Brownian Motion Applied to Partial Differential Equations

Steven M. McKay
Utah State University

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BROWNIAN MOTION APPLIED TO PARTIAL DIFFERENTIAL EQUATIONS

by

Steven M. McKay

A thesis submitted in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Mathematics

UTAH STATE UNIVERSITY

Logan, Utah

1985
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Many thanks are extended to Dr. Michael Brennan, who patiently helped me through this text. A special thanks is also extended to my wife, Shauna, who encouraged and supported me throughout my graduate career.

Steven McKay
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ABSTRACT

Brownian Motion Applied to Partial Differential Equations

by

Steven M. McKay, Master of Science
Utah State University, 1985

Major Professor: Dr. Michael Brennan
Department: Mathematics

This work is a study of the relationship between Brownian motion and elementary, linear partial differential equations. In the text, I have shown that Brownian motion is a Markov process, and that Brownian motion itself, and certain stochastic processes involving Brownian motion are also martingales. In particular, Dynkin's formula for Brownian motion was shown. Using Dynkin's formula and Brownian motion, I then constructed solutions for the classical Dirichlet problem and the heat equation, given by $\Delta u=0$, and $u_t=\frac{1}{2}\Delta u+g$, respectively. I have shown that the bounded solution is unique if Brownian motion will always exit the domain of the function once it has started at a point in the domain. The heat equation also has a unique bounded solution.

(30 pages)
CHAPTER I
INTRODUCTION

In 1827, Robert Brown, a botanist, discovered that particles in a fluid are never stationary, but are always moving in random directions. The random path that a particle may take has been named Brownian motion. Work on deriving a mathematical model for Brownian motion was done both by Einstein [6], Weiner [16], and others (see Weiner et al. [17]). As Weiner put the process on a firm mathematical basis, and dealt with it from the perspective of functions from $C[0,\infty)$, the process is sometimes called a Weiner process. For this reason, I will reserve the letter $W$ to denote Brownian motion. Since 1923, a great deal of work on Brownian motion and the theory of Stochastic Processes in general has been done. This treatise will look at some of the basic properties for Brownian motion, and will show the relationship between running a Brownian motion process and solving basic parabolic and elliptic partial differential equations. First, however, I wish to look at the analog of Brownian motion in the discrete case, a process called Random Walk.
1.1 STOCHASTIC PROCESSES: RANDOM WALKS AND MARTINGALES

In the next few sections I shall attempt to define a process called Random Walk and show some similarities to what we will do later in the text. The following discussion assumes the reader is familiar with elementary concepts of probability and measure theory. It will be necessary for what follows to apply a measure theoretic approach to Probability.

**Definition 1.1:** A probability measure $P$ on a measurable space $(\Omega, \mathcal{F})$ is a finite measure where $P(\Omega) = 1$.

**Definition 1.2:** A Stochastic Process $(X_t, t \in T)$ is a family of random variables which lie in the same probability space $(\Omega, \mathcal{F}, P)$.

As no restrictions have been placed on $t$, it can be either countably or uncountably valued. In many processes, however, $t$ takes the role of time so $T$ is usually a subset of $[0, \infty)$. There are many different examples of Stochastic Processes, but we will be limited to looking at Random Walk and Brownian motion and showing their interdependence.

**Definition 1.3:** Simple Random Walk started at $x$ is a stochastic process $(R_i, i \geq 0)$ where $R_i = x + \sum_{j=1}^{i} X_j$ and $X_1, X_2, \ldots$ are independent identically distributed Bernoulli random variables with $P(X_j = 1) = p$ and $P(X_j = -1) = q$. Unless otherwise stated, we will be
dealing with simple, symmetric Random Walk, with
\( p = q = \frac{1}{2} \).

Think of Random Walk as the random movement of a particle along the integer lattice. The above definition deals with Random Walk in one dimension, but we can extend it to \( d \) dimensions on the integer lattice similarly. Notice that once Random Walk has reached any point, the path taken to reach there has no bearing on any future movements. This is because of the independence of the \( X_j \) random variables. This is called a Markov property and will be discussed in detail later. To get a better idea of what types of properties Random Walk has, I shall state some exercises that deal with Random Walk.

**Proposition 1.4:** Let \( \{R_t, t \geq 0\} \) be Random Walk started at \( x, 0 < x < a \). If \( T_{0,a} = \inf\{t \geq 0 : R_t = 0 \text{ or } R_t = a\} \), find
\[
P_x (R_{T_{0,a}} = 0) = P (R_{T_{0,a}} = 0 | R_0 = x).
\]

Notice that
\[
P_x (R_{T_{0,a}} = 0) = P (R_{T_{0,a}} = 0 | R_0 = x, R_1 = x+1) P_x (R_1 = x+1) + P (R_{T_{0,a}} = 0 | R_0 = x, R_1 = x-1) P_x (R_1 = x-1).
\]

Now, as we stated before, Random Walk depends only upon the present and not the past. In probabilistic statements this implies
\[
P (R_{T_{0,a}} = 0 | R_0 = x, R_1 = y) = P (R_{T_{0,a}} = 0 | R_1 = y) = P_x (R_{T_{0,a}} = 0)
\]
as long as \( |x-y|=1 \). Thus, the above becomes
\[
P_x (R_{T_{0,a}} = 0) = P_{x+1} (R_{T_{0,a}} = 0) / 2 + P_{x-1} (R_{T_{0,a}} = 0) / 2.
\]
This can be thought of as a function \( f \) where
\[
f (x) = f (x+1) / 2 + f (x-1) / 2.
\]
I claim \( f (x) = Ax + b \), a linear
function. Now, \( f \) is defined inside of \((0,a)\), but as we can solve for \( f(x+1) \) and \( f(x-1) \), we can extend \( f \) over all integers. Notice that \( f(x+1) = 2f(x) - f(x-1) \). Thus, \( f(x+1) - f(x) = f(x) - f(x-1) \), and for each unit change for \( x \), the change for \( f \) is the same. This is satisfied if \( f(x) = Ax + b \).

To solve for \( A \) and \( b \), notice \( f(0) = P_x(R_t = 0) = 0 \), and \( f(a) = P_x(R_T = 0) = 0 \). This information gives \( A = -1/a \) and \( b = 1 \).

Hence, \( P_x(R_T = 0) = 1 - x/a \).

Proposition 1.5: Let \( R_t \) be defined as before with \( x > 0 \) and define \( T_0 = \inf\{t > 0 : R_t = 0\} \). Find \( P_x(T_0 < \infty) \).

What we wish to establish in this exercise is whether Random Walk is recurrent or transient. If Random Walk is recurrent, then if \( \tau = \inf\{t > 0 : R_t = x\} \), \( P_x(\tau < \infty) = 1 \). Transience implies \( P_x(\tau < \infty) < 1 \). Notice that \( \{T_0, a\} \) is an increasing sequence of exit times whose limit is \( T_0 \). Notice the event \( \{T_0 < \infty\} \) is the same as

\[
\{R_{T_0} = 0 \text{ for some } a\} = \bigcup_{a=x+1}^{\infty} \{R_{T_0} = 0\}.
\]

As the sequence of sets on the right is nested and increasing, a well known result from measure theory gives us

\[
P(T_0 < \infty) = \lim_{a \to \infty} P_x(R_{T_0} = 0) = 1.
\]

Note that the above is a very specific calculation requiring the Random Walk to be symmetric. We do not get the same result for general \( p \) and \( q \). We can also turn this calculation around and obtain \( P_0(T_x < \infty) = 1 \). Define \( \tau_x = \inf\{t > 0 : R_t = x\} \). Combining the above facts yields \( P_y(\tau_x < \infty) = 1 \), and more specifically, \( P_x(\tau_x < \infty) = 1 \). So, Random Walk is recurrent.

It can be shown with considerably more work, that Random
Walk is recurrent in dimension 2 also. However, this breaks down when the Random Walk is in dimension 3. In this case, Random walk is found to be transient. I will show this property for Brownian motion as a consequence of the solution to a certain partial differential equation. This is an interesting but intuitively unclear fact about these two random processes.

Next, I need to state the definition and some basic results of random variables called stopping times.

**Definition 1.6:** Let \((T_t, t \geq 0)\) be a set of increasing \(\sigma\)-algebras. A stopping time with respect to \((T_t)\) is a random variable \(T: \Omega \rightarrow \mathbb{R} \cup \{\infty\}\) such that the event \((T \leq t) \in T_t\).

**Definition 1.7:** Let \(T\) be defined as above. Then, define \(T_T = (A \in \mathcal{F}_t : A(T \leq t) \in T_t)\).

**Lemma 1.8:** Let \(T\) be a stopping time defined on \((T_t, t \geq 0)\). Then \(T_T\) is a \(\sigma\)-algebra.

**Proof:** Suppose \(A \in T_T\). As \(A(T \leq t) \in T_t\), the complement is also, so we have \(\overline{A}(T \leq t) = \overline{A}(T) \leq t\) which is in \(T_t\). If we intersect this with \((T \leq t)\) we obtain \(A(T \leq t)\) which must be in \(T_T\) also. Let \((A_i)\) be a sequence of sets in \(T_T\). Then as \((\cup A_i)(T \leq t) = \cup (A_i(T \leq t))\), it is a union of sets in \(T_t\) and hence is in \(T_T\). Thus, \(T_T\) is a \(\sigma\)-algebra.

**Theorem 1.9:** Let \((X_t, t \geq 0)\) be a stochastic process with continuous paths where \((T_\infty)\) is the set of \(\sigma\)-algebras generated by the process. The hitting time of a closed set is a stopping time with respect to
Proof: First I need to show an intermediate result. Suppose $O$ is an open set. Then, if $T = \inf\{t \geq 0 : X_t \in O\}$, I claim $(T(t)) \in T_t$. Notice that $(T(t)) = (X_q \in O$ for some $0 \leq u < t)$. But, as $O$ is open, and $X$ has continuous paths, we see that the above is the same event as

$(X_q \in O$ for some rational $0 \leq u < t) = \bigcup_{q} (X_q \in O)$.

Suppose $(X_t)$ is given as above, and suppose $C$ is a closed set. Let $T = \inf\{t > 0 : X_t \in C\}$. Then if $C_n = \bigcup_{x \in C} x_{1/n}$, notice that $CCC_n$, and $C_n$ is open. If $T_n$ is the hitting time of $C_n$ as above, then $T_n \leq T$ by the continuity of the paths. Also $T_n < T_{n+1}$ by the construction. Thus, $T_n$ converges upward to some $t \leq T$. If $t < \omega$, then $X_{<t}$ has hit $C_n$ before $t$ for each $n$. Thus, $X_n \in C_n$. As $T_n$ converges to $t$ we have $X_{T_n}$ converging to $X_t$ by the continuity. Thus, $X_t \in C$. So, $T \leq t$ by definition. Hence, $T_n$ converges upward to $T$. Notice that in order for $X_{<t}$ to hit $C$ by time $t$, it must hit $C_n$ before $t$ by the continuity. Thus, $(T < t) = \bigcap_{n} (T_n < t)$. But, by the above, if $T_n < t$ for all $n$, $T < t$ also, so $(\bigcap_{n} (T_n < t)) \subseteq (T < t)$. As $(T_n < t) \subseteq (T_n < t)$, we have $(T < t) = \bigcap_{n} (T_n < t)$. Thus, $(T < t) \in T_t$.

Part of the above proof comes from Durrett [4, pp. 18–19]. For a discussion of stopping times the reader is invited to see the above reference.

Definition 1.10. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, $(X_n, T_n, n \geq 0)$ is called an adapted stochastic sequence if $(T_n)$ is an increasing sequence of $\sigma$-algebras and $X_n$ is measurable with respect to $T_n$ for each $n$.

Definition 1.11: An adapted stochastic sequence
\((X_n, T_n, n \geq 0)\) is a martingale if \(E|X_n| < \infty\) and \(E(X_{n+1} | T_n) = X_n\) for all \(n\). If \(E(X_{n+1} | T_n) \geq X_n\), then \(X_n\) is called a submartingale.

**Theorem 1.12:** Random Walk is a martingale.

**Proof:** Let \(R_n = x + \sum_{i=1}^{n} X_i\) where \(P(X_i = 1) = P(X_i = -1) = 1/2\). Then

\[
E|R_n| \leq |x| + \sum_{i=1}^{n} E|X_i| = |x| + n < \infty.
\]

Let \(T_n = \sigma(R_i, 0 \leq i \leq n)\). Clearly, \((R_n, T_n, n \geq 0)\) is an adapted stochastic sequence. Thus, we have

\[
E(R_{n+1} | T_n) = E(R_n + X_{n+1} | T_n) = E(R_n | T_n) + E(X_{n+1} | T_n) = R_n + E(X_n+1 | T_n).
\]

As \(X_{n+1}\) is independent of \(X_1, \ldots, X_n\), then it can be shown that \(X_{n+1}\) is independent of \(T_n\). Thus, \(E(X_{n+1} | T_n) = E(X_{n+1}) = 0\). Thus \(R_n\) is a martingale.

**Theorem 1.13:** Let \((R_t, T_t, t \geq 0)\) be Random Walk on the integer lattice in \(\mathbb{R}^d\). If \(P\) is the matrix where the \(i,j\) entry is \(P(i,j)\), and \(f\) is a bounded measurable function from \(\mathbb{R}^d\) to \(\mathbb{R}\), then

\[
f(R_t) - \sum_{i=0}^{t-1} (P-I)f(R_i)
\]

is a martingale.

**Proof:** Notice that

\[
E\left(f(R_t) - \sum_{i=0}^{t-1} E(f(R_{i+1}) - f(R_i) | T_i) | T_{t-1}\right)
\]

\[
=E(f(R_t) | T_{t-1}) - \sum_{i=0}^{t-1} \left[E(f(R_{i+1} | T_i) - f(R_i)\right]
\]
As $R_i$ is Markov, then $E_x(f(R_{i+1})|T_i) = E_{R_i}(f(R_{i+1}))$.

Notice that

$$E_x(f(R_1)) = \sum_{j=1}^{d} \frac{(f(x+e_j) + f(x-e_j))}{2d} = Pf(x).$$

Hence,

$$\sum_{i=0}^{t-1} E(f(R_{i+1})-f(R_i)|T_i) = \sum_{i=0}^{t-1} (P-I) f(R_i)$$

and we are done.

Next I shall state some properties of martingales. As the proofs are beyond the scope of this presentation, they will not be presented here.

**Theorem 1.14**: Let $(X_n,T_n,n \geq 0)$ be an adapted stochastic sequence. Then $(X_n)$ is a martingale if and only if

$$E(X_T) = E(X_0)$$

for all bounded stopping times $T$. See Karlin and Taylor [11, pp. 261].

**Theorem 1.15**: Let $(X_n,T_n,n \geq 0)$ be a submartingale. Then,

$$P(\max_{0 \leq m \leq n} X_n \geq \lambda) \leq \lambda^{-1} E|X_m|$$

and

$$P(\min_{0 \leq n \leq m} X_n \geq \lambda) \leq \lambda^{-1} (E|X_m| + E|X_0|).$$

A proof can be found in [13, pp. 243-244].

**Theorem 1.16**: Let $(T_i)$ be an increasing sequence of $\sigma$ algebras, i.e., $T_i \subset T_{i+1}$. If $E|X| < \infty$ for some random variable $X$, then $Y_i = E(X|T_i)$ is a martingale with respect to $(T_i)$. See [2, pp. 410].
Theorem 1.17: Let \((X_n, T_n, n > 0)\) be a martingale such that \(\operatorname{sup}_{n \geq 0} |X_n| < \infty\). Then,

1) there exists \(X \in L^1\) such that \(X_n\) converges to \(X\), and

2) \(\|X_n - X\|_1\) tends to 0 and \(X_n = \mathbb{E}(X | T_n)\) if and only if for each \(\epsilon > 0\), there is \(\delta > 0\) so \(P(A) < \delta\) implies \(\mathbb{E}(|X| I_A) < \epsilon\).

The proof can be found in [4, pp. 309].
1.2 BROWNIAN MOTION: CONSTRUCTION BY RANDOM WALKS.

In Chapter 2 I will examine Brownian motion in some detail. Here I will try to show the relationship between Random Walk and Brownian motion, and examine some of its properties.

Definition 1.18: Brownian motion starting at $x$ is a stochastic process $(W_t, t \geq 0)$ satisfying the following criteria:

1) Given elements $0 < t, 0 < s$, $W_{t+s} - W_t$ is normally distributed with mean 0 and variance $s$.

2) For $0 < t_1 < t_2 < \ldots < t_n$, $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are mutually independent. This is often referred to as $(W_t, t \geq 0)$ having independent increments.

3) Brownian motion is almost surely pathwise continuous.

4) $W_0 = x$.

As $t$ is real valued, the situation is more complicated than for Random Walk. However, there is a relationship between the two processes, as the next Theorem will show.

Theorem 1.19: Let $(R_n, t, i \geq 0)$ be Random Walk where

$$R_n, t = 2^{-n} \sum_{j=1}^{i} X_j$$

where $P(X_j = 1) = P(X_j = -1) = 1/2$, and $t = i/4^n$. Then, $R_n, t$ converges in distribution to Brownian motion on the dyadics.

Proof: Notice that $(R_n, t)$ has the same properties as
usual Random Walk except it moves on the lattice of points
of the form \( k/2^n \), and it moves in shorter time intervals.

