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Minimal Nodal Domains for Strictly Elliptic Partial Differential Equations with Homogeneous Boundary Conditions

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MINIMAL NODAL DOMAINS FOR STRICTLY ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH HOMOGENEOUS BOUNDARY CONDITIONS

by

Charles E. Miller

A dissertation submitted in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematical Sciences

UTAH STATE UNIVERSITY
Logan, Utah

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ABSTRACT

Minimal Nodal Domains for Strictly Elliptic Partial Differential Equations with Homogeneous Boundary Conditions

by

Charles E. Miller, Doctor of Philosophy
Utah State University, 2006

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Department: Mathematics and Statistics

This work presents a proof of the dependence of the first eigenvalue for uniformly elliptic partial differential equations on the domain in a less abstract setting than that of Ivo Babuška and Rudolf Výborný in 1965. The proof contained here, under rather mild conditions on the boundary of the domain, \( \partial \Omega \), demonstrates that the first eigenvalue of elliptic partial differential equation

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

depends continuously on the domain in the following sense. If a sequence of domains is such that \( \Omega_i \to \Omega \) in \( \mathbb{R}^n \), then the corresponding first eigenvalues satisfy \( \lambda_i \to \lambda \) and \( \lambda \) is the first eigenvalue for

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

The work also reviews and utilizes the Sturmian comparison results of John G. Heywood, E. S. Nossair, and Charles A. Swanson. For a continuously parameterized family of domains, say \( \Omega_\mu \) with \( \mu \in I = [a, b] \), the continuous dependence of the eigenvalue on the domain combined with the Sturmian comparison results provide a theorem that insures,
under certain conditions, that the elliptic partial differential equation

\[
\begin{align*}
 Lu &= 0 \quad \text{in} \quad \Omega \\
 u &= 0 \quad \text{on} \quad \partial\Omega
\end{align*}
\]

has a solution which is positive on a nodal domain. That is, there is a least value of \(\mu \in [a, b]\) so that a positive solution \(u\) exists for

\[
\begin{align*}
 Lu &= 0 \quad \text{in} \quad \Omega_\mu \\
 u &= 0 \quad \text{on} \quad \partial\Omega_\mu
\end{align*}
\]

Beyond these results, the work contains a theorem that shows for certain types of domains, rectangles in \(\mathbb{R}^2\), among them, that there is a critical dimension smaller than which, no solution to the problem

\[
\begin{align*}
 Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
 u &= 0 \quad \text{on} \quad \partial\Omega
\end{align*}
\]

exists when the eigenvalue is fixed.

During the investigations taken up in this work, certain observations were made regarding linear approximations to eigenvalue problems in \(\mathbb{R}^2\) using a standard numerical approximation scheme. One such observation is that if a linear approximation to an eigenvalue problem contains an incorrect estimate for an eigenvalue, the resulting graphical approximation seems to betray whether or not the estimate was low or high. The observations made do not appear to exist in the literature.
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Charles E. Miller
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ON THE INTER-RELATIONSHIP BETWEEN DOMAINS AND THE FIRST EIGENVALUES OF SECOND ORDER, LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

1.1 Introduction

This paper consists of results of work done between the Fall of 2001 and 2006 under the guidance of Drs. Jerry Ridenhour and Z.-Q. Wang. The work was begun while Dr. Ridenhour was at Utah State University, and continued after his appointment as the Chairman of the Department of Mathematics and Statistics the University of Northern Iowa in 2003. The final stages of the research were overseen by Dr. Wang, here at Utah State University.

We explored the dependence of the first eigenvalue for solutions to homogeneous elliptic partial differential equations with homogeneous boundary conditions on the domain in light of certain results of Heywood, Noussair, and Swanson presented in their paper, "On the Zeros of Solutions of Elliptic Inequalities in Bounded Domains," [HNS], published in 1978. A proof of the continuous dependence of eigenvalues of elliptic partial differential equations on their domains has existed since at least 1965, where Ivo Babuška and Rudolf Výborný presented the result in their paper, "Continuous Dependence of Eigenvalues on the Domain," [BV]. We feel that our work enhances the earlier results just mentioned by providing a different proof for the continuous dependence of the first eigenvalue on the domain. The proof of Babuška and Výborný is based on properties of sequences of Hilbert spaces and is rather abstract in nature, while our proof deals with some of the intricacies of domain boundary regularity and smoothness of the coefficient functions of the partial differential equation. Initially our results along these lines were directly tied to the Heywood, Noussair, and Swanson results, but our final version of the proof of the continuous dependence of the first eigenvalue on the domain turns out to be independent of their results. Nevertheless, if solutions to various elliptic differential inequalities are of sufficient smoothness, the results of Heywood, Noussair and Swanson coupled with the continuous dependence of the first eigenvalue on the domain provide a rather general theorem whose conclusion is that there
exists a domain on which the elliptic problem

\[
\begin{align*}
L u + \lambda u &= 0 \quad \text{in} \quad \Omega \\
\quad u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

where \( \lambda = 0 \) has a solution of single sign. The results of this work appear in the first section of this document.

While investigating the continuous dependence of the first eigenvalue on the domain, we employed a standard linear approximation model to obtain graphical approximations to solutions to elliptic problems on fairly general domains in \( \mathbb{R}^2 \). We made some observations regarding eigenvalues that seemed interesting to us and we present, rather heuristically, some of these observations in Appendix C. We are not sure whether or not any of these observations appear in the literature.

Linear uniformly elliptic boundary value problems are ubiquitous throughout the literature in both mathematics and physics. In particular, Laplace’s equation

\[
\begin{align*}
\Delta u &= 0 \quad \text{in} \quad \Omega \\
\quad u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

and the associated eigenvalue problem

\[
\begin{align*}
\Delta u + \lambda u &= 0 \quad \text{in} \quad \Omega \\
\quad u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

appear directly in static or steady state problems where gradient of an unknown potential energy function gives rise to a field. Laplace’s equation is the backbone for some of the most basic problems in field theory. Laplace’s equation governs the potential function in electrostatics and magnetostatics and in gravitational problems as well. It governs the temperature distribution in solids at thermal equilibrium and the steady state current flow in solid conductors; see [Pipes], page 471. In the more general elliptic equation, the highest order term represents the diffusion of the unknown density, \( u \), (say a temperature, or a
chemical concentration) into the domain $\Omega$ see [Evans], page 295 for example. The eigenvalue problem for Laplace's equation gives rise to complete sets of orthonormal functions, i.e., Fourier series, that give series solutions, and thus approximate solutions, in a variety of coordinate systems. Our primary goal is not to study elliptic eigenvalue problems in their own right, but rather to use the eigenvalue problems to help us determine when the problem

$$\begin{cases}
Lu = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}$$

has a single signed solution on a fixed domain for a particular elliptic operator $L$.

1.2 Overview

In this section, we seek to explore the inter-relationship between domains, the first eigenvalues and their corresponding first eigenfunctions for problems of the form:

$$\begin{cases}
Lu = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases},$$

where $L$ is a linear, second order, uniformly elliptic, partial differential operator. For example,

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u$$

or

$$Lu = \sum_{i,j} D_i (a_{ij}(x) D_j u) + \sum_{i} (D_i (b_i(x) u) + c_i(x) D_i u) + d(x) u$$

in divergence form. Specifically, we will consider the following three partial differential equations, the first two being special cases of the third, where $\lambda$ is an eigenvalue:

$$\begin{cases}
\Delta u + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}; \quad \text{(pde1)}$$
\[
\begin{aligned}
\begin{cases}
Lu + \lambda u &= \Delta u + (c(x) + \lambda) u = 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
\begin{cases}
Lu + \lambda u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{cases}
\end{aligned}
\]

We will prove that for fairly general domains and relatively weak restrictions on the coefficient functions of the differential operator, \( L \), above, that the first eigenvalue is continuously dependent on the domain. The exact sense of this continuous dependence will be carefully spelled out as we proceed. Just such a general result was published in 1965, by Babuška and Výborný, ([BV]) in a setting of minimizers for stable sequences of Hilbert spaces. The proof that we will present is somewhat more concrete than the proof that they provided and certainly makes more obvious the conditions which the domains must satisfy.

In 1978, Heywood, Noussair, and Swanson, published three variants of a Sturmian type comparison theorem that applies to elliptic partial differential equations, not necessarily linear, and requires no regularity conditions on the boundary. The Sturmian comparison theorems of Heywood, Noussair, and Swanson do not directly apply to the so-called weak solutions of the partial differential equations given above. We will employ some regularity theory to show that under fairly broad and reasonable assumptions on the domains and coefficients of the partial differential equations, the hypotheses for the comparison theorems of Heywood, Noussair, and Swanson are satisfied and hence their theorems may be applied. We will review the comparison theorems of Heywood, Noussair, and Swanson and investigate the situations under which their theorems may be applied to solutions of linear, second order, uniformly elliptic partial differential equations (often, henceforth, abbreviated pde, or pdes, in the plural case).

After obtaining and reviewing the aforementioned results, we will consider partial differential eigenvalue equations on a domain whose boundary is continuously parameterized (or partially parameterized) by a parameter, say \( \mu \), that may take on certain eigenvalues, \( \lambda \), dependent on \( \mu \). We will use both of the results obtained above to investigate some
conditions under which we can say that the partial differential equation

\[
\begin{cases}
Lu = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

has a positive solution; i.e., when \( \lambda = 0 \) is the first eigenvalue of

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

In a sense to be clarified as we proceed, we will show that there is a minimal nodal domain, \( \Omega \), for the problem

\[
\begin{cases}
Lu = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

Following this, we will investigate the application of these results to partial differential equations on two dimensional domains where solutions are known explicitly. We will consider rectangular domains, circular domains, and sector domains for \( pde_1 \) and then generalize the results to both \( pde_2 \) and \( pde_3 \), where solutions are not usually known. We will continue the investigation by considering the same pdes on some domains in three dimensions: rectangular parallelepiped or "shoe-box" domains, cylindrical, spherical, and "cheese shaped" domains. The results obtained will allow us to compare first eigenvalues for the same differential equation on various domains with specific regard to the geometry of the domains and this will be discussed. We will consider the given simpler forms of the third partial differential equation on special domains, \( \Omega \), in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), in particular, and then generalize the results to more arbitrary domains of unspecified dimension. These simpler cases will, hopefully, provide insight into more complicated situations.

Various related proofs and alternate results appear in the appendices which may be found at the end of this dissertation. Broadly, the contents of the appendices consist of regularity based proofs for the continuous dependence of the first eigenvalue on the domain, proofs for the strict monotonicity of the first eigenvalue with respect to the domain and
observations pertaining to numerical approximations of solutions to eigenvalue problems.

1.3 Notation and Some General Results from the Literature

As notation for partial differential equations varies rather wildly throughout the literature, we will review much of the notation that will be used in this work. We will also make reference to certain results in the basic literature that will be used to develop our results. We will begin with notation for functions.

1.3.1 Functions and Sets

We will consider functions defined on all, or part of \( \mathbb{R}^n \) and denote such a function, say, \( u \), by any of

\[
    u = u(x) = u(x_1, x_2, \ldots, x_n).
\]

For emphasis or occasionally for clarity, we will include a reference to the independent variables, but more often than not, we will assume that this is understood. To denote a vector quantity, \( v \), in \( \mathbb{R}^n \), we will occasionally employ the component notation

\[
    v = [v_1, v_2, \ldots, v_n].
\]

For vector functions we will let

\[
    v(x) = [v_1(x), v_2(x), \ldots, v_n(x)]
    = [v_1(x_1, x_2, \ldots, x_n), v_2(x_1, x_2, \ldots, x_n), \ldots, v_n(x_1, x_2, \ldots, x_n)]
\]

when it is useful to be explicit. A normalized (unit) vector quantity, \( v \), may be denoted by the symbol \( \hat{v} \).

Throughout our discussions we will want to denote open or closed neighborhoods of particular points in \( \mathbb{R}^n \). We will use the notation \( B(r, x_0) \) to signify the open ball of radius \( r \) about the point \( x_0 \). In a similar fashion, \( \bar{B}(r, x_0) \) will denote the closed ball of radius \( r \) about the point \( x_0 \). Of particular value in our work will be sets in \( \mathbb{R}^n \) which are compact. In \( \mathbb{R}^n \) it is well known that the compact sets are precisely those that are closed and bounded. It
turns out that the domains for the partial differential equations that we will be considering will be open sets. In general, these domains will also be bounded, that is there exists an open ball of finite radius \( r > 0 \), that contains them. Although a domain, \( \Omega \), say, is not (usually) a compact set there will typically be a compact set, say \( K \), which contains it, and we will write in such a case \( \Omega \subset K \). We will also often consider the points in a set \( \Omega \) where a particular function, \( u \), is not zero. The closure of such a set is called the support of (the function) \( u \) (in the set \( \Omega \)) and we denote it as follows:

\[
spt(u) = \overline{\{x \in \Omega \mid u \neq 0\}}
\]

where we denote the closure of a set \( \Omega \) by \( \overline{\Omega} \). If a set, say \( U \), is compactly contained in another set, say \( W \), we write

\[
U \subset \subset W.
\]

This means that there exists a compact set \( K \) so that

\[
U \subset K \subset W.
\]

If, as will often be the case, the sets \( U \) and \( W \) are open then \( U \subset \subset W \) implies that there exists an open set \( V \) so that \( U \subset V \subset W \) and

\[
dist(U, V^c), dist(V, W^c) > 0
\]

where \( dist(U, V) \) signifies the distance between the sets \( U \) and \( V \), i.e.,

\[
dist(U, V) = \inf \{dist(u, v)\mid u \in U, v \in V\}.
\]

We will, in many instances, be interested in the situation when the support of a function \( u \), on a set \( \Omega \), is compactly contained in \( \Omega \). To wit

\[
spt(u) \subset \subset \Omega;
\]
specifically this implies that if

\[ K = spt(u) = \{ x \in \Omega \mid u \neq 0 \} \]

then

\[ dist(K, \Omega^c) > 0. \]

1.3.2 Notation for the Calculus

We will denote first order (often weak) partial differentiation of an appropriate function, \( u \), with respect to the variable, \( x_i \), variously as below:

\[ \frac{\partial}{\partial x_i} u = u_{x_i} = D_i u \]

and \( n^{th} \) order differentiation by

\[ \frac{\partial^n}{\partial x^n_{i_1} \ldots \partial x^n_{i_n}} u = u_{x_{i_1} \ldots x_{i_n}} = D_{i_1, i_2, \ldots, i_n} u \]

for \( n^{th} \) order differentiation with respect to the variable, \( x_i \), or, more generally,

\[ \frac{\partial^n}{\partial x^{p_1}_{i_1} \ldots \partial x^{p_k}_{i_k}} u = u_{x_{i_1} \ldots x_{i_2} \ldots x_{i_k}} = D_{1 \ldots 12 \ldots k \ldots k} u \]

where

\[ \sum_{j=1}^{k} p_j = n \]

and any (but not all) of the variables, \( x_j \), may fail to appear.

Occasionally, we will make use of multi-index notation, particularly when derivative terms of order higher than one are discussed. To wit, following Evans’ description in Appendix A, ([Evans], page 617) or the discussion of Adams and Fournier on page 2, [AF], an \( n \)-tuple of the form

\[ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \]

is a multi-index of order

\[ k = |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \]
and for a fixed multi-index, $\alpha$, we write

$$D^\alpha u(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} u.$$ 

The special meaning for $\alpha!$ is given by

$$\alpha! = a_1! a_2! \cdots a_n!$$

and, where it appears in the context of multi-index notation, the symbol

$$\left( \frac{\alpha}{\beta} \right) = \frac{\alpha!}{\beta! (\alpha - \beta)!} = \left( \frac{\alpha_1}{\beta_1} \right) \left( \frac{\alpha_2}{\beta_2} \right) \cdots \left( \frac{\alpha_n}{\beta_n} \right).$$

We will use the symbol $\nabla u$ to denote the gradient of the function $u$, i.e.,

$$\nabla u = [u_{x_1}, u_{x_2}, \ldots, u_{x_n}] = \left[ \frac{\partial}{\partial x_1} u, \frac{\partial}{\partial x_2} u, \ldots, \frac{\partial}{\partial x_n} u \right].$$

Thus,

$$|\nabla u|^2 = \sum_{i=1}^n \left( \frac{\partial}{\partial x_i} u \right)^2 = \sum_{i=1}^n (D_i u)^2;$$

it turns out that the notation $|\nabla u|^2$ will be both a useful and commonplace notation for our work.

Furthermore, we may denote the (indefinite Lesbesgue) integral of a function, $u$, where defined, by

$$\int \int \cdots \int u(x_1, x_2, \ldots x_n) dx_1 dx_2 \ldots dx_n = \int u \, dx$$

for brevity. In the case where the domain, $\Omega$, is of importance, or unclear, we shall use

$$\int \int \cdots \int_\Omega u(x_1, x_2, \ldots x_n) dx_1 dx_2 \ldots dx_n = \int_\Omega u \, dx$$

or, if the integration is over all of $\mathbb{R}^n$, we shall use

$$\int \int \cdots \int_{\mathbb{R}^n} u(x_1, x_2, \ldots x_n) dx_1 dx_2 \ldots dx_n = \int_{\mathbb{R}^n} u \, dx.$$
1.3.3 Function Spaces, Functionals, Weak Derivatives, Approximation by Smooth Functions and Embeddings

We will be working with a variety of normed Banach spaces which include $\mathbb{R}^n$, (some of) the Lesbesgue Spaces, Sobolev Spaces, and Hölder Spaces. These are denoted, respectively, by $L^p(\Omega)$, $W^{k,p}(\Omega)$, and $C^{k,\alpha}(\Omega)$ for a fixed domain $\Omega$; $p$, when it appears will usually be 2, with $0 \leq \alpha \leq 1$ and $k \in \mathbb{N}_0 = \{0, n \in \mathbb{N}\}$. The value of $k$, of course, denotes the order of differentiability. It is well known for instance that

$$L^2(\Omega) = W^{0,2}(\Omega)$$

and there exist various imbedding of these spaces, one into another, some of which will play a significant role in our discussions. Although certain theorems that we will use (and, or, prove) may apply to more general Banach spaces, we will usually only work in the Hilbert space setting, i.e., when $p = 2$. An important aspect of the Hilbert spaces is that they come equipped with an inner product and associated norm, both of which we will use.

The function spaces where we will seek solutions to the various elliptical partial differential equations are the Sobolev spaces. Specifically, we will be most interested in the first order Sobolev space, i.e., where the highest order weak derivatives that appear are those of first order – the space $W^{1,2}(\Omega)$. For a fixed domain, $\Omega$, we will consider the completion of the space of infinitely differentiable functions with compact support in $\Omega$, i.e.,

$$\left\{ u | u = \lim_{i \to \infty} \phi_i, \phi_i \in C_0^\infty(\Omega) \right\},$$

with respect to the well known first order Sobolev norm:

$$\|u\| = \|u\|_{W^{1,2}(\Omega)} = \left( \int_\Omega \sum_{i=1}^n |D_i u|^2 \, dx + \int_\Omega |u|^2 \, dx \right)^{1/2}$$

$$= \left( \int_\Omega (\nabla u)^2 \, dx + \int_\Omega u^2 \, dx \right)^{1/2}.$$
To denote that a function, \( u \), belongs to this space we write:

\[
\begin{align*}
  u & \in W^{1,2}_0(\Omega).
\end{align*}
\]

The space \( W^{1,2}_0(\Omega) \) is a subspace of \( W^{1,2}(\Omega) \). Important properties of this space include the ability to extend functions by zero to all of \( \mathbb{R}^n \), if necessary, and the ability to ignore certain delicate boundary regularity issues. See, respectively, [Evans], pg. 274, and [Frank], pg. 394, for example. In discussions and proofs, we will have some occasion to use a variety of other norms which will be identified by appropriate subscripts as needed for clarity. Some examples are listed below:

\[
\begin{align*}
  \|u\|_{L^2(\Omega)} &= \left( \int_{\Omega} |u|^2 \, dx \right)^{1/2} \quad \text{The } L^2 \text{ norm of } u \text{ on } \Omega. \\
  \|u\|_{L^2(\tilde{\Omega})} &= \left( \int_{\tilde{\Omega}} |u|^2 \, dx \right)^{1/2} \quad \text{The } L^2 \text{ norm of } u \text{ on } \tilde{\Omega} \neq \Omega. \\
  \|u\|_{L^p(\Omega)} &= \left( \int_{\Omega} |u|^p \, dx \right)^{1/p} \quad \text{The } L^p \text{ norm of } u \text{ on } \Omega. \\
  \|u\|_{W^{1,2}(\Omega)} &= \left( \int_{\Omega} |\nabla u|^2 + |u|^2 \, dx \right)^{1/2} \quad \text{The first order Sobolev norm of } u \text{ on } \Omega. \\
  \|u\|_{W^{k,p}(\Omega)} &= \left( \sum_{\alpha \leq k} \int_{\Omega} |D^\alpha u|^p \, dx \right)^{1/p} \quad \text{The } (k,p)^{th} \text{ order Sobolev norm of } u \text{ on } \Omega.
\end{align*}
\]

Note also that

\[
\|u\|_{L^p(\Omega)} = \|u\|_{W^{0,p}(\Omega)}
\]

for all values of \( p \). Occasionally, if the context is clear, a subscript pertaining to either the domain or function space or both may be suitably abbreviated or even omitted. We will also find use for the Hölder Spaces. For example the Banach space denoted by \( C^{k,\alpha}(\tilde{\Omega}) \) is the set of all functions, \( u \in C^k(\tilde{\Omega}) \), such that the norm

\[
\|u\|_{C^{k,\alpha}(\tilde{\Omega})} = \sum_{i=0}^{k} \left( \sup_{|\beta|=i} \int_{\tilde{\Omega}} |D^\beta u| + \sup_{x,y \in \tilde{\Omega}, x \neq y} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\alpha} \right), \quad 0 \leq \alpha \leq 1
\]

is finite – see Gilbarg and Trudinger, [GT], pages 52, 53, for instance, or Evans, [Evans],
at the beginning of Chapter Five, pages 240-241. In the event that $\alpha = 0$, we will write

$$C^k(\bar{\Omega}) = C^{k,0}(\bar{\Omega})$$

although the equality is not really trivial. For details see Gilbarg and Trudinger, pages 52-53, [GT]. We make note of the fact that the space, $C^{0,1}(\bar{\Omega})$ is the space of Lipschitz continuous functions – a space we will reference from time to time. In particular, if $u \in C^{0,1}(\bar{\Omega})$ then there exists $C$ such that for all $x, y \in \bar{\Omega}$

$$|u(x) - u(y)| \leq C|x - y|.$$  

A particular function $u$ belongs to a given space, say $X(\Omega)$ provided

$$\|u\|_{X(\Omega)} < \infty,$$

so if $u \in L^2(\Omega)$ we mean that $\|u\|_{L^2(\Omega)} < \infty$. Occasionally we may not quite be able to say that a function $u \in L^p(\Omega)$, but that in some sense, it is almost in $L^p(\Omega)$. In such an instance we say that a function, $u$, is locally summable if $\int_{\hat{\Omega}} |u|^p \, dx < \infty$ for each set $\hat{\Omega}$ compactly contained in $\Omega$, and we denote this by writing that $u \in L^p_{loc}(\Omega)$.

An important attribute of any Banach space is that it is complete; i.e., every Cauchy sequence converges (to an element in the space). So if a sequence $\{u_i\}_{i=1}^{\infty}$ converges to, say $u \in X(\Omega)$, where $X(\Omega)$ is a Banach space we write

$$u_i \to u \text{ as } i \to \infty,$$

meaning precisely that

$$\|u_i - u\|_{X(\Omega)} \to 0 \text{ as } i \to \infty.$$  

For a particular Banach space say $X(\Omega)$, there is an associated Banach space called the dual space of $X(\Omega)$, denoted $X^*(\Omega)$; this space is the set of bounded, linear functionals on
(the elements) of $X(\Omega)$ such that

$$v \in X^*(\Omega)$$

maps $u \in X$ into $\mathbb{R}$. That is, $v(u) \in \mathbb{R}$. It is well known that for a Hilbert space, say $H$, in general (and thus $L^2$ and $W^{k,2}$ in particular) that the space and its dual are isomorphic. Furthermore, the Riesz representation theorem gives us that each element, $f$, of the dual space $H^*(\Omega)$ may be uniquely represented as an inner product. Specifically for $f \in H^*(\Omega)$ and any $u \in H(\Omega)$ there is a unique $v \in H$, uniquely determined by $f$ such that

$$f(u) = \langle u, v \rangle$$

and

$$\|f\|_{H^*(\Omega)} = \|v\|_{H(\Omega)}.$$  

We have denoted the standard inner product on $H(\Omega)$ by $\langle \cdot, \cdot \rangle$. For the Hilbert spaces which are of interest to us, we have the following inner products. If $u, v \in L^2(\Omega)$, then

$$\langle u, v \rangle = u \cdot v = \int_\Omega uv \, dx.$$  

Similarly, if $u, v \in W^{1,2}(\Omega)$ or $u, v \in W^{k,p}(\Omega)$ then, respectively,

$$\langle u, v \rangle = \int_\Omega \sum_{i=1}^n iD_i u D_i v + uv \, dx$$

$$= \int_\Omega \nabla u \cdot \nabla v + uv \, dx$$

or

$$\langle u, v \rangle = \sum_{\alpha \leq k} \int_\Omega D_\alpha u D_\alpha v \, dx.$$  

For brevity we will usually write

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega \nabla u \nabla v \, dx.$$
It is easy to see that in these particular cases above, that the general relation

$$\|u\|^2 = \langle u, u \rangle$$

holds. In the Hilbert space setting, we will make use of the result that a bounded sequence has a weakly convergent subsequence. Specifically, if \( \{u_i\}_{i=1}^{\infty} \) is a sequence such that for all \( i = 1, 2, \ldots, u_i \in H(\Omega) \), a Hilbert space, and

$$\|u_i\|_{H(\Omega)} \leq C < \infty,$$

then for any \( v \in H^*(\Omega) \) the sequence (of real numbers \( w_i \) such that)

$$w_i = \langle u_i, v \rangle$$

has a convergent subsequence. That is, there exists \( w \in \mathbb{R} \) and a subsequence \( \{w_{ij}\}_{j=1}^{\infty} \) where

$$w_{ij} = \langle u_{ij}, v \rangle$$

and

$$|w_{ij} - w| \to \text{ as } j \to \infty.$$ 

More briefly (and somewhat inaccurately) we denote this situation by

$$u_i \rightharpoonup u.$$ 

Strictly speaking we do not know that there is an element \( u \in H(\Omega) \) that is the limit of any subsequence of \( \{u_i\} \), hence the inaccuracy of the notation, but this is standard in the literature. In the event that

$$u_i \rightarrow u$$

it is clear (by linearity) that

$$\langle u_i - u, v \rangle \rightarrow 0$$
for any $v \in H^*(\Omega)$. A good background for this material and other functional analysis material may be found in Kreyszig's book, "Introductory Functional Analysis with Applications," [Kreys]. A briefer treatment may be found in Appendices D and E in Evans, [Evans].

We assume that the reader is thoroughly familiar with the properties of ordinary and partial derivatives, hence the reader should have good intuition for the spaces of Hölder continuous functions. However, most of the time we will be working in the Sobolev spaces where the derivatives of functions do not exist in the traditional sense. The reader should recall that the Lesbesgue space $L^p(\Omega)$ is a Banach space only when functions that are equal almost everywhere are identified. That is, a function $u \in L^2(\Omega)$, for instance, has many representations and any function $v \in L^2(\Omega)$ such that

$$\int_{\Omega} u^2 dx = \int_{\Omega} v^2 dx$$

or, equivalently, such that

$$u = v \text{ a.e. (almost everywhere)}$$

implies that $u = v$. This bears a certain conceptual similarity to the idea of weak derivatives. Following a discussion on the subject of weak differentiation in Evans in Chapter Five, [Evans], we say that a function $u \in L^p(\Omega)$ has a weak first order partial derivative with respect to the variable $x_i$ provided for all functions $\phi \in C_0^\infty(\Omega)$ there exists a function $v \in L^p(\Omega)$ so that following holds:

$$\int_{\Omega} \phi_{x_i} u \, dx = -\int_{\Omega} \phi v \, dx.$$

If such is the case we (perhaps, misleadingly) write

$$v = u_{x_i}.$$
If weak derivatives exist for all the variables $x_i$, $i = 1, 2, ..., n$ we then say that

$$u \in W^{1,p}(\Omega).$$

More generally, if there exists a function $v$ so that for any multi-index $\alpha$ and every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega uD_\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega \phi v \, dx$$

we say that $v$ is the $\alpha^{th}$ weak partial derivative of $u$. Thus function $u \in W^{k,p}(\Omega)$ provided $u$ has all $k^{th}$ order and less weak derivatives defined iteratively as above in the obvious way. One of many properties of weakly differentiable functions that are similar those of functions with standard derivatives, is that if $u \in W^{k,p}(\Omega)$ and $\phi \in C_0^\infty(\Omega)$ then the product $u\phi \in W^{k,p}(\Omega)$ as well. Excellent treatments of properties of functions in the Sobolev spaces may be found in early material in both Kesavan, [Kesav], and Adams and Fournier, [AF].

