Explanation of the Fast Fourier Transform and Some Applications

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EXPLANATION OF THE FAST FOURIER TRANSFORM
AND SOME APPLICATIONS

by

Alan Kazuo Endo

A thesis submitted in partial fulfillment of the requirements for the degree of
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in
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Alan Kazuo Endo
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ABSTRACT

Explanation of the Fast Fourier Transform and Some Applications

by

Alan Kazuo Endo, Master of Science

Utah State University, 1981

Major Professor: Dr. Ronald Canfield
Department: Applied Statistics

This report describes the Fast Fourier Transform and some of its applications. It describes the continuous Fourier transform and some of its properties. Finally, it describes the Fast Fourier Transform and its applications to hurricane risk analysis, ocean wave analysis, and hydrology.

(44 pages)
CHAPTER I
INTRODUCTION

The Fourier transformation was discovered over a century ago and has become a useful technique with broad applications in engineering and many fields of science. The Fourier transformation is used in physics, mathematics, statistics, computer science, geology and sociology. Computations are very tedious to carry out by hand. With the introduction of the computer in the 1950's it might seem inevitable that the computer would be considered as a suitable vehicle for computing the Fourier transformations; however, early attempts at using the computer were quite discouraging because of the excessive amount of time needed to solve Fourier transformation problems by computer.

In 1965 J. W. Cooley of International Business Machines Corporation Research Center and J. W. Tukey of Princeton University wrote the following article: "An Algorithm for the Machine Calculation of Complex Fourier Series" (Brigham, 1974). This article by Cooley and Tukey presented the basic algorithm of an extremely fast method of numerically calculating a discrete Fourier transformation which made computer assisted solution of Fourier transformation problems practical.

In the following pages this fast algorithm for solving Fourier transformation problem, the fast Fourier Transform, shall be
described as well as some examples of its current applications in reliability theory and wave analysis.
CHAPTER II
PROPERTIES OF THE CONTINUOUS FOURIER TRANSFORM
AND FOURIER SERIES

In this chapter we shall define the continuous Fourier transforma-
tion with some of its properties and the Fourier Series.

The Fourier transformation may be defined as follows:

Definition: The Fourier transformation is a mapping from the
time space \( T' \) to the frequency space \( F \). There is associated an
inverse mapping from the frequency space \( F \) to the time space \( T' \). The

Fourier Transformation:

\[
\begin{align*}
H: & T' \rightarrow F \\
\int_{-\infty}^{\infty} h(t) \exp(-i2\pi ft) dt &= H(f) \quad (2.1)
\end{align*}
\]

and the inverse mapping

\[
\begin{align*}
H: & F \rightarrow T' \\
\int_{-\infty}^{\infty} H(f) \exp(i2\pi ft) df &= h(t) \quad (2.2)
\end{align*}
\]

The following conditions are sufficient for the existence of
a Fourier transformation.

Condition 1. If \( \int_{-\infty}^{\infty} |h(t)| \, dt < \infty \) then Equations 1 and 2 exist. \( (2.3) \)

Condition 2. If \( h(t) = L(t) \sin(2\pi ft + \alpha) \)

\[
f, R \text{ and } L(t+k) = L(t) \quad (2.4)
\]

\( \forall t \, |t| > \alpha > 0 \)
implies
\[ \int_{-\infty}^{\infty} \left| \frac{h(t)}{t} \right| \, dt \] (2.5)
then both the transformation and its inverse exist.

The Fourier transformation has the following properties:

Linearity
\[ X(f) + Y(f) = (X+Y)(f) \] (2.6)
where \( X(f) \) and \( Y(f) \) are Fourier transformations of \( x(t) \) and \( y(t) \) and
\( (X+Y)(f) \) is the Fourier transform of \([x(t)+y(t)]\).

Symmetry
\[ X(f) = X(-f) \] (2.7)
\[ x(t) = x(-t) \] (2.8)

Frequency Scaling
\[ \int_{-\infty}^{\infty} \frac{i}{k} h(t/k) \exp(-i2\pi ft) \, dt = H(fk) \] (2.9)
\[ \int_{-\infty}^{\infty} H(fk) \exp(i2\pi ft) \, df = \frac{1}{k} h(t/k) \] (2.10)

Time Shift
\[ \int_{-\infty}^{\infty} H(t-t_0) \exp(-2\pi ft) \, dt = \exp(-i2\pi ft) h(f) \] (2.11)

Even Function
If \( h(t) = h(-t) \) then \( H(f) \) is a real function
\[ \int_{-\infty}^{\infty} h(t) \cos(2\pi ft) \, dt = \text{Re}(f) \] (2.12)
\( H(f) \) is a real function denoted \( \text{Re}(f) \)

Odd function
If \( h(t) = -h(-t) \) then \( H(f) \) is an imaginary function (2.14)
\[
\int_{-\infty}^{\infty} h(t) \sin(2\pi ft) dt = I(f) \tag{2.15}
\]

\(H(f)\) is an old function denoted as \(I(f)\).

These are a few of the important properties of the Fourier Transformation.

Let us now consider two very important concepts in Fourier transformation theory. The first of these concepts is the convolution.

Definition: Given two function \(x(t)\) and \(h(t)\) such that

\[
y(t) = \int_{-\infty}^{\infty} x(n) h(t-n) dn = x(t) * h(t) \tag{2.16}
\]

then \(y(t) = x(t) * h(t)\) is called the convolution of \(x(t)\) and \(h(t)\).

The second of these important concepts is correlation which we shall now define.

Definition: Let \(x(t)\) and \(h(t)\) be two functions then the correlation function \(Z(t)\) is defined as

\[
Z(t) = \int_{-\infty}^{\infty} x(\lambda) h(t+\lambda) d\lambda \tag{2.17}
\]

Let us next state Parvseval's Theorem.

Theorem: If \(h(t)\) is a function of \(t\) and \(H(f)\) is its transformation, then

\[
\int_{-\infty}^{\infty} h^2(t) dt = \int_{-\infty}^{\infty} |H(f)|^2 df \tag{2.18}
\]

Finally a theorem that relates the F and the T spaces.

Theorem: If \(h(t)\) and \(x(t)\) are functions of time \(t\) and \(H(f)\) and \(X(f)\) their Fourier Transform are functions of frequency \(f\), then
The continuous Fourier Series is defined as follows.

