Construction and Analysis of a Family of Numerical Methods for Hyperbolic Conservation Laws with Stiff Source Terms

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CONSTRUCTION AND ANALYSIS OF A FAMILY OF NUMERICAL
METHODS FOR HYPERBOLIC CONSERVATION LAWS
WITH STIFF SOURCE TERMS

by

Cinnamon Hillyard

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of the requirements for the degree
of
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Numerical schemes for the partial differential equations used to characterize stiffly forced conservation laws are constructed and analyzed. Partial differential equations of this form are found in many physical applications including modeling gas dynamics, fluid flow, and combustion. Many difficulties arise when trying to approximate solutions to stiffly forced conservation laws numerically. Some of these numerical difficulties are investigated.

A new class of numerical schemes is developed to overcome some of these problems. The numerical schemes are constructed using an infinite sequence of conservation laws.

Restrictions are given on the schemes that guarantee they maintain a uniform bound and satisfy an entropy condition. For schemes meeting these criteria, a proof is given of convergence to the correct physical solution of the conservation law.

Numerical examples are presented to illustrate the theoretical results.
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CHAPTER 1
INTRODUCTION

Solutions to stiffly forced conservation laws have been of great interest to mathematicians, scientists, and engineers alike because of their frequent occurrence in physical applications. In this paper, we will examine some difficulties that arise when trying to solve such problems numerically and provide numerical methods constructed to overcome some of these numerical difficulties.

We will consider hyperbolic partial differential equations of the form

\[ u_t + f(u)_x = \Psi(u) \]  \hspace{1cm} (1.1)

with initial condition given by

\[ u(x, 0) = u_0(x). \]  \hspace{1cm} (1.2)

In Equation (1.1), \( u : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a function of time variable \( t \geq 0 \) and space variable \( x \in \mathbb{R} \). The function, \( f(u) : \mathbb{R} \rightarrow \mathbb{R} \), is called the flux function, and \( \Psi(u) : \mathbb{R} \rightarrow \mathbb{R} \) is a forcing function. The initial condition, \( u_0 : \mathbb{R} \rightarrow \mathbb{R}^m \), is a function of the space variable \( x \) alone.

We will be interested in cases where this system is "stiff." Equations that exhibit two or more radically different time scales are called stiff. Formally, Pember [11] classifies the conservation law in Equation (1.1) as stiff if the time scale introduced by the source term \( \Psi \) is small compared to the characteristic speed \( f' \) and some appropriate length scale.

Stiffly forced conservation laws are found in many physical problems. They arise
in modeling gases not in local thermodynamic equilibrium, aerodynamics when modeling wing flutter, meteorology when modeling weather fronts, modeling fluid flow in petroleum reservoirs, shallow water equations, kinetic theory, elasticity with memory, multi-phase transitions, linear and nonlinear waves, chromatography, and combustion problems. See [1], [4], [9], [8], [11], and [13] for discussions of these applications.

When attempting to solve these equations numerically, many difficulties arise. It is well known that many numerical methods applied to the convective part of Equation (1.1) introduce dissipative, dispersive, or other undesired numerical noise in the regions near steep gradients. For a thorough description of the numerical difficulties in approximating partial differential equations with steep gradients, see [9, pages 9–12].

Another underlying problem of Equation (1.1) is nonuniqueness of solutions. Often a number of weak solutions will satisfy the same weak form of the equation. See Smoller [15, pages 240–254] for a description of weak solutions to conservation laws. In particular, we say a function, \( u \), is a weak solution to Equation (1.1) if for all \( \phi \in C^1_0 \),

\[
(1.3) \quad \int_0^T \left( u \phi_t + f(u) \phi_x \right) dx dt + \int_0^T u_0(x) \phi(x, 0) dx = - \int_0^T \Psi \phi dx dt.
\]

Because of the nonuniqueness of solutions, additional conditions are required to pick out the "physically" correct solution. These requirements are known as entropy conditions. We also need to enforce discrete versions of the entropy condition on our numerical scheme to ensure that the numerical solution converges to the physically correct solution.

For the case when \( \Psi(u) = 0 \), many excellent finite difference numerical methods exist. These methods have been developed to overcome many of the numerical difficulties mentioned above. See [9] for a thorough description of numerical methods for conservation laws in the case \( \Psi(u) = 0 \).
When trying to generalize these schemes to equations with stiff forcing terms, new difficulties arise. The difference in time scales due to the stiff forcing function can lead to problems not unlike the classical stiffness problems in ordinary differential equations. Here, if we take the time step appropriate for the slower scale, then we may not be able to resolve the fast time scale and hence cause numerical instability around shocks. However, if the reactions are near equilibrium to begin with, using a small time step may be unnecessary.

Many robust numerical methods for stiff ordinary differential equations have been developed that allow for taking larger time steps. So, a natural method to approximate stiffly forced conservation laws would be to couple a finite difference scheme applied to the convective part of Equation (1.1),

\[
  u_t + f(u)_x = 0,
\]

with a stiff ordinary differential equation solver for

\[
  u_t = \Psi(u).
\]

These are known as operator splitting methods. However, even if the methods behave well for the individual problems, there are fundamental difficulties when they are coupled together. This situation is described in [8] by LeVeque and Yee. In general, as the stiffness of the forcing term increases (i.e., the magnitude of \(\Psi'(u)\) increases), the operator splitting methods become unstable and propagate the discontinuities at the wrong speed or not at all.

Finite difference schemes can be used alone to approximate stiffly forced conservation laws. In general, these methods require that the step sizes are taken very small. A simple
example of this type of method will be illustrated in the next chapter. Recently, Pa­
palexandris, Leonard, and Dimotakis [10] have developed another unsplit finite difference
scheme which traces corresponding invariant paths for the stiffly forced conservation law.
They obtain nice results which will be used as a comparison.

Other numerical approximations have been developed that allow for a coarser grid.
However, these schemes impose special requirements on the function $\Psi(u)$. Such properties
include requiring that $\Psi$ have only one equilibrium or that $\Psi$ is a monotone function with
respect to $u$. Some of these numerical methods for special cases of $\Psi$ are discussed in [4],
[5], [6], and [14].

A goal of this paper will be to develop and investigate a numerical finite difference
scheme that resolves the problems with a general stiff forcing term. We hope to develop
schemes that will allow us to choose the largest time step possible, yet guarantee that the
numerical method is producing accurate results and picking out the correct weak solution.

Chapter 2 starts with an analysis of a linear example of Equation (1.1). Numerical
comparisons using a simple finite different scheme against the exact solution will be given.
Analysis of a corresponding modified equation will be examined in an attempt to under­
stand the behavior of the numerical scheme and to illustrate where and why numerical
difficulties arise.

Chapter 3 discusses the development of new numerical schemes for stiffly forced con­
servation laws. This chapter will discuss how a sequence of conservation laws can be
constructed from our original conservation law and how this sequence is used to develop
numerical schemes. Some initial numerical results using one of these schemes will be given.

Chapter 4 contains an analysis of the scheme developed in Chapter 3. We will provide
conditions on the scheme that ensure certain properties of the scheme are met. Theorems and proofs will be given that lead up to and include the convergence of the schemes to an entropy satisfying solution of the stiffly forced conservation law.

Chapter 5 contains additional numerical simulations using the scheme developed in Chapter 3. Comparisons with exact solutions and other numerical methods are provided including cases with varied examples of $\Psi$ and $f$. Chapter 5 also contains concluding remarks.
CHAPTER 2
ANALYSIS OF STIFFLY FORCED CONSERVATION LAWS

In this chapter we will consider stiffly forced conservation laws of the form

\[ u_t + f(u)x = \Psi(u) \] (2.1)

with initial condition given by

\[ u(x, 0) = u_0(x) \]

as described in the introduction.

We will first consider the analytical solution of an example equation of this type. This example is presented in order to understand the nature of the analytical solutions for these types of equations and the problems that arise when trying to approximate these equations numerically. We will then illustrate these numerical difficulties with a standard first-order numerical scheme. Finally, we will attempt to explain some of the undesired behavior of the numerical schemes by looking at a modified differential equation related to our example. This effort is made in order to understand some of the difficulties and hence enable us to design more accurate and efficient numerical methods for approximating such equations.

2.1 An Analytical Solution

The simplest linear example of Equation (2.1) is

\[ u_t + au_x = \Psi(u). \] (2.2)
where $x \in \mathbb{R}$, $t > 0$, and $a > 0$ is a scalar constant. As an example we consider,

\begin{equation}
\Psi(u) = -\mu u(u - 1)(u - \frac{1}{2})
\end{equation}

where $\mu$ is a positive constant. The initial conditions are given by $u(x, 0) = U_0(x)$ for $x \in \mathbb{R}$. We will use this example since the essential numerical difficulties can be seen in this simplified case. This example is also used as a case study by LeVeque and Yee, see [8].

The exact solution may be found using the method of characteristics and separation of variables as follows. Suppose $u(x, t) = u(\tau)$ where $\tau$ is a function of $x$ and $t$. Then Equation (2.2) becomes the system of ordinary differential equations

\begin{equation}
\frac{dx}{d\tau} = a
\end{equation}

\begin{equation}
\frac{dt}{d\tau} = 1
\end{equation}

\begin{equation}
\frac{du}{d\tau} = \Psi(u).
\end{equation}

Solving the second equation in this system, along with the assumption that $\tau = 0$ at $t = 0$, we have $\tau = t$. The solution to the first equation in this system is $x = at + x_0$ where $x = x_0$ at $t = 0$.

We now solve for $u$. Separating variables and integrating Equation (2.6) gives

\begin{equation}
\int \frac{1}{\Psi(u)} du = \int dt = t + C.
\end{equation}

For our model problem with $\Psi(u)$ given in Equation (2.3), we can integrate the left-hand side of Equation (2.7) and simplify to obtain

\[ t + C = \frac{-2}{\mu} \ln \left( \frac{(1 - u)u}{(1 - 2u)^2} \right) \]

or

\begin{equation}
\frac{(1 - u)u}{(1 - 2u)^2} = \hat{C} e^{-\frac{\mu}{2}t}.
\end{equation}
Using the initial condition, we can compute
\[ (2.9) \quad \hat{C} = \frac{(1 - u_0(x_0))u_0(x_0)}{(1 - 2u_0(x_0))^2}. \]

Assuming \( u \neq 1/2 \), we simplify Equation (2.8), solve for \( u \) using the quadratic formula, and obtain
\[ u = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{1}{1 + 4\hat{C}e^{-\mu t/2}}}. \]

Note that when \( u(x_0) = 1/2 \), Equation (2.2) becomes the standard advection equation with solution given by \( u(x, t) = u_0(x - at) = 1/2 \). The final solution is given by
\[ (2.10) \quad u(x, t) = \begin{cases} 
\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{1 + 4\hat{C}e^{-\mu t/2}}} & \text{if } u_0(x_0) = u_0(x - at) < 1/2 \\
\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{1 + 4\hat{C}e^{-\mu t/2}}} & \text{if } u_0(x_0) = u_0(x - at) > 1/2 \\
\frac{1}{2} & \text{if } u_0(x_0) = u_0(x - at) = 1/2
\end{cases} \]

where \( \hat{C} \) is defined in Equation (2.9). LeVeque and Yee, [8], note that the above solution approaches
\[ (2.11) \quad u(x, t) = \begin{cases} 
0 & \text{if } u_0(x_0) < 1/2 \\
\frac{1}{2} & \text{if } u_0(x_0) = 1/2 \\
1 & \text{if } u_0(x_0) > 1/2
\end{cases} \]
as \( t \) gets large. Note that \( u = 0, u = 1/2, \) and \( u = 1 \) are equilibrium solutions of Equation (2.6). Further, \( u = 0 \) and \( u = 1 \) are stable equilibria for this system while \( u = 1/2 \) is unstable. So, for large positive values of \( \mu \), the term \( e^{-\mu t/2} \) in Equation (2.10) is small and rapid convergence towards these equilibria of the forcing function is seen.

