LINEAR OPERATORS THAT PRESERVE
QUALITATIVE MATRIX
STRUCTURES

by

Shumin Ye

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ABSTRACT

Linear Operators That Preserve Qualitative Matrix Structures

by

Shumin Ye, Doctor of Philosophy
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Major Professor: Dr. LeRoy B. Beasley
Department: Mathematics and Statistics

We characterized the group of linear operators that preserve sign-nonsingular matrices over $M_n(R)$. Then we extended these results to show that a linear operator $T$ that strongly preserves $L$-matrices over $M_{m,n}(R)$ if and only if $T$ preserves $L$-matrices and $T$ is also one to one on the set of cells. We also characterized the group of linear operators that strongly preserve $L$-matrices.

In addition, we characterized the group of linear operators that preserve super $L$-matrices, the subset of $L$-matrix. Also we investigated linear operators that preserve totally $L$-matrices, the subset of $L$-matrix.

Chapters 1 and 2 of this dissertation contain some material of the work done by other researchers on the linear preserver problems and the properties of sign-nonsingular matrices and $L$-matrix. Characterizations of linear operators in Chapters 3, 4, 5, and 6 of this dissertation are new.

(150 pages)


1.1 Linear Preservers

Let $\mathbb{K}$ be a field and $\mathcal{M} = \mathcal{M}_{m,n}(\mathbb{K})$ be the set of all $m \times n$ matrices over $\mathbb{K}$. If $T$ is a linear operator on $\mathcal{M}$ and $\mathcal{K}$ is a subset of $\mathcal{M}$, then $T$ preserves $\mathcal{K}$ if $T(X) \in \mathcal{K}$ for each $X$ in $\mathcal{K}$. If $f$ is a function on $\mathcal{M}$, then $T$ preserves $f$ if $f(T(A)) = f(A)$ for all $A$ in $\mathcal{M}$.

A common problem considered in linear algebra is called a preserver problem, that is, characterize those linear operators which "preserve" a function or a set. Many mathematicians have already done a lot of work in this area (see [1]–[35], [39]–[46], [48], [51]–[66], [69], [70], and [72]–[74]).

The classification of preservers began about 100 years ago. In 1897, Frobenius [42] characterized the linear operators on $\mathcal{M}$ which preserve certain matrix functions: those linear operators on $\mathcal{M}$ that preserve the determinant and those that preserve the characteristic polynomial.

After half a century of relative inactivity there has been renewed interest in preserver problems. That interest was sparked by the investigation of rank preservers in 1959 by Marcus and Moyls [61]. They proved:

If $F$ is a algebraically closed field and of characteristic 0 and $T$ is a rank preserver, then there exist $m \times m$ and $n \times n$ matrices $U$ and $V$, respectively, such that

$$T(A) = UAV \text{ for all } A \in \mathcal{M}_{m,n}(F) \quad (1.1.1)$$

or
We call operators of type (1.1.1) and (1.1.2) $(U,V)$-operators.

Also in 1959, Marcus and Moyls [62] found that $T$ is a rank preserver if and only if $T$ is a rank-1 preserver, that is, $T$ preserves the set of matrices whose rank is 1. In 1967, Westwick [72] generalized these results to matrices over arbitrary algebraically closed fields.

In a series of papers between 1970 and 1983, Beasley [2-6] has shown that if $T$ is a rank-$k$-preserver and $\max(m, n) \leq k + 1$, $\min(m, n) = k$, $\max(m, n) \geq 3k/2$, or $T$ is nonsingular, then $T$ is a rank preserver (and hence $T$ is a $(U,V)$-operator). Here $\mathbb{F}$ was required to be an algebraically closed field. In 1988, Beasley [13] proved that if $T$ is a rank-$k$-preserver on $M_{m,n}(\mathbb{C})$ (space of all $m \times n$ complex matrices) for any $k$, $1 \leq k \leq m$, then $T$ is $(U,V)$-operator.

The study of preservers of the set of matrices of rank at most $k$ began in 1983 with papers by Beasley [6,7]. In 1987, Botta [35] characterized linear operators that map $m \times n$ matrices to $p \times q$ matrices and that preserve rank less than or equal to one. Later, Loewy [57] and Laffey [51] obtained results concerning rank-$k$ nonincreasing maps.

A Boolean matrix is a $(0,1)$-matrix with the usual arithmetic except $1 + 1 = 1$. In 1984, Beasley and Pullman [8] studied Boolean-rank-preserving operators and Boolean-rank-1 spaces. They proved that when $\min(m,n) \geq 2$, the following properties are equivalent for all linear operators $T$ on the $m \times n$ Boolean matrices:

1. $T$ preserves all ranks;

2. $T$ preserves ranks 1 and 2;
(3) $T$ is a $(U,V)$-operator;

(4) $T$ is bijective and preserves rank 1.

In 1986, Beasley and Pullman [11] showed that the above results for Boolean matrices are true for matrices over more general "chain semirings".

For any matrix $A$, the term rank of $A$ is the minimum number, $t(A)$, of lines (rows or columns) which contain all the nonzero entries of $A$. In 1987, Beasley and Pullman [12] showed that if $T$ is a linear operator on $M_{m,n}(S)$ ($m \leq n$) where $S$ is any semiring, then the following are equivalent.

1. $T$ preserves term rank.
2. $T$ preserves both the set of matrices of term rank 1 and its complement.
3. $T$ preserves the sets of matrices of term ranks 1 and 2.
4. $T$ is a composition of one or more of the following:
   (i) transposition if $m = n$;
   (ii) $X \rightarrow PXQ$ for some fixed but arbitrary permutation matrices $P$ and $Q$;
   (iii) $X \rightarrow [a_{i,j} x_{i,j}]$ for some fixed but arbitrary matrix $A$ in $M_{m,n}(S)$ with no entry a zero divisor or zero.

In 1991, Beasley and Pullman [17] obtained more results on term rank preservers. They proved that if $F$ is any field and $T$ is a nonsingular linear operator on $M_{m,n}(F)$, then the operator $T$ preserves the set of matrices of term rank 1 if and only if $T$ is one of or a composition of the following operators.

1. $X \rightarrow X^t$ if $m = n$.
2. $X \rightarrow PXQ$ for some fixed but arbitrary permutation matrices $P$
and \( Q \) in \( M_{m,n}(\mathbb{F}) \).

(iii) \( X \rightarrow A \cdot X \) for some fixed but arbitrary matrix \( A \) in \( M_{m,n}(\mathbb{F}) \) with no zero entries.

Characterization of preservers which are not as closely related to this thesis have been appearing regularly over the past 20 years and an excellent summary of nearly all characterizations of linear preservers will soon appear in Linear and Multilinear Algebra, edited by S. Pierce with input by leaders in the field. This article includes a list of over 200 articles written on the subject.

1.2 Sign-nonsingular Matrices and L-matrices

Within the past 2 or 3 years, structural matrix theory has been an active area of research in pure and applied mathematics. In structural matrix theory, one is concerned only with the locations of the zero and nonzero entries (and perhaps the sign of the nonzero entries), and not with the magnitude of the entries. At the core of this research is the area of sign-nonsingular matrices and L-matrices (see [36]-[38], [47], [49], [50], [67], [68], and [71]).

An L-matrix is an \( m \times n \) \((m \leq n)\) real matrix \( A \) such that every \( m \times n \) real matrix which has the same \((0, +, -)\)-sign pattern as \( A \) has linearly independent rows. If \( m = n \), then \( A \) is called a sign-nonsingular matrix.

In 1984, Klee, Ladner and Manber [50] presented some properties about L-matrices, including:

\( A \in M_{m,n}(\mathbb{R}) \) is an L-matrix if and only if for every diagonal \((0,1,-1)\)-matrix \( D \neq 0 \) of order \( m \), there exists a nonzero column of \( DA \) each of whose non-zero entries has the same sign.
In 1986, Thomassen [71] proved that if $A \in M_m(\mathbb{R})$ is a sign-
nonsingular matrix, then $A$ contains at most $\frac{m^2 + 3m - 2}{2}$ nonzero entries.

In 1991, Brualdi and Shader [37] and later Brualdi, Chavey and Shader [38] investigated rectangular $L$-matrices. They obtained a decomposition theorem for $L$-matrices. They introduced new classes of subsets of $L$-matrices such as totally $L$-matrices, strong $L$-matrices and barely $L$-matrices. The maximum number of columns for matrices in each of these classes was obtained and those matrices attaining the maximum were characterized.

1.3 Sign-nonsingular Preservers and L-matrix Preservers

Many mathematicians have investigated sign-nonsingular matrices and $L$-matrices, but no one has investigated the linear operators that preserve sign-nonsingular matrices and linear operators that preserve $L$-matrices.

In this thesis, we will characterize the linear operators that preserve classes of subsets of $L$-matrices. We try to show in each case that if a linear operator $T$ preserves a class of $L$-matrices, then $T$ preserves the set of matrices of term rank 1. Then by [17], we can obtain the structure of $T$. In Chapter 3, we characterize the linear operators that preserve sign-nonsingular matrices. In Chapters 4 through 6 we extend those results to more general cases. In Chapter 4, we investigate the linear operators that preserve general $L$-matrices. We require additional restrictions to be placed on the linear operators in order to obtain a classification. In Chapter 5, we characterize the linear operators that preserve super $L$-matrices, which is a new class of $L$-matrices introduced here. In Chapter 6, we investigate the linear
operators that preserve totally $L$-matrices. The results in chapters 3, 4, 5, and 6 are new.
CHAPTER 2
DEFINITIONS AND PRELIMINARIES

In this and all succeeding chapters we restrict our matrices to \( m \times n \)
real matrices. We let \( M_{m \times n} = M_{m \times n}(\mathbb{R}) \) and \( M_n = M_{n \times n}(\mathbb{R}) \).

2.1 General Definitions and Theorems

Definition 2.1.1 The number of nonzero entries in a matrix \( A \) is denoted \( |A| \). The number of elements in a set \( S \) is also denoted \( |S| \).

Definition 2.1.2 An \( n \times m \) matrix with only one nonzero entry, say the \((i,j)\)th entry, \( E_{i,j} \), is called a cell. We also denote the \( i \)th row \( R_i = \{E_{i,1}, E_{i,2}, \ldots, E_{i,n}\} \) and \( j \)th column \( C_j = \{E_{1,j}, E_{2,j}, \ldots, E_{n,j}\} \).

Definition 2.1.3 If \( S \) is a set and \( \mathcal{Y} \) is the space spanned by \( S \), then we denote \( \mathcal{Y} = \langle S \rangle \).

Definition 2.1.4 We denote the Hadamard product of \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \) in \( M_{n \times m} \) by \( A \circ B \). That is, \( A \circ B = (a_{i,j} b_{i,j}) \).

Definition 2.1.5 If \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \) are in \( M_{n \times m} \), we say that \( B \) dominates \( A \) (written \( B \geq A \) or \( A \leq B \)) if \( b_{i,j} = 0 \) implies \( a_{i,j} = 0 \) for all \( i,j \).

Definition 2.1.6 Let \( T : M_{n \times m} \rightarrow M_{n \times m} \) be a linear operator. We say \( T \) preserves the subset \( \mathcal{X} \) of \( M_{n \times m} \) if \( T \) maps each matrix in the set \( \mathcal{X} \) to a matrix in \( \mathcal{X} \). We say \( T \) strongly preserves the subset \( \mathcal{X} \) of \( M_{n \times m} \) if \( T \) preserves both \( \mathcal{X} \) and \( M_{n \times m} \setminus \mathcal{X} \), the complement of \( \mathcal{X} \) in \( M_{n \times m} \). We call such \( T \) an \( \mathcal{X} \) preserver or an \( \mathcal{X} \) strong preserver, respectively.

Definition 2.1.7 A linear operator \( T \) is non-singular if \( T(A) = 0 \) implies
that $A = 0$.

Note that since $T$ is a real operator, if $T$ is nonsingular, $T$ is necessarily invertible.

**Definition 2.1.8** Given a matrix $A$, the column (resp., row) term rank is the minimum number of columns (resp., rows) which contain all the nonzero entries of $A$. The term rank is the minimum number, $t(A)$, of lines (columns or rows) which contain all nonzero entries of $A$.

**Definition 2.1.9** Let $A = (a_{i,j}) \in \mathbb{M}_{m,n}$. We denote by $A(j)$ the submatrix of $A$ obtained by deleting column $j$.

**Definition 2.1.10** Let $R_i = \sum_{j=1}^{n} E_i,j$. That is, $R_i$ is the matrix whose $i^{th}$ row is all ones and whose other entries are 0. We define a linear operator $T_i : \mathbb{M}_{m,n} \longrightarrow \mathbb{M}_{m,n}$ by $T_i(X) = T(X) \circ R_i$. Let

$$V_i = \langle T_i(E_{k,i}) : 1 \leq k \leq m, 1 \leq l \leq n \rangle$$

and let $V_i$ be the space spanned by $V_i$. Further, let

$$W_i = \langle E_{k,i} : T_i(E_{k,i}) \neq 0 \rangle.$$

**Definition 2.1.11** Let $C_j = \sum_{i=1}^{m} E_{i,j}$. That is, $C_j$ is the matrix whose $j^{th}$ column is all ones and other entries are 0. We define a linear operator $T^j : \mathbb{M}_{m,n} \longrightarrow \mathbb{M}_{m,n}$ by $T^j(X) = T(X) \circ C_j$. Let

$$U_j = \langle T^j(E_{k,i}) : 1 \leq k \leq m, 1 \leq l \leq n \rangle$$

and let $U_j$ be the space spanned by $U_j$. Further, let
\[ X_j = \{ E_{k,l} : T^j(E_{k,l}) \neq 0 \}. \]

We end this section with a result which is heavily used throughout this document.

Theorem 2.1.1 [Beasley and Pullman, 17, corollary 3.1.2] Suppose that \( T \) is a nonsingular linear operator on \( M = M_{m,n} \). The operator preserves the set of matrices of term rank 1 if and only if \( T \) is one of or a composition of some of the following operators.

(i) \( X \rightarrow X^t \) if \( m = n \).

(ii) \( X \rightarrow PXQ \) for some fixed but arbitrary permutation matrices \( P \) in \( M_{m,n} \) and \( Q \) in \( M_{m,n} \).

(iii) \( X \rightarrow A \circ X \) for some fixed but arbitrary matrix \( A \) in \( M_{m,n} \) with no zero entries.

2.2 Sign-nonsingular Matrices

Definition 2.2.1 A sign-nonsingular matrix is an \( n \times n \) real matrix \( A \) such that every \( n \times n \) real matrix which has the same (0, +, -)-sign pattern as \( A \) is nonsingular.

In this section we will discuss some properties of the sign-nonsingular matrices.

Lemma 2.2.1 [Brualdi, Chavey and Shader, 38] A matrix \( A \in M_n \) is a sign-nonsingular matrix if and only if there is a nonzero term in its determinant expansion and every nonzero term has the same sign.

Lemma 2.2.2 [Thomassen, 71] If \( A \in M_n \) is a sign-nonsingular matrix then

\[ |A| \leq \frac{n^2 + 3n - 2}{2}. \]
If \( A \in M_n \) and \( B \in M_n \) are sign-nonsingular matrices, then \( AB \) may not be a sign-nonsingular matrix. For example, \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) are sign-nonsingular matrices. But \( AB = O \) is not a sign-nonsingular matrix. However, a direct application of the definition of sign-nonsingular matrices establishes the following lemma.

Lemma 2.2.3 If \( A, M = (m_{i,j}) \in M_n \) with \( m_{i,j} > 0 \) (or \( m_{i,j} < 0 \)) for all \((i,j)\), then \( A \cdot M \) is a sign-nonsingular matrix if and only if \( A \) is a sign-nonsingular matrix.

If \( A, B \in M_n \) are sign-nonsingular matrices, then \( AB \) need not be a sign-nonsingular matrix. For example, if \( A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \), then \( A \) and \( B \) are sign-nonsingular. But \( AB = \begin{pmatrix} -1 & -3 \\ 3 & 1 \end{pmatrix} \) is not sign-nonsingular since the matrix \( C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \), which has the same \((0, +, -)\)-sign pattern as \( AB \), is singular. However, we have the following lemmas.

Lemma 2.2.4 If \( A \in M_n \) is a sign-nonsingular matrix and \( P \in M_n \) is a permutation matrix, then \( AP \) and \( PA \) are also sign-nonsingular matrices.

Proof. Since \( |det( PA )| = |det( AP )| = |det A| \), the lemma follows immediately.  

Lemma 2.2.5 If \( A \in M_n \) is a sign-nonsingular matrix and \( D \in M_n \) is a diagonal matrix with all nonzero entries on its main diagonal, then \( AD \) and \( DA \) are also sign-nonsingular matrices.
Proof. Since \( \det(DA) = \det D \det A \) and \( \det D \neq 0 \), \( \det A = 0 \) if and only if \( \det(AD) = 0 \) and the result follows. □

2.3 L-matrices

The following definitions were introduced by Brualdi, Chavey and Shader [38].

**Definition 2.3.1** An L-matrix is an \( m \times n \) \((m \leq n)\) real matrix \( A \) such that every \( m \times n \) real matrix with the same \((0, +, -)\)-sign pattern as \( A \) has linearly independent rows.

**Definition 2.3.2** An \( m \times n \) \((m \leq n)\) real matrix is said to be a totally L-matrix provided every submatrix of \( A \) of order \( m \) is a sign-nonsingular matrix.

**Definition 2.3.3** A strong L-matrix is an \( m \times n \) L-matrix such that every square submatrix is either a sign-nonsingular matrix or has the property that each term in its determinant expansion equals 0. If \( m = n \), then the \( m \times m \) strong L-matrix is also called a strong sign-nonsingular matrix.

**Definition 2.3.4** An \( m \times n \) \((m \leq n)\) L-matrix is a barely L-matrix provided that each of its \( m \times (n-1) \) submatrices is not an L-matrix.

Now we define a super L-matrix, a subclass of L-matrices that contains the class of totally L-matrices.

**Definition 2.3.5** An \( m \times n \) \((m \leq n)\) real matrix \( A \) is called a super L-matrix provided that for every \( i^{th} \) column of \( A \) there exists a sign-nonsingular submatrix of \( A \) of order \( m \) which contains the \( i^{th} \) column of \( A \).

In this section, we will discuss properties of L-matrices and
relationships between the various subsets of $L$-matrices.

It is easy to obtain the following three lemmas from the definitions of sign-nonsingular matrices, $L$-matrices, totally $L$-matrices, barely $L$-matrices and super $L$-matrices.

Lemma 2.3.1 If $A \in M_n$, then the following are equivalent:

1) $A$ is a sign-nonsingular matrix.
2) $A$ is an $L$-matrix.
3) $A$ is a totally $L$-matrix.
4) $A$ is a barely $L$-matrix.
5) $A$ is a super $L$-matrix.

Lemma 2.3.2 A totally $L$-matrix is a super $L$-matrix; a super $L$-matrix is an $L$-matrix.

Lemma 2.3.3 If $A \in M_{m,n}$ and $A$ contains a sign-nonsingular submatrix of order $m$, then $A$ is an $L$-matrix.

We note that the converse of Lemma 2.3.3 is not true for $m \geq 3$, that is, there are $m \times n$ $L$-matrices which do not contain any sign-nonsingular submatrix of order $m$. Let

$$S_3 = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$ 

Then $S_3$ is an $L$-matrix but $S_3$ does not contain any sign-nonsingular submatrix of order 3 since by Lemma 2.2.2, any sign-nonsingular matrix of order 3 has a zero entry [Brualdi, Chavey and Shader, 38].

Also we can see $S_3$ is a barely $L$-matrix since each of its $3 \times 3$ submatrices is not an $L$-matrix.
Every 1xn matrix with at least one nonzero entry is an L-matrix and is also a strong L-matrix. Hence $A \in M_{1,n}$ is an L-matrix if and only if $A$ is strong L-matrix. For $m \geq 2$, the above result is not true. For example, $S_3$ is a 3x4 L-matrix but it is not a strong L-matrix since every 3x3 submatrix is neither sign-nonsingular nor has the property that each term in its determinant expansion equals 0.

Every 1xn matrix with no zero entry is a super L-matrix, and is also a totally L-matrix. Hence $A \in M_{1,n}(\mathbb{R})$ is a super L-matrix if and only if $A$ is a totally L-matrix.

The following lemma helps to describe the difference between totally L-matrices and super L-matrices when $m \geq 2$.

**Lemma 2.3.4** [Brualdi, Chavey and Shader, 38, Theorem 3.3] If $A \in M_{m,m+k}$ is a totally L-matrix with $m \geq 2$, then $k \leq 2$.

Therefore, if $m \geq 2$ there is a restriction on the order of an $m \times n$ totally L-matrix. For instance, there is no $2 \times 5$ totally L-matrix. But there is no such restriction for super L-matrices. For example, let

$$Z_{m,n} = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$$

Then $Z_{m,n}$ is a super L-matrix for any $m$ and $n$ with $n \geq m$.

**Lemma 2.3.5** If $A$ is an L-matrix, then every row of $A$ has nonzero entries.

**Proof.** This follows from the definition of L-matrix. ⊙

**Lemma 2.3.6** If $A$ is a super L-matrix, then every column of $A$ has nonzero entries.
Proof. If $A$ is a super $L$-matrix and $A$ has $j^{th}$ column with all zero entries, then any submatrix of $A$ of order $m$ which contains the $j^{th}$ column of $A$ is singular. So by definition of super $L$-matrix, $A$ is not a super $L$-matrix, a contradiction.

We note that the above lemma is not true for an arbitrary $L$-matrix. For example, let $A = \begin{pmatrix} I_m & 0 \end{pmatrix}$. Then $A$ contains a sign-nonsingular submatrix of order $m$. By Lemma 2.3.4, $A$ is an $L$-matrix. But $A$ has zero columns.

Lemma 2.3.7 If $A \in \mathcal{M}_{m,n}$ is a super $L$-matrix, then

$$|A| \leq \frac{m^2 + 3m - 2}{2} + m(n-m).$$

Proof. Suppose $A \in \mathcal{M}_{m,n}$ is a super $L$-matrix and

$$|A| > \frac{m^2 + 3m - 2}{2} + m(n-m).$$

Then for any submatrix $B$ of order $m$ of $A$, we have

$$|B| > \frac{m^2 + 3m - 2}{2}$$

since the number of nonzero entries in the $n-m$ columns of $A$ is at most $m(n-m)$. By Lemma 2.2.2, $B$ is not a sign-nonsingular matrix. Hence $A$ is not a super $L$-matrix, a contradiction.

We note that the above lemma holds for totally $L$-matrices since by Lemma 2.3.2, a totally $L$-matrix is a super $L$-matrix. But the lemma does not hold for $L$-matrices. For example, the matrix $S_3$ is a $3 \times 4$ $L$-matrix.
with no zero entries. Hence, \(|S_3| = 12\) and \(\frac{m^2 + 3m - 2}{2} + m(n-m) = 11\).

We have \(|S_3| > 11\).

**Lemma 2.3.8** If \(A \in M_{m,n}\) is a super \(L\)-matrix, and \(P \in M_m\) and \(Q \in M_n\) are permutation matrices, then \(PA\) and \(AQ\) are super \(L\)-matrices.

**Proof.** We first prove that \(PA\) is a super \(L\)-matrix. Since \(A\) is a super \(L\)-matrix, for each \(j\) there is a sign-nonsingular submatrix of \(A\) of order \(m\) which contains the \(j^{th}\) column of \(A\). Denote this submatrix by \(A_j\). Since \(P\) is a permutation matrix, \(PA_j\) is sign-nonsingular and is a submatrix of \(PA\) which contains the \(j^{th}\) column of \(PA\). Hence \(PA\) is a super \(L\)-matrix.

We can similarly prove \(AQ\) is a super \(L\)-matrix.

**Lemma 2.3.9** If \(A \in M_{m,n}\) is a super \(L\)-matrix and \(M = (m_{i,j}) \in M_{m,n}\) with \(m_{i,j} > 0\) for all \((i, j)\) then \(A \circ M \in M_{m,n}\) is also a super \(L\)-matrix.

**Proof.** This follows from the definition of super \(L\)-matrix.

**Lemma 2.3.10** If \(A \in M_{m,n}\) is a super \(L\)-matrix, and \(S_1 \in M_m\) and \(S_2 \in M_n\) are diagonal matrices with all nonzero entries on the main diagonal, then \(S_1A\) and \(AS_2\) are also super \(L\)-matrices.

For \(L\)-matrices and totally \(L\)-matrices we also have three lemmas which are similar to the above three lemmas. The proofs are parallel.

**Lemma 2.3.11** If \(A \in M_{m,n}\) is an \(L\)-matrix (resp., totally \(L\)-matrix), and \(P \in M_m\) and \(Q \in M_n\) are permutation matrices, then \(PA\) and \(AQ\) are \(L\)-matrices (resp., totally \(L\)-matrices).

**Lemma 2.3.12** If \(A \in M_{m,n}\) is an \(L\)-matrix (resp., totally \(L\)-matrix) and \(M = \)
$(m_{i,j}) \in M_{m,n}$ with $m_{i,j} > 0$ (resp., $m_{i,j} < 0$) for all $(i,j)$, then $A \cdot M \in M_{m,n}$ is an $L$-matrix (resp., totally $L$-matrices).

**Lemma 2.3.13** If $A \in M_{m,n}$ is an $L$-matrix (resp., totally $L$-matrix) and $S_1 \in M_m$ and $S_2 \in M_n$ are diagonal matrices with all nonzero entries on the main diagonal, then $S_1 A \in M_{m,n}$ and $A S_2 \in M_{m,n}$ are $L$-matrices (resp., totally $L$-matrices).

**Theorem 2.3.1** [Brualdi, Chavey and Shader, 38, Lemma 3.1] Let $A$ be an $m$ by $m+1$ $L$-matrix. Let $x = (x_1, x_2, \cdots, x_{m+1})$ be the vector in which

$$x_j = \begin{cases} 0 & \text{if } A(j) \text{ is not a sign-nonsingular matrix}, \\ (-1)^{m+1+j} \text{sgn}(\text{det}(A(j))) & \text{if } A(j) \text{ is a sign-nonsingular matrix}. \end{cases}$$

If $A$ has at least one sign-nonsingular matrix of order $m$, then the matrix

$$\begin{pmatrix} A \\ x \end{pmatrix}$$

is a sign-nonsingular matrix of order $m+1$.

If $A$ is an $m$ by $m+1$ totally $L$-matrix, then every $A(j)$ is a sign-nonsingular matrix. Thus the vector $x = (x_1, x_2, \cdots, x_{m+1})$ has all $x_i \neq 0$. Also the matrix

$$\begin{pmatrix} A \\ x \end{pmatrix}$$

is a sign-nonsingular matrix of order $m+1$.

By Lemma 2.2.2, we have the following corollary.

**Corollary 2.3.1** If $A \in M_{m,m+1}$ is a totally $L$-matrix, then

$$|A| \leq \frac{(m+1)^2 + 3(m+1) - 2}{2}.$$
Lemma 2.3.14 [Brualdi, Chavey and Shader, 38] Every $m \times (m+2)$ totally $L$-matrix with $m \geq 2$ has at least two columns with exactly one nonzero entry and has exactly three nonzero entries in each row.

We look at some examples of $m \times (m+2)$ totally $L$-matrices. The matrices

$$\begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & -1 & 0 & 1
\end{pmatrix}$$

and

$$\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 1
\end{pmatrix}$$

are $2 \times 4$ and $3 \times 5$ totally $L$-matrices, respectively.
3.1 Introduction

We recall that a sign-nonsingular matrix is an \( n \times n \) real matrix \( A \) such that every \( n \times n \) real matrix which has the same \((0, +, -)\)-sign pattern as \( A \) is nonsingular. In this chapter we investigate the linear operators that preserve sign-nonsingular matrices. Throughout this chapter, we let \( \mathcal{M} = \mathcal{M}_n(\mathbb{R}) \). In section 3.2, we prove that if \( n \geq 3 \) and \( T : \mathcal{M} \to \mathcal{M} \) is a linear operator that preserves sign-nonsingular matrices, then for any matrix \( X \in \mathcal{M}, T(X) = P_i S_i (X \circ M) S_2 P_2 \) or \( T(X) = P_i S_i (X \circ M)^t S_2 P_2 \), where \( P_i \in \mathcal{M} \) \((i = 1, 2)\) is a permutation matrix, \( S_i \in \mathcal{M} \) \((i = 1, 2)\) is a diagonal matrix of ±1's and \( M = (m_{i,j}) \in \mathcal{M} \) with \( m_{i,j} > 0 \). In section 3.3, we investigate linear operators that preserve sign-nonsingular matrices for \( n = 2 \). We prove that if \( T \) is a linear operator that strongly preserves \( 2 \times 2 \) sign-nonsingular matrices, then \( T \) has the same form as when \( n \geq 3 \). Examples show that nonstrong preservers can be singular.

3.2 Sign-nonsingular Preservers \((n \geq 3)\)

In this section, we consider the linear operators preserving \( n \times n \) sign-nonsingular matrices where \( n \geq 3 \).

Let \( T : \mathcal{M} \to \mathcal{M} \) be a linear operator that preserves sign-nonsingular matrices.

In the following lemmas we use \( V_i, U_j, W_i, \) and \( X_j \) from Definitions 2.1.10 and 2.1.11.

**Lemma 3.2.1** \( \dim V_i \) (resp., \( \dim U_j \)) = \( n \) and \( |W_i| \) (resp., \( |X_j| \)) \( \geq n \).
Proof. It is evident that \( \dim V_i \leq n \). If \( \dim V_i < n \), then there exist \( r < n \) cells in \( W_i \) whose images generate \( V_i \). Suppose
\[
\langle E_1, E_2, \cdots, E_r \rangle \subseteq W_i
\]
such that
\[
V_i = \langle \langle T_1(E_1), T_1(E_2), \cdots, T_1(E_r) \rangle \rangle.
\]
Then, since \( r < n \), for some permutation matrices \( P, Q \in M \), \( P \left( \sum_{i=1}^{r} E_i \right) Q \) is an upper triangular matrix with all zeros on its main diagonal. Hence
\[
I_n + P \left( \sum_{i=1}^{r} \alpha E_i \right) Q
\]
is a sign-nonsingular matrix for any \( \alpha \)'s and so is
\[
P^t(I_n + P \left( \sum_{i=1}^{r} \alpha E_i \right) Q)Q^t = P^tQ^t + \left( \sum_{i=1}^{r} \alpha E_i \right).
\]
Therefore \( T(P^tQ^t + \left( \sum_{i=1}^{r} \alpha E_i \right) ) \) must be a sign-nonsingular matrix for any choice of \( \alpha \)'s. But for some choice of \( \alpha \)'s, \( T(P^tQ^t + \left( \sum_{i=1}^{r} \alpha E_i \right) ) \) has a zero \( i^{th} \) row and hence is not sign-nonsingular. This contradiction implies that \( \dim V_i = n \). If \( |W_i| < n \), then \( \dim V_i < n \), a contradiction. The proof is complete. 

Lemma 3.2.2 If \( \langle E_1, E_2, \cdots, E_n \rangle \subseteq W_i \) (resp., \( X_j \)) such that \( V_i = \langle \langle T_i(E_1), T_i(E_2), \cdots, T_i(E_n) \rangle \rangle \) (resp., \( X_j = \langle \langle T_j(E_1), T_j(E_2), \cdots, T_j(E_n) \rangle \rangle \)), then for each \( r, 1 \leq r \leq n \), either there is an \( E_k \in W_i \) (resp., \( X_j \)) whose nonzero entry is in row \( r \), or there is an \( E_k \in W_i \) (resp., \( X_j \)) whose nonzero entry is in column \( r \).
Proof. If not, then there exists \( \{E_1, E_2, \ldots, E_n\} \subseteq W_i \) such that \( V_i = \langle T_i(E_1), T_i(E_2), \ldots, T_i(E_n) \rangle \) and \( \sum_{i=1}^{n} E_i \) has zero row and zero column. By permuting rows and/or columns, we have, without loss of generality, that \( \sum_{i=1}^{n} E_i \) is an upper triangular matrix with all zeros on its main diagonal and \( \langle T_i(E_1), T_i(E_2), \ldots, T_i(E_n) \rangle = V_i \). Hence \( I_n + \sum_{i=1}^{n} \alpha_i E_i \) is a sign-nonsingular matrix for any \( \alpha_i \)'s. Therefore \( T(I_n + \sum_{i=1}^{n} \alpha_i E_i) \) must be a sign-nonsingular matrix for any choice of \( \alpha_i \)'s. But for some choice of \( \alpha_i \)'s, \( T(I_n + \sum_{i=1}^{n} \alpha_i E_i) \) has a zero \( i \)th row and hence is not sign-nonsingular, a contradiction.

