Four-Dimensional Non-Reductive Homogeneous Manifolds with Neutral Metrics

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A method due to É. Cartan was used to algebraically classify the possible four-dimensional manifolds that allow a (2, 2)-signature metric with a transitive group action which acts by isometries. These manifolds are classified according to the Lie algebra of the group action. There are six possibilities: four non-parameterized Lie algebras, one discretely parameterized family, and one family parameterized by \( \mathbb{R} \).
DEDICATION

To my friends and family, who have always been there for me. Especially my parents for seeing and nurturing the development of all of their children’s potential.
I'd like to thank Dr. Joe Koebbe for admitting me to the graduate program at Utah State University; Dr. Ian Anderson for teaching nearly half of my classes, and for the use of his software package Vessiot, which is used extensively throughout this thesis; and Dr. Mark Fels for teaching nearly the other half of my classes, and keeping me pointed in the right direction.

Andrew Renner
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CHAPTER 1
INTRODUCTION

In the book *Leçons sur la géométrie des espaces de Riemann* [2], É. Cartan derives a method to classify the three-dimensional simply connected Riemannian homogeneous manifolds $G/H$ which admit a group of isometries of at least four. His classification is done by specifying the pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is the Lie algebra of the Lie group $G$ and $\mathfrak{h} \subseteq \mathfrak{g}$ is the Lie algebra of the closed Lie subgroup $H \subseteq G$. His method also allows the geometry of the manifold to be studied in terms of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Cartan’s method is described in Ishihara [7], Jensen [8] and in Schmidt [14]. Ishihara [7] used Cartan’s method to complete the classification of four-dimensional simply connected homogeneous Riemannian manifolds. Jensen [8] used Cartan’s method to determine the four-dimensional simply connected homogeneous Riemannian Einstein spaces. Schmidt [14] noted that Cartan’s method generalized to pseudo-Riemannian manifolds and MacCallum [10] used this generalization to give a number of examples of homogeneous Lorentz manifolds.

In the literature, with the exception of Fels and Renner [3], we know of no examples of non-reductive homogeneous manifolds $G/H$ with a metric such that $G$ acts by isometries. Since a Riemannian homogeneous manifold $G/H$ is automatically reductive investigation of non-reductive manifolds requires the metric to be pseudo-Riemannian.

Cartan’s method is used in this thesis to determine all possible Lie algebra pairs for four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds $G/H$ where $H$ is connected and the metric has $(2,2)$-signature. The results of this thesis are summarized in [3]. In [3] the two, three and four dimensional non-reductive Lorentz manifolds are classified up to coverings.

Any manifold with a transitive smooth group action $G$ may be identified with the left cosets $G/H$ where $H$ is the isotropy group of any point from the manifold, these manifolds are said to be homogeneous. Such a manifold is also pseudo-Riemannian if $G$ acts by isometries for a pseudo-Riemannian metric $\eta$ on $G/H$. When the metric $\eta$ has $(p,q)$-
signature the isotropy subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) must be a subalgebra of \( \mathfrak{o}(p, q) \). Cartan's method starts with each subalgebra of \( \mathfrak{o}(p, q) \) as the algebra \( \mathfrak{h} \) and builds every possible Lie algebra \( \mathfrak{g} \) subject to the constraints of Theorem 2.3.3. We apply this method to the subalgebras of \( \mathfrak{o}(2, 2) \) classified by Patera et al. in [12]. This procedure leads to the following theorem.

**Theorem 1.0.1.** Let \((G/H)\) be a four-dimensional homogeneous pseudo-Riemannian manifold with a metric \( \eta \) of signature \((2, 2)\) and let \( H \) be connected. If \( G/H \) is not reductive, then the Lie algebra pair \((\mathfrak{g}, \mathfrak{h})\) is isomorphic to one of

1. The decomposable five dimensional algebra \( \mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{so}(2) \) where \( \mathfrak{so}(2) \) is the two dimensional solvable algebra. The multiplication table is

\[
[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_4, e_5] = e_4.
\]

The isotropy algebra is \( \mathfrak{h} = \{e_3 + e_4\} \).

2. The one-parameter family of five-dimensional solvable Lie algebras \( \mathfrak{g} = A_{5,30} \) as in [12]. The multiplication table is

\[
[e_1, e_5] = (\alpha + 1)e_1, \quad [e_2, e_4] = e_1, \quad [e_2, e_5] = \alpha e_2,
\]

\[
[e_3, e_4] = e_2, \quad [e_3, e_5] = (\alpha - 1)e_3, \quad [e_4, e_5] = e_4,
\]

where all values of \( \alpha \in \mathbb{R} \) are admissible. The isotropy subalgebra is \( \mathfrak{h} = \{e_4\} \).

3. The five dimensional solvable algebras \( \mathfrak{g} = A_{5,37} \) or \( \mathfrak{g} = A_{5,36} \) as in [12]. The multiplication table is

\[
[e_1, e_4] = 2e_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = e_2,
\]

\[
[e_2, e_5] = -\epsilon e_3, \quad [e_3, e_4] = e_3, \quad [e_3, e_5] = e_2,
\]

where \( \epsilon = 1 \) for \( A_{5,37} \) and \( \epsilon = -1 \) for \( A_{5,36} \). The isotropy algebra for both is \( \mathfrak{h} = \{e_3\} \).

4. The five dimensional algebra \( \mathfrak{g} = \mathfrak{sl}(2) \ltimes \mathbb{R}^2 \). The multiplication table is

\[
[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,
\]

\[
\]
The isotropy subalgebra is $\mathfrak{h} = \{e_3\}$.

v.) The six-dimensional Schrödinger algebra $\mathfrak{g} = \mathfrak{sl}(2) \ltimes \mathfrak{n}(3)$, where $\mathfrak{n}(3)$ is the three-dimensional Heisenberg algebra. The multiplication table is

$$
\begin{align*}
\end{align*}
$$

The isotropy subalgebra is $\mathfrak{h} = \{e_3 - e_5, e_5\}$.

vi.) The six-dimensional algebra $\mathfrak{g} = \mathfrak{sl}(2) \ltimes \mathbb{R}^2 \oplus \mathbb{R}$. The multiplication table is

$$
\begin{align*}
\end{align*}
$$

The isotropy subalgebra is $\mathfrak{h} = \{e_3, e_5 + e_6\}$.

In Chapter 2 we state some necessary definitions and develop the necessary theory for homogeneous pseudo-Riemannian manifolds.

In Chapter 3, we outline Cartan's method of constructing the possible Lie algebras $\mathfrak{g}$ from a subalgebra $\mathfrak{h} \subset \mathfrak{o}(p, q)$ and give the details in a basis. These calculations are illustrated by an example where the subalgebra $\mathfrak{h} \subset \mathfrak{o}(2, 2)$ leads to a reductive pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ for any constructed $\mathfrak{g}$.

In Chapter 4, the construction of the Lie algebra $\mathfrak{g}$ is reformulated in terms of the dual space $\mathfrak{g}^*$ which simplifies identifying the geometry of the manifold. We demonstrate these calculations in an example which leads to cases v.), and vi.) of Theorem 1.0.1.

In Chapter 5, Theorem 1.0.1 is proved with Chapters 6 and 7 providing the necessary calculations as Maple worksheets.

The calculations made in Chapters 6 and 7 use the Maple package Vessiot written by Dr. Ian Anderson which is available at http://www.math.usu.edu/~fg.mp.
CHAPTER 2
PRELIMINARIES

2.1 Lie Groups, Actions, and Homogeneous Manifolds

In this section, we will follow Boothby [1] to give some basic mathematical definitions essential to this thesis.

Definition 2.1.1. A Lie group $G$ is both a group and a manifold such that,

i.) the group multiplication $m : G \times G \rightarrow G$ defined as $m(g, g') = gg'$

ii.) the inverse $i : G \rightarrow G$ defined as $i(g) = g^{-1}$

are smooth maps between manifolds.

A homomorphism of Lie groups $\phi : H \rightarrow G$ is a group homomorphism which is also smooth map between manifolds.

Definition 2.1.2. A left action of a group $G$ on a set $M$ is a map $\mu : G \times M \rightarrow M$ such

i.) $\mu(g, \mu(g', x)) = \mu(gg', x)$

ii.) $\mu(e, x) = x,$

where $e \in G$ is the identity.

We will exclusively use left actions of a Lie group $G$ on a manifold $M$ such that the action $\mu : G \times M \rightarrow M$ is a smooth map between manifolds, and we will henceforth simply refer to them as actions. Furthermore, an action is effective, if $gx = x$ for all $x \in M$ implies $g = e.$ In particular, there is no loss of generality if we assume an action is effective, see page 38 of [11] for the details.

When an action $\mu : G \times M \rightarrow M$ of a group $G$ on $M$ is unambiguous, the notation will be shortened to $\mu(g, x) = gx.$ Then properties i.) and ii.) of Definition 2.1.2 are expressed as

$$g(hx) = (gh)x \quad \text{and} \quad ex = x.$$
Given any action of $G$ on $M$, then each element $g \in G$ defines two diffeomorphisms

$$\mu_g : M \to M$$

and

$$\tilde{\mu}_g : TM \to TM$$

by

$$\mu_g(x) = \mu(g, x) = gx \quad \text{and} \quad \tilde{\mu}_g(X) = (\mu_g)_* = g_*X.$$  

for $x \in M$ and $X \in TM$. Furthermore, for each point $x \in M$ we have a smooth map

$$\phi_x : G \to M$$

given by

$$\phi_x(g) = gx.$$  

**Definition 2.1.3.** A Lie subgroup $H$ of a Lie group $G$ has the properties:

i.) $H$ is a subgroup of $G$

ii.) $H$ is an immersed submanifold of $G$.

**Example 2.1.** Let $G$ act on $M$, and choose a point $x \in M$. The map $\phi_x$ determines a closed Lie subgroup $G_x \subset G$ defined as

$$G_x = \phi_x^{-1}(x) = \{g \in G | gx = x\}$$

which is called the isotropy subgroup of $x$.

**Definition 2.1.4.** An action of $G$ on $M$ is transitive if for any points $x, y \in M$ there exists an element $g \in G$ such that $gx = y$.

**Remark 1.** If $G$ acts transitively on $M$ then the isotropy subgroup for each point $x \in M$ are all conjugate in $G$.

If there exists a transitive action of $G$ on $M$, then $M$ is called a $G$-homogeneous manifold. The following theorem provides a canonical way of constructing a $G$-homogeneous manifold from a Lie group $G$ and a closed Lie subgroup $H$, see page 161 of [1] for details.

**Theorem 2.1.1.** If $H$ is a closed Lie subgroup of $G$, then the coset space $G/H$ has a unique manifold structure such

i.) $\dim G/H = \dim G - \dim H$
ii.) The map \( \pi : G \to G/H \) defined by \( \pi(g) = gH \) is smooth.

iii.) The action \( \lambda : G \times G/H \to G/H \to G/H \) defined by \( \lambda(g, [g'H]) = [gg'H] \) is smooth and transitive.

Theorem 2.1.1 provides a way to classify every \( G \)-homogeneous manifold as the following theorem shows, see page 161 of [1] for details

**Theorem 2.1.2.** Let \( \mu : G \times M \to M \) be a transitive action of \( G \) on \( M \). Then \( M \) is equivariantly diffeomorphic to \( G/H \) where \( H \) is a closed Lie subgroup of \( G \).

**Proof.** Choose any point \( x \in M \). The isotropy group \( G_x \) defined in Example 2.1.1 is a closed Lie subgroup. Hence, by Theorem 2.1.1, \( G/G_x \) is a smooth manifold. Let \( \psi([gG_x]) = gx \), this is a well defined map \( \psi : G/G_x \to M \) which is a diffeomorphism, see page 161 in [1] for the details. We define \( \lambda(g, [g'G_x]) = [gg'G_x] \) as in Theorem 2.1.1, which makes the map \( \psi \) equivariant.

By Theorem 2.1.2, a \( G \)-homogeneous manifold \( M \) is determined by the Lie group \( G \) and a closed Lie subgroup \( H \), so without any loss of generality we may assume \( M = G/H \) where \( H \) is a closed Lie subgroup of \( G \), and we will refer to this simply as a homogeneous manifold. Furthermore the quotient \( G/H \) will be assumed to be a homogeneous manifold in this manner throughout this thesis.

**Definition 2.1.5.** A real-representation of a group \( G \) is a homomorphism \( \rho : G \to GL(V) \) where \( V \) is real vector space.

**Example 2.1.2.** Choose an element \( g \in G \) and define \( \tilde{g} : G \to G \) by \( \tilde{g}(x) = gxg^{-1} \). The map \( \tilde{g} \) is smooth so we may define \( Ad : G \to GL(T_eG) \) as

\[
(2.2) \quad Ad(g) = \tilde{g}_*.
\]

The map \( Ad \) is a homomorphism, because \( Ad(gh) = (gh)_* = (g\tilde{h})_* = \tilde{g}_* \tilde{h}_* = (Ad(g))(Ad(h)) \).

Therefore, \( Ad \) is a real representation of \( G \), called the adjoint representation of \( G \).
Example 2.1.3. Let $G$ act on $M$ and choose a point $x \in M$. We define $\rho_x : G_x \to GL(T_xM)$ by

$$\rho_x(g) = g_\ast$$

where $G_x$ is the isotropy subgroup as defined in Example 2.1.1. The map $\rho_x$ is a homomorphism because $\rho_x(gg') = (gg')_\ast = g_\ast g'_\ast = \rho_x(g)\rho_x(g')$. Therefore, $\rho_x$ is a real representation of $G_x$ which is called the isotropy representation of $G_x$.

Remark 2. If $\rho : G \to GL(V)$ is a representation and $H$ is a subgroup of $G$. Then $\rho|_H : H \to GL(V)$, is representation. Furthermore, if there is a subspace $W \subset V$ such that $\rho|_H(W) \subset W$, then there is an induced representation $\psi : G \to GL(V/W)$ defined by $\psi[X] = [\rho(X)]$ where $X \in V$.

Definition 2.1.6. A homogeneous manifold $G/H$ is reductive if there exists a subspace $m$ such that $T_eG = m \oplus T_eH$ and $m$ is $Ad|_H$-invariant.

A few comments about Definition 2.1.6 are in order. By Remark 2, $Ad|_H : H \to GL(T_eG)$ is a representation. Choose an element $h \in H$ and a vector $X \in T_eH$, then $(Ad(h))X \in T_eH$; that is $T_eH$ is an invariant subspace of $T_eG$. If the homogeneous manifold $G/H$ is reductive, then there exists a subspace $m \subset T_eG$ such that $T_eG = m \oplus T_eH$ and $(Ad(h))Y \in m$ for any $Y \in m$. Then both $m$ and $T_eH$ are $Ad|_H$ invariant subspaces; that is the representation $Ad|_H$ is reducible, see page 4 of [4] for the definition of a reducible representation.

In this thesis we are interested in finding homogeneous manifolds $G/H$ that are not reductive.

2.2 Lie Algebras and Their Representations

Definition 2.2.1. A real Lie algebra $\mathfrak{g}$ is a real-vector space $\mathfrak{g}$ endowed with a function $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which satisfies

i.) $[X,Y] = -[Y,X]$
ii.) \[aX + bY, Z\] = a[X, Z] + b[Y, Z]

iii.) \[[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\]

for all \(X, Y, Z \in \mathfrak{g}\) and \(a, b \in \mathbb{R}\). Property iii) is called the Jacobi identity.

For a basis \(\{e_i\}_{i=1}^n\) of an \(n\)-dimensional Lie algebra \(\mathfrak{g}\) there are constants \(C_{ij}^k \in \mathbb{R}\) such that \([e_i, e_j] = C_{ij}^k e_k\). These numbers are called the structure constants and by ii.), determine the bracket on all of \(\mathfrak{g}\).

Remark 3. Let \(\{e_j\}_{j=1}^n\) be a basis for the Lie algebra, let \(e^i\) be a basis for \(\mathfrak{g}^*\) and define

\begin{equation}
    de^k = -\frac{1}{2}C_{ij}^k e^i \wedge e^j.
\end{equation}

Where \(\{e^i\}_{i=1}^n\) is a basis for \(\mathfrak{g}^*\) such that \(e^i(e_j) = \delta_{ij}\). In this formulation, the Jacobi identity is \(d^2 e^k = 0\) for all \(k = 1 \ldots n\).

Example 2.2.1. The set of smooth vector fields \(\mathfrak{X}(M)\) on a manifold \(M\) is an infinite dimensional Lie algebra where \([X, Y]\) is defined as \([X, Y]f = X(Yf) - Y(Xf)\) for smooth functions \(f\) on \(M\), see page 148 of [1].

Definition 2.2.2. A Lie subalgebra of a Lie algebra \(\mathfrak{g}\) is a vector subspace \(\mathfrak{h}\), such that \([X, Y] \in \mathfrak{h}\) for all \(X, Y \in \mathfrak{h}\).

Example 2.2.2. Let \(L : G \times G \to G\) be left translation on an \(n\)-dimensional Lie group \(G\), given by

\[L(g, g') \equiv gg'.\]

This action determines a set of smooth vector fields \(\mathfrak{g}\) defined by

\[\mathfrak{g} = \{X \in \mathfrak{X}(G) | (L_g)*X = X\}\]

called the left-invariant vector fields of \(G\). This set \(\mathfrak{g}\), is an \(n\)-dimensional Lie subalgebra of \(\mathfrak{X}(M)\), see page 151 of [1] for proof.

Definition 2.2.3. A homomorphism \(\phi : \mathfrak{h} \to \mathfrak{g}\) from a Lie algebra \(\mathfrak{h}\) to the Lie algebra \(\mathfrak{g}\) is a homomorphism of vector spaces such that \(\phi([X, Y]) = [\phi(X), \phi(Y)]\) for all \(X, Y \in \mathfrak{h}\).
Example 2.2.3. Let $\mathfrak{g}$ the Lie algebra of left invariant vector fields on the Lie group $G$. The vector space $T_eG$ may be made into a Lie algebra isomorphic to $\mathfrak{g}$. Choose any vector $X \in T_eG$ then there is a unique left invariant vector field $\tilde{X} \in \mathfrak{g}$ such that $(\tilde{X})_e = X$, defined by

$$(\tilde{X})_g = (L_g)_*X.$$  

This one to one correspondence between $T_eG$ and $\mathfrak{g}$ can be used to endow $T_eG$ with a Lie bracket by defining $[\cdot, \cdot] : T_eG \times T_eG \rightarrow T_eG$ as

$$[X, Y] = ([\tilde{X}, \tilde{Y}])_e.$$  

With this with this bracket $T_eG$ and $\mathfrak{g}$ are isomorphic Lie algebras and we will refer to either of these as the Lie algebra of the Lie group.

Theorem 2.2.1. Let $H$ and $G$ be Lie groups and let $\mathfrak{h}$ and $\mathfrak{g}$ be their respective Lie algebras. If $\phi : H \rightarrow G$ is a homomorphism of Lie groups, then $\phi_* : \mathfrak{h} \rightarrow \mathfrak{g}$ is a Lie algebra homomorphism, see page 151 of [1] for the details.

Corollary 2.2.2. Let $H$ be a Lie subgroup of the Lie group $G$ and let $\mathfrak{h}$ and $\mathfrak{g}$ be their respective Lie algebras. Then $\mathfrak{h}$ is isomorphic to a Lie subalgebra of $\mathfrak{g}$.

Due to this isomorphism we will identify $\mathfrak{h}$ as a subalgebra of $\mathfrak{g}$.

Example 2.2.4. Let $G$ act on $M$ and choose a point $x \in M$. From Example 2.1.1 the isotropy group $G_x$ is a Lie group. Therefore, $G_x$ has a Lie algebra $\mathfrak{g}_x$ in the manner of Example 2.2.3, which we call the isotropy algebra of $x$. Furthermore, $\mathfrak{g}_x$ is a Lie subalgebra of $\mathfrak{g}$ by Corollary 2.2.2.

Definition 2.2.4. A representation of a Lie algebra $\mathfrak{g}$ on a vector space $V$ is a homomorphism $\rho_* : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ such that $\rho_*([X, Y]) = [\rho_*(X), \rho_*(Y)] = \rho_*(X)\rho_*(Y) - \rho_*(Y)\rho_*(X).$

Example 2.2.5. Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G$ as in Example 2.2.3. The adjoint representation $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ from Equation (2.2) is smooth. Hence, by Theorem 2.2.1 there is an induced homomorphism $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ defined by

$$(2.5) \quad \text{ad}(X) = \text{Ad}_*X.$$
Therefore, \(\text{ad}\) is a representation of \(\mathfrak{g}\) which is called the \textbf{adjoint representation of the Lie algebra of} \(G\).

\textit{Example 2.2.6.} Let \(\mathfrak{g}\) be an abstract Lie algebra there is a canonical representation \(\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})\) defined by

\[
\text{ad}(X)Y = [X, Y].
\]

The map \(\text{ad}\) is a homomorphism, because

\[
[\text{ad}(X), \text{ad}(Y)]Z = \text{ad}(X)\text{ad}(Y)Z - \text{ad}(Y)\text{ad}(X)Z
\]

\[
= \text{ad}(X)[Y, Z] - \text{ad}(Y)[X, Z]
\]

\[
= [X, [Y, Z]] - [Y, [X, Z]]
\]

\[
= [X, [Y, Z]] + [Y, [Z, X]]
\]

We may use the Jacobi identity to get

\[
[X, [Y, Z]] + [Y, [Z, X]] = -[Z, [X, Y]]
\]

\[
= [[X, Y], Z]
\]

\[
= \text{ad}([X, Y])Z.
\]

Therefore, \(\text{ad}\) is a representation of \(\mathfrak{g}\) which is called the \textbf{adjoint representation of} \(\mathfrak{g}\).

\textit{Remark 4.} In fact, if \(\mathfrak{g}\) is the Lie algebra of \(G\) as in \textit{Example 2.2.3}, then the adjoint representations (2.5) and (2.6) agree, see pages 50 and 104 in [15] for the details.

\textit{Example 2.2.7.} Let \(G\) act on \(M\) and choose a point \(x \in M\). The isotropy representation \(\rho_x : G_x \to GL(T_xG)\) from equation (2.3) is smooth. Hence, by Theorem 2.2.1, there is an induced homomorphism

\[
(\rho_x)_* : \mathfrak{g}_x \to \mathfrak{gl}(T_xG),
\]

where \(\mathfrak{g}_x\) is the Lie algebra of \(G_x\) as in \textit{Example 2.2.3}. Therefore, \((\rho_x)_*\) is a representation of \(\mathfrak{g}_x\) which is called the \textbf{isotropy representation of} \(\mathfrak{g}_x\).
Remark 5. In a manner similar to Remark 2 we have that if $\rho_\ast : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then $\rho_\ast|_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{gl}(V)$, is a representation. Furthermore, if there is a subspace $W \subset V$ such that $\rho_\ast(W) \subset W$, then there is an induced representation $\psi : \mathfrak{h} \rightarrow \mathfrak{gl}(V/W)$ defined by $\psi[X] = [\rho_\ast(X)]$ where $X \in V$.

Let $\mathfrak{h}$ be a Lie subalgebra of the Lie algebra $\mathfrak{g}$ and denote this by the pair $(\mathfrak{g}, \mathfrak{h})$.

**Definition 2.2.5.** A Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is reductive if there exists a subspace $m \subset \mathfrak{g}$ such that $\mathfrak{g} = m \oplus \mathfrak{h}$ and $[m, \mathfrak{h}] \subset m$.

Similar to the comments after Definition 2.1.6, $\text{ad}|_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is a representation by Remark 5. Choose any vectors $Z, Y \in \mathfrak{h}$, then $(\text{ad}(Z))Y = [Z, Y] \in \mathfrak{h}$, that is $\mathfrak{h}$ is an $\text{ad}|_{\mathfrak{h}}$ invariant subspace. If the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is reductive there exists a subspace $m \subset \mathfrak{g}$ such that $\mathfrak{g} = m \oplus \mathfrak{h}$ and $(\text{ad}(Z))X = [Z, X] \in m$ for any $X \in m$. Then both $m$ and $\mathfrak{h}$ are $\text{ad}|_{\mathfrak{h}}$ invariant subspaces; that is $\text{ad}|_{\mathfrak{h}}$ is reducible, see [4] for the definition of a reducible representation.

We now relate Definition 2.1.6 for reductive homogeneous manifolds to Definition 2.2.5 for reductive Lie algebras with the following lemma, see page 190 of [9] for the proof.