Definition: Given \( y(t) \) is a periodic function of period \( T_0 = 1/f_0 \), \( y(t) \) is defined as a Fourier Series

\[
y(t) = a_0/2 + \sum_{n=1}^{\infty} \left( a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t) \right)
\]

where

\[
a_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \cos(2\pi n f_0 t) \, dt \quad (2.22)
\]

\[
b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \sin(2\pi n f_0 t) \, dt \quad (2.23)
\]

By the identities

\[
\cos(2\pi n f_0 t) = \frac{1}{2} [\exp(i2\pi n f_0 t) + \exp(-i2\pi n f_0 t)] \quad (2.24)
\]

\[
\sin(2\pi n f_0 t) = \frac{1}{2} [\exp(i2\pi n f_0 t) - \exp(-i2\pi n f_0 t)] \quad (2.25)
\]

then

\[
y(t) = a_0/2 + \frac{1}{2} \sum_{n=1}^{\infty} \left( a_n - ib_n \right) \exp(i2\pi n f_0 t) + \frac{1}{2} \sum_{n=1}^{\infty} \left( a_n + ib_n \right) \exp(-i2\pi n f_0 t) \quad (2.26)
\]

by Equation (2.22)
\[
a_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \cos(2 \pi n f_0 t) dt
\]
(2.27)

\[
= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \cos(2 \pi n f_0 t) \ dt
\]

\[
= a_{-n}
\]

By Equation

\[
\begin{align*}
b_n &= -\frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \sin(2 \pi n f_0 t) \ dt \\
&= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \sin[2 \pi (-n)f_0 t] \ dt \\
&= b_{-n}
\end{align*}
\]

then

\[
\sum_{n=-\infty}^{\infty} a_n \exp(-i2 \pi n f_0 t) = \sum_{n=-1}^{\infty} a_n \exp(i2 \pi n f_0 t)
\]
(2.29)

and

\[
\sum_{n=1}^{\infty} i b_n \exp(-i2 \pi n f_0 t) = -\sum_{n=-1}^{\infty} i b_n \exp(i2 \pi n f_0 t)
\]
(2.30)

By Equations 2.25, 2.26, 2.27, 2.28, becomes

\[
y(t) = a_0/2 + 1/2 \sum_{n=-\infty}^{\infty} \left( a_n - i b_n \right) \exp(i2 \pi n f_0 t)
\]
(2.31)

\[
1/2(a_n - i b_n) = 1/2(2/T_0) \int_{-T_0/2}^{T_0/2} y(t) \cos(2 \pi n f_0 t) \ dt
\]
(2.32)
By the identity

\[ \exp(-i2\pi nf_0 t) = \cos(2\pi nf_0 t) - i \sin(2\pi nf_0 t) \]  

for \( n = 0, \pm 1, \pm 2, \ldots \) Let \( \alpha_n = 1/2(a_n - ib_n) \). Then

\[ \alpha_n = 1/T_0 \int_{T_0/2}^{T_0/2} y(t) \exp(-i2\pi nf_0 t) \, dt \] 

therefore

\[ y(t) = \sum_{n=-\infty}^{\infty} d_n \exp(i2\pi nf_0 t) \] 

where

\[ \alpha_n = 1/T_0 \int_{T_0/2}^{T_0/2} y(t) \exp(-i2\pi nf_0 t) \, dt \]

The Fourier Series of a periodic triangular function is an infinite set of sinusoids. This same relationship will be shown by use of the Fourier integral.

Let us first state some important theorem and definitions.

Definition: The Periodic Triangular Function is defined

\[ y(t) = -2/T_0 \mid t \mid + 2/T_0 \quad \mid t \mid < T_0/2 \]
and

\[ y(t) = y(t + T_0 n) \text{ for } n = 0, \pm 1, \pm 2, \ldots \]  

(2.40)

i.e., \( T_0 \) is the period of \( y(t) \).

Definition: The simple Triangle function is defined

\[ h(t) = \begin{cases} -2/T_0 & |t| < T_0/2 \\ 0 & |t| > T_0/2 \end{cases} \]  

(2.41)

Definition: The Equidistant function is defined

\[ x(t) = \begin{cases} 1 & \text{for } t = n T_0 \text{ } \text{ } n = 0, \pm 1, \pm 2, \ldots \end{cases} \]  

(2.42)

Convolutions theorem: If \( H(f) \) is the Fourier transform of \( h(t) \) and \( X(f) \) is the Fourier transform of \( x(t) \) then the Fourier transform of \( h(t) \ast x(t) \) is \( H(f) \times X(f) \).

The convolution of the Simple Triangle function and the Equidistant function is the Periodic Triangle function

\[ y(t) = h(t) \ast x(t) \]  

(2.43)

The Fourier transform of the Equidistant function, \( x(t) \) is

\[ X(f) = 1/T_0 \sum_{n=-\infty}^{\infty} \delta(f - n/T_0) \]  

(2.44)

and the Fourier transform of \( h(t) \) is \( H(f) \).

By the convolution theorem, the Fourier transform of

\[ h(t) \ast x(t) = y(t) \text{ is } H(f) \times X(f) = Y(f). \]  

(2.45)

\[ Y(f) = h(f) \times X(f) = H(f) \sum_{n=-\infty}^{\infty} \delta(f - n/T_0) \]

\[ = \sum_{n=-\infty}^{\infty} H(n/T_0) \delta(f - n/T_0) \]
where \( f = n/T_0 \).

Therefore, the Fourier transform of a periodic function is \( y(t) \) is an infinite sequence with amplitudes \( 1/T_0 \delta(n/T_0) \) at \( n/T_0 \), \( n = 0, \pm 1, \pm 2, \ldots \).

However, \( y(t) \) is a periodic function with period \( T_0 \) so the Fourier Series of \( y(t) \) is

\[
y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \exp(i2\pi f_0 t)
\]

(2.46)

where \( f_0 = 1/T_0 \) and

\[
\alpha_n = \frac{1}{T_0} \int_{T_0/2}^{T_0/2} y(t) \exp(-i2\pi nf_0 t) dt
\]

(2.47)

Since \( h(t) = y(t) - T_0/2 < t < T_0/2 \), let us replace \( h(t) \) by \( y(t) \) then

\[
\alpha_n = \frac{1}{T_0} \int_{T_0/2}^{T_0/2} h(t) \exp(-i2\pi nf_0 t) dt
\]

(2.48)

\[
= \frac{1}{T_0} \delta(nf_0)
\]

\[
\alpha_n = \frac{1}{T_0} \delta(n/T_0) \quad n = 0, \pm 1, \pm 2, \ldots
\]

(2.49)

Therefore, for a periodic function, the Fourier integral and the Fourier series gives us the same coefficients.
CHAPTER III
DISCRETE FOURIER TRANSFORMATION

In this chapter we shall develop the discrete Fourier transformation from the continuous transformation.

Let us define a function to be used in later development, the Dirac Delta.

Definition: \( \delta(x) \) the Dirac Delta is defined as

\[
\delta(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \neq 0
\end{cases}
\]

(3.1)

\[
\int_{-\infty}^{\infty} \delta(x) dx = 1
\]

(3.2)

The first task in constructing a discrete analog for the Fourier Transformation is to define a sampling function.

Definition: If \( h(t) \) is a continuous function, then the sampled function is defined as follows:

\[
h(t) = \sum_{n=-\infty}^{\infty} h(nT) \delta(t-nT)
\]

(3.3)

where \( T \) is the sample interval such that at \( t = nT \) \( \hat{h}(t) = h(t) \) for \( n = 0, \pm 1, \pm 2, \ldots \).

Let us now state two important sampling theorems.