Look at \( R_n, t \) for \( 0 \leq t < 1 \). Notice that
\[
R_n, t + a - R_n, t = \sum_{j=1}^{1} X_j
\]
where \( t = k/2^n \), and \( t + s = 1/2^n \). This is independent of
\[
\sum_{j=1}^{k} X_j = R_n, t \text{ by hypothesis. So, } \{R_n, t\} \text{ has independent}
\]
increments. Also, it is clear that \( E(R_n, t) = 0 \) and
\[
\text{var}(R_n, t + s - R_n, t) = \sum_{j=k+1}^{1} E(X_j^2) \text{ by independence of } X_i \text{ and } X_j \text{ if}
\]
i and \( j \) are different. Thus,
\[
\text{var}(R_n, t + s - R_n, t) = \sum_{j=k+1}^{1} 2^{-n} = \frac{(1-k)}{2^n} = t + s - t = s.
\]

If \( n \) is large, then by the Central Limit Theorem, (see [13, pp. 26]), \( R_n, t + s - R_n, t \) is distributed approximately Normal
with mean 0 and variance \( s \). Thus, we see by taking the
limit as \( n \to \infty \), there is a process \( \{W_t, t \in D\} \) which has the
first two properties of the definition of Brownian motion
where \( D \) is the set of dyadics in \([0,1]\). This can be
extended to \( \{W_t, 0 \leq t < \omega\} \) without much trouble. Notice that
this gives restrictions that \( t \) must be dyadic, and tells us
nothing about continuity. If we connect the values of \( R_n, t \)
linearly, then it can be shown that the resulting continuous
random function converges to Brownian motion in
distribution. This is called Donsker’s theorem, and the
reader who wishes to know more can see [1, pp. 68]. As I
will give a different construction of Brownian motion in
Many times we will wish to start Brownian motion at some other place than 0. I will define a number of probability measures, \( P_x \) by the following rule:

\[
P_x(W_{t+s} \in A_1, \ldots, W_t \in A_n) = P(W_{t+s} + x \in A_1, \ldots, W_t + x \in A_n).
\]

It can be shown that \( P_x \) is a probability measure. Also, \( E_x \) is the expectation associated with \( P_x \). When I wish to start Brownian motion at \( x \), I will use \( P_x \) as the probability measure.

Like Random Walk, Brownian motion has the Markov property. This says any future event depends only upon the present. Technically this means

\[
P_x(W_{t+s} \in A_t | Q_t) = P_x(W_{t+s} \in A_t | W_t).
\]

Also, we have the equivalence of each of the following statements:

1) \( W_{t+s} - W_t \) is Brownian motion and is independent of \( Q_t \) for each \( P_x \).

2) If \( f \) is any bounded measurable function, then

\[
E_x[f(W_{t+s}) | Q_t] = E_{W_t}[f(W_s)] = \int f(y)p(s, W_t, y)\,dy.
\]

3) If \( H \) is a bounded measurable function from the class \( C \) of continuous functions to the reals, then

\[
E_x[H(W_{t+s}, \cdot) | Q_t] = E_{W_t}[H(W_s, \cdot)].
\]

Brownian motion also has a stronger property called the Strong Markov property. This says we can start the process over at any random stopping time \( T \), rather than at a finite time \( t \). See page 32 for a more formal statement.
Lemma 1.20: Let \((W_t, t \geq 0)\) be Brownian motion defined as above. Then, \((W_{t+s} - W_t, s \geq 0)\) is Brownian motion independent of \(\mathcal{A}_t\), \((-W_t, t \geq 0)\), \((\frac{1}{\sqrt{t}} W_{c t}, t \geq 0)\), and \((t W_t, t \geq 0)\) are all Brownian motion.

The proof is straightforward and is left for the reader.

Proposition 1.21: \((W_t, \mathcal{A}_t, t \geq 0)\) and \((W_t^2 - t, \mathcal{A}_t, t \geq 0)\) are martingales where \(\mathcal{A}_t\) is the \(\sigma\) algebra generated by \((W_u, 0 \leq u \leq t)\).

Proof: Let \(s, t\) be given. Then

\[
E(W_{t+s} | \mathcal{A}_t) = E(W_{t+s} - W_t + W_t | \mathcal{A}_t) = E(W_{t+s} - W_t | \mathcal{A}_t) + E(W_t | \mathcal{A}_t).
\]

But, \(W_t\) is \(\mathcal{A}_t\) measurable, and \(W_{t+s} - W_t\) is independent of \(\mathcal{A}_t\) by Lemma 1.20. Thus, the above becomes

\[
E(W_{t+s} - W_t) + W_t = 0 + W_t = W_t.
\]

Notice also, that

\[
E(W_{t+s}^2 - (t+s) | \mathcal{A}_t) = E((W_{t+s} - W_t + W_t)^2 | \mathcal{A}_t) - (t+s)
\]

\[
= E((W_{t+s} - W_t)^2) + 2W_t E(W_{t+s} - W_t) + W_t^2 - (t+s)
\]

\[
= s + W_t^2 - (t+s) = W_t^2 - t.
\]

Proposition 1.22: Let \(W_t\) be Brownian motion starting at 0. Find

a) the distribution for \(W_t\) to exit \((-a,b)\) and

b) the expected exit time of the above.

Proof: As \(W_t^2 - t\) is a martingale, then by Theorem 1.14 we have

\[
E_0(W_T^2 - T \wedge n) = E_0(W_0) = 0.
\]
Thus, as $E_0(T)\leq \max\{a^2,b^2\}$. Hence, this implies $E_0(T)$ is bounded, and $P_0(T=\infty)=0$. From Proposition 1.21, we know $W_t$ is a martingale. Let $T=\inf\{t\geq 0: W_t=-a \text{ or } W_t=b\}$. Then from Theorem 1.14, we have $E_0(W_T)E_0(W_0)=0$.

As $|W_{T\wedge n}|\leq \max\{a,b\}$ for every $n$, the Bounded Convergence Theorem gives us $E_0(W_T)=0$. But,

$$E_0(W_T)=-aP_0(W_T=-a)+bP_0(W_T=b).$$

As $P_0(W_T=b)=1-P_0(W_T=-a)$ from the boundedness of $E_0T$, we have

$$P_0(W_T=-a)=\frac{b}{a+b}. \quad \text{Similarly, } P_0(W_T=b)=\frac{a}{a+b}.$$

Again, as $E_0(W_{T\wedge n})=E_0(W_n)$, then taking the limit on the left we obtain $E_0(W_T)$ by the Bounded Convergence Theorem. Also, the limit on the right tends to $E_0(T)$ by the Monotone Convergence Theorem. Hence,

$$E_0(T)=E_0(W_T)=a^2P(W_T=-a)+b^2P(W_T=b)$$

$$=\frac{a^2b+b^2a}{a+b}=ab.$$

For some of the work in Chapter 3, I am going to need the above two results for Brownian motion in dimensions greater than 1. Brownian motion in dimension $d$ is the random vector $(W_{t_1},\ldots,W_{t_d})$ where $(W_{t})_i$ is one dimension Brownian motion, and the coordinate random variables are all independent.

**Proposition 1.23:** Let $(W_t,t\geq 0)$ be Brownian motion in $d$ dimensions starting at $x$. Then, $\|W_t\|^2-dt$ is a martingale and if $T=\inf\{t\geq 0: W_t \text{ has exited } B_0,r\}$ where $B_x,r$ is the open ball centered at $x$ with radius $r$, then $E_x(T) = (r^2-\|x\|^2)/d$ for $x \in B_0,r$.

**Proof:** Notice that $E_x(\|W_t\|^2-dt|Q_s)=$
by Proposition 1.21.

Thus, by Theorem 1.14,

\[ E_X(\|W_t\|_2^2 - d(t \wedge T)) = \|x\|^2. \]

Hence

\[ E_X(\|W_T\|_2^2) = dE_X(t \wedge T) + \|x\|^2. \]

Taking the limit of both sides, the limit can be brought inside on the left by the bounded convergence theorem, and on the right by the monotone convergence theorem. Hence

\[ E_X(\|W_T\|_2^2) = dE_X(T) + \|x\|^2. \]

As \( \|W_T\|_2^2 = r^2 \), the result follows.

Notice that the above exercise shows the time to exit a bounded set is always finite, and depends on the fact that \( x \) is inside \( B_0, r \).

I shall next consider a construction of a Stochastic process involving Brownian motion that is also Brownian motion. This process is called reflected Brownian motion, and is given by

\[ Y_t = \begin{cases} W_t & \text{if } t < T \\ 2a - W_t & \text{if } t \geq T \end{cases} \]

where \( T = \inf \{ t : W_t = a \} \).

**Theorem 1.24:** The stochastic process \( Y_t \) given above is Brownian motion.

**Proof:** I shall only provide an intuitive proof here. For greater detail, see [7, pp. 23-24]. Let \( T \) be any
stopping time. Then, we can rewrite \( W_t \) as

\[
W_t = \begin{cases} 
W_t & \text{if } t < T \\
W_T + [W_t - W_T] & \text{if } t > T.
\end{cases}
\]

Notice that for \( t > T \), \( W_t - W_T \) is Brownian motion independent of \( \mathcal{F}_T \) by the Strong Markov property and hence \( W_T \). Thus, \(-[W_t - W_T]\) is Brownian motion independent of \( W_T \) by Lemma 1.20. Using the fact that if \( (Z,X) \) and \( (Z',X) \) are equal in distribution, \( X+Z \) and \( X+Z' \) are also, we see that

\[
Y_t = \begin{cases} 
W_t & \text{if } t < T \\
W_T - [W_t - W_T] & \text{if } t > T
\end{cases}
\]

is Brownian motion.

The next few examples involve the use of \( Y_t \).

**Proposition 1.25:** Let \( T_a = \inf \{ t > 0 : W_t = a \} \). Then

\[
P_0 \langle T_a \leq t \rangle = 2P_0 \langle W_t \geq a \rangle.
\]

**Proof:** Notice \( P_0 \langle T_a \leq t \rangle = P_0 \langle \max_{0 \leq u \leq t} W_u \geq a \rangle \). Also, if \( T_a \leq t \) then \( Y_t \) has been reflected across the line \( x = a \). Hence, either \( Y_t \geq a \) or \( W_t \geq a \). So,

\[
P_0 \langle T_a \leq t \rangle = P_0 \langle Y_t \geq a \rangle + P_0 \langle W_t \geq a \rangle.
\]

But, \( P_0 \langle Y_t \geq a \rangle = P_0 \langle W_t \geq a \rangle \) by the above, and we have the result.

**Proposition 1.26** Define

\[
U(t_1, t_2) = P_0 \langle W_u = 0 \text{ for some } t_1 < u < t_2 \rangle.
\]

Show

\[
U(t_1, t_2) = \frac{2}{\pi} \cos^{-1} \left( \frac{\sqrt{t_1} - \sqrt{t_2}}{2} \right).
\]

**Proof:** To obtain this result, we need an intermediate calculation. I wish to show that

\[
P_0 \langle T_a \leq t \rangle = \int_0^t a^{-3/2} e^{-a^2/2t} dt.
\]
By the previous exercise,

\[
P_0(T_a < t) = 2P_0(W_t > a) = 2\int_{a/\sqrt{2\pi t}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2t) \, dy
\]

\[
= \frac{\sqrt{2\pi}}{a/\sqrt{2\pi t}} \int_0^{\sqrt{2\pi t}/a} \exp(-y^2/2) \, dy.
\]

Then, the density function of \( T_a \), \( f_{T_a}(t) \) is given by

\[
f_{T_a}(t) = \frac{d}{dt} \left( \frac{\sqrt{2\pi}}{a/\sqrt{2\pi t}} \int_0^{\sqrt{2\pi t}/a} \exp(-y^2/2) \, dy \right) = a t^{-3/2} \exp(-a^2/2t).
\]

Define \( H: \mathbb{C} \to \mathbb{R} \) by the rule \( H(w(.)) = 0 \) if \( w(u) = 0 \) for some \( u \) between \( 0 \) and \( t_2 - t_1 \), and \( H(w(.)) = 1 \) otherwise. Now, by the third equivalent form of the Markov property, \( V(t_1, t_2) \) can be written as \( E_0[H(W_{t_1} + (.))] \)

\[
= E_0[E_0[H(W_{t_1} + (.)) | Q_{t_1}]] = E_0[E_{W_{t_1}}[H(W(.))]]
\]

\[
= \int E_a[H(W(.))] p(t_1, 0, a) \, da.
\]

Note that if \( Y_t \) is Brownian motion reflected across \( x = a \), then

\[
E_a[H(W(.))] = P_a(W_u = 0 \text{ for } 0 \leq u \leq t_2 - t_1)
\]

\[
= P_a(Y_u = 2a \text{ for } 0 \leq u \leq t_2 - t_1) = P_0(Y_u = a \text{ for } 0 \leq u \leq t_2 - t_1)
\]

\[
= P_0(W_u = a \text{ for } 0 \leq u \leq t_2 - t_1) = P_0(T_a < t_2 - t_1).
\]

Using the above information, we obtain \( V(t_1, t_2) = \)

\[
2\int_0^{\infty} \int_0^{t_2 - t_1} \frac{a t^{-3/2} \exp(-a^2/2t)}{\sqrt{2\pi t_1}} \exp(-a^2/2t_1) \, dt \, da.
\]

The rest of the proof is an exercise in calculus and trigonometry and is left to the reader. \( \blacksquare \)
In this section I will take a look at some differential equations and sketch out a solution involving Brownian motion. First, however, I wish to look at a more simplified problem on the integer lattice in $\mathbb{R}^d$. Recall that $P(x,y)$ is the probability Random Walk moves from $x$ to $y$ in one time unit. $P$ is said to be the transition function on the integer lattice in $\mathbb{R}^d$.

**Definition 1.27:** Let $I^d$ be the integer lattice in $\mathbb{R}^d$, and let $f:I^d \rightarrow \mathbb{R}$ be given. If

$$f(x) = \sum_{y} P(x,y) f(y)$$

then $f$ is called $P$ harmonic.

If we define $P$ to be the matrix where the $i,j$ entry is $P(i,j)$, then the above statement reduces to $Pf=f$, or $(P-I)f=0$. Suppose we have an open set $G$ in $\mathbb{R}^d$ and a function $g$ defined on $\partial G$, which is the set of all boundary points of $G$. One might ask if we can find a function from the closure of $G$ to $\mathbb{R}$ such that $\Delta f = 0$ and $f(x) = g(x)$ on $\partial G$. This is called the Dirichlet problem. If a function $f$ satisfies $\Delta f = 0$ on a set $G$, $f$ is called harmonic on $G$. This is a continuous analog of the equation $(P-I)f=0$. As the lattice is shrunk as in Theorem 1.19, $P-I$ converges to the operator $\frac{1}{2}\Delta$, and the martingale in Theorem 1.13 converges in distribution to
Let $I_h^d=\{(x_1, \ldots, x_d): x_i= mh \text{ for some } m\}$. An analog of the Dirichlet Problem for $I_h^d$ follows. Let $D$ be a subset of $I_h^d$ and define $\partial D=\{y: y \notin D, \ |x-y|=h \text{ for some } x \in D\}$. Define a function $g$ from $\partial D$ to $\mathbb{R}$. Can we find a function $f$ from $\partial D$ to $\mathbb{R}$ such that $f$ is $\mathcal{P}$ harmonic and $f(x)=g(x)$ on $\partial D$? The answer is yes, given certain restrictions for $g$.

**Theorem 1.28:** Let $g$ be a bounded function from $\partial D$ to $\mathbb{R}$, and let $(R_t)$ be Random Walk on $I_h^d$. Let $T=\inf\{n>0: R_t \in \partial D\}$. Then

1) $f(x)=\mathbb{E}_x[g(R_T)1_{\{T<\infty\}}]$ is a solution of the Dirichlet Problem,
2) if $\mathbb{P}(T<\infty)=1$, then a) is the only bounded solution, and
3) if $\mathbb{P}(T=\infty)>0$, $K(x)=f(x)+\alpha \mathbb{P}(T=\infty)$ is a solution for every $\alpha$ in $\mathbb{R}$.

I will not give a proof here, but the interested reader can see Dynkin and Yushkevich [5, pp. 33-34]. I have stated the result in order to compare the solution of the problem on subsets of $I_h^d$ and $\mathbb{R}^d$.

Next, I wish to sketch the procedure for solving partial differential equations using Brownian motion. In order to do this, we will need two different results. These Theorems will be proved in a later section.

**Theorem 1.29:** Let $f$ be a bounded continuous function from $\mathbb{R}^d$ to $\mathbb{R}$ with $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_i x_j}$
continuous. Then,
\[ f(W_t) - \int_0^t \frac{1}{2} \Delta f(W_r) \, dr \]
is a local martingale. That is to say, there are stopping times \( T_n \) increasing to \( \infty \), where if \( t \) is replaced by \( T_n \wedge t \) in the above, the ensuing stochastic process is a martingale. Similarly, if \( f(t,x) \) is bounded from \((0,\infty) \times \mathbb{R}^d\) to \( \mathbb{R} \), and the partial derivatives are again continuous,
\[ f(t,W_t) - \int_0^t (\frac{\partial}{\partial t} + \frac{1}{2} \Delta f)(r,W_r) \, dr \]
is a local martingale. This result is called Dynkin's formula.

**Theorem 1.30:** Let \( (M_t, T_t, t \geq 0) \) be a continuous local martingale with \( M_0 = 0 \) and such that \( M_t \) has paths of bounded variation. Then, \( M_t \equiv 0 \).