**Lemma 2.2.3.** Let $G$ be a Lie group and $H$ be a connected closed Lie subgroup then the homogeneous manifold $G/H$ is reductive if and only if the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is reductive.

### 2.3 Pseudo-Riemannian Homogeneous Manifolds

**Definition 2.3.1.** A pseudo-Riemannian metric on a manifold $M$ is a smooth function $\eta : T_M \times T_M \rightarrow \mathbb{R}$ such that for every point $x \in M$, $\eta_x : T_xM \times T_xM \rightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear function with signature independent of the point $x$.

The metric $\eta$ is called a $(p, q)$-signature pseudo-Riemannian metric, if $\eta_x$ has signature $(p, q)$ for every point $x \in M$, in particular the metric will be called neutral if $p = q$.

**Definition 2.3.2.** A diffeomorphism $\phi : M \rightarrow M$ is an isometry for a metric $\eta$, if $\phi^*\eta = \eta$. 

Let $\phi$ be an isometry. At each point $x \in M$, the condition $\phi^* \eta = \eta$ in Definition 2.3.2 is

\[(\phi^* \eta_{\phi(x)})(X,Y) = \eta_{\phi(x)}(\phi_* X, \phi_* Y) = \eta_x(X,Y)\]

for all vectors $X, Y \in T_x M$.

An action of $G$ on $M$ is said to act by isometries if $g^* \eta = \eta$ for every element $g \in G$.

**Lemma 2.3.1.** Let $\eta$ be a $(p, q)$ signature pseudo-Riemannian metric on a manifold $M$ and let $G$ act on $M$ effectively and by isometries. Then for any point $x \in M$

i.) the isotropy representation (2.3) of $G_x$, satisfies $\rho_x : G_x \to O(p, q)$ and

ii.) the isotropy representation (2.7) of $g_x$, satisfies $(\rho_x)_* : g_x \to o(p, q)$.

Furthermore, these representations are injective.

**Proof.** Let $\eta$ be a $(p, q)$-signature pseudo-Riemannian metric on a manifold $M$, and let $G$ act on $M$ effectively and by isometries. Choose any $x \in M$, then for any element $g \in G_x$, Equation (2.8) gives

\[(g^* \eta)(X,Y) = \eta(g_* X, g_* Y) = \eta(X,Y)\]

for all vectors $X, Y \in T_x M$.

The isotropy representation from Equation (2.3) yields $g_* = \rho_x(g)$ thus

\[\eta(\rho_x(g) X, \rho_x(g) Y) = \eta(X,Y)\]

for all vectors $X, Y \in T_x M$. Therefore, $\rho_x(g) \in O(p, q)$ which proves i). By Theorem 2.2.1, we get $(\rho_x)_* \in o(p, q)$ which proves ii). Because $G$ acts effectively, $\rho_x$ is injective, see page 62 of [6] for the details. Consequently, both $\rho_x$ and $(\rho_x)_*$ are monomorphisms.

Since $\rho_x$ and $(\rho_x)_*$ are injective, $G_x$ is isomorphic to a subgroup of $O(p, q)$ and $g_x$ is isomorphic to a Lie subalgebra of $o(p, q)$.

**Definition 2.3.3.** A homogeneous manifold $G/H$ with a pseudo-Riemannian metric $\eta$ is a homogeneous pseudo-Riemannian manifold if the action $\lambda : G \times G/H \to G/H$, as defined in Theorem 2.1.1, is effective and acts by isometries.
Theorem 2.3.2. Let $G/H$ be a homogeneous manifold and, without any loss of generality, assume the action of $G$ on $G/H$ is effective. If $\eta^0 : T_e(G/H) \times T_e(G/H) \rightarrow \mathbb{R}$ is a $(p,q)$-signature symmetric bilinear form which is invariant under the isotropy representation (2.7) of $H$. There exists a unique $(p,q)$-signature pseudo-Riemannian metric $\eta$ that makes $G/H$ a homogeneous pseudo-Riemannian manifold and $\eta_e = \eta^0$.

Proof: Let $\eta^0$ be a $(p,q)$-signature symmetric bilinear form on $T_e(G/H)$ that is invariant under the isotropy representation $\rho : H \rightarrow GL(T_e(G/H))$ and let $\eta_{[g]} = (g^{-1})^* \eta^0$. Now $\eta$ is well defined because if $g' = gh$ then

$$
\eta_{[g']}(X,Y) = \eta_{[gh]}(X,Y) = ((gh)^{-1})^* \eta^0(X,Y)
= \eta^0((gh)^{-1}X,(gh)^{-1}Y)
= \eta^0((h^{-1}g^{-1})X,(h^{-1}g^{-1})Y)
= \eta^0(h^{-1}g^{-1}X,h^{-1}g^{-1}Y)
$$

By Equation (2.3) and by Lemma 2.3.1 we may write

$$
\eta^0(h^{-1}g^{-1}X,h^{-1}g^{-1}Y) = \eta^0(\rho(h^{-1})g^{-1}X,\rho(h^{-1})g^{-1}Y)
= \eta^0(g^{-1}X,g^{-1}Y)
= ((g^{-1})^* \eta^0)(X,Y)
= \eta_{[g]}(X,Y).
$$

Since $g_* : T_e(G) \rightarrow T_{[g]}G$ is an isomorphism, $\eta$ has full rank and constant signature at any point $[g] \in G/H$. Finally, $\eta$ is smooth (see page 238 in [1]) and is easily seen to be unique.

We finish this section with a theorem that is the starting point for the classification in this thesis.

Theorem 2.3.3. Let $G/H$ be an $m$-dimensional pseudo-Riemannian homogeneous space, let $\mathfrak{g}$ be the Lie algebra of $G$, and $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$. Then $\mathfrak{g} = \mathbb{R}^m \oplus \mathfrak{h}$ as a vector space and there is a map $\psi : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ such that
i.) \( \psi(Z)[X] = [(adZ)X] \)

ii.) \( \psi(Z) \in o(p, q) \)

for all \( Z \in \mathfrak{h} \) and \( [X] \in \mathfrak{g}/\mathfrak{h} \).

**Proof:** The map \( \text{ad}|\mathfrak{h} : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g}) \) is a homomorphism, furthermore \( \text{ad}|\mathfrak{h} \) leaves \( \mathfrak{h} \) invariant. By Remark 5, we may then define \( \psi : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}) \) as \( (\psi(Z))[X] = [\text{ad}(Z)X] \) where \( Z \in \mathfrak{h} \) and \( [X] \in \mathfrak{g}/\mathfrak{h} \) is the equivalence class of the vector \( X \in \mathfrak{g} \). The function defined by \( \phi[X] = \pi_*X \) is an isomorphism of \( \mathfrak{g}/\mathfrak{h} \) onto \( T_{g\mathfrak{z}}G/H \). We may then define a bilinear form \( \gamma \) on \( \mathfrak{g}/\mathfrak{h} \) by \( \gamma = \phi^*\eta \), which has the same signature as \( \eta \) since \( \phi \) is an isomorphism. Choose any \( Z \in \mathfrak{h} \) we will now show that \( \psi(Z) \in o(p, q) \).

\[

g(\psi(Z)[X], [Y]) + g([X], \psi(Z)[Y]) = \eta(\phi[(adZ)X], \phi[Y]) + \eta(\phi[X], \phi[(adZ)Y]) \\
= \eta(\pi_*((adZ)X), \pi_*Y) + \eta(\pi_*X, \pi_*((adZ)Y)) \\
= \eta((\pi_*((adZ)))X, \pi_*Y) + \eta(\pi_*X, (\pi_*((adZ)))Y)
\]

Since \( \pi_*(adZ) = (\rho_*Z)\pi_* \) we may write

\[
\eta((\pi_*((adZ)))X, \pi_*Y) + \eta(\pi_*X, (\pi_*((adZ)))Y) \eta((\rho_*Z)\pi_*X, \pi_*Y) + \eta(\pi_*X, (\rho_*Z)(\pi_*Y)) \\
= \eta((\rho_*Z)(\pi_*X), \pi_*Y) + \eta(\pi_*X, (\rho_*Z)(\pi_*Y)) \\
= \eta((\rho_*Z)(\phi[X]), \phi[Y]) + \eta(\phi[X], (\rho_*Z)(\phi[Y])) \\
= 0,
\]

because \( \rho_*Z \in o(p, q) \). Therefore \( \psi(Z) \in o(p, q) \) for all \( Z \in \mathfrak{h} \). \( \square \)
3.1 General Calculations

Cartan’s method, which is described in a basis below, constructs a list of all the Lie algebra pairs \((g, \mathfrak{h})\) such that \(g = \mathbb{R}^m \oplus \mathfrak{h}\) as a vector space and \(\text{ad}_{\mathfrak{h}}\) acts on \(\mathbb{R}^m\) by the standard representation of \(\mathfrak{h} \subset \mathfrak{o}(p,q)\) on \(\mathbb{R}^m\). On the other hand, suppose that \(G/H\) is an \(m\)-dimensional pseudo-Riemannian homogeneous manifold. By Theorem 2.3.3 the Lie algebra pair \((g, \mathfrak{h})\) for \(G/H\) has the properties that \(g = \mathbb{R}^m \oplus \mathfrak{h}\) as a vector space, \(\mathfrak{h} \subset \mathfrak{o}(p,q)\) and \(\text{ad}_{\mathfrak{h}}\) acts on \(\mathbb{R}^m = g/\mathfrak{h}\). Consequently, the Lie algebra pair \((g, \mathfrak{h})\) is on the list of Lie algebra pairs constructed by Cartan’s method.

We now write out the details of Cartan’s method in a basis. First we spell out some conventions. We will use Einstein’s summation convention: for any index that appears as a subscript and a superscript, there is an implied summation over the range of that index. Furthermore, we will fix the range of certain indices as follows

\[
\alpha, \beta, \gamma, \delta = 1 \ldots h; \quad i, j, k, l, e = 1 \ldots m; \quad r, s, t = 1 \ldots \frac{m(m-1)}{2} - h.
\]

where \(h\) is the dimension of \(\mathfrak{h}\), \(m\) is the dimension of \(\mathbb{R}^m\), and \(\frac{m(m-1)}{2} - h\) is the codimension of \(\mathfrak{h}\) in \(\mathfrak{o}(p,q)\).

Cartan’s method starts with a subalgebra \(\mathfrak{h} \subset \mathfrak{o}(p,q)\). If \(\{h_\alpha\}\) is a basis for \(\mathfrak{h}\), then

\[
[h_\alpha, h_\beta] = J^\gamma_{\alpha\beta} h_\gamma.
\]

where \(J^\gamma_{\alpha\beta} \in \mathbb{R}\) are the structure constants in this basis.

Since \(\mathfrak{h} \subset \mathfrak{o}(p,q) \subset \mathfrak{gl}(\mathbb{R}^m)\) is an \(h\)-dimensional subalgebra, \(\mathfrak{h}\) acts on \(\mathbb{R}^m\) by it’s natural representation

\[
h_\alpha(m_i) = H^k_{\alpha j} m_k
\]

where \(m_k\) is a basis for \(\mathbb{R}^m\) and \((h_\alpha)_j^k = H^k_{\alpha j}\). By simultaneously interpreting \(\mathfrak{h}\) as a matrix algebra and identifying \(H^k_{\alpha j}\) with \(h_\alpha\) we have

\[
[H^k_{\alpha i}, H^l_{\beta j}] = H^k_{\alpha i} H^l_{\beta j} - H^l_{\beta i} H^k_{\alpha j} = J^\gamma_{\alpha \beta} H^k_{\gamma j} = J^\gamma_{\alpha \beta} h_\gamma = [h_\alpha, h_\beta].
\]
We let $m_j$ and $h_\alpha$ be a basis for the vector space $\mathbb{R}^m \oplus \mathfrak{h}$ and we try to extend the representation of $\mathfrak{h}$ to a bracket on $\mathfrak{g}$ by letting

\begin{equation}
(h_\alpha, h_\beta) = J^\gamma_{\alpha \beta} h_\gamma, \quad (h_\alpha, m_j) = H^k_{\alpha j} m_k + A^\gamma_{\alpha j} h_\gamma, \quad (m_i, m_j) = B^k_{ij} m_k + C^\gamma_{ij} h_\gamma,
\end{equation}

where $A^\gamma_{\alpha j}, B^k_{ij}, C^\gamma_{ij} \in \mathbb{R}$ need to be determined by the Jacobi identities. Note that if (3.4) is a Lie algebra then the action of $\text{ad}|_{\mathfrak{h}}$ on $\mathfrak{g}/\mathfrak{h}$ is given by the natural representation of $\mathfrak{h} \subset \mathfrak{gl}(\mathbb{R}^m)$ as given in Equation (3.2). Consequently, if $(\mathfrak{g}, \mathfrak{h})$ was the Lie algebra pair for a pseudo-Riemannian manifold $G/H$ then $\mathfrak{g}$ has the Lie bracket as in (3.4) for some $A^\gamma_{\alpha j}, B^k_{ij}, C^\gamma_{ij} \in \mathbb{R}$ and the pair $(\mathfrak{g}, \mathfrak{h})$ is on the list that is constructed by Cartan’s method.

To determine all the possible Lie algebras $\mathfrak{g}$ we now only need to solve the Jacobi identities. The Jacobi identities in this basis are calculated term by term in Appendix D and ultimately give

\begin{equation}
0 = [h_\alpha, [h_\beta, h_\gamma]] + [h_\beta, [h_\gamma, h_\alpha]] + [h_\gamma, [h_\alpha, h_\beta]],
\end{equation}

which is satisfied since $\mathfrak{h}$ is a subalgebra while

\begin{align*}
0 &= [h_\alpha, [h_\beta, m_i]] + [h_\beta, [m_i, h_\alpha]] + [m_i, [h_\alpha, h_\beta]] \\
&= (H^k_{\alpha l} H^k_{\beta l} - H^k_{\beta l} H^k_{\alpha l} - H^k_{\alpha l} J^\gamma_{\alpha \beta}) m_i + (A^\delta_{\alpha k} H^k_{\beta l} + J^\delta_{\alpha \gamma} A^\gamma_{\beta l} - A^\delta_{\beta k} H^k_{\alpha l} - J^\delta_{\beta \gamma} A^\gamma_{\alpha l} - A^\delta_{\gamma l} J^\gamma_{\alpha \beta}) h_\delta
\end{align*}

simplifies to

\begin{equation}
0 = (A^\delta_{\alpha k} H^k_{\beta l} + J^\delta_{\alpha \gamma} A^\gamma_{\beta l} - A^\delta_{\beta k} H^k_{\alpha l} - J^\delta_{\beta \gamma} A^\gamma_{\alpha l} - A^\delta_{\gamma l} J^\gamma_{\alpha \beta}) h_\delta
\end{equation}

since $[H_{\alpha l}, H_{\beta l}] = J^\gamma_{\alpha \beta} H_\gamma$. Next we get

\begin{align*}
0 &= [h_\alpha, [m_i, m_j]] + [m_i, [m_j, h_\alpha]] + [m_j, [h_\alpha, m_i]] \\
&= (H^k_{\alpha l} B^k_{ij} - B^k_{ij} H^k_{\alpha l} + H^l_{\alpha j} A^\gamma_{ij} + B^l_{jk} H^k_{\alpha i} - H^l_{\gamma j} A^\gamma_{\alpha i}) m_l + (A^\delta_{\alpha k} B^k_{ij} + J^\delta_{\alpha \gamma} C^\gamma_{ij} - A^\delta_{\gamma l} A^\gamma_{ij} + A^\delta_{\gamma l} A^\gamma_{ij} - A^\delta_{\gamma l} J^\gamma_{\alpha i}) h_\delta
\end{align*}

and we finally get

\begin{align*}
0 &= [m_i, [m_j, m_k]] + [m_j, [m_k, m_i]] + [m_k, [m_i, m_j]] \\
&= (B^\delta_{\alpha l} B^\delta_{ij} - H^l_{\gamma j} C^\gamma_{ij} + C^\delta_{ij} B^\delta_{kl} - H^l_{\gamma j} C^\gamma_{kl} + B^\delta_{jk} B^\delta_{ij} - H^l_{\gamma k} C^\gamma_{ij}) m_l + (A^\delta_{\gamma l} B^\delta_{ij} - A^\delta_{\gamma l} C^\gamma_{ij} + C^\delta_{ij} B^\delta_{kl} - A^\delta_{\gamma j} C^\gamma_{kl} + C^\delta_{jk} B^\delta_{ij} - A^\delta_{\gamma k} C^\gamma_{ij}) h_\delta.
\end{align*}
Once the possible $A^\gamma_{\alpha j}, B^k_{ij}, C^\gamma_{ij} \in \mathbb{R}$ are found that make $\mathfrak{g}$ a Lie algebra and this is done for each subalgebra $\mathfrak{h}$ of $\mathfrak{o}(p, q)$ we arrive at a sufficiently large list that contains all possible Lie algebra pairs for any pseudo-Riemannian homogeneous manifold with $(p, q)$ signature. That is if $(\mathfrak{g}, \mathfrak{h})$ is the Lie algebra pair for a pseudo-Riemannian manifold $G/H$ then it must be isomorphic to one on this list.

**Remark 6.** If $(\mathfrak{g}, \mathfrak{h})$ is a Lie algebra pair from the list constructed by Cartan's algorithm then there is a Lie group $G$ and a Lie subgroup $H \subset G$ that has the pair $(\mathfrak{g}, \mathfrak{h})$ (see [1] for details) however, $H$ may or may not be closed in $G$. If $H$ is closed in $G$ then, by Theorem 2.3.2, $G/H$ is homogeneous and admits a pseudo-Riemannian metric $\eta$ such that $G$ acts by isometries that is $G/H$ is a pseudo-Riemannian homogeneous manifold.

### 3.2 Reductive Pairs

Recall that for a Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ to be reductive there must exist a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. We now use the basis used in Equation (3.4) to determine the conditions for $(\mathfrak{g}, \mathfrak{h})$ to be reductive. Choose a subspace $\mathfrak{m} \subset \mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ then a basis for $\mathfrak{m}$ may be written in the form $\tilde{\mathfrak{m}}_i = \mathfrak{m}_{i} + r^\gamma_i \mathfrak{h}_{\alpha}$ where $r^\gamma_i \in \mathbb{R}$. Consequently, the brackets are

$$[\mathfrak{h}_{\alpha}, \tilde{\mathfrak{m}}_j] = [\mathfrak{h}_{\alpha}, \mathfrak{m}_j + r^\beta_j \mathfrak{h}_{\beta}] = [\mathfrak{h}_{\alpha}, \mathfrak{m}_j] + [\mathfrak{h}_{\alpha}, r^\beta_j \mathfrak{h}_{\beta}]$$

$$= (H^k_{\alpha j} \mathfrak{m}_k + A^\gamma_{\alpha j} \mathfrak{h}_\gamma) + r^\beta_j J^\gamma_{\alpha \beta} \mathfrak{h}_\gamma = H^k_{\alpha j} \mathfrak{m}_k + (A^\gamma_{\alpha j} + r^\beta_j J^\gamma_{\alpha \beta}) \mathfrak{h}_\gamma$$

$$= H^k_{\alpha j} (\tilde{\mathfrak{m}}_k - r^\gamma_k \mathfrak{h}_\gamma) + (A^\gamma_{\alpha j} + r^\beta_j J^\gamma_{\alpha \beta}) \mathfrak{h}_\gamma = H^k_{\alpha j} \tilde{\mathfrak{m}}_k + (A^\gamma_{\alpha j} - r^\gamma_k H^k_{\alpha \gamma} + r^\beta_j J^\gamma_{\alpha \beta}) \mathfrak{h}_\gamma.$$ 

Therefore, the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ can be reductive if and only if there exists $r^\beta_j \in \mathbb{R}$ such that

$$A^\gamma_{\alpha j} = H^k_{\alpha j} r^\gamma_k - r^\beta_j J^\gamma_{\alpha \beta}.$$ 

**Remark 7.** It should also be noted, that for $(\mathfrak{g}, \mathfrak{h})$ to be reductive, it is sufficient that the Lie algebra pair $(\mathfrak{o}(p, q), \mathfrak{h})$ be reductive, see [3] for the details. However, this is not a necessary condition as the following example shows.
3.3 Construction of a Reductive Lie Algebra

Example 3.3.1. From Table A.1, subalgebra 14 is \( \mathfrak{h} = \{ h_1 = E_2, h_2 = F_1 - F_3 \} \). Although \( \mathfrak{h} \) is not reductive in \( \mathfrak{o}(2, 2) \) we will show that any Lie algebra \( \mathfrak{g} \) that satisfies Theorem 2.3.3 then the pair \( (\mathfrak{g}, \mathfrak{h}) \) will be reductive. The structure constants for \( \mathfrak{h} \) and the matrix representation \( \rho(\mathfrak{h}_\alpha) = H_{\alpha, \beta} \) give

\[
0 = J_{12}^1 = J_{12}^2,
\]

\[
-\frac{1}{2} = H_{14}^2 = H_{14}^4 = H_{22}^1 = H_{24}^1 = H_{23}^4,
\]

\[
\frac{1}{2} = H_{13}^1 = H_{11}^3 = H_{21}^2 = H_{23}^2 = H_{22}^3 = H_{24}^3,
\]

where all other \( H_{\alpha i}^j \) and \( J_{\alpha i}^j \) are zero. Let \( \{ m_i \} \) be vectors that complete \( \{ h_\alpha \} \) to a basis for \( \mathfrak{g} \) and let the structure constants \( A_{\alpha i}^j, B_{ij}^k, C_{ij}^\gamma \in \mathbb{R} \) be undetermined, as in Equation (3.4). We use Equation (3.6) from the Jacobi identity to get

\[
A_{16}^1 = A_{36}^1, \quad A_{26}^1 = A_{46}^1, \quad A_{25}^1 = A_{45}^1 + A_{36}^1, \quad A_{35}^1 = A_{15}^1 - A_{46}^1,
\]

\[
A_{16}^2 = A_{36}^2, \quad A_{26}^2 = A_{46}^2, \quad A_{25}^2 = A_{45}^2 + A_{36}^2, \quad A_{15}^2 = A_{35}^2 + A_{46}^2.
\]

We may substitute equalities (3.10) and (3.11) into Equation (3.9) which then admits the solution

\[
r_1^1 = -2A_{15}^1 + 2A_{46}^1, \quad r_2^1 = 2A_{45}^1, \quad r_3^1 = -2A_{15}^1, \quad r_4^1 = 2A_{45}^1 + 2A_{36}^1
\]

\[
r_1^2 = -2A_{35}^2, \quad r_2^2 = 2A_{45}^2, \quad r_3^2 = -2A_{35}^2 - 2A_{46}^2, \quad r_4^2 = 2A_{45}^2 + 2A_{36}^2.
\]

where \( A_{15}^1, A_{46}^1, A_{45}^1, A_{36}^1, A_{23}^1, A_{45}^2, A_{46}^2, A_{36}^2 \) are arbitrary. Let

\[
m_1 = m_1 - (2A_{15}^1 - 2A_{46}^1)h_1 - 2A_{35}^2 h_2, \quad \tilde{m}_1 = m_2 + 2A_{45}^1 h_1 + 2A_{45}^2 h_2,
\]

\[
m_3 = m_3 - 2A_{15}^1 h_1 - (2A_{35}^2 + 2A_{46}^2) h_2, \quad \tilde{m}_4 = m_4 + (2A_{45}^1 + 2A_{36}^1) h_1 + (2A_{45}^2 + 2A_{36}^2) h_2.
\]

Then the subspace \( \mathfrak{m} = \text{span}\{ \tilde{m}_i \} \) is an ad\( _{\mathfrak{h}} \) invariant subspace, regardless of the solutions to the remaining Jacobi conditions. Therefore, \( (\mathfrak{g}, \mathfrak{h}) \) is always reductive.
4.1 The Dual Formulation for Reductive Homogeneous Manifolds

We now rewrite the process of constructing $g$ in terms of a basis for the dual space $g^*$. We do this because there exists a basis for $g^*$ where the geometry of the pseudo-Riemannian manifold can be identified. Let $\{h_\alpha\}$ and $\{m_i\}$ be a basis for $g$ as in Chapter 3, and define the dual basis $\{\theta^\alpha\}$ and $\{\omega^\alpha\}$ by

$$\theta^\alpha(m_i) = \delta^\alpha_i, \quad \theta^\alpha(h_\beta) = 0, \quad \omega^\alpha(h_\beta) = \delta^\alpha_\beta, \quad \omega^\alpha(m_i) = 0.$$ 