Definition: The Sampling function \( \Delta(t) \) is defined

\[
\Delta(t) = \begin{cases} 
1 & t = nT \quad n = 0, \pm 1, \pm 2, \ldots \\
0 & t \neq nT
\end{cases}
\]

(3.4)
Definition: If $H(f)$ and $\Delta(f)$ are the Fourier transform of $h(t)$ and $\delta(t)$ respectively, the Nyquist Sampling Rate is the frequency $1/T = 2f_c$ such that $H(f) \ast \Delta(t)$ the convolution of $H(f)$ and $\Delta(t)$ will not result in aliasing (Brigham, 1974).

**Theorem:** If $H(f) = 0$ for $|f| > f_c$ then

$$\hat{h}(t) = h(nT) \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad (3.5)$$

and

$$T_c = (1/2)f_c \quad \text{and}$$

$$h(t) = T \sum_{n=-\infty}^{\infty} h(nT) \sin 2f_c (t-nT) \quad (3.6)$$

**Theorem:** If $h(t) = 0$ for $|t| > T_c$ then

$$H(f) = \frac{1}{2T_c} \sum_{n=-\infty}^{\infty} \frac{H(n/2T_c)}{(f-n/2T_c) \sin 2T_c (f-n/2T_c)} \quad (3.7)$$

where $T_c = (1/2)f_c$ and $2f_c$ is the Nyquist sampling rate.

Let us now consider $\Delta(t)$ a continuous function for all real numbers defined as follows:

**Definition:**

$$\Delta(t) = \begin{cases} 
1 & \text{if } t = kT \\
0 & \text{if } t \neq kT
\end{cases} \quad (3.8)$$

where $T$ is the sample interval and $k = 0, \pm 1, \pm 2, \ldots$ Now,

$$h(t)\Delta(t) = \begin{cases} 
h(kT) & \text{if } t = kT \\
0 & \text{if } t \neq kT
\end{cases} \quad (3.9)$$

$$h(t)\Delta(t) = h(t) \sum_{k=-\infty}^{\infty} \delta(t-kT) \quad (3.10)$$
\[ h(t) \Delta(t) = \sum_{k=-\infty}^{\infty} h(kT) \delta(t-kT) \]  
\[ \text{(3.11)} \]

Definition:

\[ x(t) = \begin{cases} 
1 & \text{if } -T/2 < t < T_0 - T/2 \\
0 & \text{otherwise} 
\end{cases} \]  
\[ \text{(3.12)} \]

therefore,

\[ h(t) \Delta(t) x(t) = \sum_{k=-\infty}^{\infty} h(kT) \delta(t-kT) x(t) \]  
\[ \text{(3.13)} \]

\[ = \sum_{k=0}^{N-1} h(kT) \delta(t-kT) \]

where \( N = T_0 / T \)

Now consider \( \Delta_1(t) \)

\[ \Delta_1(t) = \begin{cases} 
T_0 & \text{if } rT_0 = t \\
0 & \text{if } rT_0 \neq t 
\end{cases} \]  
\[ \text{(3.14)} \]

therefore

\[ \Delta_1(t) = T_0 \sum_{r=-\infty}^{\infty} \delta(t-rT_0) \]  
\[ \text{(3.15)} \]

Now let us consider

\[ [ h(t) \Delta(t) x(t) ] * \Delta_1(t) = \]

\[ = \sum_{k=0}^{N-1} h(kT) \delta(t-kT) ] * [ T_0 \sum_{r=-\infty}^{\infty} \delta(t-rT_0) ] \]  
\[ \text{(3.16)} \]
\[ \left[ h(t) \Delta (t) x(t) \right] * \Delta_1(t) = T_0 \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} h(kT) \delta(t-kT-rT_0) \]  

(3.17)

Let us denote \( \left[ h(t) \Delta_0 (t) x(t) \right] * \Delta_1(t) \) as \( \hat{h}(t) \) then \( \hat{h}(t) = h(t) \).

Therefore, we can write \( \hat{h}(t) \) as

\[ \hat{h}(t) = T_0 \sum_{r=-\infty}^{\infty} \left[ \sum_{k=0}^{N-1} h(kT) \delta(t-kT-rT_0) \right] \]  

(3.18)

Now we have developed \( \hat{h}(t) \) an approximation for \( h(t) \) which we will use in the development of the discrete Fourier transformation.

Let us now develop the formula for the discrete Fourier transformation. Assume that \( h(t) \) is a periodic function. Let the Fourier transform of the periodic function be \( H(n/T_0) \) a sequence of equidistant impulse; then \( H(n/T_0) \) is defined as follows:

\[ H(n/T_0) = \sum_{n=-\infty}^{\infty} \alpha_n \delta(f-nf_0) \quad \text{where} \quad f_0 = 1/T_0 \]  

(3.19)

and

\[ \alpha_n = \frac{1}{T_0} \int_{-T/2}^{T_0-T/2} h(t) \exp(-i2\pi nt/T_0) dt \]  

(3.20)

\( n = 0, \pm 1, \pm 2, \ldots \) By substitution we get

\[ \alpha_n = \frac{1}{T_0} \int_{-T/2}^{T_0-T/2} T_0 \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} h(kT) \delta(t-kT-rT_0) \]  

(3.21)

\[ \exp(-i2\pi nt/T_0) dt \]

which becomes

\[ \sum_{k=0}^{N-1} h(kT) \int_{-T/2}^{T_0-T/2} \exp(-i2\pi nt/T_0) \delta(t-kT) dt \]  

(3.22)

and finally becomes
\[ \sum_{k=0}^{N-1} h(kT) \exp(-i\frac{2\pi f}{T/T_0}) \]  \hspace{1cm} (3.23)

Let \( T_0 = NT \) then

\[ \alpha_n = \sum_{k=0}^{N-1} h(kT) \exp(i2\pi kn/N) \]  \hspace{1cm} (3.24)

where \( n = 0, \pm 1, \pm 2, \ldots \). The Fourier transformation of \( \hat{h}(t) \)

\[ \hat{h}(t) = T_0 \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} h(kT) \delta(t-kT-rT_0) \]  \hspace{1cm} (3.25)

is

\[ \hat{H}(n/NT) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{N-1} h(kT) \exp(-i2\pi kn/N) \]  \hspace{1cm} (3.26)

\( \hat{H}(n/NT) \) is periodic with period \( N \), therefore

\[ \hat{H}(n/NT) = \sum_{k=0}^{N-1} h(kT) \exp(-i2\pi nk/N) \]  \hspace{1cm} (3.27)

where \( n = 0, \pm 1, \pm 2, \ldots \).

The discrete Fourier transform is

\[ G(n/NT) = \sum_{k=0}^{N-1} g(kT) \exp(-i2\pi nk/N) \]  \hspace{1cm} (3.28)

where \( n = 0, 1, 2, \ldots, N-1 \).

The inverse Fourier transform is

\[ g(kT) = \frac{1}{N} \sum_{n=0}^{N-1} G(n/NT) \exp(i2\pi nk/N) \]  \hspace{1cm} (3.29)

where \( k = 0, 1, 2, \ldots, N-1 \).

Let us show that Equation (3.28) is the discrete inverse Fourier transform of Equation (3.29).
Since
\[
G(n/NT) = \sum_{k=0}^{N-1} \frac{1}{N} \sum_{r=0}^{N-1} G(r/NT) \exp(i2\pi r/N) \exp(-i2\pi nk/N)
\]  
(3.30)

\[
= \frac{1}{N} \sum_{r=0}^{N-1} G(r/NT) \left[ \sum_{k=0}^{N-1} \exp(i2\pi kr/N) \exp(-i2\pi nk/N) \right]
\]

\[
= G(n/NT)
\]

Therefore the discrete Fourier transform and the discrete inverse transform can be given as follows.

The discrete Fourier transform formula
\[
G(n/NT) = \sum_{k=0}^{N-1} g(kT) \exp(-i2\pi nk/N)
\]  
(3.32)

where \( n = 0, 1, 2, \ldots, N-1 \).

The discrete inverse Fourier transform
\[
g(kT) = \sum_{n=0}^{N-1} G(n/NT) \exp(i2\pi nk/N)
\]  
(3.33)

where \( n = 0, 1, 2, \ldots, N-1 \).

Denote a discrete Fourier transform as \( \mathcal{F}^+ \) and a discrete inverse Fourier transform as \( \mathcal{F}^- \). Some properties of the discrete Fourier transform and the discrete inverse Fourier transform will now be described.

The \( n/NT \) represents a point in the frequency space \( F \) and \( kT \) represents a points in the time space \( T' \). \( N \) represents the number of points sampled from either the frequency space or the time space.
T is the sample interval in T'. Both k and N take values 0, ±1, ±2, .... For notational convenience n shall be used in place of (n/NT) and k shall be used in place of kT.

**Linearity.** If,

\[ T^{-} [X(k)] = x(n) \]  \[ T^{-} [Y(k)] = y(n) \]

then

\[ T^{-} [X(k) + y(k)] = x(n) + y(n) \]  \[ (3.34) \]

**Symmetry**

\[ T^{-} 1/N H(k) = h(-n) \]  \[ (3.36) \]

**Time shifting**, If

\[ T^{-} (H(n)) = h(r) \]  \[ (3.37) \]

then

\[ T^{-} (h(k) \exp(-i2\pi mk/N)) = H(n-m) \text{ where } r=k-i \]  \[ (3.38) \]

**Frequency shift**, if

\[ T^{+} (h(k)) = H(r) \]  \[ (3.39) \]

then

\[ T^{+} (h(k) \exp(-i2\pi mk/N)) = H(n-m) \]  \[ (3.40) \]

where \( r = n-m \).

**Even Function**, if

\[ h_e(k) = h_e(k) \]  \[ (3.41) \]

then \( h_e(k) \) is an even function and

\[ T^{-} (h_e(k)) = \sum_{k=0}^{N-1} h_e(k) \cos(2\pi nk/N) \]  \[ (3.42) \]

where the resulting function is a real valued function.
Odd Function

If \( h_0(k) = -h_0(-k) \) then \( h_0(k) \) is an odd function and

\[
\int_{-N}^{N} H_0(k) = -i \sum_{k=0}^{N-1} h_0(k) \sin(2\pi nk/N) \tag{3.44}
\]

where the resulting function is complex valued.

Let us now consider a means by which we may decompose an arbitrary function into its odd and even parts.

Given a function \( h(k) \)

\[
h(k) = \left[ \frac{h(k)}{2} + \frac{h(-k)}{2} \right] \tag{3.45}
\]

\[
= \left[ \frac{h(k)}{2} + \frac{h(-k)}{2} \right] + \left[ \frac{h(k)}{2} - \frac{h(-k)}{2} \right]
\]

then \( \left[ \frac{h(k)}{2} + \frac{h(-k)}{2} \right] \) is an even function denoted by \( h_e(k) \) and

\( \left[ \frac{h(k)}{2} - \frac{h(-k)}{2} \right] \) is an odd function denoted by \( h_o(k) \). Therefore,

\[
h(k) = h_e(k) + h_o(k)
\]

\[
\int_{-N}^{N} [h_e(k) + h_o(k)] = H_e(n) + H_o(n) \tag{3.46}
\]
CHAPTER IV
THE FAST FOURIER TRANSFORM

In this chapter the Fast Fourier Transform (FFT) will be described. The FFT is a very fast algorithm that computes the discrete Fourier transformation. This algorithm is the discrete analog of the continuous Fourier transform. It allows for rapid computation of the discrete Fourier transformation by decreasing the number of actual complex computer additions and multiplications needed to perform this computation.

The FFT was first published by Cooley and Tukey (1965). While the origin of this algorithm is clouded in some controversy, it is accepted that most of the computer subroutines called "Fast Fourier Transform" are modifications of their algorithm (Brigham, 1974).

Let us describe the Fast Fourier Transform which was discovered by Cooley and Tukey.

Let $X(n/NT)$ be the discrete Fourier transform of $x_0(kT)$ then

$$X(n/NT) = \sum_{k=0}^{N-1} x_0(kT) \exp(-i2\pi nk/N) \tag{4.1}$$

where $n = 1, 2, 3, \ldots N-1$.

Let us now replace $n/NT$ by $n$ in $X(n/NT)$ and $kT$ by $k$ in $X_0(kT)$.

Equation (4.1) becomes

$$X(n) = \sum_{k=0}^{N-1} x_0(k) \tilde{w}^{nk} \tag{4.2}$$

where $N=2^a$ is the number of sample points used in the discrete
Fourier transform

\[ W = \exp(-i2\pi/N) \quad (4.3) \]

\( n \) and \( k \) are both written in binary form such that

\[
\begin{align*}
1 &= 1 \cdot 2^0 \\
2 &= 0 \cdot 2^0 + 1 \cdot 2^1 \\
3 &= 1 \cdot 2^0 + 1 \cdot 2^1 \\
4 &= 0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 \\
\vdots & \\
\end{align*}
\]

\[
\begin{align*}
n &= \sum_{b=1}^{a} 2^{b-1} n_{b-1} \quad \text{where } n_{b-1} = 0 \text{ or } 1 \text{ and } b = 1, 2, \ldots, a \\
\end{align*}
\]

and \( k \) can be written in binary form as follows:

\[
\begin{align*}
k &= \sum_{b=1}^{a} 2^{b-1} k_{b-1} \quad \text{where } k_{b-1} = 0 \text{ or } 1 \text{ and } b = 1, 2, \ldots, a \\
\end{align*}
\]

Therefore, Equation (4.1) becomes

\[
X \left( \sum_{b=1}^{a} 2^{b-1} k_{b-1} \right) = \sum_{k=0}^{N-1} X_0 \left( \sum_{b=1}^{a} 2^{b-1} k_{b-1} \right) W^{nk} \\
\]

or written as;

\[
X(n_{q-1}, n_{q-2}, \ldots, n_0) \\
= \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \ldots \sum_{k_a=0}^{1} X_0(k_{a-1}, k_{a-2}, \ldots, k_0) W^p \\
\]

where

\[
p = \sum_{t=1}^{b} \sum_{b=0}^{t} \left[ (2^{b-1} n_{b-1})(2^{b-t} k_{b-t}) \right] \\
\]

Therefore with this notation we can write the original equation recursively as follows:
This is the Cooley-Tukey algorithm. It is based on the fact that the number of points used in the computation is a power of 2. Data which contains exactly \(2^N\) points can use the FFT by simply employing the FFT on the \(2^N\) points; however, should the number of points be other than \(2^N\), we make the following observation.

