + \frac{\mu - \mu_0}{\sqrt{Z}} = u + \frac{\mu - \mu_0}{\sqrt{Z}}$

Where $u = \frac{\sum_{i=1}^{n} a_i (x_i - \mu)}{\sqrt{Z}} = \frac{\sum_{i=1}^{n} a_i (x_i - \mu)}{S_{\sqrt{a_i}}}$

$\sim t$ distribution
because $x_i \sim N(\mu, \sigma^2) \Rightarrow (x_i - \mu) \sim N(0, \sigma^2)$

$$\text{var} \left( \sum a_i x_i \right) = \sum a_i^2 \text{Var} (x_i) \quad \text{If } x_i \text{'s are independent.}$$

So, $\sum a_i (x_i - \mu) \sim N(0, \sum a_i^2 \sigma^2)$, $\frac{\sum a_i (x_i - \mu)}{\sqrt{\sum a_i^2}} \sim N(0, \sigma^2)$

Therefore, the conditional distribution of $u = \frac{1}{\sqrt{Z}} \sum a_i (x_i - \mu)$ given $S$, is normal with mean 0 and variance $\frac{\sum a_i^2/Z = \sigma^2/S^2}$.

The $T$ distribution is defined as following

$$T = Z/\sqrt{\nu/n} \quad \text{where } Z \sim N(0,1), \nu \sim x_{n-1}^2$$

and independent of $Z$.

$$T = y/\sigma \quad \sigma/S = y/S \quad \text{where } y \sim N(0, \sigma^2)$$

So, the conditional distribution of $u$ is normal with mean 0 and variance $\sigma^2/S^2$. Therefore, $u = \frac{1}{\sqrt{Z}} \sum a_i (x_i - \mu)$ has $t$ distribution.

We can test the hypothesis $\mu = \mu_0$, the power of the test to depend only upon $\mu - \mu_0$, not upon $\sigma^2$. In other words, this theorem can be used to obtain an unbiased test for the hypothesis $H_0$ that $\mu = \mu_0$, the power being independent of $\sigma^2$, which is supposed unknown. Let $\alpha$ be the desired size of the critical region and let $t_{n-1, \alpha/Z}$ be such that
\[ P( t_{n-1} > t_{n-1}, \alpha/Z ) = \sigma/Z. \]

Then if we reject \( H \) whenever

\[ \frac{\sum_{i=1}^{N} (\chi_i - \mu_0)}{Z} \]

we obtain an unbiased test of \( H_0 \), whose power function is \( 1 - \beta(\mu) \)

where \( \beta(\mu) = P \{ - t_{n-1}, \alpha/Z + \frac{\mu_0 - \mu}{Z} < t_{n0-1} < t_{n-1} \alpha/Z \}

\[ + \frac{\mu_0 - \mu}{Z} \} \]
CHAPTER IV
STANDARD ANALYSIS OF VARIANCE

As was explained in the preface, the analysis of variance that we are using in statistical methods these days depends upon some assumptions; namely, normality, independence, and equality of variance of the errors. Certainly, for example one-way analysis of variance, we decide whether the null hypothesis of equality of the t population means is rejected, if \( F = \frac{S^2_B}{S^2_W} \) exceeds the tabulated value of \( F \) for \( \alpha, df_1 = t_1 \) and \( df_2 = n-t \). What is \( S^2_B \)? What is \( S^2_W \)? \( S^2_B \) is a measure of the variability between (or among) the sample means. \( S^2_W \) is a measure of the within-sample variability. This method exactly depends upon above assumptions.

For example: \( S^2_B \) equals \( \frac{\sum(y_{ij} - \bar{T})^2}{t-1} \).

The formula is based on the following idea. If the null hypothesis \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 \) is true, then the populations are identical, with mean \( \mu \) and variance \( \sigma^2 \). Drawing single samples from the five populations is then equivalent to drawing five different samples from the sample population. What kind of variation might be expected for these sample means? If the variation is too great, we would reject the hypothesis that \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 \).

To discuss the variation from sample mean to sample mean, we need to do the distribution of the mean of a sample of \( n \) observations in repeated sampling. The distribution of sample means will have the same mean \( \mu \)
and variance $\sigma^2/n$. Since we have drawn five samples of $n$ observations each, we can estimate the variance of the distribution of sample means, $\sigma^2/n$, using the formula sample variance =

$$\frac{\overline{y}^2 - \left[\overline{y}^2/5\right]}{5 - 1}. $$

Note that we merely consider the $\overline{y}_5$ as a sample of five observations and calculate the "sample variance". This quantity estimates $\sigma^2/n$ and hence $\sigma_y$ (sample variance of the means) estimates $\sigma^2$. We designate this quantity as $S_{B}^2$. How about $S_{w}^2$? We assume that the five sets of measurements are normally distributed, with means given by $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$ and $\mu_5$ and a common variance $\sigma^2$. Therefore,

$$S_{w}^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2 + (n_3-1)S_3^2 + (n_4-1)S_4^2 + (n_5-1)S_5^2}{(n_1-1) + (n_2-1) + (n_3-1) + (n_4-1) + (n_5-1)}.$$

Thus, $S_{w}^2$ represents a combined estimate of the common variance $\sigma^2$, and it measures the variability of the observations within the five populations.

The analysis of variance used usually depends upon three assumptions. But fortunately, in actual experiments sometimes we don't know the populations variance, so we have possible serious problems when we attempt analysis of variance with unequal variances.
CHAPTER V
POWER OF TEST

Power of test for \( H_0: \mu = \mu_0 \). A test of the hypothesis \( H_0: \mu = \mu_0 \) was presented in which the desired level of significance was obtained by adjusting the critical region. However, even for a fixed critical region, and consequently a fixed value of \( \alpha \), the numerical value of \( \beta \) varies according to possible true values of \( \mu \). The value of \( \beta \) is large if \( \mu \) is close to \( \mu_0 \) and small if \( \mu \) is very different from \( \mu_0 \), which reflects the fact that alternatives that are only slightly different from the hypothesis will be difficult to discover and large differences will be easier to recognize.