As an example, consider the initial condition given by
\[ (2.12) \quad u(x, 0) = u_0(x) = \begin{cases} 
1 & \text{if } x < 0 \\
1 - x & \text{if } 0 < x < 1 \\
0 & \text{if } x > 1
\end{cases} \]

If \( x_0 = x - at < 1/2 \), then \( u_0(x_0) > 1/2 \). If \( x_0 = x - at > 1/2 \), then \( u_0(x_0) < 1/2 \). Using
Equation (2.10), the exact solution for the initial condition in Equation (2.12) becomes

\[
(2.13) \quad u(x, t) = \begin{cases} 
1 & \text{if } x - at < 0 \\
\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{1}{1 + 4C e^{-\mu t/2}}} & \text{if } 0 < x - at < 1/2 \\
\frac{1}{2} & \text{if } x - at = 1/2 \\
\frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{1}{1 + 4C e^{-\mu t/2}}} & \text{if } 1/2 < x - at < 1 \\
0 & \text{if } x - at > 1.
\end{cases}
\]

The evolution of this solution when \( a = 1 \) is shown in Figure 2.1. Here, it is seen that although the initial condition is relatively nice, it rapidly evolves into a solution containing steep gradients. Note that the actual solution is infinitely differentiable. However, the steep gradients introduced as \( t \) increases are indistinguishable from discontinuities or shocks on a discrete grid used in the definition of numerical methods for approximating the solution of partial differential equations.
2.2 An Example Numerical Method

As discussed in the introduction, it is well known that many numerical methods applied to the convective part of Equation (2.2) introduce dissipative, dispersive, or other undesired numerical noise in regions near steep gradients.

As an illustration of this behavior, consider the standard first-order upwinding method given by

\[
U_j^{n+1} = U_j^n - a \frac{\Delta t}{\Delta x} (U_j^n - U_{j-1}^n) + \Delta t \Psi(U_j^n)
\]

with a numerical initial condition given by

\[
U_j^0 = u(x_j, 0) = u_o(x_j)
\]

where \(u_o(x)\) is defined in Equation (2.12). Our notation will use \(U_j^n\) to represent a numerical approximation for \(u\) at the grid point \((j \Delta x, n \Delta t)\) where \(\Delta x\) and \(\Delta t\) are predetermined mesh sizes in space and time, respectively. So, \(u\) will denote a solution of the original partial differential equation, whereas \(U\) will denote the solution of the discrete approximation to that partial differential equation.

The upwinding method applied to the convective part of Equation (2.2) introduces a diffusive effect near shocks. This method is well known for this behavior in the case where \(\Psi(u) = 0\). In addition, the forcing function will attempt to draw solutions toward the stable equilibria and in effect tighten up the shock.

The stiffness of \(\Psi\) may cause instability in the scheme. Notice that the last term in Equation (2.14) is no longer an approximation on the order of the grid size if the parameter \(\Delta t \mu\) is greater than 1 in magnitude. For our investigations below, we will choose \(\Delta t < \frac{1}{\mu}\) to avoid this problem. So, for large \(\mu\), \(\Delta t\) will need to be taken very small. In addition, we
will require the standard stability condition of \( a \frac{\Delta t}{\Delta x} \leq 1 \) which will also impose limitations on the size of \( \Delta x \).

Graphically, the diffusive effect of the upwinding method is seen as "smearing" out of the shock front. This is shown in Figures 2.2, 2.3, 2.4, and 2.5:

- In Figure 2.2, \( \mu = 1 \). Here we see that although upwinding approximates the exact solution fairly closely, there is a smearing of the numerical solution near the shock edges.

- In Figures 2.3 and 2.4, we increase the stiffness parameter, \( \mu \), to 10 and let \( \Delta x = 0.1 \) be constant. In Figure 2.3, \( \Delta t = 0.05 \). The diffusion is clearly evident in this figure and the approximation of the exact solution is not very close. One approach to get a better approximation is to take a smaller time step. However, as illustrated in Figure 2.4, when \( \Delta t = 0.005 \), the approximation is still not close. It appears from these two cases that upwinding is not converging to the exact solution as \( \Delta t \to 0 \) alone.

- In Figure 2.5, we keep \( \mu = 10 \) and let \( \Delta t = 0.005 \) and \( \Delta x = 0.01 \). Here, the approximation to the actual solution is much better than that seen in Figures 2.3 and 2.4. For this case, note that the ratio \( \lambda = \frac{\Delta t}{\Delta x} = \frac{1}{2} \) is the same as in Figure 2.3 and \( \Delta t \) is the same as in Figure 2.4. This example indicates that both the time step and spatial step must be small in comparison to \( \mu \) in order to obtain a close approximation of the exact solution using the upwinding method.

In these figures, we also notice other qualitative differences between the exact solution and the numerical solution provided by upwinding. One of the clearest differences is that
Figure 2.2: Exact solution (solid line) is plotted against the approximate solution (dashed line) obtained using the upwinding method where $\mu = 1$, $\Delta x = 0.1$, $\Delta t = \frac{1}{20\mu} = 0.05$, and $\lambda = 0.5$.

Figure 2.3: Exact solution (solid line) is plotted against the approximate solution (dashed line) obtained using the upwinding method where $\mu = 10$, $\Delta x = 0.1$, $\Delta t = \frac{1}{2\mu} = 0.05$, and $\lambda = 0.5$. 
Figure 2.4: Exact solution (solid line) is plotted against the approximate solution (dashed line) obtained using the upwinding method where $\mu = 10$, $\Delta x = 0.1$, $\Delta t = \frac{1}{20\mu} = 0.005$, and $\lambda = 0.1$.

Figure 2.5: Exact solution (solid line) is plotted against the approximate solution (dashed line) obtained using the upwinding method where $\mu = 10$, $\Delta x = 0.01$, $\Delta t = 0.005$, and $\lambda = 0.5$. 
the upwinding solution appears to be moving slower than the exact solution. To illustrate
this “slowing down” of the upwinding method, we can numerically compute the speed
of the traveling wave propagated by this method. This is approximated by computing
how far the average value of the initial condition has traveled at $t = 1$. In Figure 2.6,
we fix $\lambda = 1/2$ and $\mu = 10$ and plot the speed of the numerical traveling wave solution
approximated by upwinding with different values of $\Delta x$. Here, we see that the speed of the
approximate solution approaches 1.5 as $\Delta x \to 0$. This is the speed of the exact solution
at time $t = 1$. However, as $\Delta x$ moves away from 0, the speed of the traveling wave slows
down.

Similar behavior is seen in the operator splitting methods described in [8]. These be-
have quite well for small $\mu$. However, as $\mu$ increases, the numerical solution is “completely
wrong.” The discontinuity is propagated too slowly or not at all.
2.3 Modified Equation Analysis

One way to analyze numerical schemes for Equation (2.2) is using a modified equation approach. This method, applied to the case when \( \Psi(u) = 0 \), is outlined in [9, pages 117-120]. We will follow a similar procedure here. Consider the Taylor series expansions

\[
\begin{align*}
  u(x, t + \Delta t) &= u(x, t) + \Delta t u_t(x, t) + \frac{\Delta t^2}{2} u_{tt}(x, t) + \cdots \\
  u(x - \Delta x, t) &= u(x, t) - \Delta x u_x(x, t) + \frac{\Delta x^2}{2} u_{xx}(x, t) + \cdots.
\end{align*}
\]

If we replace the discrete values, \( U_j^{n+1}, U_j^n \), and \( U_j^n \), by the continuous solution at the corresponding points, \( u(x, t + \Delta t), u(x, t), \) and \( u(x - \Delta x, t) \), then the upwinding scheme defined in Equation (2.14) becomes

\[
    u_t + \frac{\Delta t}{2} u_{tt} + \cdots = -a u_x + \frac{\Delta x}{2} u_{xx} - \frac{\Delta x^2}{3} u_{xxx} + \cdots + \Psi(u).
\]

If \( u \) satisfies Equation (2.2), then the upwinding scheme is \( O(\Delta t, \Delta x) \). However, if \( u \) satisfies the modified equation

\[
(2.16) \quad u_t + u_x - \epsilon u_{xx} = \Psi(u),
\]

then the upwinding scheme with \( \epsilon = \frac{\Delta x}{2} \) is \( O(\Delta t, \Delta x^2) \). So, upwinding will do a better job of approximating the solution of the modified equation. The term \( \epsilon u_{xx} \) in Equation (2.16) can be thought of as a "smoothing" or diffusive perturbation of our original equation.

We are interested in the exact solution to an equation in the form of Equation (2.16) and how this differs from the solution of Equation (2.10). Again, a change of coordinates is needed. First, let \( z = x - t \), then Equation (2.16) becomes

\[
(2.17) \quad u_t - \epsilon u_{zz} = \Psi(u).
\]
Next, transform out the diffusion coefficient, $\epsilon$, by setting $\xi = \frac{\xi}{\sqrt{\epsilon}}$ to obtain

$$u_t - u_{\xi\xi} = \Psi(u).$$

(2.18)

This type of partial differential equation is analyzed in Powell and Tabor [12]. The idea behind finding a solution will be to look for traveling wave solutions of Equation (2.18) with some speed $c$. Changing variables once again with $\eta = \xi - ct$ gives the ordinary differential equation

$$u_{\eta\eta} + cu_{\eta} + \Psi(u) = 0.$$

(2.19)

For $\Psi(u)$ given in Equation (2.3), this is straightforward to solve. As seen in Powell and Tabor [12], solutions must satisfy the differential equation

$$u_{\eta} = au + bu^2$$

(2.20)

which can be differentiated once to obtain

$$u_{\eta\eta} = au_{\eta} + 2bu_{\eta\eta}.$$  

(2.21)

To solve for the unknowns, $a$, $b$, and $c$, we substitute Equation (2.21) into Equation (2.19), to obtain

$$au_{\eta} + 2bu_{\eta} + cu_{\eta} - \mu u(u - 1)(u - \frac{1}{2}) = 0.$$  

Next, we substitute Equation (2.20) into the above equation, combine terms, and simplify to obtain

$$u^3(2b^2 - \mu) + u^2(3ab + bc + \frac{3}{2} \mu) + u(a^2 + ac - \frac{1}{2} \mu) = 0.$$  

This gives a system of three equations for the three unknowns:

$$2b^2 = \mu.$$  

(2.22)
The first equation in the system is easily solved to obtain

\[ b = \pm \sqrt{\frac{\mu}{2}}. \]

Eliminating \( c \) from the second and third equation of the system gives the following equation for \( a \)

\[ 2a^2b = -\frac{3a}{2} \mu - \frac{b}{2} \mu. \]

Setting \( \mu = 2b^2 \) and solving for \( a \) gives \( a = -b \) or \( a = -\frac{b}{2} \). Finally, substituting the above values of \( a \) and \( b \) into the second equation of the system, we obtain

\[ c = -3(a + b), \]

or, using the possible values of \( a \) found above, we find \( c = 0 \) or \( c = -\frac{3b}{2} \).

Now we can write out the solution of our ordinary differential equation. Equation (2.20) can be integrated to obtain

\[ u(\eta) = \frac{-a}{b - aKe^{-a\eta}} \]

where

\[ K = \frac{a + b}{a} \frac{u_0|_{\eta=0}}{u_0|_{\eta=0}} \]

is a constant dependent on the initial condition. Transforming back to the original coordinates, we obtain

\[ u(x,t) = \frac{-a}{b - aKe^{-a\left(\frac{1}{c^2}(x-t)-ct\right)}} \]
Figure 2.7: Exact solutions to the modified equation with $\epsilon = 0.1, 0.01, 0.001$ against the exact solution to the stiffly forced conservation law (solid line) at $t = 1$

where $a$, $b$, and $c$ are defined above. This gives two possible solutions. If $c = 0$, then

$-a = b = \pm \sqrt{\frac{b}{2}}$ and

$$u(x,t) = \frac{1}{1 + Ke^{\pm \sqrt{\frac{b}{2}}(x-t)}}.$$

If $c = \pm \frac{3\sqrt{2\mu}}{4}$, then

$$u(x,t) = \frac{1}{2 + Ke^{\pm \sqrt{\frac{\mu}{16\sqrt{2}}(4x-4t-3\sqrt{2\mu}t)}}}.$$  \hspace{1cm} (2.26)

As stated before, we wish to compare these solutions with that found in Equation (2.10). In particular, we are interested in the limit at $\epsilon \to 0$. Here, the two solutions should be equivalent.

To illustrate this convergence, we return to our previous example. With the initial conditions given in Equation (2.12), we graph the solution given in Equation (2.25) for $\epsilon = 0.1, 0.01$, and 0.001 against the solution for the original conservation law given in Equation (2.13). These plots are shown in Figure 2.7. Here, we see the convergence as $\epsilon \to 0$ of the solution to the modified equation to the solution of the conservation law.
As illustrated before, the upwinding scheme used to approximate Equation (2.2) actually is better at approximating the modified Equation (2.16) with $\epsilon = \frac{\Delta x}{2}$. The solution given in Equation (2.25) plotted against the numerical solution obtained using the upwinding method for Equation (2.2) is seen in Figures 2.8, 2.9, 2.10, and 2.11. As a reference, the exact solution to the original conservation law is also included in the plots. As seen here, the approximate solutions given by the upwinding scheme and the exact solution to the modified equation are almost an exact match. In addition, convergence to the exact solution is dependent on $\Delta x$ being small.
Figure 2.9: Solution to modified equation against upwinding for $\Delta x = 0.1$, $\Delta t = 0.05$, and $\mu = 10$ at $t = 2$. The exact solution is also plotted as a reference.