Let \(k\) be an even integer, then \(k\) can be written as a sum of powers of two's as follows:

Let \(a_i = 0\) or \(1\) then 
\[
    k = \sum_{i=1}^{t} a_i 2^i
\]
where \(i\) and \(t\) are positive integers.

Therefore, if we are given \(k\) points we can simply employ the Fast Fourier Transform on \(2^M\) points at a time (\(M\) is an integer greater
than zero) until we have used the FFT on all k points. For example, if we are given k = 22 points we would use the algorithm on the first 16 points; then we would use the algorithm on the next 4 points; finally, we would use the algorithm on the last 2 points.

The speed on an algorithm is a function of the number of actual additions and multiplications needed to perform a task. The direct method requires \( N^2 \) complex multiplication to perform a discrete Fourier transformation; however, the FFT requires \( Nq/2 \) complex multiplications and \( Nq \) complex additions to perform a discrete Fourier transformation on \( 2^q = N \) points. If we have \( N = 1024 \) points the number of calculations required by the usual discrete Fourier Transformation is 1,048,576 complex multiplications while the FFT requires 5120 complex multiplications and 2560 complex additions (Brigham, 1974).

Let us denote the total number of computations needed in the discrete Fourier transform as \( \text{Total}_{\text{dft}} \) and the total number of computations needed using the FFT as \( \text{Total}_{\text{fft}} \).

\[
\frac{\text{Total}_{\text{dft}}}{\text{Total}_{\text{fft}}} = \frac{N^2}{Nq+Nq/2}
\]

(4.14)

where \( 2^q = N \).

\[
\lim \frac{N^2}{Nq+Nq/2} = \lim \frac{2N}{3\log_2 N} = \lim \frac{1}{1/N} = \infty \lim \frac{N}{q+q/2} = \infty
\]

\( N \to \infty \quad N \to \infty \quad N \to \infty \quad N \to \infty \)

(4.15)

That is to say as the number of points increases the ratio becomes arbitrarily large and so the efficiency of the FFT increases as the number of points increases.
CHAPTER V
APPLICATIONS OF THE DISCRETE FOURIER TRANSFORM

The Fast Fourier Transformation has application which makes the FFT a very important breakthrough in computational mathematics. If the Fast Fourier Transform were merely an interesting algorithm void of applications it would stand as a significant contribution to numerical analysis; however, the great number of potential applications makes this discovery even more significant. In the following pages three applications of the FFT shall be given to indicate the applicability of the FFT to a broad range of problems.

The first example of an application of the FFT is in hurricane risk analysis. In "Some New Techniques for Risk Analysis," by Leon E. Borgman (1977) the FFT was used to assess the potential danger of hurricane damage.

The following model was given.

1. \( P_N(n) = P(N=n) \) is the probability that \( n \) hazardous events occur in a given time period \((0, t)\).

2. A hazardous event has probability \( P_0 \) of occurring and probability \( 1-P_0 \) of not occurring.

3. Damage \( D \) related to each damaging event is independent of damage in other situations. The distribution function and the probability density function are given as follows:

\[
F_D(d) = P(D \leq d) \text{ a single damaging event}
\]
4. Let $D_1, D_2, \ldots, D_N$ be damage from $N$ hazardous events, then

Total Damage is

$$S = D_1 + D_2 + \ldots + D_N$$

(5.2)

$$F_S(d) = P(S \leq d)$$

(5.3)

$$f_S(d) = \frac{dF_S(d)}{d}$$

(5.4)

(Borgman, 1977)

The primary objective is to determine $F_S(d)$ and $f_S(d)$ given $P_N(n)$, $F_D(d)$, and $f_D(d)$.

$F_S(d)$ and $f_S(d)$ shall be determined by the following relationship.

$$\phi_X(u) = \int_{-\infty}^{\infty} \exp(izu) f_x(x)dx$$

(5.5)

If $D_1$ and $D_2$ are damaging events the

$$f_S(d) = \int_{-\infty}^{\infty} f_{D_1}(z) f_{D_2}(d-z) dz$$

(5.6)

This is the convolution relation

$$f_S(d) = f_{D_1}(d) \ast f_{D_2}(d)$$

(5.7)

This relation implies

$$\phi_S(u) = \phi_{D_1}(u) \phi_{D_2}(u)$$

(5.8)

Since the convolution of $f_{D_1}(d)$ and $f_{D_2}(d)$ in Equation (5.7) is the product of the Fourier Transform of $f_{D_1}(d)$ and the Fourier transform of $f_{D_2}(d)$ which are $\phi_{D_1}(u)$ and $\phi_{D_2}(u)$, respectively.

If there exist $N$ IID damaging events, then
\[ f_S(d) = f_D(d) \cdot f_D(d) \cdot \ldots \cdot f_D(d) \] (5.9)

and

\[ \phi_S(u) = [ \phi_D(u) ]^N \] (5.10)

If \( S = \sum_{i=1}^{N} D_i \) where \( N \) is random then

\[ F_S(d) = \sum_{n=1}^{\infty} P(N=n) P(S \leq d/n \text{ events}) \] (5.11)

\[ = P_N(0) + \sum_{n=1}^{\infty} P_N(n) f_{S/n}(d) \]

where \( P(S \leq d \mid 0 \text{ events}) = 1 \) (5.12)

By differentiation we get

\[ f_S(d) = P_N(0) \delta(d) + \sum_{n=1}^{\infty} P_N(n) f_{S/n}(d) \] (5.13)

\[ \frac{d}{d} [f_S(d)] = \phi_S(u) \] (5.14)

\[ \frac{d}{d} [f_{S/n}(d)] = \frac{d}{d} [f_D(d) \cdot \ldots \cdot f_D(d)] = [ \phi_D(u) ]^n \] (5.15)

\[ \frac{d}{d} P_N(0) \, (d) = P_N(0) \] (5.16)

Therefore Equation (5.11) becomes the following:

\[ \phi_S(u) = P_N(0) + \sum_{n=1}^{\infty} P_N(n) [ \phi_D(u) ]^n \] (5.17)

when we take the Fourier transform of Equation (5.11)

Let \( G_N(t) = \sum_{n=0}^{\infty} P_N(n) \, t^n \), (5.18)

\( G_N(t) \) is the probability generating function of \( N \), then

\[ \phi_S(u) = G_N[\phi_D(u)] \] (5.19)
Let us now define the probability that a hazardous event is also a damaging event. Let us call it $D_0$.

\[
D_0 = \begin{cases} 
0 & \text{with probability } 1 - P_0 \\
D & \text{with probability } P_0
\end{cases}
\]  

(5.20)

then $P(D_0 \leq d_0) = P(D_0 = 0) + P(\text{damaging event})P(0 \leq D_0 \leq d)$ (Borgman, 1977)

\[
F_{D_0}(d) = (1 - P_0) + P_0 F_D(d)
\]  

(5.21)

\[
f_{D_0}(d) = (1 - P_0) \delta(d) + P_0 f_D(d)
\]  

(5.22)

by the Fourier transform we have that

\[
\phi_{D_0}(u) = (1 - P_0) + P_0 \phi_D(u)
\]  

(5.23)

Let $D_0$ be denoted as $D$ then

\[
\phi_S(u) = G_N \left[ (1 - P_0) + P_0 \phi_D(u) \right]
\]  

(5.24)

Now with this mathematical development we may determine the probability law for $S$, the total damages, using the following chain of equations.

\[
\phi_D(u) = \int_{-\infty}^{\infty} f_D(d) \exp(iud) \, d(d)
\]  

(5.25)

\[
\phi_{D_0}(u) = 1 - P_0 + P_0 \phi_D(u)
\]  

(5.26)

\[
\phi_S(u) = G_N \left[ (1 - P_0) + P_0 \phi_D(u) \right]
\]  

(5.27)

\[
f_S(d) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_S(u) \exp(-iux) \, du
\]  

(5.28)
This completes the continuous function development of this example. We shall now develop the discrete analog.

The discrete analog of the continuous model for the probability law of $S$, the total damages uses these two Fourier transforms

$$\phi(u) = f(d) \exp(iux) \, d(d) \quad (5.30)$$

and

$$f(d) = \frac{1}{2} \int_{-\infty}^{\infty} \phi(u) \exp(-iux) \, du \quad (5.31)$$

where $f(d)$ is the density for damage in a damaging event and $\phi(u)$ is the Fourier transform of $f(d)$.

Definition: $F_D\ j$ is the discrete formula for the density for damage in a damaging event $D$. $D_k$ is the discrete Fourier transform of $F_D\ j$ such that

$$\phi_D\ k = \phi[k(\Delta u)] \text{ for } 0 \leq k \leq L-1 \quad (5.32)$$

$$f_D\ j = f[j(\Delta d)] \text{ for } 0 \leq j \leq L-1 \quad (5.33)$$

where $\Delta u$ is an increment on the $u$-axis and $\Delta d$ is an increment on the $d$-axis (Borgman, 1977).

The discrete Fourier transform is

$$\phi_D\ k = \Delta d \sum_{j=0}^{L-1} f_D\ j \exp(i2\ jk/L) \quad (5.34)$$

$$f_D\ j = \Delta u/2\pi \sum_{k=0}^{L-1} \phi_D\ k \exp(-2\ jk/L) \quad (5.35)$$

where $\Delta d$ is an increment on the $d$-axis and $u$ is an increment on the $u$-axis.
The discrete convolution is

\[ f_{S,j} = f_{D_1,j} * f_{D_2,j} = \Delta d \sum_{m=0}^{L-1} f_{D_1,m} f_{D_2,j-m} \quad (5.36) \]

\( f_{S,j} \) is the probability density of \( S \), the sum of two independent damaging events \( D_1 \) and \( D_2 \) at \( j \). \( f_{D_1,j} \) is the probability density of the damaging event \( D_1 \) at \( j \). \( f_{D_2,j} \) is the probability density of the damaging event \( D_2 \) at \( j \). Finally, \( f_{D_1,m} \) and \( f_{D_2,m-j} \) are the probability density of the damaging events \( D_1 \) and \( D_2 \) at times \( m \) and \( j-m \), respectively.

The Fourier transform of \( f_{s,j} \) is

\[ \phi_{s,k} = \Delta d \sum_{j=0}^{L-1} \Delta d \sum_{m=0}^{L-1} f_{D_1,m} f_{D_2,j-m} \exp(i2\pi jk/L) \quad (5.37) \]

therefore \( f_{D,j} \) and \( F_{D,j} \) are defined as follows:

\[ F_{D,j} = F_D(j+0.5d) \quad (5.38) \]

\[ f_{D,j} = F_{D,j} - F_{D,j-1} / \Delta d \quad (5.39) \]

using \( f_D(d) \) we can define \( f_{D,j} \) and \( F_{D,j} \) as follows

\[ f_{D,j} = \begin{cases} f_D(0)/2 & \text{if } j=0 \\ f_D(j\Delta d) & \text{if } j>0 \end{cases} \quad (5.40) \]

\[ F_{D,j} = \sum_{j=0}^{L-1} f_{D,j} \exp(i2\pi jk/L) \quad (5.41) \]

then we have

\[ \phi_{s,k} = G_N \left[ 1-P_0 + P_0 \phi_{D,k} \right] \quad (5.42) \]

\[ f_{s,j} = \Delta d \sum_{m=0}^{L-1} f_{D_1,m} f_{D_2,j-m} \quad (5.43) \]
The FFT was employed in a subroutine DAMAGE which determines the potential damage of a hurricane.

Given a graph of $F_D(d)$ the distribution function of a potentially damaging event $D$ being less than or equal to $d$ and $P_N(n)$ the probability of having $n$ hazardous events and $P_0$ the probability that a hazardous event will become a damaging event. The subroutine DAMAGE will take the information described above and will determine $F_S(d)$ the distribution function of the sum of damaging events being less than or equal to $d$.

First the $F_D(d)$ is transformed into its discrete analog by the following formula

$$F_D[j] = P[D \leq (j + 0.5) \Delta d]$$

$$= F_D[(j + 0.5) \Delta d]$$

By ( ) the value of $F_D(d)$ is determined at $j = (j + 0.5) \Delta d$ for $j = 0, 1, 2, \ldots, L-1$. Then the discrete analog of $f_D(d)$ is

$$f_D[j] = \frac{F_D[j] - F_D[j-1]}{d}$$

By using the FFT the numeric analog of the characteristic function $\phi_D(u)$ is determined as follows:

$$\phi_D[k] = \Delta d \sum_{j=0}^{L-1} f_D[j] \exp(i2\pi jk/L)$$

Then

$$S_k = G_N(1 - P_0 + P_0 D^k)$$

$$= P_N(0) + \sum_{n=1}^{\infty} P_N(n) [1 - P_0 + P_0 \phi_D[k]^n]$$
however,

\[ P_N(n) = 0 \text{ for } n > N = L-1 \]  \hspace{1cm} (5.48)

Therefore

\[ \phi_S k = P_N(0) + \sum_{n=1}^{L-1} P_N(n) (1-P_0+P_0 \phi_D k)^n \]  \hspace{1cm} (5.49)

The FFT is again employed on the discrete characteristic function \( \phi_S k \) to determine \( F_S j \) the distribution function of a sum of damaging events by

\[ F_S j = u/2 \sum_{k=0}^{L-1} \phi_S k \exp(-i2 jk/L) \]  \hspace{1cm} (5.50)

Then the subroutine DAMAGE will give a \( F_S j \).

The second example of a use of the FFT is in ocean wave analysis. In Borgman (1973) the FFT was used to determine the spectra of ocean wave action during Hurricane Carla (September 1, 1961). He used a discrete analog of the \( p(f) \), the spectral density function to determine an estimate of the spectral lines at a given frequency \( f \). Then a moving average of the spectral lines were computed to produce estimates of the spectral density.

The spectral density function gives the distribution in frequency of a periodic phenomena. The spectral density function and the covariance function are Fourier transform pairs. Let us consider how this density function is used.

**Definition:** Let \( x(t) \) be a function of time space \( T' \), then for a given complex number \( \tau \) the covariance function is

\[ C(\tau) = \text{Cov} \{ x(t), x(t+\tau) \} \]

\[ = E \{ x(t), x(t+\tau) \} \]

where \( E(\ ) \) is the expectation of \( (\ ) \).
Definition: The power of $X(t)$ is the electrical analog of the following: If $X(t)$ is the voltage across a pure resistance of one ohm then

$$E(X^2) = \int_{-\infty}^{\infty} p(f) \, df \quad (5.52)$$

is proportional to the average power that will be dissipated in the resistance.

Definition: The spectral density $p(f)$ and its Fourier transform pair $C(\tau)$ is defined as follows:

$$p(f) = \int_{-\infty}^{\infty} C(\tau) \exp(-i2\pi f \tau) \, d\tau \quad (5.53)$$

$$C(\tau) = \int_{-\infty}^{\infty} p(f) \exp(i2\pi f \tau) \, df \quad (5.54)$$

where $\tau$ is a complex number and $f$ is a frequency in the frequency space $F$.

$p(f)$ called the spectral density function gives the density of the distribution of the power of $x(t)$.

Finally,

$$R(0) = \int_{-\infty}^{\infty} p(f) \, df \quad (5.55)$$

$$R(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} x(t) \, dt \quad (5.56)$$

where

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty \quad (5.57)$$

This $R(0)$ provides an average power of $x(t)$, when $x(t)$ is a continuous process.
Let us now show how the spectral density is estimated, by first making some basic definitions.

**Definition:** \( \eta(t) \) is the water elevation above the mean water level.

**Definition:** The Covariance function is

\[
C(\lambda) = E \left[ \eta(t) \eta(t+\lambda) \right]
\]

where \( \eta(t) \) is defined above.

**Definition:** The spectral density function \( p(f) \) is

\[
p(f) = \int_{-\infty}^{\infty} E \left[ \eta(t) \eta(t+\lambda) \right] \exp(-i2\pi f \lambda) \, d\lambda
\]

where \( E \left[ \eta(t) \eta(t+\lambda) \right] \) is the covariance function.

The spectral density function is the Fourier transform of the covariance that the function \( \eta(t) \) has at frequency \( f \).

Let us now develop the discrete analog of the spectral density function.

**Definition:** \( \eta_k \) is the height of the ocean wave above the mean ocean level in the \( k \)th observation of height of the ocean wave above the mean ocean level.

**Definition:** The Discrete Covariance function is

\[
\hat{C}_k = \frac{1}{N} \sum_{n=0}^{N-1} \eta_n \eta_{n+k}
\]

\[
C_k = C_{-k} = C_{n-k}
\]

and

\[
\eta_n = \eta_{n-N}
\]
Since \( C(\lambda) \) is symmetric about \( 0 \), \( p(f) \) is a real valued function defined as follows

\[
p(f) = \int_{-\infty}^{\infty} C(\lambda) \cos(2\pi f \lambda) \, d\lambda
\tag{5.63}
\]

Therefore the discrete analog of (5.53) is

\[
\hat{p}(f) = \sum_{k=-N/2+1}^{N/2} C_k \cos(2\pi f m \Delta t) \Delta t
\tag{5.64}
\]

Substituting \( f = m/N \Delta t \)

\[
\hat{p}(f_m) = \Delta t \sum_{k=0}^{N-1} \left[ \frac{1}{N} \sum_{n=0}^{N-1} \eta_n \eta_{n+k} \exp(-i2\pi mk/N) \right]
\tag{5.65}
\]

Therefore

\[
\hat{p}(f_m) = \frac{\Delta t}{N} \left[ \sum_{n=0}^{N-1} \eta_n \exp(i2\pi mn/N) \right]
\tag{5.66}
\]

\[
\sum_{k=0}^{N-1} \eta_k \exp(-i2\pi mk/N) = \bar{A}_m A_m
\tag{5.67}
\]

\[
A_m = \Delta t \sum_{n=0}^{N-1} \eta_n \cos(2\pi mn/N) - i\Delta t \sum_{n=0}^{N-1} \eta_n \sin(2\pi mn/N)
\tag{5.68}
\]

\[
A_m = U_m - iV_m
\tag{5.69}
\]

The complex conjugate of \( A_m \) is

\[
\bar{A}_m = \Delta t \sum_{n=0}^{N-1} \eta_n \cos(2\pi mn/N) + i\Delta t \sum_{n=0}^{N-1} \eta_n \sin(2\pi mn/N)
\tag{5.70}
\]

\[
\bar{A}_m = U_m + iV_m
\tag{5.71}
\]

In Borgman (1973) 4096 points were employed by the FFT. Letting \( \eta_k \) represent the wave elevation above the mean water level for \( 0 \leq n \leq N \).
\[ A_m = \Delta t \sum_{n=0}^{N-1} \eta_n \exp(-i2\pi mn/N) \quad (5.72) \]

\[ A_m \text{ is the output of } (\quad), \text{ the discrete Fourier transform. Then } \]

the product of \( A_m \) and \( \bar{A}_m \) is taken such that

\[ \hat{p}(f) = \frac{(A_m \bar{A}_m)}{N \Delta t} \quad (5.73) \]

\( p(f_m) \) represents the estimate of the spectral line for frequency \( f_m \). These estimates \( p(f_m) \) are weighted as follows, so that your estimate \( p(f_m) \) will be consistent with the covariance method of computing wave spectrum.

\[ \hat{p}(f_m) = \frac{\sum w_j \hat{p}(f_{m-j})}{\sum w_j} \quad (5.74) \]

where \( w_j = \exp(-j^2/1.8) \)

Finally, \( \hat{p}(f_m) \) becomes the estimate the spectra density function at \( f_m \)'s.

The third example of a use of the Fast Fourier Transform is in the analysis of hydrographs. Levi and Remigio (1964) describe how a discrete Fourier transform can be used to decompose a hydrograph. The decomposition of a hydrograph particularly lends itself to the use of the Fast Fourier Transform because it involves the use of a discrete Fourier transform.

The primary objective of this analysis of a hydrograph is to decompose the channel flow and effective precipitation into \( N \) simple hydrographs with maximums at \( \lambda_0, \lambda_1, \ldots, \lambda_{N-1} \), and starting times \( t_0, t_1, \ldots, t_{N-1} \). The simple hydrographs of channel flow and
effective precipitation are related by means of the instantaneous unit hydrograph.

Let us first consider the continuous model before presenting the discrete analog.

Definition: The Channel Flow is a function $C(t)$ relating the deviation from the mean due to a storm of flow volume. It may be written:

$$C(t) = \begin{cases} 0 & 0 > t \\ C(t) & 0 \leq t \leq t_1 \\ 0 & t > t_1 \end{cases}$$  \hspace{1cm} (5.75)

where $t = 0$ indicates the beginning of precipitation.

Definition: The Effective Precipitation is written:

$$E(t) = \begin{cases} 0 & 0 > t \\ E(t) & 0 \leq t \leq t_2 \\ 0 & t > t_2 \end{cases}$$  \hspace{1cm} (5.76)

where $t = 0$ indicates the beginning of effective precipitation.

Definition: The Instantaneous Unit Hydrograph $H_0(t)$ is defined as the function

$$H_0(t) = \begin{cases} 0 & t > 0 \\ H_0(t) & 0 \leq t \leq t_3 \\ 0 & t > t_3 \end{cases}$$  \hspace{1cm} (5.77)

where $H_0(t)$ has a maximum of 1 and which relates the channel flow to the effective precipitation in the following manner: Channel Flow, $C(t)$, is the Convolution of the Instantaneous Unit Hydrograph and the Effective Precipitation.
\[ C(t) = \int_{-\infty}^{t} E(\tau) H_0(t - \tau) \, d\tau \quad (5.78) \]

By the convolution property of Fourier transform

\[ C(t) = \int_{-\infty}^{t} E(\tau) H_0(t - \tau) \, d\tau = \int_{-\infty}^{\infty} \mathcal{F}[E][H_0] \exp(its) \, ds \quad (5.79) \]

Definition: The Simple Hydrograph \( H_n(t) \) is defined as follows:

\[ H_n(t) = \begin{cases} 0 & t < t_{n+3} \\ H_0(t) & t_{n+3} \leq t \leq t_{n+4} \\ 0 & t > t_{n+4} \end{cases} \quad (5.80) \]

where \( H_n(t) \) has a maximum of 1 at

\[ t \in [t_{n+3}, t_{n+4}], \text{ i.e. } H_n(t-T_n) = H_0(t). \quad (5.81) \]

In the continuous model

\[ C(t) = \sum_{n=1}^{N} \lambda_n H_n(t) \quad (5.82) \]

Applying the Fourier Transform to \( C(t) \)

\[ \mathcal{F}[C] = \int_{-\infty}^{\infty} \exp(-ist) C(t) \, dt \quad (5.83) \]

Applying the Fourier Transform to Equation (5.81)

\[ \mathcal{F}[H_n] = \exp(-i\tau_n s) \mathcal{F}[H_0] \]

Using Equation (5.81), (5.82), and (5.83)

\[ C = \sum_{n=1}^{N} \lambda_n \exp(-i\tau_n s) \mathcal{F}[H_0] \quad (5.84) \]

Therefore
\[ \frac{1}{2} [C] = \sum_{n=1}^{N} \lambda_n \exp(-i\tau_n s) \quad (5.85) \]

let

\[ E(t) = \sum_{n=1}^{N} \lambda_n \delta(t-\tau_n) \quad (5.86) \]

where \( \tau_n - \tau_{n-1} = \Delta t \) \( n = 2, 3, \ldots, N+1 \) and \( \delta(t) \) is the Dirac Delta.

Then by Dirichlet series

\[ \sum_{n=1}^{N} \lambda_n \exp(-i\tau_n s) = \int_{-\infty}^{\infty} \sum_{n=1}^{N} \lambda_n \delta(t-\tau_n) \exp(-its) \, dt \quad (5.87) \]

By the Fourier transform of the right side of Equation — we get,

\[ \int_{-\infty}^{\infty} E(t) \exp(-its) \, dt = \frac{1}{2} [E] \quad (5.88) \]

Therefore,

\[ \frac{1}{2} [C] = \frac{1}{2} [E] \quad (5.89) \]

Taking the inverse Fourier transform of the left side of Equation —

\[ \frac{1}{2} [C] = E(t) \quad (5.90) \]

where \( E(t) = \sum_{n=1}^{N} \lambda_n \delta(t-\tau_n) \) is the graph of \( N \) peaks with maximum \( \lambda_1, \lambda_2, \ldots, \lambda_N \) at times \( \tau_1, \tau_2, \ldots, \tau_N \).

Summarizing, calculate \( \frac{1}{2} [C] \), \( \frac{1}{2} [H_0] \) and \( \frac{1}{2} [C] / \frac{1}{2} [H_0] \) take

\[ \frac{1}{2} [C(t)] / \frac{1}{2} [H_0] = E(t), \text{ giving the } N \text{ initial times and density of discharge that make up } E(t). \]

Let us consider the discrete analog.

Let \( E_i = E [i(\Delta t)] \) for \( i = 0, 1, N-1 \) \quad (5.91)

and \( C_i = C [i(\Delta t)] \) for \( i = 0, 1, \ldots, N-1 \) \quad (5.92)
and $H_i = H [i(\Delta t)]$ for $i = 0, 1, \ldots N-1$ \hspace{1cm} (5.93)

Definition: $B_i(t)$ is a Box Car function where

$$0 \quad t < i\Delta t$$

$$B_i(t) = \begin{cases} 
1 & i\Delta t \leq t \leq (i+1) \Delta t \\
0 & t > (i+1) \Delta t
\end{cases}$$ \hspace{1cm} (5.94)

The appropriate continuous function can be approximated as follows:

$$E_k(k\Delta t) = \sum_{i=0}^{N-1} E_i B_i(k\Delta t) = E_k$$ \hspace{1cm} (5.95)

$$C_k(k\Delta t) = \sum_{i=0}^{N-1} C_i B_i(k\Delta t) = C_k$$ \hspace{1cm} (5.96)

$$H_0(k\Delta t) = \sum_{i=0}^{N-1} H_i B_i(k\Delta t) = H_k$$ \hspace{1cm} (5.97)

Take the discrete Fourier transform of $C_k$ and $H_k$ giving us $E_k$.

Effective Precipitation.

$$\frac{1}{N} \sum_{k=0}^{N-1} C_k \exp(-i2\pi nk/N)$$ \hspace{1cm} (5.98)

$$n = 0, 1, \ldots, N-1$$

$$\frac{1}{N} \sum_{n=0}^{N-1} H_k \exp(-i2\pi nk/N)$$ \hspace{1cm} (5.99)

Then divide Equation (5.98) by Equation (5.99) giving us

$$\frac{\{C\}[n/N(\Delta t)]}{\{H_0\}[n/N(\Delta t)]} = \frac{\{E\}[n/N(\Delta t)]}{\{H_0\}[n/N(\Delta t)]}$$ \hspace{1cm} (5.100)

Evaluating Equation (5.100) at $n = 0, 1, \ldots, N-1$.

Finally using the discrete inverse Fourier transform on Equation (5.100).
\[ E[k(\Delta t)] = \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{F}[E] \left[ \frac{n}{N(\Delta t)} \right] \exp(i2\pi nk/N) \quad (5.101) \]

Therefore, \( E_k \) is the approximated effective precipitation at times \( k = 0, 1, \ldots, N-1 \).
REFERENCES


