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The Modern Mathematics Classroom: A Collection of Virtual Manipulatives for Teachers and Students

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THE MODERN MATHEMATICS CLASSROOM: A COLLECTION
OF VIRTUAL MANIPULATIVES FOR
TEACHERS AND STUDENTS

by

David Stowell

A thesis submitted in partial fulfillment
of the requirements for the degree

of

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ABSTRACT

The Modern Mathematics Classroom: A Collection of Virtual Manipulatives for Students and Teachers

by

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An important assumption in the implementation of the National Council of Teachers of Mathematics (NCTM) Standards 2000 is that the mathematics classroom is a place where students are actively involved in the learning process. One way to foster such a learning environment is by using manipulatives. By their nature, manipulatives make the learning of mathematics a discovery-based activity. As computer use increases in the classroom, virtual manipulatives will become more important as instructional tools. Virtual manipulatives offer advantages over their traditional versions. Most important is their dynamic nature. Their dynamic capabilities provide two main benefits. First, the number of potential problems for students to explore is increased dramatically. Students can use this capability to explore an idea from many viewpoints using different problems. Second, problem
states are updated instantly and results are displayed. That is, the user can see immediately the effect of changes in problem parameters. Both of these benefits promote student engagement in the learning of mathematics. This thesis consists of a collection of such virtual manipulatives, in the form of Java applets, along with a description of their use.

(28 pages)
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David Steven Stowell
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INTRODUCTION

This thesis is part of a three-year project at Utah State University to develop a library of uniquely interactive web-based virtual manipulatives. The project is supported by a grant from the National Science Foundation (NSF). The goal of the project is to create new approaches to teaching mathematics interactively.

The Mathematical Sciences Educational Board (MSEB) of the National Research Council affirms the importance of technology in the teaching and learning of mathematics:

The availability of computers has renewed the emphasis on realistic applications, greatly simplifying the treatment of data in the classroom and permitting dynamic representations of complex processes. With the aid of computers, students can have experiences heretofore impossible in representing patterns, estimating solutions, and exploring how changes in one representation affect another. Technology... allow(s)... problems to be explored in the classroom in all their complexity. (21)

The virtual manipulatives in this thesis, as well as those in the national library, prove this to be true. They easily provide for the student “dynamic representations of complex processes.” In addition, they offer users a large number and a wide variety of problems. These dynamic capabilities provide opportunities to explore and discover the mathematical ideas associated with a problem.

MSEB continues: “Students of mathematics should be given opportunities to pose problems and advance hypotheses after they have examined a situation for the patterns and relationships it contains” (21). Thus one of the purposes of providing students with opportunities for exploration is to allow them to make conjectures and ask new questions. In this way, students become active participants in the learning
process. By providing dynamic representations of problems, the virtual manipulatives in this thesis, as well as those in the national library, encourage such student engagement.

As of this date, the team at Utah State University has developed several such web-based mathematics tutorials, all in the form of Java applets. Applets are programs that are executed through a web-browser and can be shared by many users. Those manipulatives currently in the national library can be found at http://www.matti.usu.edu. The manipulatives created for this thesis will ultimately be included in this library.

The tutorials in the library are categorized according to the corresponding NCTM Standard content area, as set forth in *Principles and Standards for School Mathematics* (NCTM). They are further categorized by the grade levels of intended users. The content areas covered in the library are Numbers and Operations, Algebra, Geometry, Measurement, and Data Analysis and Probability.

This thesis contains the following virtual manipulatives: Bar Chart, a data organization tool; Number Facts, a concentration-type game to help students with basic number facts; Ring Around 21 and Ring Around 3, addition games; Dot To Dot, a simple graphing tool to help young mathematics students learn the concepts of ordered pairs; Newton’s Method, a graphical demonstration of Newton’s Method for solving equations of one variable; and Riemann Sums, a tutorial to help students visualize the definite integral as the area under a curve.
VIRTUAL MANIPULATIVES

Bar Chart

According to the NCTM Standards for Numbers and Operations, Pre-K-2 students should be able to “count with understanding and recognize ‘how many’ in sets of objects.” (NCTM, p.78) One of the common lessons used to teach this concept is the “M&M problem.” In this problem, the teacher presents students with a number of colored candies. The student counts the candies and sorts them by color. The student then colors a bar chart representing the data obtained. Figure 1 shows Bar Chart with one entry in the first column.