Figure 2.10: Solution to modified equation (dotted line) is plotted against the approximate solution (dashed line) using the upwinding method where $\Delta x = 0.01$, $\Delta t = 0.05$, and $\mu = 10$ at $t = 1$. The exact solution (solid line) is also plotted as a reference.
Figure 2.11: Solution to modified equation (dotted line) is plotted against the approximate solution (dashed line) using the upwinding method where $\Delta x = 0.01$, $\Delta t = 0.001$, and $\mu = 100$ at $t = 1$. The exact solution (solid line) is also plotted as a reference. Note that the $x$ axis is on a smaller scale than the above graphs to illustrate the difference in solutions.
CHAPTER 3
BUILDING NUMERICAL METHODS

We saw in the previous chapter how a standard finite difference numerical method for conservation laws does a poor job when approximating stiffly forced conservation laws. Besides introducing numerical noise, traveling waves are moved at incorrect speeds or not at all. In this chapter, we will build new numerical methods to approximate stiffly forced conservation laws. These methods will attempt to resolve the effect of the stiff source term in the conservation law more accurately. We will show two initial numerical simulations using one of these methods. As we illustrate below, the results using this new method are promising.

3.1 A System of Conservation Laws

Recall that the equations we are studying have the form

\begin{equation}
\tag{3.1}
u_t + f(u)_x = \Psi(u)
\end{equation}

which can be expanded to

\begin{equation}
\tag{3.2}
u_t + f'(u) u_x = \Psi(u).
\end{equation}

We can now build a system of forced conservation laws from this. First, multiply equation (3.2) through by $\Psi'(u)$ to obtain

\begin{equation}
\tag{3.3}
\Psi'(u) u_t + \Psi'(u) f'(u) u_x = \Psi'(u) \Psi(u).
\end{equation}
The derivatives can be combined to obtain the new conservation law

\[ (3.4) \quad \Psi(u)_t + f'(u)\Psi'(u)_x = \Psi'(u)\Psi(u). \]

Using the definitions

\[ U^{(0)} = u, \]
\[ U^{(1)} = \Psi(u), \]

and

\[ U^{(2)} = \Psi'(u)\Psi(u) = U^{(1)} \frac{dU^{(1)}}{dU^{(0)}}, \]

we can write Equation (3.3) and Equation (3.4) as the system of partial differential equations:

\[ (3.5) \quad U^{(0)}_t + f'(U^{(0)})U^{(0)}_x = U^{(1)} \]
\[ (3.6) \quad U^{(1)}_t + f'(U^{(0)})U^{(1)}_x = U^{(2)}. \]

We now iterate this process. To build the next equation for our system, multiply Equation (3.6) by \( \frac{dU^{(2)}}{dU^{(1)}} \). Here, we have

\[ \frac{dU^{(2)}}{dU^{(1)}} U^{(1)}_t + f'(U^{(0)}) \frac{dU^{(2)}}{dU^{(1)}} U^{(1)}_x = U^{(2)} \frac{dU^{(2)}}{dU^{(1)}} \]

or

\[ U^{(2)}_t + f'(U^{(0)})U^{(2)}_x = U^{(3)} \]

where \( U^{(3)} = U^{(2)} \frac{dU^{(2)}}{dU^{(1)}} \). Continuing in this manner, we obtain the infinite sequence of partial differential equations

\[ (3.7) \quad U^{(k)}_t + f'(U^{(0)})U^{(k)}_x = U^{(k+1)} \]
where

\[ U^{(k+1)} = U^{(k)} \frac{dU^{(k)}}{dU^{(k-1)}} \]

for all \( k = 0, 1, 2, \ldots \).

### 3.2 A Numerical Scheme

Using the sequence of conservation laws from the previous section, we will construct numerical discretizations for stiffly forced conservation laws of the form shown in Equation (3.1). We use the method of characteristics on each conservation law as we did in Section 2.1. That is, suppose \( U^{(k)}(x, t) = U^{(k)}(\tau) \) where \( \tau \) is a function of \( x \) and \( t \).

Equation (3.7) becomes the ordinary differential equation

\[ \frac{d}{d\tau} U^{(k)} = U^{(k+1)} \]

with \( \tau = t \) and \( x = at + x_0 \) where \( a = f'(U^{(0)}) \).

First we consider the case where \( a \) is constant. Let the point \((x_j, t_{n+1})\) and the characteristic point \((x_c, t_n)\) lie on the same characteristic curve defined by the equation, \( x = at + x_0 \). Expanding about the characteristic point, we obtain

\[ U^{(0)}(x_j, t_{n+1}) = U^{(0)}(x_c, t_n) + \Delta \tau U^{(0)}_\tau(x_c, t_n) + \frac{\Delta \tau^2}{2} U^{(0)}_{\tau\tau}(x_c, t_n) \]

(3.8)

\[ + \frac{\Delta \tau^3}{3!} U^{(0)}_{\tau\tau\tau}(x_c, t_n) + \ldots \]

(3.9)

\[ = U^{(0)}(x_c, t_n) + \Delta \tau U^{(1)}(x_c, t_n) + \frac{\Delta \tau^2}{2} U^{(2)}(x_c, t_n) + \ldots \]

Equation (3.9) can be rewritten as

(3.10)

\[ \tilde{U}^{(0)}_j = \sum_{k=0}^{\infty} \frac{\Delta \tau^k}{k!} U^{(k)}(x_c, t_n) \]

where the notation

\[ \tilde{U}^{(0)}_j = U_j^{n+1} = U(x_j, t_{n+1}) \]
will be used in order not to confuse superscripts. Further, we will use the following abbreviations

\[ U_c^{(k)} = U^{(k)}(x_c, t_n) \]
\[ U_j^{(k)} = U^{(k)}(x_j, t_n). \]

The points \((x_j, t_{n+1})\) and \((x_c, t_n)\) both lie on the characteristic curve. So,

\[ x_j - at_{n+1} - x_0 = x_c - at_n - x_0. \]

Hence,

\[ x_c - x_j = -a \Delta t. \]

Using Taylor expansions again, we have

\[ U_c^{(k)} = U^{(k)}(x_c, t_n) \]
\[ = U^{(k)}(x_j, t_n) + U_x^{(k)}(\theta, t_n)(x_c - x_j) \]
\[ = U_j^{(k)} - a \Delta t U_x^{(k)}(\theta, t_n) \]

for some \(\theta\) between \(x_c\) and \(x_j\). We can approximate the derivative in the last line of this equation by

\[ U_c^{(k)} = U_j^{(k)} - a \Delta t \frac{U_{j+\frac{1}{2}}^{(k)} - U_{j-\frac{1}{2}}^{(k)}}{\Delta x}. \]

Equation (3.11) is a standard discretization for conservation laws in conservation form.

The values \(U_{j+\frac{1}{2}}^{(k)}\) can be thought of as approximations of \(U^{(k)}\) at the grid lines \(x_{j+\frac{1}{2}}\). In Koebbe [7], the background for this method of describing a discretization is discussed.

The term \(U_{j+\frac{1}{2}}^{(k)}\) is allowed to depend on any finite number of grid points at time \(t_n\).

For example, when \(f(u) = u\), the upwinding scheme is obtained by setting

\[ U_{j+\frac{1}{2}}^{(k)} = U_j^{(k)} \]
whereas the Lax Wendroff scheme is obtained by setting

\[ U_{j+\frac{1}{2}}^{(k)} = U_{j+1}^{(k)} - (U_{j+1}^{(k)} - U_{j}^{(k)}) \alpha \]

where

\[ \alpha = \frac{1}{2} + \frac{1}{2 \Delta x}. \]

We now generalize Equation (3.11) to include the case when \( f \) is nonlinear. In the case where \( k = 0 \), the scheme

\[ (3.12) \quad U_{j}^{(0)} = U_{j}^{(0)} - a \Delta t \frac{U_{j+\frac{1}{2}}^{(0)} - U_{j-\frac{1}{2}}^{(0)}}{\Delta x} \]

is consistent for \( a = f'(U_{j}^{(0)}) \), but it is well known that this method will not converge to a weak solution when \( a \) is not constant. See [8, page 123] for details. Since the term \( a \) is dependent on \( U^{(0)} \), it should be included in the discretization. That is, for \( k = 0 \) we will use

\[ (3.13) \quad U_{j}^{(0)} = \Delta t \frac{f(U_{j+\frac{1}{2}}^{(0)}) - f(U_{j-\frac{1}{2}}^{(0)})}{\Delta x} \]

to approximate the right-hand side of Equation (3.12).

For the cases when \( k \geq 1 \), \( a \) is constant with respect to the variable \( U_{j}^{(k)} \). So, the discretization

\[ (3.14) \quad U_{j}^{(k)} = \Delta t \frac{f'(U_{j}^{(0)}) \Delta t}{\Delta x} (U_{j+\frac{1}{2}}^{(k)} - U_{j-\frac{1}{2}}^{(k)}) \]

where the value of \( a \) is substituted directly into Equation (3.11) is used.

Substituting Equation (3.13) and Equation (3.14) into Equation (3.10) gives the numerical approximation for our conservation law:

\[ (3.15) \quad U_{j}^{(0)} = U_{j}^{(0)} - \frac{\Delta t}{\Delta x} (f(U_{j+\frac{1}{2}}^{(0)}) - f(U_{j-\frac{1}{2}}^{(0)})) \]

\[ + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} (U_{j}^{(k)} - \frac{f'(U_{j}^{(0)}) \Delta t}{\Delta x} (U_{j+\frac{1}{2}}^{(k)} - U_{j-\frac{1}{2}}^{(k)})) \]
where $\Delta t$ and $\Delta x$ are step sizes in time and space, respectively. For practical purposes, the infinite series in the last term will be truncated at some $K$ when the numerical scheme is implemented.

### 3.3 Numerical Comparisons

For a first look at this numerical scheme for approximating stiffly forced conservation laws, we will consider the example we used in Section 2.2. There, $f(u) = u$ and

$$
\Psi(u) = -\mu u(u - 1)(u - \frac{1}{2}).
$$

The initial condition was given by

$$
u(x, 0) = u_0(x) = \begin{cases} 
1 & \text{if } x < 0 \\
1 - x & \text{if } 0 < x < 1 \\
0 & \text{if } x > 1.
\end{cases}
$$

The exact solution is given in Equation (2.13).

Consider our scheme described in Equation (3.15) using the upwinding approximation for each level of the series. That is, let $U_j^{(k)} = U_j^{(k)}$ for all $j$ and $k$. Truncating this series at $K = 1$ gives the numerical scheme

$$
U_j^{(0)} = U_j^{(0)} - \frac{\Delta t}{\Delta x}(U_j^{(0)} - U_{j-1}^{(0)}) + \Delta t(U_j^{(1)} - \frac{\Delta t}{\Delta x}(U_j^{(1)} - U_{j-1}^{(1)}))
$$

$$
= U_j^n - \lambda(U_j^n - U_{j-1}^n) + \Psi(U_j^n) - \lambda(\Psi(U_j^n) - \Psi(U_{j-1}^n))
$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $U_j^n$ is the discrete approximation for $u(j\Delta x, n\Delta t)$. We will compare numerical results using our scheme with the results using the standard upwinding scheme described in Section 2.2. This comparison is illustrated in Figures 3.1, 3.2, 3.3, and 3.4:
• In Figure 3.1, $\lambda = 1$, $\Delta t = \Delta x = 0.1$ and $\mu = 10$. In this case, the standard upwinding method is clearly unstable. This is due to the fact that the parameter $\Delta t \mu = 1$ is too large for the standard upwinding scheme alone. However, our scheme approximates the solution adequately.

• In Figure 3.2, $\mu = 10$ and $\lambda = 1$, but $\Delta t = \Delta x = 0.01$ are taken smaller. In this figure, values from our scheme and the exact solution are indistinguishable. However, the original upwinding scheme still moves the traveling wave too slowly.

• In Figure 3.3 and 3.4, $\lambda = 1/2$, $\Delta t = 0.05$, $\Delta x = 0.1$ and $\mu = 10$. In Figure 3.3, $t = 1$ while $t = 2$ in Figure 3.4. Our scheme still does a better job in both of these cases of approximating the exact solution. The decrease in the value of $\lambda$ makes the last term in Equation (3.17) smaller. So, the effect of the stiff forcing term is dampened, and the diffusion introduced by the upwinding scheme is more evident.