By Remark 3, the structure equations of $g^*$ satisfy:

\begin{align*}
(4.1) & \quad d\theta^k = -\frac{1}{2} H^k_{ij} \omega^\alpha \wedge \theta^j - \frac{1}{2} B^k_{ij} \theta^i \wedge \theta^j \\
(4.2) & \quad d\omega^\gamma = -\frac{1}{2} A^\gamma_{ij} \omega^\alpha \wedge \theta^j - \frac{1}{2} C^\gamma_{ij} \theta^i \wedge \theta^j - \frac{1}{2} J^\gamma_{ij} \omega^\alpha \wedge \omega^\beta 
\end{align*}

where $A^\gamma_{ij}, B^k_{ij}, C^\gamma_{ij}, J^\gamma_{ij}, H^k_{ij} \in \mathbb{R}$ are the same as in Equation (3.4). For each $r$ and $\gamma$ let $\{\tilde{h}_r\}$ be a basis that completes $\{h_\gamma\}$ to a basis for $\mathfrak{o}(p, q)$ and let $\rho(h_\gamma) = H^k_{ij}, \tilde{h}_r$ be a complement in $\mathfrak{o}(2, 2)$ and $\tilde{H}^k_{ij}$ be the respective matrix representation of $\tilde{h}_r$. Let

\begin{align*}
(4.3) & \quad [h_\alpha, h_\beta] = J^\gamma_{ij} h_\gamma, \quad [h_\alpha, \tilde{h}_r] = K^\gamma_{ij} h_\gamma + \tilde{K}^t_{ij} \tilde{h}_t, \quad [\tilde{h}_r, \tilde{h}_s] = L^\gamma_{ij} h_\gamma + \tilde{L}^t_{ij} \tilde{h}_t 
\end{align*}

be the brackets for $\mathfrak{o}(p, q)$ in this basis. For each value of $i$ the matrix $B^k_{ij}$ may be assumed to be in $\mathfrak{o}(2, 2)$, see [3] for the details. Thus $B^k_{ij} = H^k_{ij}Q^\gamma_{ij} + \tilde{H}^k_{ij}P^r_{ij}$ for some unknown constants $Q^\gamma_{ij}, P^r_{ij} \in \mathbb{R}$. We may change the basis in $g^*$ by $\omega^\alpha = \tilde{\omega}^\alpha - Q^\alpha_i \theta^i$ and use equation (4.1) to get

\begin{align*}
 d\theta^k &= -\frac{1}{2} H^k_{ij} (\tilde{\omega}^\alpha - Q^\alpha_i \theta^i) \wedge \theta^j - \frac{1}{2} B^k_{ij} \theta^i \wedge \theta^j \\
 &= -\frac{1}{2} H^k_{ij} \tilde{\omega}^\alpha \wedge \theta^j - \frac{1}{2} (B^k_{ij} - H^k_{ij} Q^\alpha_i) \theta^i \wedge \theta^j \\
 &= -\frac{1}{2} H^k_{ij} \tilde{\omega}^\alpha \wedge \theta^j - \frac{1}{2} (\tilde{H}^k_{ij} P^r_{ij}) \theta^i \wedge \theta^j .
\end{align*}
Remark 8. Although not necessary for this thesis, the connection form $\omega$ and the curvature form $\Omega$ are

$$\omega^k_j = \frac{1}{2} \tilde{H}^k_{\alpha j} \xi^\alpha + \frac{1}{2} (\tilde{H}^k_{r j} P^r_i) \theta^i, \quad \Omega^k_j = d\omega^k_j - \omega^k_j \wedge \omega^k_j.$$ 

Continuing the calculations with Equation (4.1), we may, without any loss of generality, assume that we started in this new basis where $B^k_{ij} = \tilde{H}^k_{r j} P^r_i$. We examine these consequences and compare the results to [3]. After making this assumption, the coefficients of $m_i$ in Equation (3.7) of the Jacobi identities give

$$0 = H^k_{\alpha k} \tilde{H}^k_{r j} P^r_i - \tilde{H}^k_{r k} P^r_i H^k_{\alpha j} + \tilde{H}^k_{r j} P^r_i H^k_{\alpha k} + \tilde{H}^k_{r k} P^r_i H^k_{\alpha j} - \tilde{H}^k_{r j} P^r_i H^k_{\alpha k} + H^k_{\gamma k} A^\gamma_{[i j]}.$$ 

We may expand the skew symmetry to get

$$0 = H^k_{\alpha k} \tilde{H}^k_{r j} P^r_i - H^k_{\alpha k} \tilde{H}^k_{r j} P^r_i H^k_{\alpha j} + \tilde{H}^k_{r j} P^r_i H^k_{\alpha j} - \tilde{H}^k_{r j} P^r_i H^k_{\alpha k} - \tilde{H}^k_{r k} P^r_i H^k_{\alpha j} + H^k_{\gamma k} A^\gamma_{[i j]},$$

which may be rewritten as

$$0 = (H^k_{\alpha k} \tilde{H}^k_{r j} - \tilde{H}^k_{r k} H^k_{\alpha j}) P^r_i - (H^k_{\alpha k} \tilde{H}^k_{r j} - H^k_{\alpha k} \tilde{H}^k_{r j}) P^r_i + \tilde{H}^k_{r j} P^r_i H^k_{\alpha j} + H^k_{\alpha k} A^\gamma_{[i j]}.$$ 

Next, we may use the Lie bracket $[H_{\alpha}, \tilde{H}_{r}] = H^k_{\gamma k} K_{\alpha} - \tilde{H}^k_{\gamma k} K_{\alpha}$ to write

$$0 = (H^k_{\gamma j} K_{\alpha} + \tilde{H}^k_{\gamma j} K_{\alpha}) P^r_i - (H^k_{\gamma i} K_{\alpha} + \tilde{H}^k_{\gamma i} K_{\alpha}) P^r_i + \tilde{H}^k_{r j} H^k_{\alpha j} P^r_i + H^k_{\alpha k} A^\gamma_{[i j]},$$

which simplifies to

$$0 = H^k_{\gamma [i} A^\gamma_{j]} - H^k_{\gamma [i} P^r_i K_{\alpha j]} + \tilde{H}^k_{r i} H^k_{\alpha j} P^r_i - \tilde{H}^k_{r i} P^r_i K_{\alpha j]} + \tilde{H}^k_{r i} P^r_i K_{\alpha j]} - \tilde{H}^k_{r i} P^r_i K_{\alpha j]}.$$ 

Let $S^\gamma_{\alpha i} = A^\gamma_{\alpha i} - K_{\alpha i} P^r_i$ and $T^i_{\alpha i} = H^k_{\alpha i} P^r_k - \tilde{K}^t_{\alpha i} P^r_i$ and use this to write

$$0 = S^\gamma_{\alpha i} H^I_{\gamma j} - S^\gamma_{\alpha j} H^I_{\gamma i} + T^I_{\alpha i} H^t_{\gamma j} - T^I_{\alpha j} H^t_{\gamma i}.$$ 

Factoring gives

$$0 = S^\gamma_{\alpha k} (\delta^k H^I_{\gamma j} - \delta^k H^I_{\gamma i}) + T^I_{\alpha k} (\delta^k H^t_{\gamma j} - \delta^k H^t_{\gamma i}),$$

then multiplication by $\eta_{hl}$ yields

$$0 = S^\gamma_{\alpha k} (\delta^k \eta_{hl} H^I_{\gamma j} - \delta^k \eta_{hl} H^I_{\gamma i}) + T^I_{\alpha k} (\delta^k \eta_{hl} H^t_{\gamma j} - \delta^k \eta_{hl} H^t_{\gamma i}).$$
For fixed values of $t$ and $\gamma$ the matrices $H^1_{ti}, H^1_{\gamma h}$ are in $o(p, q)$. Hence, we have $\eta_{hi} H^1_{ti} + \eta_{hi} H^1_{t h} = 0$ and $0 = \eta_{hi} H^1_{\gamma i} + \eta_{hi} H^1_{\gamma h}$ which allows us to write

$$0 = S^\gamma_{ak} (\delta^k_i \eta_{hi} H^1_{\gamma j} + \delta^k_j \eta_{hi} H^1_{\gamma h}) + T^t_{ak} (\delta^k_i \eta_{hi} H^1_{ti} + \delta^k_j \eta_{hi} H^1_{t h}).$$

When we permute the indices $i, j, h$ in this equation and sum we obtain

$$0 = S^\gamma_{ak} (\delta^k_i \eta_{hi} H^1_{\gamma j} + \delta^k_j \eta_{hi} H^1_{\gamma h}) + T^t_{ak} (\delta^k_i \eta_{hi} H^1_{ti} + \delta^k_j \eta_{hi} H^1_{t h})$$

$$- S^\gamma_{ak} (\delta^k_j \eta_{hi} H^1_{\gamma i} + \delta^k_h \eta_{ji} H^1_{\gamma h}) - T^t_{ak} (\delta^k_j \eta_{hi} H^1_{ti} + \delta^k_h \eta_{ji} H^1_{t h})$$

$$+ S^\gamma_{ak} (\delta^k_h \eta_{ji} H^1_{\gamma i} + \delta^k_i \eta_{hi} H^1_{\gamma j}) - T^t_{ak} (\delta^k_h \eta_{ji} H^1_{ti} + \delta^k_i \eta_{hi} H^1_{t h}),$$

which, after cancelation yields

$$0 = 2 S^\gamma_{ak} \delta^k_i \eta_{hi} H^1_{\gamma j} + 2 T^t_{ak} \delta^k_i \eta_{hi} H^1_{t j}.$$ 

Since $H^1_{\gamma j}$ and $H^1_{t j}$ are linearly independent matrices for $t$ and $\gamma$, and both $\delta^k_i$ and $\eta_{hi}$ are non-degenerate we may conclude that $S^\gamma_{ai} = 0$ and $T^t_{ai} = 0$. That is

$$0 = A^\gamma_{ai} - K^\gamma_{ai} P^r_{i}$$

and

$$0 = P^r_{k} H^k_{ai} - \bar{K}^t_{ai} P^r_{i}.$$

Consequently, the structure constants for $g$ in this basis are

$$[h_a, h_b] = J^\gamma_{ab} h_{\gamma}, \quad [h_a, m_j] = H^k_{aj} m_k + K^r_{aj} P^r_{j} h_{\gamma}, \quad [m_i, m_j] = \bar{H}^k_{ij} P^r_{k} m_k + C^r_{i} h_{\gamma}$$

where some of the Jacobi identity remains unsolved but $P^r_{k}$ satisfies

$$0 = P^r_{k} H^k_{ai} - \bar{K}^t_{ai} P^r_{i}.$$ 

The condition for $(g, h)$ to be reductive, Equation (3.9), may be simplified to

$$K^r_{aj} P^r_{j} = r^k_{aj} H^k_{aj} - J^\gamma_{ab} r^\gamma_{bj}.$$ 

because $A = K^r_{aj} P^r_{j}$. Equation (4.6) is Equation (A.5) in [3]. It should be noted that some components of $P^r_{j}$ are known from solving Equation (4.5) and the remaining constants in Equation (4.6), $K^r_{aj}, H^k_{aj}$, and $J^\gamma_{ab}$ are all known.
4.2 Construction of a Non-reductive Lie Algebra

We now choose a subalgebra of $\mathfrak{o}(2, 2)$; we then impose Equation (4.4) and solve the remaining Jacobi identities. From Table A.1, subalgebra 10 is $\mathfrak{h} = \{h_1 = E_1 - E_3, h_2 = F_1 - F_3\}$. The Lie algebra pair $(\mathfrak{o}(2, 2), \mathfrak{h})$ is not reductive so it is possible that we could construct a Lie algebra $\mathfrak{g}$ such that $(\mathfrak{g}, \mathfrak{h})$ is not reductive. We proceed to solve equation (4.5) for $P_i^\tau$ to get

\[
P_3^2 = -2P_1^1, \quad 0 = P_3^1 = P_3^3 = P_3^4, \quad 0 = P_2^1 = P_2^2 = P_2^3 = P_2^4
\]

\[
P_4^4 = -2P_1^3, \quad 0 = P_4^1 = P_4^2 = P_4^3
\]

where all other values are free. For clarity, change variables by letting

\[
P_1^1 = p_1, \quad P_1^2 = p_2, \quad P_1^3 = p_3, \quad P_1^4 = p_4.
\]

Equation (4.6) is

\[
0 = p_1, \quad 0 = p_2 + r_3^1, \quad 0 = r_2^1, \quad 0 = r_4^1,
\]

\[
0 = p_3, \quad 0 = p_4 + r_4^2, \quad 0 = r_2^2, \quad 0 = r_3^2
\]

which if $(p_1, p_3) \neq (0, 0)$ has the solution

\[
r_2^1 = 0, \quad r_3^1 = -p_2, \quad r_4^1 = 0,
\]

\[
r_2^2 = 0, \quad r_3^2 = 0, \quad r_4^2 = -p_4.
\]

Consequently, we must assume $(p_1, p_3) \neq (0, 0)$ for $(\mathfrak{g}, \mathfrak{h})$ to not be reductive. The linear terms from the Jacobi identity yield

\[
C_{23}^2 = 0, \quad C_{24}^1 = 0, \quad C_{12}^1 = -C_{34}^1, \quad C_{12}^2 = C_{34}^2, \quad C_{13}^1 = C_{41}^2, \quad C_{23}^1 = C_{24}^2.
\]

After we substitute these values into the Jacobi identities, the simplest remaining quadratic equations give

\[
C_{34}^2 = -p_1 p_4, \quad C_{24}^2 = -2p_1 p_3, \quad C_{34}^1 = p_2 p_3.
\]
We may substitute these solutions into the remaining Jacobi identities to get:

\[ 0 = p_1(p_2 + p_4), \quad 0 = p_1(p_4 - 3p_2), \quad 0 = p_3(p_2 + p_4), \quad 0 = p_3(3p_4 - p_2), \]

which has solutions \( p_2 = 0 \) and \( p_4 = 0 \) because we have assumed that \( (p_1, p_3) \neq (0, 0) \).

These solutions simplify the Jacobi identity to only two unsatisfied equations:

\[
0 = 5C_{13}^2 p_3 - 3C_{14}^2 p_1, \\
0 = -5C_{14}^1 p_1 + 3C_{14}^3 p_3,
\]

Since we have assumed that \( (p_1, p_3) \neq (0, 0) \), we may multiply the first equation by \( p_3 \) and the second equation by \( p_1 \) then sum the results to get

\[ 0 = C_{13}^2 p_3^2 - C_{14}^1 p_1^2. \]

We have assumed that \( (p_1, p_3) \neq (0, 0) \) so this equation has the solutions \( C_{13}^2 = tp_1^2 \) and \( C_{14}^1 = tp_3^2 \) where \( t \in \mathbb{R} \). These two solutions transform the remaining two Jacobi equations to

\[
(4.7) \quad 0 = p_3(3C_{14}^2 - 5tp_1p_3) \quad \text{and} \quad 0 = -p_1(3C_{14}^2 - 5tp_1p_3)
\]

Consequently, \( C_{14}^2 = \frac{5}{3} tp_1p_3 \), since we have assumed \( (p_1, p_3) \neq (0, 0) \). Finally, we impose this solution and the Jacobi identities are satisfied. Hence, \( \mathfrak{g} \) is a Lie algebra having \( p_1, p_3, \) and \( t \) as parameters.

We will analyze this parameterized Lie algebra as two cases: either \( 0 \in \{p_1, p_3\} \neq \{0\} \) or \( 0 \not\in \{p_1, p_3\} \).

4.2.1 Case 1

If \( 0 \not\in \{p_1, p_3\} \neq \{0\} \), let \( \epsilon^a \) be a basis for \( \mathfrak{g}^* \) defined as

\[
\begin{align*}
\epsilon^1 &= -p_1\theta^3 - p_3\theta^4, \\
\epsilon^2 &= \theta^1, \\
\epsilon^3 &= p_1\omega^1 + p_3\omega^2, \\
\epsilon^4 &= p_3\theta^3 - p_1\theta^4, \\
\epsilon^5 &= (p_3^2 - p_1^2)(\frac{t}{3}\theta^1 - \theta^2) - p_3\omega^1 + p_1\omega^2, \\
\epsilon^6 &= \frac{t}{2}(p_1 + p_3)^2\theta^1 - \frac{p_3^3\omega^1 + p_1^3\omega^2}{(p_3 + p_1)^2}.
\end{align*}
\]
Let $e_b$ be the unique vectors in $g$ such that $e^a(e_b) = \delta^a_b$. In this basis, $g$ has the structure constants in \textit{vi.)} of Theorem 1.0.1. The isotropy algebra is $\mathfrak{h} = \text{span}\{\mathfrak{h}_1, \mathfrak{h}_2\}$ where

\[
\begin{align*}
\text{if } p_1 &= 0 & \mathfrak{h}_1 &= -p_3(e_5 + e_6), & \mathfrak{h}_2 &= p_3e_3. \\
\text{if } p_3 &= 0 & \mathfrak{h}_1 &= p_1e_3, & \mathfrak{h}_2 &= p_1(e_5 - e_6).
\end{align*}
\]

The automorphism $e_6 \mapsto -e_6$ makes these isotropy algebras equivalent. Consequently, we may let $\mathfrak{h} = \text{span}\{e_3, e_5 + e_6\}$, which gives us the Lie algebra pair in \textit{vi.)} of Theorem 1.0.1.

\[4.2.2 \quad \text{Case 2}\]

If $0 \notin \{p_1, p_3\}$, let $e^a$ be a basis for $g^*$ defined as

\[
\begin{align*}
e^1 &= -p_1\theta^3 - p_3\theta^4, & e^3 &= 2p_1p_3(\frac{t}{3}\theta^1 - \theta^2) - p_1\omega^1 - p_3\omega^2; & e^5 &= \sqrt{2}(p_1\omega^1 - p_3\omega^2), \\
e^2 &= -\theta^1, & e^6 &= -2p_3p_1(\frac{2t}{3}\theta^1 + \theta^2) + p_1\omega^1 + p_3\omega^2; & e^4 &= \sqrt{2}(p_1\theta^3 - p_3\theta^4).
\end{align*}
\]

Let $e_b$ be the unique vectors in $g$ such that $e^a(e_b) = \delta^a_b$. In this basis, $g$ has the structure constants in \textit{vi.)} of Theorem 1.0.1. In this basis, the isotropy subalgebra is $\mathfrak{h} = \{\mathfrak{h}_1, \mathfrak{h}_2\}$ where

\[
\begin{align*}
\mathfrak{h}_1 &= p_1(-e_3 + \sqrt{2}e_5 + e_6), & \mathfrak{h}_2 &= p_3(-e_3 - \sqrt{2}e_5 + e_6).
\end{align*}
\]

Consequently, $\mathfrak{h} = \text{span}\{\mathfrak{h}_1, \mathfrak{h}_2\} = \text{span}\{\mathfrak{h}_1/p_1 + \mathfrak{h}_2/p_3, \mathfrak{h}_1/p_1 - \mathfrak{h}_2/p_3\} = \text{span}\{e_5, e_3 - e_6\}$, which gives us the Lie algebra pair in \textit{v.)} of Theorem 1.0.1.
CHAPTER 5

PROOF OF THEOREM 1.0.1

We are now in a position to prove Theorem 1.0.1 by using each subalgebra of \( \mathfrak{o}(2, 2) \) as the isotropy algebra \( \mathfrak{h} \) and then build all possible Lie algebras \( \mathfrak{g} \) that contain \( \mathfrak{h} \) as in Theorem 2.3.3. To simplify this task we may only look at subalgebras inequivalent under inner automorphisms of \( \mathfrak{o}(2, 2) \), see [3] for details. For each inequivalent subalgebra of \( \mathfrak{o}(2, 2) \) we will use the basis provided in Table A.1 which also gives a complementary basis \( \mathfrak{r} \). We can prove Theorem 1.0.1 by considering 3 cases.

5.1 Reductive Lie Algebra Pairs \( (\mathfrak{o}(2, 2), \mathfrak{r}) \)

The first case we consider are the subalgebras

\[ 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 21, 22, 23, 27, \text{ and } 28 \]

from Table A.1. The corresponding complement \( \mathfrak{r} \) for each of these subalgebras \( \mathfrak{h} \) is an \( \text{ad}_{\mathfrak{h}} \) invariant complement in \( \mathfrak{o}(2, 2) \), so the Lie algebra pair \( (\mathfrak{o}(2, 2), \mathfrak{r}) \) is reductive. By Remark 7, the Lie algebra pair \( (\mathfrak{g}, \mathfrak{r}) \) is reductive for any constructed \( \mathfrak{g} \).

5.2 Reductive Lie Algebra Pairs \( (\mathfrak{g}, \mathfrak{r}) \)

Each of the subalgebras

\[ 14, 15, 16, 17, 18, 19, 20, 24, 25, 26, 29, 30, \text{ and } 31 \]

in Table A.1 behave similar to Example 3.3.1 in that for any Lie algebra \( \mathfrak{g} \) that we construct the Lie algebra pair \( (\mathfrak{g}, \mathfrak{r}) \) will be reductive. Choose one of these subalgebras and let \( \mathfrak{r} \) be its complement from Table A.1. We now solve Equation (4.5) for components of \( P_\mathfrak{r}^\mathfrak{g} \) and impose this solution into equation (4.4) to partially create a Lie algebra as in Equation (4.4). Then, without making any additional assumptions about \( P_\mathfrak{r}^\mathfrak{g} \), Equation (4.6) may always be solved. Consequently, the Lie algebra pair \( (\mathfrak{g}, \mathfrak{r}) \) will always be reductive, regardless of the particular Lie algebra \( \mathfrak{g} \) constructed. The Maple worksheets in Chapter 6 provide the necessary calculations for these subalgebras. The calculations for subalgebra 14, which was
done in Example 3.3.1, are redone in this manner as the first Maple worksheet in Chapter 6.

5.3 Non-reductive Lie Algebra Pairs \((\mathfrak{g}, \mathfrak{h})\)

The subalgebras

1, 2, and 10

in Table A.1 behave like Example 4.2 and lead to the classification in Theorem 1.0.1. Choose one of these subalgebras and let \(\mathfrak{h}\) be its complement from Table A.1. We now solve Equation (4.5) for components of \(P^j_i\) and impose this solution into equation (4.4) to partially create a Lie algebra as in Equation (4.4). Now, additional assumptions about \(P^j_i\) must be made to solve Equation (4.6). We then deny those assumptions about \(P^j_i\) so the Lie algebra pair \((\mathfrak{g}, \mathfrak{h})\) will not be reductive. We use this denial to help solve the remaining Jacobi identities on a case by case basis to get a parameterized family of Lie algebras. Finally, we classify these Lie algebras to get the Lie algebra pairs \((\mathfrak{g}, \mathfrak{h})\) in Theorem 1.0.1. In summary

i.) **Subalgebra 1:** The subalgebras pertaining to \(\epsilon = 1\) and \(\epsilon = -1\) are conjugate in \(GL(\mathbb{R}^4)\) and hence give equivalent Lie algebra pairs \((\mathfrak{g}, \mathfrak{h})\), see [3] for the details. Consequently, we may assume \(\epsilon = -1\) without any loss of generality. This subalgebra will give the Lie algebra pairs in sections i.), ii.), and iii.) of Theorem 1.0.1.

ii.) **Subalgebra 2:** This subalgebra will give the Lie algebra pair in iv.) of Theorem 1.0.1.

iii.) **Subalgebra 10:** This subalgebra is the one calculated in Example 4.2, which gives the Lie algebra pairs in sections v.) and vi.) of Theorem 1.0.1.

Chapter 7 provides the necessary calculations for these subalgebras as Maple worksheets.
CHAPTER 6
CALCULATIONS FOR REDUCTIVE LIE ALGEBRA PAIRS (g, h)

In this chapter the calculations required for Section 5.2 are done as Maple worksheets that use the Vessiot package and subroutines in Appendix E. The subalgebras 14, 15, 16, 17, 18, 19, 20, 24, and 31 all lead to reductive Lie algebra pairs (g, h), regardless of the remaining unsolved Jacobi identities.

6.1 Subalgebra 14

> restart;
> with(Vessiot): with(Koszul):with(Mubar):with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> read(cat(getenv(USERPROFILE),"\My\Documents\Thesis\Maple\theorems.txt")): 
> read(cat(getenv(USERPROFILE),"\My\Documents\Thesis\Maple\renner.txt")): 
Load isotropy algebra.
> h:=convert(map(collect,evalm(s14 &* Basis),{alpha,epsilon}),list);

\[ h := [E_2, F_1 - F_3] \]

Load complementary basis.
> _h:=convert(map(collect,evalm(_s14 &* Basis),{alpha,epsilon}),list);

\[ _h := [F_1 + F_3, F_2, E_1, E_3] \]

Fix a matrix representation for o22.
> Mat_Basis:=map(evalm,[op(_h),op(h)]):

> O22:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(O22):
> Isodim:=nops(h);
Calculate the $P = 0$ matrix from Equation 4.5.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix $P = 0$ consequently Equation 4.6 may always be solved by letting $r = 0$.