Suppose we wish to solve the equation \( \Delta u = 0 \) on \( G \), \( u = f \) on \( \partial G \), and \( u \) is continuous on \( \bar{G} \). This is called the classical Dirichlet problem. If \( G \) is bounded, I wish to use the above to show \( u(x) = E_x f(W_T) \) is the unique solution where \( T = \inf \{ t \geq 0 : W_t \notin \bar{G} \} \). Let \( T_r = \inf \{ t \geq 0 : W_t \in \bar{E}_{X_r} \} \). I claim without proof that
\[ u(W_{t \wedge T_r}) - \int_0^{t \wedge T_r} \frac{1}{2} \Delta u(W_s) \, ds \]
is a martingale with respect to \((Q_{t \wedge T_r}, t \geq 0)\). This will become evident in the proof of Dynkin's formula in Chapter 2. Choose \( r \) small enough so that \( \bar{B}_{X_r} \subset G \). Then, define \( H: \mathbb{C} \rightarrow \mathbb{R} \) by the following \( H(w(.)) = f(w(T)) \) where \( T \) is the first time \( w \) exits \( G \). Notice then that \( H(w(t+.)) = f(w(T)) = H(w(.)) \) for any \( t < T \). So, by the Strong Markov property,
which is a martingale by Theorem 1.16. Hence,

\[ E_X(<H(W_t)>) = E_X(<f(W_{t^\wedge T_R})>) = E_X(<H(W_{t^\wedge T_R})>) = E_X(<H(W_{t^\wedge T_R} + (<.>>)) = E_X(<H(W_{t^\wedge T_R})|\mathcal{F}_{t^\wedge T_R}) \]

\[ = E_X(f(W_{t^\wedge T_R})|\mathcal{F}_{t^\wedge T_R}) \]

This implies \[ \int_0^{t^\wedge T_R} \frac{1}{2} \Delta u(W_s) ds \] is a martingale. As it is also of bounded variation, it must be identically 0. Now, suppose \( \Delta u(x) > 0 \) for some \( x \). Then, \( \Delta u(y) > 0 \) for every \( y \in B_{x,r} \) for some \( r \). Hence, \( \Delta u(x) = 0 \). This shows \( u \) is a solution. Notice that the use of Dynkin’s formula assumes sufficient differentiability for \( u \). This is a large part of existence, and will be dealt with in Chapter 3.

Now I will show the solution is unique. Suppose \( u_1 \) is a solution to the Dirichlet problem given before where \( G \) is bounded. Define \( D_{x,r} = B_{x,r} \cap G \) where \( G_r = \{ x \in G : ||x-y|| < 1/r \} \) for \( y \in 3G \), and let \( T_n = \inf\{ t > 0 : W_t \notin D_{x,r} \} \). Notice that \( T_n \) increases up to \( T \). As \( \Delta u_1 = 0 \), then Dynkin’s formula shows that \( u_1(W_{t^\wedge T_n}) \) is a martingale. Hence by Theorem 1.14, \( E_X(u_1(W_{t^\wedge T_n})) = u_1(x) \). Taking the limit on the left as first \( t \) and then \( n \) tend to \( \infty \), then the limit can come inside by the dominated convergence theorem, and the above becomes \( E_X(u_1(W_T)) = u_1(x) \) by the continuity of \( u_1 \). As \( u_1(W_T) = f(W_T) \), it is clear that \( u_1 = u \).

Continuity does not always exist at the boundary. This is due to geometric properties of the boundary. To show what kind of problems can occur, I will give an example.
Let $D$ be the punctured disk of radius 1, that is $D = \{ x \in \mathbb{R}^2 : 0 < \|x\| < 1 \}$. I wish to find a solution of the Dirichlet problem on $\bar{D}$ with boundary information contained in $f$, where $f(x) = 1$ if $\|x\| = 1$, and $f(0) = 0$. If there is a solution, it must be $u(x) = E_x f(W_T)$ where $T = \inf \{ t > 0 : W_t \in \bar{D} \}$ by the above work. I claim that $W_T$ will always lie on the outer boundary of $D$. That is to say Brownian motion will not hit 0 before it exits $B_0,1$, as Brownian motion never hits single points. This will be shown in Theorem 3.22. Thus, $f(W_T) = 1$ with probability 1, and the solution is not continuous at 0.

The point 0 in the above example violates a criterion called regularity. Let $T = \inf \{ t > 0 : W_t \in \bar{D} \}$. If $P_0(T = 0) = 1$, 0 is said to be a regular point. I claim 0 is not a regular point. Notice that $P_0(T(t)) = P_0(\{ W_u = 0 \text{ for some } 0 < u < t \})$. Also, $\{ W_u = 0 \text{ for some } 0 < u < t \} \cap \{ W_{1/k} = 0 \text{ for some } 0 < u < t \}$ is a union of increasing, nested sets, then

$$P(T(t)) = \lim_{k \to \infty} P_0(\{ W_u = 0 \text{ for } 1/k < u < t \}) = 0.$$ 

As $t$ is arbitrary, $P_0(T(\infty)) = 1$. Thus, 0 is not a regular point in the above. It will be shown in Chapter 3 that continuity exists at the boundary for regular points.

I also wish to look for a solution for the equation $u_t = \frac{1}{2} \Delta u$ where $u$ is continuous on $\mathbb{R}^d$, and $u(0, x) = f(x)$. This is called the Heat equation. In this case we proceed similarly. It is easy to show that $v(t, x) = E_x(f(W_t))$ is a solution. Suppose $f$ is bounded. By the Markov property,

$$E_x(f(W_t) | Q_s) = E_{W_s}(f(W_{t-s})) = v(t-s, W_s).$$

So, by Theorem 1.17, $v(t-s, W_s)$ is a martingale for $s < t$. 

Dynkin's theorem then tells us that \( \int_0^t (-v_t + \frac{1}{2} \Delta v)(t-r, W_r) \, dr \) is a local martingale. As this is of bounded variation, it must be identically 0. This then shows us that \(-v_t + \frac{1}{2} \Delta v = 0\), so \( v \) satisfies the equation. Notice that the above argument depends on the smoothness of \( v \), which I have not shown here.

Suppose \( u_1 \) satisfies the heat equation. I claim \( u_1 = u \). If \( U_s = u_1(t-s, W_s) \), Dynkin's formula shows \( U_s \) is a local martingale. As \( u_1 \) is bounded, \( U_s \) is also. Thus, by Theorem 1.17, \( U_s \) converges to a limit \( U = f(W_t) \) by the continuity of \( u_1 \). Thus, \( U_s = E_x(U|Q_s) \). Setting \( s = 0 \) obtains the result.

All the details will be given in chapter 3.
CHAPTER II
BROWNIAN MOTION

2.1 DEFINITION AND EXISTENCE OF BROWNIAN MOTION

In this chapter I will discuss the properties of Brownian motion in detail. I shall start by defining the properties Brownian motion should have, and then show that the process exists.

Definition 2.1: Brownian motion is a stochastic process $(W_t, t \geq 0)$ satisfying the following criteria:

1) Given elements $0 \leq t, 0 \leq s$, $W_{t+s} - W_t$ is normally distributed with mean 0 and variance $s$.

2) For $0 \leq t_1 < t_2 < \ldots < t_n$, $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are mutually independent. This is often referred to as $(W_t, t \geq 0)$ having independent increments.

3) Brownian motion is almost surely pathwise continuous.

4) $W_0 = 0$.

We cannot assume a priori that there is a space $(\Omega, \mathcal{F}, P)$ which contains Brownian motion. Kolmogorov's Theorem, (see Appendix, Theorem A.1), tells us that there is a space $(\Omega, \mathcal{F}, P)$ so that Definition 2.1 1) and 2) are satisfied, but tells us nothing about continuity. This is the subject of the first major theorem in this section. First, however, we need the following results.
Lemma 2.2. If \( F(x_1, \ldots, x_n) = F(x) \) is the joint distribution function for \( W_{t_1}, W_{t_2}, \ldots, W_{t_n} \), then

\[
F(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} p(t_1, 0, y_1) \cdots p(t_n - t_{n-1}, y_{n-1}, y_n) dy,
\]

where \( p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp((-x-y)^2/2t) \) and \( dy = dy_1 \cdots dy_n \).

**Proof:** I propose to show that \( P(W_{t_1} \in A_1, \ldots, W_{t_n} \in A_n) = \int_{A_1} \cdots \int_{A_n} p(t_1, 0, y_1) \cdots p(t_n, y_{n-1}, y_n) dy_n \cdots dy_1 \). Define \( B_{t_i} = W_{t_i} - W_{t_i-1} \). Then \( W_0 = 0, W_{t_1} = B_{t_1} \) and \( W_{t_i} = B_{t_1} + \ldots + B_{t_i} \). Let \( g(x_1, \ldots, x_n) = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}) \).

The Jacobian of \( g \), denoted \( J_g \), is given by

\[
J_g = \begin{vmatrix}
1 & 0 & \ldots & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -1 & 1
\end{vmatrix} = 1.
\]

Thus, \( f(x) = |J_g| h(g(x)) \) where \( h \) is the joint density of \( B_1, \ldots, B_n \). So, \( P(W_{t_1} \in A_1, \ldots, W_{t_n} \in A_n) \)

\[
= \int_{A_1} \cdots \int_{A_n} f(y) dy_n \cdots dy_1
= \int_{A_1} \cdots \int_{A_n} h(y_1, y_2 - y_1, \ldots, y_n - y_{n-1}) dy_n \cdots dy_1.
\]

But by Definition 2.1.2, \( B_{t_i} \) and \( B_{t_j} \) are independent. So, the above becomes
\[ \int \cdots \int f(y_1) \cdots f(y_n - y_{n-1}) dy_n \cdots dy_1 \]
\[ = \int \cdots \int p(t_1, 0, y_1) \cdots p(t_n - t_{n-1}, y_{n-1}, y_n) dy_n \cdots dy_1 \]

as
\[ p(t, x, y) = p(t, 0, y-x) \]. If we let \( A_i = (-\infty, x_i] \), then we are done.

**Lemma 2.3.** Given a stochastic process \( (X_t, t \geq 0) \) satisfying 2.1 1) and 2), then
\[ P(\sup_{\lambda > 0} |\lambda X_s| \leq \lambda \exp(-\lambda^2/2t) \cap D_n[0, t]) \]
for \( \lambda > 0 \) where \( D \) is the set of dyadics in \( \mathbb{R} \).

**Proof:** First I will show that \( \{\exp(\theta X_t)\} \) is a submartingale. Now
\[ E[\exp(\theta X_t) | D_n] = E[\exp(\theta(X_t-X_s + X_s)) | D_n] \]
But, \( \exp(\theta X_s) \) is \( D_n \) measurable, so the above becomes
\[ \exp(\theta X_s) E[\exp(\theta(X_t-X_s)) | D_n] \]
as \( X_t-X_s \) is independent of \( D_n \). As
\[ E[\exp(\theta X_s)] = \exp(\theta^2 r/2) \]
then
\[ E[\exp(\theta X_t) | D_n] = \exp(\theta X_s) \exp(\theta^2 (t-s)/2) \exp(\theta X_s) \]
as \( t \geq s \). Define \( D_n \) to be the set of dyadics in \([0, t]\) of the form \( k/2^n \). Using Theorem 1.15, we see that
\[ P(\sup_{\lambda \in D_n} X_s \leq \lambda \exp(t \theta^2 /2 - \theta \lambda)) \]
but this last term is just \( \exp(t \theta^2 /2 - \theta \lambda) \). Notice that as \( n \) increases, \( \sup_{\lambda \in D_n} X_s \) increases also. Letting \( n \) tend to \( \infty \),
we obtain
\[ P(\sup_{\lambda \in D_n[0, t]} X_s \leq \lambda \exp(t \theta^2 /2 - \theta \lambda)) \]
that \( P(\sup_{\lambda \in D_n[0, t]} X_s \leq \lambda \exp(t \theta^2 /2 - \theta \lambda)) \) by the symmetry of
Brownian motion. Substituting \( \theta = t/\lambda \) obtains the result. \( \square \)
**Theorem 2.4:** There exists a space \((\Omega, \mathcal{F}, P)\) which contains Brownian motion.

**Proof:** Kolmogorov's Theorem guarantees a space \((\Omega, \mathcal{F}, P)\) which contains a stochastic process \((X_t, t \geq 0)\) which satisfies conditions 2.1 1) and 2). Let \(D\) be the dyadic rationals in \([0, \infty)\), and let \(D_{k,n}=[k/2^n, (k+1)/2^n] \cap D\). If we let

\[
B_n = \left( \max_{0 \leq k \leq 2^n} \sup_{r \in D_{k,n}} |X_r - X_{k2^{-n}}|/2^{n/4} \right)
\]

then a direct application of Lemma 2.4 obtains

\[
P(\max_{r \in D_{k,n}} |X_r - X_{k2^{-n}}|/2^{n/4} < 2^{n/2} \exp(-2^{n/2-1}) = n2^n \exp(-2^{n/2-1}).
\]

Thus, we have

\[
P(B_n) = \sum_{i=0}^{n2^n} 2^{i} \exp(-2^{n/2-1}) = n2^n \exp(-2^{n/2-1}).
\]

A direct application of the Ratio test shows that \(\sum_{n=0}^{\infty} P(B_n)\) converges. Let \(B = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B_i\). By the Borel-Cantelli lemma, \(P(\Omega) = 0\). Let \(\omega\) be an event outside of \(B\). Fix \(t\) and \(\epsilon\). Then there is an \(N_1\) so that \(n > N_1\) implies \(n > t\), and \(3/2^{n/4} < \epsilon\). As \(\omega\) is outside of \(B\), it can only be in finitely many \(B_i\). Thus, there is an \(N_2\) where \(n > N_2\) implies \(\omega \in B_n\). Let \(N\) be the larger of \(N_1\) and \(N_2\). For \(n > N\), let \(d_1\) and \(d_2\) be two dyadic rationals between 0 and \(t\) with \(d_1 < d_2\) and \(d_2 - d_1 < 2^{-n}\). Suppose \(d_1 \in D_{k,n}\). Then, \(d_2\) is either in \(D_{k,n}\) or \(D_{k+1,n}\). If the former is true, we have

\[
|X_{d_2} - X_{d_1}| \leq |X_{d_2} - X_{k2^{-n}}| + |X_{k2^{-n}} - X_{d_1}| \leq 2^{n/4} + 2^{-n/4} < \epsilon.
\]

If \(d_2 \in D_{k+1,n}\), we have
\[ |X - x| < \frac{|X - x|}{2^n} < (k+1)2^{-n} \]

but this is less than \(3(2^{-n/4}) \epsilon\). Thus, \(X(r)\) is uniformly continuous over the dyadic rationals on \([0, t]\) for any \(t\). Define a stochastic process \((\tilde{W}_t, t \geq 0)\) such that

\[ \tilde{W}_t(\omega) = \lim_{r \to t} (X_r(\omega)I_r(\omega)) \]

As uniformly continuous maps extend continuously to the closure of their domains, then we can conclude there is a continuous map that extends \(X_r(\omega)\) to \(\mathbb{R}\), which is the map given above. (See [14, pp. 136-137]). If we show that \((\tilde{W}_t)\) has the same finite dimensional distributions as \((X_t)\), we will be done. For \(0 < t_1 < \ldots < t_n\), let \((r_{i,j})\) be a sequence in \(D\) with \(\lim r_{i,j} = t_j\). Now, \(P(X_{r_i} \leq \xi)\)

\[ \int_{-\infty}^{x_1} \ldots \int_{-\infty}^{x_n} p(r_{i,1},0,y_1) \ldots p(r_{i,n-r_{i,n-1},y_{n-1},y_n}) dy_n \ldots dy_1 \]

by lemma 2.2. As \(p(t,x,y)\) is continuous in \(t > 0\), then

\[ \lim_{r_{i,j} \to t_j} p(r_{i,j},y_{i-1},y_i) = p(t_j,y_{i-1},y_i). \]

Thus \(\lim P(X_{r_i} \leq \xi) = P(X_{t} \leq \xi)\). Now, \(X_{r_i,j}\) approaches \(\tilde{W}_t\) by definition of \(W(\cdot, \cdot)\). For any collection \((A_i)\) of events, we know that \(\lim \sup P(A_i) \leq P(\bigcap_{i=1}^{\infty} \cup_{K=1}^{\infty} A_K)\). Let \(A_i = (X_{r_i} \leq \xi)\). Now, \(\bigcup_{K=1}^{\infty} A_K = (\omega: X_{r_{K,j}}(\omega) \leq \xi \text{ for some } j \geq i)\) and

\[ \bigcap_{i=1}^{\infty} \bigcup_{K=1}^{\infty} A_K = (\omega: \text{There exists } (k_i) \text{ such that } X_{r_{K_i}}(\omega) \leq \xi). \]

For \(\omega \in \bigcap_{i=1}^{\infty} \bigcup_{K=1}^{\infty} A_K\), it is clear that \(\omega \in (W_t \leq \xi)\). Thus, we have

\[ \lim \sup P(X_{r_i} \leq \xi) \leq P(\bigcap_{i=1}^{\infty} \bigcup_{K=1}^{\infty} A_K) \leq P(W_t \leq \xi). \]

Let \(\epsilon > 0\) be given, and
$B_i = (X_{r_i} \langle y \rangle)$ where $y = (x_1 + \epsilon, \ldots, x_n + \epsilon)$. If $\omega \in (W_t \langle x \rangle)$ then there is an $N$ so that $n > N$ implies $|W_{r_n} \langle \omega \rangle - X_{r_n} \langle \omega \rangle| < \epsilon$ for all $i$. Thus, $\omega$ is in $\cap_{i=1}^{\infty} B_k$, and so is in $\cap_{i=1}^{\infty} B_k$. Thus, $P(W_t \langle x \rangle) = P(W_t \langle x \rangle)$. Letting $\epsilon$ tend to 0, we obtain $P(X_t \langle x \rangle) = P(X_t \langle x \rangle)$. The above proof comes mainly from Billingsley [2, pp. 446-447].