In this test \( H_0: \mu = \mu_0 \), the power of the test is represented \( 1 - \beta \).

\[ \beta = P [\bar{X} \leq \text{a critical value} \mid H_0 \text{ is false}] \] power of the analysis of variance tests.

The analysis of variance technique is used to compare means of several populations. It is assumed that each of the populations has a normal distribution with a common value \( \sigma^2 \) for the variance. The means of the populations are \( \mu_1, \mu_2, \ldots, \mu_k \) and the usual hypothesis to be tested is \( \mu_1 = \mu_2 = \ldots = \mu_k \). Alternatives to this hypothesis would specify values for the \( \mu_1, \mu_2, \ldots, \mu_k \) not all the same. We can measure the dispersion of the \( \mu_i \)'s by using the variance of these quantities,

\[ \sum_{i=1}^{k} (\mu_i - \bar{\mu})^2 / k \quad \text{where} \quad \bar{\mu} = \sum_{i=1}^{k} \mu_i / k. \]
It is convenient to divide this quantity by \( \sigma^2/n \), and call it \( \phi^2 \).

\[
\phi^2 = \left[ \frac{\sum_{i=1}^{K} (\mu_i - \bar{\mu})^2}{\kappa} \right] / \sigma^2/n
\]

Where \( n \) is the number of observations from each population. This number \( \phi^2 \) can be used to measure an alternative. The probability of rejecting the hypothesis \( \mu_1 = \mu_2 = \ldots = \mu_K \) when actually they have specified unequal value can be obtained in terms of \( \phi^2 \).

In Table 5-1 are graphs giving the value of \( 1-\beta \) on the vertical scale related to \( \phi \) on the horizontal. The graphs are for two levels of significance, \( \alpha = .01 \) and .05, for eight values of \( v_1 \), the number of degrees of freedom in the denominator of the F ratio. Note that there is a different curve for each set of values \( v_1, v_2, \) and \( \alpha \).

Consider the curve of the table for \( \alpha = .05, v_1 = 3, \) and \( v_2 = 12 \). This would be used, for example, in testing the hypothesis that four \( (v_1 = 4-1 = 3) \) populations have equal means with samples of size \( n = 4 \) from each population \( (v_2 = 16-4 = 12) \) reading above \( \phi = 2 \). We see that the chance of recognizing that the four populations do not have equal means when actually \( \phi^2 = 4 \) is \( 1 - \beta = .82 \). Thus the power of the test against any set of values \( \mu_1, \mu_2, \mu_3, \mu_4 \) giving \( \phi^2 = 4 \) is .82. The alternatives are in terms of \( \phi^2 \) alone, not distinguishing among different sets of means which give the same value to \( \phi^2 \). For example, with \( \sigma^2 = 1 \) the four population means could be 50, 50, 52, 52, or 50, 50, 50, 52.31 and give \( \phi^2 = 4 \). Where \( \phi \) is a function of \( \sigma^2 \).
Table 5-1

Power and $\phi$
CHAPTER VI

BISHOP'S METHOD FOR ANALYSIS OF VARIANCE

A. The One-way Layout

The one-way layout may be set up as follows: let \( X_{ij} = \mu_i = e_{ij} \) (\( i = 1, 2, \ldots, I; j = 1, 2, \ldots \)). Where the \( \{e_{ij}\} \) are independent normal random variables with mean 0 and variance \( \sigma^2_{e_i} \). His method is to test the null hypothesis \( H_0: \mu_1 = \mu_2 = \ldots = \mu_I \) in such a way that the level and power of the test are controllable, and not dependent upon the unknown variances.

The test to be proposed is a method for:

1. Controlling \( \alpha \), the type I error, when variances are unequal.
2. Controlling \( 1-\beta \), the power of the test, when \( \delta = \Sigma(\mu_i - \bar{\mu})^2 \) is some specified size.

Hence, prior to constructing the test, we must choose \( \alpha \), \( \beta \) and \( \delta \). We assume these are preselected.

a. Procedure

Select a random sample of size \( n_0 \) from each of \( I \) levels. For the \( i \)th level let \( S_i^2 \) be the usual unbiased estimate of \( \sigma^2_i \) based on the first \( n_0 \) observations. Define \( N_i = \max\{n_0 + 1, \left[ \frac{S_i^2}{Z} \right] + 1\} \)

Where \( Z \) is to be determined from Table 7-1, using our predetermined values for \( \alpha \), \( \beta \) and \( \delta \). Use of this table will be described later.
\( N_i \) is the actual number of observations, so if \( n_0 + 1 \) is greater than \( \frac{S^2}{Z} + 1 \), then \( N_i \) is \( n_0 + 1 \). If not, we have additional observations to select, and then,

\[ N_i = \left[ \frac{S^2}{Z} + 1 \right] \]

Real numbers \( a_i \), \( i = 1, 2, \ldots, n \) and \( b_j \), \( j = n + 1, n + 2, \ldots, N \), are chosen in such a way that

\[
\begin{align*}
\sum_{i=1}^{n_0} a_i + \sum_{j=n_0+1}^{N} b_j &= 1, \\
\left( \sum_{i=1}^{n_0} a_i^2 + \sum_{j=n_0+1}^{N} b_j^2 \right) &= Z
\end{align*}
\]

\[ a_1 = a_2 = \ldots = a_{n_0} \quad b_{n_0+1} = b_{n_0+2} = \ldots = b_N \]