One significant difference between the two numerical methods can be observed in these figures. The speed of the traveling wave appears to be propagated correctly when our scheme is used while the original upwinding method is still propagating the traveling wave too slowly. This is a desirable improvement.

For our second numerical example, we will use the example found in LeVeque and Yee's paper [8]. Here, $f(u)$ and $\Psi(u)$ are as in the first example with the initial condition given by

$$u(x, 0) = u_0(x) = \begin{cases} 1 & \text{if } x < 0.3 \\ 0 & \text{if } x \geq 0.3. \end{cases}$$

The exact solution in this case is given by

$$u(x, t) = \begin{cases} 1 & \text{if } x - t < 0.3 \\ 0 & \text{if } x - t \geq 0.3. \end{cases}$$
Figure 3.1: Comparison of the numerical approximations using the standard upwinding scheme against our new scheme with upwinding at each level. Plot given at time $t = 1$ with $\lambda = 1$, $\Delta t = \Delta x = 0.1$ and $\mu = 10$. The exact solution (solid line) is also plotted for reference.

Figure 3.2: Comparison of the numerical approximations using the standard upwinding scheme against our new scheme with upwinding at each level. Plot given at time $t = 1$ with $\lambda = 1$, $\Delta t = \Delta x = 0.01$ and $\mu = 10$. The exact solution (solid line) is also plotted for reference.
Figure 3.3: Comparison of the numerical approximations using the standard upwinding scheme against our new scheme with upwinding at each level. Plot given at time $t = 1$ with $\lambda = 1/2, \Delta t = 0.5, \Delta x = 0.1$ and $\mu = 10$. The exact solution (solid line) is also plotted for reference.

Figure 3.4: Comparison of the numerical approximations using the standard upwinding scheme against our new scheme with upwinding at each level. Plot given at time $t = 2$ with $\lambda = 1/2, \Delta t = 0.5, \Delta x = 0.1$ and $\mu = 10$. The exact solution (solid line) is also plotted for reference.
We will again use our above scheme where upwinding is used at each level of the series. For this example, we use the same parameters as in [8]. We set $\Delta x = 0.02$ and $\lambda = 0.75$ and vary $\mu$. The results when $t = 0.3$ are shown in Figures 3.5, 3.6, and 3.7.

- In Figure 3.5, $\mu = 1$. The diffusion normally present with upwinding is evident, but the numerical solution appears to be traveling at the correct speed.

- In Figures 3.6 and 3.7, we increase $\mu$ to 10 and 100, respectively. The approximation is still good and traveling at the right speed. In addition, we observe that the approximation to the shock is tighter as $\mu$ increases.

These results are comparable to the those seen in LeVeque and Yee's paper [8, pages 196 and 199]. Recall that they used semi-implicit, operating splitting methods, whereas we were able to obtain the same results using a simple first-order, explicit, finite difference scheme.

The two numerical examples seen above indicate that our new numerical scheme does a good job at approximating the exact solution to our stiffly forced conservation law. We will analyze the properties of this numerical method in the next chapter.

### 3.4 Convergence of the $U^{(k)}$ Series

We will conclude this chapter by discussing the convergence of the series

\[
(3.18) \quad \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!} U^{(k)}(x_c, t_n).
\]

First, consider the following closed formula for the $U^{(k)}$'s.

**Lemma 3.1** The sequence of $U^{(k)}$'s given by

\[
U^{(0)} = u
\]
Figure 3.5: Comparison of the numerical approximation using our new scheme with upwinding at each level against the exact solution with $\mu = 1$. Plot given at time $t = 0.3$ with $\lambda = 0.75$, $\Delta t = 0.015$, and $\Delta x = 0.02$

Figure 3.6: Comparison of the numerical approximation using our new scheme with upwinding at each level against the exact solution with $\mu = 10$. Plot given at time $t = 0.3$ with $\lambda = 0.75$, $\Delta t = 0.015$, and $\Delta x = 0.02$
Figure 3.7: Comparison of the numerical approximation using our new scheme with upwinding at each level against the exact solution with $\mu = 100$. Plot given at time $t = 0.3$ with $\lambda = 0.75$, $\Delta t = 0.015$, and $\Delta x = 0.02$

$$U^{(1)} = \Psi(u)$$

and

$$U^{(k+1)} = U^{(k)} \frac{dU^{(k)}}{dU^{(k-1)}}$$

can be rewritten as

$$(3.19) \quad U^{(k+1)} = U^{(1)} \frac{dU^{(k)}}{du}$$

for $k \geq 1$.

Proof by induction. When $k = 1$, we have

$$U^{(2)} = U^{(1)} \frac{dU^{(1)}}{dU^{(0)}} = U^{(1)} \frac{dU^{(1)}}{du}$$

since $U^{(0)} = u$.

Now suppose Equation (3.19) holds for some $k$. Then,

$$U^{(k+1)} = U^{(k)} \frac{dU^{(k)}}{dU^{(k-1)}}$$
\[ (U^{(1)} \frac{dU^{(k-1)}}{du})( \frac{dU^{(k)}}{dU^{(k-1)}}) \]
\[ = U^{(1)} \frac{dU^{(k)}}{du} \]

as desired. So, Equation (3.19) holds for all \( k \). This completes our proof.

Now, we can prove the following lemma.

**Lemma 3.2** Let

\[ U^{(1)} = \Phi(U^{(0)}) = \Phi(u) = -\mu \prod_{i=1}^{l} (u - r_i) \]

be an \( l \)th degree polynomial with \( \mu \) as a positive forcing parameter. Further suppose that each root of \( U^{(1)} \), \( r_i \), is between 0 and 1. If \( U^{(k)} \) is as defined in Equation (3.19), then we have \( |U^{(k)}(u)| \leq \mu^k \) for all \( k \) and \( u \in [0, 1] \).

Proof. Consider the definition of \( U^{(1)} \) given in Equation (3.20). Since \( u \in [0, 1] \) and each root of the polynomial \( U^{(1)} \) is also between 0 and 1, we have each factor of the polynomial \( |u - r_i| \leq 1 \). So,

\[ |U^{(1)}| = |-\mu \prod_{i=1}^{l} (u - r_i)| \]
\[ = \mu \prod_{i=1}^{l} |(u - r_i)| \]
\[ \leq \mu. \]

Now consider the case where \( k = 2 \). Here,

\[ U^{(2)} = U^{(1)} \frac{dU^{(1)}}{du} \]
\[ = \mu^2 \prod_{i=1}^{l} (u - r_i) \prod_{i=1}^{l-1} (u - \hat{r}_i) \]

is a polynomial of degree \( 2l - 1 \). Applying Rolle's Theorem, we have the roots of \( \frac{dU^{(1)}}{du} \), \( \hat{r}_i \), lying between the roots of \( U^{(1)} \). That is,

\[ 0 \leq r_i \leq \hat{r}_i \leq r_{i+1} \leq 1. \]
Again the factors $|u - r_i|$ and $|u - \tilde{r}_i|$ are bounded by 1. So, we have $|U^{(2)}| \leq \mu^2$.

For clarity in notation, we rewrite

\[
U^{(1)} = -\mu \prod_{i=1}^{l} (u - r_i^{(1)}) \\
U^{(2)} = \mu^2 \prod_{i=1}^{2l-1} (u - r_i^{(2)})
\]

where the roots of these polynomials, $r_i^{(k)}$, are in the interval $[0, 1]$. We wish to show in general that

\[
U^{(k)} = (-\mu)^k \prod_{i=1}^{k(l-1)+1} (u - r_i^{(k)})
\]

with $0 \leq r_i^{(k)} \leq 1$.

We have shown this true for $k = 1$ and $k = 2$. Now suppose true for $k$. Then,

\[
U_i^{(k)} = (-\mu)^k \prod_{i=1}^{k(l-1)+1} (u - r_i^{(k)})
\]

is a polynomial of degree $k(l - 1) + 1$ with roots, $r_i^{(k)}$, between 0 and 1. So,

\[
\frac{dU^{(k)}}{du} = (-\mu)^k \frac{d}{du} \left( \prod_{i=1}^{k(l-1)+1} (u - r_i^{(k)}) \right)
\]

\[
= (-\mu)^k \prod_{i=1}^{k(l-1)} (u - \tilde{r}_i^{(k)})
\]

is a polynomial of degree $k(l - 1)$. Again applying Rolle’s Theorem gives the roots of $\frac{dU^{(k)}}{du}$ between the roots of $U^{(k)}$. So, each $\tilde{r}_i^{(k)}$ is also between 0 and 1.

Now, we have

\[
U^{(k+1)} = U^{(1)} \frac{dU^{(k)}}{du}
\]

\[
= \left( -\mu \prod_{i=1}^{l} (u - r_i^{(1)}) \right) \left( (-\mu)^k \prod_{i=1}^{k(l-1)} (u - \tilde{r}_i^{(k)}) \right)
\]

\[
= (-\mu)^{k+1} \cdot p(u)
\]
where \( p(u) \) is a polynomial of degree \( l + k(l - 1) = (k + 1)(l - 1) + 1 \) whose roots are between 0 and 1. So, we can express

\[
U^{(k+1)} = (-\mu)^{k+1} \prod_{i=1}^{l} (u - r_i^{(k+1)}).
\]

Again each factor, \( |u - r_i^{(k+1)}| \), of \( U^{(k+1)} \) is between 0 and 1. So, \( |U^{(k+1)}| \leq \mu^{k+1} \).

We have shown \( U^{(k)} \) can be written in the form of Equation (3.21), and each of its factors, \( |u - r_i^{(k)}| \), are bounded by 1. Hence, \( |U^{(k)}| \leq \mu^k \) for all \( k \). This completes our proof.

Now we can prove our desired result:

**Theorem 3.3** The series given in Equation (3.18) with \( U^{(1)} \) given by Equation (3.21) converges.

Proof. Using our bound on \( |U^{(k)}| \) found in the above lemma, our series is bounded as follows

\[
U_j^{(0)} = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!} U^{(k)}(x_c, t_n) \\
\leq \sum_{k=0}^{\infty} \frac{(\Delta t \mu)^k}{k!} \\
= e^{\Delta t \mu}
\]

and convergence is guaranteed for a fixed \( \Delta t \mu \).

Note this section was a discussion on the convergence of the infinite series used in the numerical scheme and not the convergence of the numerical scheme to the correct solution. This second type of convergence will be discussed thoroughly in the next chapter. Finally, note that the bound above explicitly depends on the stiffness parameter \( \Delta t \mu \).
CHAPTER 4
ANALYSIS OF THE NUMERICAL SCHEME

In this chapter we will analyze important properties of the numerical schemes developed in the previous chapter. These properties allow us to guarantee that the numerical approximations using our scheme converge to the correct entropy satisfying solution of the stiffly forced conservation law we are studying.

We follow the steps outlined in Smoller [15, Chapter 16] to show convergence of a discrete approximation to the solution of the conservation law. To do this, we show the following key properties. First, a uniform bound on the discrete approximation is given. Then, a discrete entropy condition for the approximate values will be constructed. Next, a bound on the total variation in both space and time of the scheme is provided. Convergence to the correct solution follows directly from these conditions.

Recall, our numerical scheme, truncated after the $K$th term, has the form

\[
\tilde{U}_j^{(0)} = U_j^{(0)} - \frac{\Delta t}{\Delta x} \left( f(U_j^{(0)} + \frac{1}{2}) - f(U_j^{(0)} - \frac{1}{2}) \right) \\
+ \sum_{k=1}^{K} \frac{\Delta t^k}{k!} \left( U_j^{(k)} - \frac{f(U_j^{(0)}))\Delta t}{\Delta x} (U_{j+\frac{1}{2}}^{(k)} - U_{j-\frac{1}{2}}^{(k)}) \right)
\]

(4.1)

where $U_{j + \frac{1}{2}}$ can depend on any number of grid line values. Hence, the numerical method can depend on any finite number of grid line values. This allows us to write Equation (4.1) in the format

\[
\tilde{U}_j^{(0)} = \sum_{k=0}^{K} \frac{\Delta t^k}{k!} G^{(k)}(U_{j-r}^{(k)}, U_{j-r-1}^{(k)}, \ldots, U_j^{(k)}, \ldots, U_{j+r}^{(k)})
\]

where $r$ is the "width" of our numerical approximation and $G^{(k)}$ is a function of the nodal
values of $U^{(k)}$. Alternately, Equation (4.1) can be written as

\begin{align}
\hat{U}_{j}^{(0)} &= \sum_{k=0}^{K} \frac{\Delta t^{k}}{k!} \sum_{i=-r}^{r} c_{j,i}^{(k)} U_{j+i}^{(k)} \\
&= \sum_{i=-r}^{r} \sum_{k=0}^{K} \frac{\Delta t^{k}}{k!} c_{j,i}^{(k)} U_{j+i}^{(k)}
\end{align}

where the coefficients, $c_{j,i}^{(k)}$, are not necessarily constant. In fact, for the case when $f$ is nonlinear, $c_{j,i}^{(0)}$ will normally depend on differences in $f$ at various grid line values.