Now, define a class of probability measures $(P_x)_{x \in \mathbb{R}}$ as follows:

$$P_x(W_{t_1} \in A_1, \ldots, W_{t_n} \in A_n) = P(W_{t_1} + x \in A_1, \ldots, W_{t_n} + x \in A_n)$$

for $t_1 < t_2 < \ldots < t_n$ and $A_1, A_2, \ldots, A_n$ Borel sets. As $W_t + x$ is measurable in $(\Omega, \mathcal{Q})$, then it is clear that $P_x$ is a measure. Note from the above that $P_x(W_t \langle y \rangle) = P(W_t \langle y-x \rangle)$. This implies that $W_t$ is Normally distributed with mean $x$ and variance $t$ under $P_x$. The mutual independence of $W_{t+s} - W_t$ and $W_t$ implies the mutual independence of $(W_{t+s} + x) - (W_t + x)$ and $(W_t + x)$. Thus, their independence remains under $P_x$. Continuity also clearly follows. Hence, we have established the following theorem.

**Theorem 2.5:** $W_t$ is Brownian motion with $W_0 = x$ on $(\Omega, \mathcal{Q}, P_x)$.

It may seem unusual to approach different starting points in this manner, but there is an advantage to this type of notation. If we define Brownian motion starting at $x$ by $W_t + x$, then we have the same probability measure for all stochastic processes, but the $\sigma$ algebra $\mathcal{Q}_t$ changes each time. In this case, though we change the measure, the $\sigma$ algebra remains the same. Note that $E_x$ is just the
expectation corresponding to $P_x$. In particular,

$$E_x(W_s) = \int y p(s, x, y) dy.$$ 

Thus, $E(\cdot)(W_s)$ is a measurable function. Later in the text we will wish to compose this with Brownian motion. In this case,

$$E_{W_t}(W_s) = \int y p(s, W_t, y) dy.$$ 

Now, we define $W_{t}(\cdot)(\omega)$ to be the function obtained by letting $t$ vary over $[0, \omega)$, and $W_t(\omega)$ be the associated value for $t$. Thus, $W_{t}(\cdot)$ can be thought of as a random function. $W_{s+}(\cdot)$ is defined similarly.

Let $C$ be the set of all continuous functions of $[0, \omega)$, and let $C$ be the $\sigma$ algebra generated by the cylinder sets, i.e.,

$$C = \sigma(\{f \in C : f(t_1) \in A_1, ..., f(t_n) \in A_n\}).$$

**Lemma 2.6:** Let $C$ and $C$ be defined as above. Then if

$$\|f - g\|_n = \sup_{0 \leq t \leq n} |f(t) - g(t)|$$

and

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{\|f - g\|_n}{2^n} \wedge 1$$

then $\rho$ is a metric and $C$ is just the borel sets generated by $\rho$.

**Proof:** Clearly $\rho \geq 0$ and if $\rho(f, g) = 0$ then $f = g$. If $f = g$, then $\|f - g\|_n = 0$ so $\rho(f, g) = 0$. As

$$|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|,$$

we can easily conclude that $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$. Thus, $\rho$
is a metric.

To finish the proof, I shall show that the generator sets for each set lie in the other. Define $B_{f, \epsilon} = \{g: p(f, g) < \epsilon\}$, and let $h \in B_{f, \epsilon}$. I claim there is an $A \in \mathcal{C}$ with $f \in \mathcal{A}B_{f, \epsilon}$. For $r = 2^{-k}$, let $A_r = (f(r) - \epsilon/2, f(r) + \epsilon/2)$. Choose $N$ so that $\sum_{i=N}^{\infty} 2^{-i} < \epsilon/2$ and let

$$A^p = \left\{ g: g(2^{-p}) \in A_{2^{-p}}, g(2.2^{-p}) \in A_{2.2^{-p}}, \ldots, g(N) \in A_N \right\}.$$

Notice that $A^p \subseteq A^q$ for $p > q$. Also, let $\bigcap_{p=1}^{\infty} A^p = \{ g: \sup_{0 \leq t \leq N} |g(t) - f(t)| < \epsilon/2 \}$ by continuity. Thus, for $g \in \bigcap_{p=1}^{\infty} A^p$, it is clear that $p(f, g) < \epsilon$. Hence, $ACB_{f, \epsilon}$. We can repeat the argument for every $g$ in $B_{f, \epsilon}$. Notice that $\bigcap_{p=1}^{\infty} A^p$ is an open set in the metric. As $\mathcal{C}$ is separable we can find a countable collection of these sets whose union is $B_{f, \epsilon}$. Thus, $B_{f, \epsilon} \in \mathcal{C}$.

Although Brownian motion has continuous paths, it can be shown that the paths are not differentiable with probability 1. So, the paths of Brownian motion are continuous, but not smooth. A proof of this result can be found in [4, pp. 6].
2.2 MARKOV PROPERTIES OF BROWNIAN MOTION

Definition 2.7: A stochastic process \((X_t, t \geq 0)\) is said to be Markov if for every Borel set \(B\), we have 
\[ P(X_{t+s} \in B | T_t) = P(X_{t+s} \in B | X_t) \]
if \(s \leq t\) and \(T_t\) is the \(\sigma\) algebra generated by \((X_r, 0 \leq r \leq t)\). If the above statement holds for a stopping time \(T\) replacing \(t\), then we say that \((X_t)\) is Strong Markov.

Theorem 2.8: Brownian motion is Markov. Furthermore each of the following statements are equivalent, and imply the Markov property:

1) \(W_{t+s} - W_t\) is Brownian motion and is independent of \(\mathcal{Q}_t\) for each \(P_x\).

2) If \(f\) is any bounded measurable function, then 
\[ E_x[f(W_{t+s}) | \mathcal{Q}_t] = E_{W_t}[f(W_s)] = \int f(y) p(s, W_t, y) dy. \]

3) If \(H\) is a bounded measurable function from the class \(C\) of continuous functions to the reals, then 
\[ E_x[H(W_{t+s}) | \mathcal{Q}_t] = E_{W_t}[H(W_s)]. \]

Proof: Suppose statement 1) is true. I wish to show 2). First, I will show

\[ E_x[g(W_t, W_{t+s} - W_t) | \mathcal{Q}_s] = \int g(W_t, y) p(s, 0, y) dy \]

for bounded \(g\) from \(\mathbb{R}^2\) to \(\mathbb{R}\). Suppose \(g(x, y) = g_1(x) g_2(y)\) for bounded functions \(g_1\) and \(g_2\). Then,
\[ \int g(x, y) p(s, 0, y) dy = g_1(x) \int g_2(y) p(s, 0, y) dy \]
and
\[ \int g(W_t, y) p(s, 0, y) dy = g_1(W_t) \int g_2(y) p(s, 0, y) dy. \]
Also,
\[ E_x[g(W_t, W_{t+s} - W_t) | Q_t] = E_x[g_1(W_t) g_2(W_{t+s} - W_t) | Q_t] \]
\[ = g_1(W_t) E_x[g_2(W_{t+s} - W_t)] \]
as \( W_t \) is \( Q_t \) measurable, and \( W_{t+s} - W_t \) is independent of \( Q_t \).

But,
\[ E_x[g_2(W_{t+s} - W_t)] = \int g_2(y) p(s, 0, y) dy, \]
so equality holds in this case. Using a standard monotone class argument, we obtain equality for all bounded \( g \) operating on \( \mathbb{R}^2 \).

Let \( f \) be a bounded measurable real valued function and let \( g(x, y) = f(x+y) \). Then,
\[ E_x[f(W_{t+s}) | Q_t] = E_x[g(W_t, W_{t+s} - W_t) | Q_t] = \int g(W_t, y) p(s, 0, y) dy = \int f(W_t + y) p(s, 0, y) dy = E_{W_t}[f(W_s)]. \]

Now, suppose statement 2). I wish to show 3). Let \( 0 < t_1 < t_2 < \ldots < t_n \) be given. Suppose \( w(\cdot) \in C \) and define a function \( K : C \rightarrow \mathbb{R} \) by the following rule:
\[ K(w(\cdot)) = \prod_{i=1}^{n} f_i(w(t_i)) \]
for given bounded real valued functions \( f_1, \ldots, f_n \). I claim that
\[ E_x[K(w_{t+\cdot}) | Q_t] = E_{W_t}[K(w_{\cdot})]. \]
Notice if \( n = 1 \) this is just 2). I shall use mathematical induction to finish the result. Suppose the above statement holds for some \( K(n) \). Then,
\[ E_x[K+1(\prod_{i=1}^{k} f_i(W_{t+t_i}) | Q_t)] \]
\[ = E_x\left\{E_x[K+1(\prod_{i=1}^{k} f_i(W_{t+t_i}) | Q_{t+t_k}) | Q_t]\right\} \]
\[ = E_x\left\{\prod_{i=1}^{k} f_i(W_{t+t_i}) E_x[f_{k+1}(W_{t+t_{k+1}}) | Q_{t+t_k}) | Q_t]\right\}. \]
By 2),
\[ E_x[f_{k+1}(W_{t+k+1}^t+t_k)] = E_{W_{t+k+1}^t}[f_{k+1}(W_{t+k+1}^t-t_k)]. \]

As the product \( f_K E(\cdot)[f_{k+1}(W_{t+k+1}^t)] \) is a function which is also bounded, by the assumed hypothesis, we have

\[ E_{W_{t+k+1}^t}\{E[\prod_{i=1}^{k+1} f_i(W_{t+i}) | Q_{t+k+1}^t]\} = E_{W_{t+k+1}^t}\{E[\prod_{i=1}^{k+1} f_i(W_{t+i}) | Q_{t+k+1}^t]\} \]

as

\[ E_x[\prod_{i=1}^{k+1} f_i(W_{t+i}) E_{W_{t+k+1}^t}[f_{k+1}(W_{t+k+1}^t-t_k)]] = E_x[\prod_{i=1}^{k+1} f_i(W_{t+i}) E_{W_{t+k+1}^t}[f_{k+1}(W_{t+k+1}^t-t_k)]] \]

Thus, by induction, the above statement is true for \( K \). By a standard monotone class argument, we can extend this to the class of bounded functions on \( C \). Next I wish to show 3) implies 1). Let \( A \in \mathcal{A}_t \), and let \( A_1, \ldots, A_n \) be in \( \mathbb{R} \). I propose to show that

\[ P_x((W_{t+s_1}^t - w_t \in A_1, \ldots, W_{t+s_n}^t - w_t \in A_n)A) = P_x(W_{s_1} \in A_1, \ldots, W_{s_n} \in A_n)P_x(A) \]

for some \( 0 < s_1 < \ldots < s_n \). Let \( f(w(t)) = \prod_{i=1}^{n} I_{A_i}(w(s_i) - w(0)) \).

Then

\[ P_x((W_{t+s_1}^t - w_t \in A_1, \ldots, W_{t+s_n}^t - w_t \in A_n)A) = E_x[\prod_{i=1}^{n} I_{A_i}(W_{t+s_i}^t - w_t)I_A] = E_x[f(W_{t+s_i}^t)I_A] \]

\[ = E_x(E_x[f(W_{t+s_i}^t) | Q_t]I_A) = E_x(E_{W_{t+s_i}^t}[f(W_{t+s_i}^t)]I_A) \]

\[ = E_x(E_{W_{t+s_i}^t}[\prod_{i=1}^{n} I_{A_i}(W_{s_i}^t)]I_A). \]

Notice, however, that

\[ E_x[\prod_{i=1}^{n} I_{A_i}(W_{s_i}^t)] = P_x(W_{s_1}^t - x \in A_1, \ldots, W_{s_n}^t - x \in A_n). \]
so we obtain
\[
P_0(w_{s_1} \in A_1, \ldots, w_{s_n} \in A_n),
\]
and we are done. 

For a general approach to the Markov property, one should see [3, pp. 2-5].

In the next theorem we will use the Markov Property to show that \((|W_t|, t>0)\) is Markov. This process is called Brownian motion reflected at the origin.

**Theorem 2.9:** \((|W_t|, t>0)\) has the Markov property.

**Proof:** Let \(A\) be a Borel set where \(A \subseteq (0, \infty)\), and let \(A' = \{x: |x| \in A\}\). Then
\[
P_x(|W_t| + s \in A | Q_t) = P_x(w_{t+s} \in A' | Q_t)
\]
\[
= P_{w_t}(w_s \in A') = P_{w_t}(|W_s| \in A).
\]
It can be shown with a little work that
\[
P_{-x}(|W_s| \in A) = P_x(|W_s| \in A).
\]
Thus, we have
\[
P_{w_t}(|W_s| \in A) = I(w_t \geq 0) P_{w_t}(|W_s| \in A) + I(w_t < 0) P_{-w_t}(|W_s| \in A).
\]

**Theorem 2.10:** Let \((W_t)\) be Brownian motion started at \(x>0\). Then, if \(T = \inf\{t>0: W_t = 0\}\), the stochastic process \((Z_t: t>0)\) given by \(Z_t = W_t I(t<T)\) is Markov.

**Proof:** Define a function \(H\) from \(C\) to \(\mathbb{R}\) such that \(H(w(\cdot)) = 1\) if \(w(s) \in A\), and \(w(u) > 0\) for \(0 \leq u \leq s\), and \(H(w(\cdot)) = 0\) otherwise. Notice that
\[ P_x(Z_{t+s} \in A | Q_t) = E_x(I(W_{t+s} \in A) I(T_{t+s} > s) | A_t) \]
\[ = E_x(I(W_{t+s} \in A) I(T_{t+s} > s) | A_t) \]
where \( S = \inf(s > 0 : W_{t+s} = 0) \). It is clear that the above is
\[ I(T_{t+s}) E_x(H(W_{t+s} | I) | A_t) = I(T_{t+s}) E_x(H(W_{t+s} | I) | A_t) \]
By the construction of \( H \), we see that this is the same as
\[ I(T_{t+s}) E_x(I(Z_s \in A)) \]
which is just \( P_{Z_t}(Z_s \in A) \).

**Theorem 2.11:** Let \( T \) be a stopping time with \( P(T(\infty) = 1) \). Then, \( (W_{t+s} - W_t, t \geq 0) \) is Brownian motion and is independent of \( A_T \).

**Proof:** I shall first show this for countably valued stopping times and then extend this to other stopping times. Suppose \( T \) takes only countably many values. Let \( \{T_i\} \) be an enumeration of the values \( T \) takes. Define \( E_j = T^{-1}(T = r_j) \). As \( T \) is \( A_T \) measurable, then \( E_j \in A_T \). As \( T \) is finite, \( \Omega = \bigcup E_j \). Clearly, \( E_i \cap E_j = \emptyset \) for \( i \neq j \). Let \( A \in A_T \).

Then, \( P_x[(W_{t+s} - W_t \in A_1, \ldots, W_{t+s} - W_t \in A_n) A] \)
\[ = \sum_{i=1}^{\infty} P_x[(W_{t+s} - W_t \in A_1, \ldots, W_{t+s} - W_t \in A_n) A E_i] \]
\[ = \sum_{i=1}^{\infty} P_x[(W_{t+s} - W_t \in A_1, \ldots, W_{t+s} - W_t \in A_n) A E_i] \]
as \( T(\omega) = r_i \) for \( \omega \in E_i \). By Lemma 1.20, \( W_{t+r_i} - W_{t+r_i} \) is independent of \( A_{r_i} \). Thus,
\[ \sum_{i=1}^{\infty} P_x[(W_{t+r_i} - W_{t+r_i} \in A_1, \ldots, W_{t+r_i} - W_{t+r_i} \in A_n) A E_i] \]
\[ = P_x(W_{t+1} - W_{t+1} \in A_1, \ldots, W_{t+n} - W_{t+n} \in A_n) \sum_{i=1}^{\infty} P_x(A E_i) \]
\[ = P_x(W_{t} \in A_1, \ldots, W_{t} \in A_n) \sum_{i=1}^{\infty} P_x(A E_i) \]
Now, let $T$ be any stopping time. Let $T_n = \left( \lceil 2^n T \rceil + 1 \right) / 2^n$ where $\lceil . \rceil$ is the greatest integer function. Clearly, for $T \in (i2^{-n}, (i+1)2^{-n})$, $T_n = (i+1)2^{-n}$, so $T_n$ is countably valued on the dyadic rationals. Now, the event $(T_n = (i+1)2^{-n})$ is the same as $(i2^{-n} < T < (i+1)2^{-n})$ which is in $\mathcal{A}_T(i+1)2^{-n}$. Thus, $T_n$ is a stopping time. For $m > n$, $(i+1)2^{-m} > (i+1)2^{-n}$, so $T_n < T_m$. Also, by the construction, $T_n$ decreases to $T$, and $\mathcal{A}_T \subset \mathcal{A}_T^{*}$ for each $n$. To see this last statement, let $A \in \mathcal{A}_T$. Then, $A(T_n < t) = A(T < t)(T_n < t)$ as $(T_n < t) \subset (T < t)$. But, $A(T < t) \in \mathcal{A}_T$. Also, $(T_n < t) \in \mathcal{A}_T$, so $A(T_n < t) \in \mathcal{A}_T$. From the above, we have

$$P_x(\langle W_{t_1 + t_n} - W_{t_1}, \ldots, W_{t_k + t_n} - W_{t_k} \rangle A) = P_0(\langle W_{t_1} - W_{t}, \ldots, W_{t_k} - W_{t} \rangle A).$$

As $T_n$ converges to $T$, then $W_{t_1 + t_n} - W_{t_n}$ converges to $W_{t_1} - W_{t}$ with probability one. Thus,

$$P_x(\langle W_{t_1 + t_1} - W_{t_1} \rangle, \ldots, W_{t_k + t_n} - W_{t_k} \rangle A) = P_x(\langle W_{t_1} - W_{t_1}, \ldots, W_{t_k} - W_{t_k} \rangle A).$$

If we let $A=\Omega$ in the above, we see that the joint distribution of $W_{t_1 + t_1} - W_{t_1}, \ldots, W_{t_k + t_n} - W_{t_k}$ is the same as $F_{t_1, \ldots, t_n}$. Thus, parts 1 and 2 of definition 2.1 follow. Continuity necessarily follows as $\langle W_t \rangle$ started with continuous paths. Thus, $W_{t_1 + t_1} - W_{t_1}$ is Brownian motion.