Lemma 3.2.3 For any \( V_i \) (resp., \( U_j \)) there exists \( k \) such that either \( R_k \subseteq W_i \) (resp., \( R_k \subseteq X_j \)) or \( E_k \subseteq W_i \) (resp., \( E_k \subseteq X_j \)) and \( \langle T_i(R_k) \rangle = V_i \) (resp., \( \langle T_i(R_k) \rangle = U_j \)) or \( \langle T_i(E_k) \rangle = V_i \) (resp., \( \langle T_i(E_k) \rangle = U_j \)).

Proof. Suppose \( E = \{E_1, E_2, \ldots, E_n\} \subseteq W_i \) and \( \langle T_i(E) \rangle = V_i \). By Lemma 3.2.2, we can assume, without loss of generality, that \( E_k \) has its nonzero entry in column \( k \). Further we can assume, by permuting rows and/or columns, that \( E_1 + \cdots + E_n \) is an upper triangular matrix, with possibly nonzero entries on the main diagonal.

Case 1. \( E_1 + E_2 + \cdots + E_n \) has a zero row.

Since \( E_1 + \cdots + E_n \) is an upper triangular matrix and \( E_1 \) has its nonzero entry in the first column, we have that \( E_1 = E_{1,1} \).

Suppose \( E_{1,k} \notin W_i \). By permuting rows and columns, we can assume that \( E_{1,n} \notin W_i \). Let

\[
D = E_{2,1} + E_{3,2} + \cdots + E_{n,n-1} + \alpha E_{1,n} + \sum_{j=1}^{n} \beta_j E_j
\]
where the $\beta_i$'s are chosen so that $T(D_\alpha) = 0$. Expanding the determinant of $D_\alpha$ on the $n^{th}$ column we have that
\[
\det D_\alpha = \alpha \det D_\alpha[2, \cdots, n|1, \cdots, n-1] + \beta_n \det D_\alpha[1, \cdots, k, \cdots, n|1, \cdots, n-1]
\]
where $E_n = E_{k,n}$.

Note here that $E_{1,n} \notin W$, $D_\alpha$ has exactly two possibly nonzero entries in the $n^{th}$ column, $\alpha$ in the $(1,n)$ position and $\beta_n$ in the $(k,n)$ position.

Now
\[
D_\alpha[2, \cdots, n|1, \cdots, n-1] = \begin{pmatrix}
1 & * \\
0 & \\
& \\
& \\
& \\
0 & \cdots & 0 & 1
\end{pmatrix}
\]
so $\det D_\alpha[2, \cdots, n|1, \cdots, n-1] = 1$. Also
\[
D_\alpha[1, \cdots, k, \cdots, n|1, \cdots, n-1] = \begin{pmatrix}
\beta_1 & * \\
1 & \\
& \\
& \\
& \\
0 & \cdots & 0 & 1 & *
\end{pmatrix} \odot \begin{pmatrix}
1 & * \\
0 & \\
& \\
& \\
& \\
0 & \cdots & 0 & 1 & *
\end{pmatrix}
\]
\[
= D_\alpha[1, \cdots, k-1|1, \cdots, k-1] \odot \begin{pmatrix}
1 & * \\
0 & \\
& \\
& \\
& \\
0 & \cdots & 1
\end{pmatrix}
\]
That is
\[
\det D_\alpha[1, \cdots, k, \cdots, n|1, \cdots, n-1] = \det D_\alpha[1, \cdots, k-1|1, \cdots, k-1].
\]
It follows that
$$\det D_\alpha = \alpha + \beta_n \det D_{\alpha \{1, \ldots, k-1 \mid 1, \ldots, k-1\}}.$$ 

Further, as above

$$\det D_{\alpha \{1, \ldots, k-1 \mid 1, \ldots, k-1\}} = \beta_{k-1} \det D_{\alpha \{1, \ldots, l \mid 1, \ldots, l\}},$$

for some $$l < k - 1$$. Continuing, we have

$$\det D_{\alpha \{1, \ldots, k, \ldots, n \mid 1, \ldots, n-1\}} = \beta_{k-1} \beta_{l-1} \cdots,$$

a single term. That is

$$\det D_\alpha = \alpha + \beta_n \beta_{k-1} \beta_{l-1} \cdots.$$ 

Choosing $$\alpha$$ of the same sign as $$\beta \beta_{k-1} \beta_{l-1} \cdots$$, or $$\alpha = 1$$ if $$\beta_n \beta_{k-1} \beta_{l-1} \cdots = 0$$, we have $$D_\alpha$$ is sign-nonsingular. But $$T_i(D_\alpha) = 0$$ so $$T(D_\alpha)$$ has a zero $$i$$th row, contradicting that $$T$$ preserves sign-nonsingular matrices. Thus $$E_{i, n} \in W_i$$, and in fact, $$E_{i, k} \in W_i$$ for all $$k$$.

Now, if $$T_i(E_{1, n})$$ and $$T_i(E_{k, n})$$ are linearly independent, then

$$(T_i(E_{1, n}), T_i(E_{k, n}), T_i(E_{l, k}), \cdots, T_i(E_{n-1, k})) \setminus T_i(E_{l, k})$$

for some $$l \neq n$$, spans $$V_i$$, but $$\{E_{1, n}', E_{k, n}', E_{l, k}', \cdots, E_{n-1, k}'\} \setminus E_{l, k}$$ all have zero entries in column $$l$$ and in the same row as $$\{E_{1, n}', \cdots, E_{n-1, k}'\}$$.

Thus $$T_i(E_{1, n})$$ and $$T_i(E_{k, n})$$ are dependent. It now follows that

$$(T_i(E_{1, n}), T_i(E_{2, n}), \cdots, T_i(E_{n, n})) = (T_i(E_{1, n}', \gamma T_i(E_{1, n}'), \cdots, \gamma T_i(E_{n, n}'))).$$

Thus $$\langle T_i(\mathbb{R}) \rangle = V_i$$.

Case 2. $$E_1 + E_2 + \cdots + E_n = I_n$$.

Suppose $$E_{k, l} \notin W_i$$ for some pair $$(k, l)$$. By permuting rows and columns we have, without loss of generality, that $$E_{i, n} \notin W_i$$. As above let

$$D_\alpha = E_{2, 1} + E_{3, 2} + \cdots + E_{n, n-1} + \alpha E_{1, n} + \sum_{j=1}^{n} \beta_{j, j'}.$$
where the $\beta_j$'s are chosen such that $T(\mathbf{D}_\alpha) = 0$. Then

$$\det \mathbf{D}_\alpha = \alpha + \beta_1 \beta_2 \cdots \beta_n.$$ 

Let $\alpha$ have the same sign as $\beta_1 \beta_2 \cdots \beta_n$ or be 1 if $\beta_1 \beta_2 \cdots \beta_n = 0$. We then have that $\mathbf{D}_\alpha$ is sign-nonsingular and yet $T(\mathbf{D}_\alpha)$ has zero $i^{th}$ row, a contradiction. Thus $E_{k,l} \in W_i$ for all $(k, l)$.

Now, if $T(E_{1,n})$ and $T(E_{n,n})$ are dependent, then

$$\{T(E_{1,1}), T(E_{1,2}), \ldots, T(E_{n-1,n-1}), T(E_{1,n})\}$$

is a basis for $\mathcal{V}_i$ and $E_{1,1} + \cdots + E_{n-1,n-1} + E_{1,n}$ has zero $n^{th}$ row. If $T(E_{1,n})$ and $T(E_{n,n})$ are independent, then for some $l \neq n$,

$$\{T(E_{1,1}), T(E_{2,2}), \ldots, T(E_{n,n}), T(E_{1,n}) \setminus T(E_{l,l})\}$$

is a basis for $\mathcal{V}_i$ and $E_{1,1} + E_{2,2} + \cdots + E_{n,n} + E_{1,n} - E_{l,l}$ has zero $l^{th}$ column. In either case, Case 1 applies and the theorem follows. ■

**Lemma 3.2.4** If $\langle T(\mathcal{R}) \rangle = \mathcal{V}_i$ (resp., $\langle T(\mathcal{E}) \rangle = \mathcal{V}_i$), then $\langle T(\mathcal{R}) \rangle \neq \mathcal{V}_r$ (resp., $\langle T(\mathcal{E}) \rangle \neq \mathcal{V}_r$) for any $r$ and $l$.

**Proof.** If not, then, without loss of generality, assume $\langle T(\mathcal{R}) \rangle = \mathcal{V}_i$ and $\langle T(\mathcal{E}) \rangle = \mathcal{V}_r$.

If $i \neq r$, then since for any $\alpha_i \neq 0$ and $\beta_i \neq 0$, the matrix

$$\sum_{i=1}^{n} \alpha_i E_{1,i} + \sum_{i=2}^{n} \beta_i E_{i,n} + \sum_{j=2}^{n-1} E_{j,j}$$

is sign-nonsingular, we can choose $\alpha_i \neq 0$ and $\beta_i \neq 0$ such that

$$T\left(\sum_{i=1}^{n} \alpha_i E_{1,i} + \sum_{i=2}^{n} \beta_i E_{i,n} + \sum_{j=2}^{n-1} E_{j,j}\right)$$

has all positive entries on the $i^{th}$ row and all entries on the $r^{th}$ row of
the same sign, either all positive or all negative, a contradiction.

If $i = r$, then $\langle T_i(\mathcal{R}_1) \rangle = \langle T_i(\mathcal{E}_n) \rangle = V_i$. Now let $j \neq i$. By Lemma 3.2.3, there exists $\mathcal{R}_k$ or $\mathcal{E}_k$ such that $\langle T_j(\mathcal{R}_k) \rangle = V_j$ or $\langle T_j(\mathcal{E}_k) \rangle = V_j$. If $\langle T_j(\mathcal{R}_k) \rangle = V_j$ for some $k$, then since $\langle T_i(\mathcal{E}_n) \rangle = V_i$ and $i \neq j$, by the above argument we get a contradiction. If $\langle T_j(\mathcal{E}_k) \rangle = V_j$ for some $k$, then since $\langle T_i(\mathcal{R}_1) \rangle = V_i$ and $i \neq j$, we get a contradiction again.

Lemma 3.2.5 If $\langle T_i(\mathcal{R}_1) \rangle = V_i$ (resp., $\langle T_i(\mathcal{E}_n) \rangle = V_i$), then for any $r \neq i$, $\langle T_r(\mathcal{R}_k) \rangle \neq V_r$ (resp., $\langle T_r(\mathcal{E}_k) \rangle \neq V_r$).

Proof. If not, then, without loss of generality, we may assume that $\langle T_i(\mathcal{R}_1) \rangle = V_i$ and $\langle T_j(\mathcal{R}_2) \rangle = V_j$. By Lemma 3.2.3 and 3.2.4, we have that for each $V_i$ ($i = 3, 4, \cdots, n$) there exists $\mathcal{R}_k$ such that $\langle T_i(\mathcal{R}_k) \rangle = V_i$.

Without loss of generality, we assume that $\langle T_3(\mathcal{R}_2) \rangle = V_3$.

Since $\langle T_1(\mathcal{R}_1) \rangle = V_1$, $\langle T_2(\mathcal{R}_1) \rangle = V_2$, and $\langle T_3(\mathcal{R}_2) \rangle = V_3$, we can choose $\alpha_{1,j} > 0$ (1 ≤ $j$ ≤ $n$), $\alpha_{2,j} > 0$ (1 ≤ $j$ ≤ $n-1$), and $\alpha_{2,n} < 0$, such that

$$|T_q\left(\sum_{i=1}^{2} \sum_{j=1}^{n} \alpha_{i,j} E_{i,j} - \sum_{k=3}^{n} E_{k,n-k+2}\right)| = n, \quad q = 1, 2, 3.$$

Then

$$T\left(\sum_{i=1}^{2} \sum_{j=1}^{n} \alpha_{i,j} E_{i,j} - \sum_{k=3}^{n} E_{k,n-k+2}\right)$$

has all nonzero entries in the first three rows. But

$$\sum_{i=1}^{2} \sum_{j=1}^{n} \alpha_{i,j} E_{i,j} - \sum_{k=3}^{n} E_{k,n-k+2}$$

is sign nonsingular, and therefore

$$T\left(\sum_{i=1}^{2} \sum_{j=1}^{n} \alpha_{i,j} E_{i,j} - \sum_{k=3}^{n} E_{k,n-k+2}\right)$$
must be sign-nonsingular. But no sign-nonsingular matrix can have three rows with all nonzero entries, a contradiction.

Lemma 3.2.6 If \( \langle T(R_i) \rangle = V_i \) (resp., \( \langle T(G_i) \rangle = V_i \)) then for any \( l \neq k, \langle T(R_l) \rangle \neq V_i \) (resp., \( \langle T(G_l) \rangle \neq V_i \)).

Proof. If not, then without loss of generality, we may assume that 
\( \langle T(R_1) \rangle = \langle T(R_2) \rangle = V_1 \) and \( \langle T(R_{i+1}) \rangle = V_i \) \((2 \leq i \leq n-1)\). Thus by Lemma 3.2.3, there exists \( R_k \) or \( G_k \) such that \( \langle T(R_k) \rangle = V_n \) or \( \langle T(G_k) \rangle = V_n \). This contradicts Lemma 3.2.4 and 3.2.5.

Lemma 3.2.7 \( |W_i| \) (resp., \( |X_j| \)) = \( n \) and there exist permutations \( \sigma \) and \( \tau \) such that \( W_i = R_{\sigma(i)} \) and \( X_j = G_{\tau(j)} \) (or transpose).

Proof. If \( |W_i| \neq n \), then by an argument similar to that in Lemma 3.2.3, we have that
\[
\langle T(R_i) \rangle = \langle T(E_{i,1}), T(E_{i,2}), T(E_{i,3}), \ldots, T(E_{i,n}) \rangle = V_i
\]
where \( l = 1, 2, \ldots, k \) with \( k \geq 2 \). This contradicts Lemma 3.2.6. Hence \( |W_i| = n \). Then by Lemma 3.2.3, \( W_i = R_k \) or \( G_k \) for some \( k \). Also by Lemma 3.2.4, if \( i \neq j \), then \( W_i \neq W_j \), and by Lemma 2.3.5, if \( W_i = R_k \) (resp. \( G_k \)) for some \( k \), then for any \( i \), \( W_i \neq G_i \) (resp. \( R_i \)). Thus there exist permutations \( \sigma \) and \( \tau \) such that \( W_i = R_{\sigma(i)} \) and \( X_j = G_{\tau(j)} \) (or transpose).

From the above lemmas, we have the following theorem immediately, since any matrix of term rank 1 is an element of \( R_i \) for some \( i \) or some \( G_j \) for some \( j \).

Theorem 3.2.1 If \( n \geq 3 \) and \( T: M \rightarrow M \) is a linear operator that
preserves sign-nonsingular matrices, then $T$ preserves the set of matrices of term rank 1.

**Theorem 3.2.2** If $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then $T$ is one-to-one on the set of cells.

**Proof.** Since $T$ preserves sign-nonsingular matrices, by Theorem 3.2.1, we have that $T$ preserves the set of matrices of term rank 1. Also by Lemma 3.2.7, $W_i = R_{\sigma(i)}$ and $X_j = \mathcal{E}_{\tau(j)}$ (or transpose). Without loss of generality, we may assume $W_i = R_i$ ($i = 1, 2, \ldots, n$) and $X_j = \mathcal{E}_j$ ($j = 1, 2, \ldots, n$).

First, since $T$ preserves the set of matrices of term rank 1, we have that $T(E) \neq 0$ for any cell $E$.

Also for any $i$ and $j$, since $E_{i,j} \in W_i \cap X_j$ we have that $T(E_{i,j}) \leq R_i \cap C_j = E_{i,j}$. Also since $T(E_{i,j}) \neq 0$ we must have that $T(E_{i,j}) = \alpha E_{i,j}$ for some $\alpha \neq 0$. Thus $T$ is one-to-one on the set of cells.

From this theorem, we obtain the following corollary.

**Corollary 3.2.1** If $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then $T$ is nonsingular.

From Theorem 3.2.1 and Corollary 3.2.1, we have that if $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then $T$ is non-singular and $T$ preserves the set of matrices of term rank 1. Then by Theorem 2.1.1 [Beasley and Pullman, 17, Corollary 3.1.2] we have the following corollary.

**Corollary 3.2.2** If $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that
preserves sign-nonsingular matrices, then for any \( X \in \mathcal{M} \), \( T(X) = P_1(X \circ M)P_2 \) or \( T(X) = P_1(X \circ M)^tP_2 \), where \( P_1, P_2 \in \mathcal{M} \) are permutation matrices and \( M = (m_{i,j}) \in \mathcal{M} \) with \( m_{i,j} \neq 0 \).

**Theorem 3.2.3** If \( n \geq 3 \) and \( T : \mathcal{M} \rightarrow \mathcal{M} \) is a linear operator that preserves sign-nonsingular matrices, then for any \( X \in \mathcal{M} \), \( T(X) = P_iS_1(X \circ M)S_2P_2 \) or \( T(X) = P_iS_1(X \circ M)^tS_2P_2 \), where \( S_i \in \mathcal{M} \) (\( i = 1, 2 \)) are diagonal matrices of \( \pm 1 \)'s, \( P_i \in \mathcal{M} \) (\( i = 1, 2 \)) are permutation matrices and \( M = (m_{i,j}) \in \mathcal{M} \) with \( m_{i,j} \neq 0 \).

**Proof.** By Corollary 3.2.2, for any \( X \in \mathcal{M} \) there exist two \( n \times n \) permutation matrices \( P_1, P_2 \) and \( M = (m_{i,j}) \in \mathcal{M} \) with \( m_{i,j} \neq 0 \) such that \( T(X) = P_1(X \circ M)P_2 \) or \( T(X) = P_1(X \circ M)^tP_2 \). First we suppose that \( T(X) = P_1(X \circ M)P_2 \).

Let \( T_1 : \mathcal{M} \rightarrow \mathcal{M} \) be a linear operator defined by

\[
T_1(X) = P_1^tT(X)P_2^t = P_1^tP_1(X \circ M)P_2^tP_2 = X \circ M
\]

for any \( X \in \mathcal{M} \), then clearly \( T_1 \) preserves sign-nonsingular matrices since \( T \) preserves sign-nonsingular matrices and \( P_1^t, P_2^t \) are permutation matrices.

Now let \( T_2 : \mathcal{M} \rightarrow \mathcal{M} \) be a linear operator defined by

\[
T_2(X) = S_1T_1(X)S_2 = S_1(X \circ M)S_2,
\]

where

\[
S_1 = \text{diag} \left( \frac{m_{1,1}}{|m_{1,1}|}, \frac{m_{2,1}}{|m_{2,1}|}, \ldots, \frac{m_{n,1}}{|m_{n,1}|} \right)
\]

and

\[
S_2 = \text{diag} \left( 1, \frac{m_{1,2}}{|m_{1,2}|}, \frac{m_{1,1}}{|m_{1,1}|}, \ldots, \frac{m_{1,n}}{|m_{1,n}|}, \frac{m_{1,1}}{|m_{1,1}|} \right).
\]

Then \( T_2 \) also preserves sign-nonsingular matrices since \( T_1 \) preserves
sign-nonsingular matrices and \( S_1, S_2 \) are diagonal matrices with nonzero entries on the main diagonal. Since \( S_1 \) and \( S_2 \) are diagonal matrices, we have that

\[
T_2(X) = S_1(X^oM)S_2 = X^o(S_1MS_2).
\]

Now let \( N = S_1MS_2 \). Then \( N = (n_{i,j}) \) with \( n_{i,j} \neq 0 \). By construction of \( N \) we also have that \( n_{i,1} > 0 \) (1\( \leq i \leq n \)) and \( n_{1,j} > 0 \) (1\( \leq j \leq n \)). Since \( T_2 \) preserves sign-nonsingular matrices, for any sign-nonsingular matrices \( A \) we should have \( T_2(A) = A^o(S_1MS_2) = A^oN \) is a sign-nonsingular matrix. If \( n_{2,2} < 0 \), we let \( A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes I_{n-2} \), then \( A \) is sign-nonsingular. Hence \( A^oN \) is a sign-nonsingular matrix. But

\[
A^oN = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & -n_{2,2} \end{pmatrix} \otimes \text{diag} (n_{3,3}, \ldots, n_{n,n})
\]

is not sign-nonsingular since matrix

\[
B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \text{diag} (n_{3,3}, \ldots, n_{n,n})
\]

which has the same (0, +, -)-sign as \( A^oN \) is a singular matrix. This contradiction implies that \( n_{2,2} > 0 \). By permuting rows and columns we can prove that all entries in \( N \) are positive. Hence \( N = (n_{i,j}) \) with \( n_{i,j} > 0 \).

Thus

\[
T(X) = P_1(X^oM)P_2 = P_1(X^o(S_1^{-1}NS_2^{-1}))P_2 = P_1S_1^{-1}(X^oN)S_2^{-1}P_2.
\]

If \( T(X) = P_1(X^oM)^tP_2 \) then by the same argument as above we can obtain that \( T(X) = P_1S_1^{-1}(X^oM)^tS_2^{-1}P_2 \). The proof is completed. \( \blacksquare \)
3.3 Sign-nonsingular Preservers (n = 2)

In this section we investigate the linear operators that preserve 2x2 sign-nonsingular matrices. Let \( T : \mathcal{M}_2 \rightarrow \mathcal{M}_2 \) be a linear operator that preserves sign-nonsingular matrices.

First, we note that \( T \) is not necessarily nonsingular. For example, let \( T : \mathcal{M}_2 \rightarrow \mathcal{M}_2 \) be a linear operator defined by \( T \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) \) for any \( A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathcal{M}_2 \). Then clearly \( T \) preserves sign-nonsingular matrices since if \( A \) is a sign-nonsingular matrix, then either \( a \neq 0 \) or \( b \neq 0 \) and \( T(A) \) is sign-nonsingular if and only if either \( a \neq 0 \) or \( b \neq 0 \). But \( T \) is a singular operator since \( T(E_{2,1} + E_{2,2}) = 0 \). This example also shows that there are sign-nonsingular preservers which do not strongly preserve sign-nonsingular matrices.

For another example, let \( T : \mathcal{M}_2 \rightarrow \mathcal{M}_2 \) be a linear operator defined by \( T \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a+b & -a+b \\ a-b & a+b \end{array} \right) \) for any \( A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathcal{M}_2 \). If \( A \) is a sign-nonsingular matrix then either \( a \neq 0 \) or \( b \neq 0 \). Now, \( \det \left( \begin{array}{cc} a+b & -a+b \\ a-b & a+b \end{array} \right) = (a + b)^2 + (a - b)^2 \) and is zero if and only if both \( a \) and \( b \) are zero. Hence \( T \) preserves sign-nonsingular matrices and \( T \) is a singular operator.

Let \( T : \mathcal{M}_2 \rightarrow \mathcal{M}_2 \) be a linear operator that preserves sign-nonsingular matrices.

We assume

\[
T(E_{1,1}) = \left( \begin{array}{cc} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{array} \right),
\]
\[ T(E_{1,2}) = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}, \]

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix} \]

and

\[ T(E_{2,2}) = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix}. \]

Where the \( \alpha_i \)'s, \( \beta_i \)'s, \( \gamma_i \)'s and \( \eta_i \)'s are real numbers.

We say that two sign patterns are equivalent if by permuting the rows and/or columns, and by multiplying by diagonal matrices of \( \pm 1 \)'s, one pattern can be transformed into the other.

We have the following three lemmas.

Lemma 3.3.1 \(|T(E_{i,j})| \neq 3\) for any \( i \) and \( j \).

Proof. Suppose that there are \( i \) and \( j \) such \(|T(E_{i,j})| = 3\). Without loss of generality, we assume \(|T(E_{1,1})| = 3\). Also without loss of generality, we may assume that \( T(E_{1,1}) = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \) with \( \alpha_1 > 0, \alpha_2 = 0, \alpha_3 > 0, \) and \( \alpha_4 > 0 \) because whenever \( T(E_{1,1}) \) has 3 nonzero entries we can permute rows and/or columns and multiply by scalar matrices to get that pattern. So we only need to show that this case never happens.

In this case

\[ T(aE_{1,1} + dE_{2,2}) = \begin{pmatrix} a \alpha_1 + d \eta_1 & d \eta_2 \\ a \alpha_3 + d \eta_3 & a \alpha_4 + d \eta_4 \end{pmatrix} \]

with any \( a \neq 0 \) and \( d \neq 0 \) is a sign-nonsingular matrix. If \( \eta_2 > 0 \), then we let \( d = 1 \) and choose \( a > 0 \) such that all entries of matrix
$T(aE_{1,1} + dE_{2,2})$ are positive. Then $T(aE_{1,1} + dE_{2,2})$ cannot be a sign-nonsingular matrix. If $\eta_2 < 0$, then we let $d = 1$ and choose $a < 0$ such that all entries of matrix $T(aE_{1,1} + dE_{2,2})$ are negative. Then $T(aE_{1,1} + dE_{2,2})$ cannot be a sign-nonsingular matrix. Hence we must have $\eta_2 = 0$. Furthermore, for $a \neq 0$, $b \neq 0$ and $d \neq 0$,

$$T(aE_{1,1} + bE_{1,2} + dE_{2,2}) = \begin{pmatrix}
\alpha_1 + \beta_1 + d\eta_1 & \beta_2 \\
\alpha_3 + \beta_3 + d\eta_3 & \alpha_4 + \beta_4 + d\eta_4
\end{pmatrix}$$

is a sign-nonsingular matrix. By the same argument as above we must have that $\beta_2 = 0$. Similarly for $a \neq 0$, $c \neq 0$ and $d \neq 0$,

$$T(aE_{1,1} + cE_{2,1} + dE_{2,2}) = \begin{pmatrix}
\alpha_1 + c\gamma_1 + d\eta_1 & c\gamma_2 \\
\alpha_3 + c\gamma_3 + d\eta_3 & \alpha_4 + c\gamma_4 + d\eta_4
\end{pmatrix}$$

is a sign-nonsingular matrix. By the same argument as above we have that $\gamma_2 = 0$. Hence for any $a$, $b$, $c$ and $d$, we have

$$T(aE_{1,1} + bE_{1,2} + cE_{2,1} + dE_{2,2}) = \begin{pmatrix}
\alpha_1 + \beta_1 + c\gamma_1 + d\eta_1 & 0 \\
\alpha_3 + \beta_3 + c\gamma_3 + d\eta_3 & \alpha_4 + \beta_4 + c\gamma_4 + d\eta_4
\end{pmatrix}.$$ 

Since for any $b \neq 0$ and $c \neq 0$ $T(bE_{1,2} + cE_{2,1})$ is a sign-nonsingular matrix we have that $b\beta_1 + c\gamma_1 \neq 0$ for any $b \neq 0$ and $c \neq 0$. Hence we have that $\beta_1 + \gamma_1 \neq 0$. Thus for any $a \neq 0$,

$$T(aE_{1,1} + E_{1,2} + E_{2,1}) = \begin{pmatrix}
\alpha_1 + \beta_1 + \gamma_1 & 0 \\
\alpha_3 + \beta_3 + \gamma_3 & \alpha_4 + \beta_4 + \gamma_4
\end{pmatrix}$$

is a sign-nonsingular matrix. But since $\beta_1 + \gamma_1 \neq 0$, we can choose $a \neq 0$ such that $a\alpha_1 + \beta_1 + \gamma_1 = 0$. Then $T(aE_{1,1} + E_{1,2} + E_{2,1})$ has a row of zeros, so it cannot be a sign-nonsingular matrix. This contradiction completes the proof.

Therefore, $|T(E_{i,j})| \neq 3$. ■
Lemma 3.3.2  If $|T(E_{i,j})| = 4$, then the sign pattern of $T(E_{i,j})$ is equivalent to \begin{pmatrix} + & + \\ - & + \end{pmatrix}.

Proof. The only 2 nonequivalent patterns with no zeros are \begin{pmatrix} + & + \\ + & + \end{pmatrix} and \begin{pmatrix} - & + \\ + & + \end{pmatrix}. If $T(E_{i,j})$ has sign pattern equivalent to \begin{pmatrix} + & + \\ + & + \end{pmatrix}, say $T(E_{1,1})$ has sign pattern equivalent to \begin{pmatrix} + & + \\ + & + \end{pmatrix}, we can choose a large so that $T(aE_{1,1} + E_{2,2})$ has sign pattern \begin{pmatrix} + & + \\ + & + \end{pmatrix}. A contradiction arises since $aE_{1,1} + E_{2,2}$ is sign-nonsingular. ■

Lemma 3.3.3  If $|T(E_{i,j})| = 2$, then the sign pattern of $T(E_{i,j})$ is equivalent to \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}.

Proof. Since only 2 nonequivalent sign patterns with 2 zeros are \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix} and \begin{pmatrix} + & 0 \\ + & 0 \end{pmatrix}, without loss of generality, we assume $|T(E_{1,1})| = 2$ and $\alpha_1 > 0$, $\alpha_2 = 0$, $\alpha_3 > 0$ and $\alpha_4 = 0$.

For $a \neq 0$ and $d \neq 0$, we consider the sign-nonsingular matrix

$$T(aE_{1,1} + dE_{2,2}) = \begin{pmatrix} a\alpha_1 + d\eta_1 & d\eta_2 \\ a\alpha_3 + d\eta_3 & d\eta_4 \end{pmatrix}.$$ 

We have either $\eta_2 \neq 0$ or $\eta_4 \neq 0$.

Case 1. $\eta_2 > 0$ and $\eta_4 > 0$.

In this case we let $d = 1$ and choose $a > 0$. Thus $T(aE_{1,1} + E_{2,2})$ has
the sign pattern \[ (+ + ) \]. This contradicts that \( T(aE_{1,1} + E_{2,2}) \) is a
sign-nonsingular matrix, so this case never happens.

Case 2. \( \eta_2 < 0 \) and \( \eta_4 < 0 \).

In this case we let \( d = -1 \) and choose \( a > 0 \). By the same argument as
Case 1, we get a contradiction, so this case never happens.

Case 3. \( \eta_2 > 0 \) and \( \eta_4 < 0 \).

If \( \eta_1 \neq 0 \), then there exists \( a \neq 0 \) such that \( a\alpha_1 + \eta_1 = 0 \). If
\( a\alpha_3 + \eta_3 = 0 \), \( T(aE_{1,1} + E_{2,2}) \) has a zero column, a contradiction since
\( aE_{1,1} + E_{2,2} \) is sign-nonsingular. Thus \( a\alpha_3 + \eta_3 \neq 0 \). For \( \varepsilon \) small,
\( (a + \varepsilon)\alpha_1 + \eta_1 \) and \( (a - \varepsilon)\alpha_1 + \eta_1 \) have opposite sign and \( (a + \varepsilon)\alpha_3 + \eta_3 \)
and \( (a - \varepsilon)\alpha_3 + \eta_3 \) have the same sign. Thus either \( T[(a + \varepsilon)E_{1,1} + E_{2,2}] \)
or \( T[(a - \varepsilon)E_{1,1} + E_{2,2}] \) is not sign-nonsingular, a contradiction since
\( (a \pm \varepsilon)E_{1,1} + E_{2,2} \) is sign-nonsingular. Thus \( \eta_1 = 0 \). Similarly \( \eta_3 = 0 \).

Now we consider the sign-nonsingular matrix

\[
T(aE_{1,1} + bE_{1,2} + dE_{2,2}) = \begin{pmatrix}
\frac{a\alpha_1 + b\beta_1}{a\alpha_3 + b\beta_3} & \frac{b\beta_2 + d\eta_2}{b\beta_4 + d\eta_4}
\end{pmatrix}
\]

where \( a \neq 0 \), \( b \neq 0 \), and \( d \neq 0 \). We must have \( \beta_2 = \beta_4 = 0 \). If not, then we
consider the following cases. If \( \beta_2 = 0 \) and \( \beta_4 \neq 0 \), we first choose \( b \)
such that \( b\beta_4 + \eta_4 > 0 \), then choose \( a > 0 \) such that \( a\alpha_1 + b\beta_1 > 0 \) and
\( a\alpha_3 + b\beta_3 > 0 \). Thus \( T(aE_{1,1} + bE_{1,2} + E_{2,2}) \) has the sign pattern \[ (+ + +) \],
a contradiction. If \( \beta_2 \neq 0 \) and \( \beta_4 = 0 \), we first choose \( b \) such \( b\beta_2 + \eta_2 < 0 \),
then choose \( a > 0 \) such that \( a\alpha_1 + b\beta_1 > 0 \) and \( a\alpha_3 + b\beta_3 > 0 \). Thus
\( T(aE_{1,1} + bE_{1,2} + E_{2,2}) \) has the sign pattern \[ (+ -) \], a contradiction.
If $\beta_2 \neq 0$ and $\beta_4 \neq 0$, we consider the number $\lambda = \max \left\{ \frac{\eta_2}{\beta_2}, \frac{\eta_4}{\beta_4} \right\}$. We let $b > \lambda$. Then since $b > \frac{\eta_2}{\beta_2}$ and $b > \frac{\eta_4}{\beta_4}$, we have that $b\beta_2 + \eta_2 > 0$ and $b\beta_4 + \eta_4 > 0$. Finally we choose $a > 0$ such that $a\alpha_1 + b\beta_1 > 0$ and $a\alpha_3 + b\beta_3 > 0$. Then $T(aE_{1,1} + bE_{1,2} + E_{2,2})$ has the sign pattern \( \left( \begin{array}{cc} + & + \\ + & + \end{array} \right) \), a contradiction. Therefore, $\beta_2 = \beta_4 = 0$. Similarly we may consider the sign-nonsingular matrix $T(aE_{1,1} + cE_{2,1} + E_{2,2})$. We can get $\gamma_2 = \gamma_4 = 0$. Thus

$$T(aE_{1,1} + bE_{1,2} + cE_{2,1} + dE_{2,2}) = \left( \begin{array}{c} a\alpha_1 + b\beta_1 + c\gamma_1 + d\eta_1 \\ a\alpha_3 + b\beta_3 + c\gamma_3 + d\eta_3 \end{array} \right).$$

If we let $a = 0$, $b = 1$, $c = 1$, and $d = 0$, we have $T(E_{1,1} + E_{2,2})$ is a sign-nonsingular matrix which has a zero column. This contradiction implies that Case 3 never happens.

