A TEST FOR DETERMINING AN APPROPRIATE MODEL
FOR ACCELERATED LIFE DATA

by
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Yuan-Who Chen
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ABSTRACT

A Test for Determining An Appropriate Model for Accelerated Life Data

by

Yuan-Who Chen, Master of Science

Utah State University, 1987

Major Professor: Dr. Ronald V. Canfield
Department: Applied Statistics

The purpose of this thesis was to evaluate a method for testing the appropriateness of accelerated life model. This method is based upon a polynomial approximation. The parameters are estimated and used for testing the appropriateness of the model.

An example illustrates the polynomial method. Real data are applied for this method. Comparison with another method demonstrates that the polynomial method is much simpler and has comparable accuracy.

(32 pages)
CHAPTER I
INTRODUCTION

Accelerated life model is commonly used in some appropriate fields for estimating the reliability of the component or the system. It is necessary to have several accelerated stress levels which are determined by the experimenter before testing. The failure times are recorded or, for those items that did not fail, the censored times are recorded. The response from the observed results are extrapolated to represent the distribution of time-to-failure under realistic stress condition.

There are many approaches to analyze the accelerated life data. All approaches require that a specific function $g(s)$ be chosen prior to testing, according to experience or physical conditions, to relate the stress variable and the response. An approach for distribution free analysis of accelerated life data at several stress levels was given by Schmoyer (1986). This approach uses a Kolmogorov-Smirnov-type test to analyze accelerated life models of the form

$$P(t,s) = \phi[g(s)t]. \quad (1.1)$$

$P(t,s)$ denotes the probability that an item stressed at level $s$ fails on or before time $t$. The $g(s)$ is a known real-valued function with an unknown parameter $c$, which describes the basic nonstochastic
relationship between stress and response, and $O$ is an unspecified continuous function. A commonly used $g(s)$ function in (1.1) is the power rule model (Levenbach 1957) for which $g(s) = s^C$ so that

$$P(t,s) = \phi(ts^C).$$  \hspace{1cm} (1.2)

Schmoyer provides a method for an interval estimation of the parameter $c$ in $g(s)$ by testing hypotheses of the form $c = c'$ against $c \neq c'$. Then, for a given significance level of the test for all values of $c'$, a confidence set of $c$ is obtained. The confidence bounds for $P(t_o,s_o)$ for given $t_o$ and $s_o$ are determined. However, Schmoyer's approach, like other known approaches, assumes that the function $g(s)$ is already defined before the testing. It introduces the risk of choosing the wrong model to test and, therefore, may lead to an incorrect result.

This thesis gives a distribution-free approach for testing the appropriateness of specific forms of $g(s)$ in (1.1). The procedure is based upon a distribution-free estimation of the cumulative density function of the time-to-failure random variable given by Canfield and Jou (1982).

Definition and theoretical background will be presented in Chapter II. Canfield and Jou's method will be extended to the accelerated life model in Chapter III. The estimation of $g(s)$ is also given in Chapter III. The following chapter will use the data of
constant-lead creep-rupture times in stainless steel (Garofalo, Whitmore, Domis, and von Gemmingen 1961) to illustrate the estimation method. There are two additional models commonly used for accelerated life analysis. These are the Arrhenius and the Eyring models (Thomas 1964), which are defined by

\[ P(t,s) = \phi(t e^{-c/s}), \]  
(1.3)

and

\[ P(t,s) = \phi(t s e^{-c/s}), \]  
(1.4)

respectively. These two models, as well as the power rule model, will be used as an example to show how to determine the function \( g(s) \) and estimate the value of the parameter \( c \). Further discussion and some computation details can be found in the last chapter.
CHAPTER II
EXPONENTIAL POLYNOMIAL APPROXIMATION
OF THE RANDOM VARIABLE

A distribution free method of approximating functions associated with time-to-failure random variables is given in Canfield and Jou (1982). This method has been further refined in Hsu (1987). The method is shown to be more accurate than the classical empirical cdf and effective in small samples. A description of the method is given in this chapter.