Let $B=\langle W_{t_1 + t_1} - W_{t_1} \in \mathcal{A}_1, \ldots, W_{t_k + t_n} - W_{t_n} \in \mathcal{A}_n \rangle$. Then, from the above work we see that $P_x(AB) = P_x(A)P_x(\langle W_{t_1} \in \mathcal{A}_1, \ldots, W_{t_n} \in \mathcal{A}_n \rangle)$. But, as $P_x(B) = P_x(\langle W_{t_1} \in \mathcal{A}_1, \ldots, W_{t_n} \in \mathcal{A}_n \rangle)$, we have $P_x(AB) = P_x(A)P_x(B)$. Thus, each set of the same form as $B$, which is called a cylinder set, is independent of every element in $\mathcal{A}_T$, then by a similar monotone class argument, we conclude that $W_{t + t} - W_t$ is independent of $\mathcal{A}_T$.

The above proof comes from [15, pp. 36-38].
Theorem 2.12: If $T$ is a stopping time, then each of the following statements are equivalent, and imply that Brownian motion is Strong Markov:

1) $W_{T+t} - W_T$ is Brownian motion and independent of $\mathcal{Q}_T$.

2) If $f$ is any bounded measurable function, then $E[f(W_{T+t}) | \mathcal{Q}_T] = E_{W_T}[f(W_t)]$.

3) If $H$ is any bounded measurable function from the class $C$ of continuous functions to the real numbers, then $E[H(W_{T+\cdot}) | \mathcal{Q}_T] = E_{W_T}[H(W_{\cdot})]$.

The proof of this Theorem is similar to the proof of Theorem 2.8.

Next, I wish to define Brownian motion in greater dimensions. Intuitively, Brownian motion in $\mathbb{R}^d$ can be thought of as a stochastic process $(W_t : t \geq 0)$ where $W_t$ is a vector $(W_{t,1}, \ldots, W_{t,d})$ with $(W_{t,i}, t \geq 0)$ being Brownian motion in $\mathbb{R}$ for each $i$. Thus, the definition for $W_t$ is similar to that of $(W_{t,i})$ with the exception that $W_{t+s} - W_t$ is distributed multivariate Normal with mean vector 0 and covariance matrix $tI$. Hence, in this case, the function $p(t, x, y)$ is given by

$$p(t, x, y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{1}{2} \|x-y\|^2 / 2t}.$$ 

The previous results for Brownian motion can then be shown for this case in a similar manner.
2.3 DYNKIN'S FORMULA:

Next I wish to develop some tools using Brownian motion that will be needed in the next chapter. First I need to show Dynkin's formula in the continuous case.

**Definition 2.13:** Let \((X_t, T_t, t \geq 0)\) be an adapted stochastic sequence. Then \(X\) is a local martingale if there is a sequence of increasing stopping times \((T_n)\) converging up to \(\infty\) such that \(X_{t \wedge T_n}\) is a martingale for each \(n\). The sequence \((T_n)\) is said to reduce \(X\).

**Theorem 2.14:** Let \(f\) be a bounded continuous function from \(\mathbb{R}^d\) to \(\mathbb{R}\) with \(\frac{\partial f}{\partial x_1}\) and \(\frac{\partial f}{\partial x_i x_j}\) continuous. Then,

\[
f(W_t) - \int_0^t \frac{1}{2} \Delta f(W_r) \, dr
\]

is a local martingale. Similarly, if \(f(t, x)\) is bounded from \([0, \infty) \times \mathbb{R}^d\) to \(\mathbb{R}\), and the partial derivatives are again continuous,

\[
f(t, W_t) - \int_0^t \left( \frac{\partial}{\partial t} + \frac{1}{2} \Delta f \right)(r, W_r) \, dr
\]

is a local martingale.

**Proof:** Suppose first that the first, second and third partial derivatives of \(f\) all exist and are bounded. It may be instructive to show this in the one dimensional case and then extend it to the more complicated one. Let \(s < t\), and let \(t_0 = s\) and \(t_i = t_{i-1} + \frac{(t-s)}{n}\) for \(1 \leq i \leq n\). Then we have
by Taylor expansion where $T_i$ is of order $O(|W_{t_{i+1}} - W_{t_i}|^3)$. As $E(X) = E(E(X|A))$, we can take the conditional expectation of each term with respect to $A_{t_i}$. In this case,

$$E(f'(W_{t_i})(W_{t_{i+1}} - W_{t_i})|A_{t_i}) = f'(W_{t_i})E(W_{t_{i+1}} - W_{t_i}) = 0,$$

and similarly,

$$E(f''(W_{t_{i+1}})(W_{t_{i+1}} - W_{t_i})^2|A_{t_i}) = f''(W_{t_i})(t-s).$$

Thus, the above becomes

$$E\left[\frac{1}{2}\sum_{i=0}^{n-1} f''(W_{t_{i+1}})(t_{i+1} - t_i)|A_{t_i}\right] + E(TA)$$

where $T = \sum T_i$. If we wish to take the limit of the above as $n$ tends to $\infty$, notice that the limit will come inside the first expectation by the bounded convergence theorem. Also, it is easy to show that $|E(T_i A)| \leq c(t_{i+1} - t_i)^{3/2}$. So as $t_{i+1} - t_i = (t-s)/n$, we have $|E(TA)| \leq c \sum_{i=0}^{n-1} n^{-3/2} = cn^{-1/2}$, which tends to $0$ as $n$ tends to $\infty$. Hence, we have the result for $d=1$. Let $f:[0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be bounded with the partial derivatives also bounded, and define $t_i$ in the same manner. Again, we wish to expand $f(t_{i+1}, W_{t_{i+1}})$ in a Taylor's
expansion about \((t_i, \omega_{t_i})\) and take conditional expectation with respect to \(\mathcal{Q}_{t_i}\). From the expansion, terms of the form

\[ \langle \omega_{t_i+1}^j \rangle - \langle \omega_{t_i}^j \rangle, \quad (t_{i+1} - t_i) \langle \omega_{t_i+1}^j \rangle - \langle \omega_{t_i}^j \rangle, \quad \text{and} \quad \langle \omega_{t_i+1}^j \rangle - \langle \omega_{t_i}^j \rangle \langle \omega_{t_i+1}^k \rangle - \langle \omega_{t_i+1}^k \rangle \]

where \(j\) is not equal to \(k\) will be 0 when the conditional expectation is applied.

Thus, \(\mathbb{E}(f(t_{i+1}, \omega_{t_{i+1}}) - f(t_i, \omega_{t_i}) | \mathcal{Q}_{t_i}) = \) \\

\[ \frac{\partial f}{\partial t}(t_i, \omega_{t_i}) (t_{i+1} - t_i) + \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}(t_i, \omega_{t_i}) (t_{i+1} - t_i)^2 + T_i \]

where \(T_i\) is the remainder term. It is not hard to show \(|T_i| \leq Cn^{-3/2}\). Taking the limit as before yields the result.

A similar argument shows

\[ \mathbb{E}_x(f(W_t) - f(W_s) | A) = \mathbb{E}_x \left[ \int_s^t \frac{1}{2} \Delta f(W_u) du | A \right]. \]

Suppose now that \(f\) has continuous, though not necessarily bounded, first and second partial derivatives.

If \(T_r = \inf \{t > 0: \omega_t \notin B_{x,r} \}\), I wish to show

\[ \mathbb{E}_x(f(W_{t \wedge T_r}) - f(W_{s \wedge T_r}) | A) = \mathbb{E}_x \left[ \int_{s \wedge T_r}^{t \wedge T_r} \frac{1}{2} \Delta f(W_u) du | A \right]. \]

Let \(\phi\) be a function on \(\mathbb{R}^d\) such that \(\phi\) has continuous derivatives, \(\phi(y) = 1\) for \(y \in B_{x,r}\), and \(\phi(y) = 0\) for \(y \in B_{x,2r}\). Then \(f\phi\) is continuous with compact support, and has continuous bounded first and second partial derivatives. Let \(\epsilon > 0\) be given. By a process called regularization, there is a function \(g\) with continuous derivatives and compact support such that \(g\) approximates \(f\) and the derivatives of \(f\) uniformly within \(\epsilon\). For more details, see [10, pp. 17-18].

As \(g\) has bounded continuous derivatives, \(g\) satisfies the criteria for the result above. Clearly, this is also true if \(t\) and \(s\) are replaced by \(t \wedge T_r\) and \(s \wedge T_r\). But,
\[
E_x \left[ g(W_{t\wedge T}) - g(W_{s\wedge T}) - \int_{s\wedge T}^{t\wedge T} \Delta g(W_u) du \right]
\]

\[
- f\phi(W_{t\wedge T}) + f\phi(W_{s\wedge T}) - \int_{s\wedge T}^{t\wedge T} \frac{1}{2} \Delta f\phi(W_u) du | I_A
\]

\[
\langle 2\epsilon P(A) + \epsilon E_x( |t\wedge T - s\wedge T | I_A) \rangle.
\]

As \( \epsilon \) is arbitrary, this shows the above result, as \( f\phi = f \) on \( B_x, \tau \). If \( s = 0 \), the above work shows

\[
f(W_t) - \int_0^t \frac{1}{2} \Delta f(W_u) du
\]

is a local martingale with reducing sequence \( (T_n) \). A similar argument finishes the proof for the second formula.

**Theorem 2.15:** Let \((M_t, T_t, \tau)\) be a continuous local martingale with \( M_0 = 0 \) and such that \( M_t \) has paths of bounded variation. Then, \( M_\tau = 0 \).

**Proof:** Let \( \tau = \inf\{t > 0 : |M_t| = k \text{ or the variation of } M_t \text{ is } k\} \). I claim \( M_{t\wedge \tau} \) is a martingale. As \( M_t \) is a local martingale there is a sequence of stopping times \((T_n)\) tending to \( \omega \) that reduce \( M \), i.e., \( M_{t\wedge T_n} \) is a martingale for each \( n \). Now, as \( T_n \wedge \tau \) is bounded, then by Theorem 1.14, we have for \( \mathcal{A} \in \mathcal{T}_S \), we have

\[
E[M_{T_n \wedge t \wedge \tau} | I_A] = E[M_{T_n \wedge s \wedge \tau} | I_A].
\]

Thus, \( M_{T_n \wedge t \wedge \tau} \) is a martingale. Notice that it is also bounded by \( k \). Thus, by Theorem 1.16, as \( n \) tends to \( \omega \), it must tend to a martingale. Note that this must be \( M_{t \wedge \tau} \).

Notice that for \( X_t = M_{t \wedge \tau} \),

\[
E[X_t^2 - X_s^2 | T_s] = E[X_t^2 | T_s] - X_s^2.
\]

But,
Fix $t > 0$ and let $t_i = it/n$ for $0 < i < n$. Then,

$$
E[X_t^2] = E\left[\sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 \right] = E\left[\sum_{i=0}^{n-1} E[(X_{t_{i+1}} - X_{t_i})^2 | T_{t_i}] \right]
$$

$$
= E\left[\sum_{i=0}^{n-1} E\left(\sum_{j=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 | T_{t_i} \right) \right] = E\left[\sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 \right].
$$

But, $\sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 \leq 2k^2$. Also, I claim this term also goes to 0 as $n$ goes to $\infty$. Choose $\epsilon > 0$, and fix $w$. Then, as $X_s(w)$ is continuous and of bounded variation, we can find $n$ large enough so that $|X_{t_{i+1}} - X_{t_i}| < \epsilon$ for each $i$. This is true as $|X_{t_{i+1}} - X_{t_i}|$ tends to 0 pointwise, but $\sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}| \leq k$ for each partition. Thus, $\sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|^2 \leq \epsilon k$. As $\epsilon$ is arbitrary, we are done. Hence by the bounded convergence theorem, we have $E(X_t^2) = 0$. As $M_t^2$ tends to $M_{t^\wedge T}$ as $k$ tends to $\infty$, Fatou's lemma gives

$$
E(M_t^2) \leq \lim \inf E(M_{t^\wedge T}^2) = 0.
$$

Thus, $M_t = 0$ almost everywhere. As $M$ is continuous, we are done.

Next I wish to use the above results to discuss the recurrence or transience of Brownian motion. If $A$ is any nonempty set in $\mathbb{R}^d$, then $L_A = \sup\{t > 0: W_t \notin A\}$ is called the last exit time of $A$. If $P_x(L_A = \infty) = 1$ for each $x$, then $A$ is said to be a recurrent set. If however, $P_x(L_A < \infty) = 1$, $A$ is said to be transient.
Theorem 2.16: Brownian motion is recurrent in dimensions 1 and 2 but transient in any dimension greater than 2.

Proof: Consider first dimension 1. I claim

\[ P_0(\lim sup_{t \to \infty} W_t = \infty) = 1. \]

Let \( T_x = \inf\{t \geq 0 : W_t = x\} \), and let \( A_s = (W_t = x \text{ for some } t \leq s) \). Then,

\[ P_0(A_s) = E_0(E_0(I_{A_s} | Q_s)) = E_0(E_{W_s}(I_{A_0}) \]

by the Markov property. But, \( E_{W_s}(I_{A_0}) = P_{W_s}(T_x < \infty) = 1 \) by Proposition 1.22. Hence,

\[ P_0(\lim_{t \to \infty} sup_{W_t > N}) = 1 \text{ for some } u \geq t = 1. \]

As \( N \) is arbitrary, we obtain the result. By symmetry of Brownian motion, \( P_0(\lim_{t \to \infty} inf_{W_t < -N}) = 1 \text{ also. This shows } P_0(L_0 = \infty) = 1. \)

Consider now dimension 2. Let \( T_r = \inf\{t \geq 0 : W_t \in \overline{B}_x, r\} \). If \( 0 < r < R \), define \( T = T_r \wedge T_R \). It is an exercise in calculus to show \( f(x) = \log(\|x\|) \) satisfies the equation \( \Delta f = 0 \) on \( B_0, R \cap \overline{B}_0, r \).

Then, by Theorem 2.14, \( f(W_t) \) is a local martingale. Thus, by Theorem 1.15, \( E_x f(W_{T^\wedge t}) = f(x) \). Suppose \( r < \|x\| < R \). Then, \( W_{T^\wedge t} \) is bounded, and I can bring the limit inside the expectation and obtain \( E_x f(W_T) = f(x) \). So,

\[ f(x) = E_x f(W_{T_r})P_x(T_r < T_R) + E_x f(W_{T_R})P_x(T_R < T_r) \]

\[ = \log r P_x(T_r < T_R) + \log R (1 - P_x(T_r < T_R)). \]

Thus,

\[ P_x(T_r < T_R) = \frac{\log |x| - \log R}{\log r - \log R}. \]

Taking the limit as \( R \) tends to \( \infty \), we find \( P_x(T_r < \infty) = 1 \). Notice that as we take the limit as \( r \) tends to 0, \( P_x(T_r < T_R) \) tends to 0. So, \( P_x(T_0 < T_R) = 0 \) and \( P_x(T_0 < \infty) = 0 \) also. Let \( A_s = (W_t \in \overline{B}_0, r \text{ for some } t \leq s) \). The same computation as above
\[ P_x(A_s) = E_x(\mathbf{E}_w(I_{A_0})) = E_x(P_w(T_w < \omega)) = 1. \]

So, \( P_x(\lim \inf \|W_t\| < r) = 1 \), and \( B_0, r \) is a recurrent set. This implies any open set is recurrent. By a similar argument, we can show \( P_0(W_t \in B_0, r \text{ for some } t) = 1 \) as Brownian motion always exits an open set. Hence, \( P_x(\lim \sup \|W_t\| = \omega) = 1. \) \( \lim t \to \infty \)

In dimension \( d \geq 3 \), I leave as an exercise to the reader that \( f(x) = \|x\|^{2-d} \) satisfies the equation \( \Delta f = 0 \) on \( B_0, R \cap B_0, r \). Thus, a similar analysis as before gives us

\[ P_x(T_R < \omega) = \frac{\|x\|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}}. \]

Letting \( R \) tend to \( \infty \) leads to \( P_x(T_R < \omega) = \frac{\|x\|^{2-d}}{r^{2-d}} < 1. \) Now, by the Strong Markov property, and using the same argument as before, we can show

\[ P_x(\inf \mathbf{s} T_N < N) = E_x(P_{w_T}(T_N < \omega)) = N^{2-d}. \]

So,

\[ \lim t \to \infty P_x(\inf \|W_t\| < N) = N^{2-d}. \]

As \( N \) is arbitrary, we have \( \lim t \to \infty P_x(\lim \sup \|W_t\| = \omega) = 0. \) Hence, \( P_x(\lim t \to \infty \|W_t\| = \omega) = 1, \) and Brownian motion is transient.

Thus, as was alluded to in Chapter 1, Brownian motion and Random walk have similar recurrence and transience relationships.
CHAPTER III
PARTIAL DIFFERENTIAL EQUATIONS

3.1 THE HEAT EQUATION

We are ready now to solve some partial differential equations using the tools acquired in Chapter 2. In Chapter 1 I stated the Dirichlet problem on a lattice. Here we will look for solutions of certain partial differential equations, the Dirichlet problem and the Heat equation, using Brownian motion. Much of the work in this chapter is modelled after work done by Durrett [4, pp. 220-229, 246-251]. I will consider first the heat equation.

**Definition 3.1:** If $u$ is a continuous function on $[0,\infty) \times \mathbb{R}^d$ such that $u_t = \frac{1}{2} \Delta u$ on $(0,\infty) \times \mathbb{R}^d$, and $u(0,x) = f(x)$, then $u$ is a solution of the Heat equation on $[0,\infty) \times \mathbb{R}^d$ with boundary information contained in $f$.