Fig. 1. Bar Chart.
Once the applet is loaded, the graph on the screen has four columns. The student changes the number of columns simply by using the “Width” button to the right of the graph. The number of possible entries in each column can be changed by using the “Height” button, also to the right of the graph.

While sorting through the data, the student builds the bar chart. To do this, the student simply clicks the mouse in the appropriate column. For example, upon finding a red candy, the student clicks the mouse in the column corresponding to the red candies. The applet updates the size of the bar accordingly, as Figure 1 shows.

At the top of each bar a label displays the number in each category as a percentage of the total. The number in each column can be displayed instead of the percentage by checking the box below the graph labeled “show numbers.” These labels demonstrate Bar Chart’s dynamic capabilities. With every change made to the graph, the applet updates the labels instantly. Students can see how changes in the total number of entries affect the percentages in each column.

Number Facts

Number Facts, as shown in Figure 2, is an interactive game that helps students “understand the meaning of addition and subtraction of whole numbers” and “understand the effects of multiplying and dividing various whole numbers” (NCTM, p. 148). This manipulative is similar to the traditional “flash cards” used in early mathematics instruction.
When the applet is loaded, an addition puzzle appears on the screen. The puzzle consists of a picture covered by sixteen colored cards. Each card displays a unique mathematical expression. The objective is to solve the puzzle by matching pairs of cards with equivalent expressions and thus uncover the picture piece by piece.

To choose a card, the student simply clicks the mouse on the desired card, after which the card will change color. Then another card is selected. If the second card matches the first, both cards will be removed from the game board. When two matching cards are removed from the board, portions of a picture are revealed, as seen in Figure 3. This provides students with a reward, reinforcing the skills being practiced. If the second card does not match the first, both cards return to their original color, the portions of the picture remain covered, and the game continues.
Fig. 3. Number Facts with two matching cards removed from the board.

When all cards have been matched, the student sees the picture completely.

Number Facts offers two main advantages over traditional flash cards. First is its versatility. That is, it can be used for highly individualized number skills practice as well as group activities. Second is the ease with which it provides practice for a variety of arithmetic skills.

Playing individually on the computer, a student can work at a comfortable pace. On their own, students can also focus on those skills for which they need the most practice. Of course, a teacher may also make a classroom activity by using a projector and having students perform while the rest of the class looks on.

To change the type of puzzle, the student simply presses the button for the
desired skill (addition, subtraction, multiplication, or division) and then pushes the “New Game” button. The magnitude of the numbers involved in the operations can be changed by using the level buttons located to the right of the puzzle. For each of the four skills, there are literally thousands of unique puzzles the applet can generate.

Ring Around 21 and Ring Around 3

Figure 4 shows Ring Around 21, an integer addition puzzle. The goal of the student using this puzzle is to position the numbers in the circles and in the intersections of the circles in such a way that the sum of the numbers in each circle is equal to 21.

Fig. 4. Ring Around 21.
When the student begins, an initial seeding of some of the numbers is provided. The seeded numbers are fixed and cannot be moved by the user. They are also displayed with a color different from the movable numbers. This allows the student to distinguish the movable numbers from the initial seeding. To position a number the user drags the desired number with the mouse and drops it onto its location. When the target sum is achieved in a given circle, the circle will change color to indicate successful positioning.

To begin a new game, the "New Game" button is pressed and a new configuration will appear on the screen. For both games, there are seven solution configurations possible. With each new game, the applet randomly selects one of the configurations.

Ring Around 3 and Ring Around 0 are extensions of Ring Around 21. Ring Around 0, a tutorial already in the national library, involves the addition of signed integers. The target sum for Ring Around 0 is zero. Ring Around 3 is a decimal addition puzzle with a target sum of 3.0. The procedures for these two applets are the same as those for Ring Around 21.

Dot to Dot

Dot to Dot is a computer tool designed to help students understand the basics of a simple coordinate system. As such, it helps Pre-K and 3-5 students meet expectations under the Geometry Standard. According to this standard, students should be able to "describe location... and make and use a coordinate system to specify
locations and describe paths" (NCTM, p.164).