Let \( T(1), T(2), \ldots, T(n) \) be the order statistics of a random sample of size \( n \) from a population with cumulative density function \( F(t) \). Pearson used an inverse expansion of a random variable \( T \) in terms of the cdf of \( T \) to evaluate the moments of the order statistics (Kendall and Stuart; 1953). Let \( t_i \) denote the value such that

\[
F(t_i) = E(F(T(i))) = i/(n+1),
\]

where \( E(F(T(i))) \) is the expected value of the cdf evaluated at the \( i \)-th order statistic. The \( T(i) \) can be expanded in a Taylor's series about \( t_i \) as

\[
T(i) = t_i + b_1(F(T(i)) - F(t_i)) + \frac{b_2}{2!} (F(T(i)) - F(t_i))^2 + \ldots + \frac{b_m}{m!} (F(T(i)) - F(t_i))^m + \ldots
\]

where \( b_m \) is the \( m \)-th derivative of \( T \) with respect to \( F \) for \( T \) given by \( t_i \).

Thus, \( T(i) \) can be expressed as powers of \( F(T(i)) \).
Assuming regularity conditions for convergence, $T(i)$ can be approximated as a finite polynomial in powers of $F(T(i))$. More generally it is shown in Hsu (1987) for sufficiently large $K$, assuming convergence that $T$ may be approximated by the following form

$$T \approx \sum_{k=0}^{K} b_k W^k,$$

(2.3)

where $W$ is any random variable.

In reliability, the hazard function $h(t)$ of a continuous distribution of life $T$ with a probability density function $f(t)$ is defined for all possible outcomes $t$ as

$$h(t) = f(t)/(1-F(t)).$$

(2.4)

It is the instantaneous failure rate at age $t$. Taking the integral of (2.4), the cumulative hazard function $H(t)$ shall be derived as

$$H(t) = - \ln(1-F(t)).$$

(2.5)

Given any random variable $T$, it can be shown that $Y = - \ln(1-F(T))$ has an exponential distribution with mean 1. Thus the standard exponential random variable has a useful association with $T$. These relationships motivate the choice $W = Y^p = (H(T))^p$, for $0 < p < 1$. Then (2.3) becomes

$$T \approx \sum_{k=0}^{K} b_k (H(T))^k,$$

(2.6)

The power $p$ provides a more general representation for $T$ in (2.6) that can be used to reduce $K$. The value of $p$ is
not critical. The estimation of the coefficients $b_k$ in equation (2.5) is considered next. By the definition of the cumulative hazard function, $H(T)$ has a standard exponential distribution. The expected value of the $i$-th order statistic of an exponential random variable is defined by

$$E_{ik} = E(W^k(i)) = \frac{n!\Gamma(kp+1)}{(n-i)!} \sum_{r=0}^{i-1} \frac{(-1)^i(n-i+r+1)^{-kp-1}}{r!(i-r-1)!}$$

(2.7)

with $E_{i0} = 1$ and $kp > 0$.

Then the equation (2.6) can be expressed by

$$T(i) = \sum_{k=0}^{K} b_k E_{ik} + \varepsilon_i,$$

(2.8)

where $E(\varepsilon_i) = 0$.

The sample order statistic $t(i)$ is the dependent variable with the value $E_{ik}$ in (2.7) as the independent variable in equation (2.8). Ordinary least squares estimation can be used to estimate the unknown coefficients.

Suppose the order statistics $t(i)$ are obtained from a test. By (2.8), $t(i)$ can be expressed as

$$t(i) = \sum_{k=0}^{K} b_k E_{ik} + \varepsilon_i,$$

where $E_{ik}$ can be obtained from (2.7). Let the sum of squares of error terms be $L$. That is,

$$L = \sum_{i} i = \sum_{i} (t(i) - \sum_{k=0}^{K} b_k E_{ik})^2$$

The solution for the unknown parameters $b_k$ is
found by taking the derivatives of $L$ with respect to the $b_{k'}$ for $k' = 1,2,\ldots,K$ and setting equal to zero.

$$\frac{\partial L}{\partial b_{k'}} = -2\left(\sum_{i} t(i)E_{ik'} - \sum_{k=1}^{K} b_{k} \sum_{i} E_{i}E_{ik'}\right) = 0$$

or

$$\sum_{k=0}^{K} b_{k} \sum_{i} (E_{i}E_{ik'}) = \sum_{i} t(i)E_{ik'}, \quad k' = 1,2,\ldots,K \quad (2.9)$$

Table 1. Breakdown Times of Insulating Fluid Between Electrodes Recorded at 38 KV Voltage