The purpose of this section will be to show $v(t,x) = E_x(f(W_t))$ is a solution given certain restrictions of $f$.

**Lemma 3.2:** If $f$ is bounded and continuous, then $v(t,x) = E_x(f(W_t))$ is continuous on $[0,\infty) \times \mathbb{R}^d$ and $v(0,x) = f(x)$.

**Proof:** Let $(t_n)$ and $(x^n)$ be sequences whose limits are $t$ and $x$, respectively. Now,
\[ \lim v(t_n, x^n) = \lim \int f(y)p(t_n, x^n, y)\,dy \]
\[ = \lim \int f(t_n)^{1/2}u+x^n)p(1,0,u)\,du \]
Now, we can bring the limit inside by the dominated convergence theorem as \( f \) is bounded, so \( \lim v(t_n, x^n) = v(t, x) \).
If \( t=0 \), notice that this is just \( f(x) \).

\[ \text{Lemma 3.3: If } f \text{ is bounded, then } v_t, v_{x_i}, \text{ and } v_{x_i}x_j \text{ for } 1 \leq i \leq j \leq d \text{ all exist and are continuous.} \]

\[ \text{Proof: Notice that } \frac{\partial}{\partial x_i}p(t, x, y) = \frac{-(x_i - y_i)}{t} p(t, x, y). \]
Also,
\[ \frac{\partial^2}{\partial x_i^2}p(t, x, y) = \left( \frac{-(x_i - y_i)^2}{t^2} - \frac{1}{t} \right) p(t, x, y). \]
Notice that both of the above are continuous. If \( |f| \leq M \),
\[ \int \left| \frac{\partial}{\partial x_i}p(t, x, y)f(y) \right| \,dy \leq M \int \left| \frac{\partial}{\partial x_i}p(t, x, y) \right| \,dy \leq M. \]
I claim also, that \( \int \frac{\partial}{\partial x_i}p(t, x, y)f(y)\,dy \) is continuous. Let
\[ x^i \rightarrow x. \] There is a \( B_{x, r} \subset \mathbb{R}^d \) and an \( N \) such that \( n \geq N \) implies \( x^n \in B_r, x \). Also, it is clear that \( \|x^i - y\| \leq r + \|x - y\| \), and \( \|x - y\| \leq r \leq \|x^i - y\| \). So, \( \frac{\partial}{\partial x_i}p(t, x, y)f(y) \) is dominated by
\[ \left| \frac{(r+x_i - y_i)}{t} \exp \left( -\frac{(r-\|x-y\|)^2}{2t} \right) \right|, \]
so the limit can come past the integral by the dominated convergence theorem, and we are done.

Now, I claim that the above is \( v_{x_i} \). Let \( e_i \) be defined as before. Then,
\[ v(t, x+he_i) - v(t, x) = \int (p(t, x+he_i,y) - p(t, x,y))f(y)\,dy \]
\[
= \int_0^h \frac{\partial}{\partial x_i} p(t, x + \theta e_i, y) f(y) d\theta dy.
\]
Dividing both sides by \(h\), and taking limits, we have
\[
\frac{\partial v}{\partial x_i} = \lim_{h \to 0} \int_0^h \frac{\partial}{\partial x_i} p(t, x + \theta e_i, y) f(y) d\theta dy
\]
\[
= \lim_{h \to 0} \int_0^h \frac{\partial}{\partial x_i} p(t, x + \theta e_i, y) f(y) dy d\theta.
\]
As the inner integral is continuous, it assumes a maximum value \(L\) on the segment from \(x\) to \(x + e_i\). Thus, the integral is bounded, and we have
\[
\lim_{h \to 0} \int_0^h \frac{\partial}{\partial x_i} p(t, x + \theta e_i, y) f(y) dy d\theta = \int_0^h \frac{\partial}{\partial x_i} p(t, x, y) f(y) dy.
\]
(See [14, pp. 102]). A similar case will show that \(\frac{\partial^2 v}{\partial x_i^2}\) exists and is continuous also.

**Theorem 3.4:** \(v(t, x) = E_x(f(W_t))\) is a solution of the Heat equation.

**Proof:** By Theorem 2.8 2), we have
\[
E_x(f(W_t)|\mathcal{F}_s) = E_{W_s}(f(W_{t-s})) = v(t-s, W_s).
\]
So, by Theorem 1.17, \(v(t-s, W_s)\) is a martingale for \(0 \leq s < t\).

Also, it easily follows from Dynkin's formula that
\[
v(t-s, W_s) - \int_0^s \left(\frac{1}{2} \Delta v - v_t\right)(t-r, W_r) dr
\]
is a local martingale. So, \(v(t, W_s) =\)
\[
(v(t, W_s) - \int_0^s \left(\frac{1}{2} \Delta v - v_t\right)(t-r, W_r) dr) + \int_0^s \left(\frac{1}{2} \Delta v - v_t\right)(t-r, W_r) dr.
\]
This implies \(\int_0^s \left(\frac{1}{2} \Delta v - v_t\right)(t-r, W_r) dr\) is a local martingale.
Thus, by Theorem 2.17, it is equal to 0 for each s, as it is of bounded variation. Note that if $\frac{1}{2}\Delta u - v_t > 0$ at some point $(t,x)$, then $\frac{1}{2}\Delta u - v_t > 0$ on some open ball $B(t, x, r)$ by continuity. Then, we can take $s$ small enough so that

$$\int_0^s \langle \frac{1}{2}\Delta u - v_t \rangle (t-r, W_r) \, dr > 0$$

with positive probability, which is a contradiction. A similar case exists for $\frac{1}{2}\Delta u - v_t < 0$. So, $\frac{1}{2}\Delta u = v_t$ and $v$ satisfies the criteria for the Heat equation.

**Theorem 3.5:** If there is a bounded solution $u$ of the Heat equation, it is of the form

$$u(t,x) = E_x \langle f(W_t) \rangle.$$

**Proof:** Let $U_s = u(t-s, W_s)$. I claim $U_s$ is a local martingale. By Dynkin's formula,

$$U_s - \int_0^s \langle \frac{1}{2}\Delta u - v_t \rangle (t-r, W_r) \, dr$$

is a local martingale. As $\frac{1}{2}\Delta u - v_t = 0$ by hypothesis, this gives the result. As $u$ is bounded, $U_s$ is also. Thus, by Theorem 1.18, $U_s$ must converge to a limit $U$. We know $U = f(W_t)$ by the continuity of $u$. By the same Theorem, as $U_s$ is bounded, $U_s = E_x[f(W_t) | Q_s]$. Thus, $U_0 = u(t,x) = E_x[f(W_t)]$, and we have the result.

Notice that in Lemma 3.2, we only needed the fact that $f$ is bounded to prove the sufficient differentiability and continuity of $v$ in $(0, \omega) \times \mathbb{R}^d$. In fact, by the nature of $p(t, x, y)$, it is fairly clear that $v$ is in the class of functions all of whose derivatives exist and are continuous. Hence it is instructive to note that for all $t > 0$, $v$ is a
very smooth function even if $f$ is discontinuous.

**Theorem 3.6:** Suppose $u$ is the bounded solution to the Heat equation of $[0,\omega) \times \mathbb{R}^d$, with the above given boundary condition, and let $t > 0$ be fixed. Then

$$\sup_{0 \leq s \leq t} u(s, x) = \sup_{x \in \mathbb{R}^d} u(0, x).$$

**Proof:** Note that for $t \in [0, \omega)$ and $x \in \mathbb{R}^d$,

$$u(t, x) = \mathbb{E}_x f(W_t) = \int f(y) p(t, x, y) dy.$$

Suppose for some $t$ and $x$ that $u(t, x) > u(0, y) = f(y)$ for all $y \in \mathbb{R}^d$. Then, if there is some $z$ where the inequality is strict, we can easily show $u(t, x) > \int f(y) p(t, x, y) dy$ by the continuity of $f$. Thus, $u(t, x) = f(y)$ for all $y \in \mathbb{R}^d$. Hence, either the maximum occurs for $t = 0$, or $u$ is constant.  

I next wish to obtain a solution to the inhomogeneous equation, \( u_t = \frac{1}{2} \Delta u + g \), where \( g \) satisfies certain restrictions. The solution will turn out to be a combination of the homogeneous solution, and a particular solution. In order to simplify the computation, the initial boundary condition, given before by \( f \), will be 0. Thus, I wish to find a continuous function \( u \) on \([0, \infty) \times \mathbb{R}^d\) where \( u(0, x) = 0 \), and \( u_t = \frac{1}{2} \Delta u + g \) for some function \( g \). I claim the function we are looking for is

\[
v(t, x) = E_x \left( \int_0^t g(t-s, W_s) \, ds \right).
\]

We will need to proceed similarly in this case as before, by showing that \( v \) is continuous and has continuous derivatives.

**Lemma 3.7:** If \( g \) is bounded, \( v(t, x) \) given above is continuous in \([0, \infty) \times \mathbb{R}^d\), and \( u(0, x) = 0 \).

**Proof:** Suppose \( |g| < M \). Then, \( E_x \left( \int_0^t g(t-s, W_s) \, ds \right) < tM \). As \( t \) tends to 0, we see that \( tM \) tends to 0 also, so

\[
\lim_{t \to 0} v(t, x) = 0 = v(0, x).
\]

Let \((x_i)\) and \((t_i)\) be sequences converging to \( x \) and \( t > 0 \), respectively. Notice that

\[
E_x \left( \int_0^t g(t-s, W_s) \, ds \right) = \int_0^t g(t-s, y) p(s, x, y) \, ds \, dy
\]

\[
= \int_0^t \int_0^t I(s) g(t-s, y) p(s, x, y) \, ds \, dy = \int_0^t I(s) E_x \left[ g(t-s, W_s) \right] ds.
\]

As \( g \) is bounded, then
\[
\lim_{n \to \infty} \int_{(0, t_n)} I(s) \, E_x \left[ g(t-s, W_s) \right] \, ds \\
= \int_{(0, t)} I(s) \lim_{n \to \infty} E_x \left[ g(t_n - s, W_s) \right] \, ds
\]

If we can show the limit can come inside the expectation, we will be done. Notice that

\[
E_x \left( g(t_n - r, W_r) \right) = \int g(t_n - r, y) p(t_n, x_n, y) \, dy
\]

As \( t-r=\epsilon >0 \), there is an \( N_1 \) so that \( |t-t_n|<\epsilon/2 \) for \( n \in N \).

Then, \( t_n - r > \epsilon /2 \), and \( 2\pi (t_n - r) - d/2 < \epsilon - d/2 \) for all \( n \in N_1 \).

Also, \( t_n - r < 3\epsilon /2 \), so \( \|x - y\|^2 \leq (2(t_n - r)) \|x - y\|^2 /3\epsilon \). There is also an \( N_2 > 0 \) so that \( \|x_n - x\| < \delta \) for \( n \in N_2 \). Thus, \( \|x_n - y\| < \delta \). The above work shows

\[
g(r, y) p(t_n - r, x_n, y) \leq C \epsilon^{-d/2} \exp \left( -\delta \|x - y\|^2 /3\epsilon \right).
\]

Thus, by the Dominated Convergence Theorem, the limit can come inside the expectation, and we are done.

\[\Box\]

**Lemma 3.8:** If \( v(t, x) \) is given as above, then \( \frac{\partial v}{\partial x_i} \) exists and is continuous.

**Proof:** I wish to show that

\[
\frac{\partial v}{\partial x_i} = \int_0^t \int g(t-s, y) \frac{\partial}{\partial x_i} p(s, x, y) \, dy \, ds.
\]

Now we see that

\[
\int_0^t \int g(t-s, y) \frac{\partial}{\partial x_i} p(s, x, y) \, dy \, ds
\]

\[
= \int_0^t E_x \left[ (x_i - (W_s)_i) g(t-s, W_s) \right] \, ds
\]

where \((W_s)_i\) is the \( i \)th component of \( W_s \). As \( |g| \leq M \),

\[
E_x \left[ |(x_i - (W_s)_i) g(t-s, W_s)| \right] \leq ME_x \left[ |x_i - (W_s)_i| \right] \leq C \sqrt{s},
\]

so
\[ \int_0^t \left| g(t-s, \gamma) \frac{\partial}{\partial x_1} p(s, x, \gamma) \right| dyds \Rightarrow \int_0^t c \sqrt{s} \, ds. \]

If I can show the above integral is continuous, then a similar argument as in Lemma 3.3 will give the result. Let \( t_n \to t \), and \( x^n \to x \). Then, for large enough \( n \), \( t_n < 2t \). So,

\[ \int_0^{t_n} \frac{1}{s} \mathbb{E}_{x^n} \left[ (x^n_i - (W_s)_i) g(t-s, W_s) \right] ds = \int_0^{2t} I(s) \frac{1}{s} \mathbb{E}_{x^n} \left[ (x^n_i - (W_s)_i) g(t-s, W_s) \right] ds. \]

Then, if we take the limit of the above, we can bring the limit inside by the Dominated convergence Theorem. Again, if we can bring the limit inside the expectation, we will be done. However, with a little work, we can dominate

\[ \frac{\partial}{\partial x_1} p(t_n - s, x^n, \gamma) \]

by an integrable function, so the limit can come inside by the Dominated Convergence Theorem.

Though it was fairly easy to show the first derivative exists, we cannot prove \( \frac{\partial^2 u}{\partial x_1^2} \) exists and is continuous without further restrictions on \( g \). If \( g \) satisfies the property that for every \( n \), there are constants \( L_n \) and \( \alpha_n \) where

\[ |g(t, x) - g(t, y)| \leq L_n \|x - y\|^{\alpha_n} \]

for \( t \), \( \|x\| \), and \( \|y\| \leq n \), then \( g \) is said to be locally Hölder continuous. We will need this restriction for the next two results.

**Lemma 3.9:** If \( g \) is bounded and locally Hölder continuous, then \( \frac{\partial^2 u}{\partial x_1^2} \) exists and

\[ \frac{\partial^2 u}{\partial x_1^2} = \int_0^t \int_0^s g(t-s, \gamma) \frac{\partial^2}{\partial x_1^2} p(s, x, \gamma) dyds. \]
Proof: Notice that
\[
\int_0^t \int p(s,x,y) \frac{\partial^2}{\partial x_i^2} p(t-s,y) \, dy \, ds = \int_0^t \mathbb{E}_x \left( \frac{(x_i - (W_s)_i)^2 - s}{s^2} g(t-s,W_s) \right) \, ds
\]
Notice \( \mathbb{E}_x \left( (x_i - (W_s)_i)^2 \right) = s \), so \( \mathbb{E}_x \left( (x_i - (W_s)_i)^2 - s \right) = 0 \), and we have
\[
\mathbb{E}_x \left( \frac{(x_i - (W_s)_i)^2 - s}{s^2} g(t-s,W_s) \right) \]
\[
\mathbb{E}_x \left( \frac{(x_i - (W_s)_i)^2 - s}{s^2} (g(t-s,W_s) - g(t,x)) \right).
\]
As \( g \) is bounded, this is finite. Then, there is an \( N \) so that
\[
\frac{1}{\sqrt{2\pi s^2}} \int_{B_0,N} \frac{|(x_i - y_i)^2 - s|}{s^2} \exp(-\|x-y\|^2/2t) \, dy \, ds
\]
\[
|g(t-s,W_s) - g(t-s,x)| \, dy \, ds^{-1/2},
\]
As \( |g(t-s,W_s) - g(t-s,x)| \leq L_N \|W_s - x\|^{\alpha_N} \) for some constants \( L_N \) and \( \alpha_N \) where \( x, y \in B_0,N \) and \( t < N \), if we replace the inside of the above expectation by its absolute value, it can be bounded by
\[
\frac{1}{\sqrt{2\pi s^2}} \int_{B_0,N} \frac{|(x_i - y_i)^2 - s|}{s^2} \|x-y\|^{\alpha_N} \exp(-\|x-y\|^2/2s) \, dy \, ds^{-1/2}.
\]
With a little work, we can show this is less than \( Cs^{-1+\alpha+N} - \frac{1}{2} \) for some constant \( C \) and where \( \alpha = \alpha_N/2 \). Thus, we have
\[
\int_0^t \mathbb{E}_x \left( \frac{|(x_i - (W_s)_i)| - s}{s^2} \, g(t-s,W_s) \right) \, ds \leq \int_0^t (Cs^{-1+\alpha+N} - \frac{1}{2}) \, ds < \infty.
\]
By a similar argument as we have done before, we can bring the limit inside and we have the continuity of
\[
\int_0^t \mathbb{E}_x \left( \frac{(x_i - (W_s)_i)^2 - s}{s^2} g(t-s,W_s) \right) \, ds.
\]
Thus, as before, we can argue that the second partial derivative is continuous, and we are done.

All we have left to show is the fact that the partial of $v$ with respect to $t$ exists and is continuous. Once we have this fact, we will have shown that $v$ is sufficiently differentiable to invoke Dynkin's formula to show that $v$ is a solution of the inhomogeneous Heat equation.