For each \( i \) populations, set constants \( a_{i1}, a_{i2}, \ldots, a_{iN_i} \)

\[
a = a_{i1} = \ldots = a_{ni} = \frac{1 - (N_i - n_0)b}{n_0}
\]

Then,

\[
\begin{align*}
\sum_{i=1}^{n_0} a_i + \sum_{j=n_0+1}^{N} b_j &= 1, \\
n_0 a + (N - n_0)b &= 1
\end{align*}
\]

Therefore, \( a = \frac{1 - (N - n_0)b}{n_0} \)

\[ b = b_1, \quad n_{0+1} = b_1, \quad n_{0+2} = \ldots = b_1, \quad N_i = \frac{1}{N_i} \left[ 1 + \sqrt{\frac{n_0(N_i - 1) - S^2}{(N_i - n_0) S^2}} \right] \]

\[ n_0 a^2 + (N - n_0)b^2 = Z/S^2 \text{ where } a = \frac{1 - (N - n_0)b}{n_0} \]

\[ n \left( \frac{1 - (N - n_0)b}{n_0} \right)^2 + (N - n_0)b^2 = Z/S^2 \]

\[ b^2 \left[ (N - n_0)^2 + n_0(N - n_0) \right] - 2(N - n_0)b + 1 - n_0 \frac{Z}{S^2} = 0 \]
\[ \frac{N b^2 - 2b + (1 - n_0 \frac{Z}{S^2})}{(N - n_0)} = 0 \]

\[
b = \frac{1}{N} \left[ 1 + \sqrt{\frac{S^2(N - n_0) - N(S^2 - n_0 Z)}{S^2(N - n_0)}} \right]
\]

\[
= \frac{1}{N} \left[ 1 + \sqrt{\frac{n_0(NZ - S^2)}{S^2(N - n_0)}} \right]
\]

For \( i = 1, \ldots, I \), compute \( \tilde{X}_i = \sum_{1 \leq i \leq I} a_{iI} X_{iI} + \sum_{i = n_0 + 1}^{N} b_{iI} X_{iI} \)

where \( X_{iI}, \ldots, X_{iI} \) is the total set of observations drawn from the \( i \)th level. Finally compute \( \tilde{F} = \frac{\sum_{i=1}^{I} (\tilde{X}_{iI} - \tilde{X}_{..})^2}{Z} \)

Where \( \tilde{X}_{..} = (1/I) \sum_{i=1}^{I} \tilde{X}_{iI} \) and \( H_0 \) is rejected if and only if \( \tilde{F} > F_{I,n_0} \).

\( F_{I,n_0} \) is the upper \( \alpha \)th percent of the null distribution of \( \tilde{F} \).

We next describe how to use the Table:

1. We first choose the number of populations for our experiment (gives \( I \) in table).
2. Choose number of observations (gives \( n \) in table).
3. Choose level of significance (gives \( \alpha \) in table).
4. Choose power, \( \delta = \sum (\mu_i - \bar{\mu})^2 \) and read off the corresponding value for \( Z \). For example, an experimenter desired the level of the test to be \(.10 \) with power of at least \(.85 \) if it is \( \frac{4}{I} \sum (\mu_i - \bar{\mu})^2 \geq 1.0 \).

It is decided that an \( n_0 \) of 15 is sufficient and thus \( Z \) is chosen from Table 7-1 to be \(.08 \) and the critical point for the test is 7.44. If the
table does not have the appropriate parameters, one uses the approximation

$$F \sim \frac{(n_0 - 1)}{(n_0 - 3)} \cdot \frac{\chi^2}{(I-1)} \cdot \left( \sum_{i=1}^{I} (\bar{\mu}_i - \bar{\mu})^2 / \lambda \right)$$

with table for the non-central chi-square (see [5]), to obtain Z.

B. The Two-way Layout

The two-way model under consideration that is usually studied in ANOVA is defined by

$$X_{ijh} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijh}$$

Where it is assumed that the \(\{e_{ijh}\}\) are independent random variables with \(e_{ijh} \sim N(0, \alpha_i^2)\) and

$$\sum_{i=1}^{I} \alpha_i = \sum_{j=1}^{J} \beta_j = \sum_{i=1}^{I} (\alpha \beta)_{ij} = \sum_{j=1}^{J} (\alpha \beta)_{ij} = 0$$

The hypothesis under consideration are

- \(H_1: \alpha_i = 0\) for all \(i\)
- \(H_2: \beta_j = 0\) for all \(j\)
- \(H_3: (\alpha \beta)_{ij} = 0\) for all \(i\) and \(j\)

We seek test of these hypothesis based on test statistics whose distributions are independent of the unknown variances.

a. Procedure

Choose \(Z_0\), obtained in a similar way to the one way case. In each cell \((i,j)\) take an initial random sample of size \(n\). Compute \(S\), the usual unbiased estimate of \(\sigma^2\), based on these first \(n\) observations,
and define $N_{ij} = \max\{n_0 + 1, \left\lfloor \frac{S^2}{Z} \right\rfloor + 1\}$

Then take $N - n$ additional observations from cell $(i, j)$ and calculate constants

$$a_{ij1}, \ldots, a_{ijn_0} = \frac{1 - (N_{ij} - n_0)b_{ij}}{n_0}$$

$$a_{ij,n_0+1} = \ldots = a_{ijN_{ij}} = b_{ij}$$

Next compute $X_{ij} = \sum_{h=1}^{N_{ij}} a_{ijh}X_{ijh}$ \hfill (B-2)

Where $X_{ij}, \ldots, X_{ij}, N_{ij}$ is the total set of observations for cell $(i, j)$.

Finally compute $\bar{X}_{ij} = \frac{1}{J} \sum_{i=1}^{J} X_{ij}$.

$$\bar{X}_{.j} = \frac{1}{I} \sum_{j=1}^{J} \bar{X}_{ij}.$$  

$$\bar{X}_{..} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \bar{X}_{ij}.$$  

Our choice of test statistics for the hypothesis in (B-1) is motivated by an interpretation of the design constant $Z$. Briefly, one may view $Z$ as playing the role of $\sigma^2/N$ in the $r$-way layout when the errors have equal variance $\sigma^2$. $\sigma^2$ is known, and $N$ observations are taken in each cell. The test statistics one would use in that case for each of the hypothesis in (B-1) are functions of the cell sample means and $\sigma^2/N$ only. The test statistics we choose have exactly the same form with (B-2) replacing the cell sample means and $Z$ replacing $\sigma^2/N$.

These test statistics may be generated along the line of Chapter IV of "The Analysis of Variance" (Scheffe). In the usual analysis of
variance model with one observation per cell, the total sum of squares may be partitioned into $2^r$ component sums of squares. Among these, there is exactly one corresponding to each main effect and to each of the possible interaction terms. When divided by $\sigma^2$ each follows a noncentral chi-square distribution (degrees of freedom and noncentrality parameter depending upon which main effect or interaction is under consideration).

In the case of $\sigma^2$ known we may take $N$ observations in each cell and reduce the problem, using the concept of sufficiency, to one observation per cell (i.e. the cell sample mean). Thus each of these sums of squares (SS) divided by $\sigma^2$ (SS/$\sigma^2$) is a function of the cell means and $\sigma^2/N$ only. To obtain the test statistic to be used in the corresponding two-stage procedure replace the cell means by the generalized cell means (B-2), and replace $\sigma^2/N$ by $Z$, in SS/$\sigma^2$ and use that resulting statistic to test the corresponding hypothesis. For example, if we want to test $H_1: \sigma_i = 0$ for all $i$, from Chapter IV of "The Analysis of Variance" (Scheffe).

$$SS_{1/\sigma^2} = \frac{\sum_{i=1}^{JN} (X_{1ij} - \bar{X}_{..})^2}{\sigma^2}$$

where $X_{1..} = \frac{1}{J} \sum_{i=1}^{I} \sum_{j=1}^{J} X_{ij}$, $X_{i..} = \frac{1}{N} \sum_{h=1}^{N} X_{ijh}$ and $X_{...} = \frac{1}{JJ} \sum_{i=1}^{I} \sum_{j=1}^{J} X_{ij}$.

so our test statistic for $H_1$, is

$$F_1 = \frac{\sum_{i=1}^{I} (\tilde{X}_{1..} - \tilde{X}_{...})^2}{Z}$$

Similarly, $H_2$ is tested using $F_2 = \frac{\sum_{j=1}^{J} (\tilde{X}_{ij..} - \tilde{X}_{...})^2}{Z}$.
Similarly, $H_2$ is tested using

$$F_2 = I \sum_{j=1}^{J} \frac{(\tilde{x}_{ij} - \tilde{x}_{...})^2}{Z}$$

and $H_3$ is tested using

$$F_3 = I \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{(\tilde{x}_{ij} - \tilde{x}_{i..} - \tilde{x}_{.j} + \tilde{x}_{...})^2}{Z}$$

The tests proceed by rejecting $H_1$, $H_2$, or $H_3$ respectively if $F_1$, $F_2$, $F_3$ exceed the upper $\alpha$th percent point of their respective null distributions. Each of these will be a level $\alpha$ test regardless of the values of the $\sigma_{ij}^2$. It is true that the power of each test is independent of the $\sigma_{ij}^2$ and controllable through the design constant $Z$. 
CHAPTER VII
MONTE CARLO METHOD FOR NULL $\chi^2$ DISTRIBUTION AND THE
STANDARD NORMAL RANDOM NUMBER GENERATOR

A. Null $\chi^2$ Distributions

Table 7-1 presents the 10%, 5%, and 1% critical points $F_{I,n_0,\alpha}$ of the null distribution of $F$ needed to test hypothesis, and the power achieved for the various combinations of $I$, $n_0$, $\delta$ and $Z$. For example, if $\delta = .1$, $n = 5$, $I = 2$ and we want to run a 5% level test, our critical point $F_{2,5,.05}$ is 8.0, and if we select $Z$ the power of our test is .613. The critical points and powers were obtained by Monte Carlo.

B. Monte Carlo Method for Critical Points (for making Table 7-1).

1. Calculate test statistic for each trial, this process was replicated 10,000 times.
2. Draw frequency histogram for each $F$ values.
3. Find a point that $\alpha$ is equal or exceed on the histogram. This point is a critical point for the $\alpha$, $I$, $n_0$.

Where $\alpha$ is the level of test, $I$ is the number of populations, and $n_0$ is the number of observations.

C. Monte Carlo Method for Powers

1. Set up $H$ and $H_1$ (specify $\mu_1$, $\mu_2$, ... $\mu_I$ ≠), difference of $\mu_1$, $\mu_2$, ... $\mu_I$ is defined by

$$\sum_{i=1}^{I} (\mu_i - \mu)^2 = \delta$$
2. Calculate the test statistic for each trial, this process was reallocated 10,000 times.

3. Count proportion rejections, this is power for Z, δ, I, n₀, α, where Z is a variance of population,

\[ \delta = \sum_{i=1}^{I} (\mu_i - \bar{\mu})^2. \]

D. Standard Normal Random Number Generator

Computer Program

```c
C FUNCTION FOR GENERATING RANDOM NORMAL NUMBERS, MEAN = 0
C STANDARD DEVIATION = 1.
C A PAIR OF VARIABLES ARE GENERATED AND RETURNED ONE AT A TIME ON ALTERNATE CALLS.
$SET OWN
DATA I/O/
$RESET OWN
IF (I.GT.O) GO TO 30
10 X=2.0*RANDOM (IR)-1.0
Y=2.0*RANDOM (IR)-1.0
S=X*X+Y*Y
IF (S.GE.((1.0)) GO TO 10
S=SORT(-2.0*ALOG(S)/S)
RNOR=X*S
$SET OWN
RNORT=Y*S
$RESET OWN
I=1
GO TO 40
30 RNOR=RNORT
I=0
40 RETURN
END
```

Figure 7-1. Standard Normal Random Number Generator.
The following program segment will generate NOBS normally, distributed values with mean AVE and standard deviation SD. [4]

```
DIMENSION X(500)
READ (5,/) NOBS
READ (5,/) AVE, SD
IA=TIME(11)
DO 5 I=1, NOBS
  5 X(I)=RNOR(IA)*SD+AVE
```
### Table 7-1

**Null F Distribution**

| n | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| u | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
| 10 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 15 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 20 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 25 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 30 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 40 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 50 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |

---

### Table 7-2

| n | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| u | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| 10 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 15 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 20 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 25 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 30 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 40 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
| 50 | .005 | .015 | .025 | .035 | .045 | .055 | .065 | .075 | .085 | .095 | .105 | .115 | .125 | .135 | .145 | .155 | .165 | .175 | .185 | .195 | .205 | .215 |
**Table 7-1**

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<th>$\lambda$</th>
<th>$\mu$</th>
<th>$\nu$</th>
<th>$\xi$</th>
<th>$\omicron$</th>
<th>$\pi$</th>
<th>$\rho$</th>
<th>$\sigma$</th>
<th>$\tau$</th>
<th>$\upsilon$</th>
<th>$\phi$</th>
<th>$\chi$</th>
<th>$\psi$</th>
<th>$\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.00</td>
<td>.01</td>
<td>.02</td>
<td>.03</td>
<td>.04</td>
<td>.05</td>
<td>.06</td>
<td>.07</td>
<td>.08</td>
<td>.09</td>
<td>.10</td>
<td>.11</td>
<td>.12</td>
<td>.13</td>
<td>.14</td>
<td>.15</td>
<td>.16</td>
<td>.17</td>
<td>.18</td>
<td>.19</td>
<td>.20</td>
<td>.21</td>
<td>.22</td>
<td>.23</td>
</tr>
</tbody>
</table>

[Table continues with more data entries in a similar format.]
CHAPTER VIII
EXPERIMENT

In this chapter, two ANOVA will be compared, namely Bishop's method and standard ANOVA. The used data adopt standard normal random number.

A. Data

FUNCTION RNOR(IR) explained in Chapter VII generates numbers which are normally distributed. However, we have to input values of the mean and the standard deviation, when we run the program. The numbers which are normally distributed are used by way of data for comparison of two ANOVA. The following section attempts to check the generator.

a. Check of Generator

One hundred means from each sample with a mean and a variance is obtained by running this program, and counted number of means which exceed C.I. $(\mu \pm Z \cdot SD/\sqrt{n})$.

Computer Program

```
FUNCTION RNOR(IR)
$SET OWN
DATA I/O/
$RESTE OWN
  IF(I.GT.0) GO TO 30
  X=2.0*RANDOM(IR)-1.0
  Y=2.0*RANDOM(IR)-1.0
  S=X*X+Y*Y
  IF(S.GE.(1.0) GO TO 10

Figure 8-1. Check of Generator
```
S = SORT(-2.0 * ALOG(S)/S)
RNOR = X*S
$SET OWN
RNORT = Y*S
$RESET OWN
I = 1
GO TO 40
30 RNOR = RNORT
I = 0
40 RETURN
END
DIMENSION X(500)
READ(5,/) NOBS
IA = TIME(11)
DO 1 J = 1, 100
DO 5 I = 1, NOBS
5 X(I) = RNOR(IA) * SD + AVE
A = 0
B = 0
DO 6 I = 1, NOBS
A = A + X(I)
B = B + X(I)**2
6 CONTINUE
AM = A/NOBS
A2 = A**2
AN = A2/NOBS
S = SORT((B - AN)/(NOBS - 1))
WRITE (6,/) AM, S
1 CONTINUE
STOP
END

Figure 8-1. (Continued)

b. Table 8-1

Result of Generator

<table>
<thead>
<tr>
<th>Variance</th>
<th>C.I. (µ ± Z·SD/√n)</th>
<th>Exceeded number</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>100 ± .693</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>100 ± .894</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>100 ± 1.058</td>
<td>2</td>
</tr>
</tbody>
</table>

Where µ = 100, number of observation = 10, and α = .05.
At the beginning, ten observations were generated from FUNCTION RNOR(IR), the mean was calculated for each trial and this process was repeated 100 times. Where the "Exceeded number" is number, that the means exceed the limit of C.I. in 100 trials. From this test we can say that our sample from FUNCTION RNOR(IR) will include 95% of the means in repeated sampling.

B. Standard ANOVA

Test procedures in the r-way layout model in ANOVA are based on the assumptions of normality, independence, and equality of the variances of errors. In my thesis, only equality of the variances of the errors will be considered, attempted ANOVA having different population variance. In this experiment one-way ANOVA was attempted, namely $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$ (level $\alpha = .05$).

a. Computer Program

```plaintext
FUNCTION RNOR(IR)
$SET OWN
DATA I/0/
$RESET OWN
IF(I.GT.O) GO TO 30
10 X=2.0*RANDOM(IR)-1.0
   Y=2.0*RANDOM(IR)-1.0
   S=X*X+Y*Y
   IF(S.GE.(1.0)) GO TO 10
   S=SORT(-2.0*ALOG(S)/S)
   RNOR=X*S
$SET OWN
   RNORT=Y*S
$REST OWN
   I=1
   GO TO 40
30 RNOR=RNORT
   I=0
40 RETURN
END
```

Figure 8-2. Standard ANOVA.
DIMENSION X(500,10),A(10),NOBS(6),AVE(6),SD(6)
FR=2.6
READ(5,/) NO
READ(5,/) (AVE(I),I=1,NO)
READ(5,/) (NOBS(I),I=1,NO)
DO 200 I2=1,5
READ(5,/) (SD(I),I=1,NO)
WRITE(6,1000) I2
1000 FORMAT(1-',///,36X,'EXAMPLE NO-',2X,I2,/) REJ=O
ACC=O
IA=TIME(11)
YOKO=1
DO 100 II=1,100
DO 22 J=1,4
DO 20 I=1,NOBS(J)
   X(I,J)=RNOR(IA)*SD+AVE(J)
20 CONTINUE
22 CONTINUE
Y=O
N=O
AG=O
B=O
S=0.0
DO 10 J=1,NO
DO 5 I=1,NOBS(J)
   S=S+X(I,J)
   AG=AG+X(IJ)*X(I,J)
   Y=Y+X(I,J)
5 CONTINUE
B=B+(Y*Y)/NOBS(J)
Y=O
N=N+NOBS(J)
IF(YOKO.EQ.0) GO TO 10
WRITE(6,2) J
2 FORMAT(1-',///,2X,'GROUP',2X,I2,/) WRITE(6,1) (X(I,J),I=1,NOBS(J)
1 FORMAT(1-',2X,8F15.10)
10 CONTINUE
YOKO=0
SG=S*S
GA=SG/N
TSS=AG-GA
SSB=B-GA
SSW=TSS-SSB
F1=SSB/3
F2=SSW/120
F=F1/F2
IF(F.GT.FR) GO TO 3
ACC=ACC+L
GO TO 4

Figure 8-2. (Continued)
3 REJ=REJ+1
4 CONTINUE
100 CONTINUE
WRITE(6,1001)
1001 FORMAT( ' ',//'///,20X.'GROUP 1',5X,'GROUP 2',5X,'GROUP 3' *
*,5X,'GROUP 4')
WRITE(6,1002) (AVE(I),I=1,4)
1002 FORMAT( ' ',10X,'MEAN',4(5X,F7.2))
WRITE(6,1003) (SD(I),I=1,4)
1003 FORMAT( ' ',12X,'SD',4(5X,F7.2))
WRITE(6,1004) (NOBS(I),I=1,4)
1004 FORMAT( ' ',8X,'NUMBER',4(5X,F7.2))
WRITE(6,1005) ACC
1005 FORMAT( ' ',///,10X,'NUMBER THAT THE HYPOTHESIS H IS *
*ACCEPTED',5X,13,2X,'PERCENT',5X,'(AT THE 5 PERCENT *
*SIG.LEV)')
WRITE(6,1006) REJ
1006 FORMAT( ' ',///,10X,'NUMBER THAT THE HYPOTHESIS H IS *
*REJECTED',5X,13,2X,'PERCENT',5X,'(AT THE 5 PERCENT *
*SIG.LEV)')
200 CONTINUE
STOP
END

Figure 8-2. (Continued)

b. Table 8-2
Combination of Variance I

<table>
<thead>
<tr>
<th>Combination of Variance</th>
<th>Pop. 1</th>
<th>Pop. 2</th>
<th>Pop. 3</th>
<th>Pop. 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>10</td>
</tr>
</tbody>
</table>

Where mean = 100 Pop. i = populations

The procedure (ANOVA) to test the hypothesis that each population has the same means and one combination (A,B,C) of variances was tried to obtain the percentage. At times it includes type I error with the generated 500 times. In the results, the variance combination A has
the largest type I errors. Therefore, hereafter I used this type of combination for the comparison. Also, this combination has the biggest probability of type I error on Table 10.4.2 in Chapter 10 in [5].

In the next experiment, ANOVA will be attempted when four population variance is increased from 1 to 40. The relation between the combination of variance and type I error is considered, where mean is 100 and level $\alpha = .05$.

Table 8-3  
Combination of Variance II

<table>
<thead>
<tr>
<th>Combination of Variance</th>
<th>Pop. 1</th>
<th>Pop. 2</th>
<th>Pop. 3</th>
<th>Pop. 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>F</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>G</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>40</td>
</tr>
</tbody>
</table>
Table 8-4
Result of Standard Test

<table>
<thead>
<tr>
<th>Combination of Variance</th>
<th>Probability of Type I Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>.03</td>
</tr>
<tr>
<td>B</td>
<td>.0333</td>
</tr>
<tr>
<td>C</td>
<td>.06</td>
</tr>
<tr>
<td>D</td>
<td>.07</td>
</tr>
<tr>
<td>E</td>
<td>.08</td>
</tr>
<tr>
<td>F</td>
<td>.0966</td>
</tr>
<tr>
<td>G</td>
<td>.0933</td>
</tr>
<tr>
<td>H</td>
<td>.1133</td>
</tr>
</tbody>
</table>
In this test, test statistics for each combination of variance was repeated 300 times. Also, the result is indicated in the following diagram:

Figure 8-3. Diagram of Test.
C. Bishop's ANOVA

Before we try the experiment using Bishop's ANOVA, we have to decide some coefficients. Table 7-1 presents the 10%, 5% and 1% critical points $F_{I,n_0}^\alpha$ of the null distribution of $\bar{F}$ needed to test the hypothesis, and the power achieved for the various combinations of $I$ (number of population), $n_0$ (original number of observations per population),

$$\delta = \left( \sum_{i=1}^{I} (\bar{\mu}_i - \bar{\mu})^2 \right),$$

$Z$ (positive number, table presents 1, .8, .6, .4, .2, .1, .08, .06, .04, .02, .01).

To equal the experiment of standard ANOVA, I chose four populations and 10 original observations. To choose the power being greater than .8, and if $\delta = \sum (\bar{\mu}_i - \bar{\mu}) \geq 1$, then $Z$ is decided .06. Therefore, the following coefficients are used:

$I = 4$
$n = 10$
$\delta = 1$
$Z = 106$
$P = .862$

Also, I want to run a 5% level test, so my critical point $F_{4,10}^{.05}$ is 10.5.

In the same manner as standard ANOVA, I ran 100 tests and counted the number committing Type I errors in the computer program.

a. Computer Program

```plaintext
FUNCTION RNOR(IR)
$SET OWN
DATA I/O/

Figure 8-4. Bishop's ANOVA.
```
$RESET OWN
  IF(I.GT.O) GO TO 30
  X=2.0*RANDOM(IR)-1.0
  Y=2.0*RANDOM(IR)-1.0
  S=X*X+Y*Y
  IF(S.GE.(1.0)) GO TO 10
  S=SORT(-2.0*ALOG(S)/S)
  RNOR=X*S
$SET OWN
  RNORT =Y*S
$RESET OWN
  I=1
  GO TO 40
30  RNOR=RNORT
  I=O
40  RETURN
END
SUBROUTINE BIS(S,SS,N2,N3,XM,Z,FA)
DIMENSION S(1000,6),SS(6),XM(6),AL(6),XI(6),XA(6)
Do 1 J=1,N#
  A3=0
  BB=SORT((N2*(XM(J)*Z-SS(J)))/((XM(J)-N2)*SS(J)))/(XM(J)-N2*BN(J))/N2
  AL(J)=(1-(XM(J)-N2*BN(J))/N2
  N1O=XM(J)
  DO 3 I=1,N1O
     IF(I.GT.N2) GO to 11
     AA=AL(J)*S(I,J)
     GO TO 12
11    AA=BN(J)*S(I,J)
12    XI(J)=A3+AA
     A3=XI(J)
3    CONTINUE
1    CONTINUE
    XX=O
    DO 4 J=1,N3
      XX=XX+XI(J)
4    CONTINUE
    XX1=XX/N3
    DO 5 J=1,N3
      XA(J)=(XI(J)-XX1)**2/Z
5    CONTINUE
    FA=O
    DO 6 J=1,N3
      FA=FA+XA(J)
6    CONTINUE
    WRITE(6,300)
300   FORMAT(·,111,·,11x,·,ss·,11x,·,NI·,10x,·,A·,12x,
               *·s·, 1sx, ·xM·)
    DO 301 J=1,N3
      WRITE(6,31) J,SS(J),XM(J),AL(J),BN(J),XI(J)
31   FORMAT(· ·,2X,I2,6X,F9.4,8X,I3,6X,F8.6,6X,F8.6,5X,F15.5)

Figure 8-4. (Continued)
CONTINUE
WRITE(6,32) FA
32 FORMAT(\"\\\\\\\",///,5X,\"F TEST=\\\\\\\",F9.4,\")
RETURN
END
SUBROUTINE SAMVAR(N3,N2,SS,S)
DIMENSION SS(6),S(1000,6)
DO 1 J=1,N3
SI=0
SII=0
DO 2 I=1,N2
SI=SI+S(I,J)
SII=SII+S(I,J)
2 CONTINUE
SI2=SI*SI
SS(J)=(SI-SI2/N2)/(N2-1)
1 CONTINUE
RETURN
END
DIMENSION S(1000,6),SS(6),AVE(6),SM(6),SD(6)
Z=.08
CP=8.68
K1=0
K2=0
YI=0
READ(5,/) N3,N2
READ(5,/) (AVE(I),I=1,N3)
READ(5,/) (SD(I),I=1,N3)
IA=TIME(11)
DO 100 JJ=1,100
DO 1 J=1,N3
DO 2 I=1,N2+1
S(I,J)=RNOR(IA)*SD(J)+AVE(J)
2 CONTINUE
1 CONTINUE
CALL SAMVAR(N3,N2,SS,S)
DO 3 J=1,N3
L2=SS(J)/Z+1
L1=N2+1
Y=L1
IF(L1.GT.L2) GO TO 5
Y=L2
DO 5 I=L1,L2
S(I,J)=RNOR(IA)*SD(J)+AVE(J)
5 CONTINUE
XM(J)=Y
IF(YI.GT.L2) GO TO 3
YI=Y
3 CONTINUE
CALL BIS(S,SS,N2,N3,XM,Z,FA)

Figure 8-4. (Continued)
IF(FA.GT.CP) GO TO 4
  K1=K1+1
GO TO 100
4 K2=K2+1
100 CONTINUE
WRITE(6,1005) K1
1005 FORMAT(1X,'NUMBER THAT THE HYPOTHESIS IS ACCEPTED',5X,I3,2X,'PERCENT',5X,'(AT THE 5 PERCENT SIG. LEV)')
WRITE(6,1006) K2
1006 FORMAT(1X,'NUMBER THAT THE HYPOTHESIS IS REJECTED',5X,I3,2X,'PERCENT',5X,'(AT THE 5 PERCENT SIG. LEV)')
STOP
END

Figure 8-4. (Continued)

b. Table 8-5

Bishop's Result

<table>
<thead>
<tr>
<th>Number of Observations</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>6.0</td>
<td>6.2</td>
<td>7.4</td>
<td>5.2</td>
<td>4.6</td>
</tr>
<tr>
<td>1000</td>
<td>5.6</td>
<td>5.8</td>
<td>6.1</td>
<td>5.1</td>
<td>5.2</td>
</tr>
<tr>
<td>1500</td>
<td>5.9</td>
<td>5.2</td>
<td>6.3</td>
<td>5.1</td>
<td>5.3</td>
</tr>
<tr>
<td>2000</td>
<td>6.1</td>
<td>5.7</td>
<td>6.0</td>
<td>5.2</td>
<td>5.5</td>
</tr>
<tr>
<td>2500</td>
<td>5.6</td>
<td>5.6</td>
<td>5.7</td>
<td>4.9</td>
<td>5.3</td>
</tr>
<tr>
<td>3000</td>
<td>5.5</td>
<td>5.7</td>
<td>5.5</td>
<td>5.1</td>
<td>5.5</td>
</tr>
<tr>
<td>3500</td>
<td>5.5</td>
<td>5.7</td>
<td>5.3</td>
<td>5.2</td>
<td>5.5</td>
</tr>
<tr>
<td>4000</td>
<td>5.4</td>
<td>5.6</td>
<td>5.4</td>
<td>5.2</td>
<td>5.7</td>
</tr>
<tr>
<td>4500</td>
<td>5.2</td>
<td>5.3</td>
<td>5.7</td>
<td>5.2</td>
<td>5.8</td>
</tr>
<tr>
<td>5000</td>
<td>5.2</td>
<td>5.3</td>
<td>5.6</td>
<td>5.3</td>
<td>5.9</td>
</tr>
</tbody>
</table>

Where the column of numbers of observation presents the number of test statistics and each column of variance ratio presents percentages of Type I errors.
c. Statistic Test for Bishop's ANOVA

\[ P = \pi_0 + Z \frac{\pi_0 (1 - \pi_0)}{n} = 0.1009 \]

\[ H_0: \pi \leq \pi_0 + 0.05 \]

\[ H: \pi > \pi_0 \]

\[ Z = \frac{P - \pi_0}{\sqrt{\pi_0 (1 - \pi_0)/n}} \]

Table 8-6

<table>
<thead>
<tr>
<th>Variance Ratio</th>
<th>Statistic Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Z_A</td>
</tr>
<tr>
<td>B</td>
<td>Z_B</td>
</tr>
<tr>
<td>C</td>
<td>Z_C</td>
</tr>
<tr>
<td>D</td>
<td>Z_D</td>
</tr>
<tr>
<td>E</td>
<td>Z_E</td>
</tr>
</tbody>
</table>

Where hypothesis test \( Z_A, Z_B, \) and \( Z_D \) was accepted, test \( Z_C \) and \( Z_E \) was rejected.
Intrinsically, the standard ANOVA and Bishop's ANOVA are not different, namely the two methods of these are based on the ratio of the variance within the population and the variance among populations. But the standard ANOVA uses a common variance and Bishop's ANOVA defines a variance by relating the power and the dispersion of \( \mu \) is for the variance with population. From my experiment, the larger a ratio of combination of variance is when we use the standard ANOVA, the more it has the desired Type I error. When we try Bishop's method, the number of Type I errors is not affected by increasing the ratio of combination of variance.

In view of the experiment, regardless of the difficulties existing in calculation and decision of \( S \), Bishop's method is better than the standard in analysis of variance with unequal variances. However, we have a question for the cases not tabled. An approximation is called for, and it is natural to consider the limiting distribution of \( \bar{F} \) as the initial sample size \( n \) approaches infinity. The limiting distribution of \( \bar{F} \) is noncentral chi-square with \( I-1 \) degrees of freedom and noncentrality parameter \( \Delta = \frac{1}{\sum_{i=1}^{I}(\mu_i - \bar{\mu})^2/Z} \), denoted by \( X^2_{I-1}(\Delta) \). However, numerical results given in [2] indicate that for small \( n \) the tails of this limiting distribution are too light to give a good approximation. The distribution of \( \bar{F} \) can be approximated...
by that of a \([(n-1)/(n-3)] \cdot X^2_{1-1} (\Delta)\) random variable (in which case \(F\) and its approximating distribution have the same expected value under \(H_0\)). As an example of how to use the limiting distribution for the one-way layout, suppose \(I = 4, n = 15\) and we desire an \(\alpha = .10\) level test with power .85 at \(S = 1.0\). We assume \(\tilde{F} \sim (14/12) \cdot X^2_3 (1/z)\) in general and \(\tilde{F} \sim (14/12/) \cdot X^2_3\) under \(H\). Thus, we seek \(c\) so that \(P [\tilde{F} > c] = .10\), hence we seek \(c\) so that \(P [(14/12) \cdot X^2_3 > c] = .10\), which implies \(c = (14/12) \cdot X^2_{3,.10}\) is the upper 10 percent point of the central chi-square distribution with 3 degrees of freedom. Thus \(c = (14/12)(6.25) = 7.29\). (Note that the Monte Carlo critical point from Table 7-1 is 7.44.) We then seek a value of \(z\) so that \(P [(14/12) \cdot X^2_3 (1/z) > 7.29] = P [X^2_3 (1/z) > 6.25] = .85\). From Table II of Haynam, Govindarajulu, and Leone [3] we have \(1/z = 10.076 \Rightarrow z = .099\).
REFERENCES