**Case 4.** $\eta_2 < 0$ and $\eta_4 > 0$.

This case is equivalent to Case 3.

**Case 5.** $\eta_2 = 0$ and $\eta_4 \neq 0$.

In this case, for any $a \neq 0$

$$T(aE_{1,1} + E_{2,2}) = \left( \begin{array}{cc} a\alpha_1 + \eta_1 & 0 \\ a\alpha_3 + \eta_3 & \eta_4 \end{array} \right)$$

is a sign-nonsingular matrix. If $\eta_1 \neq 0$, we can choose $a \neq 0$ such that $a\alpha_1 + \eta_1 = 0$. Thus $T(aE_{1,1} + E_{2,2})$ has a zero row, a contradiction. Hence $\eta_1 = 0$. Now

$$T(aE_{1,1} + bE_{1,2} + dE_{2,2}) = \left( \begin{array}{cc} a\alpha_1 + b\beta_1 & b\beta_2 \\ a\alpha_3 + b\beta_3 + d\eta_3 & b\beta_4 + d\eta_4 \end{array} \right).$$
is a sign-nonsingular matrix for any \( a \neq 0 \) and \( d \neq 0 \). If \( \beta_2 > 0 \), we let \( b = 1 \) and choose \( d \neq 0 \) such that \( \beta_4 + d\eta_4 > 0 \). Then choose \( a > 0 \) such that \( a\alpha_1 + \beta_1 > 0 \) and \( a\alpha_3 + \beta_3 + d\eta_3 > 0 \). Thus \( T(aE_{1,1} + E_{1,2} + dE_{2,2}) \) has the sign pattern \( \left( \begin{array}{cc} + & + \\ + & + \end{array} \right) \), a contradiction. If \( \beta_2 < 0 \), we let \( b = 1 \) and choose \( d \neq 0 \) such that \( \beta_4 + d\eta_4 < 0 \). Then choose \( a > 0 \) such that \( a\alpha_1 + \beta_1 > 0 \) and \( a\alpha_3 + \beta_3 + d\eta_3 > 0 \). Thus \( T(aE_{1,1} + E_{1,2} + dE_{2,2}) \) has the sign pattern \( \left( \begin{array}{cc} + & - \\ + & - \end{array} \right) \), again a contradiction. If \( \beta_2 = 0 \), then we must have \( \beta_1 = 0 \) and \( \beta_4 = 0 \), since if \( \beta_1 \neq 0 \), we can choose \( a \neq 0 \) such that \( a\alpha_1 + \beta_1 = 0 \). Then \( T(aE_{1,1} + E_{1,2} + E_{2,2}) \) has a column of zeros, a contradiction. If \( \beta_4 \neq 0 \), we can choose \( d \neq 0 \) such that \( \beta_4 + d\eta_4 = 0 \). Thus \( T(E_{1,1} + E_{1,2} + dE_{2,2}) \) has a column of zeros, a contradiction. Hence \( \beta_2 = \beta_1 = \beta_4 = 0 \). Similarly considering the matrix \( T(aE_{1,1} + cE_{2,1} + dE_{2,2}) \), we can obtain \( \gamma_2 = \gamma_1 = \gamma_4 = 0 \). Hence

\[
T(aE_{1,1} + bE_{1,2} + cE_{2,1} + dE_{2,2}) = \left( \begin{array}{cc} a\alpha_1 & 0 \\ a\alpha_3 + b\beta_3 + c\gamma_3 + d\eta_3 & d\eta_4 \end{array} \right).
\]

Now let \( a = 0 \), \( b = 1 \), \( c = 1 \), and \( d = 0 \). We have that \( T(E_{1,2} + E_{2,1}) \) has a zero row, a contradiction.

Therefore, Case 5 never happens.

Case 6. \( \eta_2 \neq 0 \) and \( \eta_4 = 0 \).

This case is equivalent to Case 5.

The proof is complete. ⊡

Now we consider linear operators which preserve 2x2 sign-nonsingular matrices and also are one-to-one on the set of cells. We have the following theorems.
Theorem 3.3.1 If \( T : M_2 \rightarrow M_2 \) is a linear operator that preserves sign-nonsingular matrices and \( T \) is also one-to-one on the set of cells, then \( T \) preserves the term rank 1 matrices.

Proof. If not, then without loss of generality, suppose \( T(E_{1,1}) = E_{1,1} \) and \( T(E_{1,2}) = E_{2,2} \). Thus either \( T(E_{2,1}) = \alpha_1 E_{1,2} \) or \( T(E_{2,2}) = \alpha_2 E_{2,1} \) for some nonzero real numbers \( \alpha_1 \) and \( \alpha_2 \), since \( T \) is one-to-one on the set of cells. But \( E_{1,1} + E_{2,2} \) is sign-nonsingular and

\[
T(E_{1,1} + E_{2,2}) = E_{1,1} + \alpha_1 E_{1,2} \quad \text{or} \quad T(E_{1,1} + E_{2,2}) = E_{1,1} + \alpha_2 E_{2,1}
\]

is not sign-nonsingular. This contradiction implies that \( T \) preserves the term rank 1 matrices. \( \blacksquare \)

By Theorem 2.1.1 [Beasley and Pullman, 17, Corollary 3.1.2], we have the following corollary:

Corollary 3.3.1 If \( T : M_2 \rightarrow M_2 \) is a linear operator that preserves sign-nonsingular matrices and \( T \) is also one-to-one on the set of cells, then for any \( X \in M_2 \), \( T(X) = P_i (X \circ M) P_2 \) or \( T(X) = (X \circ M)^t P_2 \), where \( P_i \in M_2 \) \( (i = 1, 2) \) is a permutation matrix, \( M = (m_{i,j}) \in M_2 \) with \( m_{i,j} \neq 0 \).

Corollary 3.3.2 The operator \( T : M_2 \rightarrow M_2 \) is a linear operator that preserves sign-nonsingular matrices and is also one-to-one on the set of cells if and only if for any \( X \in M_2 \), \( T(X) = P_i S_i (X \circ M) S_i^t P_2 \) or \( T(X) = (X \circ M)^t P_2 \), where \( P_i \in M_2 \) \( (i = 1, 2) \) is a permutation matrix, \( S_i \in M_2 \) \( (i = 1, 2) \) is a diagonal matrix of \( \pm 1 \)'s and \( M = (m_{i,j}) \in M_2 \) with \( m_{i,j} > 0 \).

Proof. The proof is similar to the proof of Theorem 3.2.3. \( \blacksquare \)
Theorem 3.3.2 If \( T : \mathcal{M}_2 \to \mathcal{M}_2 \) is a linear operator that preserves sign-nonsingular matrices, then \( T \) strongly preserves sign-nonsingular matrices if and only if \( T \) is one-to-one on the set of cells.

Proof. If \( T : \mathcal{M}_2 \to \mathcal{M}_2 \) is a linear operator that preserves sign-nonsingular matrices and \( T \) is also one-to-one on the set of cells, then by Corollary 3.3.2, \( T \) strongly preserves sign-nonsingular matrices.

Now we show that if \( T : \mathcal{M}_2 \to \mathcal{M}_2 \) is a linear operator that strongly preserves sign-nonsingular matrices, then \( T \) is one-to-one on the set of cells. First we will prove that \( T(E) \neq 0 \) for any cell \( E \). If not, then without loss of generality, we may assume \( T(E) = 0 \). Thus \( T(E_{2,2}) = T(E_{1,1} + E_{2,2}) \) is a sign-nonsingular matrix. Since \( T \) strongly preserves sign-nonsingular matrices, we must have that \( E_{2,2} \) is a sign-nonsingular matrix, a contradiction. Hence \( T(E) \neq 0 \) for any cell \( E \).

We next show that if \( E \) is a cell, then \( T(E) \) is also a cell. If not, then without loss of generality, we may assume \( T(E) \) is not a cell. Then since \( T(E) \neq 0 \) for any cell \( E \), we must have \( |T(E)| \geq 2 \).

If \( |T(E)| = 2 \), then since \( T \) strongly preserves sign-nonsingular matrices, we must have that \( T(E) \) is not a sign-nonsingular matrix. But by Lemma 3.3.3, the sign pattern of \( T(E_{1,1}) \) is equivalent to \( \left( \begin{array}{cc} + & 0 \\ 0 & + \end{array} \right) \). Hence we have that \( T(E_{1,1}) \) is a sign-nonsingular matrix, a contradiction. Therefore, \( |T(E)| \neq 2 \).

If \( |T(E)| = 3 \), then by Lemma 3.3.1 we get a contradiction. Therefore, \( |T(E)| \neq 3 \).

If \( |T(E)| = 4 \), then since \( T \) strongly preserves sign-nonsingular
matrices we must have that $T(E_{1,1})$ is not a sign-nonsingular matrix. But by Lemma 3.3.2, the sign pattern of $T(E_{1,1})$ is equivalent to $\left( \begin{array}{cc} + & + \\ - & + \end{array} \right)$. Hence we have that $T(E_{1,1})$ is a sign-nonsingular matrix, a contradiction. Therefore, $|T(E_{1,1})| \neq 4$.

Thus if $E$ is a cell, then $T(E)$ is also a cell.

Now we prove that $T$ is one-to-one on the set of cells. If not, then without loss of generality, we assume $T(E_{1,1}) = \alpha E_{1,1}$ and $T(E_{1,2}) = \beta E_{1,1}$ with $\alpha \neq 0$ and $\beta \neq 0$. Thus $T(\beta E_{1,1} - \alpha E_{1,2} + E_{2,2}) = \alpha \beta E_{1,1} - \alpha \beta E_{1,1} + T(E_{2,2}) = T(E_{2,2})$ is a sign-nonsingular matrix. This is a contradiction since $|T(E_{2,2})| = 1$. This completes the proof.

Corollary 3.3.3 If $T : M_2 \rightarrow M_2$ is a linear operator that strongly preserves sign-nonsingular matrices, then for any $X \in M_2$, $T(X) = P_1 S_1 (X \circ M) S_2 P_2$ or $T(X) = P_1 S_1 (X \circ M)^t S_2 P_2$, where $P_i \in M_2$ ($i = 1, 2$) is a permutation matrix, $S_i \in M_2$ ($i = 1, 2$) is a diagonal matrix of ±1’s and $M = (m_{i,j}) \in M_2$ with $m_{i,j} > 0$.

Proof. By Theorem 3.3.2, we have that $T$ is one-to-one on the set of cells. Then by Corollary 3.3.2, the result follows.
CHAPTER 4
LINEAR OPERATORS THAT PRESERVE
L-MATRICES

4.1 Introduction

Recall that an \textit{L-matrix} is an \( m \times n \) (\( m \leq n \)) real matrix \( A \) such that every \( m \times n \) real matrix with the same (0, +, -)-sign pattern as \( A \) has linearly independent rows. If \( m = n \), an \( m \times m \) L-matrix is a sign-nonsingular matrix. We have discussed sign-nonsingular preservers in Chapter 3, so throughout this chapter we assume \( m < n \). In section 4.2, we prove that if \( T \) is a linear operator that preserves L-matrices and \( T \) is also one-to-one on the set of cells, then for any \( m \times n \) real matrix \( X \), \( T(X) = P_S (X \cdot M) S P \) where \( P_i \) (\( i = 1, 2 \)) is a permutation matrix, \( S_i \) (\( i = 1, 2 \)) is a diagonal matrix of \( \pm 1 \)'s and \( M = (m_{i,j}) \) with \( m_{i,j} > 0 \). In section 4.3, we prove that \( T \) strongly preserves L-matrices if and only if \( T \) preserves L-matrices and \( T \) is also one-to-one on the set of cells.

4.2 Nonsingular L-matrix Preservers

In this section we will investigate the linear operators preserving \( m \times n \) L-matrices. Let \( T : M_{m,n} \rightarrow M_{m,n} \) be a linear operator that preserves L-matrices.

First we note that \( T \) is not necessarily a nonsingular operator. For example, if \( m = 2 \), then we define a linear operator \( T : M_{2,2n} \rightarrow M_{2,2n} \) by

\[
T\left(\begin{array}{cccc}
1 & b & a & b \\
* & * & * & * \\
a & b & 1 & 2 \\
& b & n & n
\end{array}\right) = \left(\begin{array}{cccc}
1 & -b & a & -b \\
1 & 2 & -b & 2 \\
a & b & 1 & 2 \\
& b & n & n
\end{array}\right)
\]

for any
If \( A \) is an \( L \)-matrix, then at least one of \( a_i \) or \( b_i \) for each \( i \) is nonzero and \( T(A) \) is an \( L \)-matrix if and only if at least one of \( a_i \) or \( b_i \) for each \( i \) is nonzero. Hence \( T \) preserves \( L \)-matrices. But \( T \) is singular since

\[
T \left( \sum_{i=2}^{n} E_{z,i} \right) = 0.
\]

Now we consider the linear operators that preserve \( L \)-matrices and that are also one-to-one on the set of cells. We have the following theorems.

**Theorem 4.2.1** If \( T : M_{m,n} \rightarrow M_{m,n} \) is a linear operator that preserves \( L \)-matrices and \( T \) is also one-to-one on the set of cells, then \( T \) preserves the set of matrices of row term rank 1.

**Proof.** Let \( R_i = \sum_{j=1}^{n} E_{i,j} \); that is, \( R_i \) is the \( mxn \) matrix whose \( i^{\text{th}} \) row is all ones and whose other entries are 0. Let \( G_i = T^{-1}(R_i) \). Since \( T \) is one-to-one on the set of cells, we have that \( |G_i| = n \). Without loss of generality, say \( G_i = \sum_{r=1}^{n} E_r \). We only need to show that the nonzero entries of all \( E_r \)'s lie on the same row.

Suppose not. Then either \( G_i = \sum_{r=1}^{n} E_r \) has a zero column, or \( G_i = \sum_{r=1}^{n} E_r \) has no zero column and every column has exactly one nonzero entry. If

\[
G_i = \sum_{r=1}^{n} E_r
\]

has a zero column, then there exist permutation matrices \( P \in M_m \) and \( Q \in M_n \) such that
Now we choose \( K \in M_{m,n} \) such that \( PKQ = \begin{pmatrix} I_m & 0 \end{pmatrix} \). Then \( K \) is an \( L \)-matrix and \( T(K) \) has zero \( i^{th} \) row, a contradiction. If \( G_i = \sum_{r=1}^{n} E_r \) has no zero column and every column has exactly one nonzero entry, then there exist permutation matrices \( P \in M_m \) and \( Q \in M_n \) such that

\[
PG_iQ = \begin{pmatrix}
0 & \cdots & \ast \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \ast \cdots \ast \\
m & \quad & n \quad -m
\end{pmatrix}.
\]

(4.2.1)

with the \((1,n)\) entry zero since not all nonzero entries are in the first row. We choose a matrix \( A \in M_{m,n} \) such that

\[
PAQ = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0
\end{pmatrix}.
\]

(4.2.3)

Clearly \( PAQ \) is an \( L \)-matrix. Thus we have that \( A \) is an \( L \)-matrix and \( T(A) \) has zero \( i^{th} \) row, a contradiction.

Hence the nonzero entries of all \( E_r \)'s lie on the same row. Therefore, \( T^{-1} \), and hence \( T \), preserves the set of matrices of row term rank 1.

\[ \text{Theorem 4.2.2} \]

If \( T : M_{m,n} \rightarrow M_{m,n} \) is a linear operator that preserves
\(L\)-matrices and \(T\) is also one-to-one on the set of cells, then \(T\) preserves the set of matrices of column term rank \(1\).

**Proof.** Since \(T\) is one-to-one on the set of cells, we have that \(T\) is nonsingular. By Theorem 4.2.1 we have that \(T\) preserves the set of matrices of row term rank \(1\).

Let \(C_j = \sum_{i=1}^{m} E_{i,j}\), that is, \(C_j\) is the \(m \times n\) matrix whose \(j^{th}\) column is all ones and other entries are 0. Since \(T\) is one-to-one on the set of cells, we let \(H_j = T^{-1}(C_i)\) and have \(|H_j| = m\). Without loss of generality, say \(H_j = \sum_{r=1}^{m} E_r\). We only need to prove that the nonzero entries of all \(E_r\)'s lie on the same column, for then we will have shown that \(T^{-1}\), and hence \(T\), preserves the set of matrices of column term rank \(1\).

If the nonzero entries of all \(E_r\)'s do not lie on the same column, then, without loss of generality, assume the nonzero entry of \(E_1\) lies on the first column and the nonzero entry of \(E_2\) lies on the second column. Since \(T\) preserves row term rank \(1\) matrices we must have that nonzero entries of \(E_1\) and \(E_2\) lie on different rows, say, without loss of generality, nonzero entry of \(E_1\) lies on the first row and nonzero entry of \(E_2\) lies on the second row; that is, \(E_1 = E_{1,1}\) and \(E_2 = E_{2,2}\). Now, the matrix

\[
\begin{pmatrix}
I_m & 0 \\
\end{pmatrix}
\]

is an \(L\)-matrix, but \(T \begin{pmatrix} I_m & 0 \end{pmatrix}\) has \(m\) cells whose nonzero entries lie on at most \(m-1\) columns, and hence is not an \(L\)-matrix, a contradiction. \(\blacksquare\)
By Theorem 4.2.1 and Theorem 4.2.2, we have the following corollary.

Corollary 4.2.1 If \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves \( L \)-matrices and \( T \) is also one-to-one on the set of cells, then \( T \) preserves the set of matrices of the term rank 1.

By Corollary 4.2.1, we have that if \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves \( L \)-matrices and \( T \) is also one-to-one on the set of cells, then \( T \) is non-singular and \( T \) preserves the set of matrices of the term rank 1. By Theorem 2.1.1 [Beasley and Pullman, 17, Corollary 3.1.2], we have the following corollary.

Corollary 4.2.2 If \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves \( L \)-matrices and \( T \) is also one-to-one on the set of cells, then for any \( X \in \mathcal{M}_{m,n} \), \( T(X) = P_1 (X \circ M) P_2 \), where \( P_1 \in \mathcal{M}_m \) and \( P_2 \in \mathcal{M}_n \) are permutation matrices and \( M = (m_{i,j}) \in \mathcal{M}_{m,n} \) with \( m_{i,j} \neq 0 \).

Theorem 4.2.3 If \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves \( L \)-matrices and \( T \) is also one-to-one on the set of cells, then for any \( X \in \mathcal{M}_{m,n} \), \( T(X) = P_1 S_1 (X \circ M) S_2 P_2 \), where \( P_1 \in \mathcal{M}_m \) and \( P_2 \in \mathcal{M}_n \) are permutation matrices, \( S_1 \in \mathcal{M}_m \) and \( S_2 \in \mathcal{M}_n \) are diagonal matrices of \( \pm 1 \)'s, and \( M = (m_{i,j}) \in \mathcal{M}_{m,n} \) with \( m_{i,j} > 0 \).

Proof. By Corollary 4.2.2, if \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves \( L \)-matrices and \( T \) is also one-to-one on the set of cells, then for any \( X \in \mathcal{M}_{m,n} \), \( T(X) = P_1 (X \circ M) P_2 \) where \( P_1 \in \mathcal{M}_m \) and \( P_2 \in \mathcal{M}_n \) are permutation matrices and \( M = (m_{i,j}) \) with \( m_{i,j} \neq 0 \). Let \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) be a linear operator defined by
\[ T_1(X) = p^t_1T(X)p^t_2 = p^t_1P(X \circ M)P_2p^t_2 = X \circ M. \]

Clearly \( T_1 \) preserves \( L \)-matrices, since \( T \) preserves \( L \)-matrices and \( P_1^t \), \( p_2^t \) are permutation matrices. Let \( T_2 : M_{m,n} \rightarrow M_{m,n} \) be a linear operator defined by

\[ T_2(X) = S_1T(X)S_2 = S_1(X \circ M)S_2, \]

where

\[ S_1 = \text{diag} \left( \begin{array}{ccc} m_{1,1} & m_{2,1} & \cdots & m_{m,1} \\
|m_{1,1}| & |m_{2,1}| & \cdots & |m_{m,1}| \end{array} \right) \]

and

\[ S_2 = \text{diag} \left( \begin{array}{ccc} m_{1,2} & m_{1,1} & \cdots & m_{1,n} & m_{1,1} \\
|m_{1,2}| & |m_{1,1}| & \cdots & |m_{1,n}| & |m_{1,1}| \end{array} \right). \]

Clearly \( T_2 \) preserves \( L \)-matrices, since \( T_1 \) preserves \( L \)-matrices and \( S_1 \), \( S_2 \) are diagonal matrices with all nonzero entries on the main diagonal. Also since \( S_1 \) and \( S_2 \) are diagonal matrices, we have that

\[ T_2(X) = S_1T(X)S_2 = X \circ (S_1MS_2). \]

Let \( N = S_1MS_2 \). Then \( N = (n_{i,j}) \in M_{m,n} \) with \( n_{i,j} \neq 0, n_{i,i} > 0 \) (1\( \leq i \leq m \)) and \( n_{1,j} > 0 \) (1\( \leq j \leq n \)). Since \( T_2 \) preserves \( L \)-matrices, we must have

\[ T_2(A) = A \circ (S_1MS_2) = A \circ N \]

is an \( L \)-matrix for any \( L \)-matrix \( A \). If \( n_{2,2} < 0 \), we let

\[ A = \left( \begin{array}{cccc} 1 & 1 & 0 & \cdots & 0 \\
1 & -1 & 0 & \cdots & 0 \end{array} \right) \otimes I_{m-2}. \]

Then \( A \) is an \( L \)-matrix, but
\[
A \odot N = \begin{pmatrix}
  n_{1,1} & n_{1,2} & 0 & \cdots & 0 \\
  n_{2,1} & -n_{2,2} & 0 & \cdots & 0 \\
\end{pmatrix} \odot \text{diag} (n_{3,n-m+3}, \ldots, n_{m,n})
\]

is not an \(L\)-matrix, a contradiction. Hence \(n_{2,2} > 0\). By permuting rows and columns we have that all entries in the matrix \(N\) are positive. So \(N = (n_{i,j}) \in M_{m,n}\) with \(n_{i,j} > 0\). Thus

\[
T(X) = P_1 (X \circ M) P_2 = P_1 (X (S_1^{-1} N S_2^{-1})) P_2 = P_1 S_1^{-1} (X \circ N) S_2^{-1} P_2,
\]

which completes the proof. \(\blacksquare\)

### 4.3 Strong Preservers of \(L\)-matrices

In this section we consider linear operators that strongly preserve \(L\)-matrices. We have the following theorem.

**Theorem 4.3.1** If \(T : M_{m,n} \longrightarrow M_{m,n}\) is a linear operator that strongly preserves \(L\)-matrices, then \(T\) is nonsingular.

**Proof.** If not, then there exists an \(m\) by \(n\) matrix \(A = (a_{i,j}) \neq O\) such that \(T(A) = O\). Without loss of generality, we assume \(a_{1,1} \neq 0\). Now we define an \(m\) by \(n\) matrix \(B = (b_{i,j})\) by \(b_{1,j} = 0 \; (1 \leq j \leq n)\), \(b_{i,j} = 0 \; (2 \leq i \leq n, 1 \leq j \leq n)\), \(b_{1,j} = -a_{i,j} \; (1 \leq j \leq m)\) and \(b_{i,1} + a_{i,1} \neq 0 \; (2 \leq i \leq n)\). Thus \(A + B\) is an \(L\)-matrix. So \(T(A + B) = T(A) + T(B) = T(B)\) must be an \(L\)-matrix. Since \(T\) strongly preserves \(L\)-matrices, we must have that \(B\) is an \(L\)-matrix. This is a contradiction because \(B\) has first row zero. Therefore, \(T(A) \neq O\), so \(T\) is nonsingular. \(\blacksquare\)

Now let \(T : M_{m,n} \longrightarrow M_{m,n}\) be a linear operator that strongly preserves \(L\)-matrices.

In the following lemmas, we use \(V_i', W_i', U_j', X_j\) of Definitions 2.1.10 and 2.1.11.
Lemma 4.3.1 \( \dim V_t = n \) and \( |W_t| \geq n \).

Proof. It is evident that \( \dim V_t \leq n \). If \( \dim V_t < n \), then there exist \( r < n \) cells in \( W_t \) whose images generate \( V_t \). Suppose \( \{E_1', E_2', \ldots, E_r\} \subseteq W_t \) such that \( V_t = \langle T(E_1'), T(E_2'), \ldots, T(E_r) \rangle \).

Then, since \( r < n \), for some permutation matrices \( P \in M_m \) and \( Q \in M_n \), we have that

\[
P \left( \sum_{i=1}^{r} E_i \right) Q = \begin{pmatrix} 0 & * & \cdots & * & \cdots & * \\ 0 & 0 & \cdots & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & * \end{pmatrix}.
\]

Hence \( \begin{pmatrix} I_m & O \end{pmatrix} + P \left( \sum_{i=1}^{r} \alpha_i E_i \right) Q \) is an \( L \)-matrix for any \( \alpha_i \)'s and so is

\[
P^t \begin{pmatrix} I_m & O \end{pmatrix} + P \left( \sum_{i=1}^{r} \alpha_i E_i \right) Q^t = P^t \begin{pmatrix} I_m & O \end{pmatrix} Q^t + \left( \sum_{i=1}^{r} \alpha_i E_i \right).
\]

Therefore \( T(P^t \begin{pmatrix} I_m & O \end{pmatrix} Q^t + \left( \sum_{i=1}^{r} \alpha_i E_i \right)) \) must be an \( L \)-matrix for any choice of \( \alpha_i \)'s. But for some choice of \( \alpha_i \)'s,

\[
T(P^t \begin{pmatrix} I_m & O \end{pmatrix} Q^t + \left( \sum_{i=1}^{r} \alpha_i E_i \right))
\]

has a zero \( i \)-th row and hence is not an \( L \)-matrix. This contradiction implies that \( \dim V_t = n \). If \( |W_t| < n \), then \( \dim V_t < n \), a contradiction. The proof is complete. \( \blacksquare \)

Lemma 4.3.2 If \( \{E_1', E_2', \ldots, E_n\} \subseteq W_t \) such that \( V_t = \langle T(E_1'), T(E_2'), \ldots, T(E_n) \rangle \), then for each \( j = 1, 2, \ldots, n \), there is an \( E_k \in W_t \) whose
nonzero entry lies in column $j$.

Proof. If not, then there exists $\{E_1', E_2', \cdots, E_n\} \subseteq W_i$ such that $V_i = \langle T_i(E_1'), T_i(E_2'), \cdots, T_i(E_n) \rangle$ and $\sum_{i=1}^{n}E_i$ has a zero column. Without loss of generality, we assume that $\sum_{i=1}^{n}E_i$ is a matrix of the form (4.3.1) and

$\langle T_i(E_1), T_i(E_2), \cdots, T_i(E_n) \rangle = V_i$. Hence $\left( \begin{array}{cc} I_m & 0 \\ \end{array} \right) + \sum_{i=1}^{n} \alpha_i E_i$ is an $L$-matrix for any $\alpha_i$'s. Therefore $T_i\left( \begin{array}{cc} I_m & O \\ \end{array} \right) + \sum_{i=1}^{n} \alpha_i E_i$ must be an $L$-matrix for any choice of $\alpha_i$'s. But for some choice of $\alpha_i$'s, $T_i\left( \begin{array}{cc} I_m & O \\ \end{array} \right) + \sum_{i=1}^{n} \alpha_i E_i$ has a zero $i^{th}$ row and hence is not an $L$-matrix, a contradiction.

Lemma 4.3.3 For any $V_i$ there exists $k$ such that $\mathcal{R}_k \subseteq W_i$ and $\langle T_i(\mathcal{R}_k) \rangle = V_i$.

Proof. By Lemma 4.3.1, we have that $|W_i| \geq n$. We may assume that $W_i = \{E_1', E_2', \cdots, E_q\}$ with $q \geq n$.

If $n < q < mn$, then there exists at least one cell $E \notin W_i$, say $E_{k,l}$. Since $q > n$, not all $E_r$'s in $W_i$ have nonzero entries which lie on a single row. Since $n > m \geq 2$ we can choose $n$ cells in $W_i$ whose nonzero entries do not all lie on a single row and whose images generate $V_i$, say $\{E_1', E_2', \cdots, E_n\} \subseteq W_i$ and $\langle T_i(E_1'), T_i(E_2'), \cdots, T_i(E_n) \rangle = V_i$. Without loss of generality we may assume that
with \( r \geq 2 \) and \( b_{i,n} = 0 \). There are now 4 possibilities for the location of \( E_{k,l} \notin W_i \).

If \( k,l \leq r \), we can assume, by permuting rows and columns, that \((k,l) = (1,r)\).

We define

\[
A(\beta) = \begin{pmatrix} 0^t & \beta \\ I_{r-1} & 0 \end{pmatrix} \oplus \begin{pmatrix} I_{n-r} & 0 \end{pmatrix}.
\]

If \( k,l > r \), we can assume that \((k,l) = (m,n)\) and in this case we define

\[
A(\beta) = D \oplus \begin{pmatrix} I_{m-r-1} & 0 \end{pmatrix} \oplus [\beta],
\]

where \( D \) is \( r \times r \) matrix and

\[
D_c = \begin{pmatrix} 0 & \cdots & 0 & c \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & \cdots \\ 0 & \cdots & 1 & 0 \end{pmatrix},
\]

where \( c = (-1)^{r-1} \) (so that \( \det (D + I_r) = 2 \)).

If \( k \leq r \) and \( l > r \), we can assume that \((k,l) = (1,n)\) and we define

\[
A(\beta) = \begin{pmatrix} 0^t & \beta \\ I_{m-1} \oplus 0 & 0 \end{pmatrix}.
\]
If \( k > r \) and \( 1 < l \leq r \), we can assume that \((k,l) = (r+1,r)\) and let

\[
A(\beta) = \begin{pmatrix}
0^t & 1 \\
C(\beta) & 0
\end{pmatrix}
\]

where \( C(\beta) = I_{r-1} \oplus [\beta] \oplus \left( I_{m-r-1} \mid O \right) \).

If \( l = 1 \) and \( k > r \), let

\[
A(\beta) = \begin{pmatrix}
0^t & 1 \\
C(\beta) & 0
\end{pmatrix}
\]

where

\[
C(\beta) = \begin{pmatrix}
0 & I_{m-1} \\
\beta & 0^t
\end{pmatrix} \oplus \left( I_{m-r-1} \mid O \right).
\]

In each case we choose \( \alpha_i's \) such that \( T(A(\beta) + \sum_{k=1}^{n} \alpha_i E_{k,k}) = 0 \). Note that since \( E_{k,l} \notin W_l \), the choice of \( \beta \) does not change this fact. We now choose \( \beta \) so that \( A(\beta) + \sum_{k=1}^{n} \alpha_i E_{k,k} \) is an \( L \)-matrix. But for that choice of \( \beta \),

\[
T(A(\beta) + \sum_{k=1}^{n} \alpha_i E_{k,k})\]

must be an \( L \)-matrix, a contradiction since the \( i^{\text{th}} \) row of \( T(A(\beta) + \sum_{k=1}^{n} \alpha_i E_{k,k}) \) is zero. Hence either \( q = n \) or \( q = mn \).

If \( q = n \), then \( W_i = (E_1, E_2 \cdots E_n) \) and \( V_i = \langle T_i(E_1), T_i(E_2), \cdots, T_i(E_n) \rangle \). By Lemma 4.3.2, there is an \( E_k \in W_i \) in each column. Thus if \( \sum_{r=1}^{n} E_r \neq R_k \) for any \( k \), then by the same argument as above we get a contradiction. Hence \( \sum_{r=1}^{n} E_r = R_k \) for some \( k \), that is, \( W_i = R_k \) for some \( k \).

Now suppose \( q = mn \). We will show that there is a \( k \) such that \( R_k \subseteq W_i \) and \( \langle T_i(R_k) \rangle = V_i \). If not, then since \( q > n > 2 \), we can choose \( n \) cells in
$W_i$ whose nonzero entries do not all lie on a single row and whose images generate $V_i$, say $\{E_1, E_2, \ldots, E_n\} \subseteq W_i$ and $\langle T_i(E_1), T_i(E_2), \ldots, T_i(E_n) \rangle = V_i$. Without loss of generality, we may assume $E_1 = E_{1,1}$, $E_2 = E_{2,2}$, \ldots, $E_k = E_{k,k}$ and $E_{k+1} = E_{l,k+1}$ where $l < k + 1$. That is, the matrix

$$
\sum_{i=1}^{n} E_i = \begin{pmatrix}
1 & * & & \\
& 1 & * & \\
& & \ddots & * \\
0 & 0 & & 0
\end{pmatrix}
$$

(4.3.2)

Since $|W_i| = q = mn$, we have that $E_{1,2} \in W_i$. We will show that there exists $\alpha \neq 0$ such that $T_i(E_{1,2}) = \alpha T_i(E_{2,2})$. If not, then $T_i(E_{1,2})$ and $T_i(E_{2,2})$ are linearly independent. Then

$$
\langle T_i(E_1), T_i(E_2), T_i(E_3), \ldots, T_i(E_n) \rangle = V_i.
$$

Therefore, there is an $l$ ($l \neq 2$) such that

$$
\langle T_i(E_1), T_i(E_2), T_i(E_3), \ldots, T_i(E_n) \setminus T_i(E_l) \rangle = V_i.
$$

Thus we have that

$$
\{E_1, E_{1,2}, E_{2,2}, E_3, \ldots, E_n\} \setminus E_l \subseteq W_l
$$

and

$$
\langle T_i(E_1), T_i(E_2), T_i(E_3), \ldots, T_i(E_n) \setminus T_i(E_l) \rangle = V_i.
$$

So there is no cell on the $l$\textsuperscript{th} column; this contradicts Lemma 4.3.2. Hence there exists $\alpha \neq 0$ such that $T_i(E_{1,2}) = \alpha T_i(E_{2,2})$. Thus

$$
\langle T_i(E_1), T_i(E_2), T_i(E_3), \ldots, T_i(E_n) \rangle = V_i.
$$
Similarly, we can show that there is an \( \beta \neq 0 \) such that \( T_i(E_{1,3}) = \beta T_i(E_3) \) and so on. Thus

\[
\langle T_i(\mathcal{R}_i) \rangle = \langle (T_i(E_{1,1}), T_i(E_{1,2}), T_i(E_{1,3}), \ldots, T_i(E_{1,n})) \rangle = V_i
\]

with \( \mathcal{R}_1 \subseteq W_i \). ■

Lemma 4.3.4 If \( T_i(E_{k,l}) \neq 0 \) for some \( k \) and \( l \), then \( \langle T_i(\mathcal{R}_i) \rangle = V_i \).

Proof. If \( T_i(E_{k,l}) \neq 0 \) for some \( k \) and \( l \), then \( E_{k,l} \in W_i \). We can choose \( \{E_1', E_2', \ldots, E_{n-1}'\} \subseteq W_i \) such that

\[
\langle (T_i(E_1), T_i(E_2), \ldots, T_i(E_{n-1}), T_i(E_{k,l})) \rangle = V_i.
\]

By the same argument as in the last part of the proof of Lemma 4.3.3, we have that \( \langle T_i(\mathcal{R}_i) \rangle = V_i \). ■

Lemma 4.3.5 For every \( i \), \( |W_i| = n \).

Proof. We begin by showing that there exist at least \( m-1 \) \( W_i \)'s such that \( |W_i| = n \).

First we show that there exists at least one \( W_i \) such that \( |W_i| = n \). If not, then, by Lemma 4.3.3, \( |W_i| = mn \) for every \( i \). Thus by Lemma 4.3.4, \( \langle T_i(\mathcal{R}_i) \rangle = V_i \) for any \( i \) and \( k \). Hence \( \langle T_i(\mathcal{R}_i) \rangle = V_i \) for \( i = 1, 2, \ldots, n \). So we can choose \( \alpha_{1,h} > 0 \) (\( h = 1, 2, \ldots, n \)) such that

\[
|T_i(E_{1,1} - \alpha_1 E_{1,2} + \alpha_1 E_{1,3} + \cdots + \alpha_1 E_{1,n})| = n
\]

for \( i = 1, 2, \ldots, m \). Thus

\[
|T(\alpha_1 E_{1,1} - \alpha_1 E_{1,2} + \alpha_1 E_{1,3} + \cdots + \alpha_1 E_{1,n})| = mn;
\]
that is, the matrix

\[ T(\alpha_{1,1} E_{1,1} - \alpha_{1,2} E_{1,2} + \alpha_{1,3} E_{1,3} + \cdots + \alpha_{1,n} E_{1,n}) \]  

(4.3.3)

has no zero entries.

If the matrix (4.3.3) is not an L-matrix, we can choose \( \epsilon_k \neq 0 \) (\( k = 2, 3, \cdots m \)) such that the matrix

\[ T(\alpha_{1,1} E_{1,1} - \alpha_{1,2} E_{1,2} + \alpha_{1,3} E_{1,3} + \cdots + \alpha_{1,n} E_{1,n} + \sum_{k=2}^{m} \epsilon_k E_{k,k}) \]

(4.3.4)

has the same sign pattern as the matrix (4.3.3). Thus the matrix (4.3.4) is not an L-matrix. But the matrix

\[ \alpha_{1,1} E_{1,1} - \alpha_{1,2} E_{1,2} + \alpha_{1,3} E_{1,3} + \cdots + \alpha_{1,n} E_{1,n} + \sum_{k=2}^{m} \epsilon_k E_{k,k} \]

is an L-matrix for any \( \alpha_{1,h} > 0 \) (\( h = 1, 2, \cdots n \)) and \( \epsilon_k \neq 0 \) (\( k = 2, 3, \cdots m \)). This contradicts that \( T \) preserves L-matrices.

If the matrix (4.3.3) is an L-matrix, then since \( T \) strongly preserves L-matrices, we have that the matrix

\[ \alpha_{1,1} E_{1,1} - \alpha_{1,2} E_{1,2} + \alpha_{1,3} E_{1,3} + \cdots + \alpha_{1,n} E_{1,n} \]

is an L-matrix, a contradiction since \( m \geq 2 \). Hence there exists at least one \( W_i \) such that \( |W_i| = n \).

Now we show that for \( m > 2 \), there exist at least two \( W_i \)'s such that \( |W_i| = n \). By the above, we know that there exists at least one \( W_i \) such that \( |W_i| = n \). Without loss of generality, we assume that \( W_1 = R_1 \). Thus \( \langle T(R_1) \rangle = V_1 \) and \( E_{i,j} \notin W_1 \) if \( i \neq 1 \).

Then we show that there exists at least one \( W_i \) such that \( |W_i| = n \)
where \( i = 2, 3 \cdots m \).

If not, then, by Lemma 4.3.3, \( |W_i| = mn \) for \( i = 2, 3 \cdots m \). Thus by Lemma 4.3.4, \(<T_1(R_k)\> = V_i\) where \( i = 2, 3 \cdots m \) and \( k = 1, 2 \cdots m \). Hence \(<T_1(R_2)\> = V_i\) where \( i = 2, 3, \cdots n \). Since \( W_1 = R_1 \) we first choose \( \alpha_{1,h} > 0 \) \((h = 1, 2, \cdots n)\) such that

\[
|T(a_{1,1}E_{1,1} - a_{1,2}E_{1,2} + a_{1,3}E_{1,3} + \cdots + a_{1,n}E_{1,n})| = n.
\]

We denote

\[
a_{1,1}E_{1,1} - a_{1,2}E_{1,2} + a_{1,3}E_{1,3} + \cdots + a_{1,n}E_{1,n}\quad \text{by} \quad \alpha R_1.
\]

Then we choose \( \alpha_{2,h} > 0 \) \((h = 1, 2, \cdots n)\) such that

\[
|T_1(\alpha R_1) + T_1(\alpha_{2,1}E_{2,1} + \alpha_{2,2}E_{2,2} - a_{2,3}E_{2,3} + \cdots + a_{2,n}E_{2,n})| = n.
\]

Where \( i = 2, 3, \cdots m \). As above, we denote

\[
a_{2,1}E_{2,1} + a_{2,2}E_{2,2} - a_{2,3}E_{2,3} + \cdots + a_{2,n}E_{2,n}\quad \text{by} \quad \alpha R_2.
\]

Thus

\[
T_1(\alpha R_1 + \alpha R_2) = T_1(\alpha R_1) + T_1(\alpha R_1 + \alpha R_2)
\]

for \( i = 2, 3, \cdots m \). Hence

\[
|T_1(\alpha R_1 + \alpha R_2)| = mn;
\]

that is, the matrix

\[
T_1(\alpha R_1 + \alpha R_2)
\]

has no zero entries.

If the matrix (4.3.5) is not an \( L \)-matrix, since \( m > 2 \), we can choose \( \varepsilon_k \neq 0 \) \((k = 3, 4, \cdots m)\) such that the matrix
\begin{equation}
T(\alpha_{11} R_1 + \alpha_{22} R_2 + \sum_{k=3}^{m} \varepsilon_k E_{k,k})
\end{equation}

has the same sign pattern as the matrix (4.3.5). Thus the matrix (4.3.6) is not an L-matrix. But the matrix

\begin{equation}
\alpha_{11} R_1 + \alpha_{22} R_2 + \sum_{k=3}^{m} \varepsilon_k E_{k,k}
\end{equation}

is an L-matrix for any \( \varepsilon_k \neq 0 \) (\( k = 2, 3, \cdots n \)). This contradicts that \( T \) preserves L-matrices.

If the matrix (4.3.5) is an L-matrix, then since \( T \) strongly preserves L-matrices, we must have that the matrix

\[
\alpha_{11} R_1 + \alpha_{22} R_2 = \alpha_{11} E_{1,1} - \alpha_{12} E_{1,2} + \alpha_{13} E_{1,3} + \cdots + \alpha_{1n} E_{1,n} + \alpha_{21} E_{2,1} + \alpha_{22} E_{2,2} - \alpha_{23} E_{2,3} + \cdots + \alpha_{2n} E_{2,n}
\]

is an L-matrix, a contradiction since \( m > 2 \). Hence there exists at least one \( W_i \) (\( i = 2, 3, \cdots m \)) such that \( |W_i| = n \).

Repeating the above argument, we have that there exist at least \( m-1 \) \( W_i \)’s such that \( |W_i| = n \). Without loss of generality, we may assume that \( |W_i| = n \) and \( \langle T_i (R_i) \rangle = V_i \) for each \( i, i = 1, 2, \cdots m-1 \).

Now we show that \( |W_m| = n \). If not, then \( |W_m| = mn \) and \( \langle T_m (R_i) \rangle = V_m \) for any \( i \). Thus \( \langle T_m (R_i) \rangle = V_m \). Now we consider the matrix
Since \( \langle T_{i} (\mathcal{R}) \rangle = V_{i} \) for each \( i, i = 1, 2, \cdots, m-1 \) and \( \langle T_{m-1} (\mathcal{R}) \rangle = V_{m-1} \), we can choose \( \alpha \}_{i,j} > 0 \) (\( 1 \leq i \leq m-1, 1 \leq j \leq n \)) such that \( T(A) \) has no zero entries.

If \( T(A) \) is an \( L \)-matrix, then since \( T \) strongly preserves \( L \)-matrices, we must have that \( A \) is an \( L \)-matrix, a contradiction. So \( T(A) \) is not an \( L \)-matrix. Since \( T(A) \) has no zero entries, we can choose \( \beta \neq 0 \) such that \( T(A + \beta E_{m,l}) \) has the same sign pattern as \( T(A) \). Thus \( T(A + \beta E_{m,l}) \) is not an \( L \)-matrix. But for any \( \beta \neq 0 \), \( A + \beta E_{m,l} \) is an \( L \)-matrix. This contradicts that \( T \) preserves \( L \)-matrices. Hence \( |W_{m}| = n \). The proof is complete. 

Lemma 4.3.6 If \( \langle T_{i} (\mathcal{R}) \rangle = V_{i} \) then for any \( r \neq i \), \( \langle T_{r} (\mathcal{R}) \rangle \neq V_{r} \).

Proof. By Lemma 4.3.5, since \( |W_{i}| = n \) we have that if \( \langle T_{i} (\mathcal{R}) \rangle = V_{i} \), then \( \langle T_{i} (\mathcal{R}) \rangle \neq V_{i} \) for \( k \neq l \).

Now, we prove that if \( \langle T_{i} (\mathcal{R}) \rangle = V_{i} \), then for any \( r \neq i \), \( \langle T_{r} (\mathcal{R}) \rangle \neq V_{r} \), that is if \( i \neq r \) then \( W_{i} \neq W_{r} \). If not, then without loss of generality, we may assume \( W_{1} = W_{2} = \langle E_{1,k} \rangle, k = 1, 2, \cdots, n \rangle \) and \( W_{i} = \langle E_{i-1,k} \rangle, k = 1, 2, \cdots, n \rangle \) (\( 3 \leq i \leq m \)). Then \( T(E_{m,k}) = 0 \) (\( k = 1, 2, \cdots, n \)). This is a contradiction since \( T \) is nonsingular by Theorem 4.3.1.
From the above lemmas, we have the following theorem immediately.

**Theorem 4.3.2** If $T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n}$ is a linear operator that strongly preserves $L$-matrices, then $T$ preserves the set of matrices of row term rank 1.

**Theorem 4.3.3** If $T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n}$ is a linear operator that strongly preserves $L$-matrices, then $T$ is one-to-one on the set of cells.

**Proof.** First we show $T(E)$ is a cell for any cell $E$. If not, then there exists a cell $E$ such that $|T(E)| \neq 1$. By Theorem 4.3.1, $T$ is nonsingular, so $|T(E)| \neq 0$. By Theorem 4.3.2, $T$ preserves the set of matrices of row term rank 1, so we must have that $2 \leq |T(E)| \leq n$. Without loss of generality, we suppose

$$T(E_{1,1}) \succeq E_{1,1} + E_{1,2}$$

and

$$T(E_{1,j}) \succeq E_{1,j+1} \quad (2 \leq j \leq n-1).$$

Thus

$$T(E_{1,n}) \preceq \sum_{j=1}^{n} E_{1,j} \preceq \sum_{j=1}^{n-1} T(E_{1,j}).$$

Hence for any $\alpha_k \neq 0$ ($k = 1, 2, \cdots, n-1$), $\sum_{j=1}^{n-1} T(\alpha_j E_{1,j})$ has the entries of its first row all nonzero. We can choose $\alpha_k \neq 0$ ($k = 1, 2, \cdots, n-1$) and $e \neq 0$ such that $\sum_{j=1}^{n-1} T(\alpha_j E_{1,j}) - T(e E_{1,n})$ has the same sign pattern as
\[
\sum_{j=1}^{n-1} T(\alpha_{j,1}) . \]

Now we let

\[
A = \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-m} & \cdots & \alpha_{n-2} & \alpha_{n-1} & \mathbf{e} \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-m} & \cdots & \alpha_{n-2} & \alpha_{n-1} & \mathbf{0} \\
0 & \cdots & 1 & 0 & 0 \\
0 & \vdots & \ddots & 0 \\
1
\end{pmatrix}.
\]

Clearly \( A \) is an \( L \)-matrix. So \( T(A) \) is also an \( L \)-matrix. Since

\[
\sum_{j=1}^{n-1} T(\alpha_{j,1}) - T(e_{1,n})
\]

has the same sign pattern as \( \sum_{j=1}^{n-1} T(\alpha_{j,1}) \) and \( T \)
preserves the set of matrices of row term rank 1, we have that
\( T(A - e_{1,n}) \) has the same sign pattern as \( T(A) \). So \( T(A - e_{1,n}) \) is also
an \( L \)-matrix. Since \( T \) strongly preserves \( L \)-matrices, we must have that
\( A - e_{1,n} \) is an \( L \)-matrix.

But

\[
A - e_{1,n} = \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-m} & \cdots & \alpha_{n-2} & \alpha_{n-1} & \mathbf{0} \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-m} & \cdots & \alpha_{n-2} & \alpha_{n-1} & \mathbf{0} \\
0 & \cdots & 1 & 0 & 0 \\
0 & \vdots & \ddots & 0 \\
1
\end{pmatrix}.
\]

has linearly dependent rows, hence is not an \( L \)-matrix, a contradiction. Therefore, if \( E \) is a cell, then \( T(E) \) is also a cell.

Now we show that \( T \) is one-to-one on the set of cells. If not, then
without loss of generality, we assume that \( T(E_{1,1}) = \alpha_{1} E_{1,1} \) and
\( T(E_{1,2}) = \alpha_{2} E_{1,1} \) with \( \alpha_{1} \neq 0 \) and \( \alpha_{2} \neq 0 \). Thus

\[
dim \langle \mathcal{R}_1 \rangle = dim \langle T(\mathcal{R}_1) \rangle \leq n - 1.
\]
This contradicts that \( \dim \langle R \rangle = n \). Therefore, \( T \) is one-to-one on the set of cells. ■

By Theorem 4.3.3, Corollary 4.2.1 and Theorem 4.2.3, we have the following corollaries:

**Corollary 4.3.1** If \( T : M_{m,n} \to M_{m,n} \) is a linear operator that strongly preserves \( L \)-matrices, then \( T \) preserves the set of matrices of term rank 1.

**Corollary 4.3.2** If \( T : M_{m,n} \to M_{m,n} \) is a linear operator that strongly preserves \( L \)-matrices, then for any \( X \in M_{m,n} \), \( T(X) = P S (X \circ M) S P \), where \( P_1 \in M_{m,n} \) and \( P_2 \in M_{m,n} \) are permutation matrices, \( S_1 \in M_{m,n} \) and \( S_2 \in M_{m,n} \) are diagonal matrices of \( \pm 1 \)'s, and \( M = (m_{i,j}) \in M_{2,2} \) with \( m_{i,j} > 0 \).

The characterization of \( L \)-matrix preservers gives us the following corollaries:

**Corollary 4.3.3** A linear operator \( T : M_{m,n} \to M_{m,n} \) strongly preserves \( L \)-matrices if and only if \( T \) preserves \( L \)-matrices and \( T \) is also one-to-one on the set of cells.

**Corollary 4.3.4** If \( T : M_{m,n} \to M_{m,n} \) is a linear operator that strongly preserves \( L \)-matrices, then \( T \) (strongly) preserves super \( L \)-matrices and \( T \) (strongly) preserves totally \( L \)-matrices.
5.1 Introduction

An \( m \times n \) (\( m \leq n \)) real matrix \( A \) is called a super L-matrix if for every \( j \), there exists a sign-nonsingular submatrix of \( A \) of order \( m \) which contains the \( j \)th column of \( A \). If \( m = n \), an \( m \times m \) super L-matrix is a sign-nonsingular matrix. We have discussed sign-nonsingular preservers in Chapter 3, so throughout this chapter we assume \( m < n \). In section 5.2, we will prove that if \( m \geq 3 \) and \( T \) is a linear operator that preserves super \( L \)-matrices then for any \( m \times n \) real matrix \( X \), \( T(X) = P S_1 (X \odot M) S_2 P \) where \( P_i \) \( (i = 1, 2) \) is a permutation matrix, \( S_i \) \( (i = 1, 2) \) is a diagonal matrix of \( \pm 1 \)'s and \( M = (m_{i,j}) \) with \( m_{i,j} > 0 \). In section 5.3, we will prove that if \( T \) is a linear operator that strongly preserves \( 2 \times n \) super \( L \)-matrices then \( T \) has the same form as when \( m \geq 3 \). Examples will show that nonstrong preservers can be singular.

5.2 Super L-matrix Preservers \( (m \geq 3) \)

Definition 5.2.1 A matrix \( D \in M_{m,n} \) is called an \( m \times n \) generalized diagonal matrix provided there exist permutation matrices \( P \in M_m \) and \( Q \in M_n \) such that \( PDQ = \begin{pmatrix} I_m & 0 \\ 0 & O \end{pmatrix} \).

In this section, we consider the linear operators preserving \( m \times n \) super \( L \)-matrices where \( m \geq 3 \).

Let \( T : M_{m,n} \longrightarrow M_{m,n} \) be a linear operator that preserves super \( L \)-matrices. In the following lemmas we will use \( V_i, U_j, W_i, X_j \) and \( T^j \)
defined in Definitions 2.1.10 and 2.1.11.

Lemma 5.2.1 \( \dim \mathcal{U}_j = m \) and \( |X_j| \geq m \).

Proof. It is evident that \( \dim \mathcal{U}_j \leq m \). If \( \dim \mathcal{U}_j < m \), then there exist \( r < m \) cells in \( X_j \), say \( \{E_1, E_2, \ldots, E_r\} \subseteq X_j \) such that \( \mathcal{U}_j = \langle T^j(E_1), T^j(E_2), \ldots, T^j(E_r) \rangle \). Then there exist permutation matrices \( P \in M_m \) and \( Q \in M_n \) such that

\[
P \left( \sum_{i=1}^{r} E_i \right) Q = \begin{bmatrix} 0 & \cdots & \star \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{m \times n-m}.
\]

(5.2.1)

Now we let

\[
K = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{m \times n-m}.
\]

(5.2.2)

Then \( K + P \left( \sum_{i=1}^{r} \alpha_i E_i \right) Q \) is a super \( L \)-matrix for any \( \alpha_i \)'s. By Lemma 2.3.6, we have that

\[
P^t(K + P \left( \sum_{i=1}^{r} \alpha_i E_i \right) Q)Q^t = P^tKQ^t + \sum_{i=1}^{r} \alpha_i E_i
\]

is also a super \( L \)-matrix. Therefore, \( T(P^tKQ^t + \sum_{i=1}^{r} \alpha_i E_i) \) must be a super \( L \)-matrix. But for some choice of \( \alpha_i \)'s, \( T(P^tKQ^t + \sum_{i=1}^{r} \alpha_i E_i) \) has a zero \( j^{th} \)
column. This contradicts Lemma 2.3.6. Hence \( \dim \mathcal{U}_j = m \). If \( |X_j| < m \), then \( \dim \mathcal{U}_j < m \), a contradiction. The proof is complete. \( \blacksquare \)

**Lemma 5.2.2** If \( \{E_1', E_2', \ldots, E_m'\} \subseteq X_j \) such that \( \mathcal{U}_j = \langle T^j(E_1'), T^j(E_2'), \ldots, T^j(E_m') \rangle \), then there is an \( E_k \) in each row.

**Proof.** If not, then \( \sum_{r=1}^{m} E_r \) has a zero row. Also since \( m < n \), \( \sum_{r=1}^{m} E_r \) has a zero column. By permuting rows and/or columns, we have, without loss of generality, that \( \sum_{r=1}^{m} E_r \) is a matrix of the form (5.2.1) and \( \langle T^j(E_1'), T^j(E_2'), \ldots, T^j(E_m') \rangle = \mathcal{U}_j \). Let \( K \) be a matrix of the form (5.2.2). Then

\[
K + \sum_{r=1}^{m} \alpha_r E_r
\]

is a super L-matrix for any \( \alpha_r \)'s. Therefore, \( T(K + \sum_{r=1}^{m} \alpha_r E_r) \) must be a super L-matrix for any \( \alpha_r \)'s. But for some choice of \( \alpha_r \)'s,

\[
T(K + \sum_{r=1}^{m} \alpha_r E_r)
\]

has a zero \( j \)th column and hence is not a super L-matrix, a contradiction. \( \blacksquare \)

**Lemma 5.2.3** \( |X_j| = m \).

**Proof.** Suppose \( X_j = \{E_1', E_2', \ldots, E_q\} \). By Lemma 5.2.1, we have that \( q \geq m \). If \( q > m \), then not all the nonzero entries of \( E_r \)'s in \( X_j \) lie on a column. So we can choose \( m \) cells in \( X_j \) whose nonzero entries are not all on a column, and whose images generate \( \mathcal{U}_j \), say \( \{E_1', E_2', \ldots, E_m'\} \subseteq X_j \) and \( \langle T^j(E_1'), T^j(E_2'), \ldots, T^j(E_m') \rangle = \mathcal{U}_j \). Without loss of generality, we assume that
Now we let the matrix

\[
\sum_{r=1}^{m} E_r = \begin{pmatrix}
* 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
* & 0 & \cdots & 0 \\
0 & \cdots & * & 0 \\
\end{pmatrix}.
\]

(5.2.3)

Then \( T(\sum_{r=1}^{m} \alpha_r E_r + G) \) is a super L-matrix for any \( \alpha_r \)'s. Therefore \( T(\sum_{r=1}^{m} \alpha_r E_r + G) \) must be a super L-matrix for any choice of \( \alpha_r \)'s. But for some choice of \( \alpha_r \)'s, \( T(\sum_{r=1}^{m} \alpha_r E_r + G) \) has a zero \( j^{th} \) column and hence is not a super L-matrix, a contradiction. ■

Lemma 5.2.4 \( X_j = \mathcal{E}_k \) for some \( k \).

Proof. By Lemma 5.2.2, we have that \( |X_j| = m \). Suppose \( X_j = \{E_1, E_2, \ldots, E_m\} \). We only need to show that all the nonzero entries of \( E_r \)'s lie on the same column. If not, then by Lemma 5.2.2, without loss of generality, we may assume that \( \sum_{r=1}^{m} E_r \) is a matrix of the form (5.2.3). Then by the same argument as above this lemma, we get a contradiction. Hence \( X_j = \mathcal{E}_k \) for some \( k \). ■

Lemma 5.2.5 If \( \mathcal{E}_l = X_j \), then for any \( k \neq l \), \( \mathcal{E}_k \neq X_j \), and if \( l \neq j \), then \( X_l \neq X_j \).
Proof. Since $|X_j| = m$ we have that if $e_i = X_j$, then for any $k \neq l$, $e_k \neq X_j$. Now we will show that if $X_j = e_i$, then for any $p (1 \leq p \leq m)$ there exists $\alpha_{k,i} \neq 0$ ($i = 1, 2, \ldots, p$) such that $|T^j(\sum_{k=1}^{p} \alpha_{k,i}E_{k,i})| \geq p$. If not, then without loss of generality, we may assume that $X_j = e_1$ and there exists a number $p (1 \leq p \leq m)$ such that $|T^j(\sum_{k=1}^{p} \alpha_{k,1}E_{k,1})| \leq p - 1$ for any $\alpha_{k,1} \neq 0$ ($k = 1, 2, \ldots, p$). Hence there exists $E_{r,1} \in X_j$ such that $T^j(E_{r,1}) \leq T^j(\sum_{k=1}^{p} E_{k,1})$. We may assume $r = 1$ and choose $\alpha_{2,1}, \alpha_{3,1}, \ldots, \alpha_{m,1}$ such that

$$T = \begin{pmatrix}
1 & 0 \\
\alpha_{2,1} & 1 \\
\alpha_{3,1} & 0 \\
\vdots & \vdots \\
\alpha_{m,1} & 0 & \cdots & 1 & 1 & \cdots & 1
\end{pmatrix}$$

has all zeros in its $j^{th}$ column. This contradicts that $T$ preserves super $L$-matrices.

Now, we prove that if $i \neq j$, then $X_i \neq X_j$. If not, then without loss of generality, we may assume $X_1 = X_2 = e_i$ and $X_i = e_{i-1}$ ($3 \leq i \leq n$). Then $T(E_{k,n}) = 0$ ($k = 1, 2, \ldots, m$). We choose $\alpha_{1,1}, \alpha_{2,1}, \ldots, \alpha_{m,1}$ all positive such that

$$|T^j(\sum_{k=1}^{m} \alpha_{k,1}E_{k,1})| = m$$
and

\[ |T^2(\sum_{k=1}^{m} \alpha_{k,1} E_{k,1})| = m. \]

Then we choose \( \alpha_{1,2}, \alpha_{2,2}, \ldots, \alpha_{m,2} \) all positive such that

\[ |T^3(\sum_{k=1}^{m} \alpha_{k,2} E_{k,2})| = m. \]

Further we choose \( \alpha_{1,n-m+1}, \alpha_{2,n-m+1}, \ldots, \alpha_{m,n-m+1} \) all are positive such that

\[ |T^{n-m+2}(\sum_{k=1}^{m} \alpha_{k,n-m+1} E_{k,n-m+1})| = m. \]

We continue to choose \( \alpha_{1,n-m+2}, \alpha_{2,n-m+2}, \ldots, \alpha_{m,n-m+2} \) all are positive such that

\[ |T^{n-m+3}(-\alpha_{1,n-m+2} E_{1,n-m+2} + \sum_{k=2}^{m} \alpha_{k,n-m+2} E_{k,n-m+2})| = m \]

and choose \( \alpha_{2,n-m+3}, \ldots, \alpha_{m,n-m+3} \) all are positive such that

\[ |T^{n-m+4}(-\alpha_{2,n-m+3} E_{2,n-m+3} + \sum_{k=3}^{m} \alpha_{k,n-m+3} E_{k,n-m+3})| \geq m - 1, \]

and so on. Finally, since \( T(E_{k,n}) = 0 \) \((1 \leq k \leq m)\) we take any positive \( \alpha_{m-1,n} \) and \( \alpha_{m,n} \) such that

\[ |T(-\alpha_{m-1,n} E_{m-1,n} + \alpha_{m,n} E_{m,n})| = 0. \]

Thus we obtain a matrix
Clearly $A$ is a super $L$-matrix. Hence $T(A)$ is a super $L$-matrix. But

$$|T(A)| = m(n-m) + [3m+(m-1) + \cdots + 4 + 3]$$

$$= m(n-m) + \frac{m^2 + 5m - 6}{2}$$

$$> m(n-m) + \frac{m^2 + 3m - 2}{2}$$

since $m \geq 3$. This contradicts that $T(A)$ is a super $L$-matrix by Lemma 2.3.7. The proof is complete. ■

From above lemmas we have the following theorem.

**Theorem 5.2.1** If $m \geq 3$ and $T : M_{m,n} \rightarrow M_{m,n}$ is a linear operator that preserves super $L$-matrices, then $T$ preserves the set of matrices of column term rank $1$.

**Theorem 5.2.2** If $m \geq 3$ and $T : M_{m,n} \rightarrow M_{m,n}$ is a linear operator that preserves super $L$-matrices, then for any cell $E \in M_{m,n}$, $T(E)$ is also a cell.

**Proof.** Since $T$ preserves super $L$-matrices, by Theorem 5.2.1, we have that $T$ preserves the column term rank $1$ matrices. Without loss of generality, we assume $\langle T(\mathcal{E}_j) \rangle = \langle \mathcal{E}_j \rangle$ ($j = 1, 2, \cdots, n$).
First we show that $T(E) \neq O$ for any cell $E$. If not, without loss of
generality, suppose $T(E_{1,1}) = O$. But since $T$ preserves column term rank 1
matrices and $T(E_{1,1}) = O$, while $E_{1,1}$ has column term rank 1. This is a
contradiction. Hence for any cell $E$, $T(E) \neq O$.

Now we show $T(E)$ is a cell for any cell $E$. If not, then without loss
of generality, we suppose $T(E_{m-1,n}) \geq E_{m-1,n} + E_{m-2,n}$. Since $T$ preserves
the set of matrices of column term rank 1, we can choose $\alpha_{m-1,n} > 0$ and
$\alpha_{m,n} > 0$ such that

$$T(-\alpha_{m-1,n}E_{m-1,n} + \alpha_{m,n}E_{m,n}) \geq E_{m-1,n} + E_{m-2,n} + E_{r,n}$$

where $r \neq m - 1$ and $r \neq m - 2$. That is

$$|T(-\alpha_{m-1,n}E_{m-1,n} + \alpha_{m,n}E_{m,n})| \geq 3.$$

For a proper choice of $\alpha_{i,j}$ ($i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n - 1$) we
have a matrix $A$ of the form (5.2.5) which is a super $L$-matrix. But

$$|T(A)| \geq m(n-m) + \frac{m^2 + 3m - 2}{2} + 1,$$

a contradiction to Lemma 2.3.7. The proof is complete. ■

Theorem 5.2.3 If $m \geq 3$ and $T : M_{m,n} \rightarrow M_{m,n}$ is a linear operator that
preserves super $L$-matrices, then $T$ is one-to-one on the set of cells.

Proof. By Theorem 5.2.2, if $E$ is a cell, then $T(E)$ is also a cell. Also
by Theorem 5.2.1, $T$ preserves the set of matrices of column term rank 1.
Without loss of generality, we may assume $\langle T(E) \rangle = \langle E \rangle$ ($j = 1, 2, \cdots, n$).

If $T$ is not one-to-one on the set of cells, then without loss of
generality, we may assume that $T(E_{1,1}) = \alpha_{1} E_{1,1}$ and $T(E_{2,1}) = \alpha_{2} E_{1,1}$ for some $\alpha_{1} \neq 0$ and $\alpha_{2} \neq 0$. Thus $T(E_{1,1})$ and $T(E_{2,1})$ are linearly dependent. Hence

$$\dim \mathcal{U} = \dim \langle \mathcal{E} \rangle = \dim \langle T(\mathcal{E}) \rangle$$

$$= \dim \langle T(E_{1,1}), T(E_{2,1}), \ldots, T(E_{m,1}) \rangle$$

$$= \dim \langle T(E_{1,1}), \ldots, T(E_{m,1}) \rangle$$

$$\leq m-1.$$  

This contradicts $\dim \langle \mathcal{E} \rangle = m$. Therefore, $T$ is one-to-one on the set of cells. ■

Theorem 5.2.4 If $m \geq 3$ and $T : M_{m,n} \to M_{m,n}$ is a linear operator that preserves super L-matrices, then $T$ is nonsingular.

Proof. This follows Theorem 5.2.3. ■

Theorem 5.2.5 If $m \geq 3$ and $T : M_{m,n} \to M_{m,n}$ is a linear operator that preserves super L-matrices, then $T$ preserves the set of matrices of row term rank 1.

Proof. By Theorem 5.2.1 we know that $T$ preserves the column term rank 1 matrices. Also by Theorem 5.2.3, $T$ is one-to-one on the set of cells.

Let $R_{i} = \sum_{j=1}^{n} E_{i,j}$. Let $U = T^{-1}(R_{i})$. Then by Theorem 5.2.3, $|U| = n$.

Without loss of generality, we assume $U = \sum_{r=1}^{n} E_{r}$. We only need to prove that all the nonzero entries of $E_{r}$'s lie on the same row. If not, then since $T$ preserves column term rank 1 matrices we must have that $U$ has a
nonzero entry in each column. Without loss of generality, we may assume that

\[
U = \begin{pmatrix}
1 & 0 & * \\
0 & 1 & * \\
& & \\
& & \\
0 & * & \ldots & *
\end{pmatrix}.
\]

Thus we can choose matrix \( A \) such that \( A + U \) is a super \( L \)-matrix and \(|A| \leq m - 2\). But then \( T(A + U) \) has entries in row \( i \) and at most \( m-2 \) other rows. That is \( T(A + U) \) has a zero row, a contradiction. Therefore \( T \) preserves the set of matrices of row term rank 1.

By Theorem 5.2.1 and Theorem 5.2.5 we have the following corollary.

**Corollary 5.2.1** If \( m \geq 3 \) and \( T : M_{m,n} \rightarrow M_{m,n} \) is a linear operator that preserves super \( L \)-matrices, then \( T \) preserves the set of matrices of term rank 1.

By Corollary 5.2.1 and Theorem 5.2.4, we have that if \( m \geq 3 \) and \( T : M_{m,n} \rightarrow M_{m,n} \) is a linear operator that preserves super \( L \)-matrices, then \( T \) is non-singular and \( T \) preserves the set of matrices of term rank 1.

By Theorem 2.1.1 [Beasley and Pullman, 17, Corollary 3.1.2], we have the following corollary.

**Corollary 5.2.2** If \( m \geq 3 \) and \( T : M_{m,n} \rightarrow M_{m,n} \) is a linear operator that preserves super \( L \)-matrices, then for any \( X \in M_{m,n} \), \( T(X) = P_1 (X \circ M) P_2 \), where \( P_1 \in M_m \) and \( P_2 \in M_n \) are permutation matrices and \( M = (m_{i,j}) \in M_{m,n} \) with \( m_{i,j} \neq 0 \).
Theorem 5.2.6 If \( m \geq 3 \) and \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves super \( L \)-matrices, then for any \( X \in \mathcal{M}_{m,n} \), \( T(X) = P_1 S_1 (X \circ M) S_2 P_2 \), where \( P_1 \in \mathcal{M}_m \) and \( P_2 \in \mathcal{M}_n \) are permutation matrices, \( S_1 \in \mathcal{M}_m \) and \( S_2 \in \mathcal{M}_n \) are diagonal matrices of \( \pm 1 \)'s, and \( M = (m_{i,j}) \in \mathcal{M}_{m,n} \) with \( m_{i,j} > 0 \).

Proof. By Corollary 5.2.2, we have that if \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves super \( L \)-matrices then for any \( X \in \mathcal{M}_{m,n} \), \( T(X) = P_1 (X \circ M) P_2 \) where \( P_1 \in \mathcal{M}_m \) and \( P_2 \in \mathcal{M}_n \) are permutation matrices and \( M = (m_{i,j}) \in \mathcal{M}_{m,n} \) with \( m_{i,j} \neq 0 \). Let \( T_1 : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) be a linear operator defined by

\[
T_1(X) = P_1^t T(X) P_2^t = P_1^t P_2 (X \circ M) P_1 P_2^t = X \circ M.
\]

Clearly \( T_1 \) preserves super \( L \)-matrices, since \( T \) preserves super \( L \)-matrices and \( P_1^t, P_2^t \) are permutation matrices.

Now we let \( T_2 : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) be a linear operator defined by

\[
T_2(X) = S_1 T_1(X) S_2 = S_1 (X \circ M) S_2.
\]

where

\[
S_1 = \text{diag} \left( \begin{array}{c} m_{1,1} \ 1 \\ m_{2,1} \\ \vdots \\ m_{m,1} \end{array} \right) \quad \text{and} \quad S_2 = \text{diag} \left( \begin{array}{c} 1 \\ m_{1,2} \ 1 \\ \vdots \\ m_{1,n} \ m_{1,1} \end{array} \right).
\]

Clearly \( T_2 \) preserves super \( L \)-matrices, since \( T_1 \) preserves super \( L \)-matrices and \( S_1, S_2 \) are diagonal matrices with nonzero diagonal entries.

Since \( S_1 \) and \( S_2 \) are diagonal matrices we have that
\[ T_2(X) = S_2(X \circ M)S_2 = X \circ (S_1 MS_2). \]

Let \( N = S_1 MS_2 \). Then \( N = (n_{i,j}) \in \mathcal{M}_{m,n} \) with \( n_{i,j} \neq 0, n_{i,1} > 0 (1 \leq i \leq m) \) and \( n_{1,j} > 0 (1 \leq j \leq n) \). Since \( T_2 \) preserves super \( L \)-matrices, we must have that

\[ T_2(A) = A \circ (S_1 MS_2) = A \circ N \]

is a super \( L \)-matrix for any super \( L \)-matrix \( A \). If \( n_{2,2} < 0 \), we let

\[
A = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & I_{m-3}
\end{pmatrix}
\]

Then \( A \) is a super \( L \)-matrix. But

\[
A \circ N = \begin{pmatrix}
n_{1,1} & n_{1,2} & 0 & \cdots & 0 \\
n_{2,1} & -n_{2,2} & 0 & \cdots & 0 \\
0 & 0 & n_{3,1} & \cdots & n_{3,n-m+2}
\end{pmatrix} \odot \text{diag} (n_{4,n-m+4}, \ldots, n_{m,n})
\]

is not a super \( L \)-matrix. This contradiction implies that \( n_{2,2} > 0 \). By permuting rows and columns, we have that all entries in the matrix \( N \) are positive. Hence \( N = (n_{i,j}) \in \mathcal{M}_{m,n} \) with \( n_{i,j} > 0 \). Thus

\[
T(X) = P_1(X \circ M)P_2 = P_1(X \circ (S_1^{-1}NS_2^{-1}))P_2 = P_1S_1^{-1}(X \circ N)S_2^{-1}P_2,
\]

which completes the proof.

This characterization of \( m \times n \) super \( L \)-matrix preservers gives us the following corollary:

**Corollary 5.2.3** If \( m \geq 3 \) and \( T : \mathcal{M}_{m,n} \to \mathcal{M}_{m,n} \) is a linear operator that
preserves super $L$-matrices, then $T$ (strongly) preserves $L$-matrices and $T$
(strongly) preserves totally $L$-matrices.

By the above corollary and Corollary 4.3.4, we have the following corollary.

**Corollary 5.2.4** If $m \geq 3$ and $T : M_{m,n} \rightarrow M_{m,n}$ is a linear operator,
then $T$ strongly preserves $L$-matrices if and only if $T$ preserves super $L$-matrices.

5.3 Super $L$-matrix Preservers ($m = 2$)

In this section we will investigate linear operators preserving $2 \times n$
super $L$-matrices.

Let $T : M_{2,n} \rightarrow M_{2,n}$ be a linear operator that preserves super $L$-matrices.

First we note that $T$ is not necessarily a nonsingular operator. For example, let $T : M_{2,n} \rightarrow M_{2,n}$ be a linear operator defined by

$$T \left( \begin{array}{ccc} a & \cdots & b \\ b & \cdots & a \end{array} \right) = \left( \begin{array}{ccc} a & \cdots & a - b \\ b & \cdots & b \end{array} \right)$$

for any

$$A = \left( \begin{array}{ccc} a & \cdots & * \\ b & \cdots & * \end{array} \right) \in M_{2,n}.$$

If $A$ is a super $L$-matrix, then $a$ and $b$ are not both $0$, and $T(A)$ is a super $L$-matrix if and only if either $a \neq 0$ or $b \neq 0$. Hence $T$ preserves super $L$-matrices. But $T$ is singular since $T \left( \sum_{i=2}^{n} (E_{1,i} + E_{2,i}) \right) = 0$.

Also from this example we can see that corollary 5.2.3 is not true
for \( m = 2 \); that is, if \( T : M_{2,n} \longrightarrow M_{2,n} \) is a linear operator that preserves super \( L \)-matrices, then \( T \) does not necessarily preserve \( L \)-matrices and totally \( L \)-matrices. For example, let 

\[
B = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.
\]

Then \( B \) is an \( L \)-matrix but \( T(B) = 0 \) is not an \( L \)-matrix. So \( T \) does not preserve \( L \)-matrices. Let \( C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \). Then \( C \) is an \( 2 \times 3 \) totally \( L \)-matrix but \( T(C) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) is not a totally \( L \)-matrix.

So \( T \) does not preserve totally \( L \)-matrices.

Now we consider linear operators that strongly preserve \( 2 \times n \) super \( L \)-matrices.

**Theorem 5.3.1** If \( T : M_{2,n} \longrightarrow M_{2,n} \) is a linear operator that strongly preserves super \( L \)-matrices, then \( T \) is a nonsingular operator.

**Proof.** If not, then there exists a \( 2 \times n \) matrix \( A = (a_{i,j}) \neq 0 \) such that \( T(A) = 0 \). Without loss of generality, we assume \( a_{1,1} \neq 0 \). Now we define a \( 2 \times n \) matrix \( B = (b_{i,j}) \) by 

\[
b_{1,j} = 0 \quad (1 \leq j \leq n), \quad b_{2,1} = -a_{2,1} \quad \text{and} \quad b_{2,j} + a_{2,j} \neq 0 \quad (2 \leq j \leq n).
\]

Thus \( A + B \) is a super \( L \)-matrix. So

\[
T(A + B) = T(A) + T(B) = T(B)
\]

must be a super \( L \)-matrix. Since \( T \) strongly preserves super \( L \)-matrices, we must have that \( B \) is a super \( L \)-matrix. This is a contradiction since \( B \) has all zeros in its first row. Therefore, \( T(A) \neq 0 \), so \( T \) is nonsingular.

**Theorem 5.3.2** If \( T : M_{2,n} \longrightarrow M_{2,n} \) is a linear operator that strongly preserves super \( L \)-matrices, then \( T \) preserves the set of matrices of column term rank 1.
Proof. Let $C_j = E_{1,j} + E_{2,j}$. Let $U_j = \langle T(E_{i,j}) \circ C_j, 1 \leq i \leq 2, 1 \leq j \leq n \rangle$, and let $\mathcal{U}_j$ be the space spanned by $U_j$. Further let $X_j = \{ E_{k,i} : T(E_{k,i}) \circ C_j \neq 0 \}$.

We first show that $\dim \mathcal{U}_j = 2$. It is evident that $\dim \mathcal{U}_j \leq 2$. If $\dim \mathcal{U}_j < 2$, then there exists a cell in $X_j$ whose image generates $\mathcal{U}_j$. Without loss of generality, suppose $E_{1,1} \in X_j$ and $\mathcal{U}_j = \langle T(E_{1,1}) \rangle$. Now we let

$$K = E_{2,1} + E_{1,2} + E_{1,3} + \cdots + E_{1,n}.$$ 

Then $K + \alpha E_{1,1}$ is a super $L$-matrix for any $\alpha$. Therefore, $T(K + \alpha E_{1,1})$ must be a super $L$-matrix for any $\alpha$. But for some choice of $\alpha$, $T(K + \alpha E_{1,1}) = T(K)$ has zero $j$th column. This is a contradiction by Lemma 2.3.5. Hence $\dim \mathcal{U}_j = 2$.

Second, we show that $|X_j| = 2$. Suppose $X_j = \{ E_1, E_2, \cdots, E_q \}$. If $q < 2$, then $\dim \mathcal{U}_j < 2$, a contradiction from above. If $q > 2$, then not all the nonzero entries of $E_r$'s in $X_j$ lie on a column. So we can choose 2 cells in $X_j$, say $E_1$ and $E_2$, whose nonzero entries lie in different columns and $\langle T(E_1) \circ C_j, T(E_2) \circ C_j \rangle = \mathcal{U}_j$.

Case 1. The nonzero entries of $E_1$ and $E_2$ lie on the same row.

Without loss of generality, we assume $E_1 = E_{1,1}$ and $E_2 = E_{1,2}$. Let

$$A = E_{1,3} + \sum_{k=1 \atop k \neq 3}^n E_{2,k}.$$ 

Then $\alpha_1 E_{1,1} + \alpha_2 E_{1,2} + A$ is a super $L$-matrix for any choice of $\alpha_1$ and $\alpha_2$. Thus $T(\alpha_1 E_{1,1} + \alpha_2 E_{1,2} + A)$ is a super $L$-matrix for any choice of $\alpha_1$ and $\alpha_2$. But for some choice of $\alpha_1$ and $\alpha_2$, $T(\alpha_1 E_{1,1} + \alpha_2 E_{1,2} + A)$ has zero $j$th column, a contradiction.

Case 2. The nonzero entries of $E_1$ and $E_2$ lie on different rows.
Without loss of generality, we assume $E_1 = E_{1,1}$ and $E_2 = E_{2,2}$.

Let $A = E_{1,2} + \sum_{k=1}^{n} E_{2,k}$. Then $\alpha_1 E_{1,1} + \alpha_2 E_{2,2} + A$ is a super $L$-matrix for any choice of $\alpha_1$ and $\alpha_2$. Thus $T(\alpha_1 E_{1,1} + \alpha_2 E_{2,2} + A)$ is a super $L$-matrix for any choice of $\alpha_1$ and $\alpha_2$. But for some choice of $\alpha_1$ and $\alpha_2$, $T(\alpha_1 E_{1,1} + \alpha_2 E_{2,2} + A)$ has zero $j$th column, a contradiction.

Therefore, $|X_j| = q = 2$ so that $X_j = \{E_{1,1}^j, E_{2,2}^j\}$.

Now we prove that $E_1$ and $E_2$ lie on a column. If not, then either $P(E_{1} + E_{2})Q = E_{1,1} + E_{1,2}^j$ or $P(E_{1} + E_{2})Q = E_{1,1} + E_{2,2}^j$ for some permutation matrices $P \in M_2$ and $Q \in M_n$. By the same argument as above, we have a contradiction. Hence for any $j (1 \leq j \leq n)$ there exists $k (1 \leq k \leq n)$ such that

$$U_j = \langle T(E_{1,k}^j) \circ C_j, T(E_{2,k}^j) \circ C_j \rangle = \langle T(E_{k}^j) \circ C_j \rangle.$$

Since $|X_j| = 2$, we have that if $\langle T(E_{k}) \circ C_j \rangle = U_j$, then $\langle T(E_{k}) \circ C_j \rangle \neq U_j$ for any $l \neq k$. Finally, we prove that if $i \neq j$ then $X_i \neq X_j$. If not, then without loss generality, we may assume $X_1 = X_2 = \mathcal{C}_1$ and $X_j = \mathcal{C}_{j-1}$ $(3 \leq j \leq n)$. Then $T(E_{1,n}^j) = T(E_{2,n}^j) = 0$, a contradiction since $T$ is a nonsingular operator. Hence $T$ preserves the set of matrices of column term rank 1.

**Theorem 5.3.3** If $T : M_{2,n} \rightarrow M_{2,n}$ is a linear operator that strongly preserves super $L$-matrices, then $T$ is one-to-one on the set of cells.

**Proof.** First we show $T(E)$ is a cell for any cell $E$. If not, then there exists a cell $E$ such that $|T(E)| \neq 1$. By Theorem 5.3.1, $T$ is nonsingular, so $|T(E)| \neq 0$. Also by theorem 5.3.2, $T$ preserves the set of matrices of
column term rank 1, so $|T(E)| = 2$. Without loss of generality, we may assume $T(E_{1,1}) = \alpha E_{1,1} + \beta E_{2,1}$ for some $\alpha \neq 0$ and $\beta \neq 0$. Since $T(E_{1,1}) = \alpha E_{1,1} + \beta E_{2,1}$ and $T$ preserves the set of matrices of column term rank 1, we have that $T(E_{2,1}) \leq T(E_{1,1})$. Also since for any $\gamma \neq 0$, $T(\gamma E_{2,1} + \sum_{j=1}^{n} E_{1,j})$ is a super $L$-matrix, we can choose $\gamma \neq 0$ such that $T(\gamma E_{2,1} + \sum_{j=1}^{n} E_{1,j})$ has same sign pattern as $T(\sum_{j=1}^{n} E_{1,j})$. Thus $T(\sum_{j=1}^{n} E_{1,j})$ is also a super $L$-matrix. This is a contradiction since $T$ strongly preserves super $L$-matrices and $T(\sum_{j=1}^{n} E_{1,j})$ is not a super $L$-matrix.

Therefore, if $E$ is a cell then $T(E)$ is also a cell.

Now we show that $T$ is one-to-one on the set of cells. We may assume $\langle T(E) \rangle = \langle E \rangle$ (1 ≤ j ≤ n). If $T$ is not one-to-one on the set of cells, then without loss of generality, we may assume that $T(E_{1,1}) = \alpha_{1} E_{1,1}$ and $T(E_{2,1}) = \alpha_{2} E_{2,1}$ for some $\alpha_{1} \neq 0$ and $\alpha_{2} \neq 0$. Thus

$$\dim \langle \mathbb{E} \rangle = \dim \langle T(\mathbb{E}) \rangle = \dim \langle \langle T(E_{1,1}), T(E_{2,1}) \rangle \rangle = \dim \langle \mathbb{E} \rangle = 1.$$ 

This is a contradiction since $\dim \langle \mathbb{E} \rangle = 2$. Therefore, $T$ is one-to-one on the set of cells. ■

**Theorem 5.3.4** If $T : M_{2,n} \rightarrow M_{2,n}$ is a linear operator that strongly preserves super $L$-matrices, then $T$ preserves the set of matrices of row term rank 1.

**Proof.** By Theorem 5.3.2 we know that $T$ preserves the column term rank 1 matrices. Also by Theorem 5.3.3, $T$ is one-to-one on the set of cells.
Let \( R_i = \sum_{j=1}^{n} E_{i,j} \). Let \( U = T^{-1}(R_i) \). Then by Theorem 5.3.3, \( |U| = n \).

Without loss of generality, say \( U = \sum_{j=1}^{n} E_j \). We only need to prove that all \( E_r \)'s have their nonzero entries on the same row. If not, then since \( T \) is nonsingular and preserves column term rank 1 matrices, we have that \( U \) has one nonzero entry in each column. Hence \( U \) is a super \( L \)-matrix. But \( T(U) = R_i \) is not a super \( L \)-matrix. This contradiction implies that all \( E_r \)'s have their nonzero entries on the same row, and the result follows.

By Theorem 5.3.2 and Theorem 5.3.4, we have the following corollary.

Corollary 5.3.1 If \( T : M_{2,n} \to M_{2,n} \) is a linear operator that strongly preserves super \( L \)-matrices, then \( T \) preserves the set of matrices of term rank 1.

By Corollary 5.3.1 and Theorem 5.3.1, we have that if \( T : M_{2,n} \to M_{2,n} \) is a linear operator that strongly preserves super \( L \)-matrices, then \( T \) is non-singular and \( T \) preserves the set of matrices of the term rank 1. By Theorem 2.1.1 [Beasley and Pullman, 17, Corollary 3.1.2], we have the following corollary.

Corollary 5.3.2 If \( T : M_{2,n} \to M_{2,n} \) is a linear operator that strongly preserves super \( L \)-matrices, then for any \( X \in M_{2,n} \), \( T(X) = P_1 (X \circ M) P_2 \), where \( P_1 \in M_2 \) and \( P_2 \in M_2 \) are permutation matrices and \( M = (m_{i,j}) \in M_{2,n} \) with \( m_{i,j} \neq 0 \).

Theorem 5.3.5 If \( T : M_{2,n} \to M_{2,n} \) is a linear operator that strongly preserves super \( L \)-matrices, then for any \( X \in M_{2,n} \), \( T(X) = P_1 S_1 (X \circ M) S_2 P_2 \).
where $P_1 \in \mathcal{M}_2$ and $P_2 \in \mathcal{M}_n$ are permutation matrices, $S_1 \in \mathcal{M}_2$ and $S_2 \in \mathcal{M}_n$ are diagonal matrices of ±1's, and $\mathcal{M} = (m_{i,j}) \in \mathcal{M}_{2,n}$ with $m_{i,j} > 0$.

**Proof.** The proof is parallel to the proof of Theorem 5.2.6.

This characterization of $2 \times n$ super L-matrix preservers gives us the following corollary:

**Corollary 5.3.3** If $T : \mathcal{M}_{2,n} \rightarrow \mathcal{M}_{2,n}$ is a linear operator that strongly preserves super L-matrices, then $T$ (strongly) preserves L matrices and $T$ (strongly) preserves totally L-matrices.
CHAPTER 6
LINEAR OPERATORS THAT PRESERVE
TOTALLY L-MATRICES

6.1 Introduction

Recall that an \( mxn \) real matrix \( A \) is said to be a totally \( L \)-matrix provided every submatrix of \( A \) of order \( m \) is a sign-nonsingular matrix. If \( m = n \), an \( mxm \) totally \( L \)-matrix is a sign-nonsingular matrix. We have discussed sign-nonsingular preservers in Chapter 3, so throughout this chapter we assume \( m < n \). By lemma 2.3.4, we know for \( m \geq 2 \) there is a restriction on the order of an \( mxn \) totally \( L \)-matrix. We can only have \( n = m + 1 \) or \( n = m + 2 \). So throughout this chapter we assume \( n = m + 1 \) or \( n = m + 2 \). In section 6.2, we discuss properties on \( mx(m+2) \) totally \( L \)-matrices preservers. We prove that if linear operator \( T \) strongly preserves \( mx(m+2) \) totally \( L \)-matrices, then \( T \) is nonsingular. Further, we prove that if \( T \) is a linear operator that preserves \( 2x4 \) totally \( L \)-matrices and \( T \) is also one-to-one on the set of cells, then for any \( 2x4 \) real matrix \( X \), \( T(X) = P_1 S_1 (X \cdot M) S_2 P_2' \), where \( P_1 \in M_2 \) and \( P_2 \in M_4 \) are permutation matrices, \( S_1 \in M_2 \) and \( S_2 \in M_4 \) are diagonal matrices of \( \pm 1 \)'s, and \( M = (m_{i,j}) \in M_{2,4} \) with \( m_{i,j} > 0 \). In section 6.3, we prove that if \( T \) is a linear operator that preserves \( 2x3 \) totally \( L \)-matrices, then for any \( 2x3 \) real matrix \( X \), \( T(X) = P_1 S_1 (X \cdot M) S_2 P_2' \), where \( P_1 \in M_2 \) and \( P_2 \in M_3 \) are permutation matrices, \( S_1 \in M_2 \) and \( S_2 \in M_3 \) are diagonal matrices of \( \pm 1 \)'s, and \( M = (m_{i,j}) \in M_{2,3} \) with \( m_{i,j} > 0 \).

6.2 Totally \( L \)-matrix Preservers (\( n = m + 2 \))

In this section we will investigate \( mx(m+2) \) totally \( L \)-matrices
preservers. Let \( T: \mathcal{M}_{m,m+2} \rightarrow \mathcal{M}_{m,m+2} \) be a linear operator that preserves totally \( L \)-matrices. First we note that \( T \) is not necessarily a nonsingular operator. For example, for \( m = 2 \), we let \( T: \mathcal{M}_{2,4} \rightarrow \mathcal{M}_{2,4} \) be a linear operator defined by

\[
T \begin{pmatrix} a & b & c & d \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \end{pmatrix}
\]

for any

\[
A = \begin{pmatrix} a & b & c & d \\ * & * & * & * \end{pmatrix} \in \mathcal{M}_{2,4}.
\]

If \( A \) is a totally \( L \)-matrix then, by Lemma 2.3.14, \( A \) has exactly three nonzero entries on each row, that is, only one of \( a, b, c, \) and \( d \) is zero. Also \( T(A) \) is a totally \( L \)-matrix if and only if only one of \( a, b, c, \) and \( d \) is zero. Hence \( T \) preserves totally \( L \)-matrices. But \( T \) is a singular operator since \( T(\sum_{j=1}^{4} E_{2,j}) = 0 \).

We start from an \( 2 \times 4 \) totally \( L \)-matrix \( B_{2,4} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \). Then for any \( m \geq 2 \), we can construct an \( m \times (m+2) \) totally \( L \)-matrix \( B_{m,m+2} = (b_{i,j}) \) such that \( b_{i,j} = 1 \) (\( j = 1, 2, 3 \)) and \( b_{i,j} = 0 \) (\( j = 4, 5, \ldots, m + 2 \)) [Brualdi, Chavey and Shader, 38].

Also by Lemma 2.3.14, we have that if \( A \in \mathcal{M}_{m,m+2} \) (\( m \geq 2 \)) is a totally \( L \)-matrix, then every row of \( A \) has exactly three nonzero entries.

Now we suppose \( T: \mathcal{M}_{m,m+2} \rightarrow \mathcal{M}_{m,m+2} \) is a linear operator that \textit{strongly} preserves totally \( L \)-matrices. We have the following theorem.

**Theorem 6.2.1** If \( m \geq 2 \) and \( T: \mathcal{M}_{m,m+2} \rightarrow \mathcal{M}_{m,m+2} \) is a linear operator
that strongly preserves totally L-matrices, then $T$ is nonsingular.

Proof. If not, then there exists $A = (a_{i,j}) \in M_{m,m+2}$ and $A \neq O$ such that $T(A) = O$. Without loss of generality, assume $a_{1,1} > 0$. Let $B = (b_{i,j}) \in M_{m,m+2}$ which satisfies the following conditions:

1) $A + B$ is a totally L-matrix;

2) $b_{1,1} = 0$;

3) $a_{1,j} + b_{1,j} > 0$ ($j=2,3$) and $a_{1,j} + b_{1,j} = 0$ ($j=4,\ldots,m+2$);

4) $|\{b_{1,j} : j = 1, 2, \ldots, m+2\}| \neq 3$.

Since $A + B$ is a totally L-matrix, we have that $T(A + B)$ must be a totally L-matrix. Hence $T(B) = T(A) + T(B) = T(A + B)$ is also a totally L-matrix since $T(A) = O$. Thus $B$ must be a totally L-matrix since $T$ strongly preserves totally L-matrices. But by condition 4, $|\{b_{1,j} : 1 \leq j \leq m+2\}| \neq 3$. So $B$ cannot be a totally L-matrix by Lemma 2.3.14. This contradiction implies that $T$ is nonsingular. $\blacksquare$

Now we investigate the linear operators that preserve 2x4 totally L-matrices. We suppose these linear operators are also one-to-one on the set of cells. We have the following theorems.

**Theorem 6.2.2** If $T : M_{2,4} \rightarrow M_{2,4}$ is a linear operator that preserves totally L-matrices and $T$ is also one-to-one on the set of cells, then $T$ preserves the set of matrices of row term rank 1.

Proof. Let $R_i = \sum_{j=1}^{4} E_{i,j}$ ($l = 1, 2$). Since $T$ is one-to-one on the set of cells, we have $T$ is non-singular. Let $U_i = T^{-1}(R_i)$. Then $|U_i| = 4$, say $U_i = E_1 + E_2 + E_3 + E_4$. We only need to show that the all nonzero entries
of $E_1$'s lie on the same row. If not, then every row of $U_1$ has at most three nonzero entries. So we can choose two cells, say $E_5$ and $E_6$ with $E_5 \notin \{E_r \mid 1 \leq r \leq 4\}$ and $E_6 \notin \{E_r \mid 1 \leq r \leq 4\}$, and $\alpha_i$'s where $\alpha_i = 1$ or $-1$ $(1 \leq i \leq 6)$ such that $\sum_{r=1}^{6} \alpha_r E_r$ is a totally $L$-matrix. Thus

$$T(\sum_{r=1}^{6} \alpha_r E_r) = T(\sum_{r=1}^{4} \alpha_r E_r) + T(\alpha_5 E_5) + T(\alpha_6 E_6)$$

$$= \sum_{r=1}^{4} \alpha_r E_{1,r} + T(\alpha_5 E_5) + T(\alpha_6 E_6)$$

must be a totally $L$-matrix. This is a contradiction since

$$\sum_{r=1}^{4} \alpha_r E_{1,r} + T(\alpha_5 E_5) + T(\alpha_6 E_6)$$

has more than three nonzero entries on the $i^{th}$ row. Hence all the nonzero entries of $E_1$'s lie on the same row, that is, $T$ preserves the set of matrices of row term rank 1.

Theorem 6.2.3 If $T : M_{2,4} \rightarrow M_{2,4}$ is a linear operator that preserves totally $L$-matrices and $T$ is also one-to-one on the set of cells, then $T$ preserves the set of matrices of column term rank 1.

Proof. Let $C_j = E_{1,j} + E_{2,j}$ $(j = 1, 2, 3)$ and $V_j = T^{-1}(C_j)$. Then $|V_j| = 2$ since $T$ is one-to-one on the set of cells. Without loss of generality, say $V_j = E_1 + E_2$, $T(E_1) = E_{1,j}$ and $T(E_2) = E_{2,j}$. We only need to show $E_1$ and $E_2$ are in the same column. If not, then $E_1$ and $E_2$ are in different columns, without loss of generality, assume $E_1$ is in the first column and $E_2$ is in the second column. Since $T$ preserves the set of
matrices of row term rank 1, we have that $E_1$ and $E_2$ must lie on different rows, say $E_1$ is in the first row and $E_2$ is in the second row. Thus $E_1 = E_{1,1}'$, $E_2 = E_{2,2}'$ and $T(E_{1,1}' + E_{2,2}') = C_j$. Now we consider the totally \( L \)-matrix $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}$. Since $T$ preserves totally \( L \)-matrix matrices, we have that $T(A)$ is a totally \( L \)-matrix. But $T(A)$ has zero $j^{th}$ column, a contradiction. Therefore $T$ preserves the set of matrices of column term rank 1.

By Theorem 6.2.2 and Theorem 6.2.3, we have the following corollary.

**Corollary 6.2.1** If $T : M_{2,4} \longrightarrow M_{2,4}$ is a linear operator that preserves totally \( L \)-matrices and $T$ is also one-to-one on the set of cells, then $T$ preserves the set of matrices of term rank 1.

By Theorem 2.1.1 [Beasley and Pullman, 17], we have the following corollary.

**Corollary 6.2.2** If $T : M_{2,4} \longrightarrow M_{2,4}$ is a linear operator that preserves totally \( L \)-matrices and $T$ is also one-to-one on the set of cells, then for any $X \in M_{2,4}$, $T(X) = P_1(X \circ M)P_2'$, where $P_1 \in M_2$ and $P_2 \in M_4$ are permutation matrices and $M = (m_{i,j}) \in M_{2,4}$ with $m_{i,j} \neq 0$.

**Theorem 6.2.4** If $T : M_{2,4} \longrightarrow M_{2,4}$ is a linear operator that preserves totally \( L \)-matrices and $T$ is also one-to-one on the set of cells, then for any $X \in M_{2,4}$, $T(X) = P_1S_1(X \circ M)S_2P_2'$, where $P_1 \in M_2$ and $P_2 \in M_4$ are permutation matrices, $S_1 \in M_2$ and $S_2 \in M_4$ are diagonal matrices of \( \pm 1 \)'s, and $M = (m_{i,j}) \in M_{2,4}$ with $m_{i,j} > 0$. 
Proof. By Corollary 6.2.2, for any $X \in M_{2,4}$, $T(X) = P_1 (X \circ M) P_2$, where $P_1 \in M_2$ and $P_2 \in M_4$ are permutation matrices and $M = (m_{i,j}) \in M_{2,4}$ with $m_{i,j} \neq 0$. Let $T_1 : M_{2,4} \rightarrow M_{2,4}$ be a linear operator defined by

$$T_1(X) = P_1^t T(X) P_2^t = P_1^t P_1^t (X \circ M) P_2 P_2^t = X \circ M$$

for any $X \in M_{2,4}$. Then clearly $T_1$ preserves totally $L$-matrices since $T$ preserves totally $L$-matrices and $P_1^t, P_2^t$ are permutation matrices. Let $T_2 : M_{2,4} \rightarrow M_{2,4}$ be a linear operator defined by

$$T_2(X) = S_1 T(X) S_2 = S_1 (X \circ M) S_2,$$

where

$$S_1 = \text{diag} \left( \frac{m_{1,1}}{|m_{1,1}|}, \frac{m_{2,1}}{|m_{2,1}|} \right)$$

and

$$S_2 = \text{diag} \left( 1, \frac{m_{1,2}}{|m_{1,2}|}, \frac{m_{1,1}}{|m_{1,1}|}, \frac{m_{1,3}}{|m_{1,3}|}, \frac{m_{1,1}}{|m_{1,1}|}, \frac{m_{1,4}}{|m_{1,4}|}, \frac{m_{1,1}}{|m_{1,1}|} \right).$$

Clearly $T_2$ preserves totally $L$-matrices since $T_1$ preserves totally $L$-matrices and $S_1, S_2$ are diagonal matrices with nonzero entries on the main diagonal. Also we have

$$T_2(X) = S_1 T(X) S_2 = S_1 (X \circ M) S_2 = X \circ (S_1 M S_2).$$

Let $N = S_1 M S_2$. Then $N = (n_{i,j}) \in M_{2,4}$ with $n_{i,j} \neq 0, n_{i,j} > 0 (i = 1, 2)$ and $n_{1,j} > 0 (j = 1, 2, 3, 4)$. Since $T_2$ preserves totally $L$-matrices, we have that for any $2 \times 4$ totally $L$-matrix $A$, $T_2(A) = A \circ (S_1 M S_2) = A \circ N$ is a totally $L$-matrix. If $n_{2,2} < 0$, then we let
\[ A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}, \]

which is a totally \( L \)-matrix. But

\[ A \circ N = \begin{pmatrix} n_{1,1} & n_{1,2} & 0 & n_{1,4} \\ n_{2,1} & -n_{2,2} & n_{2,3} & 0 \end{pmatrix} \]

is not a totally \( L \)-matrix since \( \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & -n_{2,2} \end{pmatrix} \) has the sign pattern

\[ \begin{pmatrix} + & + \\ + & + \end{pmatrix}. \]

This contradiction implies that \( n_{2,2} > 0 \). By permuting rows and columns we have that all entries in the matrix \( N \) are positive. Hence \( N = (n_{i,j}) \in \mathcal{M}_{2,4} \) with \( n_{i,j} > 0 \). Thus

\[ T(X) = P_1 (X \circ M) P_2 = P_1 (X \circ (S_1^{-1} N S_2^{-1})) P_2 = P_1 S_1^{-1} (X \circ N) S_2^{-1} P_1 P_2, \]

which completes the proof. \( \blacksquare \)

**Corollary 6.2.3** If \( T : \mathcal{M}_{2,4} \rightarrow \mathcal{M}_{2,4} \) is a linear operator that preserves totally \( L \)-matrices and \( T \) is also one-to-one on the set of cells, then \( T \) strongly preserves totally \( L \)-matrices.

**Proof.** This follows Theorem 6.2.4. \( \blacksquare \)

### 6.3 Totally \( L \)-matrices Preservers \((n = m + 1)\)

Let \( T : \mathcal{M}_{m,m+1} \rightarrow \mathcal{M}_{m,m+1} \) is a linear operator that preserves totally \( L \)-matrices. We will show for \( m = 2 \), \( T \) is nonsingular. Also we will give the characterization of these linear operators that preserve 2x3 totally \( L \)-matrices. First we prove some properties about 2x3 totally \( L \)-matrices.
Lemma 6.3.1 If \( A \in M_{2,3} \) is a totally L-matrix, then \( 4 \leq |A| \leq 5 \), every row has at least two nonzero entries and every column has nonzero entry.

Proof. Since every totally L-matrix is a super L-matrix, by Lemma 2.3.6, we have that every column of a totally L-matrix has nonzero entry.

If \( A = (a_{i,j}) \in M_{2,3} \) is a totally L-matrix, then by Corollary 2.3.1, we have that \( A \) has at least one zero entry. Hence \( |A| \neq 6 \). If \( A \) has two zero entries on the same row, without loss generality assume \( a_{1,1} = a_{1,2} = 0 \), then \( \left( \begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{array} \right) \) is not a sign-nonsingular matrix, a contradiction. If \( |A| < 4 \) then \( A \) has at least 3 zero entries and at least two zero entries must be on the same row (or column). So \( A \) cannot be a totally L-matrix. Therefore, \( 4 \leq |A| \leq 5 \). \( \blacksquare \)

Now we investigate the totally L-matrices preservers on \( M_{2,3} \). Let \( \text{T : } M_{2,3} \longrightarrow M_{2,3} \), be a linear operator that preserves totally L-matrices. Let \( T(E_{1,1}) = \left( \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{array} \right) \),

\[
T(x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) = \left( \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{array} \right)
\]

where \( x_i \neq 0 \) \( (i = 1, 2, 3) \) and \( \beta_j = \beta_j(x_1, x_2, x_3) \),

\[
T(E_{2,1}) = \left( \begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{array} \right)
\]

and
\[ T(E_{1,3}) = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_4 & \eta_5 & \eta_6 \end{pmatrix}. \]

We have the following lemmas.

**Lemma 6.3.2** For any \( i \) and \( j \), \( |T(E_{i,j})| \neq 6 \).

**Proof.** Suppose there are \( i \) and \( j \) such that \( |T(E_{i,j})| = 6 \). Without loss of generality, we assume \( |T(E_{1,1})| = 6 \).

We may say that

\[ T(E_{1,1}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix} \]

with \( \alpha_i \neq 0 \) (1\( \leq i \leq 6 \)). Now we suppose

\[ T(E_{1,2} + E_{2,2} + E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix}. \]

Then for any \( a \neq 0 \), \( T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3}) \) is a totally \( L \)-matrix.

We can choose \( a \neq 0 \) such that \( a\alpha_i + \beta_i \neq 0 \) (1\( \leq i \leq 6 \)). Hence

\[ |T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})| = 6 \]

for the choice of \( a \). This is a contradiction with Lemma 6.3.1. Therefore \( |T(E_{i,j})| \neq 6 \) for any cell \( E_{i,j} \). \( \blacksquare \)

**Lemma 6.3.3** For any \( i \) and \( j \), \( |T(E_{i,j})| \neq 5 \).

**Proof.** Suppose there are \( i \) and \( j \) such that \( |T(E_{i,j})| = 5 \). Without loss of generality, we assume \( |T(E_{1,1})| = 5 \) and

\[ T(E_{1,1}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & 0 \end{pmatrix} \]
with \( \alpha_i \neq 0 \) (1\( \leq i \leq 5 \)).

First we will show that for any \( x_i \)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix}
\]

with \( \beta_6 = 0 \). If there are some \( x_i \)'s such that \( \beta_6(x_1, x_2, x_3) \neq 0 \), then for any \( a \neq 0 \),

\[
T(a E_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3})
\]

is a totally \( L \)-matrix. We can choose \( a \neq 0 \) such that \( a\alpha_i + \beta_i \neq 0 \) (1\( \leq i \leq 5 \)). Hence

\[
|T(a E_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3})| = 6.
\]

This is a contradiction with Lemma 6.3.1. Hence \( \beta_6 = 0 \).

Second we show that for any \( x_i \)'s, \( \beta_3 = 0 \). If there are \( x_i \)'s such that \( \beta_3(x_1, x_2, x_3) \neq 0 \), then we can choose \( a \neq 0 \) such that \( a\alpha_3 + \beta_3 = 0 \).

Then \( T(a E_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) \) has zero 3\textsuperscript{rd} column. This is a contradiction with the fact that \( T(a E_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) \) is a totally \( L \)-matrix. Hence for any \( x_i \)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & 0 \\ \beta_4 & \beta_5 & 0 \end{pmatrix}.
\]

Now we show

\[
T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}
\]
with \( \gamma_6 = \gamma_3 = 0 \). If \( \gamma_6 \neq 0 \), then for any \( a > 0 \),

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{2,1})
\]

is a totally \( L \)-matrix. We can choose \( a > 0 \) such that

\[
|T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{2,1})| = 6,
\]

a contradiction. If \( \gamma_6 = 0 \) and \( \gamma_3 \neq 0 \), then either \( \alpha_3\gamma_3 > 0 \) or \( \alpha_3\gamma_3 < 0 \).

If \( \alpha_3\gamma_3 > 0 \), then for any \( a > 0 \)

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{2,1})
\]

is a totally \( L \)-matrix. We can choose \( a = \gamma_3/\alpha_3 > 0 \) such that

\[
a\alpha_3 - \gamma_3 = 0.
\]

Thus

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{2,1})
\]

is a totally \( L \)-matrix which has zero \( 3^{rd} \) column, a contradiction. If \( \alpha_3\gamma_3 < 0 \), then for any \( a > 0 \)

\[
T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1})
\]

is a totally \( L \)-matrix. We can choose \( a = -\gamma_3/\alpha_3 > 0 \) such that

\[
a\alpha_3 + \gamma_3 = 0.
\]

Thus

\[
T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1})
\]

is a totally \( L \)-matrix which has zero \( 3^{rd} \) column, a contradiction. So
\[ \gamma_3 = \gamma_6 = 0 \text{ and } T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & 0 \\ \gamma_4 & \gamma_5 & 0 \end{pmatrix}. \]

Since for any \( a \neq 0 \), \( T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix, by the same argument as above we have that \( \eta_3 = \eta_6 = 0 \).

Thus \( T(E_{1,3}) = \begin{pmatrix} \eta_1 & \eta_2 & 0 \\ \eta_4 & \eta_5 & 0 \end{pmatrix} \). Hence

\[ T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \]

is a totally \( L \)-matrix which has zero 3\(^{rd} \) column, a contradiction.

Therefore, \( |T(E_{i,j})| \neq 5 \) for any cell \( E_{i,j} \). \[ \blacksquare \]

Lemma 6.3.4 For any \( i \) and \( j \), \( |T(E_{i,j})| \neq 4 \).

Proof. Suppose there are \( i \) and \( j \) such that \( |T(E_{i,j})| = 4 \). Without loss of generality, we assume \( |T(E_{1,1})| = 4 \).

Case 1. \( T(E_{1,1}) \) has two zero entries in the same column.

We may say that \( T(E_{1,1}) = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_4 & \alpha_5 & 0 \end{pmatrix} \) with \( \alpha_i \neq 0 \) (\( i = 1, 2, 4, 5 \)).

First we will show that for any \( x_i \)'s,

\[ T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix} \]

with exact one of \( \beta_3 \) and \( \beta_6 \) equal zero. If there are \( x_i \)'s such that both \( \beta_3 \neq 0 \) and \( \beta_6 \neq 0 \), then for any \( a \neq 0 \),

\[ T(aE_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) \]

is a totally \( L \)-matrix. We can choose \( a \neq 0 \) such that \( a\alpha_i + \beta_i \neq 0 \).
(i = 1, 2, 4, 5). Hence

$$|T(aE_{1,1} + x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3})| = 6$$

for the choice of a, a contradiction. If there are $x_i$'s such that both $\beta_3 = 0$ and $\beta_6 = 0$, then we have that

$$T(E_{1,1} + x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3})$$

is a totally $L$-matrix which has zero $3^{rd}$ column, a contradiction again.

Hence for any $x_i$'s, either $\beta_3 \neq 0$ and $\beta_6 = 0$ or $\beta_6 \neq 0$ and $\beta_3 = 0$.

We will show that if for some fixed $x_i$'s, $\beta_3 \neq 0$ and $\beta_6 = 0$ then for any $x_i$'s $\beta_3 \neq 0$ and $\beta_6 = 0$. If there are $y_i$'s ($y_i \neq 0$, $i = 1, 2, 3$) such that $\beta_6 \neq 0$ and $\beta_3 = 0$, then for any $b \neq 0$ and $bx_i + y_i \neq 0$ ($i = 1, 2, 3$),

$$T[E_{1,1} + (bx_1 + y_1)E_{1,2} + (bx_2 + y_2)E_{2,2} + (bx_3 + y_3)E_{2,3}]$$

is a totally $L$-matrix. We can choose $b \neq 0$ and $bx_i + y_i \neq 0$ ($i = 1, 2, 3$) such that

$$|T[E_{1,1} + (bx_1 + y_1)E_{1,2} + (bx_2 + y_2)E_{2,2} + (bx_3 + y_3)E_{2,3}]| = 6,$$

a contradiction. Without loss of generality, we assume for any $x_i$'s,

$$T(x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & 0 \end{pmatrix}$$

with $\beta_3 \neq 0$.

Also we can show that $\beta_5 = 0$. If $\beta_5 \neq 0$, then for any $a \neq 0$,

$$T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})$$
is a totally $L$-matrix. We can choose $a \neq 0$ such that $a \alpha + \beta = 0$. Hence for the choice of $a$,

$$T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})$$

has two zero entries on the second row, a contradiction with the fact that

$$T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})$$

is a totally $L$-matrix. Hence $\beta = 0$. Similarly, we can show that $\beta_4 = 0$.

Therefore, for any $x_i$'s

$$T(x_1E_{1,1} + x_2E_{2,2} + x_3E_{2,3}) = \left( \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 \end{array} \right)$$

with $\beta_3 \neq 0$.

Now, we show

$$T(E_{2,1}) = \left( \begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{array} \right)$$

with $\gamma_3 = \gamma_6 = 0$. Suppose not. Then at least one of $\gamma_3$ and $\gamma_6$ is not zero.

If $\gamma_6 \neq 0$, then for any $a > 0$ and $c > 0$,

$$T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})$$

is a totally $L$-matrix. We can choose $a > 0$ and $c > 0$ such that

$$|T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})| = 6.$$

This is a contradiction with the fact that
is a totally $L$-matrix.

If $\gamma_3 \neq 0$ and $\gamma_5 = 0$, then, for any $a \neq 0$, $c \neq 0$ and $ac < 0$,

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})
\]

is a totally $L$-matrix. We can choose $a$ and $c$ such that $\beta_3 + c\gamma_3 = 0$.

Thus

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})
\]

is a totally $L$-matrix which has zero 3rd column, a contradiction.

Therefore, $\gamma_3 = \gamma_5 = 0$. Now we show that $\gamma_5 = 0$. Suppose not. If $\gamma_5 > 0$, then, without loss of generality, assume $\alpha_5 > 0$, for any $a > 0$,

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{2,1})
\]

is a totally $L$-matrix. We can choose $a > 0$ such that $a\alpha_5 - \gamma_5 = 0$. Thus

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{2,1})
\]

has two zero entries on the second row, a contradiction. If $\gamma_5 < 0$, then for any $a > 0$,

\[
T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1})
\]

is a totally $L$-matrix. We can choose $a > 0$ such that $a\alpha_5 + \gamma_5 = 0$. Thus

\[
T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1})
\]
has two zero entries on the second row, a contradiction. Similarly, 
\( \gamma_4 = 0 \). So

\[
T(E_{2,1}) = \begin{pmatrix}
\gamma_1 & \gamma_2 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

By the same argument as above, we have that

\[
T(E_{1,3}) = \begin{pmatrix}
\eta_1 & \eta_2 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has zeros second row, a contradiction. Hence this case never arises.

**Case 2.** \( T(E_{1,1}) \) has two zero entries on the same row.

We may say that \( T(E_{1,1}) = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_4 & 0 & 0
\end{pmatrix} \) with \( \alpha_i \neq 0 \) (\( i = 1, 2, 3, 4 \)).

By the same argument as Case 1, for any \( a \neq 0 \) and \( x_i \neq 0 \), we consider the totally \( L \)-matrix \( T(aE_{1,1} + x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) \). We can show that for any \( x_i \)'s,

\[
T(x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 \\
\beta_4 & \beta_5 & \beta_6
\end{pmatrix}
\]

with either \( \beta_5 = 0 \) or \( \beta_6 = 0 \). Then we can show that if for some fixed \( x_i \)'s, \( \beta_5 \neq 0 \) and \( \beta_6 = 0 \) then for any \( x_i \)'s, \( \beta_5 \neq 0 \) and \( \beta_6 = 0 \). Without loss of generality, we assume for any \( x_i \)'s,

\[
T(x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 \\
\beta_4 & \beta_5 & 0
\end{pmatrix}
\]

with \( \beta_5 \neq 0 \).

Then we show that \( \beta_3 = 0 \). If \( \beta_3 \neq 0 \), then for any \( a \neq 0 \),
is a totally $L$-matrix. We can choose $a \neq 0$ such that $a \alpha_3 + \beta_3 = 0$. Hence for the choice of $a$,

$$T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})$$

has zero 3rd column, a contradiction with the fact that

$$T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})$$

is a totally $L$-matrix. Hence $\beta_3 = 0$. Therefore, for any $x_i$'s

$$T(x_1 E_{1,1} + x_2 E_{1,2} + x_3 E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & 0 \\ \beta_4 & \beta_5 & 0 \end{pmatrix}$$

with $\beta_5 \neq 0$.

Now, we show

$$T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}$$

with $\gamma_3 = \gamma_6 = 0$. Suppose not. Then at least one of $\gamma_3$ and $\gamma_6$ is not zero.

If both $\gamma_3 \neq 0$ and $\gamma_6 \neq 0$ (or $\gamma_3 = 0$ and $\gamma_6 \neq 0$), then for any $a > 0$ and $c > 0$,

$$T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})$$

is a totally $L$-matrix. We can choose $a > 0$ and $c > 0$ such that
\[ |T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})| = 6. \]

This contradicts that \( T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1}) \) is a totally \( L \)-matrix.

If \( \gamma_3 \neq 0 \) and \( \gamma_6 = 0 \), then either \( \gamma_3 > 0 \) or \( \gamma_3 < 0 \). Without loss of generality we assume \( \alpha_3 > 0 \). If \( \gamma_3 > 0 \), then for any \( a > 0 \),

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{2,1})
\]

is a totally \( L \)-matrix. We can choose \( a > 0 \) such that \( a\alpha_3 - \gamma_3 = 0 \). Thus

\[
T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1})
\]

is a totally \( L \)-matrix which has zero 3\(^{rd}\) column, a contradiction. If \( \gamma_3 < 0 \), then we can choose \( a > 0 \), such that \( a\alpha_3 + \gamma_3 = 0 \). Thus

\[
T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1})
\]

is a totally \( L \)-matrix which has zero 3\(^{rd}\) column, a contradiction.

Therefore, \( \gamma_3 = \gamma_6 = 0 \). So

\[
T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & 0 \\ \gamma_4 & \gamma_5 & 0 \end{pmatrix}.
\]

By the same argument as above, we have that

\[
T(E_{1,3}) = \begin{pmatrix} \eta_1 & \eta_2 & 0 \\ \eta_4 & \eta_5 & 0 \end{pmatrix}.
\]

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has zero 3\(^{rd}\) column, a contradiction. Hence this case never arises.
Case 3. \( T(E_{1,1}) \) has two zero entries on the different row and column.

We may say that \( T(E_{1,1}) = \begin{pmatrix} \alpha_1 & 0 & \alpha_3 \\ \alpha_4 & \alpha_5 & 0 \end{pmatrix} \) with \( \alpha_i \neq 0 \) (\( i = 1, 3, 4, 5 \)).

By the same argument as Case 1, for any \( a \neq 0 \) and \( x_i \neq 0 \), we consider the totally \( L \)-matrix \( T(aE_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{3,3}) \). We can show that for any \( x_i \)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{3,3}) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix}
\]

with at least one of \( \beta_2 \) and \( \beta_6 \) is zero.

We will show that if for some fixed \( x_i \)'s \( \beta_2(x_1, x_2, x_3) = 0 \) and \( \beta_6(x_1, x_2, x_3) \neq 0 \), then for any \( x_i \)'s, \( \beta_2 = 0 \). If there exist \( y_i \)'s such that \( \beta_2(y_1, y_2, y_3) \neq 0 \) then \( \beta_6(y_1, y_2, y_3) = 0 \). We can choose \( b \neq 0 \) and \( b' \neq 0 \) such that \( bx_i + b'y_i \neq 0 \) (\( i = 1, 2, 3 \)) and

\[
|T[E_{1,1} + (bx_1 + b'y_1)E_{1,2} + (bx_2 + b'y_2)E_{2,2} + (bx_3 + b'y_3)E_{3,3}]| = 6,
\]

a contradiction. Similarly, we can show that if for some fixed \( x_i \)'s \( \beta_6(x_1, x_2, x_3) = 0 \) and \( \beta_2(x_1, x_2, x_3) \neq 0 \) then for any \( x_i \)'s, \( \beta_6 = 0 \).

Subcase 1. \( \beta_2(1,1,1) = 0 \) and \( \beta_6(1,1,1) \neq 0 \).

In this case for any \( x_i \)'s \( \beta_2 = 0 \). Thus for any \( x_i \)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{3,3}) = \begin{pmatrix} \beta_1 & 0 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix}.
\]

By the same argument as case 1 and 2, we can show that \( \beta_1 = \beta_3 = \beta_5 = 0 \). Therefore, for any \( x_i \)'s
Since \( \alpha_3 \neq 0 \) and \( \beta_6 (1,1,1) \neq 0 \), by the same argument as case 2 we have \( \gamma_2 = 0 \). Also since for any \( x_i \)'s \( \beta_2 \neq 0 \), we must have \( \gamma_1 = \gamma_3 = \gamma_5 = 0 \). Thus

\[
T(E_{2,1}) = \begin{pmatrix}
0 & 0 & 0 \\
\gamma_4 & 0 & \gamma_6
\end{pmatrix}
\]

Similarly,

\[
T(E_{1,3}) = \begin{pmatrix}
0 & 0 & 0 \\
\eta_4 & 0 & \eta_6
\end{pmatrix}
\]

Thus \( T(E_{1,2} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has zero first row, a contradiction. Hence this case never arises.

\textbf{Subcase 2.} \( \beta_6 (1,1,1) = 0 \) and \( \beta_2 (1,1,1) \neq 0 \).

This case is equivalent to the subcase 1. Hence this case never arises.

\textbf{Subcase 3.} \( \beta_2 (1,1,1) = \beta_6 (1,1,1) = 0 \).

In this case, first we show that for any \( x_i \)'s, \( \beta_2 = \beta_6 = 0 \). If not, then there exist \( x_i \)'s such \( \beta_2 (x_i, x_2, x_3) = 0 \) and \( \beta_6 (x_i, x_2, x_3) \neq 0 \) (or \( \beta_2 (x_i, x_2, x_3) \neq 0 \) and \( \beta_6 (x_i, x_2, x_3) = 0 \)). These are subcase 1 and 2, which never arise. Hence for any \( x_i \)'s \( \beta_2 = \beta_6 = 0 \). Thus for any \( a \neq 0 \) and \( x_i \neq 0 \), we consider the totally \( L \)-matrix

\[
T(aE_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}).
\]

We have that \( T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = 0 \).
Next we will show \( T(E_{2,1}) = \left( \begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{array} \right) \) with either \( \gamma_2 = 0 \) or \( \gamma_6 = 0 \). By the same argument as above, we have that if \( \gamma_2 = 0 \), then \( \gamma_1 = \gamma_3 = \gamma_5 = 0 \) and if \( \gamma_6 = 0 \), then \( \gamma_3 = \gamma_4 = \gamma_5 = 0 \). Thus we have

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & 0 & \gamma_6 \end{pmatrix} \quad \text{or} \quad T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Similarly,

\[
T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & 0 & \eta_6 \end{pmatrix} \quad \text{or} \quad T(E_{1,3}) = \begin{pmatrix} \eta_1 & \eta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Now we show \( |T(E_{2,1})| \leq 1 \) and \( |T(E_{1,3})| \leq 1 \). Suppose that we have

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & 0 & \gamma_6 \end{pmatrix}.
\]

We will show at least one of \( \gamma_4 \) and \( \gamma_6 \) is zero. If not, then \( \gamma_4 \neq 0 \) and \( \gamma_6 \neq 0 \). For any \( c \neq 0 \), since

\[
T(E_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3}) = \begin{pmatrix} \alpha_1 & 0 & \alpha_3 \\ \alpha_4 + c\gamma_4 & \alpha_5 & \alpha_6 \end{pmatrix}
\]

is a totally \( L \)-matrix, we must have \( \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_4 + c\gamma_4 & c\gamma_6 \end{pmatrix} \) is a sign-nonsingular matrix. If \( \alpha_1 \alpha_3 > 0 \), then we can choose \( c \neq 0 \) such that \( c\gamma_6 \) and \( \alpha_4 + c\gamma_4 \) have the same sign. Thus the sign pattern of the matrix \( \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_4 + c\gamma_4 & c\gamma_6 \end{pmatrix} \) is equivalent to \( \begin{pmatrix} ++ \\ + + \end{pmatrix} \) or \( \begin{pmatrix} + + \\ -- \end{pmatrix} \), a contradiction. If \( \alpha_1 \alpha_3 < 0 \), then we
can choose $c \neq 0$ such that $c\gamma_6$ and $\alpha_4 + c\gamma_4$ have different sign. Thus the
sign pattern of the matrix
\[
\begin{pmatrix}
\alpha_1 & \alpha_3 \\
\alpha_4 + c\gamma_4 & c\gamma_6
\end{pmatrix}
\]
is equivalent to $\begin{pmatrix} + & - \\ + & - \end{pmatrix}$ or
$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$, a contradiction. Hence at least one of $\gamma_4$ and $\gamma_6$ is zero.
Similarly, we can show if $T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & 0 \\
0 & 0 & 0 \end{pmatrix}$, then at least one of $\gamma_1$ and $\gamma_2$ is zero. Thus $|T(E_{2,1})| \leq 1$. By the same argument as above, we have $|T(E_{1,3})| \leq 1$.
Thus $T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})$ is a totally $L$-matrix but
\[
|T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})| \leq |T(E_{2,1})| + |T(E_{1,2} + E_{2,2} + E_{2,3})| + |T(E_{1,3})| \leq 1 + 0 + 1 = 2,
\]
a contradiction. So this case never arises.

Therefore, for any cell $E_{i,j}$, $|T(E_{i,j})| \neq 4$. $\blacksquare$

Lemma 6.3.5 For any $i$ and $j$, $|T(E_{i,j})| \neq 3$.

Proof. Suppose there are $i$ and $j$ such that $|T(E_{i,j})| = 3$. Without loss of generality, we assume $|T(E_{1,1})| = 3$.

Case 1. $T(E_{1,1}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\
0 & 0 & 0 \end{pmatrix}$ with $\alpha_i \neq 0$ ($i = 1, 2, 3$).

First we will show that for any $x_i$'s,
\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\
\beta_4 & \beta_5 & \beta_6 \end{pmatrix}
\]
with exact one of $\beta_4$, $\beta_5$ and $\beta_6$ is zero. If there are $x_i$'s such that $\beta_4 \neq 0$, $\beta_5 \neq 0$, and $\beta_6 \neq 0$ then for any $a \neq 0$,
\[
T(aE_{1,1} + x_1E_{1,2} + x_2E_{2,2} + x_3E_{3,3})
\]
is a totally $L$-matrix. We can choose $a \neq 0$ such that $a\alpha_i + \beta_i \neq 0$ ($i = 1, 2, 3$). Hence
\[
|T(aE_{1,1} + x_1E_{1,2} + x_2E_{2,2} + x_3E_{3,3})| = 6
\]
for the choice of $a$, a contradiction. If there are $x_i$'s such that at least two of $\beta_4$, $\beta_5$, and $\beta_6$ are zero, then we have that
\[
T(E_{1,1} + x_1E_{1,2} + x_2E_{2,2} + x_3E_{3,3})
\]
is a totally $L$-matrix which has at least two zero on the second row, a contradiction again. Hence for any $x_i$'s, exact one of $\beta_4$, $\beta_5$, and $\beta_6$ is zero.

We will show that if for some fixed $x_i$'s, $\beta_4 \neq 0$, $\beta_5 \neq 0$, and $\beta_6 = 0$ then for any $x_i$'s $\beta_4 \neq 0$, $\beta_5 \neq 0$, and $\beta_6 = 0$. If there are $y_i$'s ($y_i \neq 0$, $i = 1, 2, 3$) such that $\beta_6 \neq 0$, $\beta_4 \neq 0$, and $\beta_5 = 0$ (or $\beta_6 \neq 0$, $\beta_5 \neq 0$, and $\beta_4 = 0$), then for any $b \neq 0$ and $bx_i + y_i \neq 0$ ($i = 1, 2, 3$),
\[
T[E_{1,1} + (bx_1 + y_1)E_{1,2} + (bx_2 + y_2)E_{2,2} + (bx_3 + y_3)E_{3,3}]
\]
is a totally $L$-matrix. We can choose $b \neq 0$ such that
\[
|T[E_{1,1} + (bx_1 + y_1)E_{1,2} + (bx_2 + y_2)E_{2,2} + (bx_3 + y_3)E_{3,3}]| = 6,
\]
a contradiction. Without loss of generality, we assume for any \(x_i\)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5 \\
0
\end{pmatrix}
\]

with \(\beta_4 \neq 0\) and \(\beta_5 \neq 0\).

Also we can show that for any \(x_i\)'s \(\beta_3 = 0\). If there are \(x_i\)'s such that \(\beta_3 \neq 0\), then for any \(a \neq 0\), \(T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})\) is a totally \(L\)-matrix. We can choose \(a \neq 0\) such that \(a\beta_3 + \beta_3 = 0\). Hence for the choice of \(a\), \(T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})\) has zero 3rd column, a contradiction with \(T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3})\) is a totally \(L\)-matrix. Hence \(\beta_3 = 0\). Therefore, for any \(x_i\)'s

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix}
\beta_1 \\
\beta_2 \\
0 \\
\beta_4 \\
\beta_5 \\
0
\end{pmatrix}
\]

with \(\beta_4 \neq 0\) and \(\beta_5 \neq 0\).

Now, we show

\[
T(E_{2,1}) = \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\gamma_6
\end{pmatrix}
\]

with \(\gamma_3 = \gamma_6 = 0\).

Suppose not. Then at least one of \(\gamma_3\) and \(\gamma_6\) is not zero.

If \(\gamma_6 \neq 0\), then for any \(a > 0\) and \(c > 0\),

\[
T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})
\]

is a totally \(L\)-matrix. We can choose \(a > 0\) and \(c > 0\) such that

\[
|T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})| = 6.
\]
This is a contradiction with that

\[ T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1}) \]

is a totally \( L \)-matrix.

Now we suppose \( \gamma_3 \neq 0 \) and \( \gamma_6 = 0 \). Without loss of generality, we assume \( \alpha_3 > 0 \). If \( \gamma_3 > 0 \) and \( \gamma_6 = 0 \), then, for any \( a > 0 \),

\[ T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1}) \]

is a totally \( L \)-matrix. We can choose \( a > 0 \) such that \( a\alpha_3 - \gamma_3 = 0 \). Thus

\[ T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1}) \]

is a totally \( L \)-matrix which has zero 3\(^{rd} \) column, a contradiction. If \( \gamma_3 < 0 \) and \( \gamma_6 = 0 \), then we can choose \( a > 0 \), such that \( a\alpha_3 + \gamma_3 = 0 \). Thus

\[ T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1}) \]

is a totally \( L \)-matrix which has zero 3\(^{rd} \) column, a contradiction.

Therefore, \( \gamma_3 = \gamma_6 = 0 \). So

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & 0 \\ \gamma_4 & \gamma_5 & 0 \end{pmatrix}. \]

By the same argument as above, we have that

\[ T(E_{1,3}) = \begin{pmatrix} \eta_1 & \eta_2 & 0 \\ \eta_4 & \eta_5 & 0 \end{pmatrix}. \]

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which
has zero 3\(^{rd}\) column, a contradiction. Hence this case never arises.

Case 2. \(T(E_{1,1}) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & \alpha_6 \end{pmatrix} \) with \(\alpha_i \neq 0 (i = 1, 2, 6)\).

By the same argument as case 1, for any \(a \neq 0\) and \(x_i \neq 0\), we consider the totally \(L\)-matrix \(T(aE_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3})\). We can show that for any \(x_i\)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix}
\]

with at least one of \(\beta_3\), \(\beta_4\), and \(\beta_6\) is zero and at least one of \(\beta_4\) and \(\beta_5\) is nonzero.

First we show that for any \(x_i\)'s, \(\beta_6 = 0\). If there are \(x_i\)'s such that \(\beta_6 \neq 0\), then since \(\alpha_6 \neq 0\), we can choose \(a \neq 0\) such \(ax_6 + \beta_6 = 0\). Thus \(T(aE_{1,1} + x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3})\) is a totally \(L\)-matrix which has two zero entries on 3\(^{rd}\) column (if \(\beta_3 = 0\)) or second row (if \(\beta_4 = 0\) or \(\beta_5 = 0\)), a contradiction.

Now we show that if \(\beta_3(1,1,1) = 0\) then for any \(x_i\)'s \(\beta_3(x_1, x_2, x_3) = 0\).

If \(\beta_3(1,1,1) = 0\), \(\beta_4(1,1,1) \neq 0\), \(\beta_5(1,1,1) \neq 0\), and there are some \(x_i\)'s such that \(\beta_3(x_1, x_2, x_3) \neq 0\), then we can choose \(b \neq 0\) and \(b' \neq 0\) such that \(b + b'x_i \neq 0 (i = 1, 2, 3)\) and every entry of the totally \(L\)-matrix

\[
T(E_{1,1} + (b + b'x_1)E_{1,2} + (b + b'x_2)E_{2,2} + (b + b'x_3)E_{2,3})
\]

is not zero. This is a contradiction. If \(\beta_3(1,1,1) = 0\), \(\beta_4(1,1,1) = 0\), \(\beta_5(1,1,1) \neq 0\), and there are some \(x_i\)'s such that \(\beta_3(x_1, x_2, x_3) \neq 0\), \(\beta_4(x_1, x_2, x_3) \neq 0\), and \(\beta_5(x_1, x_2, x_3) = 0\), then we can choose \(b \neq 0\) and \(b' \neq 0\) such that \(b + b'x_i \neq 0 (i = 1, 2, 3)\) and every entry of the totally
is not zero. This is a contradiction. If \( \beta_3(1,1,1) = 0 \), \( \beta_4(1,1,1) = 0 \), 
\( \beta_5(1,1,1) \neq 0 \), and there are some \( x_i \)'s such that \( \beta_3(x_1,x_2,x_3) \neq 0 \), 
\( \beta_4(x_1,x_2,x_3) = 0 \), and \( \beta_5(x_1,x_2,x_3) \neq 0 \), then we can choose \( b \neq 0 \) and 
b' \neq 0 \) such that \( b + b'x_i \neq 0 \) and \( b\beta_5(1,1,1) + b'\beta_5(x_1,x_2,x_3) = 0 \). Thus 
the totally \( L \)-matrix 
\[ T[E_{1,1} + (b + x_1)E_{1,2} + (b + x_2)E_{2,2} + (b + x_3)E_{2,3}] \]

has two zero entries on the second row, a contradiction.

Similarly we can discuss every case. Hence if \( \beta_3(1,1,1) = 0 \) then for 
any \( x_i \)'s \( \beta_3(x_1,x_2,x_3) = 0 \).

Also we can prove that if \( \beta_3(1,1,1) \neq 0 \) then for any \( x_i \)'s 
\( \beta_3(x_1,x_2,x_3) \neq 0 \).

Subcase 1. \( \beta_3(1,1,1) = 0 \), \( \beta_4(1,1,1) \neq 0 \), and \( \beta_5(1,1,1) \neq 0 \).

In this case, we have that for any \( x_i \)'s 
\[ T(x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) = \begin{pmatrix} \beta_1 & \beta_2 & 0 \\ \beta_4 & \beta_5 & 0 \end{pmatrix} \]

with at least one of \( \beta_4 \) and \( \beta_5 \) is nonzero. First we can show that 
\( \beta_1 = \beta_2 = 0 \), since otherwise for \( a = \frac{\beta_1}{\alpha_1} \) or \( \frac{\beta_2}{\alpha_2} \)

\[ T(aE_{1,1} + x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) \]

has two zero entries on the first row, a contradiction.
Now, we show

$$T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}$$

with $\gamma_3 = \gamma_6 = 0$. Suppose not. Then at least one of $\gamma_3$ and $\gamma_6$ is not zero.

If $\gamma_3 \neq 0$, then for any $a > 0$ and $c < 0$,

$$T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})$$

is a totally $L$-matrix. We can choose $a > 0$ and $c < 0$ such that

$$|T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})| = 6,$$

a contradiction.

If $\gamma_6 \neq 0$ and $\gamma_3 = 0$, then either $\gamma_6 > 0$ or $\gamma_6 < 0$. Without loss of generality we assume $\alpha_3 > 0$. If $\gamma_6 > 0$, then for any $a > 0$,

$$T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{2,1})$$

is a totally $L$-matrix. We can choose $a$ such that $a\alpha_6 - \gamma_6 = 0$. Thus

$$T(aE_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{2,1})$$

is a totally $L$-matrix which has zero $3^{rd}$ column, a contradiction. If $\gamma_6 < 0$, then we can choose $a > 0$, such that $a\alpha_6 + \gamma_6 = 0$. Thus

$$T(aE_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + E_{2,1})$$

is a totally $L$-matrix which has zero $3^{rd}$ column, a contradiction.
Therefore, \( \gamma_3 = \gamma_6 = 0 \). So

\[
T(E_{2,1}) = \begin{pmatrix}
\gamma_1 & \gamma_2 & 0 \\
\gamma_4 & \gamma_5 & 0
\end{pmatrix} .
\]

By the same argument as above, we have that

\[
T(E_{1,3}) = \begin{pmatrix}
\eta_1 & \eta_2 & 0 \\
\eta_4 & \eta_5 & 0
\end{pmatrix} .
\]

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has zero 3\(^{rd}\) column, a contradiction.

**Subcase 2.** \( \beta_3 (1,1,1) = 0, \beta_4 (1,1,1) = 0, \) and \( \beta_5 (1,1,1) \neq 0 \).

In this case, we have that for any \( x_i \)'s

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix}
\beta_1 & \beta_2 & 0 \\
\beta_4 & \beta_5 & 0
\end{pmatrix}
\]

with at least one of \( \beta_4 \) and \( \beta_5 \) is nonzero. First as in subcase 1, we can show that \( \beta_1 = \beta_2 = 0 \).

First we will show for any \( x_i \)'s \( \beta_4 (x_1, x_2, x_3) = 0 \). Suppose there are \( x_i \)'s such that \( \beta_4 (x_1, x_2, x_3) \neq 0 \). Then, since for any \( x_i \)'s \( \beta_3 (x_1, x_2, x_3) = 0 \), we can choose \( b \neq 0 \) and \( b' \neq 0 \) such that \( b + b' x_i \neq 0 \) \((i = 1, 2, 3)\) and \( \beta_5 (b, b, b) + \beta_5 (b' x_1, b' x_2, b' x_3) \neq 0 \). Let \( y_i = b + b' x_i \) \((i = 1, 2, 3)\). We have \( y_i = 0, \beta_3 (y_1, y_2, y_3) = 0, \beta_4 (y_1, y_2, y_3) = 0 \).

\[
\beta_4 (y_1, y_2, y_3) = \beta_4 (b, b, b) + \beta_4 (b' x_1, b' x_2, b' x_3) \neq 0
\]

and
\[ \beta_5(y_1, y_2, y_3) = \beta_5(b, b, b) + \beta_5(b'x_1, b'x_2, b'x_3) \neq 0. \]

This is subcase 1 which never appears. Hence for any \( x \)'s, \[ \beta_4(x_1, x_2, x_3) = 0. \]

Therefore, for any \( x \)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{3,3}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_5 & 0 \end{pmatrix}
\]

with \( \beta_5 \neq 0. \)

Also we can show

\[
T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}
\]

with at least one of \( \gamma_3 \) and \( \gamma_4 \) is zero. If \( \gamma_3 = 0 \), then we have \( \gamma_1 = \gamma_2 = \gamma_6 = 0 \). If \( \gamma_4 = 0 \), then we have \( \gamma_1 = \gamma_5 = \gamma_6 = 0 \). Thus

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & \gamma_5 & 0 \end{pmatrix} \text{ or } T(E_{2,1}) = \begin{pmatrix} 0 & \gamma_2 & \gamma_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Similarly, we can show

\[
T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & \eta_5 & 0 \end{pmatrix} \text{ or } T(E_{1,3}) = \begin{pmatrix} 0 & \eta_2 & \eta_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

By the same argument of subcase 3 of case 3 of Lemma 6.3.5, we can show \( |T(E_{2,1})| \leq 1 \) and \( |T(E_{1,3})| \leq 1. \)

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix but

\[
|T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})|
\]
\[ \leq |T(E_{2,1})| + |T(E_{1,2} + E_{2,2} + E_{2,3})| + |T(E_{1,3})| \]

\[ \leq 1 + 1 + 1 = 3, \]

a contradiction. So this case never arises.

**Subcase 3.** \( \beta_3(1,1,1) = 0, \beta_4(1,1,1) \neq 0, \) and \( \beta_5(1,1,1) = 0. \)

This case is equivalent to the subcase 2. So this case never arises.

**Subcase 4.** \( \beta_3(1,1,1) \neq 0, \beta_4(1,1,1) \neq 0, \) and \( \beta_5(1,1,1) = 0. \)

First we can show for any \( x_i \)'s \( \beta_5(x_{1,i}x_{2,i}x_{3,i}) = 0 \) by the same argument as above. Then we can show \( \beta_2(x_{1,i}x_{2,i}x_{3,i}) = 0. \) Also since \( \beta_3(1,1,1) \neq 0 \) we have for any \( x_i \)'s \( \beta_3(x_{1,i}x_{2,i}x_{3,i}) \neq 0. \) Thus for any \( x_i \)'s

\[ T(x_{1,1}E_{1,2} + x_{2,1}E_{2,2} + x_{3,1}E_{2,3}) = \begin{pmatrix} \beta_1 & 0 & \beta_3 \\ \beta_4 & 0 & 0 \end{pmatrix} \]

with \( \beta_3 \neq 0. \)

Also we can show

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix} \]

with \( \gamma_5 = 0 \) since \( \beta_3(1,1,1) \neq 0 \) and \( \beta_4(1,1,1) \neq 0. \) Then we can show \( \gamma_6 = 0 \) and \( \gamma_2 = 0. \) Thus

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & 0 & \gamma_3 \\ \gamma_4 & 0 & 0 \end{pmatrix}. \]

Similarly, we have that

\[ T(E_{1,3}) = \begin{pmatrix} \eta_1 & 0 & \eta_3 \\ \eta_4 & 0 & 0 \end{pmatrix}. \]
Thus $T(E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})$ is a totally $L$-matrix which has zero second column, a contradiction. Hence this case never arises.

Subcase 5. $\beta_3(1,1,1) \neq 0$, $\beta_4(1,1,1) = 0$, and $\beta_5(1,1,1) \neq 0$.

This case is equivalent to the subcase 2. So this case never arises.

Therefore, case 2 never arises.

Case 3. $T(E_{1,1}) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_4 & \alpha_5 & 0 \end{pmatrix}$ with $\alpha \neq 0$ ($i = 1, 4, 5$).

First we show for any $x_i$'s at least one of $\beta_2$, $\beta_3$, and $\beta_6$ is zero, at least one of $\beta_2$ and $\beta_3$ is nonzero, and at least one of $\beta_3$ and $\beta_6$ is nonzero.

Subcase 1. $\beta_2(1,1,1) = 0$, $\beta_3(1,1,1) \neq 0$, and $\beta_6(1,1,1) \neq 0$.

First it easy to prove that for any $x_i$'s $\beta_2(x_1, x_2, x_3) = 0$ and $\beta_3(x_1, x_2, x_3) = 0$. Also we can prove for any $x_i$'s $\beta_3(x_1, x_2, x_3) \neq 0$ and $\beta_6(x_1, x_2, x_3) \neq 0$ by the same argument as above. Thus

$T(x E_{1,1} + x E_{2,2} + x E_{2,3}) = \begin{pmatrix} 0 & 0 & \beta_3 \\ \beta_4 & 0 & \beta_6 \end{pmatrix}$

with $\beta_3 \neq 0$ and $\beta_6 \neq 0$.

Also we can show

$T(E_{1,2}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}$

with $\gamma_2 = 0$ since $\beta_3 \neq 0$, and $\beta_6 \neq 0$. Then we have $\gamma_1 = \gamma_3 = \gamma_5 = 0$.

Thus we have that

$T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & 0 & \gamma_6 \end{pmatrix}$. 
Similarly, we have

\[ T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & 0 & \eta_6 \end{pmatrix}. \]

So \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has zero second column, a contradiction. Hence this case never arises.

**Subcase 2.** \( \beta_2(1,1,1) \neq 0, \beta_3(1,1,1) \neq 0, \) and \( \beta_6(1,1,1) = 0. \)

This case is equivalent to the subcase 1. So this case never arises.

**Subcase 3.** \( \beta_2(1,1,1) = 0, \beta_3(1,1,1) \neq 0, \) and \( \beta_6(1,1,1) = 0. \)

First we can see that for any \( x_i \)'s \( \beta_3(x_1, x_2, x_3) \neq 0. \) Also for any \( x_i \)'s \( \beta_6(x_1, x_2, x_3) = 0. \) If for some \( x_i \)'s \( \beta_3(x_1, x_2, x_3) \neq 0, \) then since \( \beta_3(x_1, x_2, x_3) \neq 0 \) we must have \( \beta_2(x_1, x_2, x_3) = 0. \) This is subcase 1 which never appears. Similarly, for any \( x_i \)'s \( \beta_2(x_1, x_2, x_3) = 0. \) Also we can show

\[ \beta_3(x_1, x_2, x_3) = \beta_4(x_1, x_2, x_3) = \beta_5(x_1, x_2, x_3) = 0. \]

Thus for any \( x_i \)'s

\[ T(x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) = \begin{pmatrix} 0 & 0 & \beta_3 \\ 0 & 0 & 0 \end{pmatrix} \]

with \( \beta_3 \neq 0. \)

Also we can show

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix} \]

with at least one of \( \gamma_2 \) and \( \gamma_6 \) is zero since \( \beta_3 \neq 0. \) If \( \gamma_2 = 0, \) then we
have \( y_1 = y_3 = y_5 = 0 \). Thus

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ y_4 & 0 & y_6 \end{pmatrix}.
\]

If \( y_6 = 0 \), then we have \( y_3 = y_4 = y_5 = 0 \). Thus

\[
T(E_{2,1}) = \begin{pmatrix} y_1 & y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Similarly, we have

\[
T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & 0 & \eta_6 \end{pmatrix} \quad \text{or} \quad T(E_{1,3}) = \begin{pmatrix} \eta_1 & \eta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

By the same argument of subcase 3 of case 3 of Lemma 6.3.5, we can show \( |T(E_{2,1})| \leq 1 \) and \( |T(E_{1,3})| \leq 1 \).

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix but

\[
|T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})| \\
\leq |T(E_{2,1})| + |T(E_{1,2} + E_{2,2} + E_{2,3})| + |T(E_{1,3})| \\
\leq 1 + 1 + 1 = 3,
\]

a contradiction. So this case never arises.

**Subcase 4.** \( \beta_2(1,1,1) \neq 0, \beta_3(1,1,1) = 0, \) and \( \beta_6(1,1,1) \neq 0 \).

First we can see that for any \( x_i \)'s \( \beta_3(x_1, x_2, x_3) = 0 \). Thus for any \( x_i \)'s we must have that \( \beta_2(x_1, x_2, x_3) \neq 0 \) and \( \beta_6(x_1, x_2, x_3) \neq 0 \) by the same argument as above. Then we can prove for any \( x_i \)'s \( \beta_1(x_1, x_2, x_3) = 0 \). Thus for any \( x_i \)'s,
\[ T(xE_{1,2} + xE_{2,2} + xE_{3,2}) = \begin{pmatrix} 0 & \beta_2 & 0 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix} \]

with \( \beta_2 \neq 0 \) and \( \beta_6 \neq 0 \).

Also we can show

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix} \]

with \( \gamma_3 = 0 \) since \( \beta_3 = 0 \), \( \beta_2 \neq 0 \), and \( \beta_6 \neq 0 \). Then we have

\( \gamma_1 = \gamma_2 = \gamma_6 = 0 \). Thus we have

\[ T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & \gamma_5 & 0 \end{pmatrix}. \]

Similarly, we have

\[ T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & \eta_5 & 0 \end{pmatrix}. \]

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has two zero entries on the first row, a contradiction. So this case never arises.

Therefore, case 3 never arises. The proof is complete. \( \blacksquare \)

Lemma 6.3.6 For any \( i \) and \( j \), \( |T(E_{i,j})| \neq 2 \).

Proof. Suppose there are \( i \) and \( j \) such that \( |T(E_{i,j})| = 2 \). Without loss of generality, we assume \( |T(E_{1,1})| = 2 \).

Case 1. \( T(E_{1,1}) = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) with \( \alpha \neq 0 \) (\( i = 1, 2 \)).
First it easy to see that for any \( x_i \)'s,

\[
T(x_1 E_{1,1} + x_2 E_{2,2} + x_3 E_{3,3}) = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 \\
\beta_4 & \beta_5 & \beta_6
\end{pmatrix}
\]

with at least one of \( \beta_3, \beta_4, \beta_5 \), and \( \beta_6 \) is zero, at least two of \( \beta_4, \beta_5 \), and \( \beta_6 \) are nonzero, and at least one of \( \beta_3 \) and \( \beta_6 \) is nonzero.

We will show that if for some \( x_i \)'s \( \beta_6(x_1, x_2, x_3) = 0 \), then for any \( x_i \)'s, \( \beta_6(x_1, x_2, x_3) = 0 \), \( \beta_4(x_1, x_2, x_3) \neq 0 \), and \( \beta_5(x_1, x_2, x_3) \neq 0 \). If not, then there are \( x_i \)'s and \( y_i \)'s such that \( \beta_6(x_1, x_2, x_3) = 0 \) and \( \beta_6(y_1, y_2, y_3) \neq 0 \). Since \( \beta_6(x_1, x_2, x_3) = 0 \), we must have \( \beta_3(x_1, x_2, x_3) = 0 \), \( \beta_4(x_1, x_2, x_3) \neq 0 \), and \( \beta_5(x_1, x_2, x_3) \neq 0 \). Thus we can choose \( b \neq 0 \) and \( b' \neq 0 \) such that \( bx_i + b'y_i \neq 0 \) (\( i = 1, 2, 3 \)) and

\[
|T(E_{1,1} + (bx_1 + b'y_1)E_{1,2} + (bx_2 + b'y_2)E_{2,2} + (bx_3 + b'y_3)E_{3,3})| = 6,
\]

a contradiction. So for any \( x_i \)'s \( \beta_6(x_1, x_2, x_3) = 0 \). Furthermore,

\[
\beta_3(x_1, x_2, x_3) \neq 0, \beta_4(x_1, x_2, x_3) \neq 0, \text{ and } \beta_5(x_1, x_2, x_3) \neq 0.
\]

Subcase 1. \( \beta_6(1,1,1) = 0 \).

In this case for any \( x_i \)'s \( \beta_6(x_1, x_2, x_3) = 0 \), \( \beta_3(x_1, x_2, x_3) \neq 0 \), \( \beta_4(x_1, x_2, x_3) \neq 0 \), and \( \beta_5(x_1, x_2, x_3) \neq 0 \) by above. We will show

\[
\beta_1(x_1, x_2, x_3) = \beta_2(x_1, x_2, x_3) = 0.
\]

Since for any \( a \neq 0 \) and \( b \neq 0 \),

\[
T(aE_{1,1} + bx_1 E_{1,2} + bx_2 E_{2,2} + bx_3 E_{3,3}) = \begin{pmatrix}
a\alpha_1 + b\beta_1 \\
a\alpha_2 + b\beta_2 \\
\beta_4 \\
\beta_5 \\
\beta_3
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a\alpha_1 + b\beta_1 \\
\alpha_2 + b\beta_2 \\
\beta_4 \\
\beta_5 \\
0
\end{pmatrix}
\]

...
is a totally $L$-matrix, we have that

\[
\begin{pmatrix}
a_1 x_1 + b_1 y_1 & a_2 x_2 + b_2 y_2 \\
b_1 y_1 & b_2 y_2 
\end{pmatrix}
\]

is a sign-nonsingular matrix for any $a \neq 0$ and $b \neq 0$. If there exist $x_i$'s such that either $\beta_1(x_1, x_2, x_3) \neq 0$ or $\beta_2(x_1, x_2, x_3) \neq 0$, then by the same argument as case 3 of Lemma 4.5.4 we get a contradiction. Thus for any $x_i$'s we have

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix}
0 & 0 & \beta_3 \\
\beta_4 & \beta_5 & 0
\end{pmatrix}
\]

with $\beta_3 \neq 0$, $\beta_4 \neq 0$, and $\beta_5 \neq 0$.

Also we can show

\[
T(E_{2,1}) = \begin{pmatrix}
\gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_4 & \gamma_5 & \gamma_6
\end{pmatrix}
\]

with $\gamma_6 = 0$. Then we have $\gamma_3 = \gamma_4 = \gamma_5 = 0$. Thus we have

\[
T(E_{2,1}) = \begin{pmatrix}
\gamma_1 & \gamma_2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Similarly, we have

\[
T(E_{1,3}) = \begin{pmatrix}
\eta_1 & \eta_2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Now we show that $\gamma_1 = \gamma_2 = \eta_1 = \eta_2 = 0$. If $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$, then since for any $a \neq 0$, $c \neq 0$, and $ac > 0$,

\[
T(a E_{1,1} - E_{1,2} + E_{2,2} + E_{2,3} + c E_{2,1}) = \begin{pmatrix}
a_1 x_1 + c y_1 & a_2 x_2 + c y_2 & \beta_3 \\
b_1 y_1 & b_2 y_2 & \beta_5
\end{pmatrix}
\]


is a totally $L$-matrix, we have that for any $a \neq 0$, $b \neq 0$, and $ac > 0$,

$$\begin{pmatrix} a \alpha + c \gamma_1 & a \alpha + c \gamma_2 \\ \beta_4 & \beta_5 \end{pmatrix}$$

is a sign-nonsingular matrix. If $\beta_4 \beta_5 > 0$, then we can choose $a \neq 0$, $c \neq 0$, and $ac > 0$ such that $a \alpha_1 + c \gamma_1 > 0$ and $a \alpha_2 + c \gamma_2 > 0$ (or $a \alpha_1 + c \gamma_1 < 0$ and $a \alpha_2 + c \gamma_2 < 0$). Thus the sign pattern of

$$\begin{pmatrix} a \alpha + c \gamma_1 & a \alpha + c \gamma_2 \\ \beta_4 & \beta_5 \end{pmatrix}$$

is equivalent to $\pm\hyph$ or $\pm\hyph\hyph\hyph$, a contradiction. If $\beta_4 \beta_5 < 0$, then we can choose $a \neq 0$, $c \neq 0$, and $ac > 0$ such that $a \alpha_1 + c \gamma_1 > 0$ and $a \alpha_2 + c \gamma_2 > 0$ (or $a \alpha_1 + c \gamma_1 < 0$ and $a \alpha_2 + c \gamma_2 < 0$). Thus the sign pattern of

$$\begin{pmatrix} a \alpha + c \gamma_1 & a \alpha + c \gamma_2 \\ \beta_4 & \beta_5 \end{pmatrix}$$

is equivalent to $\pm\hyph\hyph\hyph$ or $\pm\hyph\hyph\hyph\hyph$, a contradiction. Thus $\gamma_1 = \gamma_2 = 0$.

Similarly, we have $\eta_1 = \eta_2 = 0$. Hence $T(E_{2,1}) = T(E_{1,3}) = 0$.

Thus

$$T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})$$

$$= T(E_{1,2} + E_{2,2} + E_{2,3}) = \begin{pmatrix} 0 & 0 & \beta_3 \\ \beta_4 & \beta_5 & 0 \end{pmatrix}$$

is a totally $L$-matrix, a contradiction. So this case never arises.

Now we suppose $\beta_6(1,1,1) \neq 0$. First we can see for any $x_i$'s $\beta_6(x_1,x_2,x_3) \neq 0$. Because if there are $x_i$'s such that $\beta_6(x_1,x_2,x_3) = 0$, then this is equivalent to the subcase 1 which never arises. Then we can show that for any $x_i$'s, if $\beta_6(x_1,x_2,x_3) \neq 0$, then
\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 \\
\beta_4 & \beta_5 & \beta_6
\end{pmatrix}
\]

with at least one of \(\beta_3\), \(\beta_4\), and \(\beta_5\) is zero and at least one of \(\beta_4\) and \(\beta_5\) is nonzero.

**Subcase 2.** \(\beta_6(1,1,1) \neq 0, \ \beta_3(1,1,1) = 0, \ \beta_4(1,1,1) \neq 0,\) and \(\beta_5(1,1,1) \neq 0\).

First we can show for any \(x_i\)'s \(\beta_3(x_1, x_2, x_3) = 0\). Thus for any \(x_i\)'s \(\beta_6(x_1, x_2, x_3) \neq 0, \beta_1(x_1, x_2, x_3) = \beta_2(x_1, x_2, x_3) = 0\). So for any \(x_i\)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix}
0 & 0 & 0 \\
\beta_4 & \beta_5 & \beta_6
\end{pmatrix}
\]

with \(\beta_6 \neq 0\).

Also we can show

\[
T(E_{2,1}) = \begin{pmatrix}
\gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_4 & \gamma_5 & \gamma_6
\end{pmatrix}
\]

with \(\gamma_3 = 0\) since \(\beta_6(1,1,1) \neq 0, \ \beta_3(1,1,1) = 0, \ \beta_4(1,1,1) \neq 0,\) and \(\beta_5(1,1,1) \neq 0\). Then we have \(\gamma_1 = \gamma_2 = 0\). Thus we have

\[
T(E_{2,1}) = \begin{pmatrix}
0 & 0 & 0 \\
\gamma_4 & \gamma_5 & \gamma_6
\end{pmatrix}.
\]

Similarly, we have

\[
T(E_{1,3}) = \begin{pmatrix}
0 & 0 & 0 \\
\eta_4 & \eta_5 & \eta_6
\end{pmatrix}.
\]

Thus \(T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})\) is a totally \(L\)-matrix which has zero first row, a contradiction. So this case never arises.

**Subcase 3.** \(\beta_6(1,1,1) \neq 0, \ \beta_3(1,1,1) = 0, \ \beta_4(1,1,1) = 0,\) and
First we show for any $x_i$'s $\beta_3(x_1, x_2, x_3) = 0$. If there are some $x_i$'s such that $\beta_3(x_1, x_2, x_3) \neq 0$, $\beta_4(x_1, x_2, x_3) \neq 0$, and $\beta_5(x_1, x_2, x_3) = 0$, then we can choose $b \neq 0$ and $b' \neq 0$ such that $b + b'x_i \neq 0$ $(i = 1, 2, 3)$ and every entry of the totally $L$-matrix

$$T[E_{1,1} + (b + x_1)E_{1,2} + (b + x_2)E_{2,2} + (b + x_3)E_{3,2}]$$

is not zero. This is a contradiction. If there are some $x_i$'s such that $\beta_3(x_1, x_2, x_3) \neq 0$, $\beta_4(x_1, x_2, x_3) = 0$, and $\beta_5(x_1, x_2, x_3) \neq 0$, then we can choose $b \neq 0$ and $b' \neq 0$ such that $b + b'x_i \neq 0$ and $b\beta_5(1,1,1) + b'\beta_5(x_1, x_2, x_3) = 0$. Thus the totally $L$-matrix

$$T[E_{1,1} + (b + b'x_1)E_{1,2} + (b + b'x_2)E_{2,2} + (b + b'x_3)E_{3,2}]$$

has two zero entries on the second row, a contradiction.

Hence for any $x_i$'s

$$T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{3,2}) = \begin{pmatrix} \beta_1 & \beta_2 & 0 \\ \beta_4 & \beta_5 & 0 \end{pmatrix}$$

with at least one of $\beta_4$ and $\beta_5$ is nonzero. From this we can obtain that $\beta_1 = \beta_2 = 0$. Now we will show for any $x_i$'s $\beta_4(x_1, x_2, x_3) = 0$. Suppose there are $x_i$'s such that $\beta_4(x_1, x_2, x_3) \neq 0$. Then, since for any $x_i$'s $\beta_3(x_1, x_2, x_3) = 0$, we can choose $b \neq 0$ and $b' \neq 0$ such that $b + b'x_i \neq 0$ $(i = 1, 2, 3)$ and $\beta_5(b, b, b) + \beta_5(b'x_1, b'x_2, b'x_3) \neq 0$. Let $y_i = b + b'x_i$ $(i = 1, 2, 3)$. We have that $y_i \neq 0$, $\beta_3(y_1, y_2, y_3) = 0$, $\beta_4(y_1, y_2, y_3) = \beta_4(b, b, b) + \beta_4(b'x_1, b'x_2, b'x_3) \neq 0$, and $\beta_5(y_1, y_2, y_3) = \beta_5(b, b, b) + \beta_5(b'x_1, b'x_2, b'x_3) \neq 0$. This is equivalent to the subcase 2 which never
arises. Hence for any \( x_i \)'s \( \beta_i(x_1, x_2, x_3) = 0 \). Therefore, for any \( x_i \)'s

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{3,3}) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \beta_5 & \beta_6
\end{pmatrix}
\]

with \( \beta_5 \neq 0 \) and \( \beta_6 \neq 0 \).

Also we can show

\[
T(E_{2,1}) = \begin{pmatrix}
\gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_4 & \gamma_5 & \gamma_6
\end{pmatrix}
\]

with at least one of \( \gamma_3 \) and \( \gamma_4 \) is zero. If \( \gamma_3 = 0 \), then we have \( \gamma_1 = \gamma_2 = \gamma_6 = 0 \). If \( \gamma_4 = 0 \), then we have \( \gamma_1 = \gamma_5 = \gamma_6 = 0 \). Thus

\[
T(E_{2,1}) = \begin{pmatrix}
0 & 0 & 0 \\
\gamma_4 & \gamma_5 & 0
\end{pmatrix}
\quad \text{or} \quad
T(E_{2,1}) = \begin{pmatrix}
0 & \gamma_2 & \gamma_3 \\
0 & 0 & 0
\end{pmatrix}
\]

Similarly, we can show

\[
T(E_{1,3}) = \begin{pmatrix}
0 & 0 & 0 \\
\eta_4 & \eta_5 & 0
\end{pmatrix}
\quad \text{or} \quad
T(E_{1,3}) = \begin{pmatrix}
0 & \eta_2 & \eta_3 \\
0 & 0 & 0
\end{pmatrix}
\]

By the same argument of subcase 3 of case 3 of Lemma 6.3.5, we can show \( |T(E_{2,1})| \leq 1 \) and \( |T(E_{1,3})| \leq 1 \).

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{3,3} - E_{1,3}) \) is a totally \( L \)-matrix which has two zero entries on the first row (or first column), a contradiction. So this case never arises.

Subcase 4. \( \beta_6(1,1,1) \neq 0, \quad \beta_3(1,1,1) = 0, \quad \beta_4(1,1,1) \neq 0, \) and \( \beta_5(1,1,1) = 0 \).

This case is equivalent to the subcase 3. So this case never arises.

Subcase 5. \( \beta_6(1,1,1) \neq 0, \quad \beta_3(1,1,1) \neq 0, \quad \beta_4(1,1,1) \neq 0, \) and
$\beta_5(1,1,1) = 0.$

First we can show for any $x_i$'s $\beta_5(x_1, x_2, x_3) = 0$ by the same argument as above. Then we can show $\beta_3(x_2, x_2, x_3) = 0$. Also since $\beta_3(1,1,1) \neq 0$ we have for any $x_i$'s $\beta_3(x_1, x_2, x_3) \neq 0$; otherwise, it is the subcase 2, 3 or 4 which never appear. Thus for any $x_i$'s

$$T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix} \beta_1 & 0 & \beta_3 \\ \beta_4 & 0 & \beta_6 \end{pmatrix}$$

with $\beta_3 \neq 0$.

Also we can show

$$T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}$$

with $\gamma_5 = 0$ since $\beta_6(1,1,1) \neq 0$, $\beta_3(1,1,1) \neq 0$, and $\beta_4(1,1,1) \neq 0$. Then we can show $\gamma_6 = 0$ and $\gamma_2 = 0$. Thus

$$T(E_{2,1}) = \begin{pmatrix} \gamma_1 & 0 & \gamma_3 \\ \gamma_4 & 0 & 0 \end{pmatrix}.$$  

Similarly, we have

$$T(E_{1,3}) = \begin{pmatrix} \eta_1 & 0 & \eta_3 \\ \eta_4 & 0 & 0 \end{pmatrix}.$$  

Thus $T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})$ is a totally $L$-matrix which has zero second column, a contradiction. Hence this case never arises.

Subcase 6. $\beta_6(1,1,1) \neq 0$, $\beta_3(1,1,1) \neq 0$, $\beta_4(1,1,1) = 0$, and $\beta_5(1,1,1) \neq 0$.

This case is equivalent to the subcase 5. So this case never arises.
Therefore, case 1 never arises.

Case 2. \( T(E_{1,1}) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_4 & 0 & 0 \end{pmatrix} \) with \( \alpha \neq 0 \) and \( \alpha_4 \neq 0 \).

First we can show for any \( x_i \)'s at least one of \( \beta_2, \beta_3, \beta_5, \) and \( \beta_6 \) is zero, at least one of \( \beta_2 \) and \( \beta_3 \) is nonzero, at least one of \( \beta_5 \) and \( \beta_6 \) is nonzero, and at least one of \( \beta_3 \) and \( \beta_6 \) is nonzero.

Subcase 1. \( \beta_2(1,1,1) \neq 0, \beta_3(1,1,1) \neq 0, \beta_5(1,1,1) = 0, \) and \( \beta_6(1,1,1) \neq 0. \)

First it easy to see that for any \( x_i \)'s \( \beta_5(x_1, x_2, x_3) = 0. \) Thus for any \( x_i \)'s

\[
T(x_1 E_{1,1} + x_2 E_{2,2} + x_3 E_{3,3}) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & \beta_6 \end{pmatrix}.
\]

Also we can show

\[
T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}
\]

with \( \gamma_5 = 0 \) since \( \beta_2(1,1,1) \neq 0, \beta_3(1,1,1) \neq 0, \beta_5(1,1,1) = 0, \) and \( \beta_6(1,1,1) \neq 0. \) Then we have \( \gamma_2 = \gamma_4 = \gamma_6 = 0. \) Thus

\[
T(E_{2,1}) = \begin{pmatrix} \gamma_1 & 0 & \gamma_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Similarly, we have

\[
T(E_{1,3}) = \begin{pmatrix} \eta_1 & 0 & \eta_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]
So $T(E_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3})$ is a totally $L$-matrix which has two zero entries on the second row, a contradiction. Hence this case never arises.

Subcase 2. $\beta_2(1,1,1) \neq 0$, $\beta_3(1,1,1) = 0$, $\beta_5(1,1,1) \neq 0$, and $\beta_6(1,1,1) \neq 0$.

This case is equivalent to the subcase 1. So this case never arises.

Subcase 3. $\beta_2(1,1,1) \neq 0$, $\beta_3(1,1,1) = 0$, $\beta_5(1,1,1) = 0$, and $\beta_6(1,1,1) \neq 0$.

First we will show for any $x_i$'s that $\beta_3(x_1, x_2, x_3) = 0$ and $\beta_5(x_1, x_2, x_3) = 0$. If there are $x_i$'s such that $\beta_3(x_1, x_2, x_3) \neq 0$ and $\beta_5(x_1, x_2, x_3) \neq 0$, then we can choose $b \neq 0$ and $b' \neq 0$ such that $b + b'x_i \neq 0$ (for $i = 1, 2, 3$) and every entry of the totally $L$-matrix

$$T[E_{1,1} + (b + b'x_1)E_{1,2} + (b + b'x_2)E_{2,2} + (b + b'x_3)E_{3,2}]$$

is not zero. We obtain a contradiction. If there are $x_i$'s such that $\beta_3(x_1, x_2, x_3) \neq 0$ and $\beta_5(x_1, x_2, x_3) = 0$, then we can choose $b \neq 0$ and $b' \neq 0$ such that $b + b'x_i \neq 0$ (for $i = 1, 2, 3$) and $\beta_j(b, b, b) + \beta_j(b'x_1, b'x_2, b'x_3) \neq 0$ (for $j = 2, 3, 6$). Let $y_i = b + b'x_i$ (for $i = 1, 2, 3$). We get subcase 1 which never arises. Similarly, if there are $x_i$'s such that $\beta_3(x_1, x_2, x_3) = 0$ and $\beta_5(x_1, x_2, x_3) \neq 0$, then we get subcase 2 which never arises. So for any $x_i$'s, $\beta_3(x_1, x_2, x_3) = 0$ and $\beta_5(x_1, x_2, x_3) = 0$.

Then we have for any $x_i$'s $\beta_4(x_1, x_2, x_3) = \beta_5(x_1, x_2, x_3) = 0$. So for any $x_i$'s,

$$T(x_1E_{1,1} + x_2E_{2,2} + x_3E_{3,3}) = \begin{pmatrix} 0 & \beta_2 & 0 \\ 0 & 0 & \beta_6 \end{pmatrix}.$$
Also we can show

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix} \]

with either \( \gamma_3 = 0 \) or \( \gamma_5 = 0 \) since \( \beta_2(1,1,1) \neq 0 \), \( \beta_3(1,1,1) = 0 \), \( \beta_5(1,1,1) = 0 \), and \( \beta_6(1,1,1) \neq 0 \). If \( \gamma_3 = 0 \), then we have \( \gamma_1 = \gamma_2 = \gamma_6 = 0 \). If \( \gamma_5 = 0 \), then we have \( \gamma_2 = \gamma_4 = \gamma_6 = 0 \). Thus

\[ T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & \gamma_5 & 0 \end{pmatrix} \text{ or } T(E_{2,1}) = \begin{pmatrix} \gamma_1 & 0 & \gamma_3 \\ 0 & 0 & 0 \end{pmatrix}. \]

Similarly, we have

\[ T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & \eta_5 & 0 \end{pmatrix} \text{ or } T(E_{1,3}) = \begin{pmatrix} \eta_1 & 0 & \eta_3 \\ 0 & 0 & 0 \end{pmatrix}. \]

If \( |T(E_{2,1})| = 2 \), then since two nonzero entries of \( T(E_{2,1}) \) are on the same row, by case 1 we obtain a contradiction. So \( |T(E_{2,1})| \leq 1 \). Similarly, we must have \( |T(E_{1,3})| \leq 1 \). Thus

\[ T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & 0 & 0 \end{pmatrix}, \]

\[ T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_5 & 0 \end{pmatrix}, \]

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

or

\[ T(E_{2,1}) = \begin{pmatrix} 0 & 0 & \gamma_3 \\ 0 & 0 & 0 \end{pmatrix}. \]
and

\[ T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & 0 & 0 \end{pmatrix}, \]

\[ T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_5 & 0 \end{pmatrix}, \]

\[ T(E_{1,3}) = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

or

\[ T(E_{1,3}) = \begin{pmatrix} 0 & 0 & \eta_3 \\ 0 & 0 & 0 \end{pmatrix}. \]

We will show that if

\[ T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & 0 & 0 \end{pmatrix}. \]

then \( \gamma_4 = 0. \) We consider matrix

\[ T(aE_{1,1} + E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3}) \]

(or \( T(aE_{1,1} - E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3}) \)),

and choose \( a \neq 0 \) and \( c \neq 0 \) such that

\[ aE_{1,1} + E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3} \]

(or \( aE_{1,1} - E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3} \))

is a totally \( L \)-matrix and \( a\gamma_4 + c\gamma_4 = 0. \) Thus
\[ T(aE_{1,1} + E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3}) \]

(or \[ T(aE_{1,1} - E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3}) \])

is a totally \( L \)-matrix and

\[ |T(aE_{1,1} + E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3})| \leq 3 \]

(or \[ |T(aE_{1,1} - E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} + cE_{1,3})| \leq 3 \),

a contradiction with Lemma 4.5.1. Similarly, we can show if

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

then \( \gamma_1 = 0 \), if

\[ T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & 0 & 0 \end{pmatrix}, \]

then \( \eta_4 = 0 \), and if

\[ T(E_{1,3}) = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

then \( \eta_1 = 0 \).

Thus \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has zero first column), a contradiction. So this case never arises.

Subcase 4. \( \beta_2(1,1,1) = 0, \beta_3(1,1,1) \neq 0, \beta_5(1,1,1) \neq 0, \) and \( \beta_6(1,1,1) \neq 0. \)

Subcase 5. \( \beta_2(1,1,1) \neq 0, \beta_3(1,1,1) \neq 0, \beta_5(1,1,1) \neq 0, \) and \( \beta_6(1,1,1) = 0. \)
Subcase 6. \( \beta_2(1,1,1) = 0, \quad \beta_3(1,1,1) \neq 0, \quad \beta_5(1,1,1) \neq 0, \quad \text{and} \quad \beta_6(1,1,1) = 0. \)

These cases are equivalent to the subcase 1 or 3. So these cases never arise. Therefore, case 2 never arises.

Case 3. \( T(E_{1,1}) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix} \) with \( \alpha \neq 0 \) and \( \alpha \neq 0. \)

First we can show for any \( x_i \)'s at least one of \( \beta_2, \beta_3, \beta_4, \) and \( \beta_6 \) is zero, at least one of \( \beta_2 \) and \( \beta_3 \) is nonzero, at least one of \( \beta_4 \) and \( \beta_6 \) is nonzero, and at least one of \( \beta_3 \) and \( \beta_6 \) is nonzero.

Subcase 1. \( \beta_2(1,1,1) \neq 0, \quad \beta_3(1,1,1) = 0, \quad \beta_4(1,1,1) \neq 0, \quad \text{and} \quad \beta_6(1,1,1) \neq 0. \)

First it easy to see that for any \( x_i \)'s \( \beta_3(x_1,x_2,x_3) = 0. \) Thus for any \( x_i \)'s, \( \beta_1(x_1,x_2,x_3) = 0. \) So for any \( x_i \)'s,

\[
T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{3,3}) = \begin{pmatrix} 0 & \beta_2 & 0 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix}.
\]

Also we can show

\[
T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}
\]

with \( \gamma_3 = 0 \) since \( \beta_2(1,1,1) \neq 0, \quad \beta_3(1,1,1) = 0, \quad \beta_4(1,1,1) \neq 0, \quad \text{and} \quad \beta_6(1,1,1) \neq 0. \) Then we have \( \gamma_1 = \gamma_2 = \gamma_6 = 0. \) Thus

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}.
\]

Similarly, we have
\[ T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & \eta_5 & \eta_6 \end{pmatrix}. \]

So \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has two zero entries on the first row, a contradiction. Hence this case never arises.

**Subcase 2.** \( \beta_2(1,1,1) \neq 0, \beta_3(1,1,1) \neq 0, \beta_4(1,1,1) = 0, \) and \( \beta_6(1,1,1) \neq 0. \)

First it easy to see that for any \( x_i \)'s \( \beta_4(x_1, x_2, x_3) = 0. \) Thus for any \( x_i \)'s, \( \beta_1(x_1, x_2, x_3) = 0. \) So for any \( x_i \)'s,

\[ T(x_1 E_{1,2} + x_2 E_{2,2} + x_3 E_{2,3}) = \begin{pmatrix} 0 & \beta_2 & \beta_3 \\ 0 & \beta_5 & \beta_6 \end{pmatrix}. \]

Also we can show

\[ T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix} \]

with \( \gamma_4 = 0 \) since \( \beta_2(1,1,1) \neq 0, \beta_3(1,1,1) \neq 0, \beta_4(1,1,1) = 0, \) and \( \beta_6(1,1,1) \neq 0. \) Then we have \( \gamma_1 = 0. \) Thus

\[ T(E_{2,1}) = \begin{pmatrix} 0 & \gamma_2 & \gamma_3 \\ 0 & \gamma_5 & \gamma_6 \end{pmatrix}. \]

Similarly,

\[ T(E_{1,3}) = \begin{pmatrix} 0 & \eta_2 & \eta_3 \\ 0 & \eta_5 & \eta_6 \end{pmatrix}. \]

So \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix which has a zero first column, a contradiction. Hence this case never arises.
Subcase 3. \( \beta_2(1,1,1) \neq 0, \quad \beta_3(1,1,1) = 0, \quad \beta_4(1,1,1) = 0, \) and \( \beta_6(1,1,1) \neq 0. \)

First we will show for any \( x_i \)'s \( \beta_3(x_1,x_2,x_3) = 0 \) and \( \beta_4(x_1,x_2,x_3) = 0. \) If there are \( x_i \)'s such that \( \beta_3(x_1,x_2,x_3) \neq 0 \) and \( \beta_4(x_1,x_2,x_3) \neq 0, \) then we can choose \( b \neq 0 \) and \( b' \neq 0 \) such that \( b + b'x_i \neq 0 \) \( (i = 1, 2, 3) \) and every entry of the totally \( L \)-matrix

\[
T[E_{1,1} + (b + b'x_1)E_{1,2} + (b + b'x_2)E_{2,2} + (b + b'x_3)E_{2,3}]
\]

is not zero. We obtain a contradiction. If there are \( x_i \)'s such that \( \beta_3(x_1,x_2,x_3) = 0 \) and \( \beta_4(x_1,x_2,x_3) \neq 0, \) then we can choose \( b \neq 0 \) and \( b' \neq 0 \) such that \( b + b'x_i \neq 0 \) \( (i = 1, 2, 3) \) and \( \beta_j(b,b,b) + \beta_j(b'x_1,b'x_2,b'x_3) \neq 0, \) \( (j = 2, 3, 6). \) Let \( y_i = b + b'x_i \) \( (i = 1, 2, 3). \) We get subcase 1 which never arises.

Similarly, if there are \( x_i \)'s such that \( \beta_3(x_1,x_2,x_3) \neq 0 \) and \( \beta_4(x_1,x_2,x_3) = 0, \) then we get subcase 2 which never arises. So for any \( x_i \)'s \( \beta_3(x_1,x_2,x_3) = 0 \) and \( \beta_4(x_1,x_2,x_3) = 0. \) Also we can show for any \( x_i \)'s \( \beta_2(x_1,x_2,x_3) \neq 0 \) and \( \beta_6(x_1,x_2,x_3) \neq 0. \)

Then we have for any \( x_i \)'s \( \beta_1(x_1,x_2,x_3) = \beta_5(x_1,x_2,x_3) = 0. \) So for any \( x_i \)'s,

\[
T(x_1E_{1,2} + x_2E_{2,2} + x_3E_{2,3}) = \begin{pmatrix} 0 & \beta_2 & 0 \\ 0 & 0 & \beta_6 \end{pmatrix}
\]

with \( \beta_2 \neq 0 \) and \( \beta_6 \neq 0. \)

It easy to see that

\[
T(E_{2,1}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}
\]
with either \( \gamma_3 = 0 \) or \( \gamma_4 = 0 \) since \( \beta_2 \neq 0, \beta_3 = 0, \beta_4 = 0, \) and \( \beta_6 \neq 0 \). If \( \gamma_3 = 0 \), then we have \( \gamma_1 = \gamma_2 = \gamma_6 = 0 \). If \( \gamma_4 = 0 \), then we have \( \gamma_1 = \gamma_5 = \gamma_6 = 0 \). Thus

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & \gamma_5 & 0 \end{pmatrix} \quad \text{or} \quad T(E_{2,1}) = \begin{pmatrix} 0 & \gamma_2 & \gamma_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Similarly,

\[
T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & \eta_5 & 0 \end{pmatrix} \quad \text{or} \quad T(E_{1,3}) = \begin{pmatrix} 0 & \eta_2 & \eta_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

By case 1 we know that \( |T(E_{2,1})| \leq 1 \) and \( |T(E_{1,3})| \leq 1 \). Also since \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{2,3} - E_{1,3}) \) is a totally \( L \)-matrix, the only choice we have is

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T(E_{1,3}) = \begin{pmatrix} 0 & 0 & \eta_3 \\ 0 & 0 & 0 \end{pmatrix},
\]

or

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & \gamma_3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T(E_{1,3}) = \begin{pmatrix} 0 & 0 & 0 \\ \eta_4 & 0 & 0 \end{pmatrix}.
\]

Suppose

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T(E_{1,3}) = \begin{pmatrix} 0 & 0 & \eta_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Since \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{1,3}) \) is a totally \( L \)-matrix and for any \( \chi_i \)'s

\[
T(E_{2,1}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_4 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T(E_{1,3}) = \begin{pmatrix} 0 & 0 & \eta_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Since \( T(E_{2,1} + E_{1,2} + E_{2,2} + E_{1,3}) \) is a totally \( L \)-matrix and for any \( \chi_i \)'s
we must have

$$T(E_{1,2} + E_{2,2}) = \begin{pmatrix} 0 & \beta'_2 & 0 \\ 0 & 0 & \beta'_6 \end{pmatrix}$$

for some $\beta'_2 \neq 0$ and $\beta'_6 \neq 0$. Since

$$T(E_{1,2} + E_{2,2}) = \begin{pmatrix} \alpha_1 & \beta'_2 & \eta_3 \\ -\gamma_4 & \alpha_5 & \beta'_6 \end{pmatrix}$$

is a totally $L$-matrix and none of $\alpha_1$, $\alpha_5$, $\beta'_2$, and $\beta'_6$ are zero, we must have either $\gamma_4 = 0$ or $\eta_3 = 0$. Thus either $T(E_{2,1}) = 0$ or $T(E_{1,3}) = 0$, we have that $|T(E_{2,1} + E_{1,2} + E_{2,2} + E_{1,3})| = 3$, a contradiction. So this case never arises.

Subcase 4. $\beta_2(1,1,1) = 0$, $\beta_3(1,1,1) \neq 0$, $\beta_4(1,1,1) \neq 0$, and $\beta_6(1,1,1) \neq 0$.

Subcase 5. $\beta_2(1,1,1) \neq 0$, $\beta_3(1,1,1) \neq 0$, $\beta_4(1,1,1) = 0$, and $\beta_6(1,1,1) = 0$.

Subcase 6. $\beta_2(1,1,1) = 0$, $\beta_3(1,1,1) \neq 0$, $\beta_4(1,1,1) \neq 0$, and $\beta_6(1,1,1) = 0$.

These cases are equivalent to the subcase 1, 2, or 3. So these cases never arise. Therefore, case 3 never arises. The proof is complete.

Lemma 6.3.7 For any $i$ and $j$, $|T(E_{i,j})| \neq 0$.

Proof. Suppose there are $i$ and $j$ such that $|T(E_{i,j})| = 0$. Without loss of generality, we assume $|T(E_{1,1})| = 0$. So $T(E_{1,1}) = 0$. By Lemmas 6.3.2 - 6.3.6, we have that for any $i$ and $j$, $|T(E_{i,j})| \geq 1$. Thus
\[ T(E_{1,1} + E_{1,2} + E_{2,2} + E_{2,3}) \] is a totally \( L \)-matrix since \( E_{1,1} + E_{1,2} + E_{2,2} + E_{2,3} \) is. But

\[
\| T(E_{1,1} + E_{1,2} + E_{2,2} + E_{2,3}) \| \\
\leq \| T(E_{1,1}) \| + \| T(E_{1,2}) \| + \| T(E_{2,2}) \| + \| T(E_{2,3}) \| \\
\leq 0 + 1 + 1 + 1 = 3,
\]
a contradiction with Lemma 6.3.1. ■

By Lemmas 6.3.2 - 6.3.7, we know that if \( T : M_{2,3} \longrightarrow M_{2,3} \) is a linear operator that preserves totally \( L \)-matrices then for any cell \( E \), \( T(E) \) is also a cell. We can show \( T \) is one to one on the set of cells. If not, then there exist two cells, say \( E \) and \( F \) such \( T(E) \leq T(F) \) and \( T(E) \geq T(F) \). We can choose the other two cells, say \( G \) and \( H \), such that \( E + F + G + H \) is a totally \( L \)-matrix. Thus \( T(E + F + G + H) \) is a totally \( L \)-matrix but

\[
\| T(E + F + G + H) \| \leq \| T(E + F) \| + \| T(G) \| + \| T(H) \| = 1 + 1 + 1 = 3,
\]
a contradiction with Lemma 6.3.1. So we obtain the following theorem.

**Theorem 6.3.1** If \( T : M_{2,3} \longrightarrow M_{2,3} \) is a linear operator that preserves totally \( L \)-matrices then \( T \) is one to one on the set of cells.

**Theorem 6.3.2** If \( T : M_{2,3} \longrightarrow M_{2,3} \) is a linear operator that preserves totally \( L \)-matrices, then \( T \) preserves the set of matrices of row term rank 1.

**Proof.** Let \( R_i = E_{i,1} + E_{i,2} + E_{i,3} \) \((i = 1, 2)\). By Theorem 6.3.1, we know
that $T$ is one to one on the set of cells. So $T$ is nonsingular. Let $U_i = T^{-1}(R_i)$. Then $|U_i| = 3$, say $U_i = E_1 + E_2 + E_3$. We only need to show that the nonzero entries of all $E_r$'s lie on the same row. If not, then, without loss of generality, say $E_1$ and $E_2$ have nonzero entries on the first row and $E_3$ has its nonzero entry on the second row. We can choose a cell, $E_4$ with $E_4 \notin \{E_1, E_2, E_3\}$, in the second row such that $E_1 + E_2 + E_3 + E_4$ is a totally $L$-matrix. Thus

$$T(E_1 + E_2 + E_3 + E_4) = T(U_i) + T(E_4)$$

$$= R_i + T(E_4) = E_{i,1} + E_{i,2} + E_{i,3} + T(E_4)$$

must be a totally $L$-matrix. This is a contradiction since $E_{i,1} + E_{i,2} + E_{i,3} + T(E_4)$ has more than two zero entries on the same row.

Hence $T$ preserves the set of matrices of row term rank 1.

**Theorem 6.3.3** If $T : M_{2,3} \rightarrow M_{2,3}$ is a linear operator that preserves totally $L$-matrices, then $T$ preserves the set of matrices of column term rank 1.

**Proof.** Let $C_j = E_{1,j} + E_{2,j}$ ($j = 1, 2, 3$) and $V_j = T^{-1}(C_j)$. Then $|V_j| = 2$, say $V_j = E_1 + E_2$, $T(E_1) = E_{1,j}$, and $T(E_2) = E_{2,j}$. We only need to show $E_1$ and $E_2$ have their nonzero entries on the same column. If not, then $E_1$ and $E_2$ have their nonzero entries on different columns, without loss of generality, assume $E_1$ has its nonzero entry on the first column and $E_2$ has its nonzero entry on the second column. Since $T$ preserves the set of matrices of row term rank 1, we have that the nonzero entries of $E_1$ and $E_2$ must lie on different rows, say $E_1$ on the first row and $E_2$ on the
second row. Thus $E_1 = E_{1,1}$ and $E_2 = E_{2,2}$. Now we consider $T(E_{2,3})$.

Since $T$ preserves row term rank 1 matrices, and $T(E_{1,1}) = E_{1,1}$ and $T(E_{2,2}) = E_{2,2}$, we have the nonzero entry of $T(E_{2,3})$ lies on the second row. Without loss of generality, assume $T(E_{2,3}) = E_{1,k}$ ($k \neq j$). We assume $T^{-1}(E_{1,k}) = F$. Then the nonzero entry of $F$ is on the first row and $F \neq E_{1,1}$ since $T$ preserves row term rank 1 matrices and $T$ is one to one on the set of cells. Thus $E_{1,1} + E_{2,2} + E_{2,3} + F$ is a totally $L$-matrix.

Hence

$$T(E_{1,1} + E_{2,2} + E_{2,3} + F) = E_{1,1} + E_{2,2} + E_{1,k} + E_{2,k}$$

is a totally $L$-matrix, a contradiction. Therefore $T$ preserves the set of matrices of column term rank 1. 

By Theorem 6.3.2 and 6.3.3, we have the following corollary.

**Corollary 6.3.1** If $T : M_{2,3} \rightarrow M_{2,3}$ is a linear operator that preserves totally $L$-matrices, then $T$ preserves the set of matrices of term rank 1.

Thus by Theorem 6.3.1 and Corollary 6.3.1, we have that if $T : M_{2,3} \rightarrow M_{2,3}$ is a linear operator that preserves totally $L$-matrices, then $T$ is nonsingular and $T$ preserves the set of matrices of term rank 1.

By Theorem 2.1.1 [Beasley and Pullman, 17], we have the following corollary.

**Corollary 6.3.2** If $T : M_{2,3} \rightarrow M_{2,3}$ is a linear operator that preserves totally $L$-matrices, then for any $X \in M_{2,3}$, $T(X) = P_1 X P_2$, where $P_1 \in M_2$ and $P_2 \in M_3$ are permutation matrices and $M = (m_{i,j}) \in M_{2,3}$ with $m_{i,j} \neq 0$.

**Theorem 6.3.4** If $T : M_{2,3} \rightarrow M_{2,3}$ is a linear operator that preserves
totally $L$-matrices, then for any $X \in \mathcal{M}_{2,3}$, $T(X) = P_1 S_1 (X \circ M) S_2 P_2'$, where $P_1 \in \mathcal{M}_2$ and $P_2 \in \mathcal{M}_3$ are permutation matrices, $S_1 \in \mathcal{M}_2$ and $S_2 \in \mathcal{M}_3$ are diagonal matrices of $\pm 1$'s, and $M = (m_{i,j}) \in \mathcal{M}_{2,3}$ with $m_{i,j} > 0$.

**Proof.** By Corollary 6.3.2, for any $X \in \mathcal{M}_{2,3}$, $T(X) = P_1 (X \circ M) P_2'$, where $P_1 \in \mathcal{M}_2$ and $P_2 \in \mathcal{M}_3$ are permutation matrices and $M = (m_{i,j}) \in \mathcal{M}_{2,3}$ with $m_{i,j} \neq 0$. Let $T_1 : \mathcal{M}_{2,3} \rightarrow \mathcal{M}_{2,3}$ be a linear operator defined by

$$T_1 (X) = P_1^t T(X) P_2^t = P_1^t P_1 (X \circ M) P_2 P_2^t = X \circ M$$

for any $X \in \mathcal{M}_{2,3}$. Then clearly $T_1$ preserves totally $L$-matrices since $T$ preserves totally $L$-matrices and $P_1^t$, $P_2^t$ are permutation matrices. Let $T_2 : \mathcal{M}_{2,3} \rightarrow \mathcal{M}_{2,3}$ be a linear operator defined by

$$T_2 (X) = S_1 T(X) S_2 = S_1 (X \circ M) S_2$$

where

$$S_1 = \text{diag} \left( \frac{m_{1,1}}{|m_{1,1}|}, \frac{m_{2,1}}{|m_{2,1}|} \right)$$

and

$$S_2 = \text{diag} \left( 1, \frac{m_{1,2}}{|m_{1,2}|}, \frac{m_{1,1}}{|m_{1,1}|}, \frac{m_{1,3}}{|m_{1,3}|}, \frac{m_{1,1}}{|m_{1,1}|} \right).$$

Clearly $T_2$ preserves totally $L$-matrices since $T_1$ preserves totally $L$-matrices and $S_1, S_2$ are nonzero diagonal matrices. Also we have

$$T_2 (X) = S_1 T(X) S_2 = S_1 (X \circ M) S_2 = X \circ (S_1 M S_2).$$
Let \( N = S_1 M S_2 \). Then \( N = (n_{i,j}) \in M_{2,3} \) with \( n_{i,j} \neq 0 \), \( n_{i,1} > 0 \) \( (i = 1, 2) \) and \( n_{1,j} > 0 \) \( (j = 1, 2, 3) \). Since \( T_2 \) preserves totally \( L \)-matrices, we have that for any \( 2 \times 3 \) totally \( L \)-matrix \( A \),

\[
T_2(A) = A \circ (S_1 M S_2) = A \circ N
\]

is a totally \( L \)-matrix. If \( n_{2,2} < 0 \), then we let \( A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \), which is a totally \( L \)-matrix. But

\[
A \circ N = \begin{pmatrix} n_{1,1} & n_{1,2} & 0 \\ n_{2,1} & -n_{2,2} & n_{2,3} \end{pmatrix}
\]

is not a totally \( L \)-matrix since \( \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & -n_{2,2} \end{pmatrix} \) has the sign pattern \( \begin{pmatrix} + & + \\ + & + \end{pmatrix} \). This contradiction implies that \( n_{2,2} > 0 \). By permuting columns we have that all entries in the matrix \( N \) are positive. Hence \( N = (n_{i,j}) \in M_{2,3} \) with \( n_{i,j} > 0 \). Thus

\[
T(X) = P_1 (X \circ M) P_2 = P_1 (X \circ (S_1^{-1} N S_2^{-1}) P_2 = P_1 S_1^{-1} (X \circ N) S_2^{-1} P_2
\]

which completes the proof. \( \blacksquare \)

The characterization of \( 2 \times 3 \) totally \( L \)-matrices preservers give us the following corollaries.

**Corollary 6.3.3** If \( T : M_{2,3} \rightarrow M_{2,3} \) is a linear operator, then \( T \) strongly preserves totally \( L \)-matrices if and only if \( T \) preserves totally \( L \)-matrices.
Corollary 6.3.4 If $T : \mathcal{M}_{2,3} \rightarrow \mathcal{M}_{2,3}$ is a linear operator that preserves totally $L$-matrices, then $T$ preserves $L$-matrices and $T$ preserves super $L$-matrices.

Corollary 6.3.5 Let $T : \mathcal{M}_{2,3} \rightarrow \mathcal{M}_{2,3}$ be a linear operator. Then $T$ preserves totally $L$-matrices if and only if $T$ strongly preserves $L$-matrices.

Corollary 6.3.6 Let $T : \mathcal{M}_{2,3} \rightarrow \mathcal{M}_{2,3}$ be a linear operator. Then $T$ preserves totally $L$-matrices if and only if $T$ strongly preserves super $L$-matrices.
CHAPTER 7

SUMMARY

We have characterized the structure of several linear operators that preserve classes of subset of $L$-matrices. Let $T : M_{m,n} \rightarrow M_{m,n}$ be a linear operator, we have characterized the structure of $T$ when $T$, $m$ and $n$ satisfy either one of following conditions:

i) $m = n \geq 3$ and $T$ preserves sign-nonsingular matrices;

ii) $m = n = 2$ and $T$ strongly preserves sign-nonsingular matrices;

iii) $n > m \geq 2$, $T$ preserves $L$-matrices and $T$ is also one-to-one on the set of cells;

iv) $n > m \geq 2$ and $T$ strongly preserves $L$-matrices;

v) $n > m \geq 3$ and $T$ preserves super $L$-matrices;

vi) $n > m = 2$ and $T$ strongly preserves super $L$-matrices;

vii) $m = 2$, $n = 4$ and $T$ preserves totally $L$-matrices and $T$ is also one-to-one on the set of cells;

viii) $m = 2$, $n = 3$ and $T$ preserves totally $L$-matrices.

In ii) and iv), the condition "strongly preserve" is necessary. But in iii), if $n > m \geq 3$, then the condition "strongly preserve" may not be necessary. Also we could extend vii) and viii) to the general case. We have the following conjectures, with which we are still working.

If $T : M_{m,n} \rightarrow M_{m,n}$ is a linear operator, we were able to characterize the structure of $T$ when $T$, $m$ and $n$ satisfy either one of following conditions:

i) $n > m \geq 3$ and $T$ preserves $L$-matrices;

ii) $n > m = 2$, $n$ is an odd number and $T$ preserves $L$-matrices;
iii) $m \geq 3$, $n = m + 2$ and $T$ strongly preserves totally $L$-matrices;
iv) $m \geq 3$, $n = m + 1$ and $T$ preserves totally $L$-matrices.

We did not investigate linear operators that preserve barely 
$L$-matrices and strong $L$-matrices. In the near future we will characterize 
these linear preservers.
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213–224.


CURRICULUM VITAE

Shumin Ye

Personal: Born in Guiyang, Guizhou, China, February 8, 1953; Married, two children.

Education:


Professional Experience:


1981-1987 Assistant Professor in the Department of Mathematics of Wuhan University, China. Taught Linear algebra, Calculus, Ordinary differential equations, Partial differential equations, and Engineering mathematics.

1972-1974 Teacher at No. 19 High School, Guiyang, China. Taught senior high mathematics courses.

Publications:

On Equation $C \partial_t U + \partial_{txx} U = \partial_{xx} U + C^2 U$, Journal of Wuhan University (Math. Issue), No. 2, 1982.

With L. Beasley, Linear Operators Preserving Sign-nonsingular Matrices, (to appear)

With L. Beasley, Linear Operators Preserving L-matrices, (to appear)

With L. Beasley, An Inequality on Permanents of Hadamard Products, (in preparation)

Membership:

American Mathematical Society
International Linear Algebra Society
Mathematical Association of America