Thus Equation 4.6 has a solution so the Lie algebra pair $(g, h)$ is always reductive.

### 6.2 Subalgebra 15

Load isotropy algebra.

\[
h := [E3, F1 - F3]
\]

Load complementary basis.

\[
\_h := [F1 + F3, E1, E2, F2]
\]

Fix a matrix representation for $\text{so22}$. 

\[
isodim := 2
\]

\[
\text{Codim} := 4
\]
> Mat_Basis:=map(evalm,[op(_h),op(h)]):

> O22:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(O22):
> Isodim:=nops(h);

\textit{Isodim} := 2

> Codim:=6-Isodim;

\textit{Codim} := 4

Calculate the $P = 0$ matrix from Equation 4.5.

> P_matrix_4_5(Codim,Isodim,o22);

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The matrix $P = 0$ consequently Equation 4.6 may always be solved by letting $r = 0$.

Thus Equation 4.6 has a solution so the Lie algebra pair $(g,h)$ is always reductive.

6.3 Subalgebra 16

> restart;
> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected

> read(cat(getenv(USERPROFILE),"\MyDocuments\Thesis\Maple\theorems.txt"));
> read(cat(getenv(USERPROFILE),"\MyDocuments\Thesis\Maple\renner.txt"));

Load isotropy algebra.

> h:=convert(map(collect,evalm(s16 &* Basis),{alpha,epsilon}),list);

\[ h := [F2, F1 - F3] \]
Load complementary basis.
> \_h:=convert(map(collect,evalm(_s16 &*
  Basis),\{alpha,epsilon\}),list);

\_h := [F1 + F3, E1, E2, E3]

Fix a matrix representation for o22.

> Mat_Basis:=map(evalm,[op(_h),op(h)]):

> o22:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(o22):
> Isodom:=nops(h);

Isodom := 2

> Codim := 6-Isodom;

Codim := 4

Calculate the \( P = 0 \) matrix from Equation 4.5.

> P_matrix_4_5(Codim,Isodom,o22);

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The matrix \( P = 0 \) consequently Equation 4.6 may always be solved by letting \( r = 0 \).

Thus Equation 4.6 has a solution so the Lie algebra pair \((g,h)\) is always reductive.

6.4 Subalgebra 17

> restart;
> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected
Load isotropy algebra.
> h:=convert(map(collect,evalm(s17 * Basis),{alpha,epsilon}),list);

\[
h := [cE2 + F2, F1 - F3]
\]

Load complementary basis.
> _h:=convert(map(collect,evalm(_s17 * Basis),{alpha,epsilon}),list);

\[
_h := [F1 + F3, E1, E2, E3]
\]

Fix a matrix representation for \(022\).
> Mat_Basis:=map(evalm,[op(_h),op(h)]):

> 022:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(022):

> Isodim:=nops(h);

\[
Isodim := 2
\]

> Codim:=6-Isodim;

\[
Codim := 4
\]

Calculate the \(P = 0\) matrix from Equation 4.5.
> P_matrix_4_5(Codim,Isodim,o22);

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The matrix \(P = 0\) consequently Equation 4.6 may always be solved by letting \(r = 0\).

Thus Equation 4.6 has a solution so the Lie algebra pair \((g,h)\) is always reductive.
6.5 Subalgebra 19

> restart;
> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected

> read(cat(getenv(USERPROFILE),"\MyDocuments\Thesis\Maple\theorems.txt"));
> read(cat(getenv(USERPROFILE),"\MyDocuments\Thesis\Maple\renner.txt"));

Load isotropy algebra.
> h := convert(map(collect,evalm(s19 &* Basis),{alpha,epsilon}),list);

\[
h = [(E_1 - E_3)\epsilon + F_2, F_1 - F_3]
\]

Load complementary basis.
> _h:=convert(map(collect,evalm(_s19 &* Basis),{alpha,epsilon}),list);

\[
_h = [F_1 + F_3, (-E_1 + E_3)\epsilon + F_2, E_1 + E_3, E_2]
\]

Fix a matrix representation for o22.
> Mat_Basis:=map(evalm,[op(_h),op(h)]):
> O22:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(O22):
> Isodim:=nops(h);

\[
Isodim := 2
\]

> Codim:=6-Isodim;

\[
Codim := 4
\]

Calculate the \(P = 0\) matrix from Equation 4.5.
> P_matrix_4_5(Codim,Isodim,o22);

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
The matrix $P = 0$ consequently Equation 4.6 may always be solved by letting $r = 0$.

Thus Equation 4.6 has a solution so the Lie algebra pair $(g, h)$ is always reductive.

6.6 Subalgebra 20

> restart;

> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected

> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\theorems.txt"));
> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\renner.txt"));

Load isotropy algebra.

> h:=convert(map(collect,evalm(s20 &* Basis),{alpha,epsilon}),list);

\[ h := [E2 + F2, (E1 - E3)\epsilon + F1 - F3] \]

Load complementary basis.

> _h:=convert(map(collect,evalm(_s20 &* Basis),{alpha,epsilon}),list);

\[ _h := [F1 + F3, E1 + E3, E2 - F2, (-E1 + E3)\epsilon + F1 - F3] \]

Fix a matrix representation for $\text{o}22$.

> Mat_Basis:=map(evalm,subs(epsilon=1,[op(_h),op(h)]));

> 022:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):

> Lie_alg_init(022):

> Isodim:=nops(h);

\[ Isodim := 2 \]

> Codim:=6-Isodim;

\[ Codim := 4 \]

Calculate the $P = 0$ matrix from Equation 4.5.
Create the partial Lie algebra as in Equation 4.4.

```maple
> My_alg:=create_algebra_4_4(Codim,Isodim,P,o22):
> Lie_alg_init(structure_constants_array_to_Lie_algebra_data(My_alg,g))
```

Check if this must be reductive by solving Equation 4.6.

```maple
> reductive_equations_4_6(Codim,Isodim,o22);

\[
\{0, -r24, -2 p2 - 2 p1 + r14 + r13, 2 p2 + 2 p1 - r14 - r13, -r23, -2 r21, 2 r21, -r11 + r24 + r23, r12, r13 + r22, r14 + r22, -r12, -r11, r24, 2 r11, -r14 - r22, -r13 - r22, r11 - r24 - r23, r23\}
\]

```maple
> solve(%,{r22, r23, r24, r14, r11, r21, r13, r12});

\[
\{r21 = 0, r22 = -p2 - p1, r12 = 0, r24 = 0, r11 = 0, r13 = p2 + p1, r14 = p2 + p1, r23 = 0 \}
\]

Equation 4.6 has a solution thus the Lie algebra pair \((g,h)\) is always reductive.

6.7 Subalgebra 24

```maple
> restart;

> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected

> read(cat(getenv(USERPROFILE),"\MyDocuments\Thesis\Maple\theorems.txt"));
> read(cat(getenv(USERPROFILE),"\MyDocuments\Thesis\Maple\renner.txt"));

Load isotropy algebra.

```maple
> h:=convert(map(collect,evalm(s24 &*
Basis),{alpha,epsilon}),list);
```

\[
h := [E2, F2, F1 - F3]
\]
Load complementary basis.
> _h:=convert(map(collect,evalm(_s24 &* Basis),{alpha,epsilon}),list);

\[ h := [F1 + F3, E1, E3] \]

Fix a matrix representation for \( o22 \).
> Mat_Basis:=map(evalm,[op(_h),op(h)]):
> o22:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(o22):
> Isodim:=nops(h);

\[ Isodim := 3 \]
> Codim:=6-Isodim;

\[ Codim := 3 \]

Calculate the \( P = 0 \) matrix from Equation 4.5.
> P_matrix_4_5(Codim,Isodim,o22);

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

The matrix \( P = 0 \) consequently Equation 4.6 may always be solved by letting \( r = 0 \).

Thus Equation 4.6 has a solution so the Lie algebra pair \((g,h)\) is always reductive.

6.8 Subalgebra 25
> restart;
> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected
Load isotropy algebra.
> h:=convert(map(collect,evalm(s25 &*
> Basis),{alpha,epsilon}),list);

\[
h := [E3, F2, F1 - F3]
\]

Load complementary basis.
> _h:=convert(map(collect,evalm(_s25 &*
> Basis),{alpha,epsilon}),list);

\[
_h := [F1 + F3, E1, E2]
\]

Fix a matrix representation for o22.
> Mat_Basis:=map(evalm,[op(_h),op(h)]):
> o22:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(o22):
> Isodim:=nops(h);

\[
Isodim := 3
\]
> Codim:=6-Isodim;

\[
Codim := 3
\]

Calculate the \( P = 0 \) matrix from Equation 4.5.
> P_matrix_4_5(Codim,Isodim,o22);

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix \( P = 0 \) consequently Equation 4.6 may always be solved by letting \( r = 0 \).

Thus Equation 4.6 has a solution so the Lie algebra pair \((g,h)\) is always reductive.
6.9 Subalgebra 26

> restart;

> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected

> read(cat(getenv(USERPROFILE),"\My
> Documents\Thesis\Maple\theorems.txt"));
> read(cat(getenv(USERPROFILE),"\My
> Documents\Thesis\Maple\renner.txt"));:

Load isotropy algebra.

> h:=convert(map(collect,evalm(s26 &*
> Basis),{alpha,epsilon}),list);

\[
h \ := [E2 + \alpha F2, E1 - E3, F1 - F3]
\]

Load complementary basis.

> _h:=convert(map(collect,evalm(_s26 &*
> Basis),{alpha,epsilon}),list);

\[
_h 
\ := [E1 + E3, F1 + F3, F2]
\]

Fix a matrix representation for o22.

> Mat_Basis:=map(evalm, [op(_h ) ,op(h)]):

> 022:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(022):
> Isodim:=nops(h);

\[
Isodim \ := 3
\]

> Codim:=6-Isodim;

\[
Codim \ := 3
\]

Calculate the \( P = 0 \) matrix from Equation 4.5.

> P_matrix_4_5(Codim,Isodim,o22);

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
The matrix $P = 0$ consequently Equation 4.6 may always be solved by letting $r = 0$.

Thus Equation 4.6 has a solution so the Lie algebra pair $(g,h)$ is always reductive.

6.10 Subalgebra 29

```maple
restart;

with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected

> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\theorems.txt"));
> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\renner.txt"));

Load isotropy algebra.

> h:=convert(map(collect,evalm(s29 &* Basis),{alpha,epsilon}),list);

\[ h = [E2, E1 - E3, F2, F1 - F3] \]

Load complementary basis.

> _h:=convert(map(collect,evalm(_s29 &* Basis),{alpha,epsilon}),list);

\[ _h = [E1 + E3, F1 + F3] \]

Fix a matrix representation for $o22$.

> Mat_Basis:=map(evalm, [op(_h),op(h)]):
> 022:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(022):
> Isodim:=nops(h);

\[ Isodim := 4 \]

> Codim:=6-Isodim;

\[ Codim := 2 \]

Calculate the $P = 0$ matrix from Equation 4.5.
The matrix \( P = 0 \) consequently Equation 4.6 may always be solved by letting \( r = 0 \).

Thus Equation 4.6 has a solution so the Lie algebra pair \((g,h)\) is always reductive.

6.11 Subalgebra 30

```maple
restart;
> with(Vessiot):with(Koszul):with(Mubar):with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> read(cat(getenv(USERPROFILE),"\MyDocuments\Thesis\Maple\theorems.txt")): > read(cat(getenv(USERPROFILE),"\MyDocuments\Thesis\Maple\renner.txt"));

Load isotropy algebra.
> h:=convert(map(collect,evalm(s30 &* Basis),{alpha,epsilon}),list);

\[
h = [E_1, E_2, E_3, F_1 - F_3]
\]

Load complementary basis.
> _h:=convert(map(collect,evalm(_s30 &* Basis),{alpha,epsilon}),list);

\[
_h = [F_1 + F_3, F_2]
\]

Fix a matrix representation for \( \mathfrak{o}_{22} \).
> Mat_Basis:=map(evalm,[op(_h),op(h)]):
> O22:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22):
> Lie_alg_init(O22):
> Isodim:=nops(h);

\[
Isodim := 4
\]
> Codim:=6-Isodim;
```
Codim := 2

Calculate the $P = 0$ matrix from Equation 4.5.

```maple
> P_matrix_4_5(Codim,Isodim,o22);
```

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix $P = 0$ consequently Equation 4.6 may always be solved by letting $r = 0$.

Thus Equation 4.6 has a solution so the Lie algebra pair $(g,h)$ is always reductive.

6.12 Subalgebra 31

```maple
> restart;
> with(Vessiot):with(Koszul):with(Mubar):with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> read(cat(getenv(USERPROFILE), "\My Documents\Thesis\Maple\theorems.txt")) :
> read(cat(getenv(USERPROFILE), "\My Documents\Thesis\Maple\renner.txt")): Load isotropy algebra.
> h := convert(map(collect, evalm(s31 &* Basis), {alpha, epsilon}), list);
```

\[
h := [E1, E2, E3, F2, F1 - F3]
\]

Load complementary basis.

```maple
> _h := convert(map(collect, evalm(_s31 &* Basis), {alpha, epsilon}), list);
```

\[
_h := [F1 + F3]
\]

Fix a matrix representation for o22.

```maple
> Mat_Basis := map(evalm, [op(_h), op(h)]):
> o22 := matrix_algebra_to_Lie_algebra_data(Mat_Basis, o22):
```
Calculate the $P = 0$ matrix from Equation 4.5.

\[ P_{\text{matrix}_4.5}(\text{Codim}, \text{Isodim}, 022); \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix $P = 0$ consequently Equation 4.6 may always be solved by letting $r = 0$.

Thus Equation 4.6 has a solution so the Lie algebra pair $(g, h)$ is always reductive.
CHAPTER 7
CALCULATIONS FOR NON-REDUCTIVE LIE ALGEBRA PAIRS (g, h)

In this chapter the calculations required for Section 5.3 are done as Maple worksheets that use the Vessiot package and subroutines in Appendix E. The subalgebras 1, 2, and 10 lead to the subalgebras in Theorem 1.0.1.

7.1 Subalgebra 1

\[ \text{restart;} \]
\[ \text{with(Vessiot):with(Koszul):with(Mubar):with(linalg):} \]
\[ \text{Warning, the protected names norm and trace have been redefined and unprotected} \]
\[ \text{read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\theorems.txt")));} \]
\[ \text{read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\patch.txt")));} \]
\[ \text{read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\renner.txt")));} \]

Load isotropy algebra.
\[ \text{h:=convert(map(collect,innerprod(sl, Basis),\{epsilon\}),list);} \]
\[ h := [(E_1 - E_3) \epsilon + F_1 - F_3] \]

Load complementary basis.
\[ \text{\_h:=convert(map(collect,innerprod(\_s1, Basis),\{epsilon\}),list);} \]
\[ \_h := [(-E_1 + E_3) \epsilon + F_1 - F_3, F_2, (-E_1 + E_3) \epsilon + F_1 + F_3, (E_1 + E_3) \epsilon + F_1 - F_3, E_2] \]

Fix a representation of o22 adapted to this subalgebra when \( \varepsilon = 1 \).
\[ \text{Mat_Basis:=map(evalm,subs(epsilon=-1,[op(\_h),op(h)]));} \]

Initialize o22 in this basis.
\[ \text{o22:=matrix_algebra_to_Lie_algebra_data(Mat_Basis,o22);} \]
\[ \text{Lie_alg_init(o22,\{"H1","H2","H3","H4","H5","h1"\},\{"eta1","eta2","eta3} \]
\[ ","eta4","eta5","omega1\}); \]
\[ \text{Isodim:=nops(h);} \]
Codim:=6-Isodim:

Compute the $P = 0$ matrix from equation 4.5.

```maple
PO:=P_matrix_4_5(Codim,Isodim,o22);
```

\[
P_0 :=
\begin{bmatrix}
p1 & p2 & p3 & p4 \\
-2p3 + 2p4 + p6 & 0 & 2p2 + p7 + 2p5 & 2p2 + p7 \\
p5 & 0 & 0 & 0 \\
2p2 + p5 & 0 & 0 & 0 \\
p6 & 0 & p7 + 4p2 + 2p5 & p7
\end{bmatrix}
\]

Change variables in $P_0$ to make the calculations a little easier.

```maple
P:=matrix([[p1,p2,p3+p4,p4-p3],[2*p5-2*p3,0,p6+p7-p2,0],[p7-p2,0,0,0],[p2+p7,0,0,0],[2*p3+2*p5,0,p2+p6+p7,-p2+p6-p7]]);
```

Check to ensure that this is a legitimate change of variables.

```maple
subs({p1=P1,p2=P2,p3=P3,p4=P4,p5=P5,p6=P6,p7=P7},op(PO));
```

```maple
evalm(subs({p1=P1,p2=P2,p3=P3,p4=P4,p5=P5,p6=P6,p7=P7},op(PO))-P);
```

```maple
solve(convert(%,set),{P1,P2,P3,P4,P5,P6,P7});
```

\[
\{P1 = p1, P2 = p2, P3 = p3 + p4, P5 = p7 - p2, P6 = 2p3 + 2p5, P7 = -p2 + p6 - p7, P4 = p4 - p3\}
\]

Therefore the change of variables is legitimate and we may initialize the partial Lie Algebra as in Equation 4.4.

```maple
My_alg:=create_algebra_4_4(Codim,Isodim,P,o22):
```

```maple
My_alg.init(structure_constants_array_to_Lie_algebra_data(My_alg,alg),["m1","m2","m3","m4","h1"],["theta1","theta2","theta3","theta4","omega1"]) :
```

Check if this must be reductive by calculating Equation 4.6.

```maple
TBS:=reduce_equation_set(reductive_equations_4_6(Codim,Isodim,alg));
```

\[
TBS := \{2p5 + r14 - r13, p6 - p7 + r12, p6 + p7 - r12\}
\]
Solve for $p6$-values, to determine if reductive.

\[ \text{ans} := \text{subs}\left( \{r12=-p6+p7, r13=2*p5+r14\}, \text{TBS}\right); \]

\[
\text{ans} := \{0, 2p6\}
\]

To be non-reductive, $p6 \neq 0$ and use this to assist solving the Jacobi identities ($d^2 = 0$).

\[ \text{Jacobi} := \{\}; \]

\[ \text{Cobasis} := \text{frameBaseForms}(); \]

\[ \text{for} \ \theta \ \text{in} \ \text{Cobasis do} \]

\[ \text{Jacobi} := \text{Jacobi union coeff_set}(\text{ext_d}(\text{ext_d}(\theta))): \od; \]

\[ \text{Jacobi} := \text{reduce_equation_set}(\text{Jacobi}); \]

\[
\text{Jacobi} := \{2 p5 p6 - 4 p5 p7 + 2 p6 p4 - 4 p6 p3 + 2 p7 p3 - c341 + 2 p5 p2 + c121, p2 p7, -c121 - 2 p2 p4 + 4 p2 p3 + 2 p6 p4 - 4 p6 p3 + 2 p5 p2 + 2 p5 p6 - 4 p5 p7, 2 p2 p7 - c241 - c231, -c131 - c141 - 4 p3 p4 + 2 p2^2 + 4 p2 p6 - 6 p1 p7 - 6 p7^2 + 4 p5 p3, 4 p6 p3 + 2 p6 p4 + 2 p7 p3 + 2 p5 p6 + 4 p5 p7 - c241 + 2 p5 p2 - c121, c121 - 4 p2 p3 - 2 p2 p4 + 4 p6 p3 + 2 p6 p4 + 2 p5 p6 + 4 p5 p7 + 2 p5 p2, c121 p6 - 2 c121 p7 - c341 p7 - c341 p4 + 3 c241 p3 - c231 p4 + c231 p3 + 2 c121 p2, -c241 p2 + c231 p2, 2 p7^2 - c241 + c231, p2 p7 + c231 + p7^2, -p2 p7 - c241 + p7^2, 2 p2 p4 - c341 + 8 p2 p3 - 4 p7 p4 + 4 p6 p3 + 2 p7 p3, -2 p2 p4 + c341 + 8 p2 p3 - 4 p7 p4 + 4 p6 p3 - 2 p7 p3, c121 p6 + 2 c121 p7 - c341 p7 - c341 p3 - c241 p4 - 3 c231 p3 - c231 p4 + 2 c121 p2, c231 + c241 + 2 p2 p7, 2 c131 p6 - 3 c131 p7 - 2 c141 p6 - 3 c141 p7 - 2 c341 p4 - c241 p1 - c241 p2 - c241 p7 + c231 p1 + c231 p7 - c231 p2 + 2 p5 c341 - 2 c141 p2 + 2 c131 p2 + 2 c121 p3\}
\]

\[ \text{sln} := \{\}; \]

\[ \text{eqs} := \text{select}(x \rightarrow \text{type}(x, '\ast'), \text{Jacobi}); \]

\[ \text{eqs} := \{p2 p7\}\]

Thus any terms with $p2 p7$ as a factor we may simplify to zero.

\[ \text{ZERO_EQUATIONS} := \text{map}(x \rightarrow x=0, \text{eqs}); \]

\[ \text{ZERO_EQUATIONS} := \{p2 p7 = 0\}\]

\[ \text{eqs} := \text{reduce_equation_set}(\text{Jacobi}, \text{ZERO_EQUATIONS}); \]
\[ \text{Solve any linear equations.} \]

\begin{verbatim}
> eqs := select(x->type(x,linear ), eqs);
  
  eqs := \{ c241 + c231 \}

> ans := solve(eqs,\{c241\});
  
  ans := \{ c241 = -c231 \}

> sln := subs(ans,sln) union ans;
  
  sln := \{ c241 = -c231 \}

> eqs := reduce_equation_set(subs(sln,Jacobi),ZERO_EQUATIONS);
  
  eqs := \{ 2 p5 p6 - 4 p5 p7 + 2 p6 p4 - 4 p6 p3 + 2 p7 p3 - c341 + 2 p5 p2 + c121,
           -c121 - 2 p2 p4 + 4 p2 p3 + 2 p6 p4 - 4 p6 p3 + 2 p5 p2 + 2 p5 p6 - 4 p5 p7,
           p7^2 - c241, -c131 - c141 - 4 p3 p4 + 2 p2^2 + 4 p2 p6 - 6 p1 p7 - 6 p7^2 + 4 p5 p3,
           4 p6 p3 + 2 p6 p4 + 2 p7 p3 + 2 p5 p6 + 4 p5 p7 - c341 + 2 p5 p2 - c121,
           c241 + c231,
           c121 - 4 p2 p3 - 2 p2 p4 + 4 p6 p3 + 2 p6 p4 + 2 p5 p6 + 4 p5 p7 + 2 p5 p2,
           c121 p6 - 2 c121 p7 - c341 p7 - c241 p4 + 3 c231 p4 + c231 p3
           + 2 c121 p2, -c241 p2 + c231 p2, p7^2 + c231, 2 p7^2 - c241 + c231,
           2 p2 p4 - c341 + 8 p2 p3 - 4 p7 p4 + 4 p6 p3 + 2 p7 p3,
           -2 p2 p4 + c341 + 8 p2 p3 - 4 p7 p4 + 4 p6 p3 - 2 p7 p3, c121 p6 + 2 c121 p7
           - c341 p7 - c241 p3 - c241 p4 - 3 c231 p3 - c231 p4 + 2 c121 p2, 2 c131 p6
           - 3 c131 p7 - 2 c141 p6 - 3 c141 p7 - 2 c341 p4 - c241 p1 - c241 p2 - c241 p7
           + c231 p1 + c231 p7 - c231 p2 + 2 p5 c341 - 2 c141 p2 + 2 c131 p2 + 2 c121 p3 \}
\end{verbatim}

\[ \text{Solve binomial equations.} \]
> eqs := select(x->nops(x)=2, select(x->type(x,'+'), eqs), c231);

\[
eqs := \{ p7^2 + c231 \}
\]

> ans := solve(eqs, \{ c231 \});

\[
ans := \{ c231 = -p7^2 \}
\]

> sln := subs(ans, sln) union ans;

\[
sln := \{ c231 = -p7^2, c241 = p7^2 \}
\]