<table>
<thead>
<tr>
<th>i</th>
<th>t(i)</th>
<th>E(F(T(i)))</th>
<th>E1o</th>
<th>E11</th>
<th>E12</th>
<th>E13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.09</td>
<td>0.11111</td>
<td>1</td>
<td>0.31333</td>
<td>0.12500</td>
<td>0.03125</td>
</tr>
<tr>
<td>2</td>
<td>0.39</td>
<td>0.22222</td>
<td>1</td>
<td>0.48640</td>
<td>0.26786</td>
<td>0.10778</td>
</tr>
<tr>
<td>3</td>
<td>0.47</td>
<td>0.33333</td>
<td>1</td>
<td>0.63213</td>
<td>0.43452</td>
<td>0.25262</td>
</tr>
<tr>
<td>4</td>
<td>0.73</td>
<td>0.44444</td>
<td>1</td>
<td>0.77152</td>
<td>0.63452</td>
<td>0.50643</td>
</tr>
<tr>
<td>5</td>
<td>0.74</td>
<td>0.55556</td>
<td>1</td>
<td>0.91613</td>
<td>0.88452</td>
<td>0.94869</td>
</tr>
<tr>
<td>6</td>
<td>1.13</td>
<td>0.66667</td>
<td>1</td>
<td>1.07852</td>
<td>1.21786</td>
<td>1.76060</td>
</tr>
<tr>
<td>7</td>
<td>1.40</td>
<td>0.77778</td>
<td>1</td>
<td>1.28260</td>
<td>1.71786</td>
<td>3.47864</td>
</tr>
<tr>
<td>8</td>
<td>2.38</td>
<td>0.88889</td>
<td>1</td>
<td>1.60921</td>
<td>2.71786</td>
<td>8.91417</td>
</tr>
</tbody>
</table>

The solutions of (2.9) provide estimates of the $b_{k'}$'s. Using the data in table 1 and the model as follow

$T = b_{0} + b_{1}W^{1/2} + b_{2}W + b_{3}W^{2}$, (2.9) in matrix notation

$$\begin{bmatrix}
8.000 & 7.090 & 8.000 & 16.000 \\
7.090 & 7.567 & 9.634 & 22.187 \\
8.000 & 9.634 & 13.282 & 33.650 \\
16.000 & 22.187 & 33.650 & 95.895
\end{bmatrix} \begin{bmatrix}
b_{0} \\
b_{1} \\
b_{2} \\
b_{3}
\end{bmatrix} = \begin{bmatrix}
7.330 \\
8.600 \\
11.687 \\
29.310
\end{bmatrix}$$
It is not difficult to solve this system of equations. There are several computer packages available. Using MINITAB (Ryan, Joiner, and Ryan 1982) program, the solutions of the parameters $b_k$ are

\[
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
b_3 
\end{bmatrix} = \begin{bmatrix}
-0.3841 \\
1.8566 \\
-0.7559 \\
0.2055 
\end{bmatrix}
\]

The approximation polynomial in (2.6) for this data is

\[
T = -0.3841 + 1.8566w^{1/2} - 0.7559w + 0.2055w^2. \quad (2.10)
\]

The analyst is free to use trial and error in an attempt to obtain an appropriate polynomial. It is also not necessary to have complete samples to use this method of estimation. It is essentially curve fitting. If the data are censored, that portion of the curve is not estimated. The observed data and the approximation polynomial against the empirical cdf will help to judge the smoothness of the approximation polynomial.

The plot of $t(i)$ against $E(F(T(i)))$ for the data in table 1 is presented in figure 1. Table 2 gives the values of $t(i)$ and the approximated values of $t(i)$ using (2.10). Figure 2 presents both the observed $t(i)$ and the approximated values of $t(i)$ plot against $E(F(t(i)))$.

From figure 2 it appears that the agreement is close except for $t(8)$. Figure 3 plots $t(i)$ against the approximation values. The adjusted R-square is equal to
96.8%. It presents the high correlation between the observed data and the estimated values. Moreover, in model (2.10) the regression R-square is 98.9%.