Important and often complicated relationships exist between Banach spaces of interest to us. Of particular interest are the cases where one space is a subspace of another. Also of interest is the situation when one space is compactly embedded in another. It is evident that the following subspace inclusions hold:

$$C^l(\Omega) \subset C^k(\Omega) \subset W^{j,p}(\Omega) \subset L^p(\Omega)$$

where

$$0 \leq j \leq k \leq l.$$  

Also if $\tilde{\Omega} \subset \Omega$ and $X$ is a normed Banach Space, then

$$X(\tilde{\Omega}) \subset X(\Omega).$$
Less obvious, but also true, is the fact that

\[ W_0^{k,p}(\Omega) \leq W^{k,p}(\Omega). \]

It is clear that if a sequence is convergent in a space on the left hand side of an inequality in one of the strings above, that the sequence will converge in a space on the right hand side of an appropriate inequality. When spaces are compactly embedded in one another, convergence of a sequence in the embedded space provides information about the convergence of the sequence in the embedding space. This will prove useful to us. In particular, following a definition in Adams and Fournier, page 9, \([AF]\), and Evans, pages 271, 272, \([Evans]\). We have:

**Definition 1** If \(X\) and \(Y\) are normed Banach spaces then \(X\) is compactly embedded into \(Y\), written \(X \subset \subset Y\), provided

\[
\begin{align*}
(i) & \quad X \subset Y \\
(ii) & \quad \|u\|_X \leq C \|u\|_Y \text{ for all } u \in X \\
(iii) & \quad \text{each bounded sequence in } X \text{ is precompact in } Y.
\end{align*}
\]

for some fixed constant \(C\) and

Alternatively (to (iii)), if \(\{u_i\}\) is any bounded sequence in \(X\), then it has a convergent subsequence in \(Y\).

There is a great deal of literature in partial differential equations devoted to which space is compactly embedded in what. The rather complicated Sobolev embedding theorem is a modest example, and a large fraction of the book Sobolev Spaces, by Adams and Fournier is devoted to this topic. There are many regularity issues related to the compact embedding of our normed Banach spaces. We will cite theorems of particular use when needed in our work but some simpler results which introduce the notation that we will use are given below.
For \( m, \) a positive integer, and

\[ 0 < \alpha < \beta \leq 1 \]

with \( \Omega, \) bounded, we note the following results that can be found in Adams and Fournier on pages 10-12, \([AF]\):

\[ C^{m,\beta}(\bar{\Omega}) \not\subset C^{m,\alpha}(\bar{\Omega}) \not\subset C^m(\bar{\Omega}) \]

and

\[ C^m(\bar{\Omega}) \subset C^{m,\alpha}(\bar{\Omega}) \subset C^{m,\beta}(\bar{\Omega}) \].

We denote compact embedding by the symbol, \( \subset \), and the compactness of the embeddings above follows partially from the fact that we are considering only bounded domains. The notation is that of Evans. We also observe, for example, that

\[ C^{1,1}(\bar{\Omega}) \subset C^{2,0}(\bar{\Omega}) = C^2(\bar{\Omega}) \],

this follows from the fact that second order derivatives are continuous in \( C^2(\bar{\Omega}) \) and that in \( C^{1,1}(\bar{\Omega}) \), the second order difference quotients need only be Lipschitz continuous. The compact embedding, given as an inclusion above, would be written

\[ C^i(\Omega) \subset C^k(\Omega) \subset W^{j,p}(\Omega) \subset L^p(\Omega) \]

which holds for

\[ 0 \leq j \leq k \leq l. \]

The fact that imbedding are compact can easily be seen from the fact that in each case, the constant \( C = 1 \) the required subsequence being the sequence itself.

1.3.4 Domains and Their Boundaries

In general we will consider domains that are bounded, connected and open sets in \( \mathbb{R}^n \). This is to be understood unless otherwise noted. The boundary of a domain \( \Omega, \) denoted \( \partial \Omega = \bar{\Omega} \setminus \Omega, \) following McOwen \([McOwen]\), page 7. Most often we will reserve the symbol
\( \Omega \), and some minor variants, to denote our domains. Occasionally, various domains will be subscripted or superscripted as needed to uniquely identify them. We will need a number of definitions that give information on the nature of the boundary of a domain.

Occasionally, we will have to make use of some boundary regularity arguments for our bounded domains and a discussion of this material and some useful notation may be found on pages 94 and 95 of Gilbarg and Trudinger, \([GT]\).

**Definition 2** (Gilbarg and Trudinger, page 94) A bounded domain \( \Omega \) in \( \mathbb{R}^n \) and its boundary are of class \( C^{k, \alpha} \), \( 0 \leq \alpha \leq 1 \), if at each point \( x_0 \in \partial \Omega \) there is a ball \( B = B(x_0) \) and a one-to-one mapping \( \Psi \) of \( B \) onto \( D \subset \mathbb{R}^n \) such that:

\[
\begin{align*}
(i) & \quad \Psi(B \cap \Omega) \subset \mathbb{R}^n_+ \\
(ii) & \quad \Psi(B \cap \partial \Omega) \subset \mathbb{R}^n_+ \\
(iii) & \quad \Psi \in C^{k, \alpha}(B), \; \Psi^{-1} \in C^{k, \alpha}(D).
\end{align*}
\]

A boundary portion \( T \) of \( \Omega \) (\( T \subset \partial \Omega \)) is also said to be of class \( C^{k, \alpha} \) if at each point \( x_0 \in T \), there is a ball, \( B = B(x_0) \), with the above three conditions satisfied such that \( B \cap \partial \Omega \subset T \). Noteworthy is the fact that a domain \( \Omega \) is a \( C^{k, \alpha} \) domain if at each point of \( \partial \Omega \) there is a neighborhood in which \( \partial \Omega \) is the graph of a \( C^{k, \alpha} \) function of \( n-1 \) of the coordinates \( x_1, x_2, ..., x_n \). The converse also holds for \( k \geq 1 \). Additionally, a domain of class \( C^{k, \alpha} \) is also of class \( C^{j, \beta} \) as well if

\[ j + \beta < k + \alpha \]

where \( 0 \leq \alpha \) and \( \beta \leq 1 \). It is also of use to know that on bounded domains, the spaces (where \( k \geq 1 \)) \( C^{k, \alpha}(\Omega) \) and \( C^{k, \alpha}(\overline{\Omega}) \) are equivalent as functions from the former may be uniquely and continuously extended to the latter space. This is discussed on page 10 of \([AF]\) and page 94 of \([GT]\). These topics are somewhat more briefly discussed in Evans book, \([Evans]\), in Appendix C.

Our most basic definition for a domain follows.
Definition 3 (Domain in \( \mathbb{R}^n \)) A domain is a bounded, connected open set in \( \mathbb{R}^n \).

We will from time to time, restrict our otherwise arbitrary domains by some or all of the following conditions. A domain, \( \Omega \), will be a connected open set, bounded in \( \mathbb{R}^n \) so that the boundary of the closure of \( \Omega \), is the same as the boundary of \( \Omega \), i.e., \( \partial \overline{\Omega} = \partial \Omega \). In addition we may require that the Lebesgue measure (in \( \mathbb{R}^n \)) of \( \partial \Omega = 0 \), and that \( \partial \Omega \) be \( C^0 \). See Babuška and Výborný’s paper [BV], page 176, for some sketchy details. We will also require, in some instances, that our domains satisfy the segment condition of Adams and Fournier which we will give below. This condition is sufficient to guarantee both that the boundary of the domain is \( n - 1 \) dimensional and that the domain lies only on one side of the boundary (see [AF] page 84, section 4.11). The segment property is a fairly minimal restriction on a domain, but one which will prove very useful to us in some settings. We give the definition used in Adams and Fournier.

Definition 4 (Segment Condition, page 68, [AF]) A domain, \( \Omega \), has the segment condition if, for every \( x \in \partial \Omega \), there is a neighborhood \( \Omega_x \) and a non-zero vector \( y_x \) such that if \( z \in \overline{\Omega} \cap \Omega_x \), then \( z + ty_x \in \Omega \) for all \( 0 < t < 1 \).

Beyond this condition we may also require that our domains satisfy certain other conditions that will not be obvious to the reader at this stage, but that basically will insure sufficient regularity of the solutions to the pde

\[
\begin{align*}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

to apply various theorems. We give the following definition.

Definition 5 (Sufficiently Smooth Domain in \( \mathbb{R}^n \)) We will say that a connected and bounded domain \( \Omega \), is sufficiently smooth provided it satisfies the following

(i) \( \partial \Omega \) is of class \( C^{0,1} \) locally and

(ii) \( \partial \Omega = \bigcup_{j=1}^{N} U_j \) where each boundary portion \( U_j \) is of class \( C^{2,0} \).
We assume that \( N \) is finite.

This definition is similar to a notion expressed on page 8 of McOwen's book, Partial Differential Equations: Methods and Applications, [McOwen].

We will also require from time to time, that our domains satisfy the "Strong Local Lipschitz Condition" of Adams and Fournier, [AF]. This property implies, besides the afore-mentioned segment condition, that the domains will also satisfy the uniform cone condition a condition which seems to be more prevalent in the literature. We refer to the local Lipshitz condition of Adams and Fournier ([AF], pages 83-84) and refer the reader to some of their remarks concerning it.

**Definition 6 (The Strong Local Lipschitz Condition, page 83, [AF])** A domain \( \Omega \) satisfies the strong local Lipschitz condition if there exist positive numbers \( \delta \) and \( M \), a locally finite open cover \( \{U_j\} \) of \( \partial \Omega \) and for each \( j \) a real valued function \( f_j \) of \( n-1 \) variables, such that the following conditions hold:

(i) For some finite \( J \), every collection of \( J+1 \) of the sets \( U_j \) has empty intersection.

(ii) For every pair of points \( x, y \in \Omega_\delta \) where

\[
\Omega_\delta = \{ x \in \Omega | \text{dist}(x, \partial \Omega) < \delta \}
\]

such that \( |x - y| < \delta \), there exists \( j \) such that \( x, y \in V_j := \{ x \in U_j | \text{dist}(x, \partial U_j) > \delta \} \).

(iii) Each function \( f_j \) satisfies a Lipschitz condition with constant \( M \): that is, if \( \xi = (\xi_1, \xi_2, ..., \xi_{n-1}) \) and \( \rho = (\rho_1, \rho_2, ..., \rho_{n-1}) \) are in \( \mathbb{R}^{n-1} \), then

\[
|f_j(\xi) - f_j(\rho)| \leq M |\xi - \rho|.
\]

(iv) For some Cartesian coordinate system \( (\zeta_{j,1}, \zeta_{j,2}, ..., \zeta_{j,n-1}) \) in \( U_j \), \( \Omega \cap U_j \) is represented by the inequality

\[
\zeta_{j,n} < f_j(\zeta_{j,1}, \zeta_{j,2}, ..., \zeta_{j,n-1}).
\]
In their note following the definition, Adams and Fournier explain that if the domain \( \Omega \) is bounded, then this somewhat intimidating set of conditions reduces to the case that at each point \( x_0 \) on the boundary, there is a neighborhood on which the intersection of the neighborhood with the boundary is the graph of a Lipschitz continuous function. On the following page they further explain that this condition implies that the boundary also satisfies a uniform interior cone condition, and this in turn implies that the boundary satisfies the segment condition as well. It can now easily be seen that a sufficiently smooth domain, as we have defined it, satisfies the segment condition as well as a uniform interior cone condition.

In order to make use of a result of Gilbarg and Trudinger, we may also require that the boundary satisfy an exterior cone condition as well. We modify the cone condition of Adams and Fournier (see page 81-82, \([AF]\)) to delineate both the interior and exterior cases.

**Definition 7 (Cone)** We define a finite cone \( C \) with vertex at the origin, height \( \rho \), axis direction \( v \) and aperture angle \( \theta \), as follows. For each \( x, v \in \mathbb{R}^n \), \( x, v \neq 0 \), \( \rho > 0 \) and angle \( \theta \in (0, \pi] \)
\[
C = \left\{ x \in \mathbb{R}^n | x = 0 \text{ or } 0 < |x| \leq \rho, \angle (x, v) \leq \frac{\theta}{2} \right\}.
\]

We denote the cone at the point \( x \) (or with vertex \( x \), aperture angle \( \theta \) and height, \( \rho \)) as \( C_x \). A domain \( \Omega \) satisfies a cone condition (see \([AF]\), 4.9, page 82) if there exists a finite cone \( C \) such that each \( x \in \Omega \) is the vertex of a finite cone \( C_x \) contained in \( \Omega \) and congruent to \( C \). Furthermore, \( C_x \) need not be parallel to the cone \( C \), that is the axis direction \( v \neq 0 \) may be arbitrary.

Principally, we are interested in cone conditions on the boundary portion of our domains. An interior cone condition at a point \( x \in \partial \Omega \) is satisfied at if there exists a cone \( C_x \) with positive aperture angle and positive height such that \( C_x \cap \bar{\Omega} = C_x \). If a domain satisfies an interior cone condition at each point \( x \in \partial \Omega \), then the boundary of the domain is said to satisfy an interior cone condition. In similar fashion a domain satisfies an exterior cone condition if for each \( x \in \partial \Omega \) there exists a cone \( C_x \) with positive aperture angle and positive height such that \( C_x \cap \bar{\Omega}^c = C_x \). We therefore write the following definitions.
Definition 8 (Interior Cone Condition) A domain $\Omega$ satisfies an interior cone condition if at each point $x \in \partial \Omega$, there exists a cone $C_x$ with positive aperture angle and positive height such that $C_x \cap \Omega = C_x$.

Definition 9 (Exterior Cone Condition) A domain $\Omega$ satisfies an exterior cone condition if at each point $x \in \partial \Omega$, there exists a cone $C_x$ with positive aperture angle and positive height such that $C_x \cap \overline{\Omega^c} = C_x$.

A cone condition is said to be uniform if there exists for all $x \in \partial \Omega$ a $\theta^* > 0$ such that for all cones $C_x$, $\theta_x \geq \theta^* > 0$. For clarity, then, we will typically require that the boundary of a domain, $\partial \Omega$, satisfy an exterior cone condition at each point and as a consequence of satisfying the strong local Lipschitz condition above, the boundary, $\partial \Omega$, will satisfy an interior (and uniform, i.e., a minimum positive value for $\theta$) cone condition as well.

1.3.5 Operators and Bilinear Forms

Most generally, we will consider partial differential equations of the form pde3

$$
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

where $L$ is a linear, second order, uniformly elliptic, partial differential operator. Typically, as we mentioned above, we will consider only open, connected, and bounded domains, $\Omega$, in $\mathbb{R}^n$ and will make exceptions to this where appropriate. Unless otherwise noted, we will assume that the coefficient functions in pde3, for $L$ as given below satisfy the following properties: the functions $a_{ij}(x)$ are $C^0(\Omega)$ for $i,j = 1,2,\ldots,n$ and $b_i(x)$ and $c(x)$ for $i = 1,2,\ldots,n$ are $L^\infty(\Omega)$. In later discussions when we will be employing the results of Heywood, Noussair, and Swanson in particular, we will require stronger conditions, viz., $a_{ij}(x) \in C^1(\overline{\Omega})$ and $b_i(x)$, $c(x) \in C^{0,\alpha}(\overline{\Omega})$ where $0 < \alpha \leq 1$. The space $C^{0,\alpha}(\overline{\Omega})$, is the space of Hölder continuous functions with exponent $\alpha$. Thus, in such cases, where the operator $L$ is such that $a_{ij}(x) \in C^1(\Omega)$ for $i,j = 1,2,\ldots,n$, $L$ may be written in non-divergence form and
we may assume that

\[ Lu = \sum_{i,j=1}^{n} a_{ij} (x) u_{x_i x_j} + \sum_{i=1}^{n} b_i (x) u_{x_i} + c(x) u. \]

More typically, we will think of the operator \( L \) as being the divergence form given by Gilbarg and Trudinger, \([GT]\), in chapter eight of their book:

\[ Lu = \sum_{i,j} D_i (a_{ij} (x) D_j u) + \sum_i (D_i (b_i(x)u) + c_i(x) D_i u) + d(x) u. \]

It is clear in either case that given sufficient differentiability of a function \( u \in W^{k,p} (\Omega) \) that \( L \) (in either form above) is a linear map from

\[ W^{k,p} (\Omega) \to W^{k-2,p} (\Omega). \]

From time to time (especially in the context of the variational formulation of certain pdes) it may be assumed that the coefficient matrix \([a_{ij}]\) satisfies the symmetry condition \( a_{ij} = a_{ji} \). However we will always assume that the operator \( L \) satisfies both uniform bound and ellipticity conditions. That is, (following \([Evans]\), page 294) there exist positive constants, \( \Theta \) and \( \theta \), so that

\[ \Theta^2 \geq \sum_{i,j=1}^{n} |a_{ij} (x)|^2 \]

and

\[ \sum_{i,j=1}^{n} a_{ij} (x) \xi_i \xi_j \geq \theta |\xi|^2 \]

for all \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \). In addition, we assume following Gilbarg and Trudinger page 178, \([GT]\), that the remaining coefficients satisfy for all \( x \in \Omega \):

\[ \sum_{i=1}^{n} |b_i (x)| + |c(x)| \leq M < \infty. \]
in non-divergence form, or

\[ \frac{1}{\theta^2} \sum_{i=1}^{n} |b_i(x)|^2 + |c_i(x)|^2 + \frac{1}{\theta} |d(x)|^2 \leq \nu^2 < \infty \]

when \( L \) is written in divergence form. Additionally, we assume that the constant \( \nu > 0 \).

Often we will discuss simpler forms of the operator \( L \), namely,

\[ Lu = \Delta u \quad \text{and} \quad Lu = \mathbb{L}u = \Delta u + c(x)u. \]

Because these simpler forms contain the Laplacian and may be considered to be in either divergence form or non-divergence form with

\[ a_{ij}(x) = I_{i,j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \]

it is clear that

\[ \theta = \Theta = 1 \]

and we will have

\[ |c(x)| \leq M \]

or

\[ |d(x)| \leq \nu. \]

Although we will use the convention (similar to that employed in Gilbarg and Trudinger) that the operator \( L \) does not include the eigenvalue term \( \lambda u \), we may, from time to time without loss, think of an associated linear operator

\[ \bar{L}u = Lu + \lambda u \]
when convenient. Since any constant \( \lambda \) is \( C^\infty(\Omega) \) for any domain \( \Omega \), (perhaps different) coefficient bounds will exist for the new operator.

Somewhat related to the inner product in Hilbert spaces and of great utility in the treatment of partial differential equations is a particular linear functional associated with the operator \( L \) called the bilinear form. Examples relevant to our work are listed below:

\[
\begin{align*}
B_\Omega[u,v] &= \int_\Omega \nabla u \cdot \nabla v \, dx \\
B_\tilde{\Omega}[u,v] &= \int_\tilde{\Omega} \nabla u \cdot \nabla v \, dx \\
B_\Omega[u,v] &= \int_\Omega \sum_{i,j=1}^n \nabla u \cdot \nabla v - c(x) uv \, dx
\end{align*}
\]

The bilinear form for \( Lu = \Delta u \) on \( \Omega \).

The form for \( Lu = \Delta u \) on \( \tilde{\Omega} \neq \Omega \).

The bilinear form for \( Lu \) on \( \Omega \).

For the operator \( L \) in non-divergence form

\[
Lu = \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + c(x)u
\]

we have

\[
B_\Omega[u,v] = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \nabla u \cdot \nabla v + \sum_{i=1}^n b_i(x) u_{x_i} v + c(x)uv \, dx
\]

and for \( L \) in divergence form where

\[
Lu = \sum_{i,j} D_i \left( a_{ij}(x) D_j u \right) + \sum_i \left( D_i (b_i(x) u) + c_i(x) D_i u + d(x)u \right)
\]

we write

\[
B_\Omega[u,v] = \int_\Omega \sum_{j,k} a_{j,k} D_k u D_j v + \sum_j b_j u D_j v - \left( \sum_j c_j D_j u + du \right) \, v \, dx
\]

The notation is similar to that of Evans in [Evans] and Renardy or Rogers in [RR] and Gilbarg and Trudinger [GT]. Another useful quantity related to the bilinear form is the (generalized) Rayleigh quotient:

\[
J_\Omega(u) = \frac{B_\Omega[u,u]}{\|u\|_{L^2(\Omega)}^2}.
\]
The subscript of $J$ is a reference to the domain over which the integration takes place and may be omitted.

1.3.6 Solutions to Partial Differential Equations

Results in the literature (see [Evans], [McOwen] or [GT] for examples), give us the existence of unique solutions to the first (or least) eigenvalue problem, $pde_3$,

\[
\begin{aligned}
Lu + \lambda u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

under certain circumstances. The treatments are of varying generality and entertain a wide variety of conditions on the operator $L$ and the boundary of the domain, $\partial \Omega$. We will discuss this in considerable depth as we proceed, but for now, we want to broadly categorize solutions when they exist. If a function $u \in C^2(\Omega)$ solves the given pde it is called a classical solution. If a function $v$ solves the partial differential equation above (with $L$ in non-divergence form) almost everywhere in $\Omega$, then $v$ is called a strong solution after terminology found in Chapter Nine of Gilbarg and Trudinger, [GT]. On the other hand a solution of the pde as given is called a weak solution if for all test functions $\phi \in C^\infty_c(\Omega)$,

\[
B_\Omega[u, \phi] = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \nabla u \nabla \phi + \sum_{i=1}^n b_i(x) u_{x_i} \phi + c(x) u \phi \, dx
\]

for $L$ written in non-divergence form or

\[
B_\Omega[u, \phi] = \int_\Omega \sum_{j,k} a_{jk} D_k u D_j \phi + \sum_j b_j u D_j \phi - \left( \sum_j c_j D_j u + du \right) \phi \, dx
\]

for $L$ in divergence form. When solutions (first eigenfunctions, in particular) exist they exhibit important properties which we will also discuss later. Some of these properties
include the fact that the first eigenfunction is of one sign, and that the first eigenvalue is simple.

1.3.7 Miscellaneous

In general, we will be very interested in solutions to the homogeneous eigenvalue problem with a homogeneous boundary condition. When the first eigenfunction exists, it is of one sign and we call this a positive solution. A domain, $\Omega$, on which a positive solution exists or on which any non-trivial solution to the eigenvalue problem exists is called a nodal domain. We will use this terminology frequently.

1.4 Specific Noteworthy Results from the Literature

1.4.1 Results of Ivo Babuška and Rudolf Výborný

A general theorem proving the continuous dependence of the eigenvalues on the domain was given by Ivo Babuška and Rudolf Výborný in their paper "Continuous Dependence of Eigenvalues on the Domain," Czechoslovak Math. Journal, 15(90), pp. 169-178 in 1965. The results are of great generality, but the abstract nature of their proof might leave one longing for concrete examples. At any rate, we will provide a fairly concrete and general proof of the continuous dependence of the first eigenvalue on the domain. We will prove that the first eigenvalue of this partial differential equation

\[
\begin{aligned}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
\]

is continuously dependent on the domain $\Omega$ in $\mathbb{R}^n$. The proof will be given with only minor restrictions on the boundary regularity of the domain and minimal restrictions on the coefficient functions of the differential equations. Besides this, we will present some alternative arguments based on regularity theory in Appendix A as well.
1.4.2 Results of Heywood, Noussair, and Swanson

We will review and make use of some results of Heywood, Noussair, and Swanson in their paper, "On the Zeros of Solutions of Elliptic Inequalities in Bounded Domains," *Journal of Differential Equations*, Vol. 28, pp. 345-353. Academic Press, Inc. 1978. In this paper, the authors established a Sturmian type of comparison theorem for linear (and non-linear) partial differential elliptic operators of second order on the closure of the domain, irrespective of domain boundary regularity. However, the theorem was constructed using Picone's identity, and any two solutions that can be compared using their results are required to be respectively of class $C(\bar{\Omega}) \cap W^{1,2}(\Omega)$ and $C^2(\Omega) \cap W^{1,2}(\Omega)$. However, continuous and twice continuously differentiable solutions to

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

cannot always be found. We will address this issue in some detail in our review of the Heywood, Noussair, and Swanson results. The Heywood, Noussair, and Swanson paper concerns itself with non-linear pdes and establishes results in terms of certain differential inequalities; we will be concerned solely with strictly elliptic coefficient matrices and, generally speaking, ignore the non-linear results.

Many of the known solutions on specific domains can be extracted using the separation of variables technique. This approach along with the comparison results obtained will provide many of the examples and results presented later.

1.5 Continuous Dependence of the First Eigenvalue on the Domain

1.5.1 Definitions and Standard Results from the Literature

We begin by proving the continuous dependence of the first eigenvalue, $\lambda$, for the partial differential equation, \( pde3 \):

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
on the domain $\Omega$. In this section our goal is to prove that, in a suitable sense, the first eigenvalue depends in a continuous way on the domain of the partial differential equation. Several definitions will be required and we shall prove the results in the $n$ dimensional case for fairly general domains. We will reserve following subsections for some consequences of these results.

Similar results to ours were proven by Ivo Babuška and Rudolf Vyborný in their paper, "Continuous Dependence of Eigenvalues on the Domain," which appeared in 1965 ([BV]). Their proof is very abstract and the proofs we present here are more concrete in nature. We will present the proofs in the cases as outlined, by considering lower and then, upper continuity problems separately – these terms to be defined more precisely shortly. The upper continuity case is more difficult.

To begin, we will need a definition for a sequence of nested domains:

**Definition 10 (Nested Sequence of Domains)** Let $\{\Omega_i\}_{i=1}^{\infty}$ be a sequence of domains in $\mathbb{R}^n$. We say that the sequence is nested if either $\Omega_j \subset \Omega_i$ or $\Omega_i \subset \Omega_j$ for every $i < j$. For simplicity we will assume that for all $i \neq j$, $\Omega_i \neq \Omega_j$. If the containment is strict, e.g., $\Omega_i \subset \subset \Omega_j$ for every $i < j$, we will say that the sequence is strictly nested.

Beyond this definition we will need definitions for increasing and decreasing sequences of domains as well as a notion of convergence.

**Definition 11 (Increasing Sequence of Nested Domains)** We say that a nested sequence of domains $\{\Omega_i\}_{i=1}^{\infty}$ is increasing if for each $j > i$, $\Omega_j \supset \Omega_i$. As is typical we may say that the sequence is strictly increasing if the containment is strict.

In similar fashion, we have a definition for a decreasing sequence of domains.

**Definition 12 (Decreasing Sequence of Nested Domains)** A nested sequence of domains $\{\Omega_i\}_{i=1}^{\infty}$ is decreasing if for each $j > i$, $\Omega_j \subset \Omega_i$. We may say that the sequence is strictly decreasing if the containment is strict.
We will write for an increasing nested sequence of domains, \( \{ \Omega_i \}_{i=1}^{\infty} \) that

\[
\Omega_0 = \lim_{i \to \infty} \Omega_i
\]

provided

\[
\Omega_0 = \bigcup_i \Omega_i
\]

and in similar fashion for a decreasing sequence of domains \( \{ \Omega^j \}_{j=1}^{\infty} \)

\[
\Omega_0 = \lim_{j \to \infty} \Omega^j
\]

precisely when a non-empty open set \( \Omega_0 \) satisfies

\[
\Omega_0 = \left( \bigcup_j (\Omega^j)^c \right)^c
\]

or, equivalently,

\[
\Omega_0 = \text{interior} \left( \bigcap_j \Omega^j \right)
\]

where \( \Omega^c \) denotes the compliment of the open set \( \Omega \) in \( \mathbb{R}^n \). Thus \( \Omega_0 \) is open, however, it could be empty; we exclude this possibility. It should be clear that in both cases above, that the limit domain, \( \Omega_0 \), is an open set in \( \mathbb{R}^n \). It is also clear that the domain \( \Omega_0 \) is bounded in \( \mathbb{R}^n \) and non-empty in the latter case. We will assume in the former case, without so stating it, that \( \Omega_0 \) is bounded in \( \mathbb{R}^n \). We may speak of a sequence of domains for brevity, without specifying that it is nested and if the context is clear, we may even omit stating that it is either increasing or decreasing.

Because our domains are taken to be bounded and open, it follows that strict containment implies compact containment. In general, we will not place restrictions on the boundary regularity of the individual domains in the nested sequences. However, we will need to assume, in order to obtain certain of the results, that the domain which is the limit of a given sequence satisfies the segment condition of Adams and Fournier. This definition was given earlier and will be restated here for convenience.
Definition (Segment Condition) A domain, \( \Omega \), has the segment condition if for every \( x \in \partial \Omega \), there is a neighborhood \( \Omega_x \) and a non-zero vector \( y_x \) such that if \( z \in \bar{\Omega} \cap \Omega_x \), then \( z + ty_x \in \Omega \) for all \( 0 < t < 1 \).

We will now define what we mean by continuous dependence of the first eigenvalue the domain.

Definition 13 (Continuous Dependence of the First Eigenvalue) Let \( \{\Omega_i\}_{i=1}^{\infty} \) be a nested sequence of domains (either increasing or decreasing) so that

\[
\lim_{i \to \infty} \Omega_i = \Omega_0.
\]

We say that the first eigenvalue, \( \lambda_0 \), of the partial differential equation

\[
\begin{cases}
  Lu + \lambda u = 0 & \text{in } \Omega_0 \\
  u = 0 & \text{on } \partial \Omega_0
\end{cases}
\]

is continuously dependent on the domain, \( \Omega_0 \), if the following holds. For each \( i = 1, 2, \ldots \) in the sequence of partial differential equations

\[
\begin{cases}
  Lu + \lambda u = 0 & \text{in } \Omega_i \\
  u = 0 & \text{on } \partial \Omega_i
\end{cases}
\]

and the corresponding first eigenvalues, \( \lambda_i \), satisfy

\[
\lim_{i \to \infty} \lambda_i = \lambda_0.
\]

We will need to discuss the convergence of a sequence of first eigenfunctions in a suitable function space (for us \( W_0^{1,2} \)) on the domain

\[
\Omega = \sup_{j \in \{N, 0\}} \Omega_j.
\]

To simplify the proofs somewhat, we will approach the problem as one might approach
proving the continuity of a function defined on an interval. To prove that a function, \( f(x) \) is continuous at a point \( x_0 \) on an interval, \( I = (a, b) \) containing \( x_0 \), it is sufficient to show that \( f(x) \) is continuous at \( x_0 \) both from below on \( (a, x_0) \) and from above on \( (x_0, b) \). To this end we make two additional definitions.

**Definition 14 (Lower Continuous Dependence of the First Eigenvalue)** Let \( \{\Omega_i\}_{i=1}^{\infty} \) be an increasing nested sequence of domains so that

\[
\lim_{i \to \infty} \Omega_i = \Omega_0.
\]

We say that the first eigenvalue, \( \lambda_0 \), of the partial differential equation

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega_0 \\
  u = 0 & \text{on } \partial\Omega_0
\end{cases}
\]

is lower continuous on the domain, \( \Omega_0 \), if the following holds. For each \( i = 1, 2, \ldots \) in the sequence of boundary value problems

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega_i \\
  u = 0 & \text{on } \partial\Omega_i
\end{cases}
\]

and the corresponding first eigenvalues, \( \lambda_i \), satisfy

\[
\lim_{i \to \infty} \lambda_i = \lambda_0.
\]

**Definition 15 (Upper Continuous Dependence of the First Eigenvalue)** Let \( \{\Omega^i\}_{i=1}^{\infty} \) be a decreasing nested sequence of domains so that

\[
\lim_{i \to \infty} \Omega^i = \Omega_0.
\]
We say that the first eigenvalue, $\lambda_0$, of the partial differential equation

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega_0 \\
u = 0 & \text{on } \partial \Omega_0
\end{cases}
\]

is upper continuous on the domain, $\Omega_0$, if the following holds. For each $i = 1, 2, \ldots$ in the sequence of boundary value problems

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega^i \\
u = 0 & \text{on } \partial \Omega^i
\end{cases}
\]

and the corresponding first eigenvalues, $\lambda_i$, satisfy

\[
\lim_{i \to \infty} \lambda^i = \lambda_0.
\]

Given these two definitions, it is clear that any discussion of the convergence of a sequence of first eigenfunctions for the lower continuity situation will occur on $\Omega_0$ and that in the upper continuity case convergence needs to occur on $\Omega^1$. It is also clear that if the first eigenvalue $\lambda_0$ is both lower and upper continuously dependent on the domain $\Omega_0$ that it is also continuously dependent on the domain.

Before we embark on our proof of the continuous dependence of the eigenvalue on the domain, we want to review, in particular, the existence and uniqueness of solutions as well as certain regularity issues. Results on these topics can be found throughout Gilbarg and Trudinger, McOwen, Evans, Renardy and Rogers and Kesavan (respectively, [GT], [McOwen], [Evans], [RR] and [Kesav]); however, the regularity constraints on the domain vary wildly from treatment to treatment. Perhaps, the most thorough treatment of these topics for weak solutions of the given pdes may be found in Gilbarg and Trudinger, although many of their proofs contain somewhat less detail than one might want. We will be somewhat selective about what we prove here in detail, but we will provide proofs where we think they might be helpful for clarity.

Gilbarg and Trudinger discuss the existence and uniqueness of generalized, or weak
solutions to the problem

\begin{equation*}
\begin{cases}
  Lu + \lambda u = g + \sum_{i=1}^{m} D_i f_i & \text{in } \Omega \\
  u = \phi & \text{on } \partial \Omega 
\end{cases}
\end{equation*}

where \( L \) is the divergence form operator given earlier; i.e.,

\[ Lu = \sum_{i,j} D_i (a_{ij}(x) D_j u) + \sum_i (D_i (b_i(x)u) + c_i(x)D_i u) + d(x)u. \]

We will summarize their material from sections 8.1 – 8.3, in Gilbarg and Trudinger, \([GT]\).

Throughout our work, we will only be concerned with the homogeneous partial differential equation with homogeneous boundary conditions. That is we will take \( g = f_i = \phi = 0 \) for all \( i = 1, 2, \ldots, m \). In general the constraints on the operator \( L \) and the domain \( \Omega \) are minimal. We state these presently. The functions \( a_{ij}(x) \), for \( i, j = 1, 2, \ldots, n \) and \( b_i(x) \) and \( c_i(x) \) for \( i = 1, 2, \ldots, n \) and \( d(x) \) are measurable on \( \Omega \). In the general (inhomogeneous) case, the functions \( g \) and \( D_i f_i \) must be locally integrable on \( \Omega \), and \( \phi \) must be in \( W^{1,2}(\Omega) \) with \( u - \phi \in W^{1,2}_0(\Omega) \). Beyond this they require that the coefficient functions be bounded and the operator \( L \) be strictly (or uniformly) elliptic. That is,

\[ \Theta^2 \geq \sum_{i,j=1}^{n} |a_{ij}(x)|^2 \]

and

\[ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \]

for all \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \) with

\[ \frac{1}{\theta^2} \sum_{i=1}^{n} |b_i(x)|^2 + |c_i(x)|^2 + \frac{1}{\theta} |d(x)|^2 \leq \nu^2 < \infty \]
for all $x \in \Omega$. Given the associated bilinear form, $B[u, v]$, to $L$, above,

$$B_\Omega[u, v] = \int_\Omega \sum_{j,k} a_{j,k} D_k u D_j v + \sum_j b_j u D_j v - \left( \sum_j c_j D_j u + du \right) v \, dx$$

a weak solution of the pde will be a function in $W^{1,2}_0(\Omega)$ so that for all $v \in C^1_0(\Omega)$ (or, equivalently, $v \in C^\infty_0(\Omega)$)

$$B_\Omega[u, v] = \int_\Omega \sum_{j,k} a_{j,k} D_k u D_j v + \sum_j b_j u D_j v - \left( \sum_j c_j D_j u + du \right) v \, dx = \int_\Omega \left( \sum_{i=1}^m D_i f_i - g \right) v \, dx.$$

Their principle existence and uniqueness result follows from the weak maximum principle under the additional following condition:

$$\int_\Omega \left( dv - b_j \sum_j v \right) \, dx \leq 0$$

for all test functions $v \in C^1_0(\Omega)$. We state their theorem (8.3) without proof.

**Theorem 16 (Gilbarg and Trudinger, Theorem 8.3, Page 181)** Let the operator $L$ be as defined above with

$$\int_\Omega \left( dv - b_j \sum_j v \right) \, dx \leq 0$$

for all test functions $v \in C^1_0(\Omega)$. Then for $\phi \in W^{1,2}(\Omega)$ and $g, f_i \in L^2(\Omega)$ for $i = 1, 2, \ldots, n$ the Dirichlet problem

$$\begin{cases}
Lu = g + \sum_{i=1}^m D_i f_i & \text{in } \Omega \\
u = \phi & \text{on } \partial \Omega
\end{cases}$$

is uniquely solvable.

As we are interested in solutions to the homogeneous eigenvalue problem we state their existence and uniqueness results, somewhat abbreviated, for the eigenvalue problem. These
can be found on pages 182 and 183 in Section 8.2, [GT], in Theorem 8.6. Note that the requirement

\[ \int_{\Omega} \left( dv - b_j \sum_j v \right) dx \leq 0 \]

for all test functions \( v \in C_0^1(\Omega) \) is not needed here.

**Theorem 17 (See Gilbarg and Trudinger, Theorem 8.6, Pages 182)** Let the operator \( L \) be as defined above with \( \phi \in W^{1,2}(\Omega) \) and \( g, f_i \in L^2(\Omega) \) for \( i = 1, 2, \ldots, n \). Then there exists a discrete countable set \( \sigma \in \mathbb{R} \) such that if \( \lambda \notin \sigma \), the problems

\[
\begin{aligned}
Lu + \lambda u &= g + \sum_{i=1}^m D_i f_i \quad \text{in } \Omega \\
u &= \phi \quad \text{on } \partial \Omega
\end{aligned}
\]

are uniquely solvable for arbitrary \( g, f_i \) and \( \phi \). However, if \( \lambda \in \sigma \) then the subspaces of solutions for

\[
\begin{aligned}
Lu + \lambda u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

are of positive finite dimension. Furthermore, (note carefully, the change in sign convention) if

\[ \int_{\Omega} \left( dv - b_j \sum_j v \right) dx \geq 0 \]

for all test functions \( v \in C_0^1(\Omega) \), then \( \sigma \subset (0, \infty) \).

Note here that in general, the eigenvalue problem does not have a unique solution. However, we will see shortly for the first (least) eigenvalue, \( \lambda_1 \), that there is a unique solution of one sign that we typically take to be positive and that solution corresponds to the first eigenvalue which we denote \( \lambda_1 \). Also observe that if

\[ \phi = g + \sum_{i=1}^m D_i f_i \equiv 0 \]

and \( \lambda \notin \sigma \), then the unique solution is the trivial solution. In their Section 8.3, Gilbarg and
Trudinger show that under modest smoothness improvements of the coefficient functions of \( L \), that the weak solutions of the Dirichlet problem are in fact in \( W^{2,2}_{loc}(\Omega) \), not just in \( W^{1,2}(\Omega) \). We state their basic regularity result from Theorem 8.8, page 183, [GT].

**Theorem 18 (Gilbarg and Trudinger, Theorem 8.8, Page 183)** Let the operator \( L \) be as above but with coefficients \( a_{ij}(x), b_i(x) \in C^{0,1}(\bar{\Omega}) \), and \( c_i(x), d(x) \in L^{\infty}(\Omega) \) for all appropriate \( i \) and \( j \). Then if \( u \in W^{1,2}(\Omega) \) is a weak solution of

\[
\begin{aligned}
Lu &= f \quad \text{in} \quad \Omega \\
u &= \phi \quad \text{on} \quad \partial \Omega
\end{aligned}
\]

it also holds that \( u \in W^{2,2}_{loc}(\Omega) \).

Henceforth, we will assume that besides the bounds, and ellipticity condition that the coefficients of \( L \) also satisfy the smoothness properties required to employ the above theorem. Throughout the literature the various treatments of the eigenvalue problem give the results that the first eigenvalue is simple; that is there exists a unique first eigenfunction that, up to constant multiples, is unique. Additionally, the result is usually given that the first eigenfunction may be taken to be of one sign, usually positive. Although not it is not transparently so in Gilbarg and Trudinger’s brief presentation of their theorem on the existence and uniqueness of solutions to the eigenvalue problem, there are often additional constraints placed on the operator to guarantee the that the spectrum, \( \sigma \), is real valued. We will investigate this somewhat. In Section 6.5.2, "Eigenvalues of Nonsymmetric Elliptic Operators," pages 340-344, [Evans], Evans discusses eigenvalues for operators of the form

\[
\bar{L}u = \sum_{i,j} a_{ij}(x)u_{x_i,x_j} + \sum_i b_i(x)u_{x_i} + c(x)u
\]

with, for all \( i \) and \( j = 1, 2..., n \), \( a_{ij}(x) = a_{ji}(x) \). In such a case, it turns out, that the first
eigenvalue $\lambda_1$ for the problem

$$
\begin{cases}
L u + \lambda u = f & \text{in } \Omega \\
u = \phi & \text{on } \partial \Omega
\end{cases}
$$

is real, hence, as we will be primarily interested in the first eigenvalue and corresponding first eigenfunction of a given pde, our results would still hold. In general, though, as do most of the authors cited, we will assume that the operators $L$ and $\bar{L}$ are symmetric. That is, we will assume that for all $i$ and $j = 1, 2, ..., n$, $a_{ij}(x) = a_{ji}(x)$ and that $Lu$ may be written in the form

$$Lu = \sum_{i,j} D_i (a_{ij}(x) D_j u) + d(x)u$$

and that $\bar{L}u$ may be written

$$\bar{L}u = \sum_{i,j} a_{ij}(x) u_{x_i x_j} + c(x)u.$$ 

A worthwhile discussion of operators and their spectra appears in Chapter Seven, "Operator Theory" of Renardy and Rogers, pages 227-282, [RR], and some pertinent results to our work can be found in their Section 8.3, "Eigenfunction Expansions," pages 299-303 [RR]. A useful result states that the spectrum of the operators $L$ and $\bar{L}$ as restricted above, is real, discrete and countably infinite – see Theorem 8.22, page 302 in [RR]. Furthermore they conclude that the set of eigenfunctions is a complete orthonormal set. The symmetry consideration will again be important when we discuss the variational formulation of our pdes shortly.

We want to show that the first eigenfunction, $u_1$ for

$$
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

is of one sign and then that the first eigenvalue $\lambda_1$ is simple. A sketch of these proofs follow the statement of Theorem 8.38, page 214, [GT], in Gilbarg and Trudinger. Proofs also can
be found in Evans' Section 6.5, "Eigenvalues and Eigenfunctions," [Evans].

The proof that \( u_1 \) is of single sign follows from a corollary to the version of Hanack's Inequality for weak solutions. We state it here.

**Theorem 19** (Gilbarg and Trudinger's Corollary 8.21, page 199, [GT]) Let \( L \) be as above and let \( u \in W^{1,2}(\Omega) \) satisfy \( u \geq 0 \) (a.e.) in \( \Omega \). Then for any \( \hat{\Omega} \subset \subset \Omega \), we have

\[
\sup_{\hat{\Omega}} u \leq C \inf_{\Omega} u
\]

where \( C = C(n, \Theta/\theta, \nu, \hat{\Omega}, \Omega) \).

We state the single sign property of \( u_1 \) as a theorem:

**Theorem 20** Let \( u_1 \in W^{1,2}_0(\Omega) \) be a weak solution and the first eigenfunction for

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

where

\[
Lu = \sum_{i,j} D_i (a_{ij}(x)D_j u) + \sum_i (D_i (b_i(x)u) + c_i(x)D_i u) + d(x) u
\]

is strictly elliptic and satisfies the coefficient properties stated earlier. Then \( u_1 \) is of one sign.

**Proof.** Let the hypotheses of the theorem hold. Since \( u_1 \in W^{1,2}_0(\Omega) \) is a weak solution of the differential equation, we must have that

\[
B_\Omega[u_1, \phi] = \int_\Omega \sum_{j,k} a_{j,k} D_k u_1 D_j \phi + \sum_j b_j u_1 D_j \phi - \left( \sum_j c_j D_j u_1 + d u_1 \right) \phi \, dx
\]

\[
= -\lambda_1 \int_\Omega u_1 \phi \, dx
\]
for all test functions $\phi \in C_0^\infty(\Omega)$. We also have that

$$B_\Omega[u_1, u_1] = \int_\Omega \sum_{j,k} a_{j,k} D_k u_1 D_j u_1 + \sum_j b_j u_1 D_j u_1 - \left( \sum_j c_j D_j u_1 + du_1 \right) u_1 \, dx$$

$$= -\lambda_1 \int_\Omega (u_1)^2 \, dx$$

$$= -\lambda_1 \|u_1\|_{L^2(\Omega)}^2$$

since $u_i = \lim_{i \to \infty} \phi_i$ where for all $i$, $\phi_i \in C_0^\infty(\Omega)$. We observe that both

$$u_i^+ = \max\{u_1, 0\}$$

and

$$u_i^- = -\min\{u_1, 0\}$$

are in $W_0^{1,2}(\Omega)$ as well. (See Kesavan, page 61, Corollary to Theorem 2.2.5, [Kesav], for a proof of this). Since $u_1 = u_1^+ + u_1^-$ we have that

$$B_\Omega[u_1, u_1] = B_\Omega[u_1^+ + u_1^-, u_1^+ + u_1^-]$$

$$= B_\Omega[u_1^+, u_1^+] + 2B_\Omega[u_1^+, u_1^-] + B_\Omega[u_1^-, u_1^+]$$

and

$$B_\Omega[u_1, u_1] = B_\Omega[u_1^+, u_1^+] + B_\Omega[u_1^-, u_1^-]$$

$$= -\lambda_1 \left[ \|u_1^+\|_{L^2(\Omega)}^2 + \|u_1^-\|_{L^2(\Omega)}^2 \right].$$

Now suppose that $u_1^+ = 0$ on some $\bar{\Omega} \subset \subset \Omega$. Then by the corollary to Harnack's inequality cited above we would have

$$u_1^+ \equiv 0.$$

The situation with $u_1^-$ is similar. Thus we may conclude that either $u_1 = u_1^+$ or $u_1 = u_1^-$, hence $u_1$ is of one sign.
The proof that the first eigenvalue is simple is a consequence of the fact that the collection of eigenfunctions is a complete orthonormal set and the theorem above. We state this also as a theorem. The proof is similar to that found in Evans or Gilbarg and Trudinger.

**Theorem 21** Let $u_1 \in W_0^{1,2}(\Omega)$ be a weak solution and the first eigenfunction for

$$
\begin{align*}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
$$

where

$$
Lu = \sum_{i,j} D_i (a_{ij}(x) D_j u) + d(x)u
$$

is strictly elliptic and satisfies the coefficient properties stated earlier. Let the sequence

$$
\{u_i^j\}_{i=1}^{\infty}
$$

be the complete orthonormal set of eigenfunctions for $L^2(\Omega)$ where $j = 1, 2, ..., n_i$ indicates the multiplicity of each eigenvalue. Then the first eigenvalue, $\lambda_1$, is simple; i.e., if $u_1$ and $v$ satisfy

$$
\begin{align*}
Lu + \lambda_1 u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
$$

then $|u| = |kv|$ a.e. for some constant, $k$, or $v = 0$.

**Proof.** Assume that the hypotheses of the theorem hold, but there exists $v$, such that $kv \neq 0$ for some constant, $k$, as well as $u_1$, which solve

$$
\begin{align*}
Lu + \lambda_1 u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
$$

Since $u_1$ and $v$ are of both of one sign, it is impossible that

$$
\langle u_1, kv \rangle = k \int_{\Omega} u_1 v \, dx = 0.
$$
This contradicts the fact that they are orthogonal and the conclusion of the theorem follows.

At this stage we have not really shown that the least eigenvalue is actually $\lambda_1$. That is, we have yet to show that

$$\lambda_1 = \min_{i \in \mathbb{N}} \{ \lambda_i \mid \lambda_i \in \sigma(L) \}$$

where $\sigma(L)$ is the real discrete spectrum of the (symmetric) operator $L$. It is clear that if $\lambda_i \in \sigma(L)$ and $u_i$ is the corresponding eigenfunction that the Rayleigh quotient

$$J_\Omega(u_i) = \frac{B_\Omega[u_i, u_i]}{\|u_i\|^2_{L^2(\Omega)}} = \lambda_i$$

is well defined, but we do not know that the last term in the equality string

$$\lambda_1 = \min_{i \in \mathbb{N}} \{ J_\Omega(u_i) \} = \inf_{\|u\|_{L^2(\Omega)} = 1, \ u \in W_0^{1,2}(\Omega)} \{ J_\Omega(u) \}$$

is even well defined. We will prove shortly, following an argument in Evans, that

$$\lambda_1 = \min_{\|u\|_{L^2(\Omega)} = 1, \ u \in W_0^{1,2}(\Omega)} \left\{ \frac{B_\Omega[u, u]}{\|u\|^2_{L^2(\Omega)}} \right\}.$$
and this follows from the calculus of variation formulation of the partial differential equation

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

For an operator \( L \) we can define a functional

\[
I[w] = \int_{\Omega} L(p, z, x) dx \\
\int_{\Omega} L(p_1, p_2, \ldots, p_n, z, x_1, x_2, \ldots, x_n) dx
\]

so

\[
I[w] = \int_{\Omega} L(Dw, w, x) dx
\]

where \( z = w(x) \) and \( p_i = w_{x_i} \) for all \( i = 1, 2, \ldots, n \). We want to find a function \( u \) so that \( I[u] \) achieves a minimum value over some space of functions, say \( X \). If \( u \) is such that \( u = 0 \) on \( \partial \Omega \) and a minimizer of \( I \) then it solves a particular differential equation. If we define

\[
i'(\varepsilon) = I[u + \varepsilon v]
\]

for any \( v \in C_0^\infty(\Omega) \) it is clear that \( i \) has a minimum when \( \varepsilon = 0 \), so we might say that \( i'(0) = 0 \). Assuming that we could just compute \( i' \), sometimes called the first variation, we would write

\[
i'(\varepsilon) = \int_{\Omega} \sum_{j=1}^{n} \left[ L_{p_j} (Du + \varepsilon Dv, u + \varepsilon v, x) v_{x_j} \right] + L_z (Du + \varepsilon Dv, u + \varepsilon v, x) v \, dx
\]

and setting \( \varepsilon = 0 \) and integrating the sum by parts we would have

\[
i'(0) = \int_{\Omega} \left[ - \sum_{j=1}^{n} \left[ L_{p_j} (Du, u, x) x_j \right] + L_z (Du, u, x) \right] v \, dx = 0.
\]
If this were to hold for all \( v \in C^\infty_0(\Omega) \) then we would have the partial differential equation

\[
\begin{aligned}
- \sum_{j=1}^n \left[ L_{p_j} (Du, u, x)_{x_j} \right] + L_z (Du, u, x) &= 0 \quad \text{in} \quad \Omega \\
& \quad \text{on} \quad \partial \Omega, \\
\end{aligned}
\]

The first equation above is the well known Euler-Lagrange equation. It is easy to see that if

\[
L(p, z, x) = \frac{1}{2} \left[ \sum_{i,j} a_{ij}(x)p_i p_j - z^2 (c(x) + \lambda) \right]
\]

then

\[
i(\epsilon) = I[u + \epsilon v]
\]

and

\[
i(\epsilon) = \frac{1}{2} \int_\Omega \sum_{j=1}^n \left[ L_{p_j} (Du + \epsilon Dv, u + \epsilon v, x) v_{x_j} \right] - L_z (Du + \epsilon Dv, u + \epsilon v, x) v \ dx
\]

with

\[
i(\epsilon) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \left[ (a_{ji}(x)p_i + a_{ij}(x)p_i + \epsilon Dv) v_{x_j} \right] - (2u (c(x) + \lambda) + \epsilon v) v \ dx
\]

\[
= \int_\Omega \sum_{i,j=1}^n \left( a_{ij}(x) u_{x_i} + \epsilon Dv \right) v_{x_j} - (u (c(x) + \lambda) + \epsilon v) v \ dx.
\]

The last equality follows from the symmetry of the matrix \([a_{ij}]\). In minimizing \(i(\epsilon)\) we want to find a function \(u\) so that

\[
0 = i'(0) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} - (c(x) + \lambda) uv \ dx
\]

for all \( v \in C^\infty_0(\Omega) \). Note that a function \(u\) satisfying this is also a weak solution to the pde

\[
\begin{aligned}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
& \quad \text{on} \quad \partial \Omega
\end{aligned}
\]

where \(Lu\) is our familiar symmetric pde operator in divergence form. The corresponding
Euler-Lagrange equation is

\[
\begin{aligned}
0 &= -\sum_{i,j} (a_{ij}(x)u_{x_i})_{x_j} - (c(x) + \lambda) u \\
&= \sum_{i,j} (a_{ij}(x)u_{x_i})_{x_j} + (c(x) + \lambda) u \\
&= Lu + \lambda u.
\end{aligned}
\]

The boundary conditions of the pde are usually captured by the admissible set of functions \(u\) from which the minimizer comes. For us the admissible set is the functions \(u \in W^{1,2}_0(\Omega)\) where \(\|u\|_{L^2(\Omega)} = 1\). Under these observations and constraints it is easy to see that

\[
\lambda_1 = \min_{i \in \mathbb{N}} \{J_\Omega(u_i)\} = \inf_{\|u\|_{L^2(\Omega)} = 1, u \in W^{1,2}_0(\Omega)} \{J_\Omega(u)\}.
\]

The conditions under which a unique solution to the variational problem exists are rather complicated but are dependent on the coercivity and weak lower semi-continuity of the functional \(I\). The coercivity of the functional \(I\) is also related to the coercivity of the bilinear form \(B\) for the operator \(L\). Theorems in Evans, Section 8.2, pages 443-454, [Evans], specifically spell out these details. In our theorems when we use the fact that

\[
\lambda_1 = \min_{\|u\|_{L^2(\Omega)} = 1} \{J_\Omega(u)\} = \min_{\|u\|_{L^2(\Omega)} = 1, u \in W^{1,2}_0(\Omega)} \frac{B_\Omega[u, u]}{\|u\|_{L^2(\Omega)}^2}
\]

we may state or assume that \(L\) admits a variational formulation. However, we will prove a version of the statement above, following Evans’ arguments on pages 336-338, [Evans].

**Theorem 22** Let \(u_1 \in W^{1,2}_0(\Omega)\) be a weak solution and the first eigenfunction for

\[
\begin{aligned}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
\]

where

\[
Lu = \sum_{i,j} D_i (a_{ij}(x) D_j u) + d(x)u
\]
is strictly elliptic and satisfies the coefficient properties stated earlier (including the symmetry condition on \([a_{ij}]\)). Also let \(\lambda_1\) be the first eigenvalue where we have simply ordered the eigenvalues as a discrete set

\[ \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \]

Let the sequence \(\{u_i\}_{i=1}^\infty\) be the complete orthonormal set of eigenfunctions for \(L^2(\Omega)\). Then

\[ \lambda_1 = \min_{\|u\|_{L^2(\Omega)}=1, \, u \in W_0^{1,2}(\Omega)} \frac{B[u, u]}{\|u\|_{L^2(\Omega)}^2}, \]

i.e.,

\[ \lambda_1 = \min_{\|u\|_{L^2(\Omega)}=1, \, u \in W_0^{1,2}(\Omega)} \frac{\int_{\Omega} \sum_{i,j} a_{ij}(x) u_{x_i} u_{x_j} - d(x) u^2 \, dx}{\|u\|_{L^2(\Omega)}^2}. \]

**Proof.** Let the hypotheses of the theorem hold. For each \(i = 1, 2, \ldots\), we have

\[ B[u_i, u_i] = \lambda_i \]

and for \(i \neq j\) we have

\[ B[u_i, u_j] = \lambda_i \langle u_i, u_j \rangle = 0. \]

Now let \(u \in W_0^{1,2}(\Omega)\) and \(\|u\|_{L^2(\Omega)} = 1\). Since \(\{u_i\}_{i=1}^\infty\) is an orthonormal basis we may write that

\[ u = \sum_{i=1}^\infty v_i u_i \]

where

\[ v_i = \langle u, u_i \rangle = \int_{\Omega} u v_i \, dx \]

so

\[ \|u\|_{L^2(\Omega)}^2 = \left\| \sum_{i=1}^\infty u_i \int_{\Omega} u v_i \, dx \right\|_{L^2(\Omega)}^2. \]

Additionally, we can say, by the properties of Hilbert spaces, that

\[ \sum_{i=1}^\infty (v_i)^2 = \|u\|_{L^2(\Omega)}^2 = 1. \]
It is easy to see that the sequence
\[ \left\{ \frac{u_i}{\sqrt{\lambda_i}} \right\}_{i=1}^{\infty} \]
is also another orthonormal subset for \( L^2(\Omega) \) and we claim that it is, in fact, an orthonormal basis for \( W^{1,2}_0(\Omega) \) with respect to the inner product
\[ \langle u, v \rangle_B = B[u, v]. \]
as well. It is clear that
\[ B[u_i, w] = \lambda_i \langle u_i, w \rangle = 0 \]
for all \( i = 1, 2, \ldots \), and implies that \( w = 0 \) because \( \{u_i\}_{i=1}^{\infty} \) is a basis for \( L^2(\Omega) \). Therefore we can write that
\[ u = \sum_{i=1}^{\infty} v_i u_i = \sum_{i=1}^{\infty} \phi_i \frac{u_i}{\sqrt{\lambda_i}} \]
where
\[ \phi_i = \left\langle u_i, \frac{u_i}{\sqrt{\lambda_i}} \right\rangle_B = B[u, \frac{u_i}{\sqrt{\lambda_i}}]. \]
Note as well that this implies that
\[ \phi_i = v_i \sqrt{\lambda_i}. \]
Thus the series converges in \( W^{1,2}_0(\Omega) \) as well as in \( L^2(\Omega) \). We can now say that
\[ B[u, u] = B \left[ \sum_{i=1}^{\infty} v_i u_i, \sum_{i=1}^{\infty} v_i u_i \right] = \sum_{i=1}^{\infty} (v_i)^2 \lambda_i \geq \lambda_1. \]
It is also clear that if \( u = u_1 \) then equality holds.

Armed with these ideas we move on.

Throughout the following discourse we will need, on several occasions, to extend a function \( u \in W^{1,2}_0(\Omega) \), by zero to some domain \( \tilde{\Omega} \supset \Omega \) where, in fact, \( \tilde{\Omega} \) may be all of \( \mathbb{R}^n \). We will show, closely following an argument in Adams and Fournier, ([AF], Lemma 3.27, page 71), that this can be done irrespective of any boundary regularity considerations on \( \Omega \). To prove this we will make the following definition also taken from Adams and Fournier...
Definition 23 (Zero Extensions) If a function $u$ is defined on $\Omega$ let $\tilde{u}$ denote the extension to the complement $\Omega^c$ of $\Omega$ in $\mathbb{R}^n$:

$$
\tilde{u} = \begin{cases} 
  u(x) & \text{if } x \in \Omega \\
  0 & \text{if } x \in \Omega^c 
\end{cases}
$$

Lemma 24 (Extension of $u \in W_0^{k,p}(\Omega)$ to $W_0^{k,p}(\mathbb{R}^n)$) Let $u \in W_0^{k,p}(\Omega)$. If $|\alpha| \leq k$, then $D_\alpha \tilde{u} = \tilde{D}_\alpha u$ in the weak sense in $\mathbb{R}^n$. Hence $\tilde{u} \in W_0^{k,p}(\mathbb{R}^n)$.

**Proof.** Let $\{\phi_i\}_{i=1}^\infty$ be a sequence of $C_0^\infty(\Omega)$ functions converging to $u \in W_0^{k,p}(\Omega)$. For any $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\alpha$ a multi-index such that $|\alpha| \leq k$, we may compute

$$
(-1)^{|\alpha|} \int_{\mathbb{R}^n} \tilde{u} D_\alpha \psi \, dx = (-1)^{|\alpha|} \int_{\Omega} u D_\alpha \psi \, dx
$$

$$
= \lim_{i \to \infty} (-1)^{|\alpha|} \int_{\Omega} \phi_i D_\alpha \psi \, dx
$$

$$
= \lim_{i \to \infty} \int_{\Omega} (D_\alpha \phi_i) \psi \, dx
$$

$$
= \int_{\mathbb{R}^n} \psi \tilde{D}_\alpha u \, dx.
$$

Since $u$ was arbitrary, we have that

$$
\|\tilde{u}\|_{W_0^{k,p}(\mathbb{R}^n)} = \|u\|_{W_0^{k,p}(\Omega)}.
$$

Before we embark on the lower or upper continuity proofs, we wish to state and prove a lemma that will be required for both proofs. We state the proof in the general case for the operator $L$, as previously defined, and $\Omega_i$ and $\Omega_j$ bounded open sets in $\mathbb{R}^n$.

Lemma 25 (Monotonicity of the First Eigenvalue) For the partial differential equations with the operator $L$, as above and fixed, admits a variational formulation and has
respective minimizing solutions $u_i$ and $u_j$ for the problems

$$
\begin{aligned}
\begin{cases}
Lu + \lambda_i u &= 0 \text{ in } \Omega_i \\
u &= 0 \text{ on } \partial\Omega_i
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{cases}
Lu + \lambda_j u &= 0 \text{ in } \Omega_j \\
u &= 0 \text{ on } \partial\Omega_j
\end{cases}
\end{aligned}
$$

where $\Omega_i$ and $\Omega_j$ satisfy $\Omega_i \subseteq \Omega_j$, $\Omega_i \neq \Omega_j$. Then the first eigenvalues $\lambda_i$ and $\lambda_j$ satisfy, $\lambda_i \geq \lambda_j$.

**Proof.** Let the hypotheses of the lemma hold and $B$ be the associated bilinear form for the operator $L$. From facts acknowledged previously, we know that the first eigenfunctions and corresponding first eigenvalues exist. The remainder of the proof is a simple consequence of the fact the first eigenvalue, $\lambda_i$, of the problem

$$
\begin{aligned}
\begin{cases}
Lu + \lambda_i u &= 0 \text{ in } \Omega_i \\
u &= 0 \text{ on } \partial\Omega_i
\end{cases}
\end{aligned}
$$

satisfies

$$\lambda_i = \min_{\|w\|_{L^2(\Omega_i)} = 1, w \in W_0^{1,2}(\Omega_i)} \{B[w, w]\} = B[u_i, u_i]$$

and since $\Omega_i \subseteq \Omega_j$, $\lambda_j$ must satisfy

$$\min_{\|w\|_{L^2(\Omega_j)} = 1, w \in W_0^{1,2}(\Omega_i)} \{B[w, w]\} = \lambda_j \leq \lambda_i = \min_{\|w\|_{L^2(\Omega_i)} = 1, w \in W_0^{1,2}(\Omega_i)} \{B[w, w]\}.$$  

When necessary, we will refer to this lemma briefly as the monotonicity of eigenvalues lemma ignoring the misleading nature of the abbreviation. We take time to make some comments.

**Remark 26** It can be shown also that the inequality between eigenvalues is in fact strict.
That is, in the lemma above, we may conclude that $\lambda_i$ and $\lambda_j$ satisfy, $\lambda_i > \lambda_j$. A proof of this may be found in Appendix B. Under additional hypotheses on the boundary of the domain and the coefficient functions of the pde, the lemma above, and the stricter result just given, may be proved using the results of Heywood, Noussair, and Swanson. A proof of the stricter result employing the theorems of Heywood, Noussair, and Swanson is also given in the same appendix.

Throughout our lower and upper continuity proofs we will want to restrict the properties of our elliptic operator $L$ somewhat. We make the following definition that we will assume unless otherwise noted.

**Definition 27 (Uniformly Elliptic Variational Operator)** Let $L$ be a second order uniformly elliptic partial differential operator in divergence form that admits a variational formulation such that

\[
(i) \quad Lu = \sum_{i,j} D_i (a_{ij} (x) D_j u) + \sum_i (D_i (b_i (x) u) + c_i (x) D_i u) + d(x) u
\]

\[
(ii) \quad \sum_{i,j} a_{ij} (x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for all } x \in \Omega_0, \xi \in \mathbb{R}^n
\]

\[
(iii) \quad \sum_{i,j} |a_{ij} (x)|^2 \leq \Theta^2 \quad \text{and}
\]

\[
(iv) \quad \frac{1}{\theta^2} \sum_i \left| b_i (x) \right|^2 + |c_i (x)|^2 + \frac{1}{\theta} |d(x)| \leq \nu^2.
\]

We say, more briefly that $L$ is a uniformly elliptic variational operator.

**1.5.2 Lower Continuity Proof**

We will now state the general lower continuity theorem for arbitrary domains in $n$ dimensions with our general elliptic operator.

**Theorem 28 (Lower Continuity $n$ Dimensions)** Let $\{\Omega_i\}_{i=1}^{\infty}$ be an increasing sequence of nested domains with

\[
\lim_{i \to \infty} \Omega_i = \Omega_0
\]
and let $L$ be a uniformly elliptic variational operator. For any $\epsilon > 0$ there exists a value $I$ so that $0 < \lambda_i - \lambda_0 < \epsilon$ whenever $i > I$, where $\lambda = \lambda_0$ is the first eigenvalue for the pde

\[
\begin{align*}
Lu + \lambda u &= 0 \quad \text{in } \Omega_0 \\
u &= 0 \quad \text{on } \partial\Omega_0
\end{align*}
\]

That is, the eigenvalue $\lambda$ for the partial differential equation

\[
\begin{align*}
Lu + \lambda u &= 0 \quad \text{in } \Omega_b \\
u &= 0 \quad \text{on } \partial\Omega_b
\end{align*}
\]

is a lower continuously dependent function of the domain $\Omega_b$ at $b = b_0$.

**Proof.** Let the hypotheses of the theorem hold. From existence and uniqueness theorems cited earlier, we know that of each $i = 1, 2, \ldots$, there exists a first eigenfunction $u_i$ and corresponding first eigenvalue $\lambda_i$ for each problem

\[
\begin{align*}
Lu + \lambda u &= 0 \quad \text{in } \Omega_i \\
u &= 0 \quad \text{on } \partial\Omega_i
\end{align*}
\]

For the operator $L$ and its associated bilinear form $B$, $B_i$ being particular to the domain $\Omega_i$ we in fact have

\[
J_i(u) = \frac{B_i[u, u]}{\|u\|_{L^2(\Omega_i)}^2}
\]

\[
= \frac{\int_{\Omega_i} \sum_{j,k} a_{j,k} D_k u D_j u + \sum_j b_j u D_j u - \left( \sum_j c_j D_j u + d u \right) u \, dx}{\int_{\Omega_i} u^2 \, dx}.
\]

Also we have

\[
\min_{u \in W_0^{1,2}(\Omega)} J_i = \frac{B_i[u, u]}{\|u\|_{L^2(\Omega_i)}^2} = \frac{B_i[u_i, u_i]}{\|u_i\|_{L^2(\Omega_i)}^2} = \lambda_i.
\]

We state a lemma and prove the result.

**Lemma 29** Let $L$, $B$, and $J$ be defined as above on some fixed domain $\Omega$. Suppose that
$u, v \in W^{1,2}_0(\Omega)$, $\|u\|_{L^2(\Omega)} = 1$, then the quantity $J(v) \to J(u)$ as $v \to u$ in $W^{1,2}(\Omega)$.

**Proof.** As in the two dimensional case we compute

$$|J(v) - J(u)| = \left| \frac{J(v)}{\|v\|_{L^2(\Omega)}} - J(u) \right| = |B[w, w] - J(u)|$$

where

$$w = \frac{v}{\|v\|_{L^2(\Omega)}}.$$  

Using the estimates (ii) and (iii) in the hypotheses of the lower continuity theorem for $n$ dimensions we will find that

$$|J(v) - J(u)| \leq C(\theta, \Theta, \nu) \|w - u\|_{W^{1,2}(\Omega)}^2$$

where $C(\theta, \Theta, \nu)$ denotes some constant dependent on $\theta$, $\Theta$, and $\nu$. Now

$$|J(v) - J(u)| = |B[w, w] - B[u, u]|$$

$$= \left| \int_\Omega \sum_{j,k} a_{j,k} D_k w D_j w + \sum_j b_j w D_j w - \left( \sum_j c_j D_j w + w \right) w \, dx \right|$$

$$- \left( \int_\Omega \sum_{j,k} a_{j,k} D_k u D_j u + \sum_j b_j u D_j u - \left( \sum_j c_j D_j u + u \right) u \, dx \right)$$

$$\leq \left| \int_\Omega \sum_{j,k} a_{j,k} (D_k w D_j w - D_k u D_j u) \, dx \right|$$

$$+ \left| \int_\Omega \sum_j (b_j - c_j) (w D_j w - u D_j u) \, dx \right| + \left| \int_\Omega d (u^2 - w^2) \, dx \right|$$

$$\leq \Theta^2 \|\nabla (w - u)\|_{L^2(\Omega)}^2 + (\Theta^2 + \nu^2) \|\nabla (w - u)\|_{L^2(\Omega)} \|w - u\|_{L^2(\Omega)}$$

$$+ \nu^2 \|w - u\|_{L^2(\Omega)}^2.$$  

Acknowledging that the bound, $\nu$, for the lower order coefficients of $L$ is dependent upon the
ellipticity constant, $\theta$, gives the inequality

$$|J(v) - J(u)| \leq C(\theta, \Theta, v) \|w - u\|^2_{W^{1,2}(\Omega)}.$$ 

Alternatively, this may be seen as a consequence of the fact that the bilinear form, $B$, for $L$, is bounded. We now consider the quantity

$$\|w - u\|_{W^{1,2}(\Omega)} = \left\| \frac{v}{\|v\|_{L^2(\Omega)}} - u \right\|_{W^{1,2}(\Omega)}.$$ 

Now

$$\left\| \frac{v}{\|v\|_{L^2(\Omega)}} - u \right\|_{W^{1,2}(\Omega)} = \frac{1}{\|v\|_{L^2(\Omega)}} \left\| v - \|v\|_{L^2(\Omega)} u \right\|_{W^{1,2}(\Omega)} = \frac{1}{\|v\|_{L^2(\Omega)}} \left| v - \|v\|_{L^2(\Omega)} u \right| + \left\| v - \|v\|_{L^2(\Omega)} u \right\|_{W^{1,2}(\Omega)} \leq \frac{1}{\|v\|_{L^2(\Omega)}} \left( |v|_{L^2(\Omega)} + \|v - u\|_{W^{1,2}(\Omega)} \right).$$

It is clear that the second term goes to zero as $v \to u$ in $W^{1,2}(\Omega)$. Since

$$0 \leq \|v - u\|_{L^2(\Omega)} \leq \|v - u\|_{W^{1,2}(\Omega)}$$

and

$$\|u\|_{L^2(\Omega)} = 1$$

we must have that

$$\|v\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)} = 1$$

as $v \to u$ in $W^{1,2}(\Omega)$ as well. Consequently

$$\left\| \frac{v}{\|v\|_{L^2(\Omega)}} - u \right\|_{W^{1,2}(\Omega)} \to 0$$

as $v \to u$ in $W^{1,2}(\Omega)$, hence, $J(v) \to J(u)$ as $v \to u$ in $W^{1,2}(\Omega)$.

For any given $\epsilon > 0$ we can, from the lemma, find $\delta$, small enough, and $v \in \{v_i\}_{i=1}^{\infty}$ for $v_i \rightarrow u_0$ so that $v \in \{v_i\}$ for $v_i - u_0 \leq \epsilon$

whenever

$$\left\| v_0 - \frac{v}{\|v\|_{L^2(\Omega_0)}} \right\|_{W^{1,2}(\Omega_0)} < \delta.$$ 

It remains to show that the function $v$ satisfies $spt(v) \subset \Omega_i$ for every value of $i \geq I$ for some $I$ sufficiently large. We state and prove a general compactness argument.

**Lemma 30 (Standard Compactness Argument in $n$ Dimensions)** Let $v \in C^\infty_0(\Omega_0)$ and let $\{\Omega_i\}_{i=1}^{\infty}$ be an increasing sequence of nested domains with $\lim_{i \rightarrow \infty} \Omega_i = \Omega_0$.

Then there exists $I \in \mathbb{N}$ so that $spt(v) \subset \Omega_i$ for all $i \geq I$.

**Proof.** Let $S = \text{supp}(v)$. Since $v \in C^\infty_0(\Omega_0)$ and by definition the support of $v$ is compact in $\Omega_0 \cap \mathbb{R}^n$, we have that $S$ is a compact set. For each $x \in S$ choose an index $i_x \in \mathbb{N}$ so that $x \in \Omega_i$ for all $i > i_x$ and a ball about it, say $B(\epsilon_x, x)$ so that the ball is inside $\Omega_i$. By the Heine-Borel Theorem the open cover (i.e., the collection of open balls) has a finite subcover indexed by a finite collection of numbers, $i_x$. Let $I$ be the maximum such index to conclude that $spt(v) \subset \Omega_i$ for all $i \geq I$. ■

Once again, by the monotonicity of eigenvalues lemma we can write that

$$\lambda_0 \leq \lambda_i \leq J_0(v) \leq \lambda_0 + \epsilon$$

concluding the proof. ■
We remark that the compactness argument presented in last lemma of the proof just given still holds in the event that \( \Omega_0 = \mathbb{R}^n \), although we will not use this fact.

We are now ready to move on to the upper continuity proof.

1.5.3 Upper Continuity Proof

We will tackle the upper continuity situation somewhat differently than we did in the lower continuity proof. We will, as in the lower continuity case, first show that the sequence of first eigenfunctions has a convergent subsequence of functions in an appropriate space. We call this "Part One". Unfortunately, it is not clear that the limit function of the convergent subsequence satisfies the appropriate boundary conditions of the desired partial differential equation. Because of this, we will require a "Part Two" for the proof. For the first part of the proof in the \( n \) dimensional case, we will show that if \( \{ \Omega^i \}_{i=1}^{\infty} \) is a decreasing nested sequence of domains, decreasing to \( \Omega_0 \), then the sequence of first eigenfunctions \( \{ u^i \}_{i=1}^{\infty} \) has a convergent subsequence, converging to

\[ \bar{u} \in W_0^{1,2}(\Omega^1) \]

and that

\[ \lim_{i \to \infty} \lambda^i = \bar{\lambda} = \lambda_0. \]

The second part of the upper continuity proof, Part Two, will consist in showing that

\[ \bar{u} \in W_0^{1,2}(\Omega_0). \]

We will proceed as we did for the lower continuity proof for \( n \) dimensions and consider a more general uniformly elliptic operator, \( L \). We will assume that

\[ Lu = \sum_{i,j} D_i (a_{ij}(x) D_j u) + \sum_i (D_i (b_i(x) u) + c_i(x) D_i u) + d(x) u \]
where we are studying

\[
\begin{align*}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

and \( \Omega \) is a domain in \( \mathbb{R}^n \). Also, we assume that \( L \) admits a variational formulation. As in the two-dimensional rectangular case, we will consider a decreasing nested sequence of domains, \( \{\Omega^i\}_{i=1}^{\infty} \), decreasing to \( \Omega_0 \). From the existence and uniqueness argument presented earlier, we have for each \( i \), a weak solution \( u^i(x, y) \in W_0^{1,2}(\Omega^i) \) and corresponding first eigenvalue \( \lambda^i \). We will, in addition, denote the associated bilinear form for \( L \) by

\[
B_{\Omega^i}[u, v] = B^i[u, v] = \int_{\Omega^i} \sum_{j,k} a_{j,k} D_k u D_j v + \sum_{j} b_j u D_j v - \left( \sum_{j} c_j D_j u + du \right) v ~dx
\]

and the (generalized) Rayleigh quotient by

\[
J^i(u) = \frac{B^i[u, u]}{\|u\|_{L^2(\Omega^i)}^2} = \frac{\int_{\Omega^i} \sum_{j,k} a_{j,k} D_k u D_j u + \sum_{j} b_j u D_j u - \left( \sum_{j} c_j D_j u + du \right) u ~dx}{\int_{\Omega^i} u^2 dx}.
\]

These definitions follow those of Gilbarg and Trudinger in Chapter 8, [GT]. We will again make use of the fact that on any domain \( \Omega \),

\[
\min_{u \in W_0^{1,2}(\Omega)} J_\Omega = \frac{B_\Omega[u, u]}{\|u\|_{L^2(\Omega)}^2} = \lambda,
\]

where \( \lambda \) is the first eigenvalue for

\[
\begin{align*}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

Finally, we want to prove the following theorem:

**Theorem 31 (Upper Continuity \(n\) Dimensions)** Let \( \{\Omega^i\}_{i=1}^{\infty} \) be a decreasing nested sequence of domains, decreasing to \( \Omega_0 \), and let \( L \) uniformly elliptic variational operator.
For any $\epsilon > 0$ there exists a value $I$ so that $0 < \lambda_0 - \lambda^i < \epsilon$ whenever $i > I$, where $\lambda = \lambda_0$ is the first eigenvalue for the pde

$$
\begin{align*}
Lu + \lambda u &= 0 \text{ in } \Omega_0 \\
u &= 0 \text{ on } \partial\Omega_0.
\end{align*}
$$

That is, the eigenvalue $\lambda$ for the partial differential equation

$$
\begin{align*}
Lu + \lambda u &= 0 \text{ in } \Omega^b \\
u &= 0 \text{ on } \partial\Omega^b
\end{align*}
$$

is an upper continuous function of $b$ with respect to the domain $\Omega^b$ at $b = 0$ where we have $\Omega^0 = \Omega_0$ by definition.

1.5.3.1 Upper Continuity Proof, Part One

For the first part of the proof in the $n$ dimensional case, we will show that if $\{\Omega^i\}_{i=1}^\infty$ is a decreasing nested sequence of domains, decreasing to $\Omega_0$, then a sequence of first eigenfunctions $\{u^i\}_{i=1}^\infty$ where $\|u^i\|_{L^2(\Omega^i)} = 1$ has a convergent subsequence, converging to

$$
\bar{u} \in W^{1,2}_0(\Omega^1)
$$

and that

$$
\lim_{i \to \infty} \lambda^i = \bar{\lambda} = \lambda_0.
$$

The second part of the upper continuity proof, Part Two, will consist in showing that

$$
\bar{u} \in W^{1,2}_0(\Omega_0).
$$

We state the intermediate result as a theorem.

Theorem 32 (Upper Continuity Part One for $n$ Dimensions) Let $\{\Omega^i\}_{i=1}^\infty$ be a decreasing sequence of nested domains in $\mathbb{R}^n$, decreasing to $\Omega_0$, and let $L$ uniformly elliptic...
variational operator. For each $i = 1, 2, \ldots$, let $u^i$ solve

\[
\begin{aligned}
\begin{cases}
Lu + \lambda u &= 0 \quad \text{in } \Omega^i \\
u &= 0 \quad \text{on } \partial \Omega^i
\end{cases}
\end{aligned}
\]

with corresponding first eigenvalue, $\lambda^i$, and similarly, let $u_0$ be the first eigenfunction for

\[
\begin{aligned}
\begin{cases}
Lu + \lambda u &= 0 \quad \text{in } \Omega_0 \\
u &= 0 \quad \text{on } \partial \Omega_0
\end{cases}
\end{aligned}
\]

with corresponding first eigenvalue $\lambda_0$. Then a sequence $\{u^i\}_{i=1}^{\infty}$ where $\|u^i\|_{L^2(\Omega^i)} = 1$ has a convergent subsequence in $W^{1,2}_0(\Omega^i)$, i.e., $u^i \to \overline{u}$ and

\[
J(u^i) = \lambda^i \to \bar{\lambda} = J(\overline{u}) = \lambda_0
\]

as $i \to \infty$, where

\[
J(w) := J^1(w) = \frac{\int_{\Omega^i} \sum_{j,k} a_{j,k} D_k w D_j w + \sum_j b_j w D_j w - \sum_j (c_j D_j w + dw) w \, dx}{\|w\|_{L^2(\Omega^i)}^2}.
\]

**Proof.** Let $\{\Omega^i\}_{i=1}^{\infty}$ be a decreasing sequence domains in $\mathbb{R}^n$, decreasing to $\Omega_0$. Furthermore, let $u_0$ solve

\[
\begin{aligned}
\begin{cases}
Lu + \lambda u &= 0 \quad \text{in } \Omega_0 \\
u &= 0 \quad \text{on } \partial \Omega_0
\end{cases}
\end{aligned}
\]

and $\lambda_0$ be the corresponding first eigenvalue. Additionally, let $u^i$ solve

\[
\begin{aligned}
\begin{cases}
Lu + \lambda u &= 0 \quad \text{in } \Omega^i \\
u &= 0 \quad \text{on } \partial \Omega^i
\end{cases}
\end{aligned}
\]

with corresponding first eigenvalue, $\lambda^i$. Without loss of generality we will assume that

\[
\|u^i\|_{L^2(\Omega^i)} = 1
\]
for all values of \( i \), and that, following a suitable extension by zero to \( \Omega^1 \),

\[
||u^i||_{L^2(\Omega^1)} = 1
\]

as well. As in the two dimensional case by the monotonicity of eigenvalues lemma, we have that the sequence of first eigenvalues, \( \{\lambda^i\}_{i=1}^{\infty} \), is an increasing sequence of numbers as \( i \to \infty \), and, furthermore, is bounded above by \( \lambda_0 \). We may therefore write the inequality \( \lambda^i \leq \lambda_0 \). We now consider the sequence of (extended) functions \( \{u^i\}_{i=1}^{\infty} \). We claim that this is a bounded sequence of functions in \( W^{1,2}_0(\Omega^1) \). It is clear (after suitable extension) that for all \( i = 1, 2, \ldots \), that \( u^i \in W^{1,2}_0(\Omega^1) \). We claim that the sequence is also bounded in \( W^{1,2}_0(\Omega^1) \). Recall that for all \( i = 1, 2, \ldots \),

\[
\lambda^i = B^1[u^i, u^i].
\]

The higher order coefficients for the pde do not appear in the \( W^{1,2} \) norm for the \( n \) dimensional case with a more general operator \( L \) than the Laplacian. We wish to show that for all \( i = 1, 2, \ldots \)

\[
||u^i||^2_{W^{1,2}(\Omega_{a,b}^1)} = \int_{\Omega^1} (\nabla u^i)^2 + (u^i)^2 \, dx \leq C.
\]

Following an argument similar to that given for Lemma 8.4, page 181, in Gilbarg and Trudinger we compute the following.

\[
B^1[u, u] = \int_{\Omega^1} \sum_{j,k} a_{j,k} D_k u D_j u + \sum_{j} (b_j - c_j) u D_j u - \theta u^2 \, dx \geq \int_{\Omega^1} \theta |\nabla u|^2 - \frac{\theta^2}{2} |\nabla u|^2 - \theta \nu^2 |u|^2 \, dx.
\]
This holds by the following arguments. Since

\[ \frac{1}{\theta^2} \sum_i |b_i(x)|^2 + |c_i(x)|^2 + \frac{1}{\theta} |d(x)| \leq \nu^2 \]

by hypothesis, we have

\[ |d(x)| = |d| \leq \theta \nu^2 - \frac{1}{\theta} \sum_i |b_i(x)|^2 + |c_i(x)|^2 \]

and

\[ \frac{1}{\theta} \sum_i |b_i(x)|^2 + |c_i(x)|^2 - \theta \nu^2 \leq d \leq \theta \nu^2 - \frac{1}{\theta} \sum_i |b_i(x)|^2 + |c_i(x)|^2 \]

or

\[ -d \geq \frac{1}{\theta} \sum_i |b_i(x)|^2 + |c_i(x)|^2 - \theta \nu^2. \]

By the Cauchy-Schwarz inequality

\[ 2AB \leq A^2 + B^2 \]

we may write for any \( C \)

\[ CAB \geq -\frac{C}{2} (A^2 + B^2). \]

Furthermore, by the Cauchy-Schwarz inequality we have

\[ 2AB \leq A^2 + B^2 \]

and we consider the function

\[ f(A, B) = \frac{1}{2} (A^2 + B^2) - (B - A). \]

From elementary calculus techniques it is easy to see that \( f \) has a critical point at \( A = -1 \),
\( B = -1 \), that the critical point is a global minimum and that \( f(-1,1) = 0 \). A similar argument gives a similar result for

\[
g(A,B) = \frac{1}{2} \left( A^2 + B^2 \right) - (A - B).
\]

Hence, we can conclude that

\[
|B - A| \leq \frac{1}{2} \left( A^2 + B^2 \right)
\]

or

\[
B - A \geq -\frac{1}{2} \left( A^2 + B^2 \right).
\]

It is clear from (ii) in the hypothesis, that

\[
\int_{\Omega} \sum_{j,k} a_{j,k} D_j u D_k u \, dx \geq \int_{\Omega} \theta |\nabla u|^2 \, dx
\]

and it is also clear that

\[
- \int_{\Omega} \nu^2 u^2 \, dx \geq \int_{\Omega} \frac{1}{\theta} \left( \sum_i |b_i(x)|^2 + |c_i(x)|^2 \right) u^2 \, dx - \int_{\Omega} \theta \nu^2 u^2 \, dx.
\]

Therefore, it remains to show that

\[
\int_{\Omega} \sum_i (b_i - c_i) u D_j u + \frac{1}{\theta} \left( \sum_i |b_i(x)|^2 + |c_i(x)|^2 \right) u^2 \, dx
\]

\[
\geq \int_{\Omega} -\frac{\theta^2}{2} |\nabla u|^2 \, dx.
\]

Now let

\[
I = \int_{\Omega} \sum_i (b_i - c_i) u D_j u + \frac{1}{\theta} \left( \sum_i |b_i(x)|^2 + |c_i(x)|^2 \right) u^2 \, dx
\]

\[
= \int_{\Omega} \sum_i (b_i - c_i) Q u^2 D_j u + \frac{1}{\theta} \left( \sum_i |b_i(x)|^2 + |c_i(x)|^2 \right) u^2 \, dx
\]
where

$$Q = \nu \sqrt{\frac{\theta}{2}}.$$ 

By the Cauchy-Schwarz inequality we may write

$$I \geq \int_{\Omega} -\sum_i (b_i - c_i) \left( \frac{Q^2 u^2}{2} + \frac{1}{2Q^2} |\nabla u|^2 \right)$$

$$+ \frac{1}{\theta} \left( \sum_i |b_i(x)|^2 + |c_i(x)|^2 \right) u^2 \, dx$$

$$= \int_{\Omega} -\frac{1}{2} \sum_i (b_i - c_i) \left( Q^2 u^2 + \frac{1}{Q^2} |\nabla u|^2 \right)$$

$$+ \frac{1}{\theta} \left( \sum_i |b_i(x)|^2 + |c_i(x)|^2 \right) u^2 \, dx.$$ 

By the calculation following our discussion of the Cauchy-Schwarz inequality we may write that

$$I \geq \int_{\Omega} \frac{1}{4} \sum_i \left( |b_i(x)|^2 + |c_i(x)|^2 \right) \left( Q^2 u^2 + \frac{1}{Q^2} |\nabla u|^2 \right)$$

$$+ \frac{1}{\theta} \left( \sum_i |b_i(x)|^2 + |c_i(x)|^2 \right) u^2 \, dx$$

and

$$I \geq \int_{\Omega} \frac{1}{4} \sum_i \left( |b_i(x)|^2 + |c_i(x)|^2 \right) \left( \frac{1}{Q^2} |\nabla u|^2 \right)$$

$$+ \left( \frac{1}{\theta} - \frac{Q^2}{4} \right) \left( \sum_i |b_i(x)|^2 + |c_i(x)|^2 \right) u^2 \, dx.$$ 

Now substituting

$$Q = \nu \sqrt{\frac{\theta}{2}}$$
we obtain

\[
I \geq \int_{\Omega} -\frac{1}{4} \sum_i \left( |b_i(x)|^2 + |c_i(x)|^2 \right) \left( \frac{2}{\theta \nu^2} |\nabla u|^2 \right) \\
+ \left( \frac{1}{\nu} \frac{\theta \nu^2}{8} \right) \sum_i |b_i(x)|^2 + |c_i(x)|^2 u^2 dx.
\]

We now claim that without loss of generality

\[
I \geq \int_{\Omega} -\frac{1}{2 \theta \nu^2} \sum_i \left( |b_i(x)|^2 + |c_i(x)|^2 \right) |\nabla u|^2 dx.
\]

We may do this since

\[
\frac{1}{\theta} \frac{\theta \nu^2}{8} \geq 0
\]

if, and only if, \( 8 \nu \geq \theta \). If \( \theta \) as given were greater than \( 8 \nu \) then we could simply find \( \theta' \) such that

\[
0 < \theta' < \theta
\]

and recompute. Finally, since

\[
\sum_i \left( |b_i(x)|^2 + |c_i(x)|^2 \right) \leq \theta \nu^2
\]

or

\[
-\sum_i \left( |b_i(x)|^2 + |c_i(x)|^2 \right) \geq -\theta \nu^2
\]

we may conclude

\[
I \geq \int_{\Omega} -\frac{\theta^2 \nu^2}{2 \theta \nu^2} |\nabla u|^2 dx = \int_{\Omega} -\frac{\theta}{2} |\nabla u|^2 dx
\]

and the inequality

\[
B^1[u, u] \geq \int_{\Omega} \theta |\nabla u|^2 - \frac{\theta^2}{2} |\nabla u|^2 - \theta \nu^2 u^2 dx
\]

holds. Thus, given the fact that \( ||u^i||_{L^2(\Omega^i)} = 1 \) for all \( i = 1, 2, \ldots \), we may write

\[
B^1[u^i, u^i] \geq \frac{\theta}{2} ||\nabla u^i||_{L^2(\Omega^i)} - \theta \nu^2.
\]
Rearrangement of the equation gives, for any

\[ B^1[u^i, u^i] + \theta v^2 \geq \frac{\theta}{2} \| \nabla u^i \|_{L^2(\Omega)} \]

with

\[ \frac{2}{\theta} B^1[u^i, u^i] + 2v^2 \geq \| \nabla u^i \|_{L^2(\Omega)} \]

Since for all \( i \) we have that

\[ \lambda_0 \geq B^1[u^i, u^i] \]

we can conclude that

\[ \frac{2}{\theta} \lambda_0 + 2v^2 \geq \| \nabla u^i \|_{L^2(\Omega)} \].

Therefore, the sequence \( \{ u^i \}_{i=1}^\infty \) is bounded. Because, in addition we have the compact embedding,

\[ W_{0,1}^{1,2}(\Omega^1) \subset L^2(\Omega^1) \]

the sequence, \( \{ u^i \}_{i=1}^\infty \), has a convergent subsequence, (that we rename \( \{ u^i \}_{i=1}^\infty \)) and we assume that \( u^i \rightarrow \tilde{u} \) as \( i \rightarrow \infty \) in the space \( L^2(\Omega^1) \). Furthermore, because \( W_{0,1}^{1,2}(\Omega^1) \) is a subspace of the Hilbert space \( L^2(\Omega^1) \), any bounded sequence in \( W_{0,1}^{1,2}(\Omega^1) \) has a weakly convergent subsequence and

\[ u^i \rightharpoonup \tilde{u} \]

(say). That is, the weak convergence gives us that for any bounded linear functional \( f : W_{0,1}^{1,2}(\Omega^1) \rightarrow \mathbb{R} \)

\[ f(u^i) \rightarrow f(\tilde{u}) \]

as \( i \rightarrow \infty \). Since weak limits are unique, we have that \( \tilde{u} = \bar{u} \) almost everywhere. It is clear also that \( \{ \lambda^i \}_{i=1}^\infty \) has a convergent subsequence of numbers, since it is a bounded monotonic sequence; we say, perhaps after renaming the subsequence, that \( \lambda^i \rightarrow \bar{\lambda} \) as \( i \rightarrow \infty \), and we have that \( \lambda_0 \geq \bar{\lambda} \). Note that at this stage we do not yet have that \( u^i \rightarrow \bar{u} \) in \( W^{1,2}(\Omega^1) \). We
wish to show now, however, that \( J(\bar{u}) \to J(\bar{u}) \) as \( i \to \infty \) where \( \bar{\lambda} = J(\bar{u}) \). We may write

\[
J(\bar{u}) - J(u^i) = B[\bar{u}, \bar{u}] - B[u^i, u^i]
\]

\[
= \int_{\Omega} \sum_{j,k} a_{j,k} D_k \bar{u} D_j \bar{u} + \sum_j b_j \bar{u} D_j \bar{u} - \left( \sum_j c_j D_j \bar{u} + d \bar{u} \right) \bar{u} \, dx
\]

\[
- \int_{\Omega} \sum_{j,k} a_{j,k} D_k u^i D_j u^i - \sum_j b_j u^i D_j u^i
\]

\[
+ \left( \sum_j c_j D_j u^i + d u^i \right) u^i \, dx
\]

\[
= \int_{\Omega} \sum_{j,k} a_{j,k} (D_k \bar{u} D_j \bar{u} - D_k u^i D_j u^i)
\]

\[
+ \sum_j (b_j - c_j) (\bar{u} D_j \bar{u} - u^i D_j u^i) \, dx + \int_{\Omega} d \left( \bar{u}^2 - (u^i)^2 \right) \, dx.
\]

It is clear that the term

\[
\int_{\Omega} d \left( \bar{u}^2 - (u^i)^2 \right) \, dx \leq \theta \nu^2 \| \bar{u} - u^i \|_{L^2(\Omega)}^2
\]

goes to zero as \( i \to \infty \) on account of the \( L^2 \) convergence of \( u^i \) to \( \bar{u} \). Furthermore,

\[
\int_{\Omega} \sum_j (b_j - c_j) (\bar{u} D_j \bar{u} - u^i D_j u^i) \, dx \leq \frac{\theta^2 \nu^2}{2} \int_{\Omega} \bar{u} D_j \bar{u} - u^i D_j u^i \, dx
\]

\[
= \frac{\theta^2 \nu^2}{2} \int_{\Omega} \bar{u} D_j \bar{u} - \bar{u} D_j u^i + \bar{u} D_j u^i
\]

\[-u^i D_j u^i \, dx.
\]

Thus,

\[
\int_{\Omega} \sum_j (b_j - c_j) (\bar{u} D_j \bar{u} - u^i D_j u^i) \, dx \leq \frac{\theta^2 \nu^2}{2} \int_{\Omega} |\bar{u} D_j \bar{u} - \bar{u} D_j u^i|
\]

\[
+ |\bar{u} D_j u^i - u^i D_j u^i| \, dx
\]

\[
= \frac{\theta^2 \nu^2}{2} \left( \int_{\Omega} (\bar{u} D_j (\bar{u} - u^i) \, dx 
\]

\[
+ \int_{\Omega} |(\bar{u} - u^i) D_j u^i| \, dx \right).
\]
We can see that the first integral goes to zero as $i \to \infty$ due to the weak convergence of $u^i \rightharpoonup \bar{u}$ in $W^{1,2}(\Omega^1)$. The second integral goes to zero as $i \to \infty$ on account of the $L^2$ convergence and the fact that each $u^i$ is in $W^{1,2}(\Omega^1)$. We now turn our attention to the highest order terms. We have

$$
\int_{\Omega^1} \sum_{j,k} a_{j,k} (D_k \bar{u} D_j \bar{u} - D_k u^i D_j u^i) \, dx \leq \Theta^2 \int_{\Omega^1} D_k \bar{u} D_j \bar{u} - D_k u^i D_j u^i \, dx
$$

$$
\leq \Theta^2 \int_{\Omega^1} D_k \bar{u} D_j \bar{u} - D_k \bar{u} D_j u^i + D_k \bar{u} D_j u^i - D_k u^i D_j u^i \, dx
$$

$$
\leq \Theta^2 \left( \int_{\Omega^1} |D_k \bar{u} D_j \bar{u} - D_k \bar{u} D_j u^i| + |D_k \bar{u} D_j u^i - D_k u^i D_j u^i| \, dx \right)
$$

Again, the first integral goes to zero as $i \to \infty$ from the weak convergence of $u^i \rightharpoonup \bar{u}$. The last integral will require a more involved argument; we can see that spt($\bar{u}$) $\subseteq \Omega^i$ for all $i = 1, 2, \ldots$. We write

$$
\int_{\Omega^1} \sum_{j,k} a_{j,k} (D_j u^i D_k (u^i - \bar{u})) \, dx = \int_{\Omega^1} \sum_{j,k} a_{j,k} (D_j u^i D_k u^i - D_j u^i D_k \bar{u}) \, dx
$$

$$
= \lambda^i + \int_{\Omega^1} \sum_j (c_j - b_j) u^i D_j u^i + d (u^i)^2 \, dx
$$

$$
- \int_{\Omega^1} \sum_{j,k} a_{j,k} D_j u^i D_k \bar{u} \, dx,
$$

hence

$$
\int_{\Omega^1} \sum_{j,k} a_{j,k} (D_j u^i D_k (u^i - \bar{u})) \, dx = \lambda^i + \int_{\Omega^1} \sum_j (c_j - b_j) u^i D_j u^i + d (u^i)^2 \, dx
$$

$$
- \int_{\Omega^1} \sum_{j,k} a_{j,k} D_j u^i D_k \bar{u} \, dx.
$$
Now for any $\phi \in C_0^\infty(\Omega^i)$ we must have

$$\int_{\Omega^i} \sum_{j,k} a_{j,k} D_k u^i D_j \phi \, dx = \int_{\Omega^i} \sum_j \left( c_j - b_j \right) u^i D_j \phi + \left( \lambda^i + d \right) u^i \phi \, dx$$

and in particular it must hold for $\bar{u}$, and since $\bar{u}$ is the limit of functions $\phi \in C_0^\infty(\Omega_{a,b}^i)$ the equality must hold for $\bar{u} = \phi$. Now we may write

$$\int_{\Omega^i} \sum_{j,k} a_{j,k} \left( D_j u^i D_k (u^i - \bar{u}) \right) \, dx = \lambda^i + \int_{\Omega^i} \sum_j \left( c_j - b_j \right) u^i D_j u^i + d (u^i)^2 \, dx$$

$$- \int_{\Omega^i} \sum_{j,k} a_{j,k} D_j u^i D_k \bar{u} \, dx$$

so

$$\int_{\Omega^i} \sum_{j,k} a_{j,k} \left( D_j u^i D_k (u^i - \bar{u}) \right) \, dx = \lambda^i + \int_{\Omega^i} \sum_j \left( c_j - b_j \right) u^i D_j u^i + d (u^i)^2 \, dx$$

$$- \int_{\Omega^i} \sum_j \left( c_j - b_j \right) u^i D_j \bar{u} + \left( \lambda^i + d \right) u^i \bar{u} \, dx,$$

with

$$\int_{\Omega^i} \sum_{j,k} a_{j,k} \left( D_j u^i D_k (u^i - \bar{u}) \right) \, dx = \lambda^i + \int_{\Omega^i} \sum_j \left( c_j - b_j \right) u^i D_j (u^i - \bar{u}) \, dx$$

$$- \int_{\Omega^i} d u^i (u^i - \bar{u}) \, dx - \lambda^i \int_{\Omega^i} u^i \bar{u} \, dx,$$

thus

$$\int_{\Omega^i} \sum_{j,k} a_{j,k} \left( D_j u^i D_k (u^i - \bar{u}) \right) \, dx = \lambda^i - \lambda^i \int_{\Omega^i} u^i \bar{u} \, dx$$

$$+ \int_{\Omega^i} \sum_j \left( c_j - b_j \right) u^i D_j (u^i - \bar{u}) \, dx$$

$$- \int_{\Omega^i} d u^i (u^i - \bar{u}) \, dx.$$

Taking limits as $i \to \infty$ we see that the third and fourth terms go to zero by previous
arguments and that
\[ \lambda^i - \lambda \int_{\Omega^i} u^i \bar{u} \, dx \to \lambda - \lambda \| \bar{u} \|_{L^2(\Omega)} = 0. \]

We have now shown for the general uniform elliptic variational operator \( L \) that \( J(u^i) \to J(\bar{u}) = \lambda \) as \( i \to \infty \). Recalling the proof, we can see that it also holds that
\[ \| \nabla u^i - \nabla \bar{u} \|_{L^2(\Omega^i)} \to 0 \text{ as } i \to \infty. \]

Furthermore, it is clear that
\[ \bar{u} \in W^{1,2}_0(\Omega^1) \text{ as well.} \]

Hence the first part of the proof of the \( n \) dimensional upper continuity theorem is complete.

We now move on to Part Two of the upper continuity proofs.

1.5.3.2 Upper Continuity Proof, Part Two

We will now state what we would like to prove for the second part of the upper continuity problem as a theorem in the \( n \) dimensional case.

**Proposition 33** Let \( \{ \Omega^i \}_{i=1}^{\infty} \) be a decreasing sequence of nested domains in \( \mathbb{R}^n \), decreasing to \( \Omega_0 \). Furthermore let \( u \in W^{1,2}_0(\Omega^i) \) for all \( i = 1, 2, \ldots \) and let \( u \equiv 0 \) on \( \Omega^1 \setminus \Omega^i \) for all \( i = 1, 2, \ldots \). Then \( u \in W^{1,2}_0(\Omega_0) \).

It turns out that we will need a mild regularity condition on the limit domain, \( \Omega_0 \) and the domain \( \Omega^1 \) where the convergence \( u^i \to \bar{u} \) takes place. We will restate the \( n \) dimensional version of Part Two of our upper continuity theorem with this restriction. It should be noted that for the result to hold it is sufficient that for at least one value of \( i \), the domain \( \Omega^i \), have the segment property.

**Theorem 34 (Upper Continuity Part Two, \( n \) Dimensions)** Let \( \{ \Omega^i \}_{i=1}^{\infty} \) be a decreasing sequence of nested domains in \( \mathbb{R}^n \), decreasing to \( \Omega_0 \) where \( \Omega_0 \) and \( \Omega^1 \) satisfy the segment
condition. Furthermore let \( u \in W_0^{1,2}(\Omega^i) \) for all \( i = 1, 2, \ldots \) and let \( u \equiv 0 \) on \( \Omega^1 \setminus \Omega^i \) for all \( i = 1, 2, \ldots \). Then \( u \in W_0^{1,2}(\Omega_0) \).

**Proof.** The proof is a direct consequence of a Theorem of Adams and Fournier, [AF] page 159. We will quote their theorem and give our proof. We will prove the Adams and Fournier result as it is only sketched out in their work.

**Theorem 35 (Characterization of \( W_0^{k,p}(\Omega) \) by Exterior Extension, [AF])** Let \( \Omega \) be a domain with the segment property. Then a function \( u \) on \( \Omega \) belongs to \( W_0^{k,p}(\Omega) \) if and only if the zero extension \( \tilde{u} \) of \( u \) belongs to \( W_0^{k,p}(\mathbb{R}^n) \).

Let the hypotheses of the Upper Continuity Theorem, Part Two for \( n \) dimensions hold. Then, obviously, we can apply the theorem of Adams and Fournier. Since \( u \in W_0^{1,2}(\Omega^1) \), \( \tilde{u} \in W_0^{1,2}(\mathbb{R}^n) \). Furthermore, since \( u \equiv 0 \) on \( \Omega^1 \setminus \Omega^i \) for all \( i = 1, 2, \ldots \), we have that \( u \equiv 0 \) outside \( \Omega_0 \). Since \( \Omega_0 \) satisfies the segment property, \( \partial \Omega_0 \) is \( n - 1 \) dimensional and may be redefined to be zero there if need be. Hence, we may conclude that \( u \in W_0^{1,2}(\Omega_0) \).

We will now supply a proof of the theorem of Adams and Fournier, filling in details omitted in their proof.

**Proof (Characterization of \( W_0^{k,p}(\Omega) \) by Exterior Extension).** We break the proof into two parts. First we assume that if a domain \( \Omega \) has the segment property and \( u \in W_0^{k,p}(\Omega) \) then \( u \in W_0^{k,p}(\mathbb{R}^n) \). It is clear that if this is the case then \( \tilde{u} \in W_0^{k,p}(\mathbb{R}^n) \) even if \( \Omega \) does not have the segment property. The proof that a function \( u \in W_0^{k,p}(\Omega) \) may be extended by zero to \( W_0^{k,p}(\mathbb{R}^n) \) was given in the lower continuity proof, above. Now assume that \( \Omega \) has the segment property and that the zero extension \( \tilde{u} \) of \( u \) belongs to \( W_0^{k,p}(\mathbb{R}^n) \).

First we approximate \( u \) in \( W^{k,p}(\Omega) \) by constructing a cutoff function, \( f_\varepsilon \), with bounded support in \( \mathbb{R}^n \), and in fact arbitrarily close to \( \Omega \), (but not necessarily compactly contained in \( \Omega \)), so that

\[
\|u - u_\varepsilon\|_{W^{k,p}(\Omega)} \leq \varepsilon
\]

where

\[
u_\varepsilon = f_\varepsilon u.
\]
We will not prove that this can be done, however see [AF], proof of Theorem 3.22, page 68, 69, for details. Next construct the zero extension of $f_{\epsilon}, \tilde{f}_{\epsilon}$, to all of $\mathbb{R}^n$ and note that we have

$$\tilde{u}_{\epsilon} = \tilde{f}_{\epsilon}u \in W_0^{k,p}(\mathbb{R}^n).$$

Since $\Omega$ satisfies the segment condition we have for each $x \in \partial \Omega$ an open set $U_x$ containing $x$ with

$$\partial \Omega \subset \bigcup_{x \in \partial \Omega} U_x.$$

Furthermore, we define

$$K = \{x \in \Omega | u(x) \neq 0\}$$

and

$$F = \bar{K} \setminus \left( \bigcup_{x \in \partial \Omega} U_x \right).$$

We now find $U_0$ satisfying $F \subset U_0 \subset \subset \Omega$. Now since the set $\bar{K}$ is compact there exists, by the Heine-Borel theorem, a finite collection of the sets $U_x x \in \{0, x \in \partial \Omega\}$ that cover $\bar{K}$. We rename the indices of the finite collection and write

$$\bar{K} \subset U_0 \cup U_1 \cup ... \cup U_m.$$

Next we find for each $j = 0, 1, ..., m$, $V_j$ (an open set) such that

$$V_j \subset \subset U_j$$

but

$$\bar{K} \subset V_0 \cup V_1 \cup ... \cup V_m.$$

Now construct a partition of unity subject to the sets $V_j$, $j = 0, 1, ..., m$. For each $j$ we let $\psi_j$ be the sum of the finitely many functions whose supports lie in $V_j$. For details on the construction of a partition of unity, see pages 245-247 in Appendix 1, in Kesavan [Kesav],
for example. For notational convenience we rename \( \tilde{u}_\varepsilon = v \), and construct

\[
v_j = \psi_j v
\]

where \( \psi_j \) are the finitely many functions generated by the partition of unity of \( \Omega \) subject to the sets \( V_j \subset U_j, j = 0, 1, \ldots, m \). Now since \( U_0 \subset \Omega \) and

\[
spt(v_0) \subset V_0 \subset U_0 \subset \Omega
\]

we have that \( v_0 \in W_0^{k,p}(\Omega) \). Thus we can find \( \phi_0 \in C_0^\infty(\Omega) \) so that

\[
\|v_j - \phi_j\|_{W^{k,p}(\Omega)} \leq \frac{\varepsilon}{m+1},
\]

when \( j = 0 \). For the values of \( j \) such that \( 1 \leq j \leq m \), we want to find similar approximations \( \phi_j \) to \( v_j \) as well, but this is not so obvious. For each \( j \geq 1 \) we would like to find a function \( \phi_j \in C_0^\infty(\Omega) \) so that

\[
\|v_j - \phi_j\|_{W^{k,p}(\Omega)} \leq \frac{\varepsilon}{m+1}.
\]

If we could do so, we would have for

\[
\phi = \sum_{j=0}^{m} \phi_j
\]

\[
\|u - \phi\|_{W^{k,p}(\Omega)} \leq \|v_j - \phi_j\|_{W^{k,p}(\Omega)} \leq (m+1) \left( \frac{\varepsilon}{m+1} \right) = \varepsilon.
\]

Now let us define for any fixed \( j \geq 1 \),

\[
\Gamma = \bar{V}_j \cap \partial \Omega
\]

and let \( y \) be the vector \( (y \neq 0) \) associated with the set \( U_j \) in the definition of the segment condition. Next define

\[
\Gamma_t = \{x + ty | x \in \Gamma\}
\]
for some fixed $t$ satisfying

$$0 < t < \min \left\{ 1, \frac{1}{|y|} \text{dist}(V_j, \mathbb{R}^n \setminus U_j) \right\}.$$ 

Doing so gives us that

$$\Gamma_t \subset U_j \cap \Omega$$

and

$$\Gamma_t \subset \subset \Omega$$

as well since $t > 0$. We next define for the value of $j$ under consideration, the translation of $v_j$ by $t$ into $\Omega$:

$$v_{j,t}(x) = v_j(x - ty)$$

$$= \psi_j(x - ty) v(x - ty)$$

$$= \psi_j(x - ty) \tilde{u}(x - ty)$$

$$= \psi_j(x - ty) \tilde{f}(x - ty) \tilde{u}(x - ty).$$

We claim that $v_{j,t}(x) \in W_0^{k,p}(\mathbb{R}^n)$. This is so because $spt(\psi_j) \subset V_j$ and the norm of a function of compact support is invariant under (finite) translation. Furthermore, the shift, by $t$ moves $spt(\tilde{u})$ so that $spt(\tilde{u}) \subset \subset \Omega$. Since all of this holds for any fixed $t$ small enough, we claim that there is a $t$ such that

$$\|v_j - v_{j,t}\|_{W^{k,p}(\Omega)} \leq \frac{\varepsilon}{m + 1}.$$ 

This can be seen to hold, since it is clear that for a sequence $\{t_i\}_{i=1}^{\infty}$ where $t_i \to 0$ as $i \to \infty$ the sequence of functions $\{v_{j,t_i}(x)\}_{i=0}^{\infty} \to v_j(x)$. To complete the proof we pick an appropriate $t$ and let

$$\phi_j = v_{j,t}.$$
so that

\[ \|v_j - v_{j,t}\|_{W^{k,p}(\Omega)} = \|v_j - \phi_j\|_{W^{k,p}(\Omega)} \leq \frac{\epsilon}{m+1}. \]

We conclude the discussion of the continuous dependence on the first eigenvalue on the domain with a remark and a slight variation of our more general theorem. For a fixed nested sequence of domains, we only need to have the limit domain satisfy the segment property to conclude that the first eigenvalue, \( \lambda \) of a homogeneous pde with the operator \( L \) as used above, be continuously dependent on the domain. We might wish to consider more general families of domains than sequences. We state the following theorem without proof.

**Theorem 36** Let \( \{\Omega_\mu\} \) be a family of domains in \( \mathbb{R}^n \) each satisfying the segment condition, where \( \Omega_\mu \) is defined for each \( \mu \in I = [a, b] \) so that for each \( \alpha, \beta \in I \), \( \Omega_\alpha \subset \Omega_\beta \) when \( \alpha < \beta \). Furthermore, let \( L \) be a uniformly elliptic variational operator and for each \( \mu \in I \) let \( u_\mu \) solve

\[
\begin{aligned}
L u + \lambda u &= 0 \text{ in } \Omega_\mu \\
u &= 0 \text{ on } \partial \Omega_\mu
\end{aligned}
\]

with corresponding first eigenvalue, \( \lambda_\mu \). Similarly, let \( u_\Lambda \) be the first eigenfunction for

\[
\begin{aligned}
L u + \lambda u &= 0 \text{ in } \Omega_\Lambda \\
u &= 0 \text{ on } \partial \Omega_\Lambda
\end{aligned}
\]

with corresponding first eigenvalue \( \lambda_\Lambda \) where \( \Lambda \) is an arbitrary point in the open interval \( (a, b) \). Then the first eigenvalue, \( \lambda_\Lambda \), is a continuous function of \( \{\Omega_\mu\} \) at \( \mu = \Lambda \).

1.6 Some Consequences of the Results of Heywood, Noussair, and Swanson and the Continuous Dependence of the First Eigenvalue on the Domain

Because the results in this section are dependent on the results of the Sturmian type comparison theorem of Heywood, Noussair, and Swanson, we will begin this section by
reviewing their paper, "On the Zeros of Solutions of Elliptic Inequalities in Bounded Domains," and then employing their results for our purposes. In order to use their theorems we must have functions that are sufficiently smooth. In general, weak solutions to the homogeneous eigenvalue problem

\[
\begin{cases}
    Lu + \lambda u = 0 \quad \text{in} \quad \Omega \\
    u = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\]

are only of class \(W^{1,2}_0(\Omega) \cap W^{2,2}_{loc}(\Omega)\). The hypotheses of Heywood, Noussair, and Swanson require that functions be rather much "nicer" than this. We will discuss this in detail before employing their results.

1.6.1 The Heywood, Noussair, and Swanson Results

In the paper by Heywood, Noussair, and Swanson: "On the Zeros of Solutions of Elliptic Inequalities in Bounded Domains," three theorems were presented and these theorems were proven utilizing two lemmas. In addition, the proof of the first theorem, from which the other two follow, was given based on a version of Picone's Identity. We will begin by reviewing the arguments of Heywood, Noussair, and Swanson and then discussing certain regularity issues that will allow us to apply their results to solutions of eigenvalue problems.

The paper, "On the Zeros of Solutions of Elliptic Inequalities in Bounded Domains," was written to pertain to certain nonlinear operators, \(\tilde{L}\), of the form

\[
\tilde{L}u = -\sum_{i,j=1}^n D_i[\tilde{A}_{ij}(x,u)D_ju] + 2\sum_{i=1}^n \tilde{B}_i(x,u)u_{x_i} + C(x,u)u
\]

or

\[
Lu = -\sum_{i,j=1}^n A_{ij}(x,u)u_{x_i x_j} + 2\sum_{i=1}^n B_i(x,u)u_{x_i} + C(x,u)u
\]

by virtue of the fact that the coefficient matrix \(\tilde{A}_{ij}(x,u)\) is assumed to be \(C^1(\bar{\Omega} \times I)\) where \(I\) is some appropriate interval. Additionally, the functions \(u\) to which the operator \(L\) may be applied are taken to be of class \(C^2(\Omega) \cap W^{1,2}(\Omega)\). Note that for their first theorem,
we may assume that $A_{ij}(x, u)$ satisfies the symmetry condition $A_{ij}(x, u) = A_{ji}(x, u)$, (but $A_{ij}(x, u)$ need not be definite). Now if $A_{ij}(x, u)$ is symmetric, then $A_{ij}(x, u)$ is symmetric as well and the operator $L$ is a symmetric operator if, and only if $B_i$ is identically zero, or equivalently,

$$
- \sum_{i,j=1}^{n} \left( A_{ij}(x, u) \right)_{x_i} u_{x_j} + 2 \sum_{i=1}^{n} B_i(x, u) u_{x_i} = 0.
$$

We will not make much use of the nonlinear results, but will focus mainly on the special case of linear operators only of the form

$$
Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u.
$$

Although the matrices $A$ and $a$ are not necessarily assumed to be positive definite for the work of Heywood, Noussair, and Swanson, we will ignore any results for the semi-definite situation. Associated with the operator, $L$, of Heywood, Noussair, and Swanson, is the quadratic form:

$$
Q[z] = \sum_{i,j=1}^{n} A_{ij}(x, u) z_i z_j + 2 \sum_{i=1}^{n} B_i(x, u) z_i + E z_{n+1}^2;
$$

where $E$, is a continuous function of $u$ and $x$, chosen so that $Q[z]$ is positive definite. Also associated with the operator, $L$, are the functionals

$$
F[u, v] = \int_{\Omega} \sum_{i,j=1}^{n} A_{ij}(x, v) u_{x_i} u_{x_j} + 2 \sum_{i=1}^{n} B_i(x, v) u \cdot u_{x_i} + (C(x, v) + E(x, v)) u^2 dx
$$

and

$$
H_\Omega[u, v] = \int_{\Omega} \Phi[u(x), v(x); x] dx,
$$
where

\[ \Phi[u, v; x] = \sum_{i,j=1}^{n} v^2 A_{ij}(x, v) \left( \frac{u}{v} \right)_{x_i} \left( \frac{u}{v} \right)_{x_j} + 2uv \sum_{i=1}^{n} B_i(x, v) \left( \frac{u}{v} \right)_{x_i} + E(x, v)u^2. \]

In our less elaborate situation, we will take

\[ q[z] = \sum_{i,j=1}^{n} a_{ij}(x) z_i z_j + 2z_{n+1} \sum_{i=1}^{n} \frac{1}{2} b_i(x) z_i + e(x, v)z_{n+1} \]

where \( e \) is a continuous function of \( x \) and \( v \), chosen so that \( q[z] \) is positive definite. Additionally, we will have

\[ f[u, v] = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + 2 \sum_{i=1}^{n} \frac{1}{2} b_i(x) u \cdot u_{x_i} + (c(x) + e(x, v))u^2 \ dx \]

and

\[ h_{\Omega}[u, v] = \int_{\Omega} \phi[u(x), v(x); x] \ dx \]

where

\[ \phi[u, v; x] = \sum_{i,j=1}^{n} v^2 a_{ij}(x) \left( \frac{u}{v} \right)_{x_i} \left( \frac{u}{v} \right)_{x_j} + 2uv \sum_{i=1}^{n} \frac{1}{2} b_i(x) \left( \frac{u}{v} \right)_{x_i} + e(x, v)u^2. \]

It goes without saying that \( A_{ij}(x) = -a_{ij}(x) \) and that \( B_i(x) = \frac{1}{2} b_i(x) \). For the third theorem we will take for comparison

\[Mu = - \sum_{i,j=1}^{n} A_{ij}(x)u_{x_i x_j} + C(x)u\]

and

\[mu = - \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + c(x)u\]

where we define

\[g[u] = \int_{\Omega} \sum_{i,j=1}^{n} (a_{ij}(x) - A_{ij}(x)) u_{x_i x_j} + (c(x) - C(x))u^2 \ dx.\]
For convenience, we will restate the theorems and lemmas of Heywood, Noussair, and Swanson, employing the notation above, as well as their version of Picone's identity (a key element of the proofs of their theorems). Since weak solutions to the partial differential equations that we are considering are not necessarily even continuous we will discuss some of the issues related to this following the restatement of the theorems below. It is worth noting, although we will not make use of the results, that the semi-definite results for Heywood, Noussair, and Swanson will hold for these theorems as well. First we state the theorems.

**Theorem 37 (HNS Theorem 1)** "For $L$ as given above, if (1) $E(x,v)$, has been selected so that $Q[z]$ is positive definite throughout a bounded domain, $\Omega$; (2) $u \in C^0(\Omega) \cap W^{1,2}(\Omega)$ is a nontrivial function in $\Omega$ such that $u = 0$ on $\partial \Omega$; and (3) $v \in C^2(\Omega) \cap W^{1,2}(\Omega)$ is a solution of $vLv \geq 0$ in $\Omega$ such that $F[u,v] \leq 0$; then $v$ has a zero at some point in $\Omega$.

**Theorem 38 (HNS Theorem 2 - Symmetric Case)** "Suppose that $B_i(x,v) = 0$ for $i = 1,2,...n$ in $L$ as given above and $E = 0$ in $\Omega$. If $u \in C^0(\Omega) \cap C^1(\Omega)$ is a nontrivial function in $\Omega$ such that $u = 0$ on $\partial \Omega$ and $v \in C^2(\Omega) \cap W^{1,2}(\Omega)$ satisfies $vLv \geq 0$ in $\Omega$, then either $v$ has a zero at some point in $\Omega$ or else $v(x) = k \cdot u(x)$ throughout $\Omega$ for some nonzero constant $k$.

**Theorem 39 (HNS Theorem 3 - Linear Symmetric Case)** "If (1) the matrix $[A_{ij}]$ in $M$, above, is positive definite throughout $\Omega$; and (2) there exists a nontrivial solution $u \in C^0(\Omega) \cap W^{2,2}(\Omega)$ of $umu \leq 0$ in $\Omega$ such that $u$ is identically zero on $\partial \Omega$ and $g[u] \geq 0$; then every solution $v \in C^2(\Omega) \cap W^{1,2}(\Omega)$ of $vMv \geq 0$ either has a zero at some point in $\Omega$, or $v(x) = k \cdot u(x)$ throughout $\Omega$ for some nonzero constant $k$.

The lemmas used in the proofs of these three theorems and their version of Picone's identity are given below.

**Theorem 40 Lemma 41 (HNS Lemma 1)** If $u \in C^0(\Omega) \cap W^{1,2}(\Omega)$ and $u$ vanishes identically on $\partial \Omega$, then $u \in W^{1,2}_0(\Omega)$. 
Lemma 42 (HNS Lemma 2) If \( u \in W_{0}^{1,2}(\Omega) \cap W^{2,2}(\Omega) \), then

\[
\int_{\Omega} u m u d x = \int_{\Omega} u \sum_{i,j=1}^{n} -(a_{ij}(x)u_{x_{i}})_{x_{j}} + c(x)u d x
\]

\[
= \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x)u_{x_{i}x_{j}} + c(x)u^{2} d x.
\]

Theorem 43 (Picone’s Identity [HNS], equation (6), pg. 349)

\[
\Phi[u, v; x] + \sum_{i,j=1}^{n} \left( \frac{u^{2}}{v} A_{ij}(x, v) v_{x_{j}} \right)_{x_{i}}
\]

\[
= \sum_{i,j=1}^{n} A_{ij}(x, v) u_{x_{i}}u_{x_{j}} + 2 \sum_{i=1}^{n} B_{i}(x, v) u \cdot u_{x_{i}} + [C(x, v) + E(x, v)] u^{2} - \left( \frac{u}{v} \right)^{2} v L u.
\]

In general weak solutions satisfying the partial differential equation

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

(or the simpler forms, given above) are not of sufficient smoothness to allow the application of the theorems of Heywood, Noussair, and Swanson given above. Typically, solutions to these eigenvalue problems will be in \( W_{0}^{1,2}(\Omega) \) only. It is clear, on account of the use of Picone’s identity that any solution \( u \) of the pde (which according to the arguments of Heywood, Noussair, and Swanson appears as \( u \) in their version of the identity) must have at least continuous first partial derivatives and any function \( v \) in the identity, besides being continuous and non-zero, must be at least twice continuously differentiable. To remedy the situation we employ a regularity argument that can be found in Gilbarg and Trudinger. In their sixth chapter on Classical Solutions, they prove the following theorem which we may apply directly:

Theorem 44 (Gilbarg and Trudinger, 6.13, pages 106-107, [GT]) Let (the operator) \( L \) be strictly elliptic in a bounded domain \( \Omega \), with \( c(x) \geq 0 \), and let (the function) \( f \) and the coefficients of \( L \) be bounded and belong to \( C^{0,\alpha}(\Omega) \) (where \( \alpha > 0 \)). Suppose that
\( \Omega \) satisfies an exterior sphere condition at every boundary point. Then, if (the function) \( \phi \) is continuous on \( \partial \Omega \), the Dirichlet problem,

\[
\begin{aligned}
Lu &= f \quad \text{in} \quad \Omega \\
u &= \phi \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

has a (unique) solution \( u \in C^0(\overline{\Omega}) \cap C^{2,\alpha}(\Omega) \).

It is clear that all of the hypotheses of the theorem are satisfied except the exterior sphere condition. A note by Gilbarg and Trudinger following the statement of the theorem mentions that the theorem can be proven if the boundary of the domain simply satisfies an exterior cone condition. More generally, the arguments that give the result of Gilbarg and Trudinger above require that every point \( x_0 \in \partial \Omega \) be a regular point. A regular point is a point at which a continuous subharmonic function, say \( \phi_{x_0} \), exists, with \( \phi_{x_0}(x_0) = 0 \) and \( \phi_{x_0}(x) < 0 \) for all \( x \in \partial \Omega \setminus \{x_0\} \). Such a function is called a barrier function. The reader may recall that a subharmonic function, \( u \), satisfies

\[
u(\xi) \leq \frac{1}{\omega_n} \int_{|x|=1} u(\xi + rx)dS
\]

for \( r \) sufficiently small where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). Some details on this may be found in McOwen, [McOwen], section 4.3, pages 120-124 or in Gilbarg and Trudinger, [GT], Section 6.3, pages 104-105. For our work, we have that

\[
f = \phi = 0 \in C^\infty_0(\Omega)
\]

hence, any solution of the partial differential equation

\[
\begin{aligned}
Lu + \lambda \cdot u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{aligned}
\]

will be of class \( C^0(\overline{\Omega}) \cap C^{2,\alpha}(\Omega) \) provided the coefficients of the strictly elliptic operator \( L \) are in \( C^{0,\alpha}(\Omega) \) and besides the segment condition, \( \partial \Omega \) satisfies an exterior cone condition.
In subsequent discussions we will assume that any operator, and domain have the respective required properties. This allows us to use the comparison theorems of Heywood, Noussair, and Swanson directly and to compare any two solutions of the pde. Furthermore, their third theorem allows us to compare solutions to two eigenvalue problems on the same domain with the same operator $L$, but different eigenvalues, $\lambda$.

We state for convenience and without proof three simple versions of the third theorem of Heywood, Noussair, and Swanson above. We will use these results extensively in discussions that follow; they correspond to the three versions of the eigenvalue problem that we outlined above.

**Theorem 45 (GHNS)** Suppose that the operators $l$ and $L$ as defined below are such that the coefficient functions are of class $C^{0,\alpha}(\Omega)$ and the domain $\Omega$ satisfies an exterior cone condition and has the segment property. If (1) the matrix $[a_{ij}]$ in

$$
\begin{cases}
lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j} + (c(x) + \lambda) u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
$$

is positive definite throughout $\Omega$, and

$$
Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j} + (C(x) + \mu) u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
$$

with

$$g[u] = \int_{\Omega} (c(x) + \lambda - C(x) - \mu) u^2 dx \leq 0$$

and (2) there exists a nontrivial solution $u \in C^0(\bar{\Omega}) \cap W^{2,2}(\Omega)$ of $lu = 0$ in $\Omega$, then every solution $v \in C^2(\Omega) \cap W^{1,2}(\Omega)$ of $Lv = 0$ either has a zero at some point in $\Omega$, or $v(x) = k \cdot u(x)$ a.e. throughout $\Omega$ for some nonzero constant $k$.

**Corollary 46 (HNS$_1$)** Suppose the domain $\Omega$ satisfies an exterior cone condition and has
the segment property. If

\[
\begin{cases}
l u = \sum_{i,j=1}^{n} u_{x_i x_j} + \lambda u = 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\end{cases}
\]

and

\[
\begin{cases}
L u = \sum_{i,j=1}^{n} u_{x_i x_j} + \mu u = 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\end{cases}
\]

with

\[g[u] = \int_{\Omega} (\lambda - \mu) u^2 dx \leq 0\]

and there exists a nontrivial solution \( u \in C^0(\Omega) \cap W^{2,2}(\Omega) \) of \( l u = 0 \) in \( \Omega \), then every solution \( v \in C^2(\Omega) \cap W^{1,2}(\Omega) \) of \( L v = 0 \) either has a zero at some point in \( \Omega \), or \( v(x) = k \cdot u(x) \) a.e. throughout \( \Omega \) for some nonzero constant \( k \).

**Corollary 47 (HNS2)** Suppose that the operators \( l \) and \( L \) as defined below are such that the lower order coefficient functions are of class \( C^{0,\alpha}(\Omega) \) and the domain \( \Omega \) satisfies an exterior cone condition and has the segment property. If

\[
\begin{cases}
l u = \sum_{i,j=1}^{n} u_{x_i x_j} + (c(x) + \lambda) u = 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\end{cases}
\]

and

\[
\begin{cases}
L u = \sum_{i,j=1}^{n} u_{x_i x_j} + (c(x) + \mu) u = 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\end{cases}
\]

with

\[g[u] = \int_{\Omega} (\lambda - \mu) u^2 dx \leq 0\]

and there exists a nontrivial solution \( u \in C^0(\Omega) \cap W^{2,2}(\Omega) \) of \( l u = 0 \) in \( \Omega \), then every solution \( v \in C^2(\Omega) \cap W^{1,2}(\Omega) \) of \( L v = 0 \) either has a zero at some point in \( \Omega \), or \( v(x) = k \cdot u(x) \) a.e. throughout \( \Omega \) for some nonzero constant \( k \).
1.6.2 Minimal Nodal Domain Theorem

Earlier we established the continuous dependence of the first eigenvalue on the domain. We wish to examine some consequences of this fact when combined with the Sturmian type comparison theorems of Heywood, Noussair, and Swanson. We want to consider the partial differential equation

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

with suitable constraints on the operator \( L \), and a family of domains that are sufficiently regular to permit us to apply the results of Heywood, Noussair, and Swanson. Under these hypotheses we may state and prove a result that allows us to conclude that the pde

\[
\begin{cases}
Lu = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

has a positive solution. That is, the first given partial differential equation has a solution on some domain in the family for the eigenvalue \( \lambda = 0 \). We will state this more precisely in the theorem below which follows two useful definitions.

**Definition 48 (Continuously Dependent Parameterized Family of Domains)** Let \( I = [a, b] \) be a closed interval and \( \{ \Omega_\mu \} \) be a family of domains defined for every \( \mu \in I \). Furthermore let the family be increasing, that is, \( \Omega_\alpha \subset \Omega_\beta \) whenever \( \alpha < \beta \). We say that the family \( \Omega_\mu \) depends lower (respectively, upper) continuously on the parameter \( \mu \) provided that for any sequence of points \( \{ \mu_i \} \) in \( I \) that increases (decreases) to \( \mu_0 \), we have

\[
\Omega_{\mu_0} = \lim_{i \to \infty} \Omega_{\mu_i}
\]

in the senses used previously. Finally we say that the family of domains \( \{ \Omega_\mu \} \) depends continuously on the parameter \( \mu \) provided it depends both lower and upper continuously on \( \mu \).

We of course say that the continuously dependent family of domains is increasing if
\( \Omega_\alpha \subset \Omega_\beta \) whenever \( \alpha < \beta \).

We also want to use this definition to define a corresponding set of partial differential equations where we may apply the results of Heywood, Noussair, and Swanson. We make the following definition.

**Definition 49 (Regular Family of Elliptic Boundary Value Problems)** Let \( \{\Omega_\mu\} \) be a family of domains where each domain, \( \Omega_\mu \), satisfies the segment condition and the exterior cone condition and \( L \) be a second order uniformly elliptic partial differential operator in divergence form that admits a variational formulation such that

\[
\begin{align*}
(i) & \quad Lu = \sum_{i,j} D_i (a_{ij}(x) D_j u) + \sum_i (D_i (b_i(x) u) + c_i(x) D_i u) + d(x) u \\
(ii) & \quad \sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for all } x \in \Omega^b, \xi \in \mathbb{R}^n \\
(iii) & \quad \sum_{i,j} |a_{ij}(x)|^2 \leq \Theta^2 \quad \text{and} \\
(iv) & \quad \frac{1}{\theta^2} \sum_i |b_i(x)|^2 + |c_i(x)|^2 + \frac{1}{\theta} |d(x)| \leq \nu^2.
\end{align*}
\]

Also assume that the coefficient functions of \( L \), \( a_{ij}(x), b_i(x) \in C^{1,0}(\Omega) \) and \( c_i(x), d(x) \in C^{0,\alpha}(\Omega_b) \). Then the family of boundary value problems

\[
\begin{cases}
Lu = 0 \quad \text{in} \quad \Omega_\mu \\
u = 0 \quad \text{on} \quad \partial\Omega_\mu
\end{cases}
\]

is called a regular family of elliptic boundary value problems. We denote the corresponding eigenvalue problems for the family by

\[
\begin{cases}
Lu + \lambda_\mu u = 0 \quad \text{in} \quad \Omega_\mu \\
u = 0 \quad \text{on} \quad \partial\Omega_\mu
\end{cases}
\]

It is clear that a sufficiently regular family of elliptic partial differential equations as defined has a unique solution \( u_\mu \) and corresponding first eigenvalue \( \lambda_\mu \) and that it has sufficient regularity to allow the comparison theorems of Heywood, Noussair, and Swanson.
to be applied. We now state and prove the following theorem.

**Theorem 50 (Minimal Nodal Domains for Elliptic Boundary Value Problems)**

Let

\[
\begin{aligned}
    L u &= 0 \quad \text{in} \quad \Omega_\mu \\
    u &= 0 \quad \text{on} \quad \partial \Omega_\mu
\end{aligned}
\]

be a regular family of elliptic boundary value problems and \( \{ \Omega_\mu \} \) be an increasing family of domains, continuously dependent on \( \mu \in I = [a, b] \). Suppose that there exists

(i) a positive solution \( v \in W^{1,2}(\Omega_a) \cap C^2(\Omega_a) \) of \( L u = 0 \) on \( \Omega_a \) and

(ii) a nodal subdomain \( \Omega \subset \Omega_b \) of \( L u = 0 \) with solution \( w \in W^{1,2}_0(\Omega) \cap C^2(\Omega) \).

Then there exists a unique number \( \mu \in I \) such that \( \Omega_\mu \) is a nodal domain for

\[
\begin{aligned}
    L u &= 0 \quad \text{in} \quad \Omega_\mu \\
    u &= 0 \quad \text{on} \quad \partial \Omega_\mu
\end{aligned}
\]

**Proof.** Let the hypotheses of the theorem hold. By our previous results, the first eigenvalue, \( \lambda_\mu \) of the problem

\[
\begin{aligned}
    L u + \lambda_\mu u &= 0 \quad \text{in} \quad \Omega_\mu \\
    u &= 0 \quad \text{on} \quad \partial \Omega_\mu
\end{aligned}
\]

is continuously dependent on the parameter \( \mu \) and the results of Heywood, Noussair, and Swanson apply. For simplicity, we write \( \lambda = \lambda(\mu) \) to denote the continuous dependence of the eigenvalue on the domain. We consider some cases. If \( \lambda(a) = 0 \) then we are done, up to uniqueness, so we suppose that this is not the case. Then either \( \lambda(a) < 0 \) or \( \lambda(a) > 0 \). Assume the former case holds and there is a positive solution, \( u \), to

\[
\begin{aligned}
    L u + \lambda(a) u &= 0 \quad \text{in} \quad \Omega_a \\
    u &= 0 \quad \text{on} \quad \partial \Omega_a
\end{aligned}
\]

If there were such a \( u \) then the Heywood, Noussair, and Swanson comparison theorem would imply that the function \( v(x) \) restricted to \( \Omega_a \) had a zero – a contradiction. Therefore, it must
be that \( \lambda(a) > 0 \). Now in a similar fashion, we claim that \( \lambda(b) \leq 0 \). If \( \lambda(b) = 0 \) then for \( \mu = b \) there is a positive solution to

\[
\begin{align*}
Lu &= 0 \quad \text{in} \quad \Omega_\mu \\
u &= 0 \quad \text{on} \quad \partial \Omega_\mu
\end{align*}
\]

Now assume otherwise, that \( \lambda(b) > 0 \) and that a positive solution, \( u \), to

\[
\begin{align*}
Lu + \lambda(b)u &= 0 \quad \text{in} \quad \Omega_b \\
u &= 0 \quad \text{on} \quad \partial \Omega_b
\end{align*}
\]

exists. Again, the results of Heywood, Noussair, and Swanson would imply that because \( w \) is a positive solution on the nodal domain \( \Omega \subset \Omega_b \), the solution \( u \) would have zeros – again a contradiction. Together with the facts that \( \lambda(a) > 0 \) and \( \lambda(\mu) \) is continuous for \( \mu \in [a,b] \) and that the family of domains, \( \Omega_\mu \) is continuously dependent on \( \mu \), we must have that there exists a value of \( \mu \) such that \( \lambda(\mu) = 0 \). The uniqueness of \( \mu \) also follows from the Sturmian type comparison theorem of Heywood, Noussair, and Swanson – any eigensolution corresponding to a value of \( \mu^* \neq \mu \) would imply that the solution corresponding to the larger of \( \mu \) and \( \mu^* \) would have zeros. This completes the proof.

We will briefly refer to this theorem as the minimal nodal domain theorem. It is to be understood that the term "minimal" refers to the smallest domain that is nodal in a given continuously dependent parameterized family of domains and partial differential equations.

1.7 Some Consequences and Examples of the Minimal Nodal Domain Theorem

We will provide some examples and applications of the minimal nodal domain theorem and the comparison theorems of Heywood, Noussair, and Swanson. Although these results apply to \( \mathbb{R}^n \) for any \( n \), we will restrict our attention examples in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).
1.7.1 Some Applications of the Minimal Nodal Domain Theorem

To illustrate some aspects of our minimal nodal domain theorem we will give an elementary example. Suppose we want to solve

$$\begin{cases} 
Lu = \Delta u + 2u &= 0 \text{ in } \Omega_\mu \\
u &= 0 \text{ on } \partial \Omega_\mu 
\end{cases}$$

where the domain, $\Omega_\mu$, in $\mathbb{R}^2$ is of the form

$$\Omega_\mu = \{(x, y)|\sqrt{x^2 + y^2} < \mu\}$$

and $\mu \in [1, 10]$. Note that the function $c(x, y) = 2$ here. It is well known that a minimum value of $\mu$ exists for which $\Omega_\mu$ is a nodal domain. For arbitrary values of $\mu$, the positive solution to the given pde is

$$u_\mu = C \cdot J_0 \left( \frac{j_{0,1} \sqrt{2 + \lambda_\mu}}{\sqrt{2 + \lambda_\mu}} \sqrt{x^2 + y^2} \right)$$

where $C$ is an arbitrary constant, $J_0$, the zeroth order Bessel function of the first kind, $j_{0,1}$ its first zero and $\lambda_\mu$ the first eigenvalue for

$$\begin{cases} 
Lu = \Delta u + 2u + \lambda_\mu &= 0 \text{ in } \Omega_\mu \\
u &= 0 \text{ on } \partial \Omega_\mu 
\end{cases}$$

If $\lambda = 0$ then the corresponding value of $\mu \approx 1.7005$. Details on how to find both $\mu$ and the solution to this problem appear later in the work. We also observe that the function

$$v = w = \cos x \cos y$$

solves $\Delta u + 2u = 0$ and is positive on $\Omega_1 \subset \Omega_{10}$. Furthermore, since $\mu$ satisfies

$$\mu = \frac{j_{0,1}}{\sqrt{2 + \lambda}}$$
for the problem

\[
\begin{cases}
Lu = \triangle u + 2u + \lambda u &= 0 \quad \text{in } \Omega_\mu \\
u &= 0 \quad \text{on } \partial \Omega_\mu
\end{cases}
\]

we see that

\[\lambda = \frac{j_{0,1}^2}{\mu^2} - 2.\]

Because \(j_{0,1} \approx 2.4048\), we have \(\lambda \approx 3.7831\) when \(\mu = 1\) and \(\lambda \approx -1.9422\) when \(\mu = 10\). Thus, the hypotheses of the minimal nodal domain theorem are satisfied and we may conclude that a least value of \(\mu \in [1, 10]\) exists for which

\[
\begin{cases}
Lu = \triangle u + 2u &= 0 \quad \text{in } \Omega_\mu \\
u &= 0 \quad \text{on } \partial \Omega_\mu
\end{cases}
\]

has a positive solution.

We make some additional observations. First the continuous parameterization of the domains \(\Omega_\mu\) has a particularly simple form. We also note that if \(\mu \in [1, 2]\) then the hypotheses of the minimal nodal domain theorem are not satisfied as the nodal domain

\[
\Omega = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)
\]

where the positive function, \(v\), satisfies \(Lv = 0\), is not contained in

\[
\Omega_2 = \left\{(x, y)|\sqrt{x^2 + y^2} < 2\right\}
\]

since

\[\sqrt{2}\frac{\pi}{2} \approx 2.2214 > 2.\]

Simply put, the square on which \(v(x, y) = w(x, y)\) has a solution is not contained in the circle \(\Omega_2\) - its corners stick out Similarly, if \(\mu \in [2, 10]\) then the hypotheses of the theorem are not satisfied because the function \(v = \cos x \cos y\) is not positive on \(\Omega_2\). The reader may find it worthwhile to verify that for the particular choice of \(v = \cos x \cos y\), the minimum
range of values for $\mu$ for which the theorem can be applied is

$$\mu \in [a, b] = \left[\frac{\pi}{2}, \sqrt{2}\frac{\pi}{2}\right] \approx [1.5701, 2.2214].$$

If $\mu < b$ then the nodal domain for $v$, as taken, is not contained in $\Omega_b$ and if $\mu > a$ then $v$ is not positive on $\Omega_a$.

We consider a somewhat less transparent example. Consider the homogeneous partial differential equation in $\mathbb{R}^2$ below:

$$Lu = \Delta u + c(x, y)u = 0 \quad \text{in} \quad \Omega_{\mu, \varepsilon},$$

$$u = 0 \quad \text{on} \quad \partial\Omega_{\mu, \varepsilon}$$

where

$$\Omega_{\mu, \varepsilon} = \left\{(x, y) \mid \frac{x^2}{\varepsilon^2 \mu^2} + \frac{y^2}{\mu^2} < 1\right\}.$$ 

It is clear that for $\varepsilon$ fixed, the family of domains $\Omega_{\mu, \varepsilon}$ depends continuously on the parameter $\mu$. We observe that if

$$c(x, y) = \begin{cases} 
4 \left(x^2 + y^2 + \tan(x^2 + y^2)\right) & \text{for} \quad \sqrt{x^2 + y^2} \leq \frac{5\pi}{12} \approx 1.3090 \\
4 \left(\frac{5\pi}{12} + 2 + \sqrt{3}\right) & \text{for} \quad \sqrt{x^2 + y^2} > \frac{5\pi}{12}
\end{cases}$$

then for all $(x, y) \in \mathbb{R}^2$ we have that

$$0 \leq c(x, y) \leq 4 \left(\frac{5\pi}{12} + 2 + \sqrt{3}\right) \approx 20.1642.$$

A simple calculation shows that $c(x, y) \in C^{0,1}(\mathbb{R}^2)$. We now consider the eigenvalue problem

$$\Delta u + c(x, y)u + \lambda u = 0 \quad \text{in} \quad \Omega_{\mu, \varepsilon},$$

$$u = 0 \quad \text{on} \quad \partial\Omega_{\mu, \varepsilon}$$

for $\varepsilon \in [1, \infty)$, but fixed. The function $v = w = \cos(x^2 + y^2)$ satisfies $Lv = 0$ and is positive.
These claims may be verified by computation. We examine first, the situation where \( \epsilon = 1 \) – that is when the domains \( \Omega_{\mu, \epsilon} \) are circular. In this case, we have that

\[
\Omega_{\mu, 1} = \{(x, y) | x^2 + y^2 \leq \mu^2 \}
\]

and as long as

\[
a \leq \sqrt{\frac{\pi}{2}} \leq b \leq \frac{5\pi}{12}
\]

and the minimal nodal domain theorem applies. For any such interval \( I = [a, b] \), we have that the function \( v \) as defined is a positive solution on

\[
\Omega = \Omega_{\sqrt{\frac{\pi}{2}}, 1}
\]

and

\[
\Omega_{a, 1} \subset \Omega_{\sqrt{\frac{\pi}{2}}, 1} \subset \Omega_{b, 1}
\]

allowing the theorem to be applied. Of course in this special case,

\[
\Omega_{\sqrt{\frac{\pi}{2}}, 1}
\]

is the nodal domain and \( v \) the positive solution for

\[
\begin{cases}
Lu = \Delta u + c(x, y)u = 0 \quad \text{in} \quad \Omega_{\mu, 1} \\
u = 0 \quad \text{on} \quad \partial \Omega_{\mu, 1}
\end{cases}
\]

We now consider values of \( \epsilon > 1 \). For \( \epsilon \) fixed we see that

\[
\Omega_{\mu, \epsilon} = \{(x, y) | \frac{x^2}{\epsilon^2 \mu^2} + \frac{y^2}{\mu^2} < 1 \}
\]

and the domains are ellipses whose semi-minor axis length (along the y axis) is \( \mu \) and whose
semi-major axis (along the $x$ axis) is $\epsilon \mu$. In this case we may apply the minimal nodal domain theorem with the given function $v$ when

$$\sqrt{\frac{\pi}{2}} \leq b \leq \frac{5\pi}{12} \quad \text{and when} \quad a \leq \frac{1}{\epsilon} \sqrt{\frac{\pi}{2}}.$$  

For the cases where $\epsilon > 1$, a positive solution to the partial differential equation

$$L = \Delta u + c(x, y)u = 0 \quad \text{in} \quad \Omega_{\mu, \epsilon}$$
$$u = 0 \quad \text{on} \quad \partial \Omega_{\mu, \epsilon}$$

is not known, but by the minimal nodal domain theorem is known to exist.

**1.7.2 Results for $\Delta u + \lambda^2 u = 0$ on Special Two Dimensional Domains**

We will examine specific solutions to $pde_1$, 

$$\left\{ \begin{array}{l}
\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega \\
u = 0 \quad \text{on} \quad \partial \Omega
\end{array} \right.$$  

on simple geometrical domains in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$. We will use known solutions to $pde_1$ to obtain information about eigenvalues and solutions to $pde_2$

$$\left\{ \begin{array}{l}
\Delta u + (\lambda + c(x))u = 0 \quad \text{in} \quad \Omega \\
u = 0 \quad \text{on} \quad \partial \Omega
\end{array} \right.$$  

on the same domain.

The separation of variables technique, gives explicit solutions to $pde_1$ on certain special domains and these solutions provide a basis for understanding some properties of solutions to $pde_2$ on these special domains, where, in general, the solutions are not explicitly known. Specifically we are interested in determining when a solution to a pde exists that is positive on the interior of a domain, $\Omega$ and zero on $\partial \Omega$. We will begin by examining $pde_1$ for $\Omega \subset \mathbb{R}^2$ on the following special domains: 1) rectangular; 2) circular; and 3) sector. We will then
apply these results to combinations of these domains \( \Omega \subset \mathbb{R}^3 \) adding results for spherical domains. Throughout these discussions, extensive use will be made of our versions of the theorems presented in [HNS].

We will first give a theorem that shows that the coefficient function of \( u \) must be positive in

\[
\begin{cases}
Lu = \sum_{i,j=1}^{n} a_{ij} (x) u_{x_i x_j} + \sum_{i=1}^{n} b_i (x) u_{x_i} + c(x)u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases} \tag{pde3}
\]

regardless of the domain configuration or the dimension of the space.

**Theorem 51** Let \( \Omega \) be any bounded domain in \( \mathbb{R}^n \), if \( c(x) \leq 0 \) in pde3, in \( \Omega \) pde3 has only the trivial solution.

**Proof.** The proof is a consequence of the weak maximum principle, see [GT], Theorem 8.1, page 179, for example or [Evans], pages 327-330. The theorem in [GT] states that if \( u \in W^{1,2}(\Omega) \) satisfies

\[
\sum_{i,j=1}^{n} a_{ij} (x) u_{x_i x_j} + \sum_{i=1}^{n} b_i (x) u_{x_i} + c(x)u = 0
\]

then

\[
\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ \text{ and } \inf_{\partial\Omega} u^- \leq \inf_{\Omega} u.
\]

Since \( u^+ = \max(u, 0) \) and \( u^- = -\min(u, 0) \) these, together, imply that \( u = 0 \) a.e., in \( \Omega \).

As a result of this theorem, we shall, henceforth, consider the following two equations:

\[
\begin{cases}
\Delta u + \lambda^2 u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases} \tag{pde1a}
\]
and

\[
\left\{ \begin{array}{l}
\Delta u + (\lambda + c(x)) u = 0 \quad \text{in } \Omega \\
 u = 0 \quad \text{on } \partial \Omega
\end{array} \right. 
\]

(pde$_2$)

where we will assume that

\[-\infty < m \leq c(x) \leq M < \infty \text{ with } m < M.\]

1.7.2.1 Rectangular Domains

We will consider pde$_{1a}$ on rectangular domains, $\Omega$, of the form $(0,a) \times (0,b)$, that we shall denote $\Omega_{a,b}$.

For a fixed value $\lambda = \overline{\lambda} > 0$, pde$_{1a}$ has a solution

\[ u = \sin \frac{\pi}{a} x \sin \frac{\pi}{b} y \]

that may be easily found using the separation of variables technique, provided the values of $a$ and $b$ satisfy

\[ \overline{\lambda}^2 = \pi^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right). \]

It may be easily shown using elementary calculus techniques that $u$, is a positive solution (positive on $\Omega_{a,b}$ and zero on $\partial \Omega_{a,b}$) and that any such $\Omega_{a,b}$ is a nodal domain. Additional properties of this positive solution $u$, include that any constant multiple of $u$ is also a solution, and any translation, $\tilde{u} = u(x - c, y - k)$, is a positive solution on $\Omega = (c,a + c) \times (k,b + k)$. These properties may be verified by substituting $\alpha \cdot \tilde{u}$ where $\alpha \neq 0$ and $\tilde{u}$, respectively into pde$_{1a}$ given the appropriate domains.

We now consider pde$_{1a}$ with $\lambda = \overline{\lambda} > 0$, fixed. This leads us the following theorem.

**Theorem 52** Let $\lambda = \overline{\lambda} > 0$ in pde$_{1a}$, then there exists a critical value,

\[ w = \frac{\pi}{\overline{\lambda}}, \]

so that if $a \leq w$, there is no value of $b$, so that $\Omega_{a,b}$ is a nodal domain. That is, there is no
value of $b$, such that $pde_{1a}$ has a positive solution on any domain $\Omega_{a,b}$.

Proof. If

$$0 < a \leq \frac{\pi}{\lambda},$$

then

$$\lambda^2 \leq \frac{\pi^2}{a^2},$$

and for any $b > 0$,

$$\lambda^2 < \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}.$$  

Setting

$$\lambda^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}$$

in $\Delta u + \lambda^2 u = 0$, one can find, by the separation of variables technique, a positive solution,

$$u = \sin \frac{\pi}{a} x \sin \frac{\pi}{b} y,$$

satisfying the pde on $\Omega_{a,b}$. Employing the Sturmian comparison theorem, HNS, the first corollary to GHNS, gives that any solution of $\Delta u + \lambda^2 u = 0$ must either have zeros or be a multiple of $u$. Since the latter is false, we see that the pde, $\Delta u + \lambda^2 u = 0$, has no positive solution for any value of $b$. ■

We now show that for $pde_{1a}$ with $\lambda = \lambda > 0$, fixed, and

$$a > w = \frac{\pi}{\lambda},$$

there exists a unique value of $b$, so that $\Delta u + \lambda^2 u = 0$ has a positive solution on $\Omega_{a,b}$.

**Theorem 53** Let $\lambda = \lambda > 0$ in $pde_{1a}$ and

$$a > \frac{\pi}{\lambda},$$

then there exists a unique value $b$, so that $\Omega_{a,b}$ is a nodal domain.
Proof. If
\[
 a > \frac{\pi}{\bar{\lambda}}
\]
then
\[
 \bar{\lambda}^2 > \frac{\pi^2}{a^2}
\]
and there exists \( b > 0 \) so that
\[
 \bar{\lambda}^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}.
\]
Specifically,
\[
 b = \frac{a\pi}{\sqrt{\bar{\lambda}^2 a^2 - \pi^2}}.
\]
It may be easily checked that
\[
 u = \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} y \right) = \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\sqrt{\bar{\lambda}^2 a^2 - \pi^2}}{a} y \right)
\]
solves \( \Delta u + \bar{\lambda}^2 u = 0 \), and that \( u = 0 \) on
\[
 \partial \Omega_{a,b} = \partial \Omega_{a,\sqrt{\bar{\lambda}^2 a^2 - \pi^2}}.
\]
It is also easy to establish that \( u \) is positive on the interior of \( \Omega_{a,b} \). The uniqueness of the solution and the uniqueness of the value of \( b \) both follow by employing \( \text{GHNS} \), again. Suppose there exists another positive solution, say \( v \), of \( pde_{1a} \) with \( \lambda = \bar{\lambda} \), on \( \Omega_{a,b} \); then either \( v \) would be a multiple of \( u \) or it would have zeros -- a contradiction. Now suppose that there exists a value \( b < \bar{b} \) and a positive solution \( \tilde{u} \), that solves \( pde_{1a} \) with \( \lambda = \bar{\lambda} \), on \( \Omega_{a,\tilde{b}} \). In such a case, our version of Heywood, Noussair, and Swanson’s theorem would imply that the original solution, \( u \), would either be a multiple of \( \tilde{u} \), or would have zeros. Again, this is a contradiction. By employing this theorem yet again, it is easy to see that in the case \( b > \bar{b} \) any solution, \( \tilde{u} \), that solves \( pde_{1a} \) with \( \lambda = \bar{\lambda} \), on \( \Omega_{a,\tilde{b}} \) would have zeros or be a multiple of \( u \) and would thus not be a positive solution. □

In the above cases, with say, \( \bar{\lambda} = 2 \), fixed, we may plot the value of \( b \) as a function of \( a \).
The plot for this case is given below in Figure 1.

![Figure 1. Nodal Corners for Rectangles.](image)

Here

\[ a > w = \frac{\pi}{2} \approx 1.57, \]

is the critical width. The graph above may be interpreted as a plot of the allowable upper right hand corners for nodal domains of the partial differential equation

\[
\begin{array}{l}
\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega_{a,b} \\
u = 0 \quad \text{on} \quad \partial \Omega_{a,b}
\end{array}
\]

where \( \lambda = 2 \). Of course, for each positive value of \( \lambda \), there is such a graph. We will more briefly refer to these graphs in subsequent discussions as nodal corner graphs.

1.7.2.2 Circular Domains

Once again we will consider \( \text{pde} \), but we shall concern ourselves with circular domains of the form \( \Omega_{\rho} = \{(x,y) | x^2 + y^2 < \rho^2 \} \). Employing the change of coordinates

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]
\( pde_{1a} \) becomes
\[
\Delta_r u(r, \theta) + \lambda^2 u(r, \theta) = 0 \quad \text{in} \quad \Omega_p
\]
\[
u(r, \theta) = 0 \quad \text{on} \quad \partial\Omega_p
\]
\[ (pde_{1a\ circ}) \]

with
\[
\Delta_r u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.
\]

Assuming a solution of the form \( u(r, \theta) = R(r)\Theta(\theta) \), substituting this into \( pde_{1a\ circ} \) and multiplying through by
\[
\frac{r^2}{R\Theta}
\]
gives
\[
\frac{r^2}{R} R_{rr} + \frac{r}{R} R_r + \frac{\Theta_{\theta\theta}}{\Theta} + r^2 \lambda^2 = 0.
\]
\[ (pde_{1a\ R\Theta}) \]

As \( R \) and \( \Theta \) are independent we consider
\[
\Theta_{\theta\theta} + \nu^2 \Theta = 0
\]
\[ (ode_{1a\ \Theta}) \]

and
\[
r^2 R_{rr} + r R_r + (r^2 \lambda^2 - \nu^2) R = 0.
\]
\[ (ode_{1a\ R}) \]

Now \( ode_{1a\ \Theta} \) besides the trivial solution, has the solution \( \Theta = \theta_0 \), a constant, provided \( \nu = 0 \); substituting \( \Theta = \theta_0 \) into \( pde_{1a\ R\Theta} \), gives \( ode_{1a\ R} \), with \( \nu = 0 \). On the other hand, \( ode_{1a\ \Theta} \) is satisfied if \( \Theta = \sin \nu \theta \). For any such \( \nu \), \( u(r, \theta) \) has a zero along the radial line \( \theta = 0 \) and such a solution is not positive on the circular domain \( \Omega_p \). Making another change of variables, letting \( s = \lambda r \), and substituting into \( ode_{1a\ R} \) gives the well known zeroth order Bessel ordinary differential equation or ode. This equation has a general solution
\[
R(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)
\]
where \( J_0 \) and \( Y_0 \) are the zeroth order Bessel functions of the first and second kinds, respectively. Because \( R(r) \) is finite at \( r = 0 \), \( C_2 = 0 \) and the solution of \( pde_{1a\ circ} \) is given by
\[
u(r, \theta) = C_2 \theta_0 J_0(\lambda r) \]
where \( \theta_0 \), and \( C_2 \) are arbitrary constants. In order to obtain a pos-
itive solution on $\Omega_\rho$, $\lambda \rho$ must satisfy $\lambda \rho = j_{0,1}$ where $j_{0,1}$ (following the notation of [AS]) denotes the first zero of the Bessel function, $J_0(r)$. Well known properties of the Bessel functions give that $u(r, \theta)$ as defined above, is, indeed, a positive solution on $\Omega_\rho$, so we take $C_2 = \theta_0 = 1$. To summarize, a positive solution of $pde_{1a,\text{circ}}$ for $\rho$, fixed, is given by

$$u(r, \theta) = J_0 \left( \frac{j_{0,1}}{\rho} r \right)$$

and for $\lambda$, fixed, is given by

$$u(r, \theta) = J_0 (\lambda r)$$

where

$$\rho = \frac{j_{0,1}}{\lambda}$$

Employing the Sturmian comparison theorem of Heywood, Noussair, and Swanson, once again, we obtain a uniqueness theorem.

**Theorem 54** Let $\lambda = \lambda > 0$ in $pde_{1a,\text{circ}}$, then there exists a unique value

$$\rho = \frac{j_{0,1}}{\lambda},$$

so that $\Omega_\rho$ is a nodal domain.

**Proof.** The proof is similar to the uniqueness proof in the rectangular domain case above. First we obtain the solution $u(r, \theta) = J_0 (\lambda r)$ where $\rho = \frac{j_{0,1}}{\lambda}$. We then use the GHNS comparison theorem to show that for any

$$\tilde{\rho} \neq \frac{j_{0,1}}{\lambda},$$

the assumption that another positive solution exists, leads to a contradiction. ■

An implicit plot then, of $J_0 (\lambda r) = 0$ where

$$\rho = \frac{j_{0,1}}{\lambda}.$$
is shown below.

![Figure 2. Nodal Relationship for Circles.](image)

$J_0(\lambda r) = 0$ and $\rho = y = \frac{j_{0,1}}{\lambda} ; x = \lambda$.

The coordinates of the graph $(\lambda, \rho)$ represent the allowable combinations of domain radii and eigenvalues that give positive solutions on nodal domains. Additionally, it is a straightforward calculation to show that these results hold for circular domains whose centers have been translated away from the origin.

### 1.7.2.3 Sector Domains

Here, we will consider sector shaped domains of the polar form $\{ (r, \theta) \mid r < \rho, 0 < \theta < \phi \}$ and we will denote such domains using the symbol $\Omega_{\rho,\phi}$. We will consider the partial differential equation

$$\Delta_r u(r, \theta) + \lambda^2 u(r, \theta) = 0 \quad \text{in} \quad \Omega_{\rho,\phi}$$

$$u(r, \theta) = 0 \quad \text{on} \quad \partial \Omega_{\rho,\phi}$$

much as we did in the circular case immediately above. Using the separation of variables procedure as outlined in the section above, making use of the solution, $\Theta = \sin \nu \theta$ to $ode_{1a} \Theta$ and applying the appropriate boundary conditions, we find that a positive solution of $pde_{1a \: circ}$ on the sector domain $\Omega_{\rho,\phi}$, is of the form

$$u(r, \theta) = \sin \nu \theta \cdot J_{\nu} (\lambda r) .$$

In order to satisfy the boundary conditions, $\lambda \rho = j_{0,1}$, as above, and $\nu = \frac{\pi}{\phi}$. In other words,
for $\rho$ and $\phi$ fixed,

$$u(r, \theta) = \sin \frac{\pi}{\phi} \cdot J_{\phi} \left( \frac{j_{0,1}}{\rho} r \right)$$

and for $\lambda$ fixed, and any $\phi$ so that $0 < \phi \leq 2\pi$,

$$u(r, \theta) = \sin \frac{\pi}{\phi} \cdot J_{\phi} \left( \lambda r \right)$$

where

$$\rho = \frac{j_{0,1}}{\lambda}.$$ 

The uniqueness theorem for the sector case will be stated without proof as the proof is identical to that for the circular case outlined above.

**Theorem 55** Let $\lambda = \bar{\lambda} > 0$ in $pde_{1a, sect}$, then there exists a unique value

$$\rho = \frac{j_{0,1}}{\bar{\lambda}},$$

so that $\Omega_{\rho, \phi}$ is a nodal domain for any value $\phi$ so that $0 < \phi \leq 2\pi$.

An implicit polar plot of

$$J_{\phi} \left( \bar{\lambda} \rho \right) = 0$$

for $\rho = \rho(\phi)$ for $0 < \phi \leq 2\pi$ where $\bar{\lambda} = 1$ and $\bar{\lambda} = 3$ and

$$\rho = \frac{j_{0,1}}{\bar{\lambda}}$$

is given below. The graph of the circle, $\rho = 3$, is provided for reference and the graph where $\bar{\lambda} = 1$ is the larger figure denoted by the dashed line. These graphs represent the allowable
nodal corners of the sectors for $pde_{1a\ sect}$ for which one leg lies on the positive $x$ axis.

![Diagram of nodal corners for sectors](image)

Figure 3. Nodal Corners for Sectors. $\lambda = 1$: --- $\lambda = 3$: - - -.

1.7.3 Results for $\triangle u + \lambda^2 u = 0$ on Special Three-Dimensional Domains

In this section we will extend the results obtained for two-dimensional domains to three-dimensional domains where a level set of the three-dimensional domain is one of the two-dimensional domains encountered above.

1.7.3.1 Shoe Box Domains

We will consider $pde_{1a}$ on domains of the form $\Omega_{a,b,c} = (0, a) \times (0, b) \times (0, c)$ where we assume, without loss of generality that $a \leq b \leq c$. We shall refer to such domains as shoe box domains. For a fixed value $\lambda = \bar{\lambda}$, $pde_{1a}$ has a solution

$$\bar{u} = \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} y \right) \sin \left( \frac{\pi}{c} z \right)$$

that may be found using the separation of variables technique for three dimensions, assuming that the values $a$, $b$, and $c$ satisfy

$$\bar{\lambda}^2 = \pi^2 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right).$$

As in the two-dimensional case, $\bar{u}$ is a positive solution and $\Omega_{a,b,c}$ a nodal domain. As
we have done above, in the rectangular case, if we fix \( \lambda = \overline{\lambda} \), we can obtain the following theorem.

**Theorem 56** Let \( \lambda = \overline{\lambda} > 0 \) in pde\(_{1a} \) and \( a > 0 \) be fixed with \( c \geq b \geq a \) for \( \Omega_{a,b,c} \), then there exists a critical value, 

\[
    w = \frac{\pi}{\overline{\lambda}},
\]

so that if \( a \leq w \), there is no value of \( b \) (or \( c \)), so that \( \Omega_{a,b,c} \) is a nodal domain. That is, there is no value of \( b \), so that pde\(_{1a} \) has a positive solution on any domain \( \Omega_{a,b,c} \).

**Proof.** If

\[
    0 < a \leq \frac{\pi}{\overline{\lambda}},
\]

then

\[
    \overline{\lambda}^2 \leq \frac{\pi^2}{a^2},
\]

and for any \( b > 0 \),

\[
    \overline{\lambda}^2 < \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}.
\]

Setting

\[
    \lambda^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2}
\]

in \( \Delta u + \lambda^2 u = 0 \), one can find, by the separation of variables technique, a positive solution, 

\[
    u = \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} y \right) \sin \left( \frac{\pi}{c} z \right),
\]

satisfying the pde on \( \Omega_{a,b,c} \). Employing the Sturmian comparison theorem, GHNS, gives that any solution of \( \Delta u + \overline{\lambda}^2 u = 0 \) must either have zeros or be a multiple of \( u \). Since the latter is false, we see that the pde, \( \Delta u + \overline{\lambda}^2 u = 0 \), has no positive solution for any value of \( b \).

**Corollary 57** Let \( \lambda = \overline{\lambda} > 0 \) in pde\(_{1a} \), then there exists a critical area value,

\[
    A = \frac{2\pi^2}{\overline{\lambda}^2}.
\]
so that if \( ab \leq A \), there is no value of \( c \), so that \( \Omega_{a,b,c} \) is a nodal domain. That is, there is no value of \( c \), so that pde\(_{1a}\) has a positive solution on any domain \( \Omega_{a,b,c} \).

Proof. If

\[
ab \leq A = \frac{2\pi^2}{\lambda^2},
\]

then

\[
\lambda^2 \leq \frac{2ab\pi^2}{a^2b^4}.
\]

However, if \( b \geq a > 0 \), then, \( 0 \leq (b - a)^2 = a^2 - 2ab + b^2 \), and \( 2ab \leq a^2 + b^2 \). Therefore,

\[
\lambda^2 \leq \frac{\pi^2}{a^2b^4} (a^2 + b^2) = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}.
\]

Thus, following previous arguments, there is no value of \( c \), so that pde\(_{1a}\) has a positive solution on any domain \( \Omega_{a,b,c} \).

Example 58 Consider the pde,

\[
\begin{cases}
\Delta u + 9\pi^2 u = 0 & \text{in } \Omega_{a,b,c} \\
u = 0 & \text{on } \partial\Omega_{a,b,c}
\end{cases}
\]

where \( u = u(x, y, z) \) and \( 0 < a \leq b \leq c \). The theorem and corollary above give the following results – note that \( \lambda^2 = 9\pi^2 \). The first theorem gives that if

\[
a \leq \frac{1}{3} = \frac{\pi}{3\pi},
\]

there is no positive solution, and the corollary gives that if

\[
ab \leq \frac{2}{9} = \frac{2\pi^2}{9\pi^2},
\]

there is no positive solution. The first theorem does not exclude a positive solution in the case where

\[
a = b = \frac{\sqrt{2}}{3},
\]
but the corollary does.

We also have the following theorem, which will be stated without proof because it is so similar to that of the two-dimensional case.

**Theorem 59** Let \( \lambda = \bar{\lambda} > 0 \) in \( \text{pde}_{1n} \) and

\[
a > \frac{\pi}{\bar{\lambda}},
\]

\[
ab > \frac{2\pi^2}{\bar{\lambda}^2}
\]

on \( \Omega_{a,b,c} \), then there exists a unique value \( c \), so that \( \Omega_{a,b,c} \) is a nodal domain.

A surface plot of the allowable nodal corners for the case where \( \bar{\lambda} = 3\pi \) is given below.

In such a case, we must have that

\[
a > \frac{1}{3} \text{ and } ab > \frac{2}{9}.
\]

Figure 4. Nodal Corners for Shoe Box Domains. \( 9\pi^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2} \)
1.7.3.2 Cylindrical Domains

We will, herein, consider domains $\Omega_{\rho,c} = \{(x,y) | x^2 + y^2 < \rho^2\} \times (0,c)$ for $pde_{1a}$, under the coordinate transformation

$$
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta.
\end{align*}
$$

A positive solution would be of the form

$$u = \sin \left( \frac{\pi}{c} \right) J_0(\mu r)$$

provided

$$\lambda^2 = \frac{\pi^2}{c^2} + \mu^2$$

and

$$\mu \rho = J_0(1).$$

That is, $u$ would satisfy

$$\Delta_{cyl} u(r,\theta,z) + \lambda^2 u(r,\theta,z) = 0 \quad \text{in} \quad \Omega_{\rho,c}$$

$$u(r,\theta,z) = 0 \quad \text{on} \quad \partial\Omega_{\rho,c}$$

where

$$\Delta_{cyl} u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}.$$ 

As in the shoe box domain case, for fixed $\lambda = \bar{\lambda}$, there is a critical height,

$$h = \frac{\pi}{\bar{\lambda}},$$

so that if the cylinder height $c \leq h$ then $pde_{1a\_cyl}$ has only the trivial solution. Thus we have the following theorem.
Theorem 60 Let $\lambda = \frac{\pi}{\lambda} > 0$ in $\text{pde}_{1a\text{\_cyl}}$ and $c > 0$ be fixed with
\[ c \leq h = \frac{\pi}{\lambda} \]
for $\Omega_{\rho,c}$. Then there is no value of $\rho$, so that $\Omega_{\rho,c}$ is a nodal domain. That is, there is no value of $\rho$, so that $\text{pde}_{1a\text{\_cyl}}$ has a positive solution on any domain $\Omega_{\rho,c}$.

Proof. If
\[ 0 < c \leq \frac{\pi}{\lambda}, \]
then
\[ \lambda^2 \leq \frac{\pi^2}{c^2}. \]
and for any $\mu > 0$,
\[ \lambda^2 < \frac{\pi^2}{c^2} + \mu^2. \]
Setting
\[ \lambda^2 = \frac{\pi^2}{c^2} + \mu^2 \]
in $\Delta u + \lambda^2 u = 0$, one can find, by the separation of variables technique a positive solution,
\[ u = \sin\left(\frac{\pi}{c}z\right)J_0(\mu r), \]
satisfying the pde on $\Omega_{c,\rho}$. Employing the Sturmian comparison theorem, GHNS, gives us that any solution of
\[ \Delta_{\text{cyl}} u + \lambda^2 u = 0 \]
must either have zeros or be a multiple of $u$. Since the latter is false, we see that the pde,
\[ \Delta_{\text{cyl}} u + \lambda^2 u = 0, \]
has no positive solution for any value of $\mu$. ■

There is a similar restriction on the radius of the cylinder.
Theorem 61 Let $\lambda = \bar{\lambda} > 0$ in pde$_{1a\text{ cyl}}$ and

$$R = \frac{j_{0,1}}{\lambda} \geq \rho > 0$$

for $\Omega_{\rho,c}$. Then there is no value of $c$, so that $\Omega_{\rho,c}$ is a nodal domain. That is, there is no value of $c$, so that pde$_{1a\text{ cyl}}$ has a positive solution on any domain $\Omega_{\rho,c}$. Specifically,

$$R = \frac{j_{0,1}}{\lambda},$$

is a critical radius so that if $\rho < R$ then there is no nodal domain $\Omega_{\rho,c}$ for pde$_{1a\text{ cyl}}$.

Proof. If

$$0 < \rho \leq \frac{j_{0,1}}{\lambda},$$

then for any $c > 0$,

$$\lambda^2 < \frac{\pi^2}{c^2} + \frac{j_{0,1}^2}{\rho^2}.$$ Setting

$$\lambda^2 = \frac{\pi^2}{c^2} + \frac{j_{0,1}^2}{\rho^2}$$
in

$$\Delta_{cyl} u + \lambda^2 u = 0,$$

one can find, by the separation of variables technique, a positive solution,

$$u = \sin \left( \frac{\pi}{c} z \right) J_0 \left( \frac{j_{0,1} r}{\rho} \right),$$
satisfying the pde on $\Omega_{c,\rho}$. Employing the Sturmanian comparison theorem, GHNS, gives that any solution of

$$\Delta_{cyl} u + \bar{\lambda}^2 u = 0$$

must either have zeros or be a multiple of $u$. Since the latter is false, we see that the pde,

$$\Delta_{cyl} u + \lambda^2 u = 0,$$
has no positive solution for any value of c. ■

As in the shoe box case, there is a corresponding theorem which states that if the height of the cylinder is sufficient, then there is a unique nodal domain for a given value of $\lambda$. As with the above, the theorem will be stated without proof.

**Theorem 62** Let $\lambda = \bar{\lambda} > 0$ in pde$_{1a_{cyl}}$ and

$$c > h = \frac{\pi}{\lambda}.$$ 

Then there is a unique value of $\rho$, so that $\Omega_{\rho,c}$ is a nodal domain. Similarly, if

$$\rho > R = \frac{j_{0,1}}{\lambda} > 0$$

then there exists a unique value of $c$, so that $\Omega_{\rho,c}$ is a nodal domain.

An implicit plot of the relation

$$\bar{\lambda}^2 = 4 = \frac{\pi^2}{c^2} + \frac{j_{0,1}^2}{\rho^2}$$

where, obviously, $\bar{\lambda} = 2$, is provided below. Note that

$$h = \frac{\pi}{2} \approx 1.57$$

and

$$R = \frac{j_{0,1}}{2} \approx 1.202.$$ 

This may be thought of as the allowable nodal corners of the cylindrical cross-sections.
1.7.3.3 Cheese Domains

Cheese domains take the form $\Omega_{\rho, \phi, c} = \{(r, \theta) \mid r < \rho, 0 < \theta < \phi\} \times (0, c)$ in cylindrical coordinates. A positive solution, $u$, to $\text{pde}_{1a_{cyl}}$ is of the form

$$u(r, \theta, z) = \sin \left( \frac{\pi}{c} z \right) \sin \left( \frac{\pi}{\phi} \theta \right) J_{\frac{\pi}{\phi}} (\mu r)$$

where $\mu, \lambda$ and $c$ satisfy

$$\lambda^2 = \frac{\pi^2}{c^2} + \mu^2$$

as in the cylindrical domain case above. Following the same considerations as those presented above leads to the following two theorems which we state without proof.

**Theorem 63** Let $\lambda = \bar{\lambda} > 0$ in $\text{pde}_{1a_{cyl}}$ and

$$\frac{j_{\frac{\pi}{\phi}, 1}}{\lambda} \geq \rho > 0$$

on $\Omega_{\rho, \phi, c}$. Then there is no value of $c$, so that $\Omega_{\rho, \phi, c}$ is a nodal domain. That is, there is no value of $c$, so that $\text{pde}_{1a_{cyl}}$ has a positive solution on any domain $\Omega_{\rho, \phi, c}$.

**Theorem 64** Let $\lambda = \bar{\lambda} > 0$ in $\text{pde}_{1a_{cyl}}$ and $c > 0$ be fixed with $h = \pi/\bar{\lambda}$. Then there is
a unique value of $\rho$, so that $\Omega_{p,\phi,c}$ is a nodal domain. Similarly, if

$$R = \frac{j_{\frac{c}{\rho}}^{2,1}}{\lambda} \geq \rho > 0$$

then there exists a unique value of $c$, so that $\Omega_{p,\phi,c}$ is a nodal domain.

Thus, there is a critical height and critical radius for a cheese domain, $\Omega_{p,\phi,c}$, to be a nodal domain for $pde_{1a \text{ cyl}}$ with $\lambda = \bar{\lambda}$, fixed.

Two contour surface plots of the relation

$$\bar{\lambda}^2 = \frac{\pi^2}{c^2} + \frac{j_{\frac{c}{\rho}}^{2,1}}{\rho^2}$$

where $\bar{\lambda} = 2$ in cylindrical coordinates is given below. The cylindrical coordinates $(r, \theta, z)$ that lie in the surface represent the parameters $(\rho, \phi, c)$ for the positive solution to $pde_{1a \text{ cyl}},$

$$u(r, \theta, z) = \sin \left( \frac{\pi}{c} z \right) \sin \left( \frac{\pi}{\phi} \theta \right) J_{\frac{c}{\rho}}^{2,1} (\mu r) \text{ where } \mu = \frac{j_{\frac{c}{\rho}}^{2,1}}{\rho}.$$ 

Figure 6. Nodal Corners for Cheese Domains.
To verify the presence of the shape of the "hole" in the graph similar to the graph in Figure 3, we provide an $x - y$ projection of the above two dimensional surface in $\mathbb{R}^3$.

1.7.3.4 Spherical Domains

Once again we will consider $pde_{1a}$, but we shall concern ourselves with spherical domains of the form $\Omega_\rho = \{(x, y) | x^2 + y^2 + z^2 < \rho^2\}$. Employing the change of coordinates

\[
x = r \cos \theta \sin \phi \\
y = r \sin \theta \sin \phi \\
z = r \cos \phi
\]

$pde_{1a}$ becomes

\[
\begin{align*}
\Delta_{sph.} u(r, \theta, \phi) + \lambda^2 u(r, \theta, \phi) &= 0 \quad \text{in} \quad \Omega_\rho \\
u(r, \theta, \phi) &= 0 \quad \text{on} \quad \partial \Omega_\rho
\end{align*}
\]

(pde$_{1a}$ $sph$)

with

\[
\Delta_{sph.} u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2 \sin^2 \phi} u_{\theta \theta} + \frac{1}{r^2} u_{\phi \phi} + \frac{\cot \phi}{r^2} u_\phi.
\]

Assuming a solution of the form $u(r, \theta) = R(r)\Theta(\theta)\Phi(\phi)$, substituting into $pde_{1a}$ $sph$ and
multiplying through by \[ \frac{r^2}{R \Theta \Phi} \]
gives
\[
\frac{r^2}{R} R_{rr} + \frac{2r}{R} R_r + \frac{1}{\sin^2 \phi} \Theta_{\theta \theta} + \frac{1}{\Phi} \Phi_{\phi \phi} + \frac{\cot \phi}{\Phi} \Phi_{\phi} + \lambda^2 r^2 = 0.
\]
(pde1a sphR\(\Theta\Phi\))

As \( R \) and \( \Theta \) and \( \Phi \) are independent we consider
\[
\frac{r^2}{R} R_{rr} + \frac{2r}{R} R_r + \lambda^2 r^2 = \Lambda
\]
(ode1a sph\(R\))
\[
\frac{\Theta_{\theta \theta}}{\Theta} = -\mu^2
\]
(ode1a sph\(\Theta\))
and
\[
\sin^2 \phi \cdot \Phi_{\phi \phi} + \cos \phi \sin \phi \Phi_{\phi} + (\Lambda \sin^2 \phi - \mu^2) \Phi = 0.
\]
(ode1a sph\(\Phi\))

This of course is the separation of variables technique in spherical coordinates following [Myint]. The general solution of ode1a sph\(R\) is given by
\[
R(r) = C_1 \frac{1}{\sqrt{r}} J_{\frac{1}{2}} \sqrt{1 + 4\Lambda} (\lambda r) + C_2 \frac{1}{\sqrt{r}} Y_{\frac{1}{2}} \sqrt{1 + 4\Lambda} (\lambda r)
\]
where as previously used, \( J_{\nu}(x) \) and \( Y_{\nu}(x) \) are Bessel functions of the first and second kinds. Continuity of the solution requires that \( C_2 = 0 \), and because we seek real solutions, we see that
\[
\Lambda \geq -\frac{1}{4}.
\]
As in the circular case, continuity and periodicity requirements force \( \Theta \) to be constant in ode1a sph\(\Theta\). Substituting
\[
\Lambda = \frac{r^2}{R} R_{rr} + \frac{2r}{R} R_r + \lambda^2 r^2
\]
and

$$-\mu^2 = \frac{\Theta_{\theta\theta}}{\Theta}$$

into $pde_{1a \text{ sph }} R\Theta$ gives $ode_{1a \text{ sph }}$. Setting $\xi = \cos \phi$ in $ode_{1a \text{ sph }}$ leads to a solution of $\Phi$ in terms of hypergeometric functions, however, for our purposes the continuity and periodicity of $\Phi$ in similar fashion to $\Theta$, requires that $\Phi$ be constant as well. Furthermore, a constant solution to $pde_{1a \text{ sph}} R\Phi$ only occurs when $\Lambda = 0$ as well. Hence, the positive solution to the spherical Dirichlet eigenvalue problem given in $pde_{1a \text{ sph}}$ is

$$u(r, \theta, \phi) = J_{\frac{1}{2}} (\lambda r).$$

The boundary condition is satisfied if, and only if,

$$\lambda \rho = j_{\frac{1}{2}, 1}$$

as in the circular case.

Employing the Sturmian comparison theorem of $[HNS]$, once again we obtain a uniqueness theorem which we give without proof.

**Theorem 65** Let $\lambda = \bar{\lambda} > 0$ in $pde_{1a \text{ sph }}$, then there exists a unique value

$$\rho = \frac{j_{\frac{1}{2}, 1}}{\bar{\lambda}},$$

so that $\Omega_\rho$ is a nodal domain.

**1.7.4 Extension of Results for $\Delta u + \lambda^2 u = 0$ on Special Two-Dimensional Domains**

and to $\Delta u + (\lambda + c(x)) u = 0$

In this section we will consider $pde_2$, 

$$\begin{cases} 
\Delta u + (\lambda + c(x)) u = 0 & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega 
\end{cases}$$
where we will assume that

\[-\infty < m = \inf_{x \in \mathbb{R}^n} c(x) \leq c(x) \leq M = \sup_{x \in \mathbb{R}^n} c(x) < \infty\]

and that \( \Omega \) is one of the special two dimensional domains examined above. It is worth noting that \( \lambda \) may be negative in this situation and a positive solution, \( u \), still exist. As a rather trivial example with the restriction \( m < M \) relaxed, let \( c(x) = 3 \), on \( \Omega = (0, \pi) \times (0, \pi) \) then \( \lambda = -1 \), and \( u = \sin x \sin y \). For simplicity of the discussion, we will assume that \( m \) (and thus \( M \) as well) satisfy \( m > 0 \). If this is not the case useful and similar results may be obtained but the graphical representations presented here do not necessarily make sense.

1.7.4.1 Rectangular Domains

As above we will consider domains, \( \Omega \), of the form \( (0, a) \times (0, b) \), that we shall denote \( \Omega_{a,b} \).

In addition to \( pde_2 \), we shall also consider the following two partial differential equations on \( \Omega_{a,b} \):

\[
\begin{align*}
\begin{cases}
\Delta u + mu = 0 & \text{in } \Omega_{a,b} \\
u = 0 & \text{on } \partial\Omega_{a,b}
\end{cases}
\end{align*}
\]

\((pde_{2m})\)

and

\[
\begin{align*}
\begin{cases}
\Delta u + Mu = 0 & \text{in } \Omega_{a,b} \\
u = 0 & \text{on } \partial\Omega_{a,b}
\end{cases}
\end{align*}
\]

\((pde_{2M})\)

In these two equations \( m = \inf_{x \in \mathbb{R}^n} c(x) \) and \( M = \sup_{x \in \mathbb{R}^n} c(x) \). With eye to making use of previous results we define the following:

\[
\begin{align*}
w_m &= \frac{\pi}{\sqrt{m}}; \\
w_M &= \frac{\pi}{\sqrt{M}}; \\
b_m &= \frac{a\pi}{\sqrt{a^2m - \pi^2}}; \\
b_M &= \frac{a\pi}{\sqrt{a^2M - \pi^2}}.
\end{align*}
\]
We note the following properties of these values provided \( w_m < a \):

\[
\begin{align*}
    w_M < b_M \text{ and} \\
    w_m < b_m \text{ with} \\
    w_M < w_m.
\end{align*}
\]

The proof of the inequalities, above, follows. If

\[
    w_M = \frac{\pi}{\sqrt{M}} < a
\]

then

\[
    b_M = \frac{a\pi}{\sqrt{a^2M - \pi^2}}.
\]

Since both \( w_M \) and \( b_M \) are positive, \( w_M < b_M \) if, and only if, \( b_M^2 - w_M^2 > 0 \) or

\[
    \frac{a^2\pi^2}{a^2M - \pi^2} - \frac{\pi^2}{M} > 0.
\]

This holds if, and only if,

\[
    \frac{Ma^2\pi^2 - \pi^2(a^2M - \pi^2)}{M(a^2M - \pi^2)} = \frac{\pi^4}{M(a^2M - \pi^2)} > 0,
\]

which holds if, and only if, \( a^2M - \pi^2 > 0 \), or equivalently,

\[
    \frac{\pi}{\sqrt{M}} < a.
\]

Similarly, we have \( w_m < b_m \), and the fact that \( w_M < w_m \) is trivial. We also note the following relationship:

\[
    \frac{\pi^2}{a^2} \leq M - m
\]
if, and only if, \( b_M \leq w_m \). The proof is as follows:

\[
b_M \leq w_m \text{ if, and only if, } \frac{a\pi}{\sqrt{a^2M - \pi^2}} \leq \frac{\pi}{\sqrt{m}} \text{ if, and only if } \frac{a^2m}{a^2} \leq a^2M - \pi^2 \text{ which holds if, and only if } \frac{\pi^2}{a^2} \leq M - m.
\]

We summarize these results in two cases:

Case 1 \( \frac{\pi^2}{a^2} \leq M - m \) \( w_M < b_M \leq w_m < b_m \)

Case 2 \( M - m < \frac{\pi^2}{a^2} \) \( w_M < w_m < b_M < b_m \).

We will show that in either case, when \( a > w_m \) that the allowable nodal corners for \( pde_2 \) lie in the region of the \( x - y \) plane between the graphs of the two curves

\[
y = \frac{x\pi}{\sqrt{x^2M - \pi^2}}
\]

and

\[
y = \frac{x\pi}{\sqrt{x^2m - \pi^2}}.
\]

In addition, we will prove that if \( w_m > a \geq w_M \) that the allowable nodal corners lie in the region of the \( x - y \) plane bounded by the graph of the curve

\[
y = \frac{x\pi}{\sqrt{x^2M - \pi^2}},
\]

not containing the origin.

**Theorem 66** Let \( 0 < m \leq c(x,y) \leq M < \infty \) in \( pde_2 \) as assumed above, and

\[
a > w_m = \frac{\pi}{\sqrt{m}},
\]

then the nodal corner \((a, b)\) of the nodal domain \( \Omega_{a,b} \) for the positive solution to \( pde_2 \) lies
between the graphs of the two curves

\[ y = \frac{x\pi}{\sqrt{x^2 M - \pi^2}} \]

and

\[ y = \frac{x\pi}{\sqrt{x^2 m - \pi^2}}. \]

That is, the second coordinate, \( b \), of the nodal corner satisfies

\[ \frac{a\pi}{\sqrt{a^2 M - \pi^2}} < b < \frac{a\pi}{\sqrt{a^2 m - \pi^2}}. \]

**Proof.** The proof will proceed by considering cases and employing the comparison theorem of Heywood, Noussair, and Swanson, [HNS]. First, let \( 0 < m \leq c(x, y) \leq M < \infty \) in \( pde_2 \) as assumed above, and

\[ a > \omega_m = \frac{\pi}{\sqrt{m}}, \]

be given. Since in either Case 1, or Case 2, above

\[ b_M = \frac{a\pi}{\sqrt{a^2 M - \pi^2}} < b_m = \frac{a\pi}{\sqrt{a^2 m - \pi^2}}, \]

we must have exactly one of

\( (i) \quad b \leq \frac{a\pi}{\sqrt{a^2 M - \pi^2}} \) or

\( (ii) \quad \frac{a\pi}{\sqrt{a^2 M - \pi^2}} < b < \frac{a\pi}{\sqrt{a^2 m - \pi^2}} \) or

\( (iii) \quad \frac{a\pi}{\sqrt{a^2 m - \pi^2}} \leq b. \)

If \( (i) \) holds and \( u(x, y) \) is a positive solution of \( pde_2 \) on \( \Omega_{a,b} \), we consider

\[ v(x, y) = \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right). \]
The function \( v(x, y) \) solves

\[
\begin{aligned}
\Delta u + \left( \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \right) u &= 0 \quad \text{in} \quad \Omega_{a,b} \\
\Delta u + c(x)u &= 0 \quad \text{in} \quad \Omega_{a,b} \\
\end{aligned}
\]

Since

\[
b \leq \frac{a\pi}{\sqrt{a^2M - \pi^2}},
\]

we have

\[
a^2M - \pi^2 \leq \frac{a^2\pi^2}{b^2}, \quad \text{or}
\]

\[
M \leq \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}.
\]

Because \( c(x) \leq M \) we can write that

\[
c(x) \leq \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}
\]

According to our first corollary to the theorem, GHNS, of Heywood, Noussair, and Swanson we see that any solution to

\[
\begin{aligned}
\Delta u + c(x)u &= 0 \quad \text{in} \quad \Omega_{a,b} \\
u &= 0 \quad \text{on} \quad \partial\Omega_{a,b}
\end{aligned}
\]

must have a zero – for if it did not then the function \( v(x, y) \) defined above would have a zero on \( \Omega_{a,b} \) – which is clearly false. We now consider case (iii) above. By a similar argument to that just given, if

\[
\frac{a\pi}{\sqrt{a^2m - \pi^2}} \leq b
\]

holds, then

\[
\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \leq m \leq c(x).
\]

The existence of \( v(x, y) \) as defined above would together with the corollary just mentioned,
imply that any solution to

\[
\begin{cases}
\Delta u + c(x)u = 0 & \text{in } \Omega_{a,b} \\
u = 0 & \text{on } \partial\Omega_{a,b}
\end{cases}
\]

would have a zero on \( \Omega_{a,b} \). The remaining case, (ii), must hold:

\[
\frac{a\pi}{\sqrt{a^2 M - \pi^2}} < b < \frac{a\pi}{\sqrt{a^2 m - \pi^2}}
\]

which simply says that the upper right hand nodal corner of a domain \( \Omega_{a,b} \) on which

\[
\begin{cases}
\Delta u + c(x)u = 0 & \text{in } \Omega_{a,b} \\
u = 0 & \text{on } \partial\Omega_{a,b}
\end{cases}
\]

has a positive solution must lie between the graphs of

\[
y = \frac{x\pi}{\sqrt{x^2 M - \pi^2}}
\]

and

\[
y = \frac{x\pi}{\sqrt{x^2 m - \pi^2}}.
\]

We will again use the theorems of Heywood, Noussair, and Swanson to obtain a result that says that for rectangular domains, \( \Omega_{a,b} \), where \( u \) is a positive solution of the fixed pde

\[
\begin{cases}
\Delta u + c(x,y)u = 0 & \text{in } \Omega_{a,b} \\
u = 0 & \text{on } \partial\Omega_{a,b}
\end{cases}
\]

with

\[
-\infty < m = \inf_{(x,y) \in \mathbb{R}^2} c(x,y) \leq c(x,y) \leq M = \sup_{(x,y) \in \mathbb{R}^2} c(x,y) < \infty \text{ and } m < M,
\]
the height, $b$, of the nodal rectangle, $\Omega_{a,b}$, is a strictly decreasing function of the width, $a$. We state the theorem.

**Theorem 67** Let $u$ be a positive solution of

$$\begin{cases}
\Delta u + c(x, y)u = 0 & \text{in } \Omega_{a,b} \\
u = 0 & \text{on } \partial\Omega_{a,b}
\end{cases}$$

where

$$-\infty < m = \inf_{(x,y)\in\mathbb{R}^2} c(x, y) \leq c(x, y) \leq M = \sup_{(x,y)\in\mathbb{R}^2} c(x, y) < \infty \text{ and } m < M.$$  

By definition, $\Omega_{a,b}$ is a nodal domain. If $v$ is any other positive solution on a nodal domain, $\Omega_{\alpha,\beta}$, where $\alpha > a$ then $\beta < b$.

**Proof.** Let us assume the contrary for the purposes of obtaining a contradiction. That is, let $u(x, y)$ be a positive solution to the given pde on the nodal domain $\Omega_{a,b}$. Let $\alpha > a$ and $\beta \geq b$ and assume that $v(x, y)$ solves the given pde as well, on $\Omega_{\alpha,\beta}$, however. Note that in accordance with the previous theorem, we assume that both ordered pairs, $(a, b)$ and $(\alpha, \beta)$ lie between the graphs,

$$y = \frac{x\pi}{\sqrt{x^2 M - \pi^2}}$$

and

$$y = \frac{x\pi}{\sqrt{x^2 m - \pi^2}}.$$  

It is clear that in this case that $\Omega_{a,b} \subset \Omega_{\alpha,\beta}$. If $v(x, y)$ were to solve the given pde as assumed, it, along with $u(x, y)$ would be a positive solution to

$$\Delta u + c(x)u = 0 \text{ in } \Omega_{a,b}$$

on $\Omega_{a,b}$ – a contradiction to GHNS and its appropriate corollary. $\blacksquare$

It is perhaps, best to sum up the results just obtained in a graphical matter. For
example, if $m = \pi^2$ and $M = 4\pi^2$ and $c(x, y)$ is any continuous function so that

$$
\pi^2 \leq c(x, y) \leq 4\pi^2
$$

on all of $\mathbb{R}^2$, then the upper right-hand corner of a nodal rectangle, must lie between the two graphs given below. Furthermore, the upper right-hand corners of any two nodal domains for the given equation

$$
\Delta u + c(x)u = 0 \quad \text{in} \quad \Omega_{a,b}
$$

can be connected with a line segment whose slope is strictly negative. In the plot below, the curved lines represent the boundary of the region where the upper right-hand corners of the nodal domains must reside. The upper one of the two corresponds to the value $m = \pi^2$. The rectangles represent two possible nodal domains and the dashed line, the line segment connecting the upper right hand corners of those domains; the slope, of which is obviously negative.

Note that in the theorem presented directly above that the partial differential equation given

$$
\begin{cases}
\Delta u + c(x, y)u = 0 \quad \text{in} \quad \Omega_{a,b} \\
u = 0 \quad \text{on} \quad \partial\Omega_{a,b}
\end{cases}
$$
is just a special case of

\[
\begin{cases}
\Delta u + (\lambda + c(x,y))u = 0 \quad \text{in } \Omega_{a,b} \\
u = 0 \quad \text{on } \partial\Omega_{a,b}
\end{cases}
\]

It is the case where \( \lambda = 0 \). In general when \( c(x,y) \) is any bounded function – the bounds being positive or negative – on \( \mathbb{R}^2 \), for any rectangular domain \( \Omega_{a,b} \), a first eigenvalue and corresponding first eigenfunction may be found to solve the pde when \( \lambda \) is free. This is simply a consequence of the existence of a positive solution to the eigenvalue problem on a fixed domain (see [Evans], section 6.5, for example). If the eigenvalue is fixed, however, as in the case above where we took \( \lambda = 0 \), then a given rectangular domain may not have a positive solution. This topic will be treated somewhat more fully when we consider parameterized boundaries later on.

1.7.4.2 Other Special Two-Dimensional Domains

In general where we want to solve

\[
\begin{cases}
\Delta u + c(x,y)u = 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega
\end{cases}
\]

and we have

\[-\infty < m = \inf_{(x,y)\in\mathbb{R}^2} c(x,y) \leq c(x,y) \leq M = \sup_{(x,y)\in\mathbb{R}^2} c(x,y) < \infty \text{ with } m < M\]

we will have a nodal corner graph for each of the equations

\[
\begin{cases}
\Delta u + mu = 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega
\end{cases}
\]
and

\[
\begin{cases}
\Delta u + Mu = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

These graphs will place limits on the external boundaries of nodal domains for a given partial differential equation. If more than one nodal domain might exist for a fixed equation, it must hold that no domain be a subdomain of any second domain. With particular reference to the other special two dimensional domains that we have examined, we may say the following.

In the circular case for

\[
\begin{cases}
\Delta u + c(x, y)u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

where

\[
0 < m = \inf_{(x, y) \in \mathbb{R}^2} c(x, y) \leq c(x, y) \leq M = \sup_{(x, y) \in \mathbb{R}^2} c(x, y) < \infty \text{ with } m < M.
\]

There exists a unique circular domain of radius \( \rho \) with

\[
\frac{j_{0,1}}{M} \leq \rho \leq \frac{j_{0,1}}{m}
\]

on which the pde has a positive solution.

In the sector case, the values of \( M \) and \( m \) define a ribbon shaped region (similar to that in Figure 3) between which the allowable free corner of nodal sector domains must reside. A little thought, and sketching perhaps, will give some insight into the geometrical arrangement two possible sectors might take. As in the rectangular case, the more general problem

\[
\begin{cases}
\Delta u + (\lambda + c(x, y))u = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

will have a solution on any domain, provided \( \lambda \) is free and \( c(x, y) \) is bounded in \( \mathbb{R}^2 \).

1.7.5 Extension of Results for \( \Delta u + \lambda^2 u = 0 \)
on Special Three-Dimensional Domains

to \( \Delta u + (\lambda + c(x)) u = 0 \)

The extension of results for \( \Delta u + \lambda^2 u = 0 \) on the special three dimensional domains to outlined above to \( \Delta u + c(x) u = 0 \), is a straightforward exercise with repeated application of the second corollary to GHNS, the theorem of Heywood, Noussair, and Swanson. A simple combination of the results presented for special two dimensional domains gives anticipated results.

1.8 Epilogue

We have been successful in extending the results of Heywood, Noussair, and Swanson, \([HNS]\), in their paper, "On the Zeros of Solutions of Elliptic Inequalities in Bounded Domains," to linear elliptic partial differential equations on various domains. In light of applying the numerical techniques referenced in the next section of this dissertation to various domains, it seems reasonable that the results of Heywood, Noussair, and Swanson might be further expanded, to both nonlinear partial differential equations and to more general domains than those herein studied. Indeed, Heywood Noussair, and Swanson’s results already apply to quasilinear pdes and they note that further expansions are straight forward. It is possible that the proofs given above, might be reworked to extend our results to some nonlinear cases as well. Regarding more general domains, approximate graphical results that coincide with those presented above on domains that do not satisfy even the segment condition (but which have continuous boundaries) have been observed. It seems reasonable that if a weak version of Picone’s identity – the backbone of the Heywood, Noussair, and Swanson results – could be found, along with an appropriate definition for a positive function only in \( W^{1,2}(\Omega) \), but not continuous, then the results might be further expanded to more general domains.

Any of these projects may be taken up at a later time.
REFERENCES


[Maple] "Maple" is a mathematical software package that is copyrighted by Waterloo Maple, Inc.

[Matlab] "Matlab" is a mathematical software package that is copyrighted by The Mathworks, Inc.


[Vess] "Vessiot" is the name of a software package developed under the guidance of Dr. Ian M. Anderson, Utah State University, for use with the Maple software program, portions were authored by Charles E. Miller.
APPENDICES
In this section we will present some additional arguments for Part Two of the upper continuity proof. That is, we will show, as above, that if \( \{ \Omega^i \}_{i=1}^{\infty} \) is a decreasing sequence of nested domains in \( \mathbb{R}^n \), decreasing to \( \Omega_0 \), with certain properties in addition to the segment property, and \( u \in W^{1,2}_0(\Omega^i) \) with \( u \equiv 0 \) on \( \Omega^1 \setminus \Omega^i \) for all \( i = 1, 2, \ldots \), then \( u \in W^{1,2}_0(\Omega_0) \). Recall that this function, \( u \), is a function which is the limit of a sequence of first eigenfunctions for certain partial differential equations. We will construct a proof showing that under certain (additional) hypotheses on the coefficients of the operator \( L \) and additional hypotheses on the domains \( \Omega^i \) and \( \Omega_0 \) the function

\[
  u = \lim_{i \to \infty} u^i
\]

where each function \( u^i \) is the first eigenfunction for

\[
  \left\{ \begin{array}{ll}
  Lu + \lambda u &= 0 & \text{in } \Omega^i \\
  u &= 0 & \text{on } \partial \Omega^i
  \end{array} \right.
\]

is in fact in the space \( C^{0,\alpha}(\Omega_0) \). With this true, it is a simple matter to conclude that \( u \in W^{1,2}_0(\Omega_0) \), as well. The proof is simpler if the domains \( \Omega^i \) and \( \Omega_0 \) are of class \( C^2 \) in \( \mathbb{R}^n \) rather than rectangles in \( \mathbb{R}^2 \). Thus we will tackle the proof in the \( C^2 \) boundary case first. We will then construct a similar proof for the case where the boundaries of the domains \( \Omega^i \) and \( \Omega_0 \) are of class \( C^0 \) in \( \mathbb{R}^n \) with some other restrictions where additionally, each boundary \( \partial \Omega^i \) is a finite union of pieces each of which is \( C^2 \). The proof in the rectangular case will simply be a corollary of this latter proof.

Ultimately all of these proofs are dependent upon applying the Sobolev embedding theorem to the sequence of functions \( \{ u^i \} \) mentioned above. For each \( i = 1, 2, \ldots \), we will
obtain that

\[ u^i \in C^{0,\alpha}(\Omega^i). \]

Furthermore, by the (compact) embedding of \( W_0^{1,2}(\Omega^i) \) into \( C^{0,\alpha}(\Omega^i) \) and hence into \( C^{0,\alpha}(\overline{\Omega}^1) \) we will find that the limit function \( u \in C^{0,\alpha}(\overline{\Omega}^1) \), as well.

These proofs were suggested by Dr. Z.-Q. Wang.

A.1 Upper Continuity Proof for Domains with \( C^2 \) Boundaries in \( \mathbb{R}^n \)

To begin, let us consider a decreasing sequence of nested domains \( \{\Omega^i\}_{i=1}^{\infty} \) in \( \mathbb{R}^n \), decreasing to \( \Omega_0 \), where for each \( i = 0, 1, 2, \ldots, \Omega^i \) is of class \( C^2 \). We observe that it is easy to see that if a domain is of class \( C^2 \), then it satisfies interior and exterior cone conditions, the strong local Lipschitz condition and hence has the segment property. We claim that if, for each \( i = 1, 2, \ldots, u^i \) is the first eigenfunction for the problem

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega^i \\
u = 0 & \text{on } \partial\Omega^i
\end{cases}
\]

and \( u^i \to u \) as \( i \to \infty \) in \( W_0^{1,2}(\Omega^1) \cap W^{2,2}_{\text{loc}}(\Omega^1) \) then \( u \in C^{0,\alpha}(\Omega^1) \) for some \( \alpha > 0 \).

Somewhat more precisely, we state this in the following theorem.

**Theorem 68** Let \( \{\Omega^i\}_{i=1}^{\infty} \) be a decreasing sequence of nested domains in \( \mathbb{R}^n \), decreasing to \( \Omega_0 = \Omega^0 \), where for each \( i = 0, 1, 2, \ldots, \Omega^i \) is of class \( C^2 \). Furthermore, let \( u^i \) be the first eigenfunction for the problem

\[
\begin{cases}
Lu + \lambda u = 0 & \text{in } \Omega^i \\
u = 0 & \text{on } \partial\Omega^i
\end{cases}
\]

and \( u^i \to u \) as \( i \to \infty \) in \( W_0^{1,2}(\Omega^1) \cap W^{2,2}_{\text{loc}}(\Omega^1) \) where

\[
Lu = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u,
\]

\( a_{ij}(x) \in C^1(\Omega) \), \( b_i(x), c(x) \in L^\infty(\Omega) \) and \( a_{ij} = a_{ji} \) for all appropriate \( i, j = 1, 2, \ldots, n \). We
assume also that $L$ admits a variational formulation. As we have done previously, assume that the operator is strictly elliptic and the coefficient functions are bounded, say

$$\Theta \geq \sum_{i,j=1}^{n} |a_{ij}(x)|^2$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for all $x \in \bar{\Omega}^1$ and $\xi \in \mathbb{R}^n$ with

$$\sum_{i=1}^{n} |b_i(x)| + |c(x)| \leq \nu < \infty.$$ 

for all $x \in \bar{\Omega}^1$. Additionally, assume that for all $i > 0$, $u^i \equiv 0$ on $\Omega^i \setminus \Omega_0$. Then $u, u^i \in C^{0,\alpha}(\bar{\Omega}^1)$ for some $\alpha > 0$, and in fact $u \in W_{0}^{1,2}(\Omega_0)$.

**Proof.** Let $\{\Omega^i\}_{i=1}^{\infty}$ be a decreasing sequence of nested domains in $\mathbb{R}^n$, decreasing to $\Omega_0 = \Omega^0$, where for each $i = 0, 1, 2, \ldots$, $\Omega^i$ is of class $C^2$. Also let the conditions on the operator $L$ as defined in the hypotheses of the theorem hold. The properties of the operator $L$ guarantee that the partial differential equations

$$\begin{cases}
L u + \lambda u = 0 & \text{in } \Omega^i \\
\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial \Omega^i
\end{cases}$$

have respective weak (first eigenfunction) solutions $u^i \in W_{0}^{1,2}(\Omega^1) \cap W_{loc}^{2,2}(\Omega^1)$ by extension. Furthermore, by Part One of the upper continuity proof, we have that $u^i \to u$ as $i \to \infty$ in $W_{0}^{1,2}(\Omega^1) \cap W_{loc}^{2,2}(\Omega^1)$ and for all $i > 0$, $u^i \equiv 0$ on $\Omega^i \setminus \Omega_0$. It will be sufficient to show that $u^i \in C^{0,\alpha}(\bar{\Omega}^1)$ for all $i$, and for some $\alpha > 0$ since $C^{0,\alpha}(\bar{\Omega}^1)$ is a Banach space which, by definition, is complete.

To prove this we will proceed as follows: First we will show for $i$ fixed, but arbitrary, that $u^i \in W^{2,2}(\Omega^1)$, not just $W_{loc}^{2,2}(\Omega^1)$; Second we will show by the Sobolev embedding theorem that $u^i \in L^q(\Omega^1)$ for some $q > n/2$; Third and finally we will show, again, by
employing the Sobolev embedding theorem that $u^i \in C^{0,\alpha}(\Omega^1)$. The proofs of many of the details follow arguments in Evans, Section 6.3, [Evans], and Gilbarg and Trudinger, Chapter 9, on so-called strong solutions, [GT].

We will need a lemma whose proof closely follows regularity arguments presented in Evans in sections 6.3.1 (pages 310-313) and 6.3.2 (pages 317-322). We state the lemma.

**Lemma 69** Let $\Omega = \Omega^i$ for some fixed positive integer $i$ and $x_0 \in \partial \Omega$. Then for any $0 < \rho < \frac{r}{2}$ and $u = u^i$ satisfying (for $L$ as above)

$$\begin{cases} 
Lu + \lambda u = g & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

with $g \in L^2(\Omega)$, we have $u \in W^{1,2}_0(\Omega \cap B(\rho, x_0)) \cap W^{2,2}(\Omega \cap B(\rho, x_0))$.

**Proof.** Let the hypotheses hold. Following Evans closely as he argues in Section 6.3.2, we first consider the particular case where

$$\beta = B(r, x_0) = B(1, 0) \cap \mathbb{R}^n,$$

$$B(1, 0) = B(1, [0, 0, \ldots, 0])$$

and

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) | x_n > 0\}.$$

We set $\beta' = B(\frac{1}{2}, 0) \cap \mathbb{R}^n_+$ and then select a smooth cutoff function $\zeta$ so that

$$\begin{cases} 
\zeta \equiv 1 & \text{on } \beta' \\
\zeta \equiv 0 & \text{on } \mathbb{R}^n \setminus B(1, 0) \\
0 \leq \zeta \leq 1
\end{cases}$$

We observe that $\zeta$ is zero near the curved boundary of $\partial \beta$. Because $u$ is a weak solution of
the partial differential equation, we have for all \( v \in W_{0}^{1,2}(\beta) \) that

\[
\sum_{i,j=1}^{n} \int_{\beta} a_{ij}(x) u_{x_{i}} u_{x_{j}} \, dx = \int_{\beta} g v \, dx = \int_{\beta} - \left( \sum_{i=1}^{n} b_{i}(x) u_{x_{i}} + (c(x) + \lambda) u \right) v \, dx.
\]

We next choose \( h > 0 \) sufficiently small and \( k \in \{1, 2, \ldots, n-1\} \) and set

\[
v = -D_{k}^{-h} \left( \zeta^{2} D_{k}^{h} u \right)
\]

where the term \( D_{k}^{h} u \) is the \( i^{th} \) difference quotient

\[
D_{k}^{h} u := \frac{1}{h} (u(x + he_{i}) - u(x))
\]

and \( e_{i} \) is the \( i^{th} \) coordinate unit vector. We now compute

\[
v(x) = -\frac{1}{h} D_{k}^{-h} \left( \zeta^{2} [u(x + he_{k}) - u(x)] \right)
= -\frac{1}{h^{2}} \left( \zeta(x + he_{k})^{2} - [\zeta(x)]^{2} \right) [u(x + he_{k}) - u(x)]
\]

for \( x \in \beta \). Since \( u \equiv 0 \) in the hyper-plane \( x_{n} = 0 \) and \( \zeta \) is zero near the curved boundary of \( \partial \beta \) we have that \( v \in W_{0}^{1,2}(\beta) \). We define quantities to simplify our comparison of the right and left hand sides of the equation

\[
\sum_{i,j=1}^{n} \int_{\beta} a_{ij}(x) u_{x_{i}} u_{x_{j}} \, dx = \int_{\beta} g v \, dx;
\]

to wit,

\[
A := \sum_{i,j=1}^{n} \int_{\beta} a_{ij}(x) u_{x_{i}} u_{x_{j}} \, dx
\]

and

\[
B := \int_{\beta} g v \, dx = \int_{\beta} - \left( \sum_{i=1}^{n} b_{i}(x) u_{x_{i}} + (c(x) + \lambda) u \right) v \, dx.
\]

We will proceed by filling in the missing details as presented by Evans in Section 6.3.2 for estimates of the quantities \( A \) and \( B \). For the most part these may be found in Section 6.3.1,
to which he refers in the text. Now

$$A = \sum_{i,j=1}^{n} \int_{\beta} a_{ij}(x) u_{x_i} u_{x_j} \, dx$$

$$= -\sum_{i,j=1}^{n} \int_{\beta} a_{ij}(x) u_{x_i} \left[ D_k^{-h} \left( \zeta^2 D_k^h u \right) \right]_{x_j} \, dx.$$ 

Thus

$$A = \sum_{i,j=1}^{n} \int_{\beta} D_k^{-h} (a_{ij}(x) u_{x_i}) \left( \zeta^2 D_k^h u \right)_{x_j} \, dx$$

$$= \sum_{i,j=1}^{n} \int_{\beta} \left[ a_{ij}(x + he_k) D_k^h u_{x_i} + u_{x_i} D_k^h (a_{ij}(x)) \right] \left( \zeta^2 D_k^h u \right)_{x_j} \, dx.$$ 

We have used the chain rule and product rule for difference quotients of suitably differentiable functions. Applying the product rule to the last term and distributing the multiplication over the addition gives

$$A = \sum_{i,j=1}^{n} \int_{\beta} a_{ij}(x + he_k) \zeta^2 D_k^h u_{x_i} D_k^h u_{x_j} \, dx$$

$$+ \sum_{i,j=1}^{n} \int_{\beta} 2a_{ij}(x + he_k) \zeta \zeta