> eqs := reduce_equation_set(expand(simplify(subs(sln, Jacobi), ZERO_EQUATIONS)));

\[
eqs := \{ 2\ p5\ p6 - 4\ p5\ p7 + 2\ p6\ p4 - 4\ p6\ p3 + 2\ p7\ p3 - c341 + 2\ p5\ p2 + c121, -c121 - 2\ p2\ p4 + 4\ p2\ p3 + 2\ p6\ p4 - 4\ p6\ p3 + 2\ p5\ p2 + 2\ p5\ p6 - 4\ p5\ p7, -c131 - c141 - 4\ p3\ p4 + 2\ p2^2 + 4\ p2\ p6 - 6\ p1\ p7 - 6\ p7^2 + 4\ p5\ p3, 4\ p6\ p3 + 2\ p6\ p4 + 2\ p7\ p3 + 2\ p5\ p6 + 4\ p5\ p7 - c341 + 2\ p5\ p2 - c121, c121 - 4\ p2\ p3 - 2\ p2\ p4 + 4\ p6\ p3 + 2\ p6\ p4 + 2\ p5\ p6 + 4\ p5\ p7 + 2\ p5\ p2, c121\ p6 - 2\ c121\ p7 - c341\ p7 + 2\ p7^2\ p3 + 2\ c121\ p2, c121\ p6 + 2\ c121\ p7 - c341\ p7 + 2\ p7^2\ p3 + 2\ c121\ p2, 2\ c131\ p6 - 3\ c131\ p7 - 2\ c141\ p6 - 3\ c141\ p4 - 2\ p7^2\ p1 - 2\ p7^3 + 2\ p5\ c341 - 2\ c141\ p2 + 2\ c131\ p2 + 2\ c121\ p3, 2\ p2\ p4 - c341 + 8\ p2\ p3 - 4\ p7\ p4 + 4\ p6\ p3 + 2\ p7\ p3, -2\ p2\ p4 + c341 + 8\ p2\ p3 - 4\ p7\ p4 + 4\ p6\ p3 - 2\ p7\ p3 \}
\]

There are only two equations that have \( c341 \) so we may solve for it

> ans :=
> \{ convert(map(isolate, select(has, remove(has, eqs, c121), c341), c341), ' +') /2 \};

\[
ans := \{ c341 = 2\ p2\ p4 + 2\ p7\ p3 \}
\]

There are only two equations that have \( c121 \) so we may solve for it

> ans := ans union
> \{ convert(map(isolate, (select(has, remove(has, eqs, c341), c121)), c121)), ' + ' /2 \};

\[
ans := \{ c341 = 2\ p2\ p4 + 2\ p7\ p3, c121 = 4\ p2\ p3 - 4\ p6\ p3 - 4\ p5\ p7 \}
\]

> sln := subs(ans, sln) union ans;

\[
sln := \{ c231 = -p7^2, c241 = p7^2, c341 = 2\ p2\ p4 + 2\ p7\ p3, c121 = 4\ p2\ p3 - 4\ p6\ p3 - 4\ p5\ p7 \}
\]

Incorporate these solutions into the Jacobi identity.
\[ \text{eqs} := \text{reduce\_equation\_set}(\text{subs}(\text{sln}, \text{Jacobi}), \text{ZERO\_EQUATIONS}); \]

\[ \text{eqs} := \{-c131 - c141 - 4 \, p3 \, p4 + 2 \, p2^2 + 4 \, p2 \, p6 - 6 \, p1 \, p7 - 6 \, p7^2 + 4 \, p5 \, p3, -2 \, c131 \, p6 + 3 \, c131 \, p7 + 2 \, c141 \, p6 + 3 \, c141 \, p7 + 4 \, p2 \, p4^2 + 4 \, p4 \, p7 \, p3 + 2 \, p7^2 \, p1 + 2 \, p7^3 \]
- \[ -4 \, p5 \, p2 \, p4 + 4 \, p5 \, p7 \, p3 + 2 \, c141 \, p2 - 2 \, c131 \, p2 - 8 \, p2 \, p3^2 + 8 \, p6 \, p3^2, \]
- \[ 2 \, p2 \, p3 - 7 \, p4 + p6 \, p3, -p2 \, p4 + p6 \, p4 + p5 \, p2 + p5 \, p6, \]
- \[ 4 \, p6 \, p3 + p6 \, p4 + p5 \, p6 + 4 \, p5 \, p7 - p2 \, p4 + p5 \, p2 - 2 \, p2 \, p3, \]
- \[ 2 \, p5 \, p7^2 + 2 \, p7 \, p6 \, p3 + p6 \, p5 \, p7 - 2 \, p2^2 \, p3 + p6 \, p2 \, p3 + p6^2 \, p3, \]
- \[ -2 \, p5 \, p7^2 - 2 \, p7 \, p6 \, p3 + p6 \, p5 \, p7 - 2 \, p2^2 \, p3 + p6 \, p2 \, p3 + p6^2 \, p3, \]
- \[ p5 \, p6 - 4 \, p5 \, p7 + p6 \, p4 - 4 \, p6 \, p3 - p2 \, p4 + p5 \, p2 + 2 \, p2 \, p3} \]
\[ P = 0 \]

There is only one equation that has \( c341 \) and \( c141 \) so we may solve for \( c141 \) in terms of \( c341 \)

\[ \text{ans} := \text{map} \left( \text{isolate}, \text{remove} \left( \text{has}, \text{select} \left( \text{has}, \text{eqs}, c141 \right), c141 \cdot p6 \right), c141 \right); \]
\[ \text{ans} := \{c141 = -c131 - 4 \, p3 \, p4 + 2 \, p2^2 + 4 \, p2 \, p6 - 6 \, p1 \, p7 - 6 \, p7^2 + 4 \, p5 \, p3} \]

\[ \text{sln} := \text{subs} \left( \text{ans}, \text{sln} \right) \cup \text{ans}; \]
\[ \text{sln} := \{c141 = -c131 - 4 \, p3 \, p4 + 2 \, p2^2 + 4 \, p2 \, p6 - 6 \, p1 \, p7 - 6 \, p7^2 + 4 \, p5 \, p3, \]
- \[ c231 = -p7^2, c241 = p7^2, c341 = 2 \, p2 \, p4 + 2 \, p7 \, p3, \]
- \[ c121 = 4 \, p2 \, p3 - 4 \, p6 \, p3 - 4 \, p5 \, p7} \]
\[ \text{eqs} := \text{reduce\_equation\_set}(\text{subs}(\text{sln}, \text{Jacobi}), \text{ZERO\_EQUATIONS}); \]

\[ \text{eqs} := \{2 \, p2 \, p3 - 7 \, p4 + p6 \, p3, -p2 \, p4 + p6 \, p4 + p5 \, p2 + p5 \, p6, -4 \, p7^3 - 4 \, p7^2 \, p1 - 3 \, p6 \, p7^2 - 2 \, p4 \, p7 \, p3 - 3 \, p6 \, p1 \, p7 + 4 \, p5 \, p7 \, p3 + p2^2 + 3 \, p6 \, p2^2 + p2 \, p4^2 + 2 \, p2 \, p5 \, p3 - 2 \, p2 \, p3 \, p4 - c131 \, p2 - 2 \, p2 \, p3^2 + 2 \, p2 \, p6^2 - 5 \, p2 \, p4 - 2 \, p6 \, p3 \, p4 + 2 \, p6 \, p5 \, p3 + 2 \, p6 \, p3^2 - c131 \, p6, \]
- \[ 4 \, p6 \, p3 + p6 \, p4 + p5 \, p6 + 4 \, p5 \, p7 - p2 \, p4 + p5 \, p2 - 2 \, p2 \, p3, \]
- \[ 2 \, p5 \, p7^2 + 2 \, p7 \, p6 \, p3 + p6 \, p5 \, p7 - 2 \, p2^2 \, p3 + p6 \, p2 \, p3 + p6^2 \, p3, \]
- \[ -2 \, p5 \, p7^2 - 2 \, p7 \, p6 \, p3 + p6 \, p5 \, p7 - 2 \, p2^2 \, p3 + p6 \, p2 \, p3 + p6^2 \, p3, \]
- \[ p5 \, p6 - 4 \, p5 \, p7 + p6 \, p4 - 4 \, p6 \, p3 - p2 \, p4 + p5 \, p2 + 2 \, p2 \, p3} \}

Now continue to solve more of the quadratic conditions.

\[ \text{collect} \left( \text{select} \left( x \rightarrow \text{nops} \left( x \right) = 4, \text{eqs} \right), \left( p5, p4 \right) \right); \]
\[ \left\{ \left( -p2 + p6 \right) \cdot p4 + \left( p2 + p6 \right) \cdot p5 \right\} \]

Since \( p6 \neq 0 \), then \( p6 - p2 \neq 0 \) or \( p6 + p2 \neq 0 \) thus we have the solution
\( \text{ans} := \{ p_5 = t(p_2 - p_6), p_4 = t(p_2 + p_6) \} \)

\( \text{sln} := \text{subs(ans,sln)} \cup \text{ans}; \)

\( \text{sln} := \{ c_{231} = -p_7^2, c_{241} = p_7^2, c_{341} = 2p_2t(p_2 + p_6) + 2p_7p_3, c_{141} = -c_{131} - 4p_3t(p_2 + p_6) + 2p_2^2 + 4p_2p_6 - 6p_1p_7 - 6p_7^2 + 4t(p_2 - p_6)p_3, p_5 = t(p_2 - p_6), p_4 = t(p_2 + p_6), c_{121} = 4p_2p_3 - 4p_6p_3 - 4t(p_2 - p_6)p_7 \} \)

\( \text{eqs} := \text{reduce_equation_set(subs(sln,Jacobi),ZERO_EQUATIONS);} \)

\( \text{eqs} := \{ 2t^2p_7p_6 + t^2p_7p_6^2 - 2p_7p_6p_3 + 2p_2^2p_3 - p_6p_2p_3 - p_6^2p_3, -2tp_7^2p_6 + tp_7p_6^2 + 2p_7p_6p_3 + 2p_2^2p_3 - p_6p_2p_3 - p_6^2p_3, -2p_2p_3 + tp_7p_6 - p_6p_3, 2tp_7p_6 - 2p_6p_3 + p_2p_3, 4p_7^3 + 4p_7^2p_1 + 3p_6^2 + 3p_6p_3 + 3p_7 - p_2^3 - 3p_6p_2^2 - 2p_2^2t^2p_6 - 2t^2p_6^2p_2 + c_{131}p_2 + 2p_2p_3^2 - 2p_2p_6^2 + 4p_2p_3t_6 - 2p_6p_3^2 + 4p_3tp_6^2 + c_{131}p_6 \} \)

Now, consider the cubic equations.

\( \text{eqs} := \text{select}(x->\text{type}(x,\text{cubic}),\text{eqs}); \)

\( \text{eqs} := \{ -2p_2p_3 + tp_7p_6 - p_6p_3, 2tp_7p_6 - 2p_6p_3 + p_2p_3 \} \)

Two equations have a common quadratic term so we may solve for it and subtract to get

\( \text{map(isolate,factor(eqs),p_2*p_3);} \)

\( \{ p_2^2p_3 = \frac{tp_7p_6}{2} - \frac{p_6p_3^2}{2}, p_2^2p_3 = -2tp_7p_6 + 2p_6p_3 \} \)

\( \text{factor(\%[1]-\%[2]);} \)

\( 0 = \frac{5p_6(tp_7-p_3)}{2} \)

Since \( p_6 \neq 0 \) then \( p_3 = tp_7 \).

\( \text{ans} := \{ p_3 = t*p_7; \} \)

\( \text{ans} := \{ p_3 = tp_7 \} \)

\( \text{sln} := \text{simplify(subs(ans,sln)} \cup \text{ans);} \)

\( \text{sln} := \{ c_{341} = 2p_2^2t + 2tp_6p_2 + 2p_7^2t, c_{121} = 0, c_{231} = -p_7^2, c_{241} = p_7^2, c_{141} = -c_{131} - 8t^2p_7p_6 + 2p_2^2 + 4p_2p_6 - 6p_1p_7 - 6p_7^2, p_3 = tp_7, p_4 = t(p_2 + p_6), p_5 = -t(-p_2 + p_6) \} \)

\( \text{eq} := \text{op(reduce_equation_set(subs(sln,Jacobi),ZERO_EQUATIONS));} \)
\( eq := 4 p^7 + 4 t^2 p^7 p^6 + 3 p^6 p^7 + 4 p^7 p^1 + 4 t^2 p^7 p^6 + 3 p^6 p^1 p^7 - p^2^3 \\
- 2 p^2^3 t^2 p^6 - 3 p^6 p^2 - 2 t^2 p^6 p^2 + c_{131} p^2 - 2 p^2 p^6^2 + c_{131} p^6 \)

Only two equations remain, the one above and \( p^7 p^2 = 0 \), which has been used in all solutions. We solve the remaining Jacobi identities by cases. The primary criteria for our cases will be if \( p^7 = 0 \) or not. Consequently, it will be convenient to reparameterize based on \( p^7 = 0 \). Let's consider the remaining equation when \( p^7 = 0 \) and then when \( p^2 = 0 \).

\[
\text{PART1} := \text{factor}\left(\text{subs}\left(p^7 = 0, eq\right)\right);
\]

\[
\text{PART1} := -(p^2 + p^6) (2 p^2 t^2 p^6 + 2 p^2 p^6 + p^2^2 - c_{131})
\]

If \( p^7 \neq 0 \) then \( p^2 = 0 \) so

\[
\text{PART2} := \text{subs}\left(c_{131} = 0, \text{factor}\left(\text{subs}\left(p^2 = 0, eq\right)\right)\right);
\]

\[
\text{PART2} := 4 p^7 + 4 t^2 p^7 p^6 + 3 p^6 p^7 + 4 p^7 p^1 + 4 t^2 p^7 p^6 + 3 p^6 p^1 p^7
\]

The \( c_{131} p^6 \) term is duplicated, so make the substitution.

Double check we have the right two pieces.

\[
\text{PART1} + \text{PART2} - eq;
\]

\[
0
\]

To make this more manageable let's reparameterize one of the factors of PART1 as \( \beta \)

\[
\text{PARMS} := \left\{ c_{131} = p^2^2 + 2 p^2 p^6 + 2 p^2 t^2 p^6 - \beta \right\}
\]

\[
\text{PARMS} := \{ c_{131} = p^2^2 + 2 p^2 p^6 + 2 p^2 t^2 p^6 - \beta \}
\]

\[
\text{eq1} := \text{collect}\left(\text{simplify}\left(\text{subs}\left(\text{PARMS}, eq\right)\right), \{\beta\}\right);
\]

\[
eq 1 := (-p^2 - p^6) \beta + 4 p^7 + 4 t^2 p^7 p^6 + 3 p^6 p^7 + 4 p^7 p^1 + 4 t^2 p^7 p^6 + 3 p^6 p^1 p^7
\]

Modify the solutions to account for this reparameterization.

\[
\text{SLN} := \text{subs}\left(\text{PARMS}, \text{sln}\right) \cup \text{PARMS};
\]

\[
\text{SLN} := \{ c_{341} = 2 p^2^2 t + 2 t p^6 p^2 + 2 p^7^2 t, \\
c_{141} = p^2^2 + 2 p^2 p^6 - 2 p^2 t^2 p^6 + \beta - 8 t^2 p^7 p^6 - 6 p^1 p^7 - 6 p^7^2, c_{121} = 0, \\
c_{231} = -p^7^2, c_{241} = p^7^2, c_{131} = p^2^2 + 2 p^2 p^6 + 2 p^2 t^2 p^6 - \beta, p_3 = t p^7, \\
p_4 = t (p^2 + p^6), p_5 = -t (-p^2 + p^6) \}
The Jacobi identity with the new parameter is

\[
JACOBI2 := \text{collect(reduce\_equation\_set(simplify(subs(SLN,Jacobi),ZERO\_EQUATIONS)))}
\]

\[
JACOBI2 := \{4\ p7^3 + (4\ t^2\ p6 + 3\ p6 + 4\ p1)\ p7^2 + (4\ t^2\ p6^2 + 3\ p6\ p1)\ p7 + (-p2 - p6)\ \beta}\}
\]

We have two primary cases, and two secondary cases

1) \(p7 \neq 0\) then \(p2 = 0\) and we can solve for \(\beta\) since \(p6 \neq 0\).

2) \(p7 = 0\) then \((p6 + p2)\ \beta = 0\)

2b) \(\beta = 0\)

2c) \(\beta \neq 0\) then \(p6 + p2 = 0\).

7.1.1 Case 1: \(p7 \neq 0\)

Thus \(p2 = 0\) and we may solve for \(\beta\).

\[
\text{change\_frame\_to(alg)};
\]

\[
\text{Case1 := subs\{p2=0\},SLN) union \{p2=0\};}
\]

\[
\text{Case1 := \{c341 = 2\ p7^2\ t, p5 = -p6\ t, c141 = \beta - 8\ t^2\ p7\ p6 - 6\ p1\ p7 - 6\ p7^2, c131 = -\beta, p4 = p6\ t, p2 = 0, c121 = 0, c231 = -p7^2, c241 = p7^2, p3 = t\ p7\}}
\]

\[
\text{JACOBI2\_1 := reduce\_equation\_set( simplify(subs(Case1,JACOBI2)) );}
\]

\[
\text{JACOBI2\_1 := \{4\ p7^3 + 4t^2p7^2p6 + 3p6p7^2 + 4p7^2p1 + 4t^2p7p6^2 + 3p6p1p7 - p6\ \beta\}}
\]

\[
\text{ans := factor(solve(JACOBI2\_1,beta));}
\]

\[
\text{ans := \{\beta = \frac{p7(4p7^2 + 4t^2p7p6 + 3p6p7 + 4p1p7 + 4t^2p6^2 + 3p6p1)}{p6}\}}
\]

\[
\text{Case1 := simplify(subs(ans,CASE1) union ans);}
\]

\[
\text{Case1 := \{c341 = 2\ p7^2\ t, p5 = -p6\ t, p4 = p6\ t, p2 = 0, c121 = 0, c231 = -p7^2, c241 = p7^2, p3 = t\ p7, c131 = \frac{p7(4p7^2 + 4t^2p7p6 + 3p6p7 + 4p1p7 + 4t^2p6^2 + 3p6p1)}{p6}, p3 = t\ p7, c141 = \frac{p7(-4p7^2 - 4t^2p7p6 + 3p6p7 - 4p1p7 + 4t^2p6^2 + 3p6p1)}{p6}\}}
\]
> CASE1 := structure_constants_array_to_Lie_algebra_data(subs(Case1,op(My
> _alg)),case1):
> Lie_alg_init(CASE1, ["m1", "m2", "m3", "m4", "h1"], ["theta1", "theta2",
> "theta3", "theta4", "omega1"]);
>
> Lie algebra: case1
>
> check_Jacobi_identity();
>
> true

The Jacobi identity is now satisfied. Now change the basis to put this algebra into a

canonical form.
> M1 := matrix([[–p6*t, 0, 1/2*p6+p7, 1/2*p6–p7, 0], [-2*p7^2/p6, 0, 0,
> 0, 0], [p1+p7, p6, p6*t, -p6/p7], [2*p1+2*p7, -p6, p6*t, p6*t,
> -p6/p7], [-2*p6*t, 0, p6, p6, 0]]);

\[
M1 := \begin{bmatrix}
-p6 t & 0 & \frac{p6}{2} + p7 & \frac{p6}{2} - p7 & 0 \\
\frac{2 p7^2}{p6} & 0 & 0 & 0 & 0 \\
p1 + p7 & p6 & p6 t & p6 t & \frac{p6}{p7} \\
2 p1 + 2 p7 & -p6 & p6 t & p6 t & \frac{p6}{p7} \\
-2 p6 t & 0 & p6 & p6 & 0
\end{bmatrix}
\]

The change of basis is valid since this matrix is non-degenerate.
>
> det(M1);

\[-8 p7^2 p6^2\]

> change_Lie_algebra_basis(M1, case1_l):

> Lie_alg_init(%):

> Lie_bracket_mult_table();

\[
\begin{array}{c|ccccc}
\hline
 & e1 & e2 & e3 & e4 & e5 \\
\hline
e1 & 0 & 2 e2 & -2 e3 & 0 & 0 \\
e2 & -2 e2 & 0 & e1 & 0 & 0 \\
e3 & 2 e3 & -e1 & 0 & 0 & 0 \\
e4 & 0 & 0 & 0 & 0 & e4 \\
e5 & 0 & 0 & 0 & -e4 & 0 \\
\hline
\end{array}
\]

Calculate the isotropy algebra in this new basis.
> evalm(M1 &* [0,0,0,0,1]);
\[
\begin{bmatrix}
0, 0, -\frac{p6}{p7}, -\frac{p6}{p7} , 0
\end{bmatrix}
\]

Thus the isotropy is \{e3 + e4\} which gives i,j from Theorem 1.0.1.

7.1.2 Case 2: \( p7 = 0 \)

> change_frame_to(alg);

> Case2:=simplify(subs({p7=0},SLN) union {p7=0});

\[
\begin{align*}
\text{Case2} &= \{c121 = 0, c131 = p2^2 + 2 p2 p6 + 2 p2 t^2 p6 - \beta, p7 = 0, \\
c341 = 2 p2^2 t + 2 t p6 p2, c141 = p2^2 + 2 p2 p6 - 2 p2 t^2 p6 + \beta, c231 = 0, \\
c241 = 0, p3 = 0, p4 = t(p2 + p6), p5 = -t(-p2 + p6)\}
\end{align*}
\]

> JACOBI2_2:=factor(map(x->x*sign(x),simplify(subs(Case2,JACOBI2))))

\[
\begin{align*}
\text{JACOBI2}_2 &= \{(p2 + p6)\beta\}
\end{align*}
\]

> CASE2:=structure_constants_array_to_Lie_algebra_data(subs(Case2,op(My
_\text{alg})),case2);

> Lie_alg_init(CASE2,["m1","m2","m3","m4","h1"],["theta1","theta2",
"theta3","theta4","omega1"]);

\text{Lie algebra: case2}

7.1.2.1 Case 2.1: \( p7 = 0 \) and \( \beta = 0 \)

> change_frame_to(case2);

> Case21:=subs(beta=0,CASE2) union {beta=0};

\[
\begin{align*}
\text{Case21} &= \{c141 = p2^2 + 2 p2 p6 - 2 p2 t^2 p6, c131 = p2^2 + 2 p2 p6 + 2 p2 t^2 p6, \beta = 0, \\
c121 = 0, p7 = 0, c341 = 2 p2^2 t + 2 t p6 p2, c231 = 0, c241 = 0, p3 = 0, \\
p4 = t(p2 + p6), p5 = -t(-p2 + p6)\}
\end{align*}
\]

> reduce_equation_set(subs(Case21,JACOBI2_2));

\[
\begin{align*}
\end{align*}
\]

> CASE21_0:=structure_constants_array_to_Lie_algebra_data(subs(Case21,op(My
_\text{alg})),case21_0);

> Lie_alg_init(CASE21_0,["m1","m2","m3","m4","h1"],["theta1","theta2",
"theta3","theta4","omega1"]);

\text{Lie algebra: case21.0}

> check_Jacobi_identity();
The Jacobi identity is satisfied. Now change the basis to put this algebra into a canonical form.

\[
M21 := \begin{bmatrix}
-p1 & 2p6 & 0 & 0 & 0 \\
0 & 0 & 2p6 & -2p6 & 0 \\
4p6 & 0 & 0 & 0 & 0 \\
p2 & 0 & t\cdot p2 & -t\cdot p2 & 1 \\
-2t\cdot p6 & 0 & p6 & p6 & 0
\end{bmatrix}
\]

The change of basis is a valid since this matrix is non-degenerate.

\[
\text{det}(M21); \quad -32p6^4
\]

\[
\text{change_Lie_algebra_basis}(M21); \quad \texttt{[[Lie_alg, koszul, [5]], [[1, 5, 1], \frac{-p2 + p6}{p6}], [[2, 4, 1], 1], [[2, 5, 2], \frac{-p2}{p6}], [[3, 4, 2], 1],
[[3, 5, 3], \frac{-p2 + p6}{p6}], [[4, 5, 4], 1]]}
\]

Calculate the isotropy algebra in this new basis.

\[
\text{simplify( evalm(M21 &* [0,0,0,0,1]) )}; \quad [0, 0, 0, 1, 0]
\]

Thus the isotropy is \{e3\} which gives ii.) from Theorem 1.0.1 where \(\alpha = -\frac{p2}{p6}\).
\begin{verbatim}
> CASE22_0:=structure_constants_array_to_Lie_algebra_data(subs(Case22,op(My_alg)),case22_0):
> Lie_alg_init(CASE22_0, ["m1", "m2", "m3", "m4", "h1"], ["theta1", "theta2", "theta3", "theta4", "omega1"]);

Lie algebra: case22.0

> check_Jacobi_identity();

true

The Jacobi identity is satisfied. Now change the basis to put this algebra into a canonical form.

