Using Wavelets as a Computational and Theoretical Tool for Homogenization

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USING WAVELETS AS A COMPUTATIONAL AND THEORETICAL TOOL
FOR HOMOGENIZATION

by

Laura L. Watkins

A dissertation submitted in partial fulfillment
of the requirements for the degree
of
DOCTOR OF PHILOSOPHY
in
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2005
ABSTRACT

Using Wavelets as a Computational and Theoretical Tool for Homogenization

by

Laura Lee Watkins, Doctor of Philosophy
Utah State University, 2005

Since the cost of petroleum fluctuates widely, it is advisable to optimize extraction of oil and other hydrocarbon products from existing oil reserves. Because of the costs involved in recovering oil from a reservoir, predicting reservoir performance can be a useful tool for determining whether continued extraction might be profitable. This can be done using computer simulations of the physical processes involved such as pressure/head, fluid velocities, and so forth. Fluid flow within a reservoir occurs at a very small scale relative to the size of the reservoir. This size difference makes performing simulations at the physically appropriate scale unfeasible. Homogenization is a technique used in reservoir simulation to upscale small scale dependent behavior, such as a permeability tensor, to make simulation feasible. To calculate a homogenized permeability tensor, the solution to a system of uncoupled elliptic partial differential equations must be found repeatedly throughout the reservoir. Generally, the solution to the system of differential equations is approximated numerically using finite element or finite difference methods. We explore using wavelets as a means of characterizing homogenization in reservoir simulations in the search for fast algorithms for computing equivalent tensors. In addition to the analogy developed between
homogenization and wavelets, proofs of convergence results from homogenization within the wavelet characterization are considered.

(101 pages)
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Laura Lee Watkins
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CHAPTER 1

INTRODUCTION

In recent years drastic changes in the cost of various forms of energy have occurred. As of this writing gasoline prices are nearing record high levels. While we are not in immediate danger of paying a record high price of $2.94 per gallon, when adjusted for inflation, we are paying the highest prices in a decade [1]. Historically, the costs of electricity, natural gas and petroleum products have fluctuated widely. Some reasons for the fluctuations include changes in cost of crude oil, seasonal fluctuations in demand, and an imbalance of supply and demand. Additionally, the fluctuating price of petroleum can affect the price of other forms of energy [2]. Due to the volatility of the price of petroleum and other hydrocarbon products in the worldwide market it is advisable to optimize extraction of oil and other hydrocarbon products from oil reserves. One approach to optimizing production is to develop a better understanding of petroleum reservoirs by performing accurate computer simulations of the reservoirs.

Oil and gas are generally created within sedimentary rock. For this reason, hydrocarbon products are typically extracted from sedimentary formations. One type of sedimentary rock formation of great interest are river delta formations because some of the world’s richest and most prolific oil fields are found in deltaic sandstones, like the Niger River and the Mississippi River [3]. The formation of sedimentary rock generally begins in the mountains with weathering that breaks rocks into smaller particles. Plentiful rainfall in the mountains combined with the steep slopes sends streams running rapidly downhill. These streams carry rubble formed from weathering and as this rubble is carried downstream it breaks into smaller pieces. When the water approaches the lowlands its velocity decreases and the larger particles drop out. As the velocity continues to decrease, increasingly smaller particles drop out. As this deposition continues, earlier sediments are
buried deeper and deeper beneath successive layers of sediment. The increasing weight of the new material squeezes water out of the lower layers, compacting the sediments until individual grains come into contact. This is referred to as compaction. Since these layers are deposited at different times they do not necessarily have the same composition as surrounding layers and will likely vary in composition and structure. Other depositional events can also create differences between layers in the rock. Reversals of submergence and emergence from water can result in repeated sequences of layers within a formation. For example, offshore wells drilled off the U.S. Gulf coast typically have many sandstone reservoirs separated by shale zones formed by repeated emergent-submergent cycles [3]. Formations such as these may be classified as heterogeneous porous media.

Before any company invests money into a recovery project or decides whether to continue to extract oil from developed reservoirs, they would like to know if a sufficient volume of oil can be extracted from the wells to make the investment pay off. Computer simulations of the physical processes including pressure/head, fluid velocities, etc., can be used as a means of predicting reservoir performance. These simulations attempt to emulate the flow of oil through the reservoir. By examining the results of simulations, an attempt can be made to determine if a sufficient volume of oil can be extracted from the reservoir to make the investment profitable. Modelling fluid flow in reservoirs is difficult because the flow through the region is dependent upon properties such as porosity and permeability. For a rock formation to function as reservoir rock it must have open pore space within its structure to make room for the oil. Porosity is the fraction of void or empty space in the reservoir over the total volume of the reservoir. The type and amount of void space in formations varies spatially within a formation. For the porosity to be “effective” it must be interconnected to allow oil and gas to flow. Permeability is a measure of this connectedness and represents the ability of fluid to flow through a rock formation [3], [4]. Optimally, computer simulations should resolve the heterogeneities just described.

Technological advances in recent years have increased available computing power for performing reservoir simulations. With such increases in computing power it would seem
that simulating reservoir flow using available data might be possible. However, as computational capacity has increased so has the complexity of reservoir simulators. When attempting to capture the main features of fluid flow one of the most important factors is reservoir heterogeneity. These heterogeneities can be small in scale, e.g. porosity and permeability, to large scale geological features, e.g. fractures and cross bedding, that have a significant impact on fluid flow [5]. Capturing some of these characteristics in a particular region of a reservoir using a regularly spaced grid is difficult. For this reason simulators have been modified to attempt to model physical behavior more accurately than has been done in the past. For example, some simulators have been enhanced to generate flexible grids that can better represent geologic features of the reservoir (e.g. fractures, bedding, etc.) [6].

Another aspect of the computational problem is the consideration that fluid flow occurs in an interconnected network of pores. These spaces or passages that oil moves through vary in size within the rock formation. A representative example of pore size is $3 - 5\mu m$ [5]. In considering flow within a typical reservoir, the size of which is on the order of kilometers, it becomes clear that because flow occurs in extremely small spaces it is a local phenomenon. Partitioning a reservoir to capture this behavior requires a very fine computational grid (or mesh) requiring a tremendous amount of computational resources. For example, consider placing a grid on a vertical slice through a reservoir with length 5 km, width 5 km and depth 1 km where a grid block represents a $5\mu m$ by $5\mu m$ portion of the slice. The grid necessary to represent such a slice through a reservoir would have on the order of $10^{18}$ grid blocks. Rather than represent a slice, if we attempt to represent the entire reservoir in three dimensions where a grid block represents a $5\mu m$ by $5\mu m$ by $5\mu m$ block of the reservoir, the grid would need to have on the order of $10^{26}$ grid blocks. To model flow accurately we would need to resolve flow at the pore level scale. Since this scale is smaller than is computationally practical, flow simulations of many realizations at this resolution is impossible.

To make the problem of numerical simulation of flow in heterogeneous petroleum reservoirs more manageable we can average or up-scale physical parameters such as porosity
and permeability. In this work, the focus will be on averaging the permeability parameter. In two and three dimensions permeability at a particular location is characterized by a tensor, the components of which reflect the permeability or ability of fluid to flow in the coordinate directions along with anisotropy due to the case when the coordinate directions used in computer simulations are not aligned with the principal flow directions. This upscaling or averaging is used to generate an “equivalent” or “effective” permeability tensor on a coarser scale. Since this effective tensor is defined on a coarser scale, the number of grid blocks necessary for simulation is smaller making flow simulation feasible. In exchanging a fine grid for a coarse grid, the method of determining “good” effective properties of a heterogeneous region is very important in accurately simulating flow. The results of the flow simulation depend on the method of determining the effective permeability tensor, thereby affecting the accuracy of the simulation. The problem of characterizing the properties of a porous medium to obtain an accurate representation of the region has been approached in many ways. The method of averaging used should preserve as much of the fine scale character of the region as possible. Since our goal is to model flow through a region, we want to incorporate fluid flow into the averaging process. There are a number of simple methods of averaging that could be used to obtain the effective values for use in the coarse grid, e.g. the arithmetic average, the geometric average, and the harmonic average. These methods do not incorporate fluid flow in the averaging process. One method that incorporates information about flow in the averaging process, through the use of a perturbation analysis, is referred to as homogenization.

In this work, determination of the homogenized permeability tensor is dependent on finding an exact solution, or an accurate numerical approximation of this solution, of a system of differential equations referred to as the “local problem.” As the name indicates, this problem is defined and solved locally or at a fine scale. In general, the “local problem” must be solved on each heterogeneous block in the reservoir. If an exact solution is difficult to obtain, the solution can be approximated numerically by a number of methods (e.g. finite elements, finite differences, etc.). The accuracy of any such approximation is dependent
on the computational grid. The coarsest grid that still captures all of the changes in the permeability field at the microscopic scale is chosen as the original heterogeneous grid. To obtain accurate numerical approximations, however, the original grid may need to be refined, i.e., decrease the size of the grid blocks or using a finer grid. This has the effect of increasing the number of grid blocks used to represent the field. By increasing the number of grid blocks more computational effort, rather than less, is necessary to obtain accurate approximations [6].

Wavelets have been widely used to characterize data and signals in many areas of science and engineering. The main motivation for using wavelets is the search for fast algorithms for analyzing data. Additionally, wavelets have the attractive quality of being able to resolve scale information in a signal or data set. If the data being analyzed are not purely random and have some coherence or smoothness, wavelets can be useful tools [7], [8]. For example, we know sedimentary layers are formed by a river depositing material in the riverbed. If this deposition is done in a somewhat uniform manner, it seems likely that a permeability value in one location will be correlated to permeability values nearby. For this reason, and the fact that permeability values within a rock formation vary on the small scale, the use of wavelets as a tool for determining a homogenized tensor seems reasonable.

The goal of this dissertation is to present the use of wavelets as a means of characterizing homogenization in reservoir simulations. We do this in search of fast algorithms for computing equivalent tensors. In developing this characterization we make use of homogenization theory and ensure that the wavelet-based representation is theoretically sound. In addition to the analogy developed between homogenization and wavelets, the analogous proofs of convergence results from homogenization theory within the wavelet characterization will be considered.

Chapter 2 gives a brief introduction to the method of homogenization and provides a short primer of specific examples of periodic and aperiodic homogenization along with applications of the methods to assist in the development of a wavelet characterization. In Chapter 3 we briefly describe concepts related to wavelet analysis and introduce a computationally efficient method of performing wavelet analysis known as the lifting scheme.
Chapter 4 explores developing characterizations of homogenization using wavelets. Chapter 5 will identify areas of future research.
CHAPTER 2
HOMOGENIZATION

Homogenization is a mathematical method that provides a way to “upscale” differential equations. Not only does this method provide formulas for upscaling, but is also based on mathematically rigorous arguments that guarantee convergence of solutions of our model. For our purposes we consider using this method to assist in characterizing flow through a heterogeneous petroleum reservoir at a macroscopic scale. Our reservoir is considered to be a large porous region where the permeability of the rock formation may change rapidly or may even be discontinuous at the small or microscopic scale. Since the scale at which permeability changes is very small relative to the size of the reservoir, we want to derive averaged or “homogenized” models to describe the asymptotic behavior of the flow at the macroscopic scale. That is, describe the flow through an “equivalent” homogeneous medium where permeability no longer depends on the spatial variable at the microscopic scale.

To set up the problem of modelling reservoir flow, we consider partitioning a reservoir $\Omega$ of size $L$, where $L$ is measured in meters, in two spatial scales as shown in Figure 2.1. Let the variable $x$, e.g. $x = (x_1, x_2)$ as in Figure 2.1, represent length or distance on the macroscopic scale and $y$, e.g. $y = (y_1, y_2)$, represent length on the microscopic scale. In the figure, the reservoir $\Omega$ is illustrated with a grid at the macroscopic or coarse scale. Block $Y$ represents a small section of the reservoir containing the microscopic scale information.

A simple model that may be used to model flow in a porous medium, such as the petroleum reservoir, is the basic pressure/head equation

$$\begin{cases}
\nabla \cdot K \nabla h &= f \quad x \in \Omega \\
h &= 0 \quad x \in \partial\Omega.
\end{cases}$$

(2.1)
spatially throughout the reservoir, but it will be assumed to be piecewise constant over each microscopic scale block. We wish to determine $K^\#$, an equivalent coarse scale permeability tensor that is constant over each coarse scale cell.

Homogenization research considers two cases: periodic and aperiodic problems. In this work periodic problems refer to porous media when the permeability tensor is spatially periodic with some given period. Aperiodic problems refer to all other problems including permeability tensors defined via some random distribution (e.g., log-normally correlated permeabilities).

2.1 Homogenization for Periodic Permeability Tensors

A simplification of the problem of modelling flow is to assume that the reservoir has periodic structure at the small scale. This small scale structure is repeated (see Figure 2.1) throughout the reservoir. Since the repeated structure occurs at the fine scale, the size of the period is very small relative to the size of the reservoir $\Omega$. To be precise, we assume that $\Omega$ has periodic structure with period length $\epsilon$ (see Figure 2.1). $Y$ will
denote a microscopic cell, that is, periodically repeated throughout $\Omega$. Cell $Y$ is scaled to unit length, i.e. $Y = (0, 1)^d$ where $d = 1, 2, 3$. In one dimension this measure of period size is given by $\epsilon = \frac{\xi}{\nu}$. The microscopic variable $y = \frac{\xi}{\nu}$ is the fast periodic variable and macroscopic variable $x \in \Omega$ is the slow variable [9]. A superscript $\epsilon$ will be used to indicate dependence on the microscopic scale as in [10, 11] (e.g. $h^\epsilon$ will denote the solution of Equation (2.1)).

Note that Equation (2.1) is a second order elliptic partial differential equation (PDE). An alternative representation of this problem is found by transforming the second order PDE to a system of first order differential equations. Using the Darcy velocity, $v^\epsilon = -K^\epsilon \nabla h^\epsilon$, the pressure/head equation can be written as the first order system

$$\begin{cases} -\nabla \cdot v^\epsilon = f \\ v^\epsilon = -K^\epsilon \nabla h^\epsilon. \end{cases}$$

subject to zero boundary conditions. If we consider the cell size $\epsilon$ to be a sequence approaching zero, (2.2) can be related to a sequence of boundary value problems. We are interested in the behavior of the solutions of those problems; i.e., $h^\epsilon$. In homogenization we consider the case as $\epsilon$ approaches zero.

We assume that $K^\epsilon$, $h^\epsilon$ and $v^\epsilon$ are functions on two scales; microscopic and macroscopic. That is, $K^\epsilon = K^\epsilon(x, y)$, $v^\epsilon(x, y)$ and $h^\epsilon = h^\epsilon(x, y)$. A perturbation analysis is applied to the system of equations to determine a two-scale asymptotic expansion of the solutions $v^\epsilon$ and $h^\epsilon$. Due to our change of variables the spatial differentiation operator is

$$\nabla = \nabla_x + \frac{1}{\epsilon} \nabla_y.$$

Also, we assume that $h^\epsilon$ and $v^\epsilon$ can be written as expansions in $\epsilon$; that is,

$$h^\epsilon = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \ldots$$

$$v^\epsilon = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \ldots$$

These expansions are inserted into Equation (2.2). Equating like powers of $\epsilon$ we obtain first order systems of differential equations to analyze. The equations for the powers of $\epsilon^n$
for \( n = -1, 0 \) are:

**\( \epsilon^{-1} \) equations:**

\[
-\nabla_y \cdot v_0 = 0 \\
K^\epsilon \nabla_y h_0 = 0
\]

(2.3)  
(2.4)

**\( \epsilon^0 \) equations:**

\[
-(\nabla_x \cdot v_0 + \nabla_y \cdot v_1) = f \\
K^\epsilon (\nabla_x h_0 + \nabla_y h_1) = -v_0.
\]

(2.5)  
(2.6)

We neglect all terms that are multiplied by powers of \( \epsilon \) greater than zero. These terms may be neglected provided the functions \( h_i \) and \( v_i \) for \( i = 1, 2, \ldots \) remain bounded and \( \epsilon \) is chosen small enough [11].

Now to interpret these equations. Equation (2.4) implies that, provided the permeability tensor \( K^\epsilon \) is positive definite, the macroscopic pressure variable \( h_0 \) must be constant with respect to the microscopic variable \( y \), i.e., \( h_0 = h_0(x) \). Now take the gradient of (2.6) with respect to the microscopic variable to derive the equation

\[
\nabla_y \cdot K^\epsilon \nabla_x h_0 + \nabla_y \cdot K^\epsilon \nabla_y h_1 = -\nabla_y v_0.
\]

(2.7)

Using (2.3) in (2.7) we obtain the differential equation

\[
\nabla_y \cdot K^\epsilon \nabla_x h_0 = -\nabla_y \cdot K^\epsilon \nabla_y h_1
\]

(2.8)

where \( h_0 \) is the pressure at the macroscopic scale and \( h_1 \) is the pressure at the microscopic scale. At this point we express \( h_1(x, y) \) in terms of \( h_0(x) \). Using the ansatz \( \nabla_x h_1(x) = \sum_{j=1}^{d} e_j \frac{dh_0(x)}{dx_j} \), where \( e_j \) is the \( j \)th unit vector and \( d \) is the spatial dimension, permits (2.8) to be rewritten as

\[
\nabla_y \cdot K^\epsilon \nabla_y h_1 = -\sum_{j=1}^{d} \partial y_j K^\epsilon \frac{dh_0(x)}{dx_j}.
\]
Integrating this equation in $y$ leads to $h_1$ written as a scaled combination of the derivatives of $h_0$. That is,

$$h_1(x, y) = \sum_{k=1}^{d} w_k(y) \frac{dh_0}{dx_k}, \quad (2.9)$$

where $w_k(y)$ are multipliers used in computing the combination [11,12]. Substituting this representation of $h_1$ into (2.8) allows us to balance terms on each side of the differential equation and derive a partial differential equation to determine the $w_k(y)$,

$$\left\{ \begin{array}{l}
\nabla_y \cdot K^\prime \nabla_y w_k(y) = -\nabla_y \cdot K^\prime e_k \quad \text{in } Y \\
w_k(y) = 0 \quad \text{on } \partial Y 
\end{array} \right. \quad (2.10)$$

on the microscopic scale [9,11,12]. Since this equation is solved locally it is referred to as the "local problem." Solving a local problem amounts to solving a system of uncoupled elliptic PDE's. Unfortunately, an exact solution may not be possible in general. One approach is to obtain a numerical approximation of the solution using a standard finite difference or finite element method. In the periodic case, it is necessary to solve only one local problem, since the structure is assumed to be periodically repeated throughout the reservoir.

Now consider averaging (2.5) and (2.6) over the microscopic cell. This creates the system

$$\left\{ \begin{array}{l}
-<\nabla_x v_0 + \nabla_y v_1 >_Y = <f>_Y \\

K(\nabla_x h_0 + \nabla_y h_1 ) >_Y = -<v_0>_Y 
\end{array} \right. \quad (2.11)$$

where $<f>_Y = \frac{1}{Vol(Y)} \int_Y f(y) dy$ and $Vol(Y)$ is the volume of the microscopic cell. The periodicity assumption together with the fact that the order of differentiation and integration can be interchanged [11] allows the simplification of the first equation of (2.11) to

$$-\nabla_x <v_0>_Y = <f>_Y. \quad (2.12)$$

Letting $v^# = <v_0>_Y$ and $f^# = <f>_Y$, this equation is rewritten as the averaged equation

$$-\nabla_x v^# = f^#. \quad (2.13)$$
Substituting the microscopic pressure $h_1$ into the second equation of (2.11) we get

$$-v^\# = <K(\nabla_x h_0 + \nabla_y \sum_{k=1}^d w_k(y) \frac{\partial h_0}{\partial x_k})>$$

$$= <K(I + J^T)_Y \nabla_x h_0>,$$

where $J^T$ is the transpose of the Jacobian matrix of the functions $w_k(y)$. Letting

$$K^\# = <K(I + J^T)_Y>_Y,$$

the system of equations in (2.11) are rewritten as

$$
\begin{cases}
-\nabla_x \cdot v^\# &= f^\# \\
v^\# &= -K^\# \nabla_x h_0,
\end{cases}
$$

(2.16)

an "averaged" or macroscopic system of equations to model flow. Note that these equations are of the same form as the original equations and govern the asymptotic behavior of flow in an "equivalent" homogeneous medium. Note that this type of perturbation analysis does not always result in macroscopic system of the same type (see, e.g., [10]).

The homogenization process amounts to (a) solve a local problem, (b) compute the transpose of the Jacobian matrix, $J^T$, and (c) compute the homogenized permeability tensor $K^\#$ using (2.15). A useful observation relative to characterizing the homogenized tensor using wavelets is that the effective tensor $K^\#$ may be expressed as

$$K^\# = <K>_Y + <KJ^T>_Y .$$

(2.17)

This may be interpreted as the arithmetic average of $K$ on $Y$ plus some perturbation or detail determined from the solution to the local problem.

### 2.1.1 The Two-Cell Problem in One Dimension

To assist in the development of a wavelet-based characterization of the homogenization described in the previous section we consider a simple example of homogenization on a two-cell problem in one dimension. The domain is scaled to the unit interval, $Y = (0, 1)$, and divided into two intervals of equal size. The permeability coefficient function, which is constant on each subinterval, is defined as

$$k(y) = \begin{cases} 
  k_0, & 0 \leq y < \frac{1}{2} \\
  k_1, & \frac{1}{2} \leq y \leq 1 \\
  0, & \text{otherwise.}
\end{cases}$$

(2.18)
The first step is to find the solution to the local problem. In one dimension, the equation governing the local problem is given by the ordinary differential equation
\[ \frac{d}{dy} k(y) \frac{d}{dy} w(y) = -\frac{d}{dy} k(y) \]
with periodicity constraint \( w(0) = w(1) = 0 \). The problem appears easy to solve until we consider \( k(y) \) is a piecewise constant function. A variational method can be used to produce the weak solution
\[ w(y) = k_1 - \frac{k_0}{k_0 + k_1} \begin{cases} y, & 0 \leq y \leq \frac{1}{2} \\ 1 - y, & \frac{1}{2} \leq y \leq 1 \\ 0, & \text{otherwise}, \end{cases} \]
(see [11]). The piecewise or weak derivative of this function is
\[ \frac{dw}{dy}(y) = \frac{k_1 - k_0}{k_0 + k_1} \begin{cases} 1, & 0 \leq y < \frac{1}{2} \\ -1, & \frac{1}{2} < y \leq 1 \\ 0, & \text{otherwise}. \end{cases} \]
We use the one dimensional version of Equation (2.15) to find
\[
\begin{align*}
\text{k}^\# &= \int_0^1 k(y) \left(1 + \frac{dw}{dy}\right) dy = \int_0^1 k(y) dy + \int_0^1 k(y) \frac{dw}{dy} dy \\
&= \int_0^{1/2} k_0 \left(1 + \frac{k_1 - k_0}{k_1 + k_0}\right) dy + \int_{1/2}^1 k_1 \left(1 - \frac{k_1 - k_0}{k_1 + k_0}\right) dy \\
&= \frac{k_0}{2} + \frac{k_0(k_1 - k_0)}{2(k_1 + k_0)} + \frac{k_1}{2} - \frac{k_1(k_1 - k_0)}{2(k_1 + k_0)} \\
&= \frac{k_0 k_1}{k_1 + k_0} + \frac{k_0 k_1}{k_1 + k_0} \\
&= \frac{2k_0 k_1}{k_0 + k_1}.
\end{align*}
\]
From this calculation we see that the coefficient produced by the homogenization operator is the harmonic average of the two cells. This result is "physically" correct in the sense that in one dimension the appropriate physical average is the harmonic average [10, 11].

2.1.2 The Two-Cell Problem in Two Dimensions

Now we consider a more difficult problem; extending the two-cell problem to two dimensions and finding the weak solution so that a homogenized tensor can be determined. The motivation for doing this is to use the weak solution of the local problem to assist
Fig. 2.2: Example of a permeability distribution in two dimensions. In (a) we see that \( Y = [0, 1] \times [0, 1] \) can be divided into four sub-regions or quadrants-\( I, II, III, IV \). In each sub-region permeability is constant, represented by \( k_I, k_{II}, k_{III}, k_{IV} \). This structure is then repeated periodically throughout the entire region as represented in (b).

In developing a wavelet characterization of the homogenized tensor in two dimensions. In two dimensions the periodically repeated cell is defined on \( Y = [0, 1] \times [0, 1] \). \( Y \) is divided into four sub-regions or quadrants of equal size—defined as Region \( I, II, III, \) and \( IV \). The permeability tensor is assumed to be constant in each quadrant (see Figure 2.2(a)) and the structure is periodically repeated throughout the reservoir in the periodic case, (see Figure 2.2 (b)). Using \( y_1 \) and \( y_2 \) as the microscopic variables the permeability tensor in \( Y \) given by

\[
k(y) = \begin{cases} 
  k_I, & 0 \leq y_1 \leq \frac{1}{2}, \quad 0 \leq y_2 \leq \frac{1}{2}, \\
  k_{II}, & \frac{1}{2} \leq y_1 \leq 1, \quad 0 \leq y_2 \leq \frac{1}{2}, \\
  k_{III}, & 0 \leq y_1 \leq \frac{1}{2}, \quad \frac{1}{2} \leq y_2 \leq 1, \\
  k_{IV}, & \frac{1}{2} \leq y_1 \leq 1, \quad \frac{1}{2} \leq y_2 \leq 1.
\end{cases} \quad (2.20)
\]
In each quadrant of our domain the permeability is represented by the tensor

$$k_i = \begin{pmatrix} k_{xx}^i & k_{xy}^i \\ k_{xy}^i & k_{yy}^i \end{pmatrix}$$

where $i = I, II, III, \text{ and } IV$. Each component of this tensor represents the ability of fluid to flow relative to the coordinate directions, $y_1$ represented by $x$ and $y_2$ represented by $y$. Usually the tensor is symmetric; i.e. $k_{xy}^i = k_{yx}^i$.

**2.1.2.1 Homogenization of the 2 x 2 Cell in Two Dimensions**

As in the one dimensional case we would like to determine the analytic solution of the local problem. In two dimensions determining this solution is more complicated. It can, however, be done. Since the heterogeneous tensor $K(x)$ is assumed to be piecewise constant in the local cell, we use an alternate way to solve the problem. A possible solution form can be justified by a degree of freedom argument. Our goal is to compute a single effective permeability tensor from four tensors at a finer scale. Assuming piecewise constant permeabilities in each of the cells our goal is to compute a single, constant tensor to use on the entire cell. Assuming symmetry of the permeability tensors at the microscopic scale, the true effective tensor has three fixed values, $k_{xx}, k_{yy}$, and $k_{xy} = k_{yx}$. The four tensors used as input in any homogenization process have a total of $4 \times 3 = 12$ fixed values. The difference in the number of values between the input tensors and the effective tensor is $12 - 3 = 9$ giving 9 degrees of freedom to play with in determining an appropriate algorithm for computing an effective tensor. Note this degree of freedom argument assumes piecewise constants and symmetry; more complicated configurations would require more analysis to determine an appropriate approach to the problem.

To develop our method we start with the periodic local problem

$$\nabla \cdot K(x) \nabla w_i = 0,$$

for $i = 1, 2$. The periodicity constraint dictates the value of the solution $w(y_1, y_2)$ at nodes $(0, 0), (0, 1), (1, 0), (1, 1)$, i.e.,

$$w(0, 0) = w(0, 1) = w(1, 0) = w(1, 1).$$
The lack of uniqueness allows us to choose $w(0, 0) = 0$. To construct a weak solution to the local problem we define one-dimensional piecewise linear functions

$$
\phi(y) = \begin{cases} 
4y, & 0 \leq y \leq \frac{1}{2}, \\
4 - 4y, & \frac{1}{2} \leq y \leq 1, 
\end{cases} \tag{2.21}
$$

$$
\psi(y) = \begin{cases} 
4, & 0 \leq y \leq \frac{1}{2}, \\
-4, & \frac{1}{2} \leq y \leq 1, 
\end{cases} \tag{2.22}
$$

$$
\chi(y) = \begin{cases} 
1, & 0 \leq y \leq 1 \\
0, & \text{otherwise}
\end{cases} \tag{2.23}
$$

which can be used to construct basis functions

$$
\eta_1(y_1, y_2) = \phi(y_1)\chi(y_2)
$$

$$
\eta_2(y_1, y_2) = \phi(y_1)\psi(y_2)
$$

$$
\eta_3(y_1, y_2) = \psi(y_1)\phi(y_2)
$$

$$
\eta_4(y_1, y_2) = \chi(y_1)\phi(y_2).
$$

Via the concept of tensor products, the form of the weak solution is then assumed to be

$$
w = \begin{pmatrix} w_1 \\
w_2 \end{pmatrix}
$$

where

$$
w_1(y_1, y_2) = \alpha_{1,1} \eta_1(y_1, y_2) + \alpha_{1,2} \eta_2(y_1, y_2) + \alpha_{1,3} \eta_3(y_1, y_2) + \alpha_{1,4} \eta_4(y_1, y_2). \tag{2.24}
$$

The $\alpha_{i,j}$'s in (2.24) are unknown constants that need to be determined.

We are looking for a solution in the function space $V = \text{span} \{\eta_1, \eta_2, \eta_3, \eta_4\}$. To determine the weak solution utilizing these basis functions the derivatives

$$
\frac{\partial \eta_1}{\partial y_1} = \psi(y_1)\chi(y_2) \quad \frac{\partial \eta_1}{\partial y_2} = 0
$$

$$
\frac{\partial \eta_2}{\partial y_1} = \psi(y_1)\psi(y_2) \quad \frac{\partial \eta_2}{\partial y_2} = 0
$$

$$
\frac{\partial \eta_3}{\partial y_1} = 0 \quad \frac{\partial \eta_3}{\partial y_2} = \psi(y_1)\psi(y_2)
$$

$$
\frac{\partial \eta_4}{\partial y_1} = 0 \quad \frac{\partial \eta_4}{\partial y_2} = \chi(y_1)\psi(y_2). \tag{2.25}
$$
are needed. Note the integral of each of these derivatives is zero on the local problem domain. A cyclic relationship exists between the derivatives of the basis functions:

\[(\psi(y_1)\chi(y_2))^2 = (\psi(y_1)\psi(y_2))^2 = (\chi(y_1)\psi(y_2))^2 = \chi(y_1)\chi(y_2)\]  \hspace{1cm} (2.26)

\[(\psi(y_1)\chi(y_2))(\psi(y_1)\psi(y_2)) = (\chi(y_1)\psi(y_2))\]  \hspace{1cm} (2.27)

\[(\psi(y_1)\chi(y_2))(\chi(y_1)\psi(y_2)) = (\chi(y_1)\psi(y_2))\]  \hspace{1cm} (2.28)

\[(\psi(y_1)\psi(y_2))(\chi(y_1)\psi(y_2)) = (\chi(y_1)\psi(y_2))\]  \hspace{1cm} (2.29)

The integral of the last three products is zero. Therefore our three unique derivatives of the basis functions are orthogonal on the domain of the local problem. This reduces the amount of work necessary to compute the equations related to the variational problem.

Having the solution form (2.24) we determine the weak solution to Equation (2.10) by solving the systems of equations defined by

\[\langle K\nabla w_i, \eta_j \rangle = -\langle Ke_i, \nabla \eta_j \rangle\]  \hspace{1cm} (2.30)

where \(i = 1, 2\) and \(j = 1, 2, 3, 4\). Note there are two systems of equations to solve; e.g. \(i = 1, 2\). The integral for an arbitrary \(j\) is written as

\[\langle K\nabla w_i, \nabla \eta_j \rangle = \int_Y \left\{ \left( k_{xx} \frac{\partial w_i}{\partial y_1} + k_{xy} \frac{\partial w_i}{\partial y_2} \right) \frac{\partial \eta_j}{\partial y_1} \right\} dy_1 dy_2\]

\[+ \left( k_{yx} \frac{\partial w_i}{\partial y_1} + k_{yy} \frac{\partial w_i}{\partial y_2} \right) \frac{\partial \eta_j}{\partial y_2} \right\} dy_1 dy_2\]

\[= \int_Y k_{xx} \left( \alpha_{i,1} \frac{\partial \eta_1}{\partial y_1} + \alpha_{i,2} \frac{\partial \eta_2}{\partial y_1} \right) \frac{\partial \eta_j}{\partial y_1} dy_1 dy_2\]

\[+ \int_Y k_{xy} \left( \alpha_{i,3} \frac{\partial \eta_1}{\partial y_2} + \alpha_{i,4} \frac{\partial \eta_2}{\partial y_2} \right) \frac{\partial \eta_j}{\partial y_2} dy_1 dy_2\]

\[+ \int_Y k_{yx} \left( \alpha_{i,1} \frac{\partial \eta_1}{\partial y_1} + \alpha_{i,2} \frac{\partial \eta_2}{\partial y_1} \right) \frac{\partial \eta_j}{\partial y_1} dy_1 dy_2\]

\[+ \int_Y k_{yy} \left( \alpha_{i,3} \frac{\partial \eta_1}{\partial y_2} + \alpha_{i,4} \frac{\partial \eta_2}{\partial y_2} \right) \frac{\partial \eta_j}{\partial y_2} dy_1 dy_2\]
This is computed for each \( j \) and each \( i \). Using the partial derivatives from (2.25) this form can be expressed

\[
\langle K \nabla w_i, \nabla \eta_j \rangle = \int_Y k_{xx} (\alpha_{i,1} \psi(y_1) \chi(y_2) + \alpha_{i,2} \psi(y_1) \psi(y_2)) \frac{\partial \eta_j}{\partial y_1} dy_1 dy_2 \\
+ \int_Y k_{xy} (\alpha_{i,3} \psi(y_1) \psi(y_2) + \alpha_{i,4} \chi(y_1) \psi(y_2)) \frac{\partial \eta_j}{\partial y_1} dy_1 dy_2 \\
+ \int_Y k_{yx} (\alpha_{i,1} \psi(y_1) \chi(y_2) + \alpha_{i,2} \psi(y_1) \psi(y_2)) \frac{\partial \eta_j}{\partial y_2} dy_1 dy_2 \\
+ \int_Y k_{yy} (\alpha_{i,3} \psi(y_1) \psi(y_2) + \alpha_{i,4} \chi(y_1) \psi(y_2)) \frac{\partial \eta_j}{\partial y_2} dy_1 dy_2.
\]

Upon integrating this expression it is clear the coefficients in the system of equations are quite messy. To allow concise representation we define the following expressions which are composed of tensor components from the four quadrants:

\[\bar{k}_- = k^{I} + k^{II} + k^{III} + k^{IV}\] (2.31)
\[\bar{k} = k^{I} - k^{II} + k^{III} - k^{IV}\] (2.32)
\[\tilde{k} = k^{I} + k^{II} - k^{III} - k^{IV}\] (2.33)
\[\tilde{k} = k^{I} - k^{II} - k^{III} + k^{IV}\] (2.34)

where \( \cdot = x \) or \( y \), for the two coordinate directions. Note \( x \) and \( y_1, y_2 \) are used interchangeably to make notation less clumsy. Expressions (2.31) – (2.32) can be applied to the full tensors \( K_I, ..., K_{IV} \), as well. An important observation here is the above representation is equivalent to multiplying

\[
H_4 \begin{pmatrix} K_I \\ K_{II} \\ K_{III} \\ K_{IV} \end{pmatrix} = \begin{pmatrix} \bar{K} \\ \bar{K} \\ \bar{K} \end{pmatrix}
\]

(2.35)

where \( H_4 \) is the Hadamard operator of order 4

\[
H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 \end{pmatrix}.
\] (2.36)
This structure is created by the choice of basis functions. To see this, consider any non-zero partial derivative of a basis function, say

\[ \frac{\partial \eta_1}{\partial y_1} = \psi(y_1) \chi(y_2), \]

defined on the unit square. The definition, by quadrant, of this function would be

\[
\begin{align*}
\frac{\partial \eta_1}{\partial y_1} &= \begin{cases}
1 & 0 \leq y_1 \leq \frac{1}{2}, \ 0 \leq y_2 \leq \frac{1}{2} \\
-1 & \frac{1}{2} < y_1 \leq 1, \ 0 \leq y_2 \leq \frac{1}{2} \\
1 & 0 \leq y_1 \leq \frac{1}{2}, \ \frac{1}{2} < y_2 \leq 1 \\
-1 & \frac{1}{2} < y_1 \leq 1, \ \frac{1}{2} < y_2 \leq 1
\end{cases}
\]

An alternate representation of this function would be via a row vector in which the value in the i-th position provides the function definition in the i-th quadrant; i.e.

\[ \frac{\partial \eta_1}{\partial y_1} = \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}. \]

Note using this representation \( \frac{\partial \eta_1}{\partial y_1} \) corresponds to the second row of \( H_4 \). All of the non-zero partial derivatives of our basis, in vector form, correspond to one of rows two through four of \( H_4 \). This relationship assists in our development of a transform method.

Using our concise notation we obtain a system of four equations in four unknowns defined by

\[
A \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \\ \alpha_{i,3} \\ \alpha_{i,4} \end{pmatrix} = B
\]

where the entries of \( A \) are

\[
\begin{align*}
A(1, 1) &= A(2, 2) = k_{xx} \\
A(1, 2) &= A(2, 1) = k_{xx} \\
A(1, 3) &= A(3, 1) = k_{xy} \\
A(1, 4) &= A(4, 1) = k_{xy} \\
A(2, 3) &= A(3, 2) = k_{xy} \\
A(2, 4) &= A(4, 2) = k_{xy} \\
A(3, 3) &= A(4, 4) = k_{yy} \\
A(3, 4) &= A(4, 3) = k_{yy}
\end{align*}
\]
for $i = 1, 2$. The coefficient matrix $A$ is the same for all $i$. The coefficients of the weak solution, $\alpha_{i,j}$ are dependent on the right-hand side $B$. For $i = 1$

$$B = \begin{pmatrix} k^0_{xx} \\ k^0_{xx} \\ k^0_{yx} \\ k^0_{yx} \end{pmatrix}.$$  

For $i = 2$

$$B = \begin{pmatrix} k^0_{xy} \\ k^0_{xy} \\ k^0_{yy} \\ k^0_{yy} \end{pmatrix}.$$  

Once the solutions to the systems of equations are determined the weak solution $w$ is known. We can now determine the Jacobian transpose needed to compute the homogenized coefficient via the formula

$$K^\# = \int_Y K(I + J^T)$$  

$$= \int_Y K + \int_Y KJ^T$$  

where

$$J = \begin{pmatrix} \nabla w_1^T \\ \nabla w_2^T \end{pmatrix}.$$  

This provides the ingredients for the analytic formula for $K^\#$.

Previously no connection was made between the first row of $H_4$ and the computation of $K^\#$. We noted the partial derivatives of the basis functions correspond to rows two through four of the operator. These play a role in the computation of $\int_Y KJ^T$. The importance of the first row may be seen in the computation of $\int_Y K$:

$$\int_Y K dy = \int_I K dy + \int_{II} K dy + \int_{III} K dy + \int_{IV} K dy.$$  

Since $K$ is piecewise constant on our dyadic grid, the tensor $K$ can be considered to be multiplied by a function which is one on the unit square. In terms of (2.21) -(2.23) and a row vector, respectively, this function can be expressed

$$\chi(y_1)\chi(y_2) = (1 \quad 1 \quad 1 \quad 1),$$
or row one of $H_4$. Thus,

$$\int_Y K \chi(y_1) \chi(y_2) dy = \int_I K \chi(y_1) \chi(y_2) dy + \int_{II} K \chi(y_1) \chi(y_2) dy + \int_{III} K \chi(y_1) \chi(y_2) dy + \int_{IV} K \chi(y_1) \chi(y_2) dy$$

$$= \int_I K_1 dy + \int_{II} K_{II} dy + \int_{III} K_{III} dy + \int_{IV} K_{IV} dy$$

$$= \frac{1}{4} \bar{K}.$$

### 2.1.2.2 A Simple Example: Stratified Coefficient Case

To illustrate the method we consider a special case in which the the homogenized tensor is known. In the stratified coefficient case $k_{xy} = k_{yx} = 0$. Additionally, we assume

$$k_{xx}^I = k_{yy}^I = a = k_{yy}^I = k_{yy}^{III}$$  \hspace{1cm} (2.41)

$$k_{xx}^{II} = k_{xx}^{IV} = b = k_{yy}^{II} = k_{yy}^{IV}.$$  \hspace{1cm} (2.42)

This tensor represents stratification in the $x$ direction. In this case the parameters are

$$k_{xx}^- = k_{yy}^- = 2(a + b)$$  \hspace{1cm} (2.43)

$$k_{xx}^o = k_{yy}^o = 2(a - b)$$  \hspace{1cm} (2.44)

and all others are zero. Using these values to produce the weak solution generates the nonzero coefficient

$$\alpha_{1,1} = \frac{b - a}{a + b}.$$  \hspace{1cm} (2.45)

All other $\alpha_{i,j}$'s are zero. The weak solution is

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{b-a}{a+b} \psi(y_1) \chi(y_2) \\ 0 \end{pmatrix}$$

and

$$J^T = \begin{pmatrix} \frac{b-a}{a+b} \psi(y_1) \chi(y_2) & 0 \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.46)
Recall the homogenized coefficient is computed via

\[
K^\# = \int_Y K (I + J^T) dy \\
= \int_Y K dy + \int_Y K J^T dy.
\]

The first integral in \(K^\#\) is the sum of the integral in each quadrant; i.e.

\[
\int_Y K dy = \int_I K^I dy + \int_{I_1} K^{I_1} dy + \int_{I_{III}} K^{I_{III}} dy + \int_{I_{IV}} K^{I_{IV}} dy.
\]

which produces the tensor

\[
\begin{pmatrix}
\frac{a+b}{2} & 0 \\
0 & \frac{a+b}{2}
\end{pmatrix}.
\]

(2.47)

The second integral produces

\[
\int_Y K J^T dy = \int_I K^{I^T} dy + \int_{I_{I_1}} K^{I_{I_1}} J^T dy + \int_{I_{III}} K^{I_{III}} J^T dy + \int_{I_{IV}} K^{I_{IV}} J^T dy
\]

giving the tensor

\[
\begin{pmatrix}
\frac{(b-a)^2}{2(a+b)} & 0 \\
0 & 0
\end{pmatrix}.
\]

(2.48)

Now we sum the two tensors to construct

\[
K^\# = \begin{pmatrix}
\frac{2ab}{b+a} & 0 \\
0 & \frac{a+b}{2}
\end{pmatrix}.
\]

(2.49)

This generates a tensor in which the homogenized value in the \(x\) direction, \(k_{xx}^\#\), is the harmonic average and \(k_{yy}^\#\), the homogenized value in the \(y\) direction, is the arithmetic average. These are the physically appropriate values for this case.

### 2.2 Aperiodic Homogenization

In reality, periodicity is usually not a valid assumption, especially if we try to upscale permeability on coarse grid blocks each of which may have different fine scale structure. As noted in the periodic case, the assumption of periodicity allows computation of the solution of the local problem once, for all grid blocks. In the aperiodic case, the local
solution must be computed on each block in the domain. Also, convergence of the solution is a more complicated issue.

In this case, we consider reservoir \( \Omega \subset \mathbb{R}^d, \quad d = 1, 2, 3, \) to be a bounded domain containing \( N \) different heterogeneous regions. The reservoir may be partitioned using these regions denoting the \( m \)-th region as \( \Omega_m, m = 1, \ldots, N \). As before we assume the tensor \( K \) contains permeabilities that are piecewise constant within each region of the reservoir. Recall that to find the homogenized tensor in the periodic case we had to solve the local problem on the period cell \( Y \). That solution was necessary for computing the homogenized tensor. The partitions \( \Omega_m \) can be considered analogous to the periodic cell \( Y \). In the aperiodic case, the local problem for \( \Omega_m \) is written in the form

\[
-\nabla \cdot [K(x)\nabla w_i] = 0 \quad \text{in} \quad \Omega_m
\]

\[
w_i = x_i \quad \text{on} \quad \Omega_m
\]

for \( i = 1, 2, 3 \) and where \( x_i \) is the \( i \)-th coordinate [6]. Since the structure of \( \Omega \) was not periodic, the solution to the local problem must be computed for \( m = 1, 2, 3, \ldots, N \). Once the solutions are determined the homogenized tensor is computed using, for example,

\[
(K_m)_{i,j}^\# = \frac{1}{|\Omega_m|} \int_{\Omega_m} K(x) \nabla w_{m,i} \cdot \nabla w_{m,j} \, dx,
\]

for \( 1 \leq i, j \leq d \). This computes an equivalent tensor which is symmetric and positive definite, but may be anisotropic. In many cases the off-diagonal terms of this tensor are significant. Note that full permeability tensors make sense physically because the principal flow directions do not necessarily align with the axes of the grid system [6].

### 2.2.1 The Two-Cell Problem in One Dimension

As in the periodic case, we consider calculating a homogenized coefficient for a two-cell problem in one dimension. The boundary conditions for the local problem in this case are \( w(0) = 0 \) and \( w(1) = 1 \). Solving the local problem we find that in the one dimensional case the aperiodic problem has the same weak solution as the periodic problem, see (2.19).
Using the one dimensional version of (2.51) we find

\[
\begin{align*}
k^\# &= \int_0^1 k(y) \left( 1 + \frac{dw}{dy} \right) \left( 1 + \frac{dw}{dy} \right) dy \\
&= \int_0^1 k(y) \left( 1 + 2 \frac{dw}{dy} + \left( \frac{dw}{dy} \right)^2 \right) dy \\
&= \int_0^{1/2} k_0 \left( 1 + 2 \frac{k_1 - k_0}{k_0 + k_1} + \left( \frac{k_1 - k_0}{k_0 + k_1} \right)^2 \right) dy + \int_{1/2}^1 k_1 \left( 1 - 2 \frac{k_1 - k_0}{k_0 + k_1} + \left( \frac{k_1 - k_0}{k_0 + k_1} \right)^2 \right) dy \\
&= \frac{k_0}{2} + \frac{k_0(k_1 - k_0)}{k_0 + k_1} + \frac{k_0(k_1 - k_0)^2}{2(k_0 + k_1)^2} + \frac{k_1}{2} - \frac{k_1(k_1 - k_0)}{k_0 + k_1} + \frac{k_1(k_1 - k_0)^2}{2(k_0 + k_1)^2} \\
&= \frac{k_0 + k_1}{2} \left( \frac{k_1 - k_0}{k_0 + k_1} \right)^2 + \frac{(k_1 - k_0)^2}{2(k_0 + k_1)} \\
&= \frac{k_0 + k_1}{2} \left( \frac{k_1 - k_0}{k_0 + k_1} \right)^2 + \frac{(k_1 - k_0)^2}{2(k_0 + k_1)} \\
&= \frac{2 k_0 k_1}{k_0 + k_1}. \quad (2.53)
\end{align*}
\]

This establishes that in one dimension periodic and aperiodic homogenization of the two-cell problem produces the same coefficient—the harmonic average.

### 2.2.2 The Two-Cell Problem in Two Dimensions

Extending this concept to two dimensions we seek to establish a weak solution to (2.50). We approach determining an approximate solution in a similar manner to that of periodic homogenization. The aperiodic problem in two dimensions is

\[
\begin{align*}
\nabla K \nabla w_i &= 0 \quad \text{on } Y \\
w_i &= y_i \quad \text{on } \partial Y
\end{align*}
\]

for \( i = 1, 2 \). Using the same basis functions as we did with periodic homogenization we search for a weak solution in \( V = \text{span}\{\eta_1, \ldots, \eta_4\} \). We assume a solution of the form

\[
w_i = \alpha_{i,1} \eta_1 + \alpha_{i,2} \eta_2 + \alpha_{i,3} \eta_3 + \alpha_{i,4} \eta_4
\]

for \( i = 1, 2 \).
To determine the weak solution we must solve the system of equations determined by

\[
\langle K \nabla w_i, \eta_j \rangle = -\langle Ke_i, \nabla \eta_j \rangle
\]  

(2.54)

where \( i = 1, 2 \) and \( j = 1, 2, 3, 4 \). Note (2.54) can be rewritten

\[
\int_Y (\nabla K \nabla w_i) \eta_j \, dy = -\int_Y (\nabla Ke_i) \eta_j \, dy.
\]  

(2.55)

Integrating the equation we find

\[
\int_{\partial Y} (K \nabla w_i) \eta_j \, dy - \int_Y (K \nabla w_i) \nabla \eta_j \, dy = -\int_{\partial Y} (Ke_i) \eta_j \, dy + \int_Y Ke_i \nabla \eta_j \, dy.
\]  

(2.56)

In the periodic case the integrals on the boundary were zero. The boundary conditions of the aperiodic case produce non-zero contribution from both of the integrals.

To compute the integrals along the boundary of \( Y \) we must parameterize each side of the unit square. Let \( \partial Y = C_1 + C_2 + C_3 + C_4 \), where \( C_i = (x(t), y(t)) \). The parameterization used is

\[
\begin{align*}
C_1 &= (t, 0) \quad 0 \leq t \leq 1 \\
C_2 &= (1, t-1) \quad 1 < t \leq 2 \\
C_3 &= (3-t, 1) \quad 2 < t \leq 3 \\
C_4 &= (0, 4-t) \quad 3 < t \leq 4.
\end{align*}
\]  

(2.57)

For example the first integral in (2.56) can be expressed

\[
\int_{\partial Y} (K \nabla w_i) \eta_j \, dy = \int_{C_1} (K \nabla w_i) \eta_j \, dy + \int_{C_2} (K \nabla w_i) \eta_j \, dy + \int_{C_3} (K \nabla w_i) \eta_j \, dy + \int_{C_4} (K \nabla w_i) \eta_j \, dy.
\]

We compute these integrals along each piece of the boundary via

\[
\int_{C_i} (K \nabla w_i) \eta_j \, dy = \int_{t_0}^{t_1} \left( (K \nabla w_i) \eta_j \right)(t) \cdot \left( x'(t), y'(t) \right) \, dt
\]

Similar operations are performed for \( \int_{\partial Y} (Ke_i) \eta_j \, dy \).
Computing the value of these integrals for values of \( j = 1, 2, 3, 4 \) leads to the system of equations

\[
A \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \\ \alpha_{i,3} \\ \alpha_{i,4} \end{pmatrix} = B
\]  \hspace{1cm} (2.58)

where

\[
A = \begin{pmatrix} k_{xx}^- & k_{xx}^- & k_{xy}^- & k_{xy}^- \\ k_{xx}^- & k_{xx}^- & k_{xy}^- & k_{xy}^- \\ k_{xy}^- & k_{xy}^- & k_{yy}^- & k_{yy}^- \\ k_{yx}^- & k_{yx}^- & k_{yy}^- & k_{yy}^- \end{pmatrix}
\]

for \( i = 1, 2 \). The coefficients of the weak solution, \( \alpha_{i,k} \), are dependent on the right-hand side \( B \). For \( i = 1 \)

\[
B = \begin{pmatrix} -k_{xx}^- + k_{xx}^- \\ -k_{xx}^- + 4k_{xx}^- \\ -k_{yx}^- - 4k_{xy}^- \\ -k_{yx}^- - k_{xy}^- \end{pmatrix}
\]

For \( i = 2 \)

\[
B = \begin{pmatrix} -k_{xy}^- + k_{xy}^- \\ -k_{xy}^- + 4k_{xy}^- \\ -k_{yy}^- - 4k_{yy}^- \\ -k_{yy}^- - k_{yy}^- \end{pmatrix}
\]

Once the \( \alpha_{i,k} \)'s are known the weak solution \( w \) and \( J^T \) are computed. The homogenized coefficient

\[
K^\# = \int_Y K(I + J^T)
\]  \hspace{1cm} (2.59)

can then be computed.

2.3 Extension of the Two-Cell Problem to Three Dimensions

Now we outline how to extend our method for the two-cell problem to three dimensions. These ideas can readily be extended to higher dimensions. To motivate the idea consider the two-cell problem in three dimensions. The tensor \( K \) would be defined on the unit cube
and contain eight samples. To standardize notation for the discussion the unit cube is divided into octants on a dyadic grid and a permeability tensor is defined for each octant. For example, the definition for $K$ is

$$
\begin{align*}
K_I & \quad 0 \leq y_1 \leq \frac{1}{2}, \quad 0 \leq y_2 \leq \frac{1}{2}, \quad 0 \leq y_3 \leq \frac{1}{2} \\
K_{II} & \quad \frac{1}{2} < y_1 \leq 1, \quad 0 \leq y_2 \leq \frac{1}{2}, \quad 0 \leq y_3 \leq \frac{1}{2} \\
K_{III} & \quad 0 \leq y_1 \leq \frac{1}{2}, \quad \frac{1}{2} < y_2 \leq 1, \quad 0 \leq y_3 \leq \frac{1}{2} \\
K_{IV} & \quad \frac{1}{2} < y_1 \leq 1, \quad \frac{1}{2} < y_2 \leq 1, \quad 0 \leq y_3 \leq \frac{1}{2} \\
K_V & \quad 0 \leq y_1 \leq \frac{1}{2}, \quad 0 \leq y_2 \leq \frac{1}{2}, \quad \frac{1}{2} < y_3 \leq 1 \\
K_{VI} & \quad \frac{1}{2} < y_1 \leq 1, \quad 0 \leq y_2 \leq \frac{1}{2}, \quad \frac{1}{2} < y_3 \leq 1 \\
K_{VII} & \quad 0 \leq y_1 \leq \frac{1}{2}, \quad \frac{1}{2} < y_2 \leq 1, \quad \frac{1}{2} < y_3 \leq 1 \\
K_{VIII} & \quad \frac{1}{2} < y_1 \leq 1, \quad \frac{1}{2} < y_2 \leq 1, \quad \frac{1}{2} < y_3 \leq 1.
\end{align*}
$$

The goal in this case is to compute an averaged tensor from the eight tensors at the finer scale. This averaged tensor is to be stored in place of one of the original tensors leaving seven tensors available for storage of details removed in the averaging process.

Recall in the computation of $K^#$ there were two integrals. The first $\int_Y K \, dy$ representing the average of $K$ over the domain. The second was $\int_Y K J^T \, dy$, where $J^T$ was the Jacobian transpose of the weak solution. In previous examples we have seen that information about the weak solution is contained in the second integral. To develop the form for our weak solution we define the Hadamard operator of order 8:

$$
H_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1
\end{pmatrix}.
$$

Each row of this operator is understood to be function in three dimensions defined on the unit cube. Now we consider the result of applying this operator to a vector representing
the eight samples \( K \); i.e.

\[
H_8 \begin{pmatrix}
K_1 \\
K_{I1} \\
K_{III} \\
K_{IV} \\
K_V \\
K_{VI} \\
K_{VII} \\
K_{VIII}
\end{pmatrix} = K_*.
\] (2.62)

The vector \( K_* \) contains expressions similar to those in (2.31)-(2.34). The purpose of these expressions is to simplify the representation of the weak solution. For example, in two dimensions \( \tilde{K} \) was useful in determining the average of \( K \) on the domain and \( K_*(1) \) will serve a similar purpose. The additional entries of \( K_* \) are used in computing \( \int_{\Omega} K J^T dy \). Looking toward a wavelet-based transform these entries are useful in representing the detail.

Crucial to determining a weak solution is the choice of the basis. This is where the consideration of \( H_8 \) becomes important. Consider each row of \( H_8 \) as being a function defined on the unit cube. Thus, for \( H_8(i,j) \) the \( j \) indicates the octant on which that component is defined defined. The first row of \( H_8 \) is composed of ones implying a constant value of one in all octants of the unit cube. In terms of functions (2.21)-(2.23), \( H_8(1,1..8) = \chi(y_1)\chi(y_2)\chi(y_3) \). In row two the entries change from 1 to -1 in the \( y_1 \) direction, but remain constant in the \( y_2 \) and \( y_3 \) directions. Thus, \( H_8(2.1..8) = \psi(y_1)\chi(y_2)\chi(y_3) \). Continue performing similar analysis on the remaining rows of the operator.

Recall the rows of the Hadamard operator were useful in representing the piecewise partial derivatives of the \( \eta_i \) and, therefore, \( \nabla w_i \) and \( J \). From the rows of \( H_8 \) we can determine a form for the weak solution. This is accomplished by choosing basis functions whose partial derivatives are represented by rows two through seven of \( H_8 \). For example, since \( H_8(2,1..8) = \psi(y_1)\chi(y_2)\chi(y_3) \) we choose a function with this partial derivative; i.e.

\[
\eta_1 = \phi(y_1)\chi(y_2)\chi(y_3).
\]
For \( H_8(3,1..8) = \psi(y_1)\psi(y_2)\chi(y_3) \) there are two functions we can construct with this partial derivative

\[
\begin{align*}
\eta_2 &= \phi(y_1)\psi(y_2)\chi(y_3) \\
\eta_3 &= \psi(y_1)\phi(y_2)\chi(y_3).
\end{align*}
\]

In three dimensions there are twelve basis functions computed this way. The remaining basis functions are

\[
\begin{align*}
\eta_4 &= \chi(y_1)\phi(y_2)\chi(y_3) \\
\eta_5 &= \chi(y_1)\chi(y_2)\phi(y_3) \\
\eta_6 &= \phi(y_1)\chi(y_2)\psi(y_3) \\
\eta_7 &= \psi(y_1)\chi(y_2)\phi(y_3) \\
\eta_8 &= \phi(y_1)\psi(y_2)\psi(y_3) \\
\eta_9 &= \psi(y_1)\phi(y_2)\psi(y_3) \\
\eta_{10} &= \psi(y_1)\psi(y_2)\phi(y_3) \\
\eta_{11} &= \chi(y_1)\phi(y_2)\psi(y_3) \\
\eta_{12} &= \chi(y_1)\psi(y_2)\phi(y_3).
\end{align*}
\]

Using this basis, we assume weak solutions of the form

\[
w_i = \sum_{j=1}^{12} \alpha_{i,j} \eta_j
\]

for \( i = 1, 2 \) and 3. Now that the basis is established, the weak solution is found via a system of equations produced by (2.30) using appropriate boundary conditions. The solution to this system of equations is not included in this work due to the difficulty of developing a concise representation of the quantities contained in the coefficient matrix \( A \) and the vector \( B \).
2.4 Theoretical Results

Now the method of homogenization has been reviewed, we outline some of the theory that supports the method. We begin by establishing definitions and notation, as well as stating useful lemmas and theorems. This will be useful later in proving analogous theorems within the wavelet analogy.

2.4.1 Definitions and Convergence Results

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a bounded open domain. Let $\alpha, \beta$ be two positive constants such that $0 < \alpha \leq \beta$. $M_S(\alpha, \beta, \Omega)$ is the set of all possible symmetric permeability tensors defined on $\Omega$ with uniform coercivity constant $\alpha$ and $L^\infty(\Omega)$-bound $\beta$. Denoting the set of $N \times N$ symmetric matrices by $M_S^{N \times N}$, we define

$$
M_S(\alpha, \beta, \Omega) = \{ K(x) \in \mathcal{L}^\infty(\Omega, M_S^{N \times N}) \mid \alpha|\xi|^2 \leq K(x)\xi \cdot \xi \leq \beta|\xi|^2 \}
$$

for any $\xi \in \mathbb{R}^d$.

An integrable function is said to be $L_1$ approximable (can be approximated) on a domain $\Omega$ by functions of some class $F$ if, for any given $\epsilon > 0$, there is a function $\phi \in F$ such that $\int |f - \phi| < \epsilon$, [13, p. 65]. $L^2(\Omega)^N$ is the Lebesgue space of square integrable functions defined as

$$
L^2(\Omega)^N = \{ u \mid u : \Omega \to \mathbb{R}^N, \int_{\Omega} |u(x)|^2 \, dx < \infty \}
$$

with norm

$$
\| u \|_{L^2(\Omega)^N} = \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{1/2}
$$

where

$$
|u| = |< u_1, ..., u_N > | = \left( \sum_{i=1}^{N} |u_i|^2 \right)^{1/2}.
$$

A useful property of integrable functions is stated in the following theorem.

Theorem 2.4.1 [13, p. 66] Any integrable function is approximable (can be approximated by) by a step function.
Theorem 2.4.2 \[14, \text{p. 156}\] For \(1 \leq p \leq \infty\) the space \(L_p(S, Y)\) is a normed vector space. If \(S\) is a finite measure space, then \(L_p(S, Y)\) is embedded in \(L_q(S, Y)\) for \(1 \leq q \leq p \leq \infty\), and the embedding mapping is continuous.

Consider a fixed Banach space \(E\) equipped with a norm \(\| \cdot \|\). Define \(D = D(\mathbb{R}, \mathbb{C})\) the linear space of all infinitely many times differentiable functions \(\psi : \mathbb{R} \to \mathbb{C}\) with compact support. Let \(f : D \to E\) be a linear vector-valued functional. Any continuous linear functional \(f : D \to E\) is called a distribution on \(\mathbb{R}\). Denotes \(D' = D'(\mathbb{R}, E)\) the space of all distributions \(f\). We say that a sequence \(f_k \in D'\) converges to \(f \in D'\) if \(\langle \psi, f_k \rangle \to \langle \psi, f \rangle\) for all \(\psi \in D\).

\(H^1(\Omega)\) is the Hilbert space defined as

\[
H^1(\Omega) = \{ u | u \in L^2(\Omega), \nabla u \in (L^2(\Omega))^N \}
\]

where

\[
\| u \|^2_{H^1(\Omega)^N} = \| u \|^2_1 + \| u \|^2_0.
\]

In this definition

\[
\| u \|^2_0 = \| u \|^2_{L^2(\Omega)} \quad \text{and} \quad \| u \|^2_1 = \| \nabla u \|^2_{(L^2(\Omega))^N}.
\]

\(H^1_0(\Omega)\) is a subspace of \(H^1(\Omega)\) defined as

\[
H^1_0(\Omega) = \{ u | u \in H^1(\Omega) \text{ and } u \text{ vanishes on } \partial \Omega \}.
\]

Friedrichs inequality implies that \(\| u \|^2_0\) and \(\| u \|^2_1\) are equivalent norms. This allows \(\| u \|^2_1\) to be taken as an equivalent norm on \(H^1_0(\Omega)\) and will be used as the norm of choice in this space. The dual space of \(H^1_0(\Omega)\) is denoted \(H^{-1}(\Omega)\) and consists of all continuous linear functionals on \(H^1_0(\Omega)\). Define the inner product notation \((f, g) = \int_\Omega f g \, dx\).

Convergence with respect to norm in Hilbert spaces, i.e., strong convergence, will be denoted by \(\to\). Weak convergence is denoted by \(\rightharpoonup\). For example, if

\[
\lim_{\epsilon \to 0} (f, u^\epsilon) = (f, u^0), \quad \forall f \in H^{-1}(\Omega),
\]
we write

\[ u^\epsilon \to u^0 \text{ in } H^1_0(\Omega) \]

as \( \epsilon \to 0 \). Similarly, if for any \( \phi \in L^2(\Omega) \) we have

\[ \lim_{\epsilon \to 0} (w^\epsilon, \phi) = (w^0, \phi) \]

then \( w^\epsilon \to w^0 \) in \( L^2(\Omega) \).

**Theorem 2.4.3** [{15, p. 631}] A bounded sequence \( \{f_n\} \) in a Hilbert space \( H \) has a weakly convergent subsequence.

A useful property of weakly convergent sequences is that they are necessarily bounded in norm. A more general form of weak convergence, weak-* convergence, is also needed. For \( w^\epsilon, w^0 \in L^1(\Omega) \), if the sequence \( w^\epsilon \) is bounded in \( L^1(\Omega) \) and

\[ \lim_{\epsilon \to 0} \int \omega^\epsilon \phi \, dx = \int \omega^0 \phi \, dx \]

holds for any \( \phi \in C^\infty_0 \), then we write \( w^\epsilon \rightharpoonup w^0 \). It is important to note that a weak-* limit is unique.

Often we need to find the limit of the scalar product \( p^\epsilon \cdot v^\epsilon \) where \( p^\epsilon, v^\epsilon \to 0 \) in \( L^2(\Omega) \). Weak convergence does not allow us to easily pass to the limit of the scalar product. To do this additional properties are needed which “compensate” for the lack of strong convergence. Passing to the limit is possible provided:

**Lemma 2.4.4** [{16, p. 4}] Let \( p^\epsilon, v^\epsilon \) be vector fields in \( L^2(\Omega) \) such that

\[ p^\epsilon \to p^0, \quad v^\epsilon \to v^0 \quad \text{in} \quad L^2(\Omega). \]

Additionally, if \( \text{div } p^\epsilon \to f^0 \) in \( H^{-1}(\Omega) \) and \( \text{curl } v^\epsilon = 0 \) then

\[ p^\epsilon \cdot v^\epsilon \rightharpoonup^* p^0 \cdot v^0 \]
When using a variational method to determine a weak solution to our problem, the \Lax-Milgram lemma guarantees the existence and uniqueness of the solution of the problem. This is done by establishing the solvability of an equation of the form

\[ Lu = f. \]

The weak form of this problem is: \textit{Find} \( u \in V \), \textit{where} \( V \) \textit{is a real Hilbert space},

\[ L(u, v) = (f, v) \quad (2.63) \]

for all \( v \in V \). Recall we want to solve the Dirichlet problem

\[ \nabla(K \nabla h) = f \]

for \( u \in H^1_0([0, 1]) \). The weak formulation of our problem is:

\textit{For any given} \( f \in H^{-1}(\Omega) \) \textit{find an element} \( u \in H^1_0(\Omega) \) \textit{such that}

\[ (K \nabla u, \nabla v) = (f, v) \quad (2.64) \]

for all \( v \in H^1_0(\Omega) \).

\textbf{Lemma 2.4.5} \textbf{Lax-Milgram} \cite[p. 8]{16} \textit{The weak form of the problem (2.63) has a solution} \( u \), \textit{which is unique and satisfies the estimate}

\[ \| u \|_1 \leq \nu_1^{-1} \| f \|_{H^{-1}(\Omega)}. \]

\textit{In other words, the bounded coercive operator} \( L \) \textit{is an isomorphism between the spaces} \( H^1_0(\Omega) \) \textit{and} \( H^{-1}(\Omega) \), \textit{the norm of the inverse operator being not larger than} \( \nu_1^{-1} \).

A Sobolev space is defined as

\[ W^{m,p}(\Omega) := \{ f \in L^p(\Omega) \| \partial^\alpha f \in L^p \text{ for all multi-index } \alpha, |\alpha| \leq m \} \]

for \( m \) \textit{a nonegative integer and} \( p \geq 1 \).

The Sobolev and Rellich-Kondrachov embedding theorems and the Lebesgue dominated convergence theorem are useful in proving \( G \)-convergence.
Theorem 2.4.6 Sobolev and Rellich-Kondrachov [17, p. 41] Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary.

i) If $1 \leq p < n$, then

$$W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for every } 1 \leq q \leq np/(n-p)$$

and the imbedding is compact for every $1 \leq q < np/(n-p)$.

ii) If $p = n$, then

$$W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for every } 1 \leq q \leq \infty$$

and the imbedding is compact.

iii) If $p > n$, then

$$W^{1,p}(\Omega) \subset C(\bar{\Omega})$$

and the imbedding is compact.

Some important properties of periodic functions follow. Let $W^{1,p}_{loc}(Y)$ denote the subset of $W^{1,p}(Y)$ of all the functions with mean value zero which have the same trace on opposite faces of $Y$.

Denote $f^\# = \int_Y f(y) dy$ for $Y$ the unit interval.

Lemma 2.4.7 [17] Let $f \in W^{1,p}_{\#}(Y)$. Then $f$ can be extended by periodicity to an element of $W^{1,p}_{loc}(\mathbb{R}^n)$

Lemma 2.4.8 [17] Let $g \in L^q(Y; \mathbb{R}^n)$ such that $\int_Y g \ dv = 0$ for every $v \in W^{1,p}_{\#}(Y)$. Then $g$ can be extended by periodicity to an element of $L^q_{loc}(\mathbb{R}^n; \mathbb{R}^n)$, still denoted by $g$ such that $-\text{div} \ g = 0$ in $D'(\mathbb{R}^n)$.
2.4.2 $G$-Convergence

A notion of convergence associated with sequences of symmetric, second-order, elliptic operators is referred to as $G$-convergence. The $G$ means Green since this type of convergence roughly corresponds to the convergence of the associated Green functions [9, 12]. In homogenization theory the main result of $G$-convergence is a compactness theorem which states that, for any bounded and uniformly coercive sequence of coefficients of a symmetric, second-order, elliptic equation there exists a subsequence and a $G$-limit (i.e., homogenized coefficient) such that for any source term in $L^2(\Omega)$, the corresponding subsequence of solutions converges to the solution of the homogenized equation. This means the physical properties of the heterogeneous reservoir can be approximated well by the properties of a homogeneous or homogenized reservoir if the size of the heterogeneities is small compared to the overall size of the region [12].

For a given source term $f(x) \in L^2(\Omega)$ the Lax-Milgram lemma implies the equation

\[
\left\{ \begin{array}{l}
-\nabla \cdot K^\varepsilon \nabla u^\varepsilon = f \\
\quad u^\varepsilon = 0 \quad x \in \partial \Omega,
\end{array} \right. \tag{2.68}
\]

admits a unique solution $u^\varepsilon$ in the space $H^1_0(\Omega)$. $G$-convergence of operators associated with the sequence $K^\varepsilon$ is defined as the convergence of corresponding solutions $u^\varepsilon$.

**Definition 2.4.1** [12, p. 230] The sequence of tensors $K^\varepsilon(x)$ is said to $G$-converge to a limit $K^\#$, as $\varepsilon \to 0$, if for any source term $f \in L^2(\Omega)$ in (2.68) the sequence of solutions $u^\varepsilon$ converges weakly in $H^1_0(\Omega)$ to a limit $u$ which is the unique solution of the homogenized equation associated with $K^\#$

\[
\left\{ \begin{array}{l}
-\nabla \cdot K^\# \nabla u = f \\
\quad u = 0 \quad x \in \partial \Omega,
\end{array} \right. \tag{2.69}
\]

Using this notion of convergence we now state the crucial theorem previously mentioned.

**Theorem 2.4.9** [12, p. 230] For any sequence $K^\varepsilon$ in $M_S(\alpha, \beta, \Omega)$, there exists a subsequence and a homogenized limit $K^\#$, belonging to $M_S(\alpha, \beta, \Omega)$, such that $K^\varepsilon$ $G$-converges to $K^\#$. 

The following theorem pertains to a property which appears to be a natural requirement on the homogenized matrix [16, p. 14]. This theorem and its proof, are included for completeness, as well as to motivate the idea of writing a wavelet based proof. It is important to recall the previous assumption that $K \in M_S(\alpha, \beta, \Omega)$. That is, $K^\epsilon$ is symmetric, an element of $L^\infty(\Omega)$ and satisfies a coercivity condition. Additionally, $K$ is assumed to be $Y$ periodic, i.e. $K \in M_S(\alpha, \beta, Y)$, and note $L^2([0, 1])$ is a dense subset of $H^{-1}([0, 1])$.

**Theorem 2.4.10** [17, p. 12] Let $K^\epsilon \in M_S(\alpha, \beta, Y)$ and $\epsilon$ be a sequence of positive numbers converging to 0. Assume $f^\epsilon \to f^\#$ in $H^{-1}([0, 1]^d)$, e.g. for $f^\epsilon, f^\# \in L^2([0, 1]^d)$, $f^\epsilon \to f^\#$ in $L^2([0, 1]^d)$ [16, p. 14] Let $u^\epsilon$ and $u^\#$ be solutions to the boundary value problems:

$$u^\epsilon \in H^1_0([0, 1]^d) : \quad -\nabla (K^\epsilon \nabla u^\epsilon) = f^\epsilon \quad (2.70)$$

$$u^\# \in H^1_0([0, 1]^d) : \quad -\nabla (K^\# \nabla u^\#) = f^\#. \quad (2.71)$$

Then,

$$u^\epsilon \rightharpoonup u^\# \quad \text{weakly in } H^0_0([0, 1]^d) \quad (2.72)$$

$$K^\epsilon \nabla u^\epsilon \rightharpoonup K^\# \nabla u^\# \quad \text{weakly in } L^2([0, 1]^d). \quad (2.73)$$

**Proof**: Denote $[0, 1]^d = \Omega$. Recall $K^\epsilon = K(\frac{\cdot}{\epsilon})$ for all $x$ in $\mathbb{R}$. The weak formulation of the sequence of boundary value problems (2.70) becomes

$$\int_{\Omega} (K^\epsilon \nabla u^\epsilon) \nabla v \, dx = \int_{\Omega} f^\epsilon \, v \, dx \quad (2.74)$$

$$u^\epsilon \in H^1_0(\Omega), \quad (2.75)$$

for all $v \in H^1_0(\Omega)$. Taking $\phi = u^\epsilon$ in (2.74) we have

$$\int_{\Omega} K^\epsilon(x) \nabla u^\epsilon \cdot \nabla u^\epsilon \, dx \quad (2.76)$$

The coercivity constraint on $K^\epsilon(x)$ means we can write

$$\alpha \int_{\Omega} |\nabla u^\epsilon|^2 \, dx \leq \int_{\Omega} K^\epsilon(\nabla u^\epsilon \cdot \nabla u^\epsilon) \, dx. \quad (2.77)$$
Additionally, by (2.76) we can write

\[
\int_{\Omega} K^\varepsilon(x) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx = \int_{\Omega} f^\varepsilon \, u^\varepsilon \, dx 
\]

(2.78)

\[
\leq \| f^\varepsilon \|_{H^{-1}(\Omega)} \| u^\varepsilon \|_{H^1_0(\Omega)} 
\]

(2.79)

\[
\leq C_1 \| u^\varepsilon \|_{H^1_0(\Omega)} 
\]

(2.80)

where \( C_1 \) is independent of \( \varepsilon \). Combining (2.77) and (2.80) and noting

\[
\alpha \int_{\Omega} |\nabla u^\varepsilon|^2 \, dx = \alpha \| u^\varepsilon \|_1^2
\]

we find

\[
\| u^\varepsilon \|_1 \leq \frac{C_1}{\alpha},
\]

where \( C_1 \) is independent of \( \varepsilon \).

Now consider the vector \( q^\varepsilon(x) = K^\varepsilon \nabla u^\varepsilon \) on \( \Omega \). Note

\[
\| q^\varepsilon \|_0 = \| K^\varepsilon \nabla u^\varepsilon \|_0 
\]

\[
\leq \| K^\varepsilon \|_0 \| u^\varepsilon \|_1 
\]

\[
\leq C_2,
\]

where \( C_2 \) is independent of \( \varepsilon \). Since \( u^\varepsilon \) and \( q^\varepsilon \) are bounded sequences they possess convergent subsequences, called \( u^\varepsilon \) and \( q^\varepsilon \) respectively. Denote the weak limit points of these sequences as \( u^* \in H^1_0(\Omega) \) and \( q^* \in L^2(\Omega) \). That is,

\[
u^\varepsilon \rightharpoonup u^* \text{ in } H^1_0(\Omega) \tag{2.81}
\]

\[
K^\varepsilon \nabla u^\varepsilon \rightharpoonup q^* \text{ in } L^2(\Omega). \tag{2.82}
\]

Now take (2.74) and pass to the limit as \( \varepsilon \) approaches 0 for any fixed \( \phi \in H^1_0(\Omega) \) to get

\[
\int_{\Omega} (q^* \cdot \nabla \phi) \, dx = \int_{\Omega} f^\varepsilon \phi \, dx \tag{2.83}
\]
Fix \( v \in H_0^1(\Omega) \). Due to the convergence of \( f^\varepsilon \) to \( f^# \) we may pass to the limit and find that
\[
\int_\Omega q^* \nabla v \, dx = \int_\Omega f^# \, v \, dx \tag{2.84}
\]
for all \( v \in H_0^1(\Omega) \). Assume \( q^* = K^# \nabla u^* \) a.e. in \( \Omega \). This assumption will be proven in the following lemma. Equation (2.84) shows that \( u^* \) satisfies the weak form of the boundary value problem. By passing to the limit we see that \( K^# \nabla u^* \) satisfies the weak form of the homogenization boundary value problem. If, in addition, \( K^# \) satisfies the same ellipticity condition (see Lemma (2.4.12) the solution is unique; i.e. \( u^* = u^# \)).

The following lemma establishes our assumption.

**Lemma 2.4.11** ([17, p. 14]) \( q^* = K^# \nabla u^* \) a.e. in \( [0,1] \).

**Proof**: Consider the local problem
\[
K^\varepsilon \nabla w_i = -K^\varepsilon e_i \tag{2.85}
\]
\[
w_i \in H_\#(Y) \tag{2.86}
\]
where \( Y \) is the period cell in \( \Omega \) and \( H_\#(Y) \) are the \( Y \)-periodic members of \( H_0^1(\Omega) \). Denote the \( Y \)-periodic extension of \( w_i \) to \( \mathbb{R}^d \) by \( w_i \). By Lemma 2.4.7 \( w_i \in H_{\text{loc}}^1(\mathbb{R}) \). For each \( i \) define the sequence of functions
\[
w_i^\varepsilon = x_i + \varepsilon w_i \left( \frac{x}{\varepsilon} \right) \tag{2.87}
\]
\[
= (e_i, x) + \varepsilon w^i \left( \frac{x}{\varepsilon} \right) \tag{2.88}
\]
for a.e. \( x \in \mathbb{R}^d \). The periodicity property of \( w_i^\varepsilon \) implies
\[
w_i^\varepsilon \to x_i \text{ in } L^2(\Omega)
\]
\[
\nabla w_i^\varepsilon \to e_i \text{ in } L^2(\Omega; \mathbb{R}^d)
\]
as \( \varepsilon \to 0 \). By Lemma 2.4.8, with \( g(y) = K^\varepsilon(y) (e_i + \nabla w_i^k(y)) \), the functions \( w_i^\varepsilon \) satisfy the equations
\[
-\nabla (K^\varepsilon(x) \nabla w_i^\varepsilon) = 0 \text{ in } D' (\mathbb{R}^d). \tag{2.89}
\]
Multiplying this equation by any \( v \in H^1_0(\Omega) \) and integrating we have

\[
\int_\Omega K^\varepsilon \nabla w_i^\varepsilon \nabla v dx = 0. \tag{2.90}
\]

To avoid difficulties with boundary conditions, let \( \phi \in C^{\infty}_0(\omega) \) and let us rewrite (2.74) with \( v = \phi w_i^\varepsilon \in H^1_0(\Omega) \) as

\[
\int_\Omega K^\varepsilon \nabla u^\varepsilon (\nabla \phi) w_i^\varepsilon dx + \int_\Omega K^\varepsilon \nabla u^\varepsilon (\nabla w_i^\varepsilon) \phi dx = \int_\Omega f^\varepsilon (\phi w_i^\varepsilon) dx. \tag{2.91}
\]

Now rewrite (2.90) with \( v = \phi u^\varepsilon \in H^1_0(\Omega) \) to obtain

\[
\int_\Omega K^\varepsilon \nabla w_i^\varepsilon (\nabla \phi) u^\varepsilon dx + \int_\Omega K^\varepsilon \nabla w_i^\varepsilon (\nabla u^\varepsilon) \phi dx = 0. \tag{2.92}
\]

The symmetry of \( K^\varepsilon \) means

\[
\int_\Omega (K^\varepsilon \nabla w_i^\varepsilon)(\nabla u^\varepsilon) \phi dx = \int_\Omega (K^\varepsilon \nabla u^\varepsilon)(\nabla w_i^\varepsilon) \phi dx.
\]

Subtracting (2.92) from (2.91) we obtain

\[
\int_\Omega (K^\varepsilon \nabla u^\varepsilon \nabla \phi) w_i^\varepsilon dx - \int_\Omega (K^\varepsilon \nabla w_i^\varepsilon)(\nabla \phi) u^\varepsilon dx = \int_\Omega f^\varepsilon \phi w_i^\varepsilon dx
\]

for every \( \phi \in C^{\infty}_0(\Omega) \). Now we can pass to the limit as \( \varepsilon \to 0 \) since each term is a scalar product, in \( L^2(\Omega; \mathbb{R}^d) \), of an element converging weakly and another element which is converging strongly. That is,

\[K^\varepsilon \nabla u^\varepsilon \to K^* \nabla u^*
\]

weakly in \( L^2(\Omega) \) and

\[(\nabla \phi) w_i^\varepsilon \to (\nabla \phi) x_i
\]

strongly in \( L^2 \). (Note \( \nabla \phi \) is fixed.) Moreover,

\[
(K^\varepsilon \nabla w_i^\varepsilon)_j(x) = \sum_{k=1}^d K^\varepsilon_{jk} \frac{\partial w_i^\varepsilon}{\partial x_k}
\]

\[
= \sum_{k=1}^d K_{jk} (\cdot) \left( \delta_{k,i} + \frac{\partial w_i}{\partial y_k} \right) \left( \frac{x}{\varepsilon} \right).
\]
for $j = 1, \ldots, d$. Hence,

$$(K^\epsilon \nabla w^\epsilon)_j \rightarrow \int_Y \left( K_{j,i}(y) + \sum_{k=1}^{d} K_{j,k}(y) \frac{\partial w^i}{\partial y_k}(y) \right) dy$$

$$= K_{j,k}^\#$$

weakly in $L^2$. The fact that $u^\epsilon \rightharpoonup u^*$ in $H^1_0(\Omega)$ in conjunction with Rellich's Theorem implies

$$((\nabla \phi) u^\epsilon) \rightarrow (\nabla \phi) u^*$$

strongly in $L^2(\Omega)$. Since $K_{i,j}^\epsilon \in L^\infty(\mathbb{R}^d)$, the sequence $w_i^\epsilon \rightharpoonup x_i$ in $H^1_0(\Omega)$ and by assumption $f^\epsilon \rightharpoonup f$ in $H^{-1,2}([0,1])$, we can assert that

$$\int_\Omega \left( \sum_{k=1}^{d} q_k^\epsilon (\nabla_k \phi) x_i - \sum_{k=1}^{d} K_{k,i}^\# (\nabla_k \phi) u^* \right) dx = \langle f, \phi x_i \rangle.$$ (2.93)

By (2.84) taking $v = \phi x_i$, this equation becomes

$$\int_\Omega \sum_{k=1}^{d} (q_k^\epsilon x_i - K_{k,i}^\# u^*) (\nabla_k \phi) dx = \int_\Omega \sum_{k=1}^{d} q_k^\epsilon \nabla_k (\phi x_i) dx$$

for every $\phi \in C_0^\infty([0,1])$ and for $i = 1, \ldots, d$

$$\int_\Omega \left( q_i^\epsilon - \sum_{k=1}^{d} K_{k,i}^\# \nabla_k u^* \right) \phi dx = 0$$

for every $\phi \in C_0^\infty([0,1])$. This implies

$$q_i^* = \sum_{k=1}^{d} K_{k,i}^\# \nabla_k u^*$$

a.e. on $\Omega$. By symmetry of $K^\#$, see next lemma,

$$K^* \nabla u^* = K^\# \nabla u^*.$$

Recall the homogenization operator is uniquely defined and the uniqueness of the solution to (4.6) we can conclude the convergences

$$u^\epsilon \rightharpoonup u^\# \text{ in } H^1_0(\Omega)$$

$$K^\epsilon \nabla u^\epsilon \rightharpoonup K^\# \nabla u^\# \text{ in } L^2(\Omega)$$

hold for the sequences and not only for the extracted subsequence.
Lemma 2.4.12  [17, p. 18] Let $K^\varepsilon(x) : \mathbb{R}^d \to M_{d\times d}$ be a function in $M_S(\alpha, \beta, Y)$. Let $K^\#$ be the constant matrix defined by

$$K^\# = \int_Y K^\varepsilon(y) \left( I + J^T \right) dy.$$ 

Then, $K^\#$ is still symmetric and satisfies the same ellipticity condition as $K^\varepsilon(x)$; i.e.,

(i) \hspace{1cm} K^\#_{i,j} = K^\#_{j,i} \hspace{1cm} \forall i, j = 1, \ldots, d$

(ii) \hspace{1cm} (K^\# \xi, \xi) = \sum_{i,j=1}^d K^\#_{i,j} \xi_i \xi_j \geq \alpha |\xi|^2 \hspace{1cm} \forall \xi \in \mathbb{R}^d.$

Proof: We prove (i) first. Consider the local problem

$$\int_Y K^\varepsilon(y) (e_k + \nabla w_k(y)) \cdot \nabla v(y) dy = 0 \hspace{1cm} w_k \in H^1_\#(Y).$$

Fix $j$ and $s$ in \{1, \ldots, d\} and let $v = w_s$ to obtain

$$\int_Y K^\varepsilon(y) (e_k + \nabla w_k(y)) \cdot \nabla w_s(y) dy = 0.$$

Now add to both sides of this equation the quantity

$$\int_Y \left( K^\varepsilon(y) \left( e_k + \nabla w^k(y) \right), e_s \right) dy$$

to obtain

$$\int_Y K^\varepsilon(y) (e_k + \nabla w_k(y)) (e_s + \nabla w_s(y)) dy = \int_Y K^\varepsilon(y) \left( e_k + \nabla w^k(y) \right) e_s dy$$

which is equivalent to

$$= \int_Y \left( K^\varepsilon_{s,k}(y) + \sum_{j=1}^d K^\varepsilon_{s,j}(y) \frac{\partial w^k}{\partial y_j}(y) \right) dy$$

$$= K^\#_{s,k}.$$

Thus,

$$K^\#_{s,k} = \int_Y K^\varepsilon(y) \left( e_k + \nabla w^k(y) \right) (e_s + \nabla w^s(y)) dy.$$
Since $K^\xi(y)$ is symmetric it is clear $K^\xi_{s,k} = K^\xi_{k,s}$ and $K^\xi$ is symmetric.

Now to show (ii). Given $\xi \in \mathbb{R}^d$, define the sequence of functions

$$v^\xi(x) = \sum_{k=1}^d \xi_k w_k^\epsilon \left( \frac{x}{\epsilon} \right)$$

for a.e. in $\mathbb{R}^d$ where

$$w_k^\epsilon = x_k + \epsilon \, w^k \left( \frac{x}{\epsilon} \right)$$

for $k = 1, ..., d$. Recall

$$w_i^\epsilon \to x_i \quad L^2(\mathbb{R}^d)$$

$$\nabla w_i^\epsilon \to e_i \quad L^2(\Omega; \mathbb{R}^d)$$

$$(K^\xi \nabla w_i^\epsilon)_j \to K^\xi_{j,i} \quad L^2(\Omega)$$

implying

$$v^\epsilon \to \sum_{k=1}^n \xi_k x_k = (\xi, x) \text{ in } L^2(\Omega)$$

and

$$\nabla v^\epsilon = \sum_{k=1}^d \xi_k \nabla w_k^\epsilon(x)$$

$$\to \sum_{k=1}^d \xi_k e_k \text{ in } L^2(\Omega)$$

establishing

$$\nabla v^\epsilon \to \xi.$$
Additionally,

\[
(K^e \nabla v^e(x))_i = \left( K^e \nabla \sum_{k=1}^{d} \xi_k w_k^e(x) \right)_i
\]

\[
= \left( K^e \left( \sum_{k=1}^{d} \xi_k \nabla w_k^e(x) \right) \right)_i
\]

\[
= \left( \sum_{k=1}^{d} \xi_k (K^e \nabla w_k^e(x)) \right)_i
\]

\[
\rightarrow \sum_{k=1}^{d} \xi_k \left( K^e_{i,k} \right)
\]

\[
= \left( K^e \xi \right)_i. \tag{2.94}
\]

for all \( i = 1 \ldots n \). Moreover, for \( \phi \in C_0^\infty(\Omega) \) we have

\[-\nabla (K^e \nabla v^e) = 0 \quad \text{in } D'(\mathbb{R}^d).\]

Now consider

\[
\int_\Omega (K^e(x) \nabla v^e, \nabla v^e) \phi \, dx = -\int_\Omega (K^e \nabla v^e, \nabla \phi) \, v^e \, dx - \int_\Omega \nabla (K^e \nabla v^e) \phi \, v^e \, dx
\]

\[
= -\int_\Omega (K^e \nabla v^e, \nabla \phi) \, v^e \, dx.
\]

By (2.94) the last integral converges to

\[
-\int_\Omega \left( K^e \xi, \nabla \phi \right) v^e \, dx = -\int_\Omega \left( K^e \xi, \nabla \phi \right) (\xi, x) \, dx
\]

\[
= \int_\Omega \left( K^e \xi, \xi \right) \, \phi \, dx.
\]

Thus,

\[
\int_\Omega (K^e(x) \nabla v^e \nabla v^e) \phi \, dx \rightarrow \int_\Omega \left( K^e \xi \cdot \xi \right) \phi \, dx \tag{2.95}
\]

for all \( \phi \in C_0^\infty(\Omega) \). Note ellipticity of \( K^e(x) \) implies

\[
\int_\Omega (K^e(x) \nabla v^e, \nabla v^e) \phi \, dx \geq \int_\Omega \alpha |\nabla v^e|^2 \, dx
\]

for all \( \phi \in C_0^\infty(\Omega) \), \( \phi \geq 0 \). By passing to the limit as \( \epsilon \) approaches zero, (2.95) and the weak lower semicontinuity of the norm in \( L^2(\Omega) \) implies

\[
\int_\Omega \left( K^e(x) \xi \cdot \xi \right) \phi \, dx \geq \int_\Omega \alpha |\xi|^2 \phi \, dx,
\]
for all $\phi \in C_0^\infty(\Omega), \phi \geq 0$, implying

$$(K^\# \xi, \xi) \geq \alpha |\xi|^2.$$ 

\[ \]

2.4.3 $H$-Convergence

$H$-convergence is the term applied to the joint convergence of the solutions and the flows [12]. The $H$ in $H$-convergence stands for "homogenization" [9,12]. When speaking of $G$ or $H$-convergence it is important to clarify the definitions. French authors (e.g. F. Murat and L. Tartar) tend to refer to $G$-convergence as the convergence of solutions and $H$-convergence as convergence of solutions and flows. Russian authors (e.g. Jikov, Kozlov and Oleinik) refer to convergence of solutions and flows as strong $G$-convergence [16].

Definition 2.4.2 [12, p. 232] The sequence of tensors $K^\epsilon(x)$ is said to $H$-converge to a limit $K^\#$, as $\epsilon \to 0$, if for any source term $f \in L^2(\Omega)$ in (2.68), the sequence of solutions $u^\epsilon$ converges weakly in $H^1_0(\Omega)$ to a limit $u$, and the sequence of fluxes (or flows) $K^\epsilon \nabla u^\epsilon$ converges weakly in $(L^2(\Omega))^N$ to $K^\# \nabla u$ where $u$ is the unique solution of the homogenized equation (2.69) associated with $K^\#$.

As with $G$-convergence there is an important theorem establishing the existence of the $H$-limit.

Theorem 2.4.13 [12, p. 232] For any sequence $K^\epsilon$ in $M(\alpha, \beta, \Omega)$, there exists a subsequence and homogenized limit $K^\#$, belonging to $M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$ such that $K^\epsilon$ $H$-converges to $K^\#$.

Note for $H$-convergence the limit belongs to the set $M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$ rather than $M(\alpha, \beta, \Omega)$. This is due to the non-symmetry of sequence $K^\epsilon$ causing instability with respect to $H$-convergence. In physical terms, this means that macroscopic diffusive effects can be realized due to microscopic convective phenomena [12]. No proof of $H$-convergence is provided here because Theorem 2.4.10 included the notion of $H$-convergence. That is, the weak convergence of solutions and flows.
CHAPTER 3
WAVELET NOTATION AND PROPERTIES

Wavelets have been widely used to characterize signals in many areas of science and engineering because of the ability to identify scales within data. This makes the method an attractive tool for analyzing scale information within a signal or data. In this chapter, notation will be set, as well as a brief description of the concept of a multiresolution analysis (MRA), wavelets, and the lifting scheme, a computational method of performing wavelet transformations, will be given. For details, refer to [7], [8].

3.1 Scaling Functions and Multiresolution Analyses

A brief introduction to the concept of MultiResolution Analysis (MRA) is that MRA is a method for performing discrete wavelet analysis and synthesis on a signal. The method is recursive and ideal for computations. The idea is to start with a signal, \( k^N \), sampled at regular intervals. This signal is then split into a “blurred” or averaged version of the signal, \( k^{N-1} \), at a coarser scale and the corresponding “detail”, \( d^{N-1} \), removed at that scale. The process is repeated producing a sequence \( k^N, k^{N-1}, k^{N-2}, \ldots \) of more and more smeared versions of the original signal along with a sequence of details \( d^{N-1}, d^{N-2}, \ldots \) removed at every scale. The original signal, \( k^N \), can be recovered after \( N \) iterations by taking the signal at the \( N^{th} \) iteration or level and adding in the “details” that have been removed, i.e., \( k^N = k^0 + d^{N-1} + d^{N-2} + \ldots + d^1 \), [7].

Development of an MRA begins with scaling functions. A scaling function is a basic function, \( \phi(x) \), that serves as a “potential” for generating a function referred to as a mother wavelet, \( \psi(x) \). It can generally be thought of as a “bump” function that has some width \( W \) centered near \( x = 0 \). Translates and dyadic dilates of a scaling function are defined by

\[
\phi_{m,n}(x) = \phi(2^{-m}x - n),
\]  

(3.1)
where \( m \) represents scale or dilation and \( n \) represents the translation. The translated and dilated versions of the scaling function \( \phi(x) \) can be used to sample the signal at various positions and scales. The scaling function must satisfy some conditions. Among those conditions are orthonormality within the zero scale and an averaging property, \( \hat{\phi}(0) = 1 \), where \( \hat{\phi}(\xi) \) is the Fourier Transform of \( \phi \). Examples of scaling functions can be found in Figure 3.1.

Scaling functions are used to generate function spaces \( V_m \). For a fixed \( m \in \mathbb{Z} \), the space \( V_m \) is the closed subspace of \( L^2(\Omega) \) spanned by \( \{ \phi_{m,n} : n \in \mathbb{Z} \} \). That is,

\[
V_m = \left\{ f = \sum_n \phi_{m,n} u_n : \|f\|^2 = \sum_n |u_n|^2 < \infty \right\}.
\]

It should be clear from properties of scaling functions the set \( \{ \phi_{m,n} : n \in \mathbb{Z} \} \) forms an orthonormal basis for the space \( V_m \). The space \( V_m \) is used to specify a scale or resolution on \( L^2(\Omega) \). As \( m \to \infty \) the scale grows “finer” and as \( m \to -\infty \) the scale grows “coarser”. Consequently, a chain of subspaces is generated such that

\[
\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots
\]

When the domain \( \Omega \) is bounded there exists a coarsest space \( V_0 \) and the chain of subspaces can be expressed

\[
V_0 \subset V_1 \subset V_2 \cdots
\]
with appropriate definition. Functions in $L^2(\Omega)$ that are smooth and slowly varying may be represented on a coarser scale than a function which is highly oscillatory and has steep gradients. The decomposition of $L^2(\Omega)$ into a chain of subspaces, where

$$\bigcap_m V_m = \{0\} \quad \text{and} \quad \bigcup_m V_m = L^2(\Omega),$$

is referred to as a MRA.

### 3.2 Wavelet Spaces

Once a multiresolution analysis is established it is used to construct wavelets. We begin by considering the orthogonal complement of $V_m$ in $V_{m+1}$:

$$W_m = \{ f \in V_{m+1} : \langle f, g \rangle = 0 \quad \text{for all} \quad g \in V_m \}.$$

This is to say $V_{m+1} = W_m \oplus V_m$. Recall $W_m \subset V_{m+1}$ and $W_m$ is orthogonal to $V_m$; therefore, $W_m$ is orthogonal to $W_{m+1}$. This implies the spaces $W_m$ are mutually orthogonal, unlike the spaces $V_m$.

Now we define the wavelets associated with the MRA. The mother wavelet $\psi$ belongs to $W_0$, just as the scaling function $\phi$ belongs to $V_0$. The mother wavelet is defined via the two-scale relation

$$\psi_{0,0}(x) = \sum_n g_n \phi(2x - n).$$

The translates and dilates of the mother wavelet, known as wavelets, are given by

$$\psi_{m,n}(x) = \psi(2^m x - n). \quad (3.2)$$

and form an orthonormal basis for $L^2(\mathbb{R})$. The space $W_m$ is used to represent the "detail" component of the space $V_{m+1}$. Thus, wavelets of $W_m$ capture the "oscillatory" and quickly varying components of the functions in $V_{m+1}$.

Every signal $f^{m+1} \in V_{m+1}$ has a unique decomposition $f^{m+1} = f^m + d^m$ where $f^m \in V_m$ and $d^m \in W_m$. This means each signal can be decomposed into an "average" at
the next coarser level and the "detail" removed by moving to the coarser level. This is our first glimpse of the homogenization/wavelet connection. Compare this idea with (2.17).

The homogenized tensor can be interpreted as the sum of an "average" and a term that could be considered as "detail."

Denote the orthogonal projection operator onto $V_m$ as $P_m$ and $W_m$ as $Q_m$. Based on the relationship between the spaces we can define the relationship $Q_m = P_{m+1} - P_m$ between the operators. An equivalent representation is $P_{m+1} = P_m + Q_m$. As a consequence for each $f \in L^2(\Omega)$ the projection of $f$ into $V_{m+1}$ can be expressed as

$$P_{m+1}f = P_m f + Q_m f$$
$$= P_m f + \sum_{n=0}^{2^m-1} <f, \psi_{m,n}> \psi_{m,n}.$$ 

Thus, $L^2(\Omega)$ can be decomposed into $L^2(\Omega) = V_0 \oplus_{m \geq 0} W_m$.

An example of a scaling function and its corresponding mother wavelet is the characteristic function and Haar wavelet (see Figure 4.4). The definitions of these functions are:

$$\phi(x) = \begin{cases} 
1, & 0 \leq x \leq 1, \\
0, & \text{otherwise}, 
\end{cases}$$
and

\[
\psi(x) = \begin{cases} 
1, & 0 \leq x \leq \frac{1}{2}, \\
-1, & \frac{1}{2} \leq x \leq 1, \\
0, & \text{otherwise}. 
\end{cases}
\]

3.3 The Lifting Scheme

The lifting scheme is a simple method of constructing wavelets that can be adapted to intervals, domains, surfaces, weights, and irregular samples. Additionally, this method leads to a faster, in-place calculation of the wavelet transform. For more details see [8].

3.3.1 The Haar Wavelet: A Simple Example

To explore the idea of the lifting scheme we begin by considering two neighboring samples, \(a\) and \(b\), of a signal. Since they are neighboring samples it is not unreasonable to assume there is some correlation between the samples. We take advantage of correlation between the samples via a well-known, simple linear transformation which replaces \(a\) and \(b\) by their average \(s\) and their difference \(d\). If the samples are correlated then \(s\) is a good approximation of both \(a\) and \(b\) and the detail removed will be small. This transform is given by:

\[
s = \frac{a + b}{2} \tag{3.3}
\]
\[
d = b - a. \tag{3.4}
\]

It is important to observe this idea resembles the concept of two-cell homogenization in one dimension. Note that no information has been lost in the transformation. \(a\) and \(b\) can always be recovered via the transformation:

\[
a = s - \frac{d}{2} \tag{3.5}
\]
\[
b = s + \frac{d}{2}. \tag{3.6}
\]

These formulas can be found by inverting a 2x2 matrix. The key behind the Haar wavelet transform is this simple observation.
Now extend this idea to a signal \( s_n \) containing \( 2^n \) samples given by

\[
 s_n = \{ s_{n,l} | 0 \leq l \leq 2^n - 1 \}. 
\]

We apply the average and difference transforms to each pair of even and odd indexed values, i.e., \( a = s_{n,2l} \) and \( b = s_{n,2l+1} \). There are \( 2^{n-1} \) such pairs and we denote the results of those transforms by

\[
 s_{n-1,j} = \frac{s_{n,2l} + s_{n,2l+1}}{2} \quad \text{and} \quad d_{n-1,j} = s_{n,2l+1} - s_{n,2l}.
\]

Since \( s_{j-1} \) and \( d_{j-1} \) each contain half as many numerical values as \( s_j \) we can store these values within the signal \( s_j \). We store the new “averages” \( s_{j-1} \) in the even locations and the “details” \( d_{j-1} \) in the odd locations. That is, the original signal, which contained the information

\[
\{ s_{n,0}, s_{n,1}, s_{n,2}, s_{n,3}, \ldots, s_{n,2^n-2}, s_{n,2^n-1} \},
\]

is replaced by the signal

\[
\{ s_{n-1,0}, d_{n-1,0}, s_{n-1,1}, d_{n-1,1}, \ldots, s_{n-1,2^{n-1}-1}, d_{n-1,2^{n-1}-1} \}.
\]

The averaging process is repeated on the coarser signal \( s_{n-1} \) creating a coarser signal \( s_{n-2} \) and so forth. Eventually after \( n \) applications of this process we will have \( n \) detail signals \( d_j \) with \( 0 \leq j \leq n - 1 \). Each of these detail signals containing \( 2^j \) coefficients. There would be one averaged signal \( s_0 \) containing one coefficient \( s_{0,0} \). This coefficient is the arithmetic average of all the samples in the original signal. The array which contained the original signal would now contain

\[
\{ s_{0,0}, d_{n-1,0}, d_{n-2,0}, d_{n-1,1}, ..., d_{n-2,2^{n-2}-1}, d_{n-1,2^{n-1}-1} \}.
\]

This is referred to as the Haar transform.

It should be clear that this method requires no additional memory. We can think of the Haar transform as applying an \( N \times N \) matrix, where \( N = 2^n \), to the signal \( s_n \). When we
consider that a general linear transformation of an $N$ vector is $O(N^2)$, the Haar transform as outlined here has the remarkable computational efficiency of $O(N)$. The method is even remarkable when compared to the Fast Fourier Transform which is $O(N \log N)$. It is the structure of a wavelet transform which allows switching to and from the wavelet representation in $O(N)$ operations [8].

3.3.2 The General Lifting Method

Consider starting with a signal or sample $s_j$ containing $2^j$ values. We want to use wavelets to transform that signal into a coarser “average” signal $s_{j-1}$ and a “detail” signal $d_{j-1}$. When looking to transform the signal into an averaged signal, we may choose what type of average we wish to preserve. The Haar transform outlined above preserved the arithmetic average. We will be interested in building, via a lifting scheme analogy, a transform method that preserves the harmonic average in one dimension. Building a wavelet transform utilizing lifting consists of three steps: split, predict, and update.

The splitting step divides the signal into two disjoint sets of samples. For example, one group consists of even indexed samples $s_{2l}$ and the other consists of odd indexed samples $s_{2l+1}$. This can be abstractly represented as passing signal $s_j$ through a routine

$$\text{Split}(s_j) := (\text{even}_{j-1}, \text{odd}_{j-1})$$

The new signals even$_{j-1}$ and odd$_{j-1}$ contain half as many samples as the original signal, i.e., $2^{j-1}$ samples.

Once the signal has been split, the next step is prediction (P). Since the even and odd sets of data were interspersed, if the original signal has local correlation the even and odd sets of data will be correlated. That is, we should be able to predict with reasonable accuracy the values in one set using the values contained in the other set. If we use the even set to predict the odd values, a formula for the prediction step $P(\text{even}_{j-1})$ must be determined. This formula uses the even samples to predict the values of the odd samples. These predicted values will, in general, vary from the true value of the odd samples. So
we form a detail signal that contains the difference between the odd samples and their corresponding predicted value. That is, the detail signal \( d_{j-1} \) is

\[
d_{j-1} = \text{odd}_{j-1} - P(\text{even}_{j-1}).
\]

A key property of the lifting scheme is that coarser signals possess the same "average" value as the original signal. This "average" is dictated by the application in which the lifting scheme is being used. For example, we wish to perform a wavelet transform on a signal \( K \) to produce a homogenized value \( K^\# \). Because of our application the appropriate "average" in one dimension for the transform to preserve would be the harmonic average. The update step \((U)\) ensures this "average" is preserved by updating the even indexed samples using the differences from the prediction step to create the coarser signal, i.e.

\[
s_{j-1} = \text{even}_{j-1} + U(d_{j-1}).
\]

For convenience each time through the lifting scheme the previous signal \( s_j \) is considered to be two signals. One signal containing the even indexed samples even\(j-1\) and the other signal containing odd indexed samples odd\(j-1\). The signal even\(j-1\) is overwritten with the "updated" values and becomes the sample signal for the next iteration. The signal odd\(j-1\) is replaced with the "details" removed in the updating process. This is all done in the space occupied by the original signal \( s_j \). The inverse scheme can be immediately built by reversing the order of operations and flipping the signs. Thus, the lifting scheme possesses advantages such as efficiency and ease of the inversion of the process [8].
CHAPTER 4
CHARACTERIZING HOMOGENIZATION USING WAVELETS

In homogenization we replace a heterogeneous permeability tensor $K(x)$ defined on the unit interval $[0,1]$ by an effective homogeneous permeability tensor, $K^\#$. The heterogeneous tensor is assumed to be coercive, and bounded. Denote $K(x)$ by $K^\epsilon(x)$, where the superscript $\epsilon$ is used to indicate the dependence on microscopic scale structure. A crucial component of performing homogenization is determining the solution to the local problem, see (2.10), on the local cell. This solution is used to compute the homogenized tensor $K^\#$, via (2.15). In using a wavelet based method for computing an approximation of $K^\#$ we apply a transform, which incorporates the solution to the local two-cell problem, see sections (2.1.2) and (2.2.2), to an approximation of $K^\epsilon$ to create an approximation of $K^\#$.

4.1 The One-Dimensional Case

To illustrate the idea of characterizing the computation of $K^\#$ using wavelets, we will consider an example that uses two-cell homogenization in the one dimensional case. Suppose $K(x)$ is sampled at $2^N$ evenly spaced intervals in $[0,1]$, see Figure 4.1. Taking advantage of the dyadic structure we take neighboring permeability values and replace them with an "effective" or average value for the two cells, see Figure 4.2. The top row is computed pairwise from values on the bottom row. This creates a set of $2^{N-1}$ average permeabilities.

<table>
<thead>
<tr>
<th>$k_{N,0}$</th>
<th>$k_{N,1}$</th>
<th>$k_{N,2}$</th>
<th>$k_{N,3}$</th>
<th>$\ldots$</th>
<th>$k_{N,2^N-2}$</th>
<th>$k_{N,2^N-1}$</th>
</tr>
</thead>
</table>

Fig. 4.1: A visualization of a signal containing $2^N$ permeabilities obtained by sampling $K^\epsilon(x)$ on an evenly spaced grid.
Fig. 4.2: The signal containing $2^N$ evenly spaced permeabilities is used to create a new signal containing half as many or $2^{N-1}$ values. The values in the new signal are generated by averaging the values in the previous signal.

The process can be repeated $N$ times, thereby creating smaller sets of averaged values defined on successively coarser dyadic meshes until one averaged value has been generated which is an effective value defined on the entire domain, see Figure 4.3. This resembles the process used in the Haar transform.

4.1.1 Wavelet Representation via One-Dimensional Transform

We utilize a wavelet transform method that incorporates the solution to the two-cell problem as a means of approximating $K^\#$. Homogenized values are computed based on $K_N(x)$ and referred to as $K^\#_{N-1}(x)$, an averaged signal at a coarser scale. The sampled signal $K_N(x)$ is an approximation of $K^\epsilon$ defined by

$$K_N(x) = \begin{cases} 
  k_{N,0}, & 0 \leq x < 2^{-N} \\
  k_{N,1}, & 2^{-N} \leq x < 2 \cdot 2^{-N} \\
  k_{N,2}, & 2 \cdot 2^{-N} \leq x < 3 \cdot 2^{-N} \\
  \vdots \\
  k_{N,2^N-1}, & (2^N - 1) \cdot 2^{-N} \leq x \leq 1,
\end{cases} \quad (4.1)$$

where each $k_{N,j}$ is a sample from the original signal on the specified interval. The averaged signal $K_{N-1}(x)$ is defined as

$$K_{N-1}(x) = \begin{cases} 
  k_{N-1,0}, & 0 \leq x < 2^{-N+1} \\
  k_{N-1,1}, & 2^{-N+1} \leq x < 2 \cdot 2^{-N+1} \\
  k_{N-1,2}, & 2 \cdot 2^{-N+1} \leq x < 3 \cdot 2^{-N+1} \\
  \vdots \\
  k_{N-1,2^N-1-1}, & (2^N-1) \cdot 2^{-N+1} \leq x \leq 1,
\end{cases} \quad (4.2)$$
Fig. 4.3: The process of dyadically averaging the values of neighboring cells is completed $N$ times generating one averaged value to represent the domain.

Let $I_{N,j} = [j2^{-N}, (j+1)2^{-N})$ and $\chi_{N,j}$ be the characteristic function on $I_{N,j}$. This framework allows $K_N$ to be expressed as

$$K_N(x) = \sum_{j=0}^{2^{N-1}} k_{N,j} \chi_{N,j}.$$ 

Since $K^c$ is a bounded, piecewise defined function on $[0,1]$ it is a member of $L^2([0,1])$. Any $L^2([0,1])$ function can be approximated to arbitrary accuracy by a piecewise constant function as $N$ approaches infinity, [18, p. 11].

The values contained in $K_{N-1}(x)$ are computed via the following three-step transform, [11]. The transform is given by:

$$k_{N-1,j} = \frac{k_{N,2j} + k_{N,2j+1}}{2}$$
$$d_{N-1,j} = -\frac{(k_{N,2j+1} - k_{N-1,j})^2}{k_{N-1,j}}$$
$$k_{N-1,j} = k_{N-1,j} + d_{N-1,j}.$$
This transform is used because it preserves the harmonic average on a dyadic grid. As previously noted the physically appropriate average for modelling flow in one dimension is the harmonic average. This transform is iterated creating functions of the form

\[
K_l^#(x) = \begin{cases}
  k_{l,0}^#, & 0 \leq x < 2^{-l} \\
  k_{l,1}^#, & 2^{-l} \leq x < 2 \cdot 2^{-l} \\
  k_{l,2}^#, & 2 \cdot 2^{-l} \leq x < 3 \cdot 2^{-l} \\
  \vdots \\
  k_{l,2^l-1}^#, & (2^l - 1) \cdot 2^{-l} \leq x \leq 1.
\end{cases}
\]  

(4.3)

where \( l = N, N - 1, N - 2, \ldots, 2, 1, 0, j = 0, 1, 2, \ldots, 2^l - 1 \), using the transform

\[
k_{l,j} = \frac{k_{l+1,2j} + k_{l+1,2j+1}}{2}
\]

\[
d_{l,j} = \frac{(k_{l+1,2j+1} - k_{l,j})^2}{k_{l,j}}
\]

(4.4)

When \( l = 0 \), we obtain the signal

\[
K_0^# = \left\{ k_{0,0}^#, \ 0 \leq x \leq 1 \right\},
\]

containing an "average" of the permeabilities contained in the original heterogeneous tensor.

4.1.2 Convergence Properties

The transform defined in (4.4) was created to compute the harmonic average of neighboring cells on a dyadic grid. The process repeats until the value \( k_{0,0}^# \) is produced containing the harmonic average of the \( 2^N \) values in the original signal \( K_N \), i.e.

\[
K_0^# = \left\{ \frac{2^N}{\sum_{i=0}^{2^N-1} \frac{1}{k_{N,i}}}, \ 0 \leq x \leq 1 \right\}.
\]

Lemma 4.1.1 Let the signal \( K_N \) containing \( 2^N \) equally spaced samples be defined \( K_N = \{k_0, k_1, k_2, \ldots, k_{2^N-1}\} \). Define the operator \( H_N : \mathbb{R}^{2^N} \rightarrow \mathbb{R}^{2^N-1} \) as

\[
H_N (k_0, k_1, k_2, \ldots, k_{2^N-1}) = \left\{ \frac{2}{k_0 + 1/k_1}, \frac{2}{k_1 + 1/k_2}, \ldots, \frac{2}{k_{2^N-2} + 1/k_{2^N-1}} \right\}
\]
Repeating this process $N$ more times produces

$$H_1 \ldots H_{N-1}H_N (k_0, k_1, k_2, \ldots, k_{2N-1}) = \left\{ \frac{2^N}{\sum_{i=0}^{2N-1} \frac{1}{k_i}} \right\}$$

containing the harmonic average of $K_N$.

**Proof:** By Induction, for $N = 1$ consider a signal containing 2 samples. Let $K_1 = \{k_0, k_1\}$. Computing $K_0$ we find

$$K_0 = \left\{ \frac{2}{\frac{1}{k_0} + \frac{1}{k_1}} \right\}.$$ 

Now assume that a signal containing $2^{N-1}$ samples can be averaged in the outlined manner $N - 1$ times to produce the harmonic average

$$K_0 = \left\{ \frac{2^{N-1}}{\sum_{i=0}^{2^{N-1}} \frac{1}{k_i}} \right\}.$$ 

Let $K_N = \{k_0, k_1, k_2, \ldots, k_{2N-1}\}$, which contains $2^N$ samples, be averaged to obtain the signal

$$K_{N-1} = \left\{ \frac{2}{\frac{1}{k_0} + \frac{1}{k_1}}, \frac{2}{\frac{1}{k_3} + \frac{1}{k_4}}, \ldots, \frac{2}{\frac{1}{k_{2N-2}} + \frac{1}{k_{2N-1}}} \right\},$$

which contains $2^{N-1}$ values. By assumption this can be averaged $N - 1$ more times to obtain

$$K_0 = \left\{ \frac{2^{N-1}}{\sum_{i=0}^{2^{N-1}} \frac{1}{k_{N-1,i}}} \right\}$$

where $k_{N-1,i}$ denotes the $i$-th element of $K_{N-1}$. Note that

$$k_{N-1,i} = \frac{2}{\frac{1}{k_{2i}} + \frac{1}{k_{2i+1}}}$$

where $k_{2i}$ and $k_{2i+1}$ are elements of the original signal $K_N$. This implies that $\frac{1}{k_{N-1,i}}$ can be rewritten as

$$\frac{1}{k_{N-1,i}} = \frac{1}{2} \left( \frac{1}{k_{2i}} + \frac{1}{k_{2i+1}} \right).$$
Thus,

\[ \sum_{i=0}^{2^N-1} \frac{1}{k_{N-1,i}} = \sum_{i=0}^{2^N-1} \frac{1}{2} \left( \frac{1}{k_{2i}} + \frac{1}{k_{2i+1}} \right) = \frac{1}{2} \sum_{i=0}^{2^N-1} \frac{1}{k_i}. \]

Therefore,

\[ K_0 = \left\{ \frac{\sum_{i=0}^{2^N-1} 1}{\sum_{i=0}^{2^N-1} \frac{1}{k_i}} \right\}. \]

This lemma shows the transformation produced the desired result, the harmonic average of the entire signal.

**Theorem 4.1.2** In one dimension \( K_0^{\#} \), the approximation to the homogenized coefficient, converges to the homogenized coefficient \( K^{\#} \), \( \left( \int_0^1 \frac{1}{K(x)} dx \right)^{-1} \), in \( L^2([0,1]) \).

**Proof:** Let \( 0 < \delta \ll 1 \) be given. Note the function \( \frac{1}{K_m(x)} \) can be expressed

\[ \sum_{k=0}^{2^m-1} \frac{1}{k_{m,k}} \chi_{m,k}. \]

This observation can be used to write

\[ \int_0^1 \frac{1}{K_m(x)} dx = \int_0^1 \left( \sum_{k=0}^{2^m-1} \frac{1}{k_{m,k}} \chi_{m,k} \right) dx = \left( \sum_{k=0}^{2^m-1} \frac{1}{k_{m,k}} \right) 2^{-m} = \left( \frac{1}{K_{0,0}} \right) = \frac{1}{K_0^{\#}}. \]

This implies

\[ K_0^{\#} = \left( \int_0^1 \frac{1}{K_m(x)} dx \right)^{-1}. \]
Now consider

\[ \| K_0^\# - K^\# \|_0 = \left\| \left( \int_0^1 \frac{1}{K_m(x)} \, dx \right)^{-1} - \left( \int_0^1 \frac{1}{K^\epsilon(x)} \, dx \right)^{-1} \right\|_0 \]

\[ = \left\| \left( \int_0^1 \frac{1}{K^\epsilon(x)} \, dx \right) - \left( \int_0^1 \frac{1}{K_m(x)} \, dx \right) \left( \int_0^1 \frac{1}{K^\epsilon(x)} \, dx \right)^{-1} \left( \int_0^1 \frac{1}{K_m(x)} \, dx \right)^{-1} \right\|_0 \]

\[ \leq \left\| \int_0^1 \frac{1}{K^\epsilon(x)} \, dx - \frac{1}{K_m(x)} \, dx \right\|_0 \left\| \left( \int_0^1 \frac{1}{K^\epsilon(x)} \, dx \right)^{-1} \right\|_0 \left\| K^\# \right\|_0 \left\| K^\# \right\|_0 \]

Note \( K_m(x) \) and \( K^\epsilon(x) \) are bounded coercive elements of \( L^2([0, 1]) \) such that \( K_m(x) \rightarrow K^\epsilon(x) \) in \( L^2([0, 1]) \). Thus there is a number \( M \) such that for all \( m > M \)

\[ \| K_m(x) - K^\epsilon(x) \|_0 < \frac{\delta}{\| (K^\epsilon(x))^{-1} \|_0 \| (K_m(x))^{-1} \|_0 \left\| K^\# \right\|_0 \left\| K^\# \right\|_0} \]

This establishes that for \( m > M \)

\[ \| K_0^\# - K^\# \|_0 < \delta \]

While this convergence result is nice to know, this information alone is not enough to validate the method. We really want to see how our wavelet approach compares with the perturbation approach in theory. In homogenization theory \( G \)-convergence, or weak convergence of the solutions to the heterogenous equations to the solution of the homogeneous equation in \( H_0^1 \) is desired. So the desired means of convergence is for \( K_m \) \( G \)-converge to \( K^\# \) or even better \( K_m \) \( H \)-converges to \( K^\# \). To accomplish this there are some crucial observations to be made.
Recall $K^\varepsilon$ can be said to $G$-converge to $K^\#$ in $M(\alpha, \beta; [0, 1])$, if and only if for all $f^\# \in H^{-1}([0,1])$ the solutions $u_\varepsilon$ and $u^\#$ of

\[
\begin{align*}
&u_\varepsilon \in H^1_0([0,1]) : \quad -\nabla (K^\varepsilon \cdot \nabla u_\varepsilon) = f^\varepsilon \\
&u^\# \in H^1_0([0,1]) : \quad -\nabla (K^\# \cdot \nabla u^\#) = f^\#
\end{align*}
\]

where $f^\varepsilon \to f^\#$ in $H^{-1}([0,1])$ satisfy $u_\varepsilon \to u^\#$ in $H^1_0([0,1])$, see Theorem 2.4.11 and [6].

To analyze our scheme we assume a problem in which $K^\varepsilon$ $G$-converges to $K^\#$ and consider four Dirichlet problems associated with this problem:

\[
\begin{align*}
&u_\varepsilon \in H^1_0([0,1]) : \quad -\nabla (K^\varepsilon \cdot \nabla u_\varepsilon) = f^\varepsilon \quad (4.5) \\
&u^\# \in H^1_0([0,1]) : \quad -\nabla (K^\# \cdot \nabla u^\#) = f^\# \quad (4.6) \\
&u_m \in H^1_0([0,1]) : \quad -\nabla (K_m \cdot \nabla u_m) = f_m \quad (4.7) \\
&u^\#_0 \in H^1_0([0,1]) : \quad -\nabla (K^\#_0 \cdot \nabla u^\#_0) = f^\#_0 \quad (4.8)
\end{align*}
\]

Note the first pair of equations are the traditional homogenization equations. The second pair of equations are differential equations based on the use of the approximation $K_m$ as the permeability tensor at the fine scale. In this pair of equations, we use a sample of $f^\varepsilon$ called $f_m$ and its averaged value, $f^\#_m$. Define $f_m = \sum_{k=0}^{2^m-1} f_{m,k} \chi_{m,k}$ where $f_{m,k}$ is the sample of $f^\varepsilon$ at the right endpoint of $I_{m,k}$. By Theorem 2.4.8 we know $f^\varepsilon \to f^\#$ and $f_m \to f^\#_m$ in $L^2([0,1])$.

To establish the desired $G$-convergence in one dimension we begin by establishing $f_m \to f^\#$ in $L^2([0,1])$. Since $f^\varepsilon$ is an element of $L^2([0,1])$ and integrable, by Theorem 2.4.1 it is clear that $f^\varepsilon$ is approximable by $f_m$. This implies for $0 < \delta \ll 1$ there exists a number $M$ such that for all $m > M$

\[
\int_{[0,1]} |f_m(y) - f^\varepsilon(y)| < \delta.
\]

Since $[0, 1]$ is a finite interval we can establish

\[
\int_{[0,1]} |f_m(y) - f^\varepsilon(y)|^2
\]

can be made arbitrarily small, as well. Therefore, $f_m \to f^\varepsilon$ in $L^2([0,1])$. 

Lemma 4.1.3 \( f_m \rightharpoonup f^\# \) in \( L^2([0,1]) \).

Proof: Let \( 0 < \delta \ll 1 \) be given. Let \( \phi \in L^2([0,1]) \) and consider

\[
\begin{align*}
(f_m - f^\#, \phi) &= (f_m - f^\varepsilon + f^\varepsilon - f^\#, \phi) \\
&= (f_m - f^\varepsilon, \phi) + (f^\varepsilon - f^\#, \phi).
\end{align*}
\]

Since strong convergence implies weak convergence and \( f_m \to f^\varepsilon \) in \( L^2 \) there exists a number \( M_1 \) such that for all \( m > M_1, \ (f_m - f^\varepsilon, \phi) < \frac{\delta}{2} \). From homogenization theory we know \( f^\varepsilon \rightharpoonup f^\# \) in \( L^2([0,1]) \), meaning there is a number \( M_2 \) such that \((f^\varepsilon - f^\#, \phi) < \frac{\delta}{2} \).

Let \( M = \max(M_1, M_2) \). Then, for all \( m > M \)

\[
\begin{align*}
(f_m - f^\#, \phi) &\leq \delta + \frac{\delta}{2} \\
&= \delta.
\end{align*}
\]

Lemma 4.1.4 \( u_m \) converges to \( u^\varepsilon \) in \( H^1_0([0,1]) \).

Proof: Consider the variational equation for the difference of Equations (4.5) and (4.7)

\[
(K_m \nabla u_m - K^\varepsilon \nabla u^\varepsilon, \nabla \phi) = (f_m - f^\varepsilon, \phi).
\]

Manipulate this equation by adding and subtracting the term \( K_m \nabla u^\varepsilon \) and choose \( \phi \) to be \( u_m - u^\varepsilon \). We find

\[
(K_m \nabla u_m - K^\varepsilon \nabla u^\varepsilon, \nabla \phi) = (f_m - f^\varepsilon, \phi)
\]

becomes

\[
(K_m (\nabla u_m - \nabla u^\varepsilon) - (K^\varepsilon - K_m) \nabla u^\varepsilon, \nabla (u_m - u^\varepsilon)) = (f_m - f^\varepsilon, (u_m - u^\varepsilon)).
\]

Now we use the definition of inner product to rewrite the equation to obtain

\[
\int_0^1 K_m (\nabla u_m - \nabla u^\varepsilon)^2 \, dx = \int_0^1 (K^\varepsilon - K_m) \nabla u^\varepsilon (\nabla u_m - \nabla u^\varepsilon) \, dx
\]

\[
= \int_0^1 (f_m - f^\varepsilon) (u_m - u^\varepsilon) \, dx
\]
Adding the second term to the right hand side of the equation we have

\[
\int_0^1 K_m (\nabla u_m - \nabla u^\varepsilon)^2 \, dx = \int_0^1 (f_m - f^\varepsilon) (u_m - u^\varepsilon) \, dx + \int_0^1 (K^\varepsilon - K_m) \nabla u^\varepsilon (\nabla u_m - \nabla u^\varepsilon) \, dx.
\]

Due the coercivity of $K_m$ we can observe

\[
\alpha \| u_m - u^\varepsilon \|_1^2 \leq \int_0^1 (f_m - f^\varepsilon)(u_m - u^\varepsilon) \, dx + \int_0^1 (K^\varepsilon - K_m) \nabla u^\varepsilon (\nabla u_m - \nabla u^\varepsilon) \, dx.
\]

Using the Cauchy-Schwartz inequality and Poincaré's inequality, we find

\[
\alpha \| u_m - u^\varepsilon \|_1^2 \leq \| f_m - f^\varepsilon \|_0 \| u_m - u^\varepsilon \|_0 + \| K^\varepsilon - K_m \|_0 \| u^\varepsilon \|_1 \| u_m - u^\varepsilon \|_1 \\
\leq \| f_m - f^\varepsilon \|_0 (C \| u_m - u^\varepsilon \|_1) + \| K^\varepsilon - K_m \|_0 \| u^\varepsilon \|_1 \| u_m - u^\varepsilon \|_1.
\]

This implies

\[
\| u_m - u^\varepsilon \|_1 \leq \alpha^{-1} (C \| f_m - f^\varepsilon \|_0 + \| K^\varepsilon - K_m \|_0 \| u^\varepsilon \|_1).
\]

We know $f_m \to f^\varepsilon$ and $K_m \to K^\varepsilon$ in $L^2([0, 1])$, as well as, $u^\varepsilon \in H^1_0$. Thus, there exists a number $M$ such that for all $m > M$

\[
\| f_m - f^\varepsilon \|_0 < \frac{\delta \alpha}{2C}
\]

\[
\| K^\varepsilon - K_m \|_0 < \frac{\delta \alpha}{2\| u^\varepsilon \|_1}
\]

and, therefore,

\[
\| u_m - u^\varepsilon \|_1 \leq \delta.
\]
Theorem 4.1.5 \( K_m \) \( G \)-converges to \( K^\# \). That is, for all \( f^\# \in L^2([0,1]) \) the solutions \( u_m \) and \( u^\# \) of
\[
\begin{align*}
  u_m &\in H^1_0([0,1]) : & -\nabla (K_m \nabla u_m) &= f_m \\
  u^\# &\in H^1_0([0,1]) : & -\nabla (K^\# \nabla u^\#) &= f^\#
\end{align*}
\]
where \( f_m \to f^\# \) in \( L^2([0,1]) \) satisfy \( u_m \to u^\# \) in \( H^1_0([0,1]) \).

Proof: We have already established that \( f_m \) converges to \( f^\# \) in \( L^2([0,1]) \). It must now be shown that \( u_m \to u^\# \) in \( H^1_0([0,1]) \). Let \( 0 < \delta \ll 1 \) be given. Using the definition of weak convergence in \( H^1_0 \), let \( \phi \in H^{-1}([0,1]) \) and consider
\[
| (u_m - u^\#, \phi) | = | (u_m - u^\epsilon + u^\epsilon - u^\#, \phi) | \\
\leq | (u_m - u^\epsilon, \phi) | + | (u^\epsilon - u^\#, \phi) |.
\]
The Cauchy-Schwartz inequality indicates
\[
| (u_m - u^\epsilon, \phi) | \leq \| u_m - u^\epsilon \|_1 \| \phi \|.
\]
\( K_m \) satisfies the same coercivity condition as \( K^\epsilon \) because \( K_m \) consists of samples of \( K^\epsilon \). Since \( u_m \) converges to \( u^\epsilon \) in \( H^1_0([0,1]) \) there is a number \( M_1 \) such that for all \( m \) larger
\[
\| u_m - u^\epsilon \|_1 \leq \frac{\delta}{2\| \phi \|}.
\]
From traditional homogenization theory we know that \( u^\epsilon \) converges weakly to \( u^\# \) in \( H^1_0 \), which means there is a number \( M_2 \) such that for all \( m > M_2 \)
\[
| (u^\epsilon - u^\#, \phi) | < 2^{-m} \\
\leq \frac{\delta}{2}.
\]
Let \( M = \max(M_1, M_2) \) and we know
\[
| (u_m - u^\#, \phi) | \leq | (u_m - u^\epsilon, \phi) | + | (u^\epsilon - u^\#, \phi) | \\
\leq \| u_m - u^\epsilon \|_1 \| \phi \| + \| (u^\epsilon - u^\#, \phi) | \\
\leq \frac{\delta}{2} + \frac{\delta}{2}.
\]
establishing the weak convergence of \( u_m \) to \( u^\# \).

If we make an additional assumption that \( K^\varepsilon \) \( H \)-converges to \( K^\# \), we can show that \( K_m \) \( H \)-converges to \( K^\# \). Recall that \( G \)-convergence involved weak convergence of the solutions to the problems (4.5)-(4.6). \( H \)-convergence adds the additional requirement of weak convergence of the flows; i.e.

\[
K^\varepsilon \nabla u^\varepsilon \rightharpoonup K^\# \nabla u^\#
\]

in \( (L^2([0,1]))^n \).

**Theorem 4.1.6** \( K_m(x) \) \( H \)-converges to \( K^\# \). That is, for all \( f^\# \in L^2([0,1]) \) the solutions \( u_m \) and \( u^\# \) of

\[
\begin{align*}
  u_m &\in H^1_0([0,1]) : \quad -\nabla (K_m \nabla u_m) = f_m \\
  u^\# &\in H^1_0([0,1]) : \quad -\nabla (K^\# \nabla u^\#) = f^\#
\end{align*}
\]

where \( f_m \rightarrow f^\# \) in \( L^2([0,1]) \) satisfy

\[
\begin{align*}
  u_m &\rightharpoonup u^\# \quad \text{in} \quad H^1_0([0,1]) \\
  K_m \nabla u_m &\rightharpoonup K^\# \nabla u^\# \quad \text{in} \quad L^2([0,1]).
\end{align*}
\]

**Proof:** Having already established

\[
\begin{align*}
  f_m &\rightarrow f^\# \quad \text{in} \quad L^2([0,1]) \\
  u_m &\rightarrow u^\# \quad \text{in} \quad H^1_0([0,1])
\end{align*}
\]

we must now show

\[
K_m \nabla u_m \rightharpoonup K^\# \nabla u^\# \quad \text{in} \quad L^2([0,1]).
\]

Let \( 0 < \delta \ll 1 \) be given. Using the definition of weak convergence in \( L^2([0,1]) \), let \( \phi \in L^2([0,1]) \) and consider

\[
\left| \left( K_m \nabla u_m - K^\# \nabla u^\#, \phi \right) \right|. \tag{4.9}
\]
Rewriting expression (4.9) and using the triangle inequality we find
\[\| (K_m \nabla u_m - K_m \nabla u^\epsilon + K_m \nabla u^\epsilon - K^\epsilon \nabla u^\epsilon - K^\# \nabla u^\#, \phi) \|
\leq \| (K_m \nabla (u_m - u^\epsilon), \phi) \| + \| (K_m - K^\epsilon) \nabla u^\epsilon, \phi) \| + \| (K^\epsilon \nabla u^\epsilon - K^\# \nabla u^\#, \phi) \|.
\]

The Cauchy-Schwarz inequality establishes
\[\| (K_m \nabla (u_m - u^\epsilon), \phi) \| + \| (K_m - K^\epsilon) \nabla u^\epsilon, \phi) \| + \| (K^\epsilon \nabla u^\epsilon - K^\# \nabla u^\#, \phi) \| \leq \| K_m \|_0 \| u_m - u^\epsilon \|_1 \| \phi \|_0 + \| K_m - K^\epsilon \|_0 \| u^\epsilon \|_1 \| \phi \|_0 + \| (K^\epsilon \nabla u^\epsilon - K^\# \nabla u^\#, \phi) \|.
\]

Since \( u_m \to u^\epsilon \), there is a number \( M_1 \) such that for all \( m > M_1 \)
\[\| u_m - u^\epsilon \|_1 \leq \frac{\delta}{3 \| K_m \|_0 \| \phi \|_0}.
\]

Also, \( K_m \to K^\epsilon \) and therefore a number \( M_2 \) where for all \( m > M_2 \)
\[\| K_m - K^\epsilon \|_0 \leq \frac{\delta}{3 \| u^\epsilon \|_1 \| \phi \|_0}.
\]

Finally, we assumed \( K^\epsilon H\)-converges to \( K^\# \) meaning there is a number \( M_3 \) such that for \( m > M_3 \)
\[\| (K^\epsilon \nabla u^\epsilon - K^\# \nabla u^\#, \phi) \| < 2^{-m} < \frac{\delta}{3}.
\]

Let \( M = \max(M_1, M_2, M_3) \). Then, for all \( m > M \) we have established
\[\| (K_m \nabla u_m - K^\# \nabla u^\#, \phi) \| < \delta
\]

establishing that
\[K_m \nabla u_m \to K^\# \nabla u^\# \text{ in } L^2([0,1]).
\]

This completes the proof.

The traditional method for computing the homogenized coefficient involves a perturbation method in which the solution to the local problem is computed on a microscopic cell of size \( \epsilon \) and the cell size is sent to zero. To solve the local problem a finite element method may be used. Once the solution to the local problem has been determined \( K^\# \) may be computed. The wavelet based method is a simpler approach for computing \( K^\# \). It is based on computing a harmonic average of \( K^\epsilon(x) \) on a dyadic grid. We have shown the wavelet approach can be used to compute an equivalent homogenized value as the number of samples goes to infinity.
4.1.3 Numerical Illustration

In the previous section we established that in one dimension $K_m$ converges to $K^\#$. We now explore, through examples, the convergence of the homogenized value computed via the wavelet based transform $K_m^\#$ to the standard value $K^\#$, the harmonic mean of the original signal.

To assist in visualizing the convergence we start with a signal containing 30 values, called $K^\epsilon$, defined on the interval $[0, 1]$. The number of samples chosen is not a power of 2. This is done so the dyadic grid on which $K^\epsilon$ is sampled will not align with the intervals on which $K_m$ is defined. Before discussing the method of sampling it is necessary to define notation. Let the $i$-th interval of $K^\epsilon$ be denoted $I_i = \left[\frac{i}{30}, \frac{i + 1}{30}\right]$ where $i = 0, 1, 2, ..., 29$ and use $K_i$ to denote the $i$-th value of the original signal $K^\epsilon$. The sampled signal $K_m$ is defined on intervals $I_j = \left[\frac{j}{2^m}, \frac{j + 1}{2^m}\right]$, where $j = 0, 1, 2, ..., 2^m - 1$. The notation $K_{m,j}$ is used to denote the $j$-th value of $K_m$.

Now to define the method for sampling $K^\epsilon$ to create $K_m$. When comparing the intervals for the two signals we note that they will never be identical. Since the intervals $I_i$ and $I_j$ are both refinements of the interval $[0, 1]$, for each $j$ there is an $i$ such that the left endpoint of $I_j$ lies in $I_i$; i.e.

$$\frac{i}{30} \leq \frac{j}{2^m} < \frac{i + 1}{30}.$$

Using a simple left-hand rule to sample the signal we let $K_{m,j} = K_i$.

Now we explore the results of our method. To test the method a number of signals containing 30 values were generated to use as input for our transform method. Table 4.1 contains the results obtained running the transform method on one of the signals. Note the first column contains the value of $m$. This indicates the sample signal contains $2^m$ values. The second column contains the harmonic average of the original signal, $K^\#_{\text{orig}}$. The third column contains the harmonic average of the sample signal, $K^\#_m$. Next we have the average computed by running the transform method on the sample signal, $K^\#_0$. Note the values in columns three and four are identical. This is expected. The wavelet transform was
Table 4.1: This table displays the results of running our wavelet transform method on a signal containing 30 samples. The columns of the table are: the value of $m$—the sampled signal contains $2^m$ values, $K_{\text{orig}}^\#$—the harmonic average of the original signal, $K_{\text{sample}}^\#$—the harmonic average of the sample, $K_0^\#$—the average resulting from performing the transform on the sample, and the absolute error between $K_{\text{orig}}^\#$ and $K_0^\#$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$K_{\text{orig}}^#$</th>
<th>$K_{\text{sample}}^#$</th>
<th>$K_0^#$</th>
<th>Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.42390</td>
<td>1.76813</td>
<td>1.76813</td>
<td>0.337933</td>
</tr>
<tr>
<td>2</td>
<td>1.42390</td>
<td>1.22563</td>
<td>1.22563</td>
<td>0.198259</td>
</tr>
<tr>
<td>3</td>
<td>1.42390</td>
<td>1.37764</td>
<td>1.37764</td>
<td>0.046246</td>
</tr>
<tr>
<td>4</td>
<td>1.42390</td>
<td>1.47287</td>
<td>1.47287</td>
<td>0.048971</td>
</tr>
<tr>
<td>5</td>
<td>1.42390</td>
<td>1.44117</td>
<td>1.44117</td>
<td>0.017277</td>
</tr>
<tr>
<td>6</td>
<td>1.42390</td>
<td>1.40964</td>
<td>1.40964</td>
<td>0.014252</td>
</tr>
<tr>
<td>7</td>
<td>1.42390</td>
<td>1.42091</td>
<td>1.42091</td>
<td>0.002981</td>
</tr>
<tr>
<td>8</td>
<td>1.42390</td>
<td>1.42686</td>
<td>1.42686</td>
<td>0.002965</td>
</tr>
<tr>
<td>9</td>
<td>1.42390</td>
<td>1.42496</td>
<td>1.42496</td>
<td>0.001068</td>
</tr>
<tr>
<td>10</td>
<td>1.42390</td>
<td>1.42300</td>
<td>1.42300</td>
<td>0.000899</td>
</tr>
<tr>
<td>11</td>
<td>1.42390</td>
<td>1.42371</td>
<td>1.42371</td>
<td>0.000187</td>
</tr>
<tr>
<td>12</td>
<td>1.42390</td>
<td>1.42408</td>
<td>1.42408</td>
<td>0.000185</td>
</tr>
</tbody>
</table>

Fig. 4.4: These plots illustrate the convergence of the method as $m$ increases. The first graph illustrates convergence using number of samples versus absolute error. To create the graph on the right the $\log_2$ of the data was plotted along with a least squares fit of that data.
Table 4.2: This table contains the slopes for the least square line fit to the logarithmic data.

<table>
<thead>
<tr>
<th>Slope of Least Squares Line</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

developed to preserve the harmonic average of the signal on which it was used. The last column contains the absolute error between $K_{\text{orig}}^\#$, the harmonic average of the original signal, and $K_{\text{avg}}^\#$, the average obtained via the transform method. Note that as $m$ increases we can observe a decrease in the error made by using our transform. For all simulations using $2^{12}$ samples, we found the absolute error was $O(10^{-4})$.

The error can also be analyzed graphically. Figure 4.4 contains two graphs illustrating behavior of the error. The plot on the left shows number of samples versus absolute error. We can see a significant drop in error. To further analyze our data we took the logarithm base 2 \( \log_2 \) of the number of samples and the error to create the second graph. The points on this graph correspond to the points on the previous graph but on a logarithmic scale. Note the data seems to have a linear trend. A least squares fit to the data produced a line with slope $-0.936825$ and $y$-intercept of $-1.20785$. Table 4.2 contains the slopes of the least square line for ten simulations. The slopes of these lines are in the neighborhood of -1 implying the error is on the order of $2^{-m}$.

4.2 The Two-Dimensional Case

Our goal is given a tensor for the two-cell problem in two dimensions, $K^r$, develop an efficient means of computing an homogenized tensor $K^\#$ and the details necessary to
reconstruct the original tensor. That is, a lifting scheme in two dimensions. The idea is to take a spatially defined tensor

\[ K^\epsilon(x) = \begin{pmatrix} K_{III}(x) & K_{IV}(x) \\ K_I(x) & K_{II}(x) \end{pmatrix} \] (4.10)

where \( x \in \mathbb{R}^2 \) and overwrite it with an homogenized tensor for the region and details removed in the homogenization process. The ability to perform the necessary computations in place is an important component of our transform method, just as this attribute was important in the one-dimensional case.

4.2.1 Wavelet Representation via Two-Dimensional Transform

Before we outline the method we must establish our notation. We assume that \( K^\epsilon(y) \) is defined on the unit square. As was done in one dimension, we dyadically refine the unit square by imposing \( 2^N \) intervals in both the \( y_1 \) and \( y_2 \) directions. This creates \( 2^{2N} \) square regions on which \( K^\epsilon \) can be sampled. Denote an arbitrary region spatially as

\[ R_{i,j} = \{(y_1, y_2) | \frac{i-1}{2^N} \leq y_1 < \frac{i}{2^N} \text{ and } \frac{j-1}{2^N} \leq y_2 < \frac{j}{2^N} \}, \]

where \( i = 1, \ldots, 2^N \) and \( j = 1, \ldots, 2^N \). The notation is easier to work with if one index is used to indicate a particular region and a second index is used to indicate scale.

At the \( N \)-th scale we enumerate the regions by starting at the origin and increment the \( y_1 \) direction first then the \( y_2 \) direction; i.e.

\[
\begin{align*}
R_{N,1} &= R_{1,1} \\
R_{N,2} &= R_{1,2} \\
&\vdots \\
R_{N,2^N} &= R_{1,2^N} \\
R_{N,2^N+1} &= R_{2,1} \\
&\vdots \\
R_{N,2^{N+1}} &= R_{2,2^N},
\end{align*}
\]
and so forth. The general relationship at the $N$ scale between the spatially defined region, $R_{i,j}$, and the enumerated region is given by

$$R_{i,j} = R_{N,(j-1)2^N+i}$$

The tensor $K^e$ is sampled on each of these regions creating

$$K_N(y) = \begin{cases} 
k_{N,1}, & y \in R_{N,1} 
k_{N,2}, & y \in R_{N,2} 
k_{N,3}, & y \in R_{N,3} 
\vdots 
k_{N,2^N}, & y \in R_{N,2^N} \end{cases}$$

where $y = (y_1, y_2)$ and $k_{N,i}$ is a sample of the heterogeneous tensor on $R_{N,i}$ for $i = 1, ..., 2^N$. We now have an approximation of $K^e(y)$ given by

$$K_N(y) = \sum_{i=1}^{2^N} k_{N,i} R_{N,i}$$

where $R_{N,i}$ is analogous to the characteristic function in one dimension. Note $K_N$ is a characteristic function in two dimensions.

We begin with a two cell problem; that is, $N = 2$. In this case there are four regions in the domain. The outline of our method is to overwrite the spatially defined tensor $K_N$ with two intermediary tensors before producing the homogenized and detail tensors which characterize the region. Symbolically this can be seen as

$$\left( \begin{array}{cc} K_{N}^{III} & K_{N}^{IV} \\ K_{N}^{I} & K_{N}^{II} \end{array} \right) \Rightarrow \left( \begin{array}{cc} K^o & \hat{K} \\ \tilde{K} & \tilde{K} \end{array} \right) \Rightarrow \left( \begin{array}{cc} d_1 & d_2 \\ K_{avg} & d_3 \end{array} \right) \Rightarrow \left( \begin{array}{cc} d_1 & d_2 \\ K^# & d_3 \end{array} \right). \quad (4.11)$$

To assist in explaining the algorithm the matrix containing tensor $K_N$ will be referred to as $K$. All work will be done using matrix notation. This notation can be converted to vector notation, if desired. Note for the two cell problem $K$ is actually a $4 \times 4$ matrix. The indexing for $K$ will run from the upper left corner to the lower right corner. This means tensor $K_I$ occupies the portion of $K$ defined by $K(3..4,1..2)$; that is, the third and fourth rows of $K$ and the first and second columns of $K$.

The first step in the transform process is

$$\left( \begin{array}{cc} K_{N}^{III} & K_{N}^{IV} \\ K_{N}^{I} & K_{N}^{II} \end{array} \right) \Rightarrow \left( \begin{array}{cc} K^o & \hat{K} \\ \tilde{K} & \tilde{K} \end{array} \right). \quad (4.12)$$
The tensors $K^o, \tilde{K}, \tilde{\tilde{K}},$ and $\tilde{\tilde{K}}$ are used in the computation of $K^#$. This conversion is accomplished using the system of equations

\[
\begin{pmatrix}
\tilde{K} \\
K^o \\
\tilde{\tilde{K}} \\
\tilde{\tilde{K}}
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1
\end{pmatrix}
\begin{pmatrix}
K_I \\
K_{II} \\
K_{III} \\
K_{IV}
\end{pmatrix}.
\] (4.13)

Note that $\tilde{K}$ is stored in $K(3..4,1..2)$, $\tilde{\tilde{K}}$ in $K(3..4,3..4)$, $K^o$ in $K(1..2,1..2)$ and $\tilde{\tilde{K}}$ in $K(1..2,3..4)$. Additionally, observe this is the same operation used in (2.35) to produce our weak solution for the two-cell problem.

The next step is

\[
\begin{pmatrix}
K^o & \tilde{K} \\
\tilde{K} & \tilde{\tilde{K}}
\end{pmatrix} \Rightarrow \begin{pmatrix}
d_1 \\
d_2 \\
d_3
\end{pmatrix}.
\] (4.14)

First compute $K_{\text{avg}} = \frac{1}{4} \tilde{K}$. In terms of the matrix $K$, $K_{\text{avg}}$ is stored in $K(3..4,1..2)$. So the actual computations are

\[
\begin{align*}
K(3,3) & = \frac{1}{4} K(3,3) \\
K(3,4) & = \frac{1}{4} K(3,4) \\
K(4,3) & = \frac{1}{4} K(4,3) \\
K(4,4) & = \frac{1}{4} K(4,4)
\end{align*}
\] (4.15) (4.16) (4.17) (4.18)

In the averaging process details are lost about permeability relative to the coordinate directions “xx” and “yy”, as well as other directions, i.e. those denoted “xy”, which characterize anisotropic behavior. Note that if any entry in $K_{\text{avg}}$ is zero the original entries of the tensors were zero. Thus no detail will be lost in the averaging process. Assuming that none of the entries are zero we outline the method for computing the three detail tensors, $d_1, d_2,$ and $d_3$, to retain that information. The detail tensors are considered to be of the form

\[
d_i = \begin{pmatrix}
d_{ix}^i & d_{xy}^i \\
d_{xy}^i & d_{yy}^i
\end{pmatrix}
\]

with indexing similar to $K$. 
In place of $K_0$, which is located in $K(1..2,1..2)$, we put $d_1$. This tensor contains information lost about the $x$-direction. Due to assumed symmetry, only three of the locations are needed to contain this information. That is entry for $d_1(2,1)$ is the same as $d_1(1,2)$. We store the value once. This frees up the location $d_1(2,1)$ to store $k^{py}_{x}$, which contains no information relevant to the $x$-direction but is needed for reconstruction of the original tensor. The computations are defined with the notion of $d_1$ being a two by two tensor, as well as give the computations in terms of the storage in $K$.

The computations in terms of the detail tensor $d_1$ are

\begin{align*}
    d_1(2,1) &= d_1(2,2) \\
    d_1(2,2) &= \frac{1}{16 K_{avg}(1,1)}(d_1(1,2))^2 \\
    d_1(1,2) &= \frac{1}{16 K_{avg}(1,1)}d_1(1,2)d_1(1,1) \\
    d_1(1,1) &= \frac{1}{16 K_{avg}(1,1)}(d_1(1,1))^2
\end{align*}

and in terms of matrix $K$

\begin{align*}
    K(2,1) &= K(2,2) \\
    K(2,2) &= \frac{1}{16 K_{avg}(1,1)}(K(1,2))^2 \\
    K(1,2) &= \frac{1}{16 K_{avg}(1,1)}K(1,2)K(1,1) \\
    K(1,1) &= \frac{1}{16 K_{avg}(1,1)}(K(1,1))^2.
\end{align*}

Tensor $d_2$ replaces $\hat{K}$, which is located in $K(1..2,3..4)$, and contains information about the non-coordinate directions. The computations are

\begin{align*}
    d_2(1,2) &= \frac{1}{16 K_{avg}(1,2)}d_2(1,1)d_2(2,2) \\
    d_2(1,1) &= \frac{1}{16 K_{avg}(1,2)}d_2(1,1)d_2(2,1) \\
    d_2(2,2) &= \frac{1}{16 K_{avg}(1,2)}d_2(2,2)d_2(2,1) \\
    d_2(2,1) &= \frac{1}{16 K_{avg}(1,2)}(d_2(2,1))^2
\end{align*}
or

\begin{align*}
K(1, 4) &= \frac{1}{16} K(3, 2) K(1, 3) K(2, 4) \quad (4.31) \\
K(1, 3) &= \frac{1}{16} K(3, 2) K(1, 3) K(2, 3) \quad (4.32) \\
K(2, 4) &= \frac{1}{16} K(3, 2) K(2, 4) K(2, 3) \quad (4.33) \\
K(2, 3) &= \frac{1}{16} K(3, 2) (K(2, 3))^2. \quad (4.34)
\end{align*}

Finally replace \( \tilde{K} \), located in \( K(3..4, 3..4) \), with \( d_3 \) via

\begin{align*}
\begin{aligned}
d_3(2, 1) &= d_3(1, 1) \quad (4.35) \\
d_3(1, 1) &= \frac{1}{16} K_{\text{avg}}(2, 2) (d_3(1, 2))^2 \quad (4.36) \\
d_3(1, 2) &= \frac{1}{16} K_{\text{avg}}(2, 2) d_3(1, 2) d_3(2, 2) \quad (4.37) \\
d_3(2, 2) &= \frac{1}{16} K_{\text{avg}}(2, 2) (d_3(2, 2))^2 \quad (4.38)
\end{aligned}
\end{align*}

and alternatively

\begin{align*}
\begin{aligned}
K(4, 3) &= K(3, 3) \quad (4.39) \\
K(3, 3) &= \frac{1}{16} K(4, 2) (K(3, 4))^2 \quad (4.40) \\
K(3, 4) &= \frac{1}{16} K(4, 2) K(3, 4) K(4, 4) \quad (4.41) \\
K(4, 4) &= \frac{1}{16} K(4, 2) (K(4, 4))^2. \quad (4.42)
\end{aligned}
\end{align*}

The final step in the transform method is to replace \( K_{\text{avg}} \) with the homogenized tensor; i.e.

\[ \left( \begin{array}{cc} d_1 & d_2 \\ K_{\text{avg}} & d_3 \end{array} \right) \Rightarrow \left( \begin{array}{cc} d_1 & d_2 \\ K^\# & d_3 \end{array} \right). \quad (4.43) \]

Note that \( K^\# \) is computed via \( \int_V K dy + \int_V K J^T dy \) where the first term is the arithmetic average and the second term is the detail. In terms of our method the detail is \( D = (-1)d_1 + (-1)d_2 + (-1)d_3 \). \( K^\# = K_{\text{avg}} + D \) will be located in \( K(3..4, 1..2) \) and is computed
as follows

\[
K^\#(1, 1) = K_{\text{avg}}(1, 1) + \sum_{i=1}^{3} (-1)d_i(1, 1) \quad (4.44)
\]

\[
K^\#(1, 2) = K_{\text{avg}}(1, 2) + \sum_{i=1}^{3} (-1)d_i(1, 2) \quad (4.45)
\]

\[
K^\#(2, 1) = K_{\text{avg}}(2, 1) + (-1)d_1(1, 2) + (-1)d_2(2, 1) + (-1)d_3(1, 2) \quad (4.46)
\]

\[
K^\#(2, 2) = K_{\text{avg}}(2, 2) + \sum_{i=1}^{3} (-1)d_i(2, 2) \quad (4.47)
\]

and relative to the matrix \( K \) we have

\[
K(3, 1) = K(3, 1) - K(1, 1) - K(1, 3) - K(3, 3) \quad (4.48)
\]

\[
K(3, 2) = K(3, 2) - K(1, 2) - K(1, 4) - K(3, 4) \quad (4.49)
\]

\[
K(4, 1) = K(4, 1) - K(1, 2) - K(2, 3) - K(3, 4) \quad (4.50)
\]

\[
K(4, 2) = K(4, 2) - K(2, 2) - K(2, 4) - K(4, 4). \quad (4.51)
\]

4.2.2 The Inverse Transform

The inverse transform is to take matrix containing \( K^\#, d_1, d_2, \) and \( d_3 \) and overwrite it with the original tensor which was defined on the spatial domain. That is, we wish to accomplish

\[
\begin{pmatrix}
    d_1 & d_2 \\
    K^\# & d_3
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
    d_1 & d_2 \\
    K_{\text{avg}} & d_3
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
    K^o & \hat{K} \\
    \hat{K} & \hat{K}
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
    K_{N}^{III} & K_N^{IV} \\
    K_N^{III} & K_N^{IV}
\end{pmatrix}. \quad (4.52)
\]

To accomplish this we note we must essentially reverse the steps used to create \( K^\# \).

That is,

\[
\begin{pmatrix}
    d_1 & d_2 \\
    K^\# & d_3
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
    d_1 & d_2 \\
    K_{\text{avg}} & d_3
\end{pmatrix}. \quad (4.53)
\]
Since \( K^\# = K_{avg} + D \) this means that \( K_{avg} = K^\# - D \). \( K_{avg} \) is computed first because it is necessary for the reconstruction of \( K^o, \hat{K}, \tilde{K}, \) and \( \check{K} \). We compute \( K_{avg} \) via

\[
K_{avg}(1,1) = K^\#(1,1) + \sum_{i=1}^{3} d_i(1,1) \quad (4.54)
\]
\[
K_{avg}(1,2) = K^\#(1,2) + \sum_{i=1}^{3} d_i(1,2) \quad (4.55)
\]
\[
K_{avg}(2,1) = K^\#(2,1) + d_1(1,2) + d_2(2,1) + d_3(1,2) \quad (4.56)
\]
\[
K_{avg}(2,2) = K^\#(2,2) + \sum_{i=1}^{3} d_i(2,2) \quad (4.57)
\]

and relative to the matrix \( K \)

\[
K(3,1) = K(3,1) + K(1,1) + K(1,3) + K(3,3) \quad (4.58)
\]
\[
K(3,2) = K(3,2) + K(1,2) + K(1,4) + K(3,4) \quad (4.59)
\]
\[
K(4,1) = K(4,1) + K(1,2) + K(2,3) + K(3,4) \quad (4.60)
\]
\[
K(4,2) = K(4,2) + K(2,2) + K(2,4) + K(4,4). \quad (4.61)
\]

The next step is

\[
\begin{pmatrix}
    d_1 & d_2 \\
    K_{avg} & d_3
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
    K^o & \hat{K} \\
    \tilde{K} & \check{K}
\end{pmatrix}.
\quad (4.62)
\]

\( K^o \) is reconstructed from \( d_1 \) via

\[
d_1(1,1) = \sqrt{16K_{avg}(1,1)d_1(1,1)} = k_{xx}^o
\]
\[
d_1(1,2) = 16K_{avg}(1,1) \frac{d_1(1,2)}{d_1(1,1)} = k_{xy}^o
\quad (4.63)
\]
\[
d_1(2,1) = d_1(2,1) = k_{y}^o
\]
\[
d_1(2,2) = d_1(1,2) = k_{xy}^o.
\]

In terms of matrix \( K \)

\[
K(1,1) = \sqrt{16K(3,1)K(1,1)} \quad (4.64)
\]
\[
K(1,2) = 16K(3,1) \frac{K(1,2)}{K(1,1)} \quad (4.65)
\]
\[
K(2,2) = K(2,1) \quad (4.66)
\]
\[
K(2,1) = K(1,2). \quad (4.67)
\]
From $d_2$ we determine $\hat{K}$

\[
\begin{align*}
    d_2(2,1) & = \sqrt{16K_{avg}(1,2)d_2(2,1)} = \hat{k}_{xy} \\
    d_2(2,2) & = 16K_{avg}(1,2)\frac{d_2(2,2)}{d_2(2,1)} = \hat{k}_{yy} \\
    d_2(1,1) & = 16K_{avg}(1,2)\frac{d_2(1,1)}{d_2(2,1)} = \hat{k}_{xx} \\
    d_2(1,2) & = d_2(2,1) = \hat{k}_{xy}
\end{align*}
\]

or

\[
\begin{align*}
    K(2,3) & = \sqrt{16K(3,2)K(2,3)} \\
    K(2,4) & = 16K(3,2)\frac{K(2,4)}{K(2,3)} \\
    K(1,3) & = 16K(3,2)\frac{K(1,3)}{K(2,3)} \\
    K(1,4) & = K(2,3)
\end{align*}
\] (4.68)

Next $d_3$ is used to compute $\tilde{K}$

\[
\begin{align*}
    d_3(2,2) & = \sqrt{16K_{avg}(2,2)d_3(2,2)} = \tilde{k}_{yy} \\
    d_3(1,2) & = 16K_{avg}(2,2)\frac{d_3(1,2)}{d_3(2,2)} = \tilde{k}_{xy} \\
    d_3(1,1) & = d_3(2,1) = \tilde{k}_{xx} \\
    d_3(2,1) & = d_3(1,2) = \tilde{k}_{xy}
\end{align*}
\]

and alternatively

\[
\begin{align*}
    K(4,4) & = \sqrt{16K(4,2)K(4,4)} \\
    K(3,4) & = 16K(4,2)\frac{K(3,4)}{K(4,4)} \\
    K(3,3) & = K(4,3) \\
    K(4,3) & = K(3,4).
\end{align*}
\] (4.73)

Finally, we compute $\bar{K}$ from $K_{avg}$ using the fact $K_{avg} = \frac{1}{4}k$. That is,

\[
\begin{align*}
    \bar{K}(1,1) & = 4K_{avg}(1,1) = \bar{k}_{xx} \\
    \bar{K}(1,2) & = 4K_{avg}(1,2) = \bar{k}_{xy} \\
    \bar{K}(2,1) & = 4K_{avg}(2,1) = \bar{k}_{xy} \\
    \bar{K}(2,2) & = 4K_{avg}(2,2) = \bar{k}_{yy}.
\end{align*}
\] (4.74)
or

\[
K(3, 3) = 4K(3, 3) \quad (4.79) \\
K(3, 4) = 4K(3, 4) \quad (4.80) \\
K(4, 3) = 4K(4, 3) \quad (4.81) \\
K(4, 4) = 4K(4, 4). \quad (4.82)
\]

The final step of the transform is

\[
\begin{pmatrix}
K^o \hat{K} \\
\hat{K} \hat{K}
\end{pmatrix} \Rightarrow
\begin{pmatrix}
K_{III} & K_{IV} \\
K_I & K_{II}
\end{pmatrix} \quad (4.83)
\]

This transformation back to the spatial domain is completed by computing

\[
\begin{pmatrix}
K_I \\
K_{II} \\
K_{III} \\
K_{IV}
\end{pmatrix} = \frac{1}{4}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
K^o \\
\hat{K}
\end{pmatrix}. \quad (4.84)
\]

4.2.2.1 A Simple Example: Stratified Coefficient Case

So the results of this method may be compared with the periodic results in two dimensions we assume the values

\[
K_{xx}^I = K_{xx}^{III} = a = K_{yy}^I = K_{yy}^{III} \\
K_{xx}^{II} = K_{yy}^{IV} = b = K_{yy}^{II} = K_{yy}^{IV} \quad (4.85)
\]

representing isotropic stratification in the x-direction. An important observation is that since the original tensor is isotropic and stratified in the x-direction we would expect the detail lost in the averaging process for the y-direction to be zero, as well as \(d_2\) being zero.

The transform begins with the matrix

\[
K = \begin{pmatrix}
a & 0 & b & 0 \\
0 & a & 0 & b \\
a & 0 & b & 0 \\
0 & a & 0 & b
\end{pmatrix}. \quad (4.86)
\]
K is transformed using (4.13) to obtain

\[
K = \begin{pmatrix}
2a - 2b & 0 & 0 & 0 \\
0 & 2a - 2b & 0 & 0 \\
2a + 2b & 0 & 0 & 0 \\
0 & 2a + 2b & 0 & 0
\end{pmatrix}.
\] (4.87)

K now contains \( K_0 \) in \( K(1..2, 1..2) \), \( \tilde{K} \) in \( K(3..4, 1..2) \), \( \tilde{K} \) in \( K(1..2, 3..4) \) and \( \tilde{K} \) in \( K(3..4, 3..4) \). From these components we compute the tensors \( K_{avg}, d_1, d_2, \) and \( d_3 \). To compute \( K_{avg} \) we divide the entries pertaining to \( \tilde{K} \) by 4. Using (4.19) - (4.35) we compute the detail signals

\[
d_1 = \begin{pmatrix}
\frac{(a-b)^2}{2(a+b)} & 0 \\
\frac{(a-b)^2}{2(a+b)} & 0
\end{pmatrix},
\] (4.88)

\[
d_2 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\] (4.89)

and

\[
d_3 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\] (4.90)

As noted in the outline of the transform these are computed and stored in place, thus the current form of the matrix is

\[
K = \begin{pmatrix}
\frac{(a-b)^2}{2(a+b)} & 0 & 0 & 0 \\
\frac{(a-b)^2}{2(a+b)} & 0 & 0 & 0 \\
2a - 2b & 0 & 0 & 0 \\
2a - 2b & 0 & 0 & 0
\end{pmatrix}.
\] (4.91)

As previously observed, \( d_2 \) and \( d_3 \) are zero.

The transform is complete once \( K^\# \) is computed via (4.47); i.e.

\[
K = \begin{pmatrix}
\frac{(a-b)^2}{2(a+b)} & 0 & 0 & 0 \\
\frac{(a-b)^2}{2(a+b)} & 0 & 0 & 0 \\
2a - 2b & 0 & 0 & 0 \\
2ab & 0 & 0 & 0
\end{pmatrix}.
\] (4.92)

Note that \( K^\# \) is located at \( K(3..4, 1..2) \) with \( K^\#_{xx} \) containing the harmonic average and \( K^\#_{yy} \) containing the arithmetic average as expected. The original tensor \( K \) can be reconstructed in place by reversing the process.
4.2.3 Convergence Properties

The means of sample $K^e$ previously outlined is similar to the method used in one dimension. In two dimensions $K_m(x)$ is a step function in two dimensions created by sampling $K^e(x)$ on squares determined by a dyadic grid. These squares are analogous to the intervals in one dimension. It is clear to see $K_m \rightarrow K^e$ in $L^2([0, 1])^2$. Recall that this was an important component in establishing $K_m \overset{C}{\rightarrow} K^e$ in one dimension.

Another important piece of information in establishing $G$-convergence was the convergence of the forcing function. If we sample $f^e$ using a step function; i.e., $f_m(x) = \sum_{i=1}^{2^N} f_{m,i} R_{m,i}$ where $f_{m,i}$ is a sample of $f^e(x)$ on $R_{m,i}$, we know that $f_m \rightarrow f^e$ in $L^2$. Since the methods of sampling used in two dimensions are analogous to those used in one dimension the proofs presented in the one dimensional case are easily extended to the two dimensional case.

4.2.4 Numerical Illustration

Uy we explore the accuracy of this method by comparing the results obtained via our two cell wavelet transform to the results of benchmark problems. It is important to verify that our method is producing reasonable results. To accomplish this we will compare the results generated by our method to the results of standard problems that have been computed and documented. The two standard test problems are the periodic symmetric cell problem and the periodic inverted L cell problem.

To establish whether the results generated by the transform are reasonable we need the following theorem.

**Theorem 4.2.1** [6, p. 8] Assume $K^e(x) \in M(\alpha, \beta, \Omega)$. Let $N$ be the number of partitions of $\Omega$. Then the following are true:

1) $K^# \in M(\alpha, \beta, \Omega)$

2) For $m = 1, ..., N$, we have the estimate

$$K_{-1,\Omega_m} \leq K^# \leq K_{1,\Omega_m}$$
Table 4.3: The table represents the entries in an $8 \times 8$ array for the Symmetric Cell problem before the two-dimensional wavelet transform is applied.

\[
\begin{array}{cccccccc}
10 & 0 & 10 & 0 & 10 & 0 & 10 & 0 \\
0 & 10 & 0 & 10 & 0 & 10 & 0 & 10 \\
10 & 0 & 1 & 0 & 1 & 0 & 10 & 0 \\
0 & 10 & 0 & 1 & 0 & 10 & 0 & 10 \\
10 & 0 & 1 & 0 & 1 & 0 & 10 & 0 \\
0 & 10 & 0 & 1 & 0 & 10 & 0 & 10 \\
10 & 0 & 10 & 0 & 10 & 0 & 10 & 0 \\
0 & 10 & 0 & 10 & 0 & 10 & 0 & 10 \\
\end{array}
\]

where $K_{-1,\Omega_m}$ denotes the harmonic average, i.e.

\[
K_{-1,\Omega_m} = \left( \frac{1}{\|\Omega_m\|} \int_{\Omega_m} K^{-1}(x) dx \right)^{-1}
\]

and $K_{1,\Omega_m}$ denotes the arithmetic average

\[
K_{1,\Omega_m} = \frac{1}{\|\Omega_m\|} \int_{\Omega_m} K^e(x) dx
\]

The theorem establishes the homogenized tensor should lie between the values of the arithmetic average and the harmonic average of $K^e(x)$. This provides bounds for the homogenized tensor useful in determining the validity of our method.

In testing the transform we will consider a region which is a $4 \times 4$ grid containing 16 cells. The cells will be populated by tensors of type I

\[
K^e(x) = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}
\]  

(4.93)

or type II

\[
K^e(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]  

(4.94)

The transform will be run on the region and the results analyzed.
Table 4.4: The entries in this table represent the 8x8 array for the Symmetric Cell problem after the two-dimensional wavelet transform applied to the array in Table (4.3).

\[
\begin{array}{cccc}
0.653 & 0 & 0 & 0.653 \\
9.000 & 0 & 0 & 0 \\
7.097 & 0 & 0 & 7.097 \\
0 & 7.097 & 9 & 0.653 \\
0.653 & 0 & 0 & 0.653 \\
9.000 & 0 & 0 & 0 \\
7.097 & 0 & 0 & 7.097 \\
0 & 7.097 & -9 & 0.653 \\
\end{array}
\]

4.2.4.1 Symmetric Cell Problem

For the symmetric cell problem we populated the grid, using a I or II to indicate the tensor used, in the following configuration

\[
\begin{array}{cccc}
I & I & I & I \\
I & II & II & I \\
I & II & II & I \\
I & I & I & I \\
\end{array}
\]

(4.96)

In terms of computational storage this represents an 8x8 array.

The array begins with entries indicated in Table (4.3). Recall the homogenized tensor is located in the cell at the lower left corner of the array. The other entries in the array are used to reconstruct the original tensor. Our method produces an homogenized tensor containing the values

\[
K^# = \begin{pmatrix} 7.09677 & 0 \\ 0 & 7.09677 \end{pmatrix}.
\]

(4.98)

The following averages were computed to have bounds against which to compare our results. First we compute the arithmetic average, which is an upper bound for an homogenized tensor, to find

\[
K^A_# = \begin{pmatrix} 7.75 & 0 \\ 0 & 7.75 \end{pmatrix}.
\]

(4.99)

Next compute the harmonic average, the lower bound for \( K^# \),

\[
K^H_# = \begin{pmatrix} 3.0769 & 0 \\ 0 & 3.0769 \end{pmatrix}.
\]

(4.100)
Table 4.5: The table represents the entries in an 8x8 array for the Inverted-L problem before the two-dimensional wavelet transform is applied.

\[
\begin{array}{cccccccc}
10 & 0 & 10 & 0 & 10 & 0 & 10 & 0 \\
0 & 10 & 0 & 10 & 0 & 10 & 0 & 10 \\
10 & 0 & 1 & 0 & 1 & 0 & 10 & 0 \\
0 & 10 & 0 & 1 & 0 & 1 & 0 & 10 \\
10 & 0 & 10 & 0 & 1 & 0 & 10 & 0 \\
0 & 10 & 0 & 10 & 0 & 1 & 0 & 10 \\
10 & 0 & 10 & 0 & 1 & 0 & 10 & 0 \\
0 & 10 & 0 & 10 & 0 & 1 & 0 & 10 \\
\end{array}
\] (4.102)

Note that our results lie within the appropriate bounds. One final comparison can be made to an homogenized tensor produced using a more traditional method, periodic conforming finite element method [19],

\[
K_{CFE}^\# = \begin{pmatrix} 6.65 & 0 \\ 0 & 6.65 \end{pmatrix}.
\] (4.101)

By way of comparison the values returned for $K_{xx}^\#$ and $K_{yy}^\#$ vary from the same components in the conforming finite element method by approximately 0.45. It appears the two-dimensional wavelet transform produces reasonable results in this benchmark case.

4.2.4.2 Inverted-L Problem

For inverted-L problem we populated the grid, using tensors of type I or II, in the following configuration

\[
\begin{array}{cccc}
I & I & I & I \\
I & II & II & I \\
I & I & II & I \\
I & I & II & I \\
\end{array}
\] (4.103)

The array begins with entries indicated in Table (4.5). The homogenized tensor located in the cell at the lower left corner of the array is

\[
K^\# = \begin{pmatrix} 5.8595 & 0 \\ 0 & 7.4090 \end{pmatrix}.
\] (4.105)
Table 4.6: The entries in this table represent the 8x8 array for the Inveted-L problem after the two-dimensional wavelet transform applied to the array in Table (4.5).

\[
\begin{array}{cccc|cccc}
0.653 & 0 & 0.653 & 0 & 0 & 0 & 0 & 0 \\
9.000 & 0 & -9.00 & 0 & 0 & 0 & 0 & 0 \\
0.485 & 0 & 0 & 0 & 0 & 0 & 0.653 & 0 \\
4.500 & 0 & 0 & 0 & 9.000 & 0 & 0 & 0.653 \\
0 & 0 & 0 & 0 & 3.68 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -18.0 & 0 & 0 & 0 \\
5.8595 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7.4090 & 0 & 0 & 8.82 & 0.015 & 0 & 0 \\
\end{array}
\]

(4.104)

The arithmetic average of this problem is

\[
K_A^\# = \begin{pmatrix}
7.75 & 0 \\
0 & 7.75
\end{pmatrix},
\]

while the harmonic average

\[
K_H^\# = \begin{pmatrix}
3.0769 & 0 \\
0 & 3.0769
\end{pmatrix}.
\]

One last tensor to use for comparison is that produced by a periodic conforming finite element method ([19]),

\[
K_{CFE}^\# = \begin{pmatrix}
5.5741 & 0 \\
0 & 6.9488
\end{pmatrix}.
\]

(4.107)

Note that again our homogenized tensor lies within the appropriate bounds. Additionally, comparing \(K^\#\) to \(K_{CFE}^\#\) the values returned for \(K_{xx}^\#\) and \(K_{yy}^\#\) vary from the same components in the conforming finite element method by approximately 0.29 and 0.46, respectively. Again reasonable results in a benchmark case have been obtained using the wavelet transform method.

4.2.4.3 ELF Data Problem

The ELF Data test problem is a relatively simple, yet realistic test for homogenization methods [19]. The configuration of the problem for this case is based on a 30 by 30 grid. The data contains the permeability tensor for 900 cells. The test involves using the 30 by 30 grid to create a 6 by 6 homogenized grid for the porous medium.
Table 4.7: This table contains the arithmetic average, the harmonic average and our wavelet average of $K_{xx}$ for each cell in the 6 by 6 grid generated from the original 30 by 30 grid.

$K_{xx}$ Arithmetic Average

<table>
<thead>
<tr>
<th></th>
<th>10.00</th>
<th>8.42</th>
<th>1.68</th>
<th>0.10</th>
<th>0.64</th>
<th>7.84</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5.97</td>
<td>9.21</td>
<td>0.10</td>
<td>0.10</td>
<td>0.50</td>
<td>7.84</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>6.40</td>
<td>2.94</td>
<td>0.10</td>
<td>4.06</td>
<td>4.85</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>3.20</td>
<td>7.84</td>
<td>0.75</td>
<td>8.42</td>
<td>1.68</td>
</tr>
<tr>
<td></td>
<td>0.96</td>
<td>0.46</td>
<td>9.28</td>
<td>10.00</td>
<td>8.42</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>0.35</td>
<td>0.89</td>
<td>6.04</td>
<td>10.00</td>
<td>4.92</td>
<td>0.32</td>
</tr>
</tbody>
</table>

$K_{xx}$ Wavelet Average

<table>
<thead>
<tr>
<th></th>
<th>10.00</th>
<th>7.85</th>
<th>1.57</th>
<th>0.31</th>
<th>0.78</th>
<th>6.53</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5.12</td>
<td>8.96</td>
<td>0.31</td>
<td>0.31</td>
<td>0.57</td>
<td>6.71</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>4.93</td>
<td>1.64</td>
<td>0.31</td>
<td>2.28</td>
<td>3.05</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>1.83</td>
<td>6.79</td>
<td>0.78</td>
<td>7.36</td>
<td>0.57</td>
</tr>
<tr>
<td></td>
<td>1.21</td>
<td>0.40</td>
<td>0.15</td>
<td>10.00</td>
<td>7.36</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>0.57</td>
<td>1.04</td>
<td>4.83</td>
<td>10.00</td>
<td>3.57</td>
<td>0.49</td>
</tr>
</tbody>
</table>

$K_{xx}$ Harmonic Average

<table>
<thead>
<tr>
<th></th>
<th>10.00</th>
<th>0.59</th>
<th>0.012</th>
<th>0.10</th>
<th>0.22</th>
<th>3.16</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.82</td>
<td>1.12</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>3.16</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>2.17</td>
<td>0.15</td>
<td>0.10</td>
<td>0.17</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.25</td>
<td>3.16</td>
<td>0.14</td>
<td>0.59</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td>0.74</td>
<td>0.16</td>
<td>5.81</td>
<td>10.00</td>
<td>0.59</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>0.13</td>
<td>0.48</td>
<td>2.02</td>
<td>10.00</td>
<td>1.04</td>
<td>0.13</td>
</tr>
</tbody>
</table>

As before the results of our method should lie between the arithmetic and harmonic averages for the region. Table (4.7) contains the arithmetic average, the harmonic average and our wavelet based average of $K_{xx}$ for each cell in the 6 by 6 grid. Now we turn our attention to Table (4.8) containing the arithmetic average, the harmonic average and our wavelet based average of $K_{yy}$ for each cell in the 6 by 6 grid. The entries for $K_{xy}$ are zero’s and therefore the arithmetic average is zero for each cell on the 6 by 6 grid. If we compare the averages for corresponding cells note our wavelet average satisfies the appropriate inequalities and therefore has produced reasonable results.
Table 4.8: This table contains the arithmetic average, the harmonic average and our wavelet average for each cell in the 6 by 6 grid generated from the original 30 by 30 grid.

$K_{yy}$ Arithmetic Average

<table>
<thead>
<tr>
<th></th>
<th>10.00</th>
<th>8.42</th>
<th>1.68</th>
<th>0.10</th>
<th>0.64</th>
<th>7.84</th>
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<td>5.97</td>
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<td>4.85</td>
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<tr>
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<td>3.20</td>
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<td>0.75</td>
<td>8.42</td>
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<tr>
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<td>8.42</td>
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<td>0.35</td>
<td>0.89</td>
<td>6.04</td>
<td>10.00</td>
<td>4.92</td>
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</table>

$K_{yy}$ Wavelet Average

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<thead>
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<th>10.00</th>
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<th>1.13</th>
<th>0.31</th>
<th>0.78</th>
<th>7.71</th>
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<td>0.31</td>
<td>0.57</td>
<td>7.71</td>
</tr>
<tr>
<td></td>
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<td>2.75</td>
<td>0.31</td>
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<td>4.76</td>
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<tr>
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<td>6.79</td>
<td>0.78</td>
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<tr>
<td></td>
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<td>5.85</td>
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<td>4.67</td>
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$K_{yy}$ Harmonic Average

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<tr>
<th></th>
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<th>0.012</th>
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<th>0.22</th>
<th>3.16</th>
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<tr>
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<tr>
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</tr>
<tr>
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CHAPTER 5

CONCLUSION

In an attempt to examine fluid flow in a region $\Omega$, we set out to create a computationally efficient and accurate wavelet-based scheme for computing an homogenized permeability tensor $K^#$ on the macroscopic scale to represent permeability at the fine scale. Recall the computation of this tensor involved determining a weak solution to the local problem. Traditional methods for computing the weak solution include finite element methods. Once the weak solution was determined the transpose of the Jacobian Matrix was used in the computation of the tensor $K^#$.

In this paper approached the problem from a different perspective. We defined our local problem in terms of a two cell problem in one, two, or three dimensions. We then derived weak solutions to the local problem in one and two dimensions. The framework for the solution in three dimensions was established but not derived. A concise representation of the of this weak solution still needs to be developed.

In one and two dimensions the Jacobian transpose of the weak solutions were crucial in the development a scheme for computing the homogenized tensor. Note that our scheme was based on the lifting scheme utilizing the Haar basis. Once the form of the transpose, for the two-cell problem, was determined the homogenized tensor could be computed. From that homogenized tensor a lifting scheme was developed to determine $K^#$. This still must be done for the two-cell problem in three dimensions.

Note our method is different from more traditional methods in that we consider a two-cell problem at the microscopic scale. Our solutions were based on assumptions that allowed us to construct a basis. A weak solution to the local problem could then be computed via this basis for an arbitrary tensor $K^t(x)$. This eliminates the necessity of solving the local problem for each subregion in the domain. This arbitrary weak solution
is used in the computation of $K^\#$, the homogenized tensor. Our assumptions dictated the appropriate wavelet-basis was the Haar basis. As we saw our method generated reasonable homogenized tensors. At this point we should consider changing some of our assumptions. This would allow consideration of other wavelet bases. A comparison of the efficiency and accuracy the method relative to the wavelet-basis would be important information.
REFERENCES


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(480) E-Mail: laura.watkins@.edu

Education

<table>
<thead>
<tr>
<th>Degree</th>
<th>Institution</th>
<th>Date of Graduation</th>
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<tr>
<td>B.S., Mathematics</td>
<td>Utah State University</td>
<td>June 1995</td>
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<tr>
<td>Minors: Physics, Computer Science</td>
<td>Logan, UT</td>
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<tr>
<td>M.S., Mathematics</td>
<td>Utah State University</td>
<td>March 1998</td>
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<tr>
<td>Ph.D., Mathematical Sciences</td>
<td>Utah State University</td>
<td>Summer 2005</td>
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Awards

2004–2005 Project ACCCESS Fellow, Jointly Sponsored by AMATYC and MAA
April 2002 Graduate Student Teacher Award, Department of Mathematics and Statistics, Utah State University
April 2000 Graduate Student Teacher Award, Department of Mathematics and Statistics, Utah State University
May 1997 Graduate Student Teacher Award, Department of Mathematics and Statistics, Utah State University

Teaching Experience

August 2002–Current **Mathematics Instructor**, Department of Mathematics, Glendale Community College, Responsibilities: Complete responsibility for a variety of courses from the remedial level to Differential Equations. This included preparing lectures, classroom activities, holding office hours, writing and grading exams and assigning grades at the end of the semester. Serving on committees as assigned.
August 1999-August 2002 Teaching Assistant, Department of Mathematics and Statistics, Utah State University, Responsibilities: Complete responsibility for courses in Calculus, Business Calculus, and Introduction to Logic and Geometry. This included preparing lectures, holding office hours, writing and grading quizzes and exams, and assigning grades at the end of the semester.

October 1997-March 1998 Teaching Assistant, Department of Mathematics and Statistics, Utah State University, Mathematics and Statistics, Utah State University, Responsibilities: Complete responsibility for courses in Calculus. This included preparing lectures, holding office hours, writing and grading quizzes and exams, and assigning grades at the end of the semester.

June 1995-June 1997 Teaching Assistant, Department of Mathematics and Statistics, Utah State University, Responsibilities: Complete responsibility for courses in Introduction to Algebra, College Algebra and Calculus. This included preparing lectures, holding office hours, writing and grading quizzes and exams, and assigning grades at the end of the semester.

Related Experience

June 1997-August 1997, June 1999-August 1999 Research Experience for Undergraduates (REU) Mentor, Department of Mathematics and Statistics, Utah State University, Responsibilities: Mentor undergraduate students working on research projects in non-linear dynamical systems. Duties included meeting with students to assist in assessing progress, advise students, and proofread reports written to illustrate their work.

March 1998-May 1999 Research Assistant, Department of Mathematics and Statistics, Utah State University, Responsibilities: The research was done for Thiokol Corporation. I worked on determining complete solutions to the Navier equation for spherical particles using vector spherical harmonics. This work was part of a larger project to develop a constitutive theory for solid rocket fuel.
Publications/Papers


Talks


"Making a Connection Between Homogenization and Wavelets", at the Industrial Mathematics Seminar, Utah State University, October 29, 2001.

"Methods for Characterization of Fluvial-Deltaic Reservoirs", at the Spring Meeting of the Intermountain Section of the MAA, Logan, Utah, April 12, 1997.

"Characterization of Fluvial-Deltaic Reservoirs with Archetypes", at the Graduate Student Club of the Department of Mathematics and Statistics, Utah State University, February 19, 1997.

Research

A computer code provided by Mobil Oil Company was modified to generate realizations of permeability values. This has been done as part of a project with the Utah Geologic Survey. The project involves characterization of sedimentary bedforms typically found in hydrocarbon reservoirs.
As part of a project to develop a constitutive theory for solid rocket fuel, I determined complete solutions to the Navier equation for spherical particles using vector spherical harmonics.

My current research lies in using wavelets as a means of characterizing the solution to a second order elliptic partial differential equation.

**Professional Memberships**

American Mathematical Society (AMS)

Mathematical Association of America (MAA)

American Mathematics Association of Two Year Colleges (AMATYC)