For consistency, we require $\sum_{i=-r}^{r} c_{j,i}^{(k)} = 1$ for each $k$. Further, we will assume that $c_{j,i}^{(k)} \geq 0$ for all $i$, $j$, and $k$. It turns out that many methods, such as Lax Wendroff, violate this second condition. However, this is a sufficient condition that a standard finite difference scheme be uniformly bounded. See [8, pages 178–179] for details on this type of representation of a numerical scheme for the case when $\Psi = 0$.

We make a comment here about notation. Recall the definitions

$$U_{j}^{(0)} = U_{j}^{n} \approx u(j \Delta x, n \Delta t)$$

and

$$\hat{U}_{j}^{(0)} = U_{j}^{n+1} \approx u(j \Delta x, (n + 1) \Delta t).$$

It should also be noted that the terms $U_{j}^{(0)}$ and $U_{j}^{0}$ represent different things. The notation $U_{j}^{(0)}$ is as defined above whereas $U_{j}^{0}$ represents the initial condition. That is,$$
U_{j}^{0} = u(j \Delta x, 0).$$

These notations will be used without further discussion.

For our proofs below, we will assume that the forcing term is a polynomial with roots between 0 and 1. That is,$$
U^{(1)} = \Psi(U^{(0)}) = \Psi(u) = -\mu \Pi_{i=1}^{m} (u - r_{i})$$
as defined earlier. Without loss in generality, assume \( r_1 \) is the smallest root of \( U^{(1)} \) and \( r_m \) is the largest root of \( U^{(1)} \). We are now ready to prove the desired properties on our scheme.

### 4.1 Uniform Bound

The first property we show is that there exists a uniform bound on the numerical values. We give conditions for this property in the following lemma.

**Lemma 4.1** For all \( j \) and \( n \geq 0 \) we have

\[
    r_1 \leq U_j^n \leq r_m
\]

whenever

\[
    r_1 \leq U_j^0 \leq r_m
\]

and

\[
    (e^{\Delta t \mu} - 1) C_{j,i} \leq c_{j,i}^{(0)}
\]

where

\[
    C_{j,i} = \max_{k \geq 1} \{ c_{j,i}^{(k)} \}.
\]

**Proof.** We inductively assume that \( r_1 \leq U_j^{(k)} \leq r_m \) for all \( j \) at time \( t_n \). We will show this still holds at time \( t_{n+1} \).

First, consider the \( i \)th term

\[
    \sum_{k=0}^{K} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} U_j^{(k)} = c_{j,i}^{(0)} U_j^{(0)} + \sum_{k=1}^{K} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} U_j^{(k)}
\]

in the right-hand side of Equation (4.3). We first show that this term is bounded between \( c_{j,i}^{(0)} r_1 \) and \( c_{j,i}^{(0)} r_m \).
Recall, from Theorem 3.3, that when \( U^{(1)} \) is a polynomial as described above, then \( U^{(k)} = U^{(1)} \frac{dU^{(k-1)}}{du} \). So, every root of \( U^{(1)} \) is also a root of \( U^{(k)} \). Further, it was shown that the roots of \( U^{(k)} \) lie between the roots of its predecessors. So, the smallest and largest roots of \( U^{(1)} \) are also the smallest and largest roots of \( U^{(k+1)} \). Therefore,

\[
U^{(k)} = (-\mu)^k \prod_{i=1}^{k} (u - r_i^{(k)})
\]

\[
= (-\mu)^k (u - r_1) P^{(k)}(u)
\]

where \( P^{(k)}(u) \) is a polynomial bounded in absolute value by 1 for \( u \in (0, 1) \) since all its roots lie between 0 and 1.

Now we have

\[
\sum_{k=0}^{K} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} U^{(k)}_{j+i} = c_{j,i}^{(0)} U_{j+i}^{(0)} + (U_{j+i}^{(0)} - r_1) \sum_{k=1}^{K} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} (-1)^k P^{(k)}(U_{j+i}^{(0)})
\]

\[
\geq c_{j,i}^{(0)} U_{j+i}^{(0)} - (U_{j+i}^{(0)} - r_1) \sum_{k=1}^{K} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} |P^{(k)}(U_{j+i}^{(0)})|
\]

\[
\geq c_{j,i}^{(0)} U_{j+i}^{(0)} - (U_{j+i}^{(0)} - r_1) (e^{\Delta t \mu} - 1) C_{j,i}
\]

\[
\geq c_{j,i}^{(0)} U_{j+i}^{(0)} - c_{j,i}^{(0)} (U_{j+i}^{(0)} - r_1)
\]

\[
= c_{j,i}^{(0)} r_1.
\]

Note that we have used the fact that \( U_{j+i}^{(0)} \geq r_1, c_{j,i}^{(k)} \geq 0, \) and the definition of \( C_{j,i} \geq \max_k \{ c_{j,i}^{(k)} \} \) in this calculation. This gives a lower bound for each term in the numerical scheme.

On the other hand, \( U^{(k)} \) can also be factored as

\[
U^{(k)} = (-\mu)^k (u - r_m) P^{(k)}(u)
\]
where \( p^{(k)}(u) \) is also a polynomial bounded by 1 for \( u \in (0, 1) \). For this case we have

\[
\sum_{k=0}^{K} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} U_{j+i}^{(k)} = c_{j,i}^{(0)} U_{j+i}^{(0)} - (r_m - U_{j+i}^{(0)}) \sum_{k=1}^{K} \frac{(\Delta t \mu)^k}{k!} c_{j,i}^{(k)} (-1)^k p^k(U_{j+i}^{(0)}) \leq c_{j,i}^{(0)} U_{j+i}^{(0)} + (r_m - U_{j+i}^{(0)}) \sum_{k=1}^{K} \frac{(\Delta t \mu)^k}{k!} c_{j,i}^{(k)} p^k(U_{j+i}^{(0)}) \]

This gives us an upper bound for each term in the series.

We now show the desired bound for our scheme:

\[
\tilde{U}_j^{(0)} = U_j^{n+1} = U_j^{n+1} = \sum_{i=-r}^{r} \sum_{k=0}^{K} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} U_{j+i}^{(k)} \geq \sum_{i=-r}^{r} c_{j,i}^{(0)} r_1 = r_1 \sum_{i=-r}^{r} c_{j,i}^{(0)} = r_1
\]

and

\[
U_j^{n+1} = \sum_{i=-r}^{r} \sum_{k=0}^{K} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} U_{j+i}^{(k)} \leq \sum_{i=-r}^{r} c_{j,i}^{(0)} r_m = r_m
\]

since \( \sum_{i=-r}^{r} c_{j,i}^{(0)} = 1 \) as stated before. This proves the lemma.
Note that the case where \( c_{j,i}^{(0)} = 0 \) implies that \( c_{j,i}^{(k)} = 0 \) for all \( k \). Otherwise, the restriction given in Equation (4.4) of the hypothesis would require \( \Delta t = 0 \).

We give an example that illustrates the type of restriction this lemma imposes on a scheme. We consider the simple example of our scheme discussed in the previous chapter. Here, \( f(u) = u, \Psi(u) = -\mu u(u - 1)(u - \frac{1}{2}) \), and \( K = 1 \), and we use the upwinding scheme at each level. Our scheme in this case has the form

\[
\begin{align*}
U_j^{(0)} &= U_j^{(0)} - \lambda(U_j^{(0)} - U_{j-1}^{(0)}) \\
&\quad + \Delta t(U_j^{(1)} - \lambda(U_j^{(1)} - U_{j-1}^{(1)}))
\end{align*}
\]

where \( \lambda = \frac{\Delta t}{\Delta x} \). We can rewrite this in the format of Equation (4.3) as

\[
\begin{align*}
U_j^{(0)} &= c_{j,0}^{(0)}U_j^{(0)} + c_{j,-1}^{(0)}U_{j-1}^{(0)} \\
&\quad + \Delta t(c_{j,0}^{(1)}U_j^{(1)} + c_{j,-1}^{(1)}U_{j-1}^{(1)})
\end{align*}
\]

where

\[
\begin{align*}
c_{j,-1}^{(0)} &= c_{j,-1}^{(1)} = \lambda, \\
c_{j,0}^{(0)} &= c_{j,0}^{(1)} = 1 - \lambda, \quad \text{and} \\
c_{j,1}^{(0)} &= c_{j,1}^{(1)} = 0.
\end{align*}
\]

Here, \( c_{j,i}^{(k)} \geq 0 \) whenever \( 0 \leq \lambda \leq 1 \). Further, the condition

\[
(e^{\Delta t\mu} - 1)C_{j,i} \leq c_{j,i}^{(0)}
\]

is satisfied for \( e^{\Delta t\mu} \leq 2 \) since \( C_{j,i} = c_{j,i}^{(0)} \) for all \( i \) in this example. So, the hypothesis of the previous lemma requires that the parameter \( \Delta t\mu \leq \ln 2 \) in order to guarantee a uniform bound on our scheme.
In contrast, consider the basic upwinding scheme

\[ \tilde{U}_j^{(0)} = U_j^{(0)} - \lambda(U_j^{(0)} - U_{j-1}^{(0)}) + \Delta t U_j^{(1)} \]

which can also be written in the format of Equation (4.3) as

\[ \tilde{U}_j^{(0)} = c_{j,0}^{(0)} U_j^{(0)} + c_{j,-1}^{(0)} U_{j-1}^{(0)} + \Delta tc_{j,0}^{(1)} U_j^{(1)} \]

where

\[ c_{j,-1}^{(0)} = \lambda \]
\[ c_{j,0}^{(0)} = 1 - \lambda \]
\[ c_{j,1}^{(0)} = 0 \]

but

\[ c_{j,-1}^{(1)} = 0 \]
\[ c_{j,0}^{(1)} = 1 \]
\[ c_{j,1}^{(1)} = 0. \]

Again, \( c_{j,i}^{(k)} \geq 0 \) whenever \( \lambda \leq 1 \). However, the condition

\[ (e^{\Delta t\mu} - 1)C_{j,i} \leq c_{j,i}^{(0)} \]

when \( i = 0 \) implies

\[ e^{\Delta t\mu} \leq 2 - \lambda. \]

If we choose \( \lambda \) close to 1, then \( \Delta t\mu \) must be chosen close to 0. This is an undesirable condition since \( \lambda \) is often chosen close to 1.
4.2 Entropy Condition

As discussed in the introduction, one problem with approximating stiffly forced conservation laws is that their solutions are not necessarily unique. A family of weak solutions may exist that satisfy the differential equation. To pick out the correct solution, we require that the numerical solution satisfy an additional entropy condition.

Our entropy condition will provide a bound on the discrete spatial derivatives. This will be needed in the following convergence proofs. We will provide an entropy condition for the classes of our numerical scheme in which upwinding is used at each level of the discretization. That is, we will assume our numerical scheme has the form

\[
\tilde{U}_j^{(0)} = U_j^{(0)} - \frac{\Delta t}{\Delta x} (f(U_j^{(0)}) - f(U_{j-1}^{(0)})) \\
+ \sum_{k=1}^{K} \frac{\Delta t^k}{k!} (U_j^{(k)} - f'(U_j^{(0)}) \frac{\Delta t}{\Delta x} (U_j^{(k)} - U_{j-1}^{(k)}))
\]

for some \(K\). The entropy condition for this scheme is given in the following theorem.

**Lemma 4.2** Let \(U_j^n\) be a numerical approximation of a stiffly forced conservation law of the form defined in Equation (4.5). Then, there exists an upper bound for

\[
z_j^n = \frac{U_j^{n+1} - U_j^n}{\Delta x}
\]

as \(t \to \infty\) whenever

\[
\Delta t \mu \leq \ln(1 + \frac{C}{2B})
\]

where

\[
C = \min\{\frac{1}{2\lambda}, f''(u)\}
\]

and

\[
B = \max\{f''(u)\}
\]
for all $u$ between 0 and 1.