> M22 := matrix([[1/4*2^2-(1/2)/p6*abs(beta)-(1/2)*p1,
> -1/2*2^2-(1/2)*abs(beta)-(1/2), 0, 0, 0],
> 1/2*2^2-(1/2)*abs(beta)-(1/2), -1/2*2^2-(1/2)*abs(beta)-(1/2), 0, [p6, 0, p6*t, -p6*t, -1], [-2*p6*t, 0, p6, p6, 0], [2^2-(1/2)*abs(beta)-(1/2), 0, 0, 0]]);

\[
M22 := \begin{bmatrix}
\frac{1}{4} \sqrt{2} \sqrt{\beta} \frac{p1}{p6} & -\frac{1}{2} \beta \frac{p1}{p6} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} \beta & -\frac{1}{2} \beta & 0 \\
p6 & 0 & p6 t & -p6 t & -1 \\
-2 p6 t & 0 & p6 & p6 & 0 \\
\beta & 0 & 0 & 0 & 0
\end{bmatrix}
\]

%1 := \sqrt{2} \sqrt{\beta}

The change of basis is a valid since this matrix is non-degenerate.

> det(M22);

\(-\sqrt{2} |\beta|^{3/2} p6\)

> CASE22:=change_Lie_algebra_basis(M22,case22):
> Lie_alg_init(CASE22):
> Lie_bracket_mult_table();
\end{verbatim}
Calculate the isotropy algebra in this new basis.

> evalm( M22 &* [0,0,0,0,1]);

\[ [0, 0, -1, 0, 0] \]

Thus the isotropy is \{e3\} which gives iii.) from Theorem 1.0.1 where \( \varepsilon = -\frac{\beta}{|\beta|} \).

7.2 Subalgebra 2

> restart;

> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected

> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\theorems.txt"));
> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\patch.txt"));
> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\renner.txt"));

Load isotropy subalgebra.

> h:=convert(map(collect,evalm(s2 &* Basis),{alpha,epsilon}),list);

\[ h := [-E1 + E3] \]

Load complementary basis.

> _h:=convert(map(collect,evalm(_s2 &* Basis),{alpha,epsilon}),list);

\[ _h := [E1 + E3, E2, F1, F2, F3] \]

Fix a representation of \( s_{22} \) adapted to this subalgebra.
Initialize o22 in this basis.

\[ \text{Mat}_\text{Basis} := \text{map} (\text{evalm}, [\text{op} (_\text{h}), \text{op} (\text{h})]) : \]

\[ \text{Lie}_\text{alg}_\text{init}(\text{o22}) ; \]

\[ \text{Isodim} := \text{nops} (\text{h}) ; \]

\[ \text{Codim} := 6 - \text{Isodim} ; \]

Compute the \( P \) matrix from equation 4.5.

\[ P_0 := \text{P}\text{-matrix}_4_5 (\text{Codim}, \text{Isodim}, \text{o22}) ; \]

Change variables in \( P0 \) to make the calculations easier.

\[ P := \text{matrix} ([[p1, 0, 0, p2], [1/3*p3-1/3*p7, 2*p2, -2*p1, 1/3*p4-1/3*p9], [p5+p9, 0, 0, p6-p3], [p7+p3, 0, 0, p8], [p9-p5, 0, 0, p10]]) ; \]

Check to ensure that this is a legitimate change of variables.

\[ \text{subs} ([p1=P1, p2=P2, p3=P3, p4=P4, p5=P5, p6=P6, p7=P7, p8=P8, p9=P9, p10=P10] \]

\( P := \text{evalm} (\text{subs} ([p1=P1, p2=P2, p3=P3, p4=P4, p5=P5, p6=P6, p7=P7, p8=P8, p9=P9, p10=P10] \)

\[ \text{solve} (\text{convert} (\%, \text{set}), [P1, P2, P3, P4, P5, P6, P7, P8, P9, P10]) ; \]

\{ P9 = p9 - p5, P1 = p1, P2 = p2, P6 = p6 - p3, P3 = \frac{p3}{3} - \frac{p7}{3}, P4 = \frac{p4}{3} - \frac{p9}{3}, P10 = p10, P8 = p8, P7 = p7 + p3, P5 = p5 + p9 \}
Therefore this is a legitimate change of variables and we may initialize the partial Lie Algebra as in Equation 4.4.

```plaintext
> My_alg:=create_algebra_4_4(Codim, Isodim, P, o22):
> Lie_alg_init(structure_constants_array_to_Lie_algebra_data(My_alg,alg
> ), ["m1","m2","m3","m4","h1"],["theta1", "theta2", "theta3", "theta4", "omega1"]):
```

Check if this must be reductive by calculating Equation 4.6.

```plaintext
> TBS:=reduce_equation_set(reductive_equations_4_6(Codim, Isodim,
> alg));
TBS := \{p1, p2, p3 - p7 - 3 r13, -p4 + p9 - 3 r12\}
> subs(r12=-1/3*p4+1/3*p9,r13=1/3*p3-1/3*p7,TBS);
{0, p1, p2}
```

To be non-reductive, \( p1 \neq 0 \) or \( p2 \neq 0 \). We use this to assist in solving the Jacobi identities \( d^2 = 0 \)

```plaintext
> Jacobi:={}:
> Cobasis:=frameBaseForms():
> for theta in Cobasis do
>    Jacobi:= Jacobi union coeff_set(ext_d(ext_d(theta)));
> od:
> Jacobi:=reduce_equation_set(Jacobi);
```

There is only one equation that has c231 as a term so we may solve for it.

\[
\text{ans} := \{\text{convert(map(isolate, select(has, remove(has, Jacobi, [c341,c131,c241]),c231),'+'))}/4\};
\]

\[
\text{ans} := \{c231 = \frac{17 p1 p9}{24} - \frac{p2 p3}{24} - \frac{p2 p7}{3} - \frac{3 p1 p8}{8} - \frac{3 p2 p6}{8} + \frac{3 p2 p10}{8} + \frac{p1 p4}{24}\}
\]

There are only two equations that have c241 as a term so we may solve for it.

\[
\text{ans} := \text{ans union}\{\text{convert(map(isolate, select(has, remove(has, Jacobi, [c341,c131,c121]),c241),'+'))}\};
\]

\[
\text{ans} := \{c241 = \frac{p9 p4}{6} + \frac{7 p9^2}{6} + \frac{p9 p8}{2} - \frac{5 p7 p6}{4} + \frac{5 p3^2}{12} + \frac{5 p7 p3}{6} - \frac{2 p3 p10}{3} - \frac{5 p7 p10}{6} + \frac{p10^2}{6} - \frac{p3 p6}{6},\]
\[
c231 = \frac{17 p1 p9}{24} - \frac{p2 p3}{24} - \frac{p2 p7}{3} - \frac{3 p1 p8}{8} - \frac{3 p2 p6}{8} + \frac{3 p2 p10}{8} + \frac{p1 p4}{24}\}
\]
There are only two equations that have $c_{131}$ as a term so we may solve for it.

```latex
\begin{align*}
\text{ans} &:= \text{ans union }\left\{\text{convert\left(map\left(isolate, select\left(has, remove\left(has, Jacobi, \{c_{341}, c_{231}, c_{241}\}\right), c_{131}\right), c_{131}\right), '+')}\right\};
\end{align*}
```

\begin{align*}
\text{ans} &:= \{c_{241} = -\frac{9 p^4}{6} + \frac{7 p^2}{2} - \frac{5 p^6}{6} - \frac{p^6}{4} + \frac{5 p^3}{12} + \frac{5 p^7 p^3}{6} - \frac{2 p^3 p^10}{3} - \frac{5 p^7 p^10}{6} + \frac{p^{10^2}}{4} - \frac{3 p^6}{6}, c_{131} = -\frac{5 p^3 p^6}{6} + \frac{p^6}{4} - \frac{p^6 p^10}{2} + \frac{7 p^3}{12} + \frac{5 p^3 p^10}{6} + \frac{p^{10^2}}{4} - \frac{5 p^5 p^9}{6} + \frac{3 p^5 p^8}{2} - \frac{p^7 p^6}{6} + \frac{p^7 p^3}{6} + \frac{7 p^7 p^10}{6}, c_{231} = \frac{17 p^1 p^9}{24} - \frac{p^2 p^3}{3} - \frac{p^2 p^7}{8} - \frac{3 p^1 p^8}{8} - \frac{3 p^2 p^6}{8} + \frac{3 p^2 p^10}{8} + \frac{p^1 p^4}{24}\}.
\end{align*}
```

There are only two equations that have $c_{341}$ as a term so we may solve for it.

```latex
\begin{align*}
\text{ans} &:= \text{ans union }\left\{\text{convert\left(map\left(isolate, select\left(has, remove\left(has, Jacobi, \{c_{321}, c_{341}, c_{231}, c_{241}\}\right), c_{121}\right), c_{121}\right), '+')}\right\};
\end{align*}
```

\begin{align*}
\text{ans} &:= \{c_{241} = -\frac{9 p^4}{6} + \frac{7 p^2}{2} - \frac{5 p^6}{6} - \frac{p^6}{4} + \frac{5 p^3}{12} + \frac{5 p^7 p^3}{6} - \frac{2 p^3 p^10}{3} - \frac{5 p^7 p^10}{6} + \frac{p^{10^2}}{4} - \frac{3 p^6}{6}, c_{131} = -\frac{5 p^3 p^6}{6} + \frac{p^6}{4} - \frac{p^6 p^10}{2} + \frac{7 p^3}{12} + \frac{5 p^3 p^10}{6} + \frac{p^{10^2}}{4} - \frac{5 p^5 p^9}{6} + \frac{3 p^5 p^8}{2} - \frac{p^7 p^6}{6} + \frac{p^7 p^3}{6} + \frac{7 p^7 p^10}{6}, c_{231} = \frac{17 p^1 p^9}{24} - \frac{p^2 p^3}{3} - \frac{p^2 p^7}{8} - \frac{3 p^1 p^8}{8} - \frac{3 p^2 p^6}{8} + \frac{3 p^2 p^10}{8} + \frac{p^1 p^4}{24}\}.
\end{align*}
```

There are only two equations that have $c_{121}$ as a term so we may solve for it.

```latex
\begin{align*}
\text{ans} &:= \text{ans union }\left\{\text{convert\left(map\left(isolate, select\left(has, remove\left(has, Jacobi, \{c_{321}, c_{341}, c_{131}, c_{241}\}\right), c_{121}\right), c_{121}\right), '+')}\right\};
\end{align*}
```

\begin{align*}
\text{ans} &:= \{c_{341} = \frac{p^3 p^4}{12} - \frac{p^4 p^6}{12} + \frac{p^4 p^{10}}{12} - \frac{11 p^3 p^9}{12} + \frac{7 p^9 p^6}{12} - \frac{7 p^9 p^{10}}{12} + \frac{p^3 p^8}{12} + \frac{p^8 p^6}{4} - \frac{p^8 p^{10}}{4} + \frac{p^5 p^6}{2} + \frac{p^5 p^3}{2} - \frac{p^5 p^{10}}{2} - \frac{2 p^7 p^9}{3} + \frac{2 p^7 p^8}{3}\}.
\end{align*}
```
ans := {c241 = \frac{-p9}{6} + \frac{7p9}{6} + \frac{p9p8}{2} - \frac{5p7p6}{6} - \frac{p6^2}{4} + \frac{5p3^2}{12} + \frac{5p7p3}{6} - \frac{2p3p10}{3}}
- \frac{5p7p10}{6} + \frac{p10^2}{4} - \frac{p3p6}{6}, c131 = \frac{-5p3p6}{6} + \frac{p6^2}{4} - \frac{2p6p10}{2} + \frac{7p3^2}{12}
+ \frac{5p3p10}{6} + \frac{p10^2}{4} + \frac{p5p4}{6} - \frac{7p5p9}{6} + \frac{3p5p8}{2} - \frac{p7p6}{6} + \frac{p7p3}{6} + \frac{p7p10}{6},
c231 = \frac{17p1p9}{24} - \frac{p2p3}{12} - \frac{p2p7}{6} + \frac{3p1p8}{6} - \frac{3p2p6}{6} + \frac{3p2p10}{6} + \frac{p1p4}{24},
c341 = \frac{p3p4}{12} - \frac{p4p6}{12} + \frac{p4p10}{12} - \frac{11p3p9}{12} + \frac{7p9p6}{12} - \frac{7p9p10}{12} + \frac{p3p8}{12}
+ \frac{p8p6}{4} - \frac{p8p10}{4} - \frac{p5p6}{4} + \frac{p5p3}{4} - \frac{p5p10}{4} - \frac{2p7p9}{3} + \frac{2p7p8}{3}, c121 =
\frac{5p9p6}{12} - \frac{5p3p9}{4} - \frac{5p9p10}{12} + \frac{p3p4}{6} + \frac{p7p4}{12} + \frac{p4p6}{12} - \frac{p4p10}{12} - \frac{p7p9}{12}
+ \frac{p3p8}{2} + \frac{p7p8}{2} - \frac{p8p6}{4} - \frac{p8p10}{4} - \frac{p5p6}{2} + \frac{p5p3}{2} - \frac{p5p10}{2} + \frac{p5p10}{2},
> sin := ans;
sin := {c241 = \frac{-p9}{6} + \frac{7p9}{6} + \frac{p9p8}{2} - \frac{5p7p6}{6} - \frac{p6^2}{4} + \frac{5p3^2}{12} + \frac{5p7p3}{6} - \frac{2p3p10}{3}}
- \frac{5p7p10}{6} + \frac{p10^2}{4} - \frac{p3p6}{6}, c131 = \frac{-5p3p6}{6} + \frac{p6^2}{4} - \frac{2p6p10}{2} + \frac{7p3^2}{12}
+ \frac{5p3p10}{6} + \frac{p10^2}{4} + \frac{p5p4}{6} - \frac{7p5p9}{6} + \frac{3p5p8}{2} - \frac{p7p6}{6} + \frac{p7p3}{6} + \frac{p7p10}{6},
c231 = \frac{17p1p9}{24} - \frac{p2p3}{12} - \frac{p2p7}{6} + \frac{3p1p8}{6} - \frac{3p2p6}{6} + \frac{3p2p10}{6} + \frac{p1p4}{24},
c341 = \frac{p3p4}{12} - \frac{p4p6}{12} + \frac{p4p10}{12} - \frac{11p3p9}{12} + \frac{7p9p6}{12} - \frac{7p9p10}{12} + \frac{p3p8}{12}
+ \frac{p8p6}{4} - \frac{p8p10}{4} - \frac{p5p6}{4} + \frac{p5p3}{4} - \frac{p5p10}{4} - \frac{2p7p9}{3} + \frac{2p7p8}{3}, c121 =
\frac{5p9p6}{12} - \frac{5p3p9}{4} - \frac{5p9p10}{12} + \frac{p3p4}{6} + \frac{p7p4}{12} + \frac{p4p6}{12} - \frac{p4p10}{12} - \frac{p7p9}{12}
+ \frac{p3p8}{2} + \frac{p7p8}{2} - \frac{p8p6}{4} - \frac{p8p10}{4} - \frac{p5p6}{2} + \frac{p5p3}{2} - \frac{p5p10}{2} + \frac{p5p10}{2},
> eqs := reduce_equation_set(subs(sin, Jacobi));
Currently, consider only the quadratic equations.

> eqs := select(x->type(x,quadratic),eqs);

Pick the following equations.

> ans1:=(-p8+p4+p9)*p1+(3*p6-3*p10+3*p3)*p2;

\[
\text{ans1} := (-p8 + p4 + p9) p1 + (3 p6 - 3 p10 + 3 p3) p2
\]

> ans2:=(-3*p8+3*p9+3*p4)*p1+(-p10+p3+p6)*p2;

\[
\text{ans2} := (-3 p8 + 3 p9 + 3 p4) p1 + (-p10 + p3 + p6) p2
\]

Add these equations together.

> collect(primpart(simplify(convert(ans1 union ans2,'+'))),{p1,p2});

\[
(-p8 + p4 + p9) p1 + (-p10 + p3 + p6) p2
\]

We have assumptions about \( p1 \) and \( p2 \) so let’s look at equations with those two parameters as factors.

> select(x->type(x,`*`),collect(eqs,{p1,p2}));

\[
((-p8 + p4 + p9) p2, (-p6 - p3 + p10) p1)
\]

Since \( p1 \neq 0 \) or \( p2 \neq 0 \) we may conclude that \( p8 = p4 + p9 \) and \( p10 = p3 + p6 \).

> ans:={p8=p4+p9,p10=p3+p6};

\[
\text{ans} := \{p8 = p4 + p9, p10 = p3 + p6\}
\]

> sln:=subs(ans,sln) union ans;
Consider the remaining quadratic equations.

> eqs1 := select(x -> type(x, quadratic), eqs);

eqs1 := \{-p2 p5 - p1 p7, p1 p9 - p2 p3, p1 p6 - p2 p4, p1 p7 + p2 p5\}

Again, since p1 \neq 0 or p2 \neq 0 we get the general solutions p7 = r p2 and p5 = -r p1, p9 = s p2 and p3 = s p1, p6 = t p2 and p4 = t p1.
\[
\text{ans} := \{p7 = r \cdot p2, p5 = -r \cdot p1, p9 = s \cdot p2, p3 = s \cdot p1, p6 = t \cdot p2, p4 = t \cdot p1\};
\]
\[
\text{ans} := \{p6 = t \cdot p2, p4 = t \cdot p1, p7 = r \cdot p2, p5 = -r \cdot p1, p9 = s \cdot p2, p3 = s \cdot p1\}
\]
\[
\text{sln} := \text{simplify}\left(\text{subs}(\text{ans}, \text{sln}) \cup \text{ans}\right);
\]
\[
\text{sln} := \{p6 = t \cdot p2, p4 = t \cdot p1, p7 = r \cdot p2, p5 = -r \cdot p1, p9 = s \cdot p2, p3 = s \cdot p1, p10 = s \cdot p1 + t \cdot p2, p8 = t \cdot p1 + s \cdot p2, c241 = \frac{5s^2p2^2}{3} - \frac{5r p2^2 t}{3}, c131 = \frac{5s^2p1^2}{3} - \frac{5r p1^2 t}{3}\},
\]
\[
c231 = \frac{2p1sp2}{3} - \frac{p2^2r}{3} - \frac{p1^2t}{3}, c341 = -\frac{5s^2p1p2}{3} + \frac{5r p1 p p2}{3},
\]
\[
c121 = -\frac{5s^2p1p2}{3} + \frac{5r p1 p p2}{3}\}
\]

Again, since \( p1 \neq 0 \) or \( p2 \neq 0 \) the remaining Jacobi will allow us to always solve for \( c141 \).
\[
\text{eqs} := \text{factor}\left(\text{reduce_equation_set}\left(\text{simplify}\left(\text{subs}(\text{sln}, \text{Jacobi})\right)\right)\right);
\]
\[
\text{eqs} := \{p1(\]
-4tp1^2s^2 + 4t^2p1^2r + 8p2s^3p1 - 8p2s p p1 t - 9c141 - 4p2^2 r s^2 + 4p2^2 r^2 t), p2(\]
-4tp1^2s^2 + 4t^2p1^2r + 8p2s^3p1 - 8p2s p p1 t - 9c141 - 4p2^2 r s^2 + 4p2^2 r^2 t)\} \}
\]
\[
\text{ans} := \text{solve}\left(\text{eqs}[2], \{c141\}\right);
\]
\[
\text{ans} := \{c141 = \frac{4tp1^2s^2}{9} + \frac{4t^2p1^2r}{9} + \frac{8p2s^3p1}{9} - \frac{8p2s p p1 t}{9} - \frac{4p2^2 r s^2}{9} + \frac{4p2^2 r^2 t}{9}\}
\]
\[
\text{sln} := \text{simplify}\left(\text{subs}(\text{ans}, \text{sln}) \cup \text{ans}\right);
\]
\[
\text{sln} := \{p6 = t \cdot p2, p4 = t \cdot p1, p7 = r \cdot p2, p5 = -r \cdot p1, p9 = s \cdot p2, p3 = s \cdot p1, p10 = s \cdot p1 + t \cdot p2, p8 = t \cdot p1 + s \cdot p2, c241 = \frac{5s^2p2^2}{3} - \frac{5r p2^2 t}{3}, c131 = \frac{5s^2p1^2}{3} - \frac{5r p1^2 t}{3}, c141 = \frac{4tp1^2s^2}{9} + \frac{4t^2p1^2r}{9} + \frac{8p2s^3p1}{9} - \frac{8p2s p p1 t}{9} - \frac{4p2^2 r s^2}{9} + \frac{4p2^2 r^2 t}{9},
\]
\[
c231 = \frac{2p1sp2}{3} - \frac{p2^2r}{3} - \frac{p1^2t}{3}, c341 = -\frac{5s^2p1p2}{3} + \frac{5r p1 p p2}{3},
\]
\[
c121 = -\frac{5s^2p1p2}{3} + \frac{5r p1 p p2}{3}\}
\]

The Jacobi identity is satisfied.
\[
\text{simplify}\left(\text{subs}(\text{sln}, \text{Jacobi})\right);
\]
Now change the basis to put this algebra into a canonical form.

This change of basis is always non-singular.

Calculate the isotropy algebra in this new basis.