**Lemma 3.10:** Let $g$ be given as in Lemma 3.9. Then, $\frac{\partial v}{\partial t}$ exists and is continuous.

**Proof:** I wish to show

$$\frac{\partial v}{\partial t} = g(t,x) + \int_0^t \int \frac{\partial}{\partial t} p(t-r,x,y) g(r,y) dydr.$$  

First, I wish to point out some elementary facts that will be used later. Notice that

$$\frac{\partial}{\partial t} \int p(t-r,x,y) g(r,y) dy = \int \frac{\partial}{\partial t} p(t-r,x,y) g(r,y) dy$$

by similar arguments as before. Also, a simple calculation shows

$$\frac{\partial}{\partial t} p(t-r,x,y) = \frac{1}{2} \Delta p(t-r,x,y).$$

By the proof of Lemma 3.9, it is easy to see that

$$\int \frac{1}{2} \Delta p(t-r,x,y) g(r,y) dy \lessdot C_1 (t-r)^{-1+\alpha} + C_2 (t-r)^{-1/2},$$

so

$$\int \frac{\partial}{\partial t} p(t-r,x,y) g(r,y) dy \lessdot C_1 (t-r)^{-1+\alpha} + C_2 (t-r)^{-1/2}.$$  

Now, suppose $t < N$, and let $h > 0$ be given. Notice that

$$\frac{v(t+h,x) - v(t,x)}{h} = \int_0^{t+h} \int_0^t g(r,y) p(t+h-r,x,y) dyds - \int_0^t \int g(r,y) p(t-r,x,y) dydr.$$
\[ \int_{t}^{t+h} g(r, y) p(t+r-x, y) \, dy \, dr \]

\[ + \frac{1}{h} \int_{0}^{h} g(r, y) [p(t-r-x, y) - p(t-r, x)] \, dy \, dr. \]

Now, by a change of variable, the first integral becomes

\[ \frac{1}{h} \int_{0}^{h} g(t-r, x-y) \, p(1, 0, u) \, du \, dr. \]

Taking the limit as \( h \) tends to 0, this becomes \( g(t, x) \).

Notice the second integral becomes

\[ \int_{0}^{t} g(r, y) \frac{\partial}{\partial t} p(t-r, x, y) \, dy \, dr \]

\[ = \int_{0}^{t} \frac{\partial}{\partial t} g(r, y) p(t-r, x, y) \, dy \, dr \]

where \( t < t' < t+h \). But, this is integrable. Let \( \epsilon > 0 \) be given.

Then, there is a \( t_{\epsilon} \) such that

\[ \int_{t_{\epsilon}}^{t} \left| \frac{\partial}{\partial t} g(r, y) p(t-r, x, y) \right| \, dy \, dr < \epsilon, \]

provided \( h \) is small enough. Also, it is easy to see by continuity that

\[ \left| \frac{\partial}{\partial t} g(r, y) p(t-r, x, y) \right| \, dy \, dr < \frac{\epsilon}{t} \]

for sufficiently small \( h \) and for \( r < t_{\epsilon} \). Thus, for small enough \( h \),

\[ \int_{0}^{t} \frac{\partial}{\partial t} g(r, y) p(t-r, x, y) \, dy \, dr \]

\[ < 3 \epsilon. \]

As \( \epsilon \) was arbitrary, then \( \int_{0}^{t} \frac{\partial}{\partial t} g(t-r, x, y) \, dy \, dr \)

\[ \int_{0}^{t} \frac{\partial}{\partial t} p(t-r, x, y) \, dy \, dr. \]

With work similar to that done before, it can be shown that continuity follows.

For more detail of the arguments for Lemmas 9 and 10 with a slightly different proof, the reader should see Friedman [8,
Theorem 3.11: Suppose \( v(t,x) = \mathbb{E}_x \left( \int_0^t g(t-r, \mathbf{W}_r) \, dr \right) \) where \( g \) is bounded and Hölder continuous. Then, \( v \) satisfies \( v_t - \frac{1}{2} \Delta v = g \) on \((0, \infty) \times \mathbb{R}^d\).

Proof: Define \( H: \mathcal{C} \rightarrow \mathbb{R} \) such that
\[
H(w(.)) = \int_0^{t-s} g(t-s-r, w(r)) \, dr.
\]
Then, \( H(w(s+.)) = \int_s^t g(t-r, w(r)) \, dr \). Then by the third form of the Markov property, we have
\[
\mathbb{E}_x \left( \int_s^t g(t-r, \mathbf{W}_r) \, dr \right) = \mathbb{E}_x \left( \int_0^t g(t-s-r, \mathbf{W}_r) \, dr \right) | \mathcal{A}_s,
\]
But,
\[
\mathbb{E}_x \left( \int_s^t g(t-r, \mathbf{W}_r) \, dr \right) = \mathbb{E}_x (H(\mathbf{W}_{s+}.)) | \mathcal{A}_s) = \mathbb{E}_x (H(\mathbf{W}_s)).
\]
So, the above becomes
\[
\int_0^s g(t-r, \mathbf{W}_r) \, dr + \mathbb{E}_x \left( \int_0^{t-s} g(t-s-r, \mathbf{W}_r) \, dr \right).
\]
As the left hand side is a martingale by Theorem 1.17, then the right hand side is also. Hence,
\[
v(t-s, \mathbf{W}_s) + \int_0^s g(t-r, \mathbf{W}_r) \, dr = v(t-s, \mathbf{W}_s) - \int_0^s (-v_t + \frac{1}{2} \Delta v)(t-r, \mathbf{W}_r) \, dr.
\]
As \( v(t-s,W_s) - \int_0^s (-v_t + \frac{1}{2} \Delta v + g)(t-r, W_r) dr \) is a local martingale by Dynkin's formula, then \( \int_0^s (-v_t + \frac{1}{2} \Delta v + g)(t-r, W_r) dr \) must be a local martingale. As the above is also of bounded variation, \(-v_t + \frac{1}{2} \Delta v + g = 0\), so we are done.

**Theorem 3.12:** If \( u \) is a bounded solution of the inhomogeneous Heat equation given above, then \( u \) must be of the form given in Theorem 3.11.

**Proof:** Suppose \( u(t,x) \) is continuous and bounded in \([0, \infty) \times \mathbb{R}^d\) with \( u(0,x) = 0 \) and \( u_t = \frac{1}{2} \Delta u + g \) on \((0, \infty) \times \mathbb{R}^d\). Let

\[
U_s = u(t-s, W_s) + \int_0^s g(t-r, W_r) dr.
\]

I claim \( U_s \) is a local martingale for \( 0 \leq s < t \). Notice that

\[
u(t-s, W_s) - \int_0^s (-u_t + \frac{1}{2} \Delta u)(s-r, W_r) dr
\]

is a local martingale by Dynkin's formula. Replacement of \(-u_t + \frac{1}{2} \Delta u\) by \(-g\) obtains the result. As \( u \) and \( g \) are both bounded, \( U_s \) is bounded and by Theorem 1.18 converges to a limit \( U \) as \( s \) tends to \( t \). By the continuity, we have

\[
U = u(0, W_t) + \int_0^t g(t-r, W_r) dr
\]

\[
= \int_0^t g(t-r, W_r) dr.
\]

Hence we obtain

\[
U_s = \mathbb{E}_x(U | A_s) = \mathbb{E}_x\left( \int_0^t g(t-r, W_r) dr | A_s \right),
\]

and
\[ U_0 = u(t,x) = \mathbb{E}_x \left( \int_0^t g(t-r, W_r) \, dr \right). \]

**Theorem 3.13:** Suppose \( g(\cdot, \cdot) \geq 0 \) on \([0, \infty) \times \mathbb{R}^d \). Then if \( u \) is a bounded solution, \( u(\cdot, \cdot) \geq 0 \) on \([0, \infty) \times \mathbb{R}^d \).

The proof is trivial. Notice that this says that the maximum or minimum of \( u \) lies on the boundary, as \( u = 0 \) when \( t = 0 \). Suppose we wish to find a solution of the inhomogeneous heat equation where \( u(0, x) = f(x) \). Then, the previous work tells us that

\[ u(t,x) = \mathbb{E}_x \left( f(W_t) \right) + \mathbb{E}_x \left( \int_0^t g(t-r, W_r) \, dr \right) \]

is the unique bounded solution. Again, if \( g \leq 0 \), then for fixed \( t \), \( \sup_{s \leq t} u(s,x) = \sup_{x \in \mathbb{R}^d} u(0,x) \).
Next I wish to look at the homogeneous solution $u$ that satisfies $\Delta u = 0$ on some open set $G$ where $u$ is continuous in $\bar{G}$ and $u(x) = f(x)$ on $\partial G$. I claim a solution is of the form $v(x) = E_x f(W_T)$ where $T = \inf\{t > 0 : W_t \in \partial G\}$.

The procedure for this type of equation is the same as before. I will show continuity and sufficient differentiability of $v$, and then use Dynkin's formula to show $v$ satisfies the hypothesis.

**Lemma 3.14:** Let $v(x) = E_x f(W_T)$. Then every derivative of $v$ exists and is continuous.

**Proof:** Let $x$ be any point in $G$. As $G$ is open, there is a ball $B_{x,r}$ about $x$ such that $\bar{B}_{x,r} \subset G$. Define $T_r = \inf\{t > 0 : W_t \in \partial B_{x,r}\}$. Let $H : \mathbb{C} \to \mathbb{R}$ by the rule $H(w(.)|w(T)) = f(w(T))$ for $T = \inf\{t : w(t \in \partial G)\}$. Notice that if $t < T$ then $H(w(t+.)|w(T)) = f(w(t+S))$ where $S = \inf\{s : w(t+s) \in \partial G\}$. But, $t + S = T$, so $H(w(t+.)|w(T)) = f(w(T))$. Hence,

$$v(x) = E_x f(W_T) = E_x E_x\left[f(W_T) | Q_{T_r}\right]$$

$$= E_x E_x\left[H(W_{T_r+}(.)) | Q_{T_r}\right] = E_x E_x\left[H(w(.))\right]$$

by the third form of the Strong Markov property. Notice that this just says Brownian motion can run until it hits the boundary of $B_{x,r}$ and then start over again. Thus,
\[ \nu(x) = \int_{\partial B_{x,r}} \nu(y) \, d\pi(y) \] where \( \pi(y) \) is the distribution measure of \( W_{t,r} \). I claim that \( \pi(y) \) is a uniform measure, i.e., the probability of \( W_{t,r} \) hitting any set \( A \) on the surface of \( B_{x,r} \) is proportional to the area of \( A \). This is clear by the distribution of the increments of \( W_t \). As \( W_{t+s} - W_s \) is Normally distributed with mean 0 and variance \( s \), and the density function only depends on the square of the norm in \( \mathbb{R}^d \), the distribution does not change on rotation. Hence, rotation of \( W_{t,r} \) does not change the distribution, and it is clear the distribution is uniform. I wish to use this fact to show the sufficient differentiability of \( \nu \).

Let \( \psi \) be a nonnegative function vanishing on \([r^2, \infty)\) where all the derivatives exist and are continuous. Then, \( \psi(\|x-y\|^2) \) vanishes outside \( B_{x,r} \). Define a function \( g(x) \) such that

\[ g(x) = \int_{\partial B_{x,r}} \psi(\|x-y\|^2) \nu(y) \, dy = \int_{B_{x,r}} \psi(\|x-y\|^2) \nu(y) \, dy. \]

Note that

\[ \frac{g(x+he_i) - g(x)}{h} = \frac{\frac{1}{h} \int_{B_{x,2r}} \left[ \psi(\|y-(x+he_i)\|^2) - \psi(\|y-x\|^2) \right] \nu(y) \, dy}{h} \]

for sufficiently small \( h \). Thus, by the bounded convergence theorem, we have

\[ \frac{\partial g}{\partial x_i}(x) = \int_{B_{x,2r}} \psi(\|y-x\|^2) \nu(y) \, dy. \]

Thus the differentiability of \( g \) depends upon the differentiability of \( \psi \). Notice that continuity of \( g' \) follows by the bounded convergence theorem and the
continuity of $\phi'$. A similar process for higher order derivatives follows. Now, by a change of variable, we see that

$$g(x) = \int_{B_0, r} \phi(\|z\|^2)v(x+z)dz$$

$$= \int_0^r \int_{\partial B_0, s} C\phi(s^2)v(x+z)s^{d-1}d\sigma(z)ds$$

$$= \int_0^r C\phi(s^2)s^{d-1} \left[ \int_{\partial B_0, s} v(x+z)d\sigma(z) \right]ds$$

$$= \int_0^r C\phi(s^2)s^{d-1}v(x)ds = C'v(x).$$

Thus, as $C'$ does not depend on $x$, the differentiability of $v$ depends on the differentiability of $g$, and we are done.

Note that the above proof depends heavily on the existence of an infinitely differentiable function $\phi$ which vanishes on $[r^2, \infty)$. Thus, to complete the argument, I need to show one exists. Let $\phi(x) = \exp(1/(x-r^2))1_{(-\infty, r^2)}$. It is clear that any derivative of $\exp(1/(x-r^2))$ contains terms of the form $(x-r^2)^{-a}\exp(1/(x-r^2))$ for some $a>1$. The only place where there might be a discontinuity is at $x=r^2$. But, as terms of the form above tend to 0 as $x$ approaches $r^2$, we see continuity exists.

I next wish to address the problem of continuity at the boundary. Continuity of $v(x)$ at the boundary is a geometric property depending on the shape of the boundary called regularity.

**Definition 3.15:** A point $x \in \partial G$ is said to be a regular point if $P_x(T=0)=1$. The open set $G$ is said to be regular if every point in $\partial G$ is regular.

**Theorem 3.16:** Let $G$ be an open set, and let $f$ be
bounded and continuous on $\partial G$. Then, if $G$ is regular, $v$ is continuous on $\partial G$.

**Proof:** I claim that $P_x(T \leq t)$, $t > 0$, is a lower semicontinuous function in $x$. Notice that

$$P_x(W_S \in \tilde{G} \text{ for some } s, \epsilon < s \leq t)$$

$$= \int P_y(T \leq t-s)p(\epsilon, x, y)dy \cdot \int p(\epsilon, x, y)dy.$$  

Let $x^n$ tend to $x$. Then

$$\lim_{n \to \infty} \int P_y(T \leq t-s)p(\epsilon, x^n, y)dy = \int P_y(T \leq t-s)p(\epsilon, x, y)dy$$

by the dominated convergence theorem as before. So,

$P_x(W_S \in \tilde{G} \text{ for some } \epsilon < s \leq t)$ is continuous. As $\epsilon$ tends to 0, we see that the above class of functions is increasing, and as an increasing limit of continuous functions must be lower semicontinuous, we see that $P_x(T \leq t)$ is lower semicontinuous.

Now, let $(x^n)$ be a sequence in $G$ such that $x^n \to y$ where $y \in \partial G$, and $y$ is a regular point. By the properties of lower semicontinuous functions, we have $\lim_{n \to \infty} \int P_y(T \leq t) = P_y(T \leq t) = 1$.

Also, as $\lim_{n \to \infty} P_x (T \leq t) = 1$, we have $\lim_{n \to \infty} P_x (T \leq t) = 1$ for any $t > 0$.

Let $\epsilon > 0$ be given. Then choose $t$ small enough to make

$$P_0(\max_{0 \leq s \leq t} |W_s| \geq \delta/2) < \epsilon.$$  

Then we have

$$\lim_{n \to \infty} \int P_{x^n}(W_T \in B_Y, \delta) \geq \int P_{x^n}(T \leq t, \sup_{0 \leq s \leq t} |W_s - x^n| \geq \delta/2)$$

$$\geq \int P_{x^n}(T \leq t) - P_0(\sup_{0 \leq s \leq t} |W_s| \geq \delta/2) \geq 1 - \epsilon$$

as $P(AB) \geq P(A) - P(A \cup B)$ for any two sets $A$ and $B$. Letting $\epsilon$ tend to 0 shows $\lim_{n \to \infty} \int P_{x^n}(W_T \in B_Y, \delta) \to 1$. As $f$ is bounded and continuous, then for each $\delta$ there is a $\gamma$ so that

$$P_{x^n}(f(W_T) \in B_{f}(\gamma), \gamma)$$

tends to 1 as $n \to \infty$. Letting $\delta \to 0$, $\gamma$
goes to 0 by the continuity of \( f \), and \( \lim_{x \to 0} E_n(f(W_t)) \) tends to \( f(y) \). As \( f(y) = E_y f(W_t) \), the proof is complete.

In Chapter 1 we found that a deleted point in an open set is not regular. I would next like to give a sufficient, though not necessary criterion for regularity.

**Theorem 3.17:** Let \( G \) be an open set, and let \( y \in \partial G \).

If there is a cone \( V \) with vertex \( y \) and \( r > 0 \) such that \( \forall B_y, r \subseteq G \), then \( y \) is a regular point.

**Proof:** Let \( T = \inf \{ t > 0 : W_t \in V \} \). By spherical symmetry of the Normal distribution, \( P_y(W_t \in V) = p > 0 \) which only depends on the size of \( V \). Also it must be the same for every \( t \) by the same reason. This shows \( P_y(T < t) \sim p \). As \( P_y(T = 0) = \lim_{t \to \infty} P_y(T < t) \) by the previous Theorem, I claim \( P_y(T = 0) = 1 \). This can be shown by Blumenthal's zero one law. In essence, if \( Q = \bigcap \{ A_t, t > 0 \} \), Blumenthal's zero one law states \( P_x(A) \epsilon (0,1) \) for all \( A \in Q \). Clearly, \( (T = 0) \in Q \), so if we show Blumenthal's zero-one law, we will be done. First, I claim \( E_x(Y|Q) = E_x(Y) \) for \( Y \) such that \( E_x|Y| < \infty \). Suppose \( Y = \prod_{i=1}^{n} f_i(W_{t_i}) \) where \( 0 < t_1 < \ldots < t_n \), and each \( f_i \) is bounded and continuous. Choose \( \epsilon \) such that \( 0 < \epsilon < t_1 \). Then, if \( A \in Q \), \( A \in Q_\epsilon \) also. So, by the Markov property, we have

\[
E_x \left[ \prod_{i=1}^{n} f_i(W_{t_i} + \epsilon) I_A \right] = E_x \left[ E_x \left( \prod_{i=1}^{n} f_i(W_{t_i} + \epsilon) | A_\epsilon \right) I_A \right] = E_x \left[ E_{W_\epsilon} \left( \prod_{i=1}^{n} f_i(W_{t_i}) \right) I_A \right].
\]

As \( \prod_{i=1}^{n} f_i \) is bounded, then taking the limit of both sides via the dominated convergence theorem yields
\[ E_x \left[ \prod_{i=1}^{n} f_i(w_{t_i}) I_A \right] = E_x \left[ E_x \left( \prod_{i=1}^{n} f_i(w_{t_i}) I_A \right) \right]. \]

A standard monotone class argument then gives the result.

Now, let \( A \in \mathcal{A} \). Then, \( I_A \) is \( \mathcal{A} \) measurable, so \( I_A = E_x(I_A|\Omega) \). But, by the above, \( E_x(I_A|\Omega) = E_x(I_A) = P_x(A) \). As \( I_A \) takes only the values 0, and 1, we are done. Thus, Brownian motion starting at \( y \) will instantly hit \( V \), and by the continuity, will instantly hit \( V \cap B_y, r \) for any \( r \).

It is interesting to note that if regularity is approached from an analytic viewpoint, the concept is harder to grasp. In fact, \( G \) is sometimes defined to be regular if the bounded solution \( v \) is continuous at the boundary. Regularity can also be approached by barrier functions. A function \( w \) is a barrier function for \( x \in \partial G \) if \( w \) is defined on \( \Omega \setminus \partial G \) for some open set \( \Omega \) containing \( x \) and such that \( \Delta w \geq 0 \), \( w > 0 \), and \( \lim_{y \to x} w(y) = 0 \). It can be shown that \( x \) is a regular boundary point if and only if there is a barrier at \( x \). For more information, the reader can see [9, pp. 168-170]. Thus we note that the probabilistic definition of regularity is not only easier to state and use, but it gives us an intuitive feel for the criterion. That is, a point \( y \) on \( \partial G \) is regular if \( W(\cdot) \) starting at \( y \) hits \( \partial G \) instantly. Often \( G \) is restricted to have a continuously differentiable boundary in order overcome the regularity restriction. But, from the viewpoint of this text, a general, regular open set requires no more work than a set with a smooth boundary.

**Theorem 3.18:** Let \( G \) be an open bounded set in \( \mathbb{R}^d \) and let \( f \) be bounded. Then, \( v(x) \) given above satisfies \( \Delta v = 0 \) on \( G \). If \( G \) is regular, then \( v \) is continuous on \( \bar{G} \).
Proof: The continuity has already been shown previously. Thus, we have only to show $\Delta u = 0$ on $G$. Let $r > 0$ be given so that $B_x, r \subseteq G$. Then, by Dynkins formula,

$$v(W_{t \wedge T_r}) - \int_0^{t \wedge T_r} \frac{1}{2} \Delta v(W_s) ds$$

is a martingale with respect to $(\Theta_{t \wedge T_r}, t \geq 0)$ where $T_r$ is defined as before. Notice that

$$v(W_{t \wedge T_r}) = E_x f(W_T) = E_x (f(W_T) | \Theta_{t \wedge T_r})$$

and so $v(W_{t \wedge T_r})$ is a martingale by Theorem 1.6. Hence,

$$\int_0^{t \wedge T_r} \frac{1}{2} \Delta v(W_s) ds$$

is a martingale which is also of bounded variation, so $\int_0^{t \wedge T_r} \frac{1}{2} \Delta v(W_s) ds = 0$. Suppose $\Delta v(x) > 0$. Then, there is an $r > 0$ small enough so that $\Delta v(y) > 0$ for $y \in B_x, r$.

Thus, $\int_0^{t \wedge T_r} \frac{1}{2} \Delta v(W_s) ds > 0$, which is a contradiction. A similar case applies for $\Delta v(x) < 0$. Hence, $v$ satisfies the Dirichlet problem.

Theorem 3.19: If there is a bounded solution of the equation $\Delta u = 0$ above where $G$ is bounded, then it must be $u(x) = E_x f(W_T)$.

Proof: Suppose $u$ satisfies the above Dirichlet problem. Let $G_n = \{x \in G : \|x - y\| < 1/n \text{ for all } y \in \partial G \}$, and let $D_n = B_0, n \cap G_n$, and $T_n = \inf(t > 0 : W_t \in \partial D_n)$. Notice that $(T_n)$ is an increasing sequence of stopping times whose limit is $T$. As $\Delta u \equiv 0$, then Dynkin's formula shows $v(W_{s \wedge T_n})$ is a martingale and so by
Theorem 1.14: \( E_x u(W_{t \wedge T_n}) = u(x) \) as \( t \wedge T_n \) is a bounded stopping time. Now, I wish to take the limit of the above as \( t \) tends to \( \infty \). However, as \( u \) is bounded and continuous, I can bring the limit inside to obtain \( E_x u(W_T) = u(x) \). Taking the limit as \( n \) tends to \( \infty \), I can bring it inside by the same reason, and so have \( E_x u(W_T) = u(x) \). However, as \( W_T \) is on \( \mathcal{G} \), \( u(W_T) = f(W_T) \), and the result follows.

Theorem 3.20: Let \( G \) be an open regular subset of \( \mathbb{R}^d \), and suppose \( f \) is bounded. Then, if \( P_x(T < \infty) = 1 \) for each \( x \) in \( G \), we have \( u(x) \) given above as the unique solution of \( \Delta u = 0 \), \( u \) is continuous in \( \overline{G} \), and \( u(x) = f(x) \) on \( \partial G \).

Proof: Note that the only place in the above two proofs that the boundedness of \( G \) was used was to imply \( P_x(T < \infty) = 1 \) for all \( x \) in \( G \). Thus, the same proofs follow as before.

Theorem 3.21: Suppose \( G \) and \( f \) are as in Theorem 3.18 and \( P_x(T < \infty) < 1 \) on a set of positive measure. Then,

1) \( v(x) = E_x f(W_T 1_{T < \infty}) + P_x(T = \infty) \) is also a bounded solution, and

2) if \( u(x) \) is a bounded solution of \( \Delta u = 0 \), then \( u \) is of the form

\[
  u(x) = E_x f(W_T 1_{T < \infty}) + \alpha P_x(T = \infty).
\]

Proof: Let \( h(x) = P_x(T = \infty) = E_x[I(T = \infty)] \). We see that \( h \) has the property

\[
  h(x) = \int_{\partial B_{x, r}} h(y)d\pi(y)
\]
by a similar proof as before. Hence,
by Dynkin's formula, we have
\[ h(W_t \wedge T) - \int_0^{t \wedge T} \frac{1}{2} \Delta h(W_s) \, ds \]
is a local martingale. Suppose \( \Delta h(x) > 0 \) for some \( x \). Then, by the continuity, we can pick \( r > 0 \) so \( \Delta h(y) > 0 \) for \( y \in B_{x,r} \cap \mathbb{C}^G \). Let \( T_r = \inf\{t > 0 : W_t \in B_{x,r}\} \). Then,
\[ E_x[h(W_{t \wedge T_r}) - \int_0^{t \wedge T_r} \frac{1}{2} \Delta h(W_s) \, ds] = h(x) \]
by Theorem 1.15. Now, as \( h \) and \( \Delta h \) are bounded on \( B_{x,r} \), if we take the limit of the above as \( t \) tends to \( \omega \), we have
\[ E_x[h(W_{T_r}) - \int_0^{T_r} \frac{1}{2} \Delta h(W_s) \, ds] = h(x). \]

But, as \( E_x[h(W_{T_r})] = h(x) \) by the above, we have
\[ E_x(\frac{1}{2} \int_0^{T_r} \Delta h(W_s) \, ds) = 0, \]
which is a contradiction to the assumed hypothesis. The case for \( \Delta h(x) < 0 \) is similar. Thus, \( \Delta h = 0 \).

Let \( y \in \partial G \). Notice that \( P_x(T = \omega) \leq P_x(T > 1) \). But, as \( x \) approaches \( y \), \( P_x(T > 1) \) approaches 0 as \( P_y(T = 0) = 1 \). So, \( h(x) \) tends to 0 as \( x \) tends to \( y \). This shows \( h \) is a solution of the above equation where \( f \equiv 0 \).

Suppose \( u \) is a bounded solution. Then, \( u \) is continuous in \( \bar{G} \). Let \( D_n \) and \( T_n \) be given as before. Suppose \( v \) is the bounded solution of the Dirichlet problem on \( D_n \) with boundary data given by \( u \). Then, \( v(x) = E_x(u(W_{T_n})) \) by Theorem 3.19. But, as \( u \) satisfies the Dirichlet problem on \( G \), it also satisfies the Dirichlet problem on \( D_n \). So, by uniqueness properties, \( u = v \) on \( \bar{D}_n \). Thus, for fixed \( x \), as \( n \) increases, \( E_x(u(W_{T_n})) \) is constant. Now, \( u \) is bounded, so by the dominated convergence theorem,
\[ \lim_{n \to \infty} E_x \left( u(\mathcal{W}_T^n) I_{T<n} \right) = E_x \left( \lim_{n \to \infty} u(\mathcal{W}_T^n) I_{T<n} \right) = E_x \left[ u(\mathcal{W}_T^n) I_{T<\infty} \right] + \lim_{n \to \infty} E_x \left[ u(\mathcal{W}_T^n) I_{T=\infty} \right]. \]

Let \( h(x) = \lim_{n \to \infty} \left[ u(\mathcal{W}_T^n) I_{T<\infty} \right] \). Notice that \( h \) is bounded as \( u \) is. Also, as

\[
E_x \left[ u(\mathcal{W}_T^n) I_{T=\infty} \right] = \left( E_x \left[ E_x \left( u(\mathcal{W}_T^n) I_{T<n} \right) I_{T=\infty} |\mathcal{Q}_t \right] \right) + E_x \left[ u(\mathcal{W}_T^n) I_{T=\infty} \right] = E_x \left[ E_x \left( u(\mathcal{W}_T^n) I_{T<n} \right) I_{T=\infty} |\mathcal{Q}_t \right] + E_x \left[ u(\mathcal{W}_T^n) I_{T=\infty} \right],
\]

an application of the dominated convergence theorem gives

\[ h(x) = E_x \left[ h(\mathcal{W}_T^n) I_{T<n} \right]. \]

I wish to show \( h(x) = \alpha P_x(T=\infty) \) for some constant \( \alpha \) independent of \( x \). Notice that if \( |u| \leq M \), \( h(\mathcal{M}_x(T=\infty)) \geq 0 \) and satisfies the criteria above, so I can assume \( h \) is nonnegative.

Extend \( h \) to \( \mathbb{R}^d \) by setting \( h=0 \) on \( \tilde{G} \). Then,

\[ E_x h(\mathcal{W}_t^n) \geq E_x \left[ h(\mathcal{W}_t^n) I_{T<n} \right] = h(x) \]

on \( \tilde{G} \). Also, \( E_x h(\mathcal{W}_t^n) \geq 0 = h(x) \) on \( \tilde{G} \) as \( h \geq 0 \). Hence, \( E_x h(\mathcal{W}_t^n) \geq h(x) \) on \( \mathbb{R}^d \). Again, by use of the Markov property,

\[ E_x h(\mathcal{W}_{n+1}^t) = E_x \left[ E_x \left[ h(\mathcal{W}_{n+1}^t) |\mathcal{Q}_n \right] \right] = E_x \left[ E_x (E_x h(\mathcal{W}_{n+1}^t) |\mathcal{Q}_n) \right]. \]

Let \( g_t(x) = E_x h(\mathcal{W}_t^n) \). Then, by the above, \( \{g_n \} \) is an increasing sequence of functions and so must converge to some function \( g \) on \( \mathbb{R}^d \) as \( n \) tends to \( \infty \). I claim \( g \) is constant. Let \( x,y \in G \) be given. Then, for \( t>0 \),

\[ |g_t(y) - g_t(x)| = \]
\[ \int |p(t,x,z) - p(t,y,z)| h(z) \, dz \]
\[ \approx M \int |p(t,x,z) - p(t,y,z)| \, dz \]
\[ = M \int |p(1,0,u) - p(1,t^{-1/2}(x-y),u) | \, du. \]

As \( p(t,x,y) \) is continuous and integrable, this goes to 0 by the dominated convergence theorem. Hence \( g(x) = g(y) = \alpha \). As \( g(x) \geq h(x) \), \( h(x) \leq \alpha \) for every \( x \). So, \( h(x) \leq \alpha P_x(T \geq t) \). As this is true for any \( t \), \( h(x) \leq \alpha P_x(T = \infty) \). But, \( \mathbb{E}_x [h(W_t) 1_{T > t}] = \mathbb{E} [h(W_t) 1_{T > t}] \leq \alpha P_x(T > t) \).

As this is true for all \( t \), then \( \alpha \leq h(x) + \alpha P_x(T = \infty) \), so \( h(x) \leq \alpha P_x(T = \infty) \), and we are done. \( \Box \)

The latter part of the above proof comes from Port and Stone [12, pp. 116-118].

From the last theorem, it is interesting to see that while there are no unique bounded solutions as was found in the solution for the Heat equation, all the bounded solutions are of the same form. Also, as before, the sufficient differentiability of \( v \) was dependent only on the boundedness of \( f \). Thus, even if the boundary condition is very discontinuous, the solution is smooth everywhere in \( G \).

As the bounded solution \( u \) to the Dirichlet problem is continuous on \( \overline{G} \) it may be useful to ask where \( u \) attains its maximum. The next theorem will give a partial answer.

**Theorem 3.22:** Let \( G \) be a regular, connected, open set in \( \mathbb{R}^d \), and let \( u(x) \) be a bounded solution to the Dirichlet problem where \( u(x) = f(x) \) on \( \partial G \) with \( f \) bounded. Then,

\[ \max_{x \in G} u(x) = \max_{x \in \partial G} u(x). \]
Proof: Suppose there is an \( x \) in \( G \) such that \( u(x) \leq u(z) \) for all \( z \) in \( G \), and let \( y \) be any point in \( G \). As \( u \) satisfies the Dirichlet problem, \( \Delta u = 0 \). Thus, by Dynkin's formula \( u(W_{t \wedge T_\rho}) \) is a martingale. Notice that \( E_x u(W_{t \wedge T_\rho}) = u(x) \) as \( t \wedge T_\rho \) is a bounded stopping time. Thus, by the dominated convergence theorem, taking the limit as \( t \) tends to \( \infty \) obtains \( E_x u(W_{T_\rho}) = u(x) \). Thus, \( u(y) = u(x) \) for all \( y \in \partial B_x, r \) as the maximum of \( u \) is at \( x \). As \( G \) is connected, we can find a finite sequence \( (x_1, \ldots, x_K) \) in \( G \) and a sequence \( (r_1, \ldots, r_K) \) where \( x_{j+1} \in \partial B_{x_j}, r_j \), \( x_1 \in \partial B_{x}, r \), and \( y \in \partial B_{x_k}, r_k \). Also, we can force \( B_{x_i}, r_i \) inside of \( G \). Thus, by the above, \( u(y) = u(x) \). So, either the maximum of the function lies along the boundary, or \( u \) is constant. \( \blacksquare \)
REFERENCES


APPENDIX
In this section, I wish to state some properties and theorems which are used in this text. I first wish to review some fundamental concepts of conditional expectation. Let $\mathcal{T}$ be a $\sigma$-algebra and let $\mathcal{T}_0$ be a sub $\sigma$-algebra of $\mathcal{T}$. If the event $\{X \in A\}$ is in $\mathcal{T}$ for every Borel set $A$, then $X$ is said to be $\mathcal{T}$ measurable. Now, the conditional expectation of $X$ given $\mathcal{T}_0$, denoted $E(X|\mathcal{T}_0)$, is defined to be that unique random variable which is $\mathcal{T}_0$ measurable and where $E[X|A] = E[E(X|\mathcal{T}_0)I_A]$ for all $A \in \mathcal{T}_0$, provided $E|X| < \infty$.

Next, I will state without proof a Theorem due to Kolmogorov which discusses the possibility of existence of stochastic processes. The interested reader can see Billingsley, 1979, p. 433-8, or Vardahan, 1980, p. 279-82.

**Theorem A.1:** Let $\mathcal{T}$ be any nonempty set. For each finite ordered subset $(t_1, \ldots, t_d) = t$ of $\mathcal{T}$, let $P_t$ be a probability on the Borel $\sigma$-algebra in $\mathbb{R}^d$. If

1) $P_t(B) = P_{t_1, t_{d+1}}(B \times \mathbb{R})$ for any Borel set $B$ in $\mathbb{R}^d$, and

2) $F_\phi(x_1, \ldots, x_d) = F_t(x_{\phi(1)}, \ldots, x_{\phi(d)})$ where $\phi$ is a permutation on $(1, 2, \ldots, d)$, and $F$ is the distribution function formed by the above probability measure, then there is a probability space $(\Omega, \mathcal{F}, P)$ and a collection of random variables $(X_t : t \in \mathcal{T})$ in $(\Omega, \mathcal{F})$ such that $P_t(B) = P((X_{t_1}, \ldots, X_{t_d}) \in B)$ for all Borel sets $B$ of $\mathbb{R}^d$.

**Definition A.2:** A collection of sets $M$ which is closed under monotone unions and intersections is
called a monotone class.

**Theorem A.3:** Let $T$ be an algebra and let $M(T)$ be the smallest monotone class containing $T$. If $\sigma(T)$ is the $\sigma$ algebra generated by $T$, then $M(T) = \sigma(T)$.

The above theorem is given without proof, but is used to show that a property which holds on an algebra can be extended to the $\sigma$ algebra if it holds for a monotone class. For more information, see Billingsley, 1979, p. 34-5.