Proof. We begin with some definitions. Let

$$\dot{u}_{j}^{(k)} = \begin{cases} U_{j}^{n} - \lambda(f(U_{j}^{n}) - f(U_{j-1}^{n})) & \text{if } k = 0 \\ U_{j}^{(k)} - \lambda f'(U_{j}^{n})(U_{j}^{(k)} - U_{j-1}^{(k)}) & \text{if } k \geq 1 \end{cases}$$

be the discretization at each time level of the associated conservation laws. Define

$$z_{j}^{n} = \frac{U_{j+1}^{n} - U_{j}^{n}}{\Delta x}, \quad \text{(difference for nodal values)}$$

$$z_{j}^{(k)} = \frac{U_{j+1}^{(k)} - U_{j}^{(k)}}{\Delta x}, \quad \text{(difference for each } U^{(k)})$$

and

$$z_{j}^{(k)} = \frac{\dot{u}_{j+1}^{(k)} - \dot{u}_{j}^{(k)}}{\Delta x} \quad \text{(difference for associated conservation laws)}$$

as approximations to the spatial derivatives.

We will bound each of these differences.

$$z_{j}^{(0)} = \frac{\dot{u}_{j+1}^{(0)} - \dot{u}_{j}^{(0)}}{\Delta x}$$

$$= \frac{1}{\Delta x} \left( U_{j+1}^{n} - \lambda(f(U_{j+1}^{n}) - f(U_{j}^{n})) - U_{j}^{n} + \lambda(f(U_{j}^{n}) - f(U_{j-1}^{n})) \right)$$

$$= z_{j}^{n} - \frac{\lambda}{\Delta x} \left( f'(U_{j}^{n})(U_{j+1}^{n} - U_{j}^{n}) + \frac{f''(\theta_{1})}{2}(U_{j+1}^{n} - U_{j}^{n})^{2} \right)$$

$$+ \frac{\lambda}{\Delta x} \left( -f'(U_{j}^{n})(U_{j-1}^{n} - U_{j}^{n}) - \frac{f''(\theta_{2})}{2}(U_{j-1}^{n} - U_{j}^{n})^{2} \right)$$

$$= z_{j}^{n} - \lambda f'(U_{j}^{n})z_{j}^{n} - \Delta t \frac{f''(\theta_{1})}{2}(z_{j}^{n})^{2} + \lambda f'(U_{j}^{n})z_{j-1}^{n} - \Delta t \frac{f''(\theta_{2})}{2}(z_{j-1}^{n})^{2}$$

$$= (1 - \lambda f'(U_{j}^{n}))z_{j}^{n} + \lambda f'(U_{j}^{n})z_{j-1}^{n} - \Delta t \frac{f''(\theta_{1})}{2}(z_{j}^{n})^{2} + f''(\theta_{2})(z_{j-1}^{n})^{2}$$

$$\leq (1 - \lambda f'(U_{j}^{n}))z_{j}^{n} + \lambda f'(U_{j}^{n})z_{j-1}^{n} - \frac{\Delta t C}{2}((z_{j}^{n})^{2} + (z_{j-1}^{n})^{2})$$

since $C \leq \min\{f''\}$. Next, applying the mean value theorem,

$$z_{j}^{(k)} = \frac{U_{j+1}^{(k)} - U_{j}^{(k)}}{\Delta x}$$
\[
\frac{dt(k)}{dt}(\theta)(U^n_{j+1} - U^n_j)
\]
\[
\leq \mu^k|z^n_j|
\]

for some \(\theta\) between \(U^n_j\) and \(U^n_{j+1}\). So,

\[
z^{(k)}_j = \frac{\hat{u}^{(k)}_{j+1} - \hat{u}^{(k)}_j}{dx}
\]
\[
= \frac{1}{dx} \left( U^{(k)}_{j+1} - \lambda f'(U^n_{j+1})(U^n_j - U^n_{j+1}) - U^n_j + \lambda f'(U^n_j)(U^n_j - U^n_{j-1}) \right)
\]
\[
= z^{(k)}_j - \lambda f'(U^n_{j+1})z^{(k)}_j + \lambda f'(U^n_j)z^{(k)}_{j-1}
\]
\[
= z^{(k)}_j - \lambda (f'(U^n_j) + f''(\theta_3)(U^n_{j+1} - U^n_j))z^{(k)}_j + \lambda f'(U^n_j)z^{(k)}_{j-1}
\]
\[
= (1 - \lambda f'(U^n_j))z^{(k)}_j + \lambda f'(U^n_j)z^{(k)}_{j-1} - \Delta tf''(\theta_3)z^n_j z^{(k)}_j
\]
\[
\leq (1 - \lambda f'(U^n_j))\mu^k|z^n_j| + \lambda f'(U^n_j)\mu^k|z^n_{j-1}| + \Delta tB\mu^k(z^n_j)^2
\]
since \(B = \max\{f''\} \).

Finally, consider \(z^{n+1}_j\). We have

\[
z^{n+1}_j = \frac{U^{n+1}_{j+1} - U^{n+1}_j}{dx}
\]
\[
= \sum_{k=0}^K \frac{\Delta t^k}{k!} \frac{\hat{u}^{(k)}_{j+1} - \hat{u}^{(k)}_j}{dx}
\]
\[
= z^{(0)}_j + \sum_{k=1}^K \frac{\Delta t^k}{k!} z^{(k)}_j
\]
\[
\leq (1 - \lambda f'(U^n_j))z^n_j + \lambda f'(U^n_j)z^n_{j-1} - \frac{\Delta tc^2}{2}((z^n_j)^2 + (z^n_{j-1})^2)
\]
\[
+ \left( (1 - \lambda f'(U^n_j))|z^n_j| + \lambda f'(U^n_j)|z^n_{j-1}| + \Delta tB(z^n_j)^2 \right) \sum_{k=1}^K \frac{\left(1 + \lambda f'(U^n_j)\right)}{k!}
\]
\[
\leq (1 - \lambda f'(U^n_j))(z^n_j + (e^{\Delta t\mu} - 1)|z^n_j|)
\]
\[
+ (\lambda f'(U^n_j))(z^n_{j-1} + (e^{\Delta t\mu} - 1)|z^n_{j-1}|)
\]
\[
\leq (4.7)
\]
\[
\leq (1 - \lambda f'(U^n_j))(z^n_j + (e^{\Delta t\mu} - 1)|z^n_j|)
\]
\[
+ (\lambda f'(U^n_j))(z^n_{j-1} + (e^{\Delta t\mu} - 1)|z^n_{j-1}|)
\]
\[
- \frac{\Delta tc^2}{2}((z^n_j)^2 + (z^n_{j-1})^2) + \Delta tB(e^{\Delta t\mu} - 1)(z^n_j)^2.
\]
We will bound this difference term by term. Let \( m_j^n = \max\{z_j^n, z_{j-1}^n, 0\} \). For the first term, we consider two cases. First, suppose \( z_j^n < 0 \). We label the first term in Equation (4.7) as \( T_1 \) and bound it as follows:

\[
T_1 = z_j^n + (e^{\Delta t \mu} - 1)|z_j^n| \\
= -|z_j^n| + (e^{\Delta t \mu} - 1)|z_j^n| \\
= (e^{\Delta t \mu} - 2)|z_j^n| \\
\leq 0 \leq e^{\Delta t \mu} m_j^n
\]

since \( e^{\Delta t \mu} - 2 \leq 0 \) by hypothesis. For the other case, suppose \( z_j^n > 0 \). Then,

\[
T_1 = z_j^n + (e^{\Delta t \mu} - 1)z_j^n \\
= e^{\Delta t \mu} z_j^n \\
\leq e^{\Delta t \mu} m_j^n.
\]

Similarly, the second term is also bounded,

\[
T_2 = z_{j-1}^n + (e^{\Delta t \mu} - 1)|z_{j-1}^n| \leq e^{\Delta t \mu} m_j^n,
\]

regardless of the sign of \( z_{j-1}^n \).

We now consider the last set of terms in this equation. Here,

\[
T_3 = \frac{-\Delta t C}{2}((z_j^n)^2 + (z_{j-1}^n)^2) + \Delta t B(e^{\Delta t \mu} - 1)(z_j^n)^2 \\
= -\Delta t\left(C - B(e^{\Delta t \mu} - 1)\right)(z_j^n)^2 - \frac{\Delta t C}{2}(z_{j-1}^n)^2 \\
\leq -\Delta t \hat{C}((z_j^n)^2 + (z_{j-1}^n)^2)
\]

where

\[
\hat{C} = \frac{C}{2} - B(e^{\Delta t \mu} - 1) \leq \frac{C}{2}.
\]
The coefficient, $\hat{C}$, is positive since $\Delta t \mu \leq \ln(1 + \frac{C}{2b})$ as required by the hypothesis.

We also consider cases in bounding $T_3$. First, suppose $m_j^n = 0$. Then,

$$T_3 \leq 0 = -\Delta t \hat{C}(m_j^n)^2.$$  

Next, suppose $m_j^n = z_j^n$. Here,

$$T_3 \leq -\Delta t \hat{C}(z_j^n)^2 = -\Delta t \hat{C}(m_j^n)^2.$$  

The case where $m_j^n = z_{j-1}^n$ is identical.

Now we have

$$z_j^{n+1} = (1 - \lambda f'(U_j^n))T_1 + \lambda f'(U_j^n)T_2 + T_3$$

$$\leq (1 - \lambda f'(U_j^n))e^{\Delta t \mu}m_j^n + \lambda f'(U_j^n)e^{\Delta t \mu}m_j^n - \Delta t \hat{C}(m_j^n)^2$$

$$= e^{\Delta t \mu}m_j^n - \Delta t \hat{C}(m_j^n)^2.$$  

Next, we seek a uniform bound on $z_j^{n+1}$ for all $j$. Let $M^n = \max_j \{m_j^n\}$. Consider the function $g(y) = e^{\Delta t \mu}y - \Delta t \hat{C}y^2$. So, $g'(y) = e^{\Delta t \mu} - 2\Delta t \hat{C}y \geq 0$ whenever $y \leq \frac{e^{\Delta t \mu}}{2\Delta t \hat{C}}$. That is, $g$ is increasing for $y \leq \frac{e^{\Delta t \mu}}{2\Delta t \hat{C}}$. Notice that

$$z_j^n \leq \frac{1}{\Delta x} = \frac{\lambda}{\Delta t} \leq \frac{1}{2\Delta t \hat{C}} \leq \frac{e^{\Delta t \mu}}{2\Delta t \hat{C}}$$

for all $j$ since $\hat{C} \leq C$ by hypothesis and $\hat{C} \leq C$. So, $z_j^n \leq M^n \leq \frac{e^{\Delta t \mu}}{2\Delta t \hat{C}}$. Therefore,

$$z_j^{n+1} \leq g(m_j^n) \leq g(M^n)$$

$$= e^{\Delta t \mu}M^n - \Delta t \hat{C}(M^n)^2.$$  

for all $j$. So, we can set

(4.8) $M^{n+1} = e^{\Delta t \mu}M^n - \Delta t \hat{C}(M^n)^2.$
We now have a relationship on the $M^n$'s similar to the discrete dynamical system for the logistic equation.

We conclude the proof by discussing the dynamics of this system. We follow arguments found in any book discussing dynamical systems. See, for example, [3, pages 81-84].

Note that $M^n$ is between 0 and $\frac{e^{\Delta t \mu}}{2\Delta t C}$ for all $n$. So, the values on the right-hand side of Equation (4.8) are always positive. This system has two equilibria:

$$M_1 = 0$$

and

$$M_2 = \frac{e^{\Delta t \mu} - 1}{\Delta t C}.$$  \hspace{1cm} (4.9)

The point $M_1 = 0$ is stable for $e^{\Delta t \mu} < 1$, and the equilibrium point $M_2$, given in Equation (4.9), is stable for $1 < e^{\Delta t \mu} < 3$ which is already guaranteed. So the sequence of $M^n$'s converges to $M_2$.

Finally, we note that this equilibrium is bounded. As $\Delta t \to 0$, we see that $\dot{C} \to C/2$ and

$$M_2 \to \frac{2\mu}{C}.$$  

This is a desirable entropy condition since it is expected that the steepness of the spatial derivative will be controlled by the ratio of the stiffness parameter and the convexity of the flux function. This completes our proof.

We comment here that the above entropy proof is a generalization of the entropy proof given for the Lax Friedrich's scheme when $\Psi = 0$ described in [15, Lemma 16.3]. Here, we chose the scheme with upwinding at each time step for our analysis instead of the Lax Friedrich's scheme for two reasons. First, all of our numerical simulations in the next
and previous chapter use upwinding at each level. Second, the upwinding scheme fits the format of our discretization whereas the Lax Friedrich’s scheme is not easily written in the format with $U_{j+\frac{1}{2}}^n$.

We will illustrate the restrictions given by the above lemma with the following example. Consider the case where $f(u) = \frac{u^2}{2}$. This case is commonly known as Burgers’ equation. In this case, $B = f''(u) = 1$ and $C = \min\{f''(u), \frac{1}{2}\lambda\} = \min\{1, 1/2\}$. So, the condition in Equation (4.6) requires that

$$\Delta t \mu \leq \ln(1.25)$$

for this flux function.

We must also consider the restrictions required for a uniform bound. Here, we have $c_{j,0}^{(k)} = 1 - \lambda f'(U_j^n)$ and $c_{j,-1}^{(k)} = \lambda f''(U_j^n)$ for all $k \geq 1$. So,

$$(e^{\Delta t \mu} - 1)C_{j,-1} = (e^{\Delta t \mu} - 1)\lambda f'(U_j^n)$$

$$\leq .25\lambda U_j^n$$

$$\leq \frac{1}{2} \lambda (U_j^n + U_{j-1}^n)$$

$$= \frac{1}{2} \lambda \left( \frac{U_j^n U_{j-1}^n - (U_{j-1}^n)^2}{U_j^n - U_{j-1}^n} \right)$$

$$= \lambda \frac{f(U_j^n) - f(U_{j-1}^n)}{U_j^n - U_{j-1}^n}$$

$$= c_{j,-1}^{(0)}$$

and

$$(e^{\Delta t \mu} - 1)C_{j,0} = (e^{\Delta t \mu} - 1)(1 - \lambda f'(U_j^n))$$

$$\leq \frac{1}{2} (1 - \lambda U_j^n)$$

$$\leq \frac{1}{2} (1 - \lambda U_j^n) + \frac{1}{2}(1 - \lambda U_{j-1}^n)$$
\[
\frac{1}{2} (U^n_j + U^n_{j-1})
\]
\[
= 1 - \frac{1}{2} \frac{(U^n_j)^2 - (U^n_{j-1})^2}{U^n_j - U^n_{j-1}}
\]
\[
= 1 - \lambda \frac{f(U^n_j) - f(U^n_{j-1})}{U^n_j - U^n_{j-1}}
\]
\[
= c^{(0)}_{j-0}.
\]

Therefore, the uniform bound conditions are also satisfied by this restriction.

4.3 Total Variation Bounds

We have given requirements to guarantee that our scheme is uniformly bounded and satisfies an entropy condition. This ensures that we have an upper bound on the spatial derivative as \( t \) gets large. That is, we can bound

\[
\frac{U^n_{j+1} - U^n_j}{\Delta x} \leq C_1
\]

for \( n \geq n_0 \).

We are now prepared to prove the lemmas concerning the total variation bounds on our scheme. The two lemmas will provide us with the tools to guarantee that our numerical scheme converges to a measurable function \( u \). In the next section we will show that the function \( u \) is a weak solution to our original conservation law defined in Equation (2.1).

In the following lemmas, let \( M \) be a uniform bound on \( U^n_j \) and \( X \) be length of the interval \([-x, x]\) for which our problem is defined.

**Lemma 4.3 (Space Estimate)** There is a constant \( c \) depending on \( M, X \), and some initial time \( t_0 \), independent of \( \Delta t \) and \( \Delta x \), such that

\[
\sum_{|b| \leq J} |U^n_{j+1} - U^n_j| \leq c
\]

where \( J = \frac{X}{2\Delta x} \).
Proof. Let \( t_0 \) be given so that \( \frac{U_{n+1}^n - U_n^n}{\Delta x} \leq C_1 \) whenever \( n\Delta t \geq t_0 \). Now set \( v_j^n = U_j^n - C_1 x_j \). Then,

\[
v_{j+1}^n - v_j^n = U_{j+1}^n - U_j^n - C_1 \Delta x \\
\leq \Delta x C_1 - C_1 \Delta x \\
\leq 0.
\]

So,

\[
\sum_{|j| \leq J} |U_{j+1}^n - U_j^n| \leq \sum_{|j| \leq J} |v_{j+1}^n - v_j^n| + \sum_{|j| \leq J} C_1 \Delta x \\
= - \sum_j (v_{j+1}^n - v_j^n) + C_1 \Delta x (2J + 1) \\
\leq 2 \max_j |v_j^n| + C_1 (X + \Delta x) \\
\leq 2M + C_2 X \\
= c.
\]

This completes our proof.

Lemma 4.4 (Time Estimate) If \( \lambda \geq \delta > 0 \) and \( \Delta t, \Delta x \leq 1 \), then there exists a constant \( L \geq 0 \) independent of \( \Delta t \) and \( \Delta x \) such that if \( k \geq p \) we have

\[
\sum_{|j| \leq J} |U_j^k - U_j^p| \Delta x \leq L(k - p) \Delta t.
\]

Proof. Recall our scheme of the form

\[
\bar{U}_j^{(0)} = \sum_{k=0}^{K} \sum_{i=-r}^{r} \frac{\Delta t^k}{k!} c_{j,i}^{(k)} U_j^{(k)}
\]

Then,

\[
U_{j+1}^n - U_j^n = \bar{U}_j^{(0)} - U_j^{(0)}
\]
Letting \( c \) be the bound found in Lemma 4.3, we get

\[
\sum_{|j| \leq J} |U^{n+1}_j - U^n_j| \leq \sum_{i=-r}^r c^{(0)}_{j,i} \sum_{w=0}^{i-1} \sum_j |U^{(k)}_{j+w+1} - U^{(0)}_{j+w}| \sum_{j} + \sum_{k=1}^K \sum_{i=-r}^r \frac{\Delta t^k}{k!} \sum_j \mu^k \\
\leq \sum_{i=-r}^r c^{(0)}_{j,i} \sum_{w=0}^{i-1} c + \sum_{k=1}^K \sum_{i=-r}^r \frac{\Delta t^k}{k!} \mu^{k \lambda 2J} \\
= c \sum_{i=-r}^r c^{(0)}_{j,i} |i - 1| + \mu \sum_{k=1}^K \sum_{i=-r}^r \frac{(\Delta t \mu)^{k-1}}{k!} (2J \Delta t) \\
\leq c(r^2 + r + 1) + X \lambda \mu \sum_{i=-r}^r \sum_{k=1}^K \frac{1}{k!} \\
\leq c(r^2 + r + 1) + 2rX \lambda \mu e \\
= \tilde{L}.
\]

Finally, we obtain

\[
\sum_j |U_{n+1}^k - U_j^p| \Delta x \leq \sum_{i=1}^{k-p} \sum_j |U_{j+i}^{p+i} - U_{j+i-1}^p| \Delta x \\
= (k - p) \sum_j |U_{j+i}^{p+i} - U_{j+i-1}^p| \Delta x \\
= (k - p) \tilde{L} \Delta x \\
\leq (k - p) \frac{\tilde{L}}{\delta} \Delta t \\
= L(k - p) \Delta t
\]

as desired. This completes our proof.
We now have bounds on the total variation of our numerical scheme in both space and time.

### 4.4 Convergence

We construct a family of functions \( \{ U_{\Delta x, \Delta t} \} \) from the numerical approximation using our scheme with a given \( \Delta x \) and \( \Delta t \) by defining

\[
U_{\Delta x, \Delta t}(x, t) = U^n_j, \quad \text{if} \quad j\Delta x \leq x < (j+1)\Delta x, \quad n\Delta t \leq t < (n+1)\Delta t.
\]

That is, the value of \( U_{\Delta x, \Delta t} \) in the rectangle \( j\Delta x \leq x < (j+1)\Delta x, \quad n\Delta t \leq t < (n+1)\Delta t \) is the value of the difference approximation at the point \( (j\Delta x, n\Delta t) \).

Now, we have the results to show that there exists a sequence, \( \{ U_i = U_{\Delta x_i, \Delta t_i} \} \), that converges to a measurable function \( u(x, t) \). An application of Helley's theorems and standard diagonalization arguments give convergence. See Smoller [15, Lemma 16.7] for a complete proof. We can also refer to DiPerna's paper [2] where it is shown that an entropy condition and bounds on the total variation are sufficient to guarantee that the numerical approximation converges to a function \( u \).

Now, we will show that the function, \( u \), is the desired solution to our conservation law. DiPerna's paper also gives us this result and guarantees the uniqueness of \( u \). We provide the simplified proof below for completeness.

Consider the numerical scheme we developed written in conservation form

\[
\begin{align*}
\bar{U}_j^{(0)} &= U_j^{(0)} - \frac{\Delta t}{\Delta x} (f(U_{j+\frac{1}{2}}^{(0)}) - f(U_{j-\frac{1}{2}}^{(0)})) \\
&\quad + \sum_{k=1}^{K} \frac{\Delta t^k}{k!} (U_j^{(k)} - \frac{a_j \Delta t}{\Delta x} (U_{j+\frac{1}{2}}^{(k)} - U_{j-\frac{1}{2}}^{(k)})).
\end{align*}
\]

(4.10)

where the \( a_j = f'(U_j^{(0)}) \).
Lemma 4.5 Let $\Delta x_i \to 0$ as $i \to \infty$, and suppose that for a test function $\phi \in C^1_0$

$$\lim_{i \to \infty} \int_{-\infty}^{\infty} [U_i(x, 0) - u_0(x)] \phi(x, 0) \, dx = 0.$$ 

Then, $u$ is a weak solution of the stiffly forced conservation law given in Equation (2.2).

Proof. We will show that the limit function $u(x, t)$ satisfies the weak form of the conservation law, i.e., for all $\phi \in C^1_0$ with compact support,

$$\int_0^\infty \int_{-\infty}^{\infty} [\phi, u + \phi_x f(u)] \, dx \, dt + \int_{-\infty}^{\infty} [\phi(x, 0)u(x, 0)] \, dx = -\int_0^\infty \int_{-\infty}^{\infty} \phi(x, t) \Psi(u(x, t)) \, dx \, dt.$$ 

Let $\phi$ be an arbitrary test function. First, we multiply the numerical scheme in Equation (4.10) by $\phi_j^n = \phi(x_j, t_n)$ and rearrange a few terms to obtain

$$\phi_j^n (U_j^{n+1} - U_j^n) + \lambda \phi_j^n (f(U_j^{(0)})_j + \phi_x f(u)) =$$

$$\sum_{k=1}^{K} \frac{\Delta t^k}{k!} \phi_j^n (U_j^{(k)} - a_j \lambda (U_j^{(k)} - U_j^{(k)}))$$

where $a_j = f'(U_j^{(0)})$ and $\lambda = \frac{\Delta t}{\Delta x}$ as before.

To approximate the integral, we sum over all $j$ and $n \geq 0$. We will use a "summation by parts" simplification which amounts to recombining the terms in each sum. That is, the first term will be rearranged as follows:

$$\sum_{n=0}^{\infty} \phi_j^n (U_j^{n+1} - U_j^n) = (\phi_j^0 U_j^1 - \phi_j^0 U_j^0) + (\phi_j^1 U_j^2 - \phi_j^1 U_j^1) + \cdots$$

$$= -\phi_j^0 U_j^0 + (\phi_j^0 U_j^1 - \phi_j^1 U_j^1) + (\phi_j^1 U_j^2 - \phi_j^2 U_j^2) + \cdots$$

$$= -\phi_j^0 U_j^0 + \sum_{n=1}^{\infty} (\phi_j^{n-1} - \phi_j^n) U_j^n.$$
Similarly, the second term in the above equation can be rearranged to obtain

\[
\sum_{j=-\infty}^{\infty} \phi_j^n (f(U_{j+\frac{1}{2}}^{(0)}) - f(U_{j-\frac{1}{2}}^{(0)})) \\
= \cdots \phi_{-1}^n (f(U_{-\frac{1}{2}}^{(0)}) - f(U_{-\frac{3}{2}}^{(0)})) + \phi_0^n (f(U_{\frac{1}{2}}^{(0)}) - f(U_{-\frac{1}{2}}^{(0)})) \\
+ \phi_1^n (f(U_{\frac{1}{2}}^{(0)}) - f(U_{\frac{3}{2}}^{(0)})) + \cdots \\
= \cdots + (\phi_{-1}^n - \phi_0^n) f(U_{-\frac{1}{2}}^{(0)}) + (\phi_0^n - \phi_1^n) f(U_{\frac{1}{2}}^{(0)}) + \cdots \\
= - \sum_{j=-\infty}^{\infty} (\phi_{j+1}^n - \phi_j^n) f(U_{j+\frac{1}{2}}^{(0)}).
\]

We now sum Equation (4.11) over all \(j\) and \(n\) and use the simplifications above to obtain

\[
- \sum_{j=-\infty}^{\infty} \phi_j^0 U_j^0 - \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} (\phi_j^n - \phi_j^{n-1}) U_j^n \\
- \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} (\phi_{j+1}^n - \phi_j^n) f(U_{j+\frac{1}{2}}^{(0)}) \\
= \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=1}^{K} \frac{\Delta t^k}{k!} \phi_j^n \{U_j^{(k)} - a_j \lambda (U_{j+\frac{1}{2}}^{(k)} - U_{j-\frac{1}{2}}^{(k)})\}.
\]

Note that each of these sums is in fact a finite sum since \(\phi\) has compact support.

Multiplying through by \(-\Delta x\), and rearranging a few terms in this equation gives

\[
\Delta t \Delta x \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \phi_j^n - \phi_j^{n-1} \frac{U_j^n}{\Delta t} \\
+ \Delta t \Delta x \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi_{j+1}^n - \phi_j^n}{\Delta x} f(U_{j+\frac{1}{2}}^{(0)}) \\
+ \Delta x \sum_{j=-\infty}^{\infty} \phi_j^0 U_j^0 \\
= -\Delta t \Delta x \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \phi_j^n U_j^{(1)} + \Delta t^2 \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} a(U_{j+\frac{1}{2}}^{(1)} - U_{j-\frac{1}{2}}^{(1)}) \\
- \Delta t^2 \Delta x \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=2}^{K} \frac{\Delta t^{k-1}}{k!} \phi_j^n \{U_j^{(k)} - a_j \lambda (U_{j+\frac{1}{2}}^{(k)} - U_{j-\frac{1}{2}}^{(k)})\}.
\]
Now let $\Delta t, \Delta x \to 0$. Using the convergence of $U_j^{(0)} \to u$ and the smoothness of $\phi$, the first term converges to

$$\int_0^\infty \int_{-\infty}^\infty \phi_t(x,t) u(x,t) \, dx \, dt.$$ 

The second term also relies on the consistency of $U_j^{(0)}$ with $u$. Here, we obtain

$$\int_0^\infty \int_{-\infty}^\infty \phi_x(x,t) f(u(x,t)) \, dx \, dt.$$ 

Using the assumption that the initial data is close to $u(x,0)$, the third term converges to

$$\int_{-\infty}^\infty \phi(x,0) u(x,0) \, dx.$$ 

Similarly the fourth term converges to

$$\int_0^\infty \int_{-\infty}^\infty \phi(x,t) \Psi(u(x,t)) \, dx \, dt.$$ 

The remaining terms all are bounded and have an additional power of $\Delta t$. Thus they converge to 0. This completes the proof.

This theorem guarantees that the limit of our numerical scheme is in fact a weak solution to the conservation law. Additionally, the requirement that the numerical approximation satisfy a discrete version of the entropy condition guarantees that the approximations will converge to the correct entropy satisfying solution as $\Delta t$ and $\Delta x$ go to zero.

The results of our work allow us to state the following theorem:

**Theorem 4.6** Let $u_0 \in L_\infty(\mathbb{R})$ and let $f \in C^2(\mathbb{R})$ with $f'' > 0$. Then, there exists a solution $u$ of Equation (1.1), (1.2) with the following properties:

- $|u(x,t)| \leq ||u_0||_\infty = M$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. 

There is a constant $E$ depending only on $M$, $\min\{f''\}$, $\max\{|f'|\}$, and the parameter $\Delta t \mu$ such that for every $a > 0$, $t > t_0$, and $x \in \mathbb{R}$

$$\frac{u(x + a, t) - u(x, t)}{a} < E.$$
CHAPTER 5
NUMERICAL EXAMPLES

This chapter contains numerical simulations using the numerical scheme we developed in Chapter 3. In these examples, we use the class of schemes where the upwinding discretization is taken at each level of the scheme. That is, our scheme is of the form

\[
\begin{align*}
\dot{U}_j^{(0)} &= U_j^{(0)} - \frac{\Delta t}{\Delta x} (f(U_j^{(0)}) - f(U_{j-1}^{(0)})) \\
&\quad + \sum_{k=1}^{K} \frac{\Delta t^k}{k!} (U_j^{(k)} - f'(U_j^{(0)})\frac{\Delta t}{\Delta x} (U_j^{(k)} - U_{j-1}^{(k)}))
\end{align*}
\]

(5.1)

for some value of \( K \). We will compare results obtained using our scheme with known solutions as well as numerical simulations and examples given in reference papers.

5.1 Example 1: Linear Forcing Function

The first example we consider is discussed by Tang [16]. Here, we consider the stiffly forced conservation law

\[ u_t + u_x = -\mu u \]

with initial condition \( u_0(x) \). The exact solution for this conservation law is given by

\[ u(x, t) = e^{-\mu t}u_0(x - t). \]

For our simulation, we use the initial condition \( u_0 = \sin(2\pi x) \). The numerical results using our scheme with \( K = 1 \) are seen in Figures 5.1, 5.2, and 5.3.

- In Figure 5.1, the parameters are set to \( \Delta t = 0.05 \) as well as for \( \Delta t = 0.005 \). In both cases, we see that the
solution is moving at the correct speed. However, in the case where \( \Delta t \) is smaller, the solution is tending to pull closer to the single equilibrium, \( u = 0 \).

- To verify that the numerical solution maintains the correct speed, we provide Figure 5.2 which plots the solution when \( t = 0.2 \). Using the same setup as in Figure 5.1, we see that both solutions are traveling at the right speed, but again the smaller \( \Delta t \) allows the solution to be drawn to the equilibrium.

- For comparison, we also ran this simulation truncating our scheme after \( K = 2 \) instead. The result is seen in Figure 5.3 for the case when \( \Delta t = 0.05 \) and \( t = 0.1 \). In this example, using another level of the scheme provides better results in approximating the exact solution.
Figure 5.2: Exact solution (solid line) for Example 1 is plotted against two approximate solutions for different values of $\Delta t$ at $t = 0.2$. Here, $\mu = 10$, $\Delta x = 0.1$, $\Delta t = 0.05$ and 0.005

Figure 5.3: Exact solution (solid line) for Example 1 is plotted against two approximate solutions using the numerical scheme truncated at different values of $K$. Here, $\Delta t = 0.05$, $t = 1$, $\mu = 10$, and $\Delta x = 0.1$
5.2 Example 2: Nonlinear Flux Function and Rarefaction Solution

The next two examples come from the paper of Papalexandris, Leonard, and Dimotakis [10]. For these examples, we assume the conservation law has the form

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = -\mu u(u - 1). \]

Note that in this case the sign of \( \mu \) determines which equilibrium, \( u = 0 \) or \( u = 1 \), is stable.

Our first simulation assumes the initial condition

\[ u_0(x) = \begin{cases} 
1 & \text{if } 0 < x < 1 \\
0 & \text{if } x > 1 \text{ or } x < 0
\end{cases} \]

For this simulation, we use the same parameter settings as are found in this paper. Here, \( \mu = 1 \), \( \lambda = .6 \), and \( \Delta x = .02 \). For this value of \( \mu \), \( u = 0 \) is the stable equilibrium. Figure 5.4 gives the solution at times \( t = 0, 1, 3, \) and \( 5 \). The results are identical to those found in [10, Figure 3] and closely match the exact solutions. It is not surprising that both our solution and the solution provided by the Papalexandris scheme behave well in this case since the stiffness parameter, \( \mu = 1 \), is small.

5.3 Example 3: Nonlinear Flux Function and Counter Example

Next, we consider another example from [10]. Again, we assume the conservation law has the form given in Equation (5.2). Here, the initial condition is

\[ u_0(x) = \begin{cases} 
1 & \text{if } x < 1 \\
0 & \text{if } x > 1
\end{cases} \]

and \( \mu < 0 \) so that \( u = 1 \) is the stable equilibrium. The numerical simulations are given in Figure 5.5. Results are plotted for different values of \( \mu \) and using the same parameters as...
in this paper, \( t = 2, \mu = 100, 34, \text{ and } 14, \lambda = 0.6, \text{ and } \Delta x = 0.02 \).

In each case, we see that the numerical solution is moving the wave front faster than the exact solution. We expect such behavior in the case where the parameter \( \Delta t \mu \) is greater than \( \ln(1.25) \) since we are using Burgers flux function in this example. However, the increase in the speed of the traveling wave front for the smaller values of \( \mu \) were unexpected. Similar problems with incorrect wave speeds were seen in [10, Figure 4]. We give a partial explanation of this behavior in the next example by examining its modified equation.

### 5.4 Example 4: Modified Equation

For this final example, we return to the example given in LeVeque and Yee [8] as described in Section 3.3. There, we saw accurate results using our numerical scheme truncated at \( K = 1 \).

We again consider the case where \( \mu = 100 \). In our previous numerical simulations, we
saw in Figure 3.7 that the numerical method was moving the solution slightly faster than the exact solution. We attempt to remedy this problem with the following approaches.

First, we reduced $\Delta t$ so that the value $\Delta t \mu$ is within the bound required by the analysis to guarantee convergence to the correct solution. We still get a slight overshoot in this case as we did with Papalexandris’ examples. The results are given in Figure 5.6.

In an attempt to explain the increase in speed, we look at the modified equation for our scheme,

$$
\tilde{U}_j^{(0)} = U_j^{(0)} - \frac{\Delta t}{\Delta x}(U_j^{(0)} - U_{j-1}^{(0)}) \\
(5.3)
+ \Delta t(U_j^{(1)} - \frac{\Delta t}{\Delta x}(U_j^{(1)} - U_{j-1}^{(1)})) \\
= U_j^n - \lambda(U_j^n - U_{j-1}^n) \\
(5.4)
+ \psi(U_j^n) - \psi(U_{j-1}^n).
$$

As in Section 2.3, we replace the discrete values, $U_j^n$, with the Taylor approximations to
Figure 5.6: Numerical solutions for Example 4 for different values of $\Delta t$. Plots given at time $t = 0.3$ with $\lambda = 0.75$, $\Delta x = 0.02$ and $\mu = 100$

the continuous values, $u(x_j, t_n)$, in our numerical scheme. Here we have

\[
\begin{align*}
    u_t &+ \frac{\Delta t}{2} u_{tt} + \cdots = \\
    &- u_x + \frac{\Delta x}{2} u_{xx} - \frac{\Delta x^2}{3} u_{xxx} + \cdots \\
    &+ \Psi(u) - \Delta t(\Psi(u)_x + \Psi(u)_{xx} \frac{\Delta x^2}{2} + \cdots).
\end{align*}
\]

Now, $u_t = -u_x + \Psi(u)$ and $\Delta t \Psi_x = \Delta t(u_{tx} + u_{xx}) = \Delta t u_{tx} + \Delta x \lambda u_{xx}$. Using these equalities, we simplify this expansion to

\[
\begin{align*}
    \frac{\Delta t}{2} u_{tt} + \cdots & = \Delta x (\frac{1}{2} - \lambda) u_{xx} - \frac{\Delta x^2}{3} u_{xxx} + \cdots \\
    &- \Delta t (u_{tx} + \Psi(u)_{xx} \frac{\Delta x^2}{2} + \cdots).
\end{align*}
\]

Notice, if $\lambda = \frac{1}{2}$, this method is $O(\Delta t, \Delta x^2)$.

Using this information, we rerun the simulation for the LeVeque example by taking $\Delta x$ slightly smaller so that $\lambda = 1/2$. The results are given in Figure 5.7. We have plotted the approximations given when $\lambda = 0.75$ and $\lambda = 0.5$. We see that choosing $\lambda = 0.5$ gives
Figure 5.7: Numerical solutions for Example 4 for different values of $\lambda$ and $K$. Plots given at time $t = 0.3$ with $\Delta x = 0.02$ and $\mu = 100$

a better approximation.

One more simulation is also included in this figure for comparison. We give the approximation with $K = 2$ and $\lambda = 1/2$. In this case, we see that using an extra level in the numerical simulation makes little difference.

We conclude this example by plotting the numerical wave speeds for this simulation both when $\lambda = .75$ and when $\lambda = .5$. This is seen in Figure 5.8. Here, we see that as $\Delta x$ increases, the speed when $\lambda = .75$, pulls away from the numerical wave speed of the exact solution. However, the speeds when $\lambda = .5$ are a close match to the wave speed of the exact solution. Note that the wave speed of the exact solution varies with the grid size. For example, if $\Delta x = .1$, then the exact solution would be represented as 1 at $x = 0.5$ and then jump to 0 at $x = 0.6$. So, the average value for this representation would be at $x = 0.55$. 
5.5 Concluding Remarks

In this paper we have developed a family of numerical schemes for stiffly forced conservation laws using an infinite series of differential equations. We have provided an analysis of these schemes and given requirements that guarantee that schemes satisfying these requirements converge to the physically correct weak solution. The numerical simulations in this chapter indicate that these schemes accurately approximate such equations.
REFERENCES


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