This manipulative presents a piece of virtual graph paper to the user, along with several buttons used to draw objects on the grid (see Figure 5). A set of points is also given. The goal of the student is to connect the given points to draw some type of geometric figure.

Fig. 5. Dot to Dot with a line segment drawn on the grid.

To draw a point, the student presses the “Dot” button and then clicks the mouse on the grid in the desired location. Because this applet is intended for young students, only points with whole number coordinates can be drawn. The applet draws the point at the closest intersection of grid lines.

Given any two points on the graph, the student can draw a line segment. To do this, the student presses the “Line Segment” button, clicks the mouse on one
endpoint and drags the mouse to the other endpoint. The line appears on the screen as the mouse is dragged. The student repeats this process to draw a figure.

After the student has drawn a figure, all or portions of the figure can be colored. This is done by pressing the “Select” button and then clicking the mouse on all of the vertices of the polygon to be colored. When done with the selection process, the student selects a color from the color bar and presses the “Fill” button.

The order in which the vertices are selected is an important element in the coloring of the figure. It also demonstrates the dynamic nature of this manipulative. The polygon to be colored is constructed as the user moves from one vertex to the next. For example, if the points 2A, 5C, 2C, and 5A are selected, in this order, and the “Fill” button is pressed, the polygon with vertices 2A, 5C, 2C, and 5A will be colored. This polygon, a crossed quadrilateral, is shown in Figure 6.

Fig. 6. Dot to Dot with a crossed quadrilateral.
If the student were to select theses same vertices, but in different order, 2A, 5A, 5C, and 2C, for example, the resultant polygon would be the rectangle shown in Figure 6. Through this process, students can gain an understanding of the idea of paths connecting points on a coordinate system. They see how to construct polygons by connecting the vertices with line segments. This feature also allows students to color only part of a figure, if desired.

Fig. 7 Dot to Dot with a rectangle.

Riemann Sums

When presenting the idea of approximations to areas under curves, traditional calculus texts are limited by space in the number and types of examples used. This virtual manipulative overcomes some of these limitations. When using the Riemann
Sums applet, a student can explore this topic with nearly any function of one variable. Riemann Sums computes an approximation to the integral $\int f(x) \, dx$. The student chooses between the left endpoint rectangle, right endpoint rectangle, and midpoint rectangle approximations. Figure 8 shows Riemann Sums for the function with $f(x) = \sin(x)$ with $n = 19$ on the interval $[-5.11, 4.94]$.

To begin, the user enters a function of $x$, using the toolbar located above the edit box for such symbols as the square root, exponents, and fractions. After clicking on the "Graph" button, a graph of the entered function appears in the graph window. The graph window is set to $[-10,10]$ by $[-10,10]$ initially, but can be changed by entering the desired values in the appropriate boxes, located below the edit box. The graph window can also be changed by clicking and dragging the mouse in the graph.
This procedure draws a “zoom box” in the window and resizes the window accordingly. When the function is drawn, the rectangles associated with the left endpoint rectangle approximation are also displayed in the graph window. The default interval [a,b] = [-2,2] is also shown on the graph, represented by two solid, vertical lines. The applet computes Δx = \( \frac{b-a}{n} \) with the default value \( n = 4 \).

With the curve, the rectangles, and the interval in the graphing window, the student can explore what happens when different parameters are changed. Using the \( n \)-slider below the graph, the user can increase or decrease \( n \), thus changing the value of \( Δx \). The new approximation to the area is immediately computed with each new value of \( n \). This new value is displayed on the screen, and the new rectangles are drawn. The user can change the values of \( a \) and \( b \) by using the appropriate sliders located below the graph.

The type of approximation used in the computation can also be changed. This is done simply by clicking the mouse in the appropriate checkbox. When a new type of approximation is chosen, the new approximating rectangles are immediately drawn to the screen.

This high level of interactivity provides students the opportunity to explore a number of questions. Examples of these could be the following: What happens with different types of symmetry? The user can graph an odd function, such as \( \sin x \). By changing the values of the endpoints of the interval, \( a \) and \( b \), the student can observe what happens when \( a = b \). Under what conditions will a left endpoint approximation be an overapproximation? The student can graph a strictly increasing function, such as...
lnx. By changing the type of approximation used, the student can discover how characteristics of the function influence the closeness of the approximation.

The efficiency of this tool is also of benefit to users. Instead of having to sketch a graph of a function, along with approximating rectangles, the applet draws the curve along with these rectangles efficiently and accurately. Thus, more time can be spent exploring concepts and ideas.

**Newton’s Method**

The Newton’s Method manipulative combines the graphics and function editing capabilities of Riemann Sums to give a step-by-step visual representation of Newton’s Method for solving equations of one variable.

One begins by typing the desired function, \( f(x) \), into the edit box and pressing the graph button or the enter key. To begin the Newton iterations, the user must provide an initial guess, \( x_0 \), for the solution of the equation. The guess box is located on the panel below the graph window. When the initial value for \( x \) has been entered, the user presses the “Initial Guess” button. An invalid initial guess, such as a character instead of a number, results in an error message.

Figure 9 shows Newton’s Method with one iteration for the function \( f(x) = x^2 - 3 \). For this example the initial guess is \( x_0 = 3 \). The student begins the iterations by pressing the “Step” button. The applet then graphically demonstrates the Newton iteration by drawing a vertical line from the point \( (x_0, 0) \) to the point \( (x_0, f(x_0)) \). Next, the line tangent to \( f(x) \) through the point \( (x_0, f(x_0)) \) is drawn. As the applet
draws the lines, the current value of $x_i$ appears in the iterations window. At any time, the user can change the view window, zooming in if desired to look closer at the approximations. The procedure is the same as that for Riemann Sums.

![Newton's Method with one iteration](image)

Fig. 9. Newton’s Method with one iteration.

When the first iteration is complete, the applet updates the value of $x$, using the Newton formula, $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$. The next iteration is can be viewed by pressing the “Step” button and the process is repeated.

This manipulative, like Riemann Sums, overcomes some limitations of traditional texts. Examples can be graphed for nearly any function. Teachers and students can use this to graphically depict Newton’s method efficiently and accurately. This allows more time for in-depth discussion and discovery, since the student spends less time with the details of graphing.
The interactive nature of this tool allows students to explore many ideas associated with Newton's Method. For example, the question may be asked, will Newton’s method always work? If not, under what conditions will the method fail to converge to a solution? The student can graph a function such as \( f(x) = \tan^{-1}(x) \). By starting the iterations with different initial guesses, the student can make conjectures about which values of \( x_0 \) cause the method to diverge and which cause it to converge.
DESIGN AND IMPLEMENTATION ISSUES

All the manipulatives in this thesis were created using the Java programming language. As Java applets, the manipulatives can be embedded into an html web page where they can be accessed by anyone with web-browsing capabilities. The use of Java also ensures platform independence, i.e., the manipulatives can be run on any operating system with the minimum web-browsing requirements.

In order to create dynamic and engaging manipulatives, there are several things to be considered. First, the interface should be simple and easy to use. It is a challenge to balance the functionality of the applet with the desire to keep the interface clean and simple. Another consideration is the amount of control given to the user. To keep the manipulative simple and directed to a single mathematical idea, certain user actions must be restricted. These restrictions, though, need to be balanced with the freedom given to the user to explore and discover. The manipulatives in this thesis are designed to achieve this balance.

During the programming process, IBM Visual Age for Java was used as the development environment. This environment provides two main advantages over others. First, using a large code-base library, it is necessary to keep all code organized and readily accessible. Visual Age does exactly this. New code updates are easily imported into the environment. Second, the environment provides incremental compiling of source code. This feature allows the programmer to observe the effects of small changes in commands without having to recompile all other code. The
programmer can make changes easily and efficiently. This also will allow future versions to be updated easily.
CONCLUSION

The manipultives presented in this thesis, and others found in the national library are effective teaching and learning tools. Their dynamic nature is the key to their effectiveness. The number and types of possible problems these manipulatives can offer students provide flexibility. Students can proceed at their own pace while obtaining valuable feedback from the computer. Because the manipulatives generate problems efficiently, users can move beyond procedural details of many problems and explore ideas and concepts. While using these manipulatives, students and teachers can see immediately how small changes in inputs affect a given problem, promoting student engagement in the learning process.
REFERENCES


APPENDIX
BIBLIOGRAPHY