Figure 1. Plot of $t(i)$ against $E(F(T(i)))$
Table 2. The Values of $t(i)$ and the Approximated $t(i)$

<table>
<thead>
<tr>
<th>i</th>
<th>$t(i)$</th>
<th>Approximated $t(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.09</td>
<td>0.16694</td>
</tr>
<tr>
<td>2</td>
<td>0.39</td>
<td>0.36969</td>
</tr>
<tr>
<td>3</td>
<td>0.47</td>
<td>0.52544</td>
</tr>
<tr>
<td>4</td>
<td>0.73</td>
<td>0.66601</td>
</tr>
<tr>
<td>5</td>
<td>0.74</td>
<td>0.80995</td>
</tr>
<tr>
<td>6</td>
<td>1.13</td>
<td>0.97943</td>
</tr>
<tr>
<td>7</td>
<td>1.40</td>
<td>1.22070</td>
</tr>
<tr>
<td>8</td>
<td>2.38</td>
<td>1.69891</td>
</tr>
</tbody>
</table>
Figure 2. Multi-Plot of $E(F(T(i)))$ against $t(i)$ and the Approximated $t(i)$. Points A Denote $t(i)$ and B Denote Approximated $t(i)$. 
Figure 3. Plot of $t(i)$ against the Approximated $t(i)$ with Adjusted R-Square = 96.8%.
CHAPTER III
POLYNOMIAL APPROXIMATION FOR
THE ACCELERATED LIFE MODEL

The survival function $R(t)$ for a life distribution is the probability of (or population fraction) surviving beyond age $y$.

$$R(t) = Pr(T>t) = 1 - F(t),$$

where $T$ is the time-to-failure random variable. In accelerated life data analysis, there are $m$ ordered stress levels, $s_1 < s_2 < \ldots < s_m$. For each stress level $n_1, n_2, \ldots, n_m$ items are put on test respectively. The survival function $R$ for the accelerated life model is assumed to be $R_j$ under stress $s_j$ related to the function under a reference stress $s_0$ as

$$R_j(t) = R_0(g(s_j)t).$$ (3.1)

This form (3.1) is referred to as the accelerated life model (Cox and Oakes; 1983). The function $g(s_j)$ defines the relationship between stress $s_j$ and its response. For convenience in this work the reference stress $s_0$ will be the first level observed.

Let $T_j$ be a time-to-failure random variable for stress $s_j$, and let $T_0$ be the reference. Then (3.1) is equivalent to

$$T_j = g(s_j)T_0.$$ (3.2)

By (2.6), $T_0$ in (3.2) can be approximated by a polynomial with a sufficiently large $K$. That is,
The polynomial approximation method mentioned in the last chapter can be used equally well on the random variable \( Y_j = \ln T_j \) as on \( T_j \). Then (3.2) can be written into

\[
Y_j = \ln g(s_j) + \ln T_0. 
\]  

(3.4)

The random variable \( Y_j \) can be approximated by the form:

\[
Y_j \approx \ln g(s_j) + b_0 + \sum_{k=1}^{K} b_k W^k, 
\]  

(3.5)

where \( W = H(\ln(T_0)) \).

In this accelerated life model, there are \( m \) stress levels \( s_1 < s_2 < \ldots < s_m \) with sample size \( n_1, n_2, \ldots, n_m \), respectively. The expected value of the \( i \)-th order statistic on stress level \( s_j \) is obtained from (2.6) and denoted

\[
E_{ijk} = \frac{n_j! \Gamma(kp+1)}{(n_j-i)!} \sum_{r=0}^{i-1} \frac{(-1)^r (n_j-i+r+1)^{-kp-1}}{r!(i-r-1)!} 
\]  

(3.6)

Now, the expected value of the \( i \)-th order statistic \( Y(i)j \) of the random variable \( Y_j \) for stress \( s_j \) is

\[
E(Y(i)j) = \ln g(s_j) + b_0 + \sum_{k=1}^{K} b_k E_{ijk}, 
\]  

(3.7)

where \( E_{ijk} \) is given by (3.6). Thus \( Y(i)j \) in (3.5) can be expressed as the follows:

\[
Y(i)j = \ln g(s_j) + b_0 + \sum_{k=1}^{K} b_k E_{ijk} + \xi_{ij} 
\]  

(3.8)
Let
\[ G_j = \ln g(s_j) + b_0. \] (3.9)
The least squares method will be used to estimate the parameters, \( b_0, b_1, \ldots, b_K \) and \( G_j, \ j=1,2,\ldots,m. \) Substituting (3.9) into (3.7) results in
\[ Y(i)j = G_j + \sum_{k=1}^{K} b_k E_{ijk} + \epsilon_{ij}. \] (3.10)
Then
\[ \epsilon_{ij} = Y(i)j - G_j - \sum_{k=1}^{K} b_k E_{ijk}. \]