```
[0, 0, 1, 0, 0]
```
Thus the isotropy is \{e3\} this leads to iv. in Theorem 1.01.

7.3 Subalgebra 10

> restart;
> with(Vessiot):with(Koszul):with(Mubar):with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected
> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\theorems.txt"));
> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\patch.txt"));
> read(cat(getenv(USERPROFILE),"\My Documents\Thesis\Maple\renner.txt"));

Load isotropy algebra.
> \(h:=\text{convert(\text{map(collect,evalm(s10 \&* Basis)},\{alpha,epsilon\}),list);}
\)

\(h := [E1 - E3, F1 - F3] \)

Load complementary basis.
> \(_h:=\text{convert(\text{map(collect,evalm(_s10 \&* Basis)},\{alpha,epsilon\}),list);}
\)

\(_h := [E1 + E3, E2, F1 + F3, F2] \)

Fix a representation of o22 adapted to this subalgebra.
> \(\text{Mat\_Basis:=\text{map(evalm,[op(\_h),op(h)])}};\)

Initialize o22 in this basis.
> \(\text{o22:=matrix\_algebra\_to\_Lie\_algebra\_data(Mat\_Basis,o22);}\)
> \(\text{Lie\_alg\_init(o22);}\)

\(\text{Lie algebra : o22}\)

> \(\text{Isodim:=nops(h);}\)
> \(\text{Codim:=6-Isodim;}\)

Compute the \(P\) matrix from equation 4.5.
\[
\begin{bmatrix}
p_1 & 0 & 0 & 0 \\
p_2 & 0 & -2p_1 & 0 \\
p_3 & 0 & 0 & 0 \\
p_4 & 0 & 0 & -2p_3 \\
\end{bmatrix}
\]

Fix a solution since particular solutions are runtime dependant.

\[
P := \begin{bmatrix}
p_1 & 0 & 0 & 0 \\
p_2 & 0 & -2p_1 & 0 \\
p_3 & 0 & 0 & 0 \\
p_4 & 0 & 0 & -2p_3 \\
\end{bmatrix}
\]

Initialize the partial Lie Algebra as in Equation 4.4.

Solve for \( r \)-values, to determine when it is reductive.

\[
TBS := \{p_3, r_{12}, r_{14}, p_1, r_{22}, r_{23}, p_2 + r_{13}, p_4 + r_{24}\}
\]

\[
\text{subs}\{r_{12}=0,r_{13}=-p_2,r_{23}=0,r_{24}=-p_4,r_{22}=0,r_{14}=0\},TBS);
\]

To be reductive \( p_1 \) and \( p_3 \) must be zero, so assume the contrary: \( p_1 \neq 0 \) or \( p_3 \neq 0 \) and use this to assist in solving the Jacobi identity (\( d^2 = 0 \))

\[
\text{JACOBI} := \{\}
\]

\[
\text{Cobasis} := \text{frameBaseForms}();
\]

\[
\text{for theta in Cobasis do}
\]

\[
\text{JACOBI} := \text{JACOBI} \text{ union coeff_set} (\text{ext_d} (\text{ext_d} (\text{theta}))) ; \text{ od}:
\]

\[
\text{JACOBI} := \text{reduce_equation_set} (\text{JACOBI});
\]
JACOBI := \{ 2 p1 p3 + c231, 3 c121 p1 - c341 p1, c231 - c242, 3 c122 p3 + c342 p3, 2 p1 p3 + c242, -2 c342 - 3 p1 p2 + 3 p1 p4, c232 p3, c241 p1, -c342 + c122, 3 p3 p4 - p3 p2, -c341 + 2 p3 p2 + c121, -2 c341 - 3 p3 p2 + 3 p3 p4, -5 c141 p1 - p2 c341 + 3 c131 p3, -2 c122 + 3 p1 p2 + p1 p4, c342 p1 - c232 p2 + p4 c232 - c122 p1, -3 p1 p2 + p1 p4, -c131 + c142, c341 + c121, c241, c232, -5 c132 p3 + p4 c342 + 3 c142 p1, c341 p3 + c241 p4 - p2 c241 + c121 p3, c342 + 2 p1 p4 + c122, -2 c121 + 3 p3 p4 + p3 p2\}

Solve the linear equations.

\[
\text{eqs := select(x->type(x,linear), JACOBI);}
\]

\[
\text{eqs := \{ c231 - c242, -c342 + c122, -c131 + c142, c341 + c121, c241, c232 \}}
\]

\[
\text{ans := solve(eqs,\{c231, c232, c241, c131, c121, c122\});}
\]

\[
\text{ans := \{ c232 = 0, c131 = c142, c121 = -c341, c231 = c242, c241 = 0, c122 = c342 \}}
\]

\[
\text{sln := remove(x->evalb(rhs(x)=lhs(x)), solve(ans,}
\]

\[
\text{\{c231, c232, c241, c131, c121, c122\}));}
\]

\[
\text{sln := \{ c232 = 0, c131 = c142, c121 = -c341, c231 = c242, c241 = 0, c122 = c342 \}}
\]

\[
\text{eqs := reduce_equation_set(subs(sln, JACOBI));}
\]

\[
\text{eqs := \{ 2 p1 p3 + c242, -c341 + p3 p2, c342 + p3 p2, c341 p1, -5 c141 p1 - p2 c341 + 3 c142 p3, -2 c342 + 3 p1 p2 + p1 p4, 2 c341 + 3 p3 p4 + 3 p3 p2, 3 c34 p4 - p3 p2, -2 c341 - 3 p3 p2 + 3 p3 p4, c342 p3, -3 p1 p2 + p1 p4, c342 + p1 p4, -5 c132 p3 + p4 c342 + 3 c142 p1 \}}
\]

Solve the simplest quadratic Jacobi Identities.

\[
\text{eqs := remove(x->type(x,'*'), factor(select(x->evalb(nops(x)=2), eqs))));}
\]

\[
\text{eqs := \{ 2 p1 p3 + c242, -c341 + p3 p2, c342 + p1 p4 \}}
\]

\[
\text{ans := solve(eqs,\{c341, c242, c342\});}
\]

\[
\text{ans := \{ c341 = p3 p2, c242 = -2 p1 p3, c342 = -p1 p4 \}}
\]

\[
\text{sln := subs(ans, sln) union ans;}
\]

\[
\text{sln := \{ c121 = -p3 p2, c231 = -2 p1 p3, c232 = 0, c122 = -p1 p4, c341 = p3 p2, c242 = -2 p1 p3, c131 = c142, c342 = -p1 p4, c241 = 0 \}}
\]

\[
\text{eqs := factor(reduce_equation_set(simplify(subs(sln, JACOBI))) );}
\]
Consider the remaining quadratic equations.

\[ \text{remove}(x \rightarrow \text{type}(x, \text{cubic}), \text{select}(x \rightarrow \text{type}(x, '\ast'), \text{eqs})); \]


Since \( p1 \neq 0 \) or \( p3 \neq 0 \) we must have \( p2 = 0 \) and \( p4 = 0 \).

\[ \text{ans} := \{p2=0, p4=0\}; \]

\[ \text{ans} := \{p2 = 0, p4 = 0\} \]

\[ \text{sln} := \text{subs}(\text{ans}, \text{sln}) \cup \text{ans}; \]

\[ \text{sln} := \{c231 = -2 p1 p3, c232 = 0, c242 = -2 p1 p3, c131 = c142, c121 = 0, c341 = 0, c342 = 0, p2 = 0, c241 = 0, c122 = 0, p4 = 0\} \]

\[ \text{eqs} := \text{subs}(\text{sln}, \text{JACOBI}) \setminus \{0\}; \]

\[ \text{eqs} := \{-5 c132 p3 + 3 c142 p1, -5 c141 p1 + 3 c142 p3\} \]

\[ \text{eqsV} := \text{convert}(\text{eqs}, \text{list}); \]

\[ \text{eqsV} := \{-5 c132 p3 + 3 c142 p1, -5 c141 p1 + 3 c142 p3\} \]

We may multiply one equation by \( p3 \) and the other by \( p1 \) and sum to get

\( -5 c132 p3^2 + 5 c141 p1^2 \). Since \( p1 \neq 0 \) or \( p3 \neq 0 \) we have \( c132 = t p1^2 \) and \( c141 = t p3^2 \).

\[ \text{ans} := \{c141 = t \ast p3^2, c132 = t \ast p1^2\}; \]

\[ \text{ans} := \{c141 = t p3^2, c132 = t p1^2\} \]

\[ \text{sln} := \text{subs}(\text{ans}, \text{sln}) \cup \text{ans}; \]

\[ \text{sln} := \{c231 = -2 p1 p3, c232 = 0, c242 = -2 p1 p3, c131 = c142, c121 = 0, c341 = 0, c342 = 0, c141 = t p3^2, c132 = t p1^2, p2 = 0, c241 = 0, c122 = 0, p4 = 0\} \]

\[ \text{factor}(\text{subs}(\text{sln}, \text{JACOBI})); \]

\[ \{0, 0 \ast (5 p1 t p3 + 3 c142), p1 (-5 p1 t p3 + 3 c142)\} \]
Since $p1 \neq 0$ or $p3 \neq 0$ we have $c142 = \frac{5 \cdot p1 \cdot p3}{3}$.

```maple
> ans := \{c142 = \frac{5 \cdot p1 \cdot p3}{3}\};

ans := \{c142 = \frac{5 \cdot p1 \cdot p3}{3}\}
```

```maple
> sln := subs(ans, sln) union ans;

sln := \{c231 = -2 \cdot p1 \cdot p3, c232 = 0, c142 = \frac{5 \cdot p1 \cdot p3}{3}, c131 = \frac{5 \cdot p1 \cdot p3}{3}, c242 = -2 \cdot p1 \cdot p3, c121 = 0, c341 = 0, c342 = 0, c141 = t \cdot p3^2, c132 = t \cdot p1^2, p2 = 0, c241 = 0, c122 = 0, p4 = 0\}
```

The Jacobi identity is satisfied.

```maple
> subs(sln, JACOBI);

\{0\}
```

```maple
> Case0 := sln;

CASE0 := map(factor, helmsimp(subs(Case0, structure_constants_array_to_Lie_algebra_data(My_alg, Case0))));

Lie algabra: case0
```

```maple
> check_Jacobi_identity();

true
```

Now change the basis to put this algebra into a canonical form.

### 7.3.1 Case 1: $p1 = 0$ or $p3 = 0$

```maple
> change_frame_to(case0);