Let \( L = \sum_{i} \sum_{j} \epsilon_{ij}^2 = \sum_{i} \sum_{j} (Y(i)j - G_j - \sum_{k=1}^{K} b_k E_{ijk})^2. \)

Taking the derivatives of \( L \) with respect to those unknown parameters \( G_j \) and \( b_k \) for appropriate \( j \) and \( k, \) and setting these derivatives equal to zero individually results in the following equation.
\[ \frac{\partial L}{\partial G_j} = -2 \left( \sum_{i} Y(i)j - n_j G_j - \sum_{k=1}^{K} b_k \sum_{i} E_{ijk} \right) = 0 \] (3.11)
and
\[ \frac{\partial L}{\partial b_k} = -2 \left( \sum_{i} \sum_{j} Y(i)j E_{ijk} \right) - \sum_{i} n_j G_j \sum_{j} E_{ijk} - \sum_{k=1}^{K} b_k \sum_{i} \sum_{j} E_{ijk} E_{ijk} = 0 \] (3.12)
The equations of (3.11) and (3.12) are equivalent to
\[ n_j G_j + \sum_{k=1}^{K} b_k \sum_{i} E_{ijk} = \sum_{i} Y(i)j \] (3.13)
and

$$\sum_j n_j G_j \sum_i E_{ijk}' - \sum_{k=1}^K b_k \sum_i \sum_j E_{ijk} E_{ijk}' = \sum_i \sum_j Y(i) j E_{ijk}'$$ \hspace{1em} (3.14)

There are \( m \) stress levels for the test and \( k \) appropriate terms for the approximation polynomial. The \((m+k)\) numbers of unknown parameters are to estimate. All the solutions can be obtained from \((m+k)\) equations in (3.13) and (3.14).

By (3.9), a plot of \( G_j \) against \( s_j \) may be used to confirm the functional form of \( g(s) \). After choosing the appropriate model of \( g(s) \), the parameters can be estimated. In next chapter an example is used to illustrate this method, using the constant-lead creep-rupture times data from Garofalo et al. (1961).
CHAPTER IV
EXAMPLE AND DETAILS

Table 3 is from Garofalo et al. (1961) containing creep-rupture times in hours for specimens of stainless steel at different levels in 1,000 psi. The four groups of data for the corresponding stresses are complete data.

Table 3. Creep-Rupture Times for Specimens of Stainless Steel

<table>
<thead>
<tr>
<th>Initial Stress (1,000 psi)</th>
<th>Rupture Time (hours) statistic order (i)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>28.84</td>
<td>1,267</td>
</tr>
<tr>
<td>31.63</td>
<td>170</td>
</tr>
<tr>
<td>34.68</td>
<td>76</td>
</tr>
<tr>
<td>38.02</td>
<td>22</td>
</tr>
</tbody>
</table>

The sample size of each stress $s_j$ is equal to six. Because of the equal sample sizes, the expected value of the $i$-th order statistic at stress $s_j$ with power $k_p$ in (3.6) has the relationships $E_{i1k} = E_{i2k} = E_{i3k} = E_{i4k}$. These expected values can be obtained from the FORTRAN computer program presented in Appendix.

Table 4 presents the logarithm values of table 3 data. Table 5 are these expected values of the $i$-th order statistic with $k_p$ power at each stress $s_j$. In an unbalanced samples case, each group of expected
values are different.

Table 4. The Values of Table 3 in Logarithm

<table>
<thead>
<tr>
<th>Initial Stress (1,000 psi)</th>
<th>28.84</th>
<th>31.63</th>
<th>34.68</th>
<th>38.02</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rupture Time (hours) (ORDER i)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7.1444</td>
<td>5.1358</td>
<td>4.3307</td>
<td>3.0910</td>
</tr>
<tr>
<td>2</td>
<td>7.4006</td>
<td>5.5491</td>
<td>4.4659</td>
<td>3.6109</td>
</tr>
<tr>
<td>3</td>
<td>7.4134</td>
<td>5.5797</td>
<td>4.5644</td>
<td>3.6636</td>
</tr>
<tr>
<td>4</td>
<td>7.4437</td>
<td>6.3456</td>
<td>4.7449</td>
<td>3.7136</td>
</tr>
<tr>
<td>5</td>
<td>7.4872</td>
<td>6.3869</td>
<td>4.8040</td>
<td>3.7377</td>
</tr>
<tr>
<td>6</td>
<td>7.7985</td>
<td>6.6580</td>
<td>4.8828</td>
<td>3.7612</td>
</tr>
</tbody>
</table>

Table 5. The Expected Value of i-th Order Statistic with kp Power for Each Stress

<table>
<thead>
<tr>
<th>POWER (kp)</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>ORDER (i)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.3618</td>
<td>0.1667</td>
<td>0.0556</td>
</tr>
<tr>
<td>2</td>
<td>0.5690</td>
<td>0.3667</td>
<td>0.2022</td>
</tr>
<tr>
<td>3</td>
<td>0.7527</td>
<td>0.6167</td>
<td>0.5106</td>
</tr>
<tr>
<td>4</td>
<td>0.9431</td>
<td>0.9500</td>
<td>1.1439</td>
</tr>
<tr>
<td>5</td>
<td>1.1705</td>
<td>1.4500</td>
<td>2.5939</td>
</tr>
<tr>
<td>6</td>
<td>1.5203</td>
<td>2.4500</td>
<td>7.4939</td>
</tr>
</tbody>
</table>

With the data in table 4 and 5, the coefficients and constant terms in (3.13) and (3.14) can be expressed in matrix notation. Solving the system of equations
results in the estimated parameters $G_j$, for $j = 1, 2, 3,$ and 4, and $b_k$ for $k = 1, 2,$ and 3.

The system of equations expressed in matrix notation is shown as follows:

$$
\begin{bmatrix}
0 & 6 & 0 & 0 & 5.32 & 6.00 & 12.00 & G_1 \\
0 & 0 & 6 & 0 & 5.32 & 6.00 & 12.00 & G_2 \\
0 & 0 & 0 & 6 & 5.32 & 6.00 & 12.00 & G_3 \\
5.32 & 5.32 & 5.32 & 5.32 & 22.37 & 28.20 & 64.11 & G_4 \\
6.00 & 6.00 & 6.00 & 6.00 & 28.20 & 38.20 & 94.42 & b_1 \\
12.00 & 12.00 & 12.00 & 12.00 & 64.11 & 94.42 & 258.00 & b_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
44.688 \\
35.655 \\
27.793 \\
21.578 \\
117.404 \\
134.316 \\
272.658 \\
\end{bmatrix}
$$

The solution of the system of equations in (3.13) and (3.14) in matrix form is

$$
\begin{bmatrix}
G_1 \\
G_2 \\
G_3 \\
G_4 \\
b_1 \\
b_2 \\
b_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
6.324 \\
4.819 \\
3.508 \\
2.472 \\
2.311 \\
-1.112 \\
0.094 \\
\end{bmatrix}
$$

The above regression gives the adjusted multiply correlation coefficient $R^2$ as equal to 98.0%. Thus the model

$$
Y_j = G_j + 2.311W^{1/2} - 1.112W + 0.0941W^2
$$

is an appropriate approximation polynomial.

After confirming the approximation polynomial, the estimated values $G_j$ can be used to investigate the
form of \( g(s) \). Consider the forms \( s^c \), \( e^{-c/s} \), and \( se^{-c/s} \) corresponding to (1.2), (1.3) and (1.4), respectively. These are used as examples to illustrate the use of \( G_j \) to determine the best functional form of \( g(s) \).

By (3.9), \( G_j \) is equal to \( \ln g(s_j) + b_0 \). By taking the logarithms of \( s^c \), \( e^{-c/s} \) and \( se^{-c/s} \), respectively, it follows the \( G_j \) has the forms

\[
G_j = b_0 + c \ln(s_j), \quad (4.2)
\]

\[
G_j = b_0 - c/s_j, \quad (4.3)
\]

and

\[
G_j = b_0 - c/s_j + \ln(s_j). \quad (4.4)
\]

Equation (4.2) indicates a linear relationship between \( G_j \) and \( \ln(s_j) \). From (4.3) it is seen that there is a linear relationship between \( G_j \) and \( 1/s_j \). Equation (4.4) can be modified to express a linear relationship between \( G_j - \ln(s_j) \) and \( 1/s_j \).

Figures 4, 5, and 6 show that all of these three equations are good models representing \( g(s) \) for the data in table 3. The adjusted R-squares are equal to 98.4\%, 99.4\%, and 99.5\%, respectively.

The parameter \( c \) in (4.2), (4.3), or (4.4) can also be estimated from the values of \( G_j \). In figure 6 the equation of the regression line for (4.4) is

\[
(G_j - \ln(s_j)) = -14.1 + 490(1/s_j) \quad (4.5)
\]

The slope in (4.5) is the estimated value of \( c \) and the constant term estimates \( b_0 \). Some further discussions about the estimation of parameters in function \( g(s) \) will
be presented in the next chapter.

Figure 4. Plot of $G_j$ against $\ln(s_j)$ with Adjusted $R^2$ Equal to 98.4%. The Equation of Regression Line Is $G_j = 53.598 - 14.12 \ln(s_j)$.
Figure 5. Plot of $G_j$ against $1/s_j$ with Adjusted R-Square Equal to 99.4%. The Equation of Regression Line Is $G_j = -9.62 + 457 (1/s_j)$. 
Figure 6. Plot of \((G_j - \ln(s_j))\) against \(1/s_j\) with Adjusted R-Square = 99.5%. The Regression Line Is \((G_j - \ln(s_j)) = -14.1 + 490(1/s_j)\).
In this chapter some suggestions for reducing the computational burden of the analysis are given. Then the effect of using polynomials for the accelerated life model is discussed. For the purpose of comparison, the analysis of the Garofalo et al. (1961) data as given by Schmoyer (1986) is repeated in this section using the method developed in Chapter IV. The power rule model for accelerated effect, i.e., \( g(s) = s^c \) is used. The parameter \( c \) is estimated and the lower 90% lower confidence bound for the 0.25 quantile at stress level 22,000 psi is also estimated. These results are compared with Schmoyer's outcomes. Finally, problems for further research are suggested.

In this thesis estimation requires computation of the expected value of a power of the \( i \)-th order statistic of an exponentially distributed random variable. The computation of the value \( E_{ijk} \) is constrained by the computer's round-off errors and overflow. If sample size is greater than 30, then the \( E_{ijk} \) may not be calculated accurately.

The value \( E_{ijk} \) is equal to \( E(-\ln(1-F(T(i))) \), which may be approximated by \( -\ln(1-E(F(T(i))) \). Since \( E(F(T(i))) = i/(n+1) \) it is recommended that \( -\ln(1-i/n+1) \) be substituted for \( E_{ijk} \) in (3.6) for large sample size.
Then the computational problem will be alleviated.

The polynomial used in estimation is not unique. However, the resulting estimates for the different models are very close. For example, consider the approximation for Garofalo's data given in Chapter IV. The polynomial used in that chapter is of the form:

\[ Y_j = G_j + b_1 W^{1/2} + b_2 W + b_3 W^2, \]  

where \( W = H(t) \). Consider another polynomial model

\[ Y_j = G_j + b_1 W + b_2 W^2 + b_3 W^3. \]  

The solution for these unknown parameters in this case is shown in table 6 with the results of model (5.1) from Chapter IV for comparison.

Table 6. Comparison of Polynomial Coefficients

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Model (5.1)</th>
<th>Model (5.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>6.324</td>
<td>6.765</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>4.819</td>
<td>5.260</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>3.508</td>
<td>3.950</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>2.472</td>
<td>2.914</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>2.311</td>
<td>1.653</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>-1.112</td>
<td>-0.829</td>
</tr>
<tr>
<td>( b_3 )</td>
<td>0.094</td>
<td>0.115</td>
</tr>
</tbody>
</table>

For polynomial models (5.1) and (5.2) the adjusted \( R^2 \) are both equal to 98.0%. Consider \( q \) within the interval \((1/7, 6/7)\). Figure 7 shows the hazard function plots for model (5.1) and (5.2) estimated from the data in table 3. The model (5.1) is smoother than
Figure 7. Multi-Plot of $H(t_q)$ against $t_q$ Given in Model (5.1) and Model (5.2). Points A Are for Model (5.1) And Points B Are for Model (5.2).
This illustrates the importance of plotting then choosing a model.

Consider also the effect of model choice on estimation of the acceleration function \( g(s) \). Let \( g(s) = se^{-c/s} \). By (4.4), \( G_j - \ln(s_j) \) is linear in \( 1/s_j \), i.e.,

\[
G_j - \ln(s_j) = b_0 + c/s_j.
\]

Using the \( G_j \) values from table 6 and the associated stress levels \( s_j \), the regression lines for model (5.1) and (5.2) are given by

\[
(G_j - \ln(s_j)) = -14.1 + 490(1/s_j)
\]

and

\[
(G_j - \ln(s_j)) = -13.7 + 490(1/s_j),
\]

respectively. The slopes and intercepts of these two models are essentially the same. The difference in intercepts is compensated for the remaining model term. Thus it doesn't affect the work of extrapolating to a lower stress. For other \( g(s) \) models the results are the same. Thus both (5.1) and (5.2) are appropriate for data in table 3.

The power model \( g(s) = s^c \) has been used in Schmoyer's (1986) analysis of the data in table 3. His estimate of the parameter \( c \) is 13.91 with 95% confidence interval (13.2548, 15.0862). Using polynomial approximation method for the same data and \( g(s) = s^c \), the point estimate of the parameter \( c \) is 14.120 of which is within the confidence interval.

Schmoyer extrapolated the response to stress at 22,000 psi. From his result, a 90% lower confidence
bound for the 0.25 quantile of time-to-failure at stress 22,000 psi is 31,014 hours.

An approximated confidence bound for the quantiles of X can also be derived by using the polynomial method as follows. Let $X_q$ be the q-th quantile of the random variable X. Let Y be the number of observations in a random sample of size n less than or equal to $X_q$. It follows that Y has a binomial distributed with the parameters q and n, that is, $Y : B(q,n)$, where n is the sample size. For data in Table 3, 24 observations were used estimate the model coefficients. So n is equal to 24 and q is 0.25. From binomial table the probability $Pr(4 < Y) = 0.88$. It is close to 0.90. The empirical quantile associated with $X(i)$ is the expected value of the cdf of X evaluated at $X(i)$ which is

$$E(F(X(i))) = i/(n+1).$$

In the term $\ln(s^C) + b_0$ the value of s is equal to 22, $b_0 = 53.598$, and $c = 14.120$. Then the lower 90% confidence bound for the 0.25 quantile is obtained from the polynomial model (5.1)

$$T_q(LCB) = 9.95248 + 2.311(-\ln(1-4/25))^{1/2}$$

$$- 1.112(-\ln(1-4/25)) + 0.0941(-\ln(1-4/25))^2.$$

The 90% LCB at the 0.25 quantile at stress 22,000 psi is 45,544 hours using the polynomial model (5.1). This polynomial approximation method is much simpler.

The analysis presented in this paper presents an experimental design problem. It is noted in Chapter IV
that accelerated life models (4.2), (4.3), and (4.4) fit the data in table 3 equally well. Thus the four stress levels used together are not adequate for determining the best model. The purpose of an accelerated test is to provide information in a more realistic stress environment. This requires extrapolation. The three models considered will not extrapolate the same. Therefore, it is useful to consider the problem of allocating resources to most efficiently estimate the g(s) function. For example, given resources for 24 observations, how many stress levels should be used? This problem is suggested for further research.
REFERENCES


Thomas, R. E. (1964), "When Is a Life Test Truly Accelerated?," Electronic Design, 12, 64-70.
This Program is for computing the expected value of the i-th order statistic of an standard exponential random variable at j-th stress level with pk power.

```
DIMENSION E(10,30)
OPEN(UNIT=1,FILE='THE.DAT',STATUS='NEW')
ID=0
PRINT*, 'ENTER Nj'
READ(5,*) Nj
10 PRINT*, 'ENTER PK, GAMMAK'
READ(5,*) PK, GAMMAK
FACN=1
ID=ID+1
DO 1 I=1,Nj
FACN=FACN*(Nj-I+1.)
FACD=1.
IF (I.EQ.1) GO TO 3
DO 4 L=1,I-1
FACD=FACD*L
4 CONTINUE
3 SIGN=-1.
SUM=0.
DO 5 J=0,I-1
SIGN=SIGN*(-1.)
SUM=SUM+SIGN/((Nj-I+J+1.)*FACD)**(PK+1.)
XIJ=(I-J-1.)
IF(XIJ.EQ.0.) XIJ=1.
FACD=FACD*(J+1.)/XIJ
5 CONTINUE
E(ID,I)=SUM*FACN*GAMMAK
1 CONTINUE
PRINT *, 'ENTER 1 TO CONTINUE'
READ (5,*) IC
IF (IC.EQ.1) GO TO 10
DO 6 I=1,Nj
WRITE (1,100) (E(K,I),K=1,ID)
100 FORMAT (7(X,E15.8))
6 CONTINUE
STOP
END
```