M1 := matrix([[0, 0, -p1, -p3, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, p1, p3], [0, 0, p3, -p1, 0, 0], [1/3*t*(p3-p1)*(p3+p1), -(p3-p1)*(p3+p1), 0, 0, -p3-3/(p3-2+p1-2), -p1-3/(p3-2+p1-2)]]);
```
\[
M1 := \begin{bmatrix}
0 & 0 & -p1 & -p3 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p1 & p3 \\
0 & 0 & p3 & -p1 & 0 & 0 \\
\frac{t(p3 - p1)(p3 + p1)}{3} & -(p3 - p1)(p3 + p1) & 0 & 0 & -p3 & p1 \\
\frac{t(p3^4 + p1^4)}{2(p3^2 + p1^2)} & 0 & 0 & 0 & -\frac{p3^3}{p3^2 + p1^2} & -\frac{p1^3}{p3^2 + p1^2}
\end{bmatrix}
\]

The change of basis is a valid since the matrix is non-degenerate.

\[
\text{det}(M1) = p3^6 - p3^4 p1^2 - p1^4 p3^2 + p1^6
\]

\[
\text{factor}(\text{simplify}(\text{change\_Lie\_algebra\_basis}(M1, \text{case1}), \{p1*p3\}))
\]

\[
\text{Lie\_alg\_init}(%);
\]

\[
\text{Lie\_bracket\_mult\_table}();
\]

Calculate the isotropy algebra in this new basis.

\[
V1 := \text{matrix}(6, 2, [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]);
\]

\[
V1 := \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
\text{simplify}(\text{subs}(p3=0, \text{evalm}(M1 \&* \text{matrix}(6, 2, [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]))))
\]
Thus the span is \{e3,e5+e6\}, \{e3,e5-e6\}, however \(e6 \Rightarrow -e6\) is an automorphism. So we have \{e3,e5+e6\} which gives vi.) from Theorem 1.0.1.

7.3.2 Case 2: \(p2 \neq 0\) and \(p4 \neq 0\)

\[
\begin{bmatrix}
0 & 0 & -p1 & -p3 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
2/3*t*p1*p3 & -2*p1*p3 & 0 & 0 & -p1 & -p3 \\
0 & 0 & 2-(1/2)*p1 & 2-(1/2)*p3 & 0 & 0 \\
-4/3*t*p1*p3 & -2*p1*p3 & 0 & 0 & p1 & p3 \\
\end{bmatrix}
\]

\(M2 := \) The change of basis is valid since this matrix is non-degenerate thus

\[
\begin{align*}
32 p3^3 p1^3 \\
\text{det}(M2); \\
\text{CASE} := \text{change_Lie_algebra_basis}(	ext{evalm}(M2), \text{case}); \\
\text{Lie_alg init}(\text{CASE}); \\
\text{Lie_bracket_mult_table}();
\end{align*}
\]
Calculate the isotropy algebra in this new basis.

\[
V := \text{simplify}(\text{map}(\text{factor}, \text{simplify}(\text{evalm}(M2 \times \text{matrix}(6, 2, [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1]))) ));
\]

\[
V := \begin{bmatrix}
0 & 0 & -p1 & -p3 & 0 & 0 \\
p1 & p3 & 0 & 0 & \sqrt{2} p1 & -\sqrt{2} p3 \\
p1 & p3 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
V1 := \text{col}(V, 1);
V1 := [0, 0, -p1, 0, \sqrt{2} p1, p1]
\]

\[
V2 := \text{col}(V, 2);
V2 := [0, 0, -p3, 0, -\sqrt{2} p3, p3]
\]

\[
\text{map}(\text{factor}, \text{simplify}(\text{evalm}(p3 \times V1 + p1 \times V2)));
[0, 0, -2 p1 p3, 0, 0, 2 p1 p3]
\]

\[
\text{map}(\text{factor}, \text{simplify}(\text{evalm}(p3 \times V1 - p1 \times V2)));
[0, 0, 0, 2 \sqrt{2} p1 p3, 0]
\]

Thus the isotropy is \{e3 - e6, e5\} which gives \(v\) from \textit{Theorem 1.0.1}. 
REFERENCES


APPENDICES
APPENDIX A
SUBALGEBRAS OF $\mathfrak{o}(2,2)$

The Lie algebra $\mathfrak{o}(2,2)$ may be represented as $\mathfrak{o}(2,2) = \{ H \in \mathfrak{gl}(4) \mid H^\top \eta + \eta H = 0 \}$ where $\eta$ is any bilinear form with (2,2) signature. For the calculations in this thesis it is advantageous use the form

$$\eta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$  

For this representation, we will use the following matrices as a basis.

$$E_1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad E_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$  

$$F_1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad F_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad F_3 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$  

The basis for each inequivalent subalgebra of $\mathfrak{o}(2,2)$ under inner automorphism is given in the following table. The basis for the complementary subspace $\mathfrak{h}$ used in the proof of Theorem 1.0.1 is also given. For a comparison of the classification in Table A.1 with the classification by Patera, et al., [13] and with the classification by Ghanam and Thompson [5] see Appendix B and Appendix C.
### Table A.1. Subalgebras of $\mathfrak{o}(2, 2)$

<table>
<thead>
<tr>
<th>#</th>
<th>Basis $\mathfrak{h}$</th>
<th>Complement $\mathfrak{j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$F_1 - F_3 + \epsilon(E_1 - E_3)$</td>
<td>$E_1 - E_3 + F_1 - F_3, F_2, E_1 - E_3 + F_1 + F_3, - E_1 - E_3 + F_1 - F_3, E_2$</td>
</tr>
<tr>
<td>2</td>
<td>$E_1 - E_3$</td>
<td>$E_1 + E_3, E_2, F_1, F_3$</td>
</tr>
<tr>
<td>3</td>
<td>$E_2$</td>
<td>$E_1, E_3, F_1 + rE_2, F_2 + sE_2, F_3 + tE_2$</td>
</tr>
<tr>
<td>4</td>
<td>$E_3$</td>
<td>$E_1, E_2, F_1 + rE_3, F_2 + sE_3, F_3 + tE_3$</td>
</tr>
<tr>
<td>5</td>
<td>$F_1 - F_3 - E_2$</td>
<td>$E_1, F_1 - F_3, E_3, F_1 + t(F_1 - F_3 - E_2), F_2$</td>
</tr>
<tr>
<td>6</td>
<td>$F_1 - F_3 + \epsilon E_3$</td>
<td>$E_1, E_2, \epsilon(F_1 - F_3), F_1 + t(F_1 - F_3 + \epsilon E_3), F_2$</td>
</tr>
<tr>
<td>7</td>
<td>$E_2 + cF_2$</td>
<td>$E_1, E_2 + t(E_2 + cF_2), E_3, F_1, F_3$</td>
</tr>
<tr>
<td>8</td>
<td>$E_2 - cF_3$</td>
<td>$E_1, E_2 + t(E_2 - cF_3), E_3, F_1, F_2$</td>
</tr>
<tr>
<td>9</td>
<td>$E_3 + cF_3$</td>
<td>$E_1, E_2, E_3 + t(E_3 + cF_3), F_1, F_2$</td>
</tr>
<tr>
<td>10</td>
<td>$E_1 - E_3, F_1 - F_3$</td>
<td>$E_1 + E_3, E_2, F_1 + F_3, F_2$</td>
</tr>
<tr>
<td>11</td>
<td>$E_2, F_2$</td>
<td>$E_1, E_3, F_1, F_3$</td>
</tr>
<tr>
<td>12</td>
<td>$E_2, F_3$</td>
<td>$E_1, E_3, F_1, F_2$</td>
</tr>
<tr>
<td>13</td>
<td>$E_3, F_2$</td>
<td>$E_1, E_2, F_1, F_2$</td>
</tr>
<tr>
<td>14</td>
<td>$E_2, F_1 - F_3$</td>
<td>$E_1, E_3, F_2, F_1 + F_3$</td>
</tr>
<tr>
<td>15</td>
<td>$E_3, F_1 - F_3$</td>
<td>$F_1 + F_3, E_1, E_2, E_3$</td>
</tr>
<tr>
<td>16</td>
<td>$F_2, F_1 - F_3$</td>
<td>$F_1 + F_3, E_1, E_2, E_3$</td>
</tr>
<tr>
<td>17</td>
<td>$F_2 + cE_2, F_1 - F_3$</td>
<td>$F_1 + F_3, E_1, E_2, E_3$</td>
</tr>
<tr>
<td>18</td>
<td>$F_2 - cE_3, F_1 - F_3$</td>
<td>$F_1 + F_3, E_1, E_2, E_3$</td>
</tr>
<tr>
<td>19</td>
<td>$F_2 + \epsilon(E_1 - E_3), F_1 - F_3$</td>
<td>$F_1 + F_3, F_2, E_1 + E_3, E_2$</td>
</tr>
<tr>
<td>20</td>
<td>$E_2 + F_2, F_1 - F_3 + \epsilon(E_1 - E_3)$</td>
<td>$F_1 + F_3, E_1 + E_3, E_2 - F_2, F_1 - F_3 - \epsilon(E_1 - E_3)$</td>
</tr>
<tr>
<td>21</td>
<td>$E_1, E_2, E_3$</td>
<td>$F_1, F_2, F_3$</td>
</tr>
<tr>
<td>22</td>
<td>$E_1 - F_1, E_2 + F_2, E_3 - F_3$</td>
<td>$E_1 + t(E_1 - F_1), E_2 + t(E_2 + F_2), E_3 + t(E_3 - F_3)$</td>
</tr>
<tr>
<td>23</td>
<td>$E_1 + F_1, E_2 + F_2, E_3 + F_3$</td>
<td>$E_1 + t(E_1 + F_1), E_2 + t(E_2 + F_2), E_3 + t(E_3 + F_3)$</td>
</tr>
<tr>
<td>24</td>
<td>$E_2, F_2, F_1 - F_3$</td>
<td>$E_1, E_3, F_1 + F_3$</td>
</tr>
<tr>
<td>25</td>
<td>$E_3, F_2, F_1 - F_3$</td>
<td>$E_1, E_2, F_1 + F_3$</td>
</tr>
<tr>
<td>26</td>
<td>$E_2 + \alpha F_2, E_1 - E_3, F_1 - F_3$</td>
<td>$E_1 + E_3, F_1 + F_3, F_2$</td>
</tr>
<tr>
<td>27</td>
<td>$E_1, E_2, E_3, F_3$</td>
<td>$F_1, F_2$</td>
</tr>
<tr>
<td>28</td>
<td>$E_1, E_2, E_3, F_2$</td>
<td>$F_1, F_3$</td>
</tr>
<tr>
<td>29</td>
<td>$E_2, E_1 - E_3, F_2, F_1 - F_3$</td>
<td>$E_1 + E_3, F_1 + F_3$</td>
</tr>
<tr>
<td>30</td>
<td>$E_1, E_2, E_3, F_1 - F_3$</td>
<td>$E_1 + E_3, E_2$</td>
</tr>
<tr>
<td>31</td>
<td>$E_1, E_2, E_3, F_2, F_1 - F_3$</td>
<td>$E_1 + E_3$</td>
</tr>
</tbody>
</table>
APPENDIX B
ANALYSIS OF PARAMETERS

In [5], Thompson and Ghanam classify the subalgebras of $\mathfrak{o}(2,2)$ in a representation where

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. $$

In [5] the basis for each subalgebra of $\mathfrak{o}(2,2)$ is given explicitly as matrices. Instead of doing this to specify the subalgebras of $\mathfrak{o}(2,2)$, we will use the basis

$$D_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

for $\mathfrak{o}(2,2)$.

The subalgebras listed in [5] are often parameterized but some parameters values duplicate other algebras on that list. We will analyze what restrictions need to be placed on the parameters to eliminate any duplicates. To do this we note that $\mathfrak{o}(2,2) \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2)$ where $\{C_i\}$ and $\{D_i\}$ are a basis for $\mathfrak{sl}(2)$. Consequently, any subalgebra has a symmetric counterpart given by $D_j \mapsto C_j$ and $C_j \mapsto D_j$, thus we will analyze only one of these symmetric algebras, and freely make assumptions obtainable by this symmetry. In addition,
the following inner automorphisms will often be used.

(B.1) \( \text{Ad}(\frac{\pi}{2}C_1) : C_2 \mapsto -C_2 \) and \( C_3 \mapsto -C_3 \).

(B.2) \( \text{Ad}(\frac{\pi}{2}D_1) : D_2 \mapsto -D_2 \) and \( D_3 \mapsto -D_3 \).

(B.3) \( \text{Ad}(\frac{\ln|\alpha|}{2}C_2) : C_1 + C_3 \mapsto |\alpha|(D_1 + D_3) \).

(B.4) \( \text{Ad}(\frac{\ln|\alpha|}{2}D_2) : D_1 + D_3 \mapsto |\alpha|(D_1 + D_3) \).

It should be noted that the list of subalgebras in [5] is missing three subalgebras and subalgebra 25 is not an algebra.

B.1 The Analysis

B.1.1 Subalgebras 1

For subalgebras 1, \( h_{\alpha\beta} = \text{span}\{(\beta + \alpha)D_1 + (\beta - \alpha)C_1\} \), as a matrix we have

\[
\eta = \begin{bmatrix}
0 & \alpha & 0 & 0 \\
-\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \beta \\
0 & 0 & -\beta & 0 \\
\end{bmatrix}.
\]

We may assume that \( \alpha \neq \pm \beta \), otherwise this subalgebra is equivalent to subalgebra 6. By symmetry, we may further assume \( |\beta + \alpha| \geq |\beta - \alpha| \) and then we can rescale to get

(B.5) \( h_c = \text{span}\{D_1 + cC_1\} \) where \( c = \frac{\beta - \alpha}{\beta + \alpha} \) and \( 0 < |c| \leq 1 \).

B.1.2 Subalgebras 2

For subalgebras 2, \( h_{\alpha\beta} = \text{span}\{(\alpha - \beta)D_2 + (\alpha + \beta)C_2\} \), we may assume that \( \alpha \neq \pm \beta \), otherwise this subalgebra is equivalent to subalgebra 6. By symmetry and automorphisms (B.1) or (B.2), we may further assume that \( |\alpha - \beta| \leq |\alpha + \beta| \) and \( \alpha + \beta \) and \( \alpha - \beta \) both have the same sign. Then we can rescale to get the subalgebra

(B.6) \( h_c = \text{span}\{D_2 + cE_2\} \) where \( 0 < c = \frac{\alpha + \beta}{\alpha - \beta} \leq 1 \).
B.1.3 Subalgebras 4

For subalgebras 4, \( h_{\alpha\beta} = \text{span}\{\alpha D_2 + \beta C_1\} \), we may assume \( \alpha \neq 0 \) and \( \beta \neq 0 \) otherwise this subalgebra is equivalent to subalgebra 6 and subalgebra 7 respectively. By automorphism (B.2) we may further assume \( \alpha \) and \( \beta \) have the same sign. We may then rescale to get the subalgebra

\[(B.7) \quad h_c = \text{span}\{D_2 + cE_1\} \text{ where } 0 < c = \frac{\beta}{\alpha}\]

B.1.4 Subalgebras 5

For subalgebras 5, \( h_\alpha = \text{span}\{D_1 + D_3 + \alpha C_1\} \), we may assume \( \alpha \neq 0 \) otherwise this subalgebra is equivalent to subalgebra 9. We can apply automorphism (B.2) to get \( |\alpha|(D_1 + D_3) + \alpha C_1 \). Then we can rescale to get the subalgebra

\[(B.8) \quad h_\epsilon = \text{span}\{D_1 + D_3 + \epsilon C_1\} \text{ where } \epsilon = \pm 1\]

B.1.5 Subalgebras 8

For subalgebras 8, \( h_\alpha = \text{span}\{D_1 + D_3 - \alpha C_2\} \), we may assume \( \alpha \neq 0 \) otherwise this subalgebra is equivalent to subalgebra 9. By automorphism (B.1) we may assume \( \alpha > 0 \) and we can apply automorphism (B.2) to get \( |\alpha|(D_1 + D_3) - \alpha C_2 \). Then we can rescale to get the subalgebra

\[(B.9) \quad h_\epsilon = \text{span}\{D_1 + D_3 - C_2\}\]

B.1.6 Subalgebras 11

For subalgebras 11, \( h_\alpha = \text{span}\{C_2 + \alpha D_2, C_1 + C_3\} \) we may assume \( \alpha \neq 0 \) otherwise this subalgebra is equivalent to subalgebra 10. By automorphism (B.2), we may further
assume that $\alpha > 0$. Thus we get the subalgebra

(B.10) \[ h_c = \text{span}\{C_2 + cD_2, C_1 + C_3\} \text{ where } 0 < c = \alpha. \]

B.1.7 Subalgebras 12

For subalgebras 12, $h_\alpha = \text{span}\{C_1 + C_3, C_2 + \alpha D_1\}$, we may assume $\alpha \neq 0$ otherwise this subalgebra is equivalent to subalgebra 10. Thus we get the subalgebra

(B.11) \[ h_\alpha = \text{span}\{C_1 + C_3, C_2 + \alpha D_1\} \text{ where } c = \alpha \neq 0. \]

B.1.8 Subalgebras 26

For subalgebras 26, $h_\alpha = \text{span}\{C_2 + \alpha D_2, C_1 + C_3, D_1 + D_3\}$, we don't need to make any assumptions. Thus we get the subalgebra

(B.12) \[ h_\alpha = \text{span}\{C_2 + \alpha D_2, C_1 + C_3, D_1 + D_3\} \text{ where } \alpha \in \mathbb{R}. \]

B.2 Analysis of Subalgebra 25

We will spell out why subalgebra 25 is not a Lie algebra. The basis given in [5] are the matrices

\[ X_1 = \begin{bmatrix} 0 & 0 & 1 & \alpha \\ 0 & 0 & \alpha & 1 \\ 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}. \]

Now if we calculate $[X_1, X_3]$ we get

\[ \begin{bmatrix} 0 & 0 & 1 & \alpha \\ 0 & 0 & \alpha & 1 \\ 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2\alpha & 0 \\ 0 & 0 & 0 & 2\alpha \\ -2\alpha & 0 & 0 & 0 \\ 0 & 2\alpha & 0 & 0 \end{bmatrix}. \]

Which is not a linear combination of $\{X_1, X_2, X_3\}$, therefore this is not a Lie algebra.
APPENDIX C

COMPARISON OF CLASSIFICATIONS FOR SUBALGEBRAS OF $\mathfrak{o}(2,2)$

Patera, et al., [13] classified the inequivalent subalgebras of $\mathfrak{o}(2,2)$, as did Ghanam and Thompson [5], in a representation where

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$ 

Both papers use the fact that $\mathfrak{o}(2,2) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and chose a basis based on this. The basis used in [13] is denoted by $\{A_i, B_j\}$ where $\{A_i\}$ and $\{B_j\}$ are each a basis for $\mathfrak{sl}(2)$. The basis used in [5] will be denoted by $\{C_i, D_j\}$ where $\{C_i\}$ and $\{D_j\}$ are each a basis for $\mathfrak{sl}(2)$. The matrix representations for these bases are given below.

$$2A_1 = D_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad 2B_1 = -C_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$2A_2 = D_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad 2B_2 = C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$2A_3 = D_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad 2B_3 = -C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

To identify the subalgebras enumerated in Table A.1, with those in these classifications we need to do some calculations. Conjugation of the basis matrices in Table A.1 by

$$\frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

gives the basis matrices for the classification in [13]. Furthermore, the basis vectors for the subalgebras in Table A.1 are directly mapped to the basis vectors for the subalgebras classified in [13]. The identification of subalgebras in the $\{C_i, D_i\}$ basis with subalgebras
in the \( \{A_i, B_i\} \) basis is more involved. We will use the results of Appendix B and the automorphisms (B.1),(B.2),(B.3), and (B.4) listed there to identify the subalgebras in \([13]\) in the classification in \([5]\). The following tables identify the equivalent subalgebras in the different classifications. The columns in the following tables are to be interpreted as follows:

i.) The number of the subalgebra as in Table A.1.

ii.) The classification of this subalgebra in \([13]\) for specified parameter.

iii.) The subalgebra in the basis \( \{A_i, B_i\} \) and \( \{C_i, D_i\} \).

iv.) The automorphisms that map the basis to the basis used in the classification in \([5]\).

v.) The basis for the subalgebra as classified in \([5]\).

vi.) The numerical classification of this subalgebra in \([5]\) for the specified parameter.

The subalgebras \( e_{1,4}, e_{2,23} \) for \( \epsilon = 1 \) and \( e_{2,8} \) from \([13]\) have no equivalent subalgebra in \([5]\). Consequently, these three subalgebras do not appear in this table the denoted with \( ^1 \). Furthermore, in \([5]\) subalgebra 25 is not an algebra, thus does it does not appear in this table.
### Table C.1. Comparison of One Dimensional Subalgebras of $\mathfrak{o}(2, 2)$

<table>
<thead>
<tr>
<th>#</th>
<th>PSWZ</th>
<th>Basis</th>
<th>Automorphisms</th>
<th>Basis</th>
<th>Parameter</th>
<th>GT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e_{1,3}$ $\epsilon = 1$, $e_{1,4}$ $\epsilon = -1$,</td>
<td>$B_1 - B_3 + \epsilon(A_1 - A_3)$ $= C_1 - C_3 - \epsilon(D_1 - D_3)$</td>
<td>$\text{Ad}(\frac{\alpha}{2}C_1), \text{Ad}(\frac{\beta}{2}D_1)$</td>
<td>$C_1 + C_3 - \epsilon(D_1 + D_3)$,</td>
<td>$\epsilon = -1^\dagger$</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>$e_{1,10}$</td>
<td>$A_3 - A_1$ $= D_1 - D_3$</td>
<td>$\text{Ad}(\frac{\alpha}{2}D_1)$</td>
<td>$D_1 + D_3$</td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>$e_{1,11}$</td>
<td>$A_2$ $= D_2$</td>
<td>$D_2$</td>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>$e_{1,1}$</td>
<td>$A_3$ $= D_1$</td>
<td>$D_1$</td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>$e_{1,6}$</td>
<td>$B_1 - B_3 - A_2$ $= C_1 - C_3 - D_2$</td>
<td>$\text{Ad}(\frac{\alpha}{2}C_1)$</td>
<td>$C_1 + C_3 - D_2$</td>
<td>$\epsilon = \frac{\alpha}{</td>
<td>\alpha</td>
</tr>
<tr>
<td>6</td>
<td>$e_{1,7}$ $\epsilon = -1$, $e_{1,8}$ $\epsilon = 1$,</td>
<td>$B_1 - B_3 + \epsilon A_3$ $= C_1 - C_3 + \epsilon D_1$</td>
<td>$\text{Ad}(\frac{\alpha}{2}C_1)$</td>
<td>$C_1 + C_3 + \epsilon D_1$</td>
<td>$\epsilon = \frac{\alpha}{</td>
<td>\alpha</td>
</tr>
<tr>
<td>7</td>
<td>$e_{1,2}$ $\epsilon = 1$, $e_{1,5}$ $\epsilon &gt; 0$,</td>
<td>$A_2 + \epsilon B_2$ $= D_2 + \epsilon C_2$</td>
<td>$D_2 + \epsilon C_2$</td>
<td></td>
<td>$c = \frac{\alpha + \beta}{\alpha - \beta}$</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>$e_{1,9}$ $\epsilon &gt; 0$</td>
<td>$A_2 - \epsilon B_3$ $= D_2 + \epsilon C_1$</td>
<td>$D_2 + \epsilon C_1$</td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>$e_{1,12}$ $\epsilon \neq 0, \pm 1$, $e_{1,13}$ $\epsilon = -1$, $e_{1,14}$ $\epsilon = 1$,</td>
<td>$A_3 + \epsilon B_3$ $= D_1 - \epsilon C_1$</td>
<td>$D_1 - \epsilon C_1$</td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
Table C.2. Comparison of Two Dimensional Subalgebras of o(2, 2)

<table>
<thead>
<tr>
<th>#</th>
<th>PSWZ</th>
<th>Basis</th>
<th>Automorphisms</th>
<th>Basis</th>
<th>Parameter</th>
<th>GT</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$e_{2,1}$</td>
<td>$A_1 - A_3, B_1 - B_3$</td>
<td>$D_3 - D_1, C_1 - C_3$</td>
<td>$\text{Ad}(\frac{\pi}{2}C_1), \text{Ad}(\frac{\pi}{2}D_1)$</td>
<td>$-D_1 - D_3, C_1 + C_3$,</td>
<td>17</td>
</tr>
<tr>
<td>11</td>
<td>$e_{2,2}$</td>
<td>$A_2, B_2$</td>
<td>$D_2, C_2$</td>
<td>$D_2, C_2$,</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$e_{2,5}$</td>
<td>$A_2, B_3$</td>
<td>$D_2, -C_1$,</td>
<td>$D_1, -C_1$,</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>$e_{2,6}$</td>
<td>$A_3, B_3$</td>
<td>$D_1, -C_1$,</td>
<td>$D_1, -C_1$,</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$e_{2,3}$</td>
<td>$A_2, B_1 - B_3$</td>
<td>$D_2, C_1 - C_1$</td>
<td>$\text{Ad}(\frac{\pi}{2}C_1)$</td>
<td>$D_2, C_1 + C_3$,</td>
<td>18</td>
</tr>
<tr>
<td>15</td>
<td>$e_{2,4}$</td>
<td>$A_3, B_1 - B_3$</td>
<td>$D_1, C_1 - C_3$</td>
<td>$\text{Ad}(\frac{\pi}{2}C_1)$</td>
<td>$D_1, C_1 + C_3$,</td>
<td>19</td>
</tr>
<tr>
<td>16</td>
<td>$e_{2,7}$</td>
<td>$B_2, B_1 - B_3$</td>
<td>$C_2, C_1 - C_3$</td>
<td>$\text{Ad}(\frac{\pi}{2}C_1)$</td>
<td>$-C_2, C_1 + C_3$,</td>
<td>10</td>
</tr>
<tr>
<td>17</td>
<td>$e_{2,10}$</td>
<td>$c = 1, e_{2,11}$ 0 $&lt; c \neq 1$,</td>
<td>$B_2 + cA_2, B_1 - B_3$</td>
<td>$C_2 + cD_2, C_1 - C_3$</td>
<td>$\text{Ad}(\frac{\pi}{2}C_1)$</td>
<td>$-C_2 + cD_2, C_1 + C_3$,</td>
</tr>
<tr>
<td>18</td>
<td>$e_{2,12}$</td>
<td>$c \neq 0$,</td>
<td>$B_2 - cA_3, B_1 - B_3$</td>
<td>$C_2 - cD_1, C_1 - C_3$</td>
<td>$\text{Ad}(\frac{\pi}{2}C_1)$</td>
<td>$-C_2 - cD_1, C_1 + C_3$,</td>
</tr>
<tr>
<td>19</td>
<td>$e_{2,13}$</td>
<td>$\epsilon = \pm 1$</td>
<td>$B_2 + \epsilon(A_1 - A_3), B_1 - B_3$</td>
<td>$C_2 - \epsilon(D_1 - D_3), C_1 - C_3$</td>
<td>$\text{Ad}(\frac{\pi}{2}C_1), \text{Ad}(\frac{\pi}{2}D_1)$</td>
<td>$-C_2 - \epsilon(D_1 + D_3), C_1 + C_3$,</td>
</tr>
<tr>
<td>20</td>
<td>$e_{2,8}$</td>
<td>$\epsilon = 1$,</td>
<td>$B_2 + B_2, B_1 - B_3 + \epsilon(A_1 - A_3)$</td>
<td>$C_2 + D_2, C_1 - C_3 - \epsilon(D_1 - D_3)$</td>
<td>$\text{Ad}(\frac{\pi}{2}C_1), \text{Ad}(\frac{\pi}{2}D_1)$</td>
<td>$-C_2 - D_2, C_1 + C_3 - \epsilon(D_1 + D_3)$,</td>
</tr>
</tbody>
</table>
Table C.3. Comparison of Three Dimensional Subalgebras of $\mathfrak{o}(2,2)$

<table>
<thead>
<tr>
<th>#</th>
<th>PSWZ</th>
<th>Basis</th>
<th>Automorphisms</th>
<th>Basis</th>
<th>Parameter</th>
<th>GT</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>$\epsilon_{3,1}$</td>
<td>$A_1, A_2, A_3$</td>
<td>$D_3, D_2, D_1$</td>
<td>$D_3, D_2, D_1$,</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>$\epsilon_{3,8}$</td>
<td>$A_1 - B_1, A_2 + B_2, A_3 - B_3$</td>
<td>$C_3 + D_3, C_2 + D_2, C_1 + D_1$</td>
<td>$C_3 + D_3, C_2 + D_2, C_1 + D_1$,</td>
<td>†</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>$\epsilon_{3,9}$</td>
<td>$A_1 + B_1, A_2 + B_2, A_3 + B_3$</td>
<td>$D_3 - C_3, C_2 + D_2, D_1 - C_1$</td>
<td>$D_3 - C_3, C_2 + D_2, D_1 - C_1$,</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>$\epsilon_{3,2}$</td>
<td>$A_2, B_2, B_1 - B_3$</td>
<td>$\text{Ad}(\frac{\alpha}{2} C_1)$</td>
<td>$D_2, -C_2, C_1 + C_3$,</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>$\epsilon_{3,4}$</td>
<td>$A_3, B_2, B_1 - B_3$</td>
<td>$\text{Ad}(\frac{\alpha}{2} C_1), \text{Ad}(\frac{\beta}{2} D_1)$</td>
<td>$D_1, -C_2, C_1 + C_3$,</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>$\epsilon_{3,3}$</td>
<td>$\alpha = 0$,</td>
<td>$\text{Ad}(\frac{\alpha}{2} C_1), \text{Ad}(\frac{\beta}{2} D_1)$</td>
<td>$-D_2 - \alpha C_2$,</td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>$\epsilon_{3,5}$</td>
<td>$\alpha = 1$,</td>
<td></td>
<td></td>
<td>$-D_1 - D_3, C_1 + C_3$,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\epsilon_{3,6}$</td>
<td>$\alpha = -1$,</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\epsilon_{3,7}$</td>
<td>$\alpha \neq 0, \pm 1$,</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
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</table>
### Table C.4. Comparison of Four Dimensional Subalgebras of $o(2, 2)$

<table>
<thead>
<tr>
<th>#</th>
<th>PSWZ</th>
<th>Basis</th>
<th>Automorphisms</th>
<th>Basis</th>
<th>Parameter</th>
<th>$\mathrm{GT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td>$e_{4,1}$</td>
<td>$A_1, A_2, A_3, B_3$ [= D_3, D_2, D_1, -C_1]</td>
<td></td>
<td>$D_3, D_2, D_1, -C_1$</td>
<td></td>
<td>28</td>
</tr>
<tr>
<td>28</td>
<td>$e_{4,4}$</td>
<td>$A_1, A_2, A_3, B_2$ [= D_3, D_2, D_1, C_2]</td>
<td></td>
<td>$D_3, D_2, D_1, C_2$</td>
<td></td>
<td>27</td>
</tr>
<tr>
<td>29</td>
<td>$e_{4,2}$</td>
<td>$A_2, A_1 - A_3, B_2, B_1 - B_3$ [= D_2, D_3 - D_1, C_2, C_1 - C_3]</td>
<td>$\text{Ad}(\frac{\pi}{2} C_1), \text{Ad}(\frac{\pi}{2} D_1)$</td>
<td>$-D_2, -D_1 - D_3, -C_2, C_1 + C_3$</td>
<td></td>
<td>30</td>
</tr>
<tr>
<td>30</td>
<td>$e_{4,3}$</td>
<td>$A_1, A_2, A_3, B_1 - B_3$ [= D_3, D_2, D_1, C_1 - C_3]</td>
<td>$\text{Ad}(\frac{\pi}{2} C_1)$</td>
<td>$D_3, D_2, D_1, C_1 + C_3$</td>
<td></td>
<td>29</td>
</tr>
</tbody>
</table>

### Table C.5. Comparison of Five Dimensional Subalgebras of $o(2, 2)$

<table>
<thead>
<tr>
<th>#</th>
<th>PSWZ</th>
<th>Basis</th>
<th>Automorphisms</th>
<th>Basis</th>
<th>Parameter</th>
<th>$\mathrm{GT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>$e_{5,1}$</td>
<td>$A_1, A_2, A_3, B_2, B_1 - B_3$ [= D_3, D_2, D_1, C_2, C_1 - C_3]</td>
<td>$\text{Ad}(\frac{\pi}{2} C_1)$</td>
<td>$D_3, D_2, D_1, -C_2, C_1 + C_3$</td>
<td></td>
<td>31</td>
</tr>
</tbody>
</table>
APPENDIX D
JACOBI IDENTITIES FOR $g$

\[
[h, [h, m]] = [h, H^h_{jk} m + \Delta_j^h h] = H^h_{jk} H^k_{il} m + H^h_{jk} \Delta_i^h h + \Delta_j^h \Delta_j^{il} h
\]

\[
[h, m], h = -[h, H^h_{jk} m + \Delta_j^h h] = -H^h_{jk} H^k_{il} m - H^h_{jk} \Delta_i^h h - \Delta_j^h \Delta_j^{il} h
\]

\[
[m, [h, h]] = [m, J^h_{jk} h, h] = -J^h_{jk} [h, m]
\]

\[
[m, [m, m]] = [m, B^m_{jk} m + C^m_{jk} h] = B^m_{jk} H^m_{il} m + B^m_{jk} \Delta_i^m h + C^m_{jk} \Delta_j^{il} h
\]

\[
[m, [m, h]] = [m, H^m_{jk} m + \Delta_j^m h] = -H^m_{jk} B^m_{il} m - H^m_{jk} \Delta_i^m h + \Delta_j^m \Delta_j^{il} h
\]

\[
[m, [m, h]] = [m, C^m_{jk} m + \Delta_j^m h] = H^m_{jk} B^m_{il} m + H^m_{jk} \Delta_i^m h - \Delta_j^m \Delta_j^{il} h
\]

\[
[m, [m, m]] = [m, B^m_{jk} m + C^m_{jk} h] = B^m_{jk} B^m_{il} m + B^m_{jk} \Delta_i^m h - C^m_{jk} H^m_{il} m - C^m_{jk} \Delta_j^{il} h
\]

\[
[m, [m, m]] = [m, B^m_{jk} m + C^m_{jk} h] = B^m_{jk} B^m_{il} m + B^m_{jk} \Delta_i^m h - C^m_{jk} H^m_{il} m - C^m_{jk} \Delta_j^{il} h
\]

\[
[m, [m, m]] = [m, B^m_{jk} m + C^m_{jk} h] = B^m_{jk} B^m_{il} m + B^m_{jk} \Delta_i^m h - C^m_{jk} H^m_{il} m - C^m_{jk} \Delta_j^{il} h
\]

\[
[m, [m, m]] = [m, B^m_{jk} m + C^m_{jk} h] = B^m_{jk} B^m_{il} m + B^m_{jk} \Delta_i^m h - C^m_{jk} H^m_{il} m - C^m_{jk} \Delta_j^{il} h
\]
APPENDIX E
MAPLE SUBROUTINES

## Utility to reduces a set of polynomials that differ by both sign and a multiples

```
reduce_equation_set:=proc()
local ans;
if nargs = 2 then ans:= simplify( args[1], args[2]);
  else ans:= args[1];
end if;
an := map(x-> sign(x)*primpart(x),ans);
return(expand(ans minus {0}));
end proc;
```

## P matrix 4 5

```
P_matrix_4_5:=proc(Codim, Isodim, alg_name)
local basis, PM, i, j, k, K, EQS, TEMP, SLN, NEWP, vars, p_subs, count;
change_frame_to(alg_name);
basis:=frameBaseVectors();

##### Define P
```
PM:=array(1..Codim,1..4):
for k from 1 to Codim do
  for j from 1 to 4 do
    PM[k,j]:=p||k||j:
  end do:
end do:

#### CALCULATE "-K" from ad(h) on h
for i from 1 to Isodim do
  K||(Codim+i):=quotient_ad(basis[Codim+i],basis[1..Codim],basis[7-Isodim..6]):
end do:

#### SET UP EQUATION 4.5
EQS:={}:
for i from 1 to Isodim do
  TEMP:=evalm( PM&* Mat_Basis[Codim+i] - K||(Codim+i) &* PM):
  EQS:=EQS union convert(TEMP,set):
end do:

#### SOLVE EQUATION 4.5 for P
SLN:=solve(EQS, convert(PM,set)):
NEWP:=subs(SLN, op(PM)):

#### RELABEL COMPONENTS of P
vars:={}:
for k from 1 to Codim do
for j from 1 to 4 do
  if NEWP[k,j] = p||k||j then
    count:=count+1:
    vars:=vars union {p||count}:
    p_subs:=p_subs union {p||k||j=p||count}:
  end if:
end do:
end do:

return(subs(p_subs, op(NEWP)));
end proc;

##
## Create Lie Algebra using P in accordance with Equation 4.4
#######################################
create_algebra_4_4:=proc(Codim, Isodim, PM ,alg_name)
local My_alg, epsilon, temp, i, j, delta, gam, r, k;
change_frame_to(alg_name);
My_alg:=array(sparse,1 .. 4+Isodim,1 .. 4 +Isodim,1 .. 4+Isodim):

### ASSIGN "H" (ISOTROPY REPRESENTATION) for epsilon from 1 to Isodim do temp:=Mat_Basis[Codim+epsilon]:
for i from 1 to 4 do
  for j from 1 to 4 do
    My_alg[epsilon+4,i,j]:= temp[j,i]:
    My_alg[i,4+epsilon,j]: = -temp[j,i]:
  end do:
end do:
end do:
end do:

##### ASSIGN "J" (ISOTROPY SUBALGEBRA)
for epsilon from 1 to Isodim do
  for delta from epsilon+1 to Isodim do
    for gam from 1 to Isodim do
      My_alg[epsilon+4,delta+4,gam+4]
      :=structure_constant(Codim+epsilon,Codim+delta,Codim+gam,022):
      end do:
    end do:
  end do:
end do:

##### ASSIGN "B" Coefficients
for i from 1 to 4 do
  for j from 1 to 4 do
    for k from j+1 to 4 do
      for r from 1 to Codim do
        My_alg[j,k,i] := My_alg[j,k,i] + 
        PM[r,j]*Mat_Basis[r][i,k] - PM[r,k]*Mat_Basis[r][i,j]:
      end do:
      My_alg[k,j,i] :=-My_alg[j,k,i]:
    end do:
  end do:
end do:
end do:

##### ASSIGN "A" Coefficients
for delta from 1 to Isodim do
  for epsilon from 1 to Isodim do

end do:

for j from 1 to 4 do
    for r from 1 to Codim do
            PM[r,j] * structure_constant(Codim+epsilon,r,Codim+delta,022):
    end do:
    My_alg[j,epsilon+4,delta+4] := -My_alg[epsilon+4,j,delta+4]:
end do:
end do:
end do:

#### FILL IN UNKNOWN "C" Coefficients
for epsilon from 1 to Isodim do
    for j from 1 to 4 do
        for k from j+1 to 4 do
            My_alg[j,k,epsilon+4] := cl lj
                I
                lkl
                I
                epsilon:
            My_alg[k,j,epsilon+4] := -My_alg[j,k,epsilon+4]:
        end do:
    end do:
end do:
return op(My_alg);
end proc;

##
## Utility for the reductive equations P

EQ_LD:=proc( M_form, Iso_number )
local LDF, i, j, k, epsilon, basis:
basis:=frameBaseVectors();
LDF:=array(1..4,1..4):
for i from 1 to 4 do
  for j from 1 to 4 do
    LDF[i,j]:=Lie_derivative(basis[4+Iso_number],M_form[i,j]):
  end do:
end do:
LDF:= LDF &mplus ( Mat_Basis[Codim+Iso_number] &mmult3 M_form
                    &mminus (M_form &mmult4 Mat_Basis[Codim+Iso_number])):
return op(LDF):
end proc:

##
## Make the reductive Equation 4.6
##################################
reductive_equations_4_6:=proc(Codim, Isodim,Alg_name)
local rdu,epsilon,temp,i, RDE,REQS,j,cobasis;
rdu:=array(1..Isodim,1..4):
cobasis:=frameBaseForms();
for epsilon from 1 to Isodim do
  temp:= cobasis[4+epsilon]:
  for i from 1 to 4 do rdu[epsilon,i]:=r||epsilon||i:
    temp:=temp &plus (rdu[epsilon,i] &mult cobasis[i])
  end do:
  if epsilon =1 then RDE:= Mat_Basis[Codim+epsilon] &mmult2 temp:
  else RDE:= RDE &mplus (Mat_Basis[Codim+epsilon] &mmult2 temp ):
end if:
end do:

### FOR REDUCTIVE NEED TO SOLVE ALL red|\epsilon = 0 for the r variables

for \epsilon from 1 to Isodim do
  \text{red}||\epsilon:=\text{EQ\_LD}(\text{RDE}, \epsilon);
end do:

\text{REQS:}={}:

for \epsilon from 1 to Isodim do
  for i from 1 to 4 do
    for j from 1 to 4 do
      \text{REQS:}=\text{REQS union coeff\_set}(\text{red}||\epsilon[i,j]):
    end do:
  end do:
end do:

return \text{REQS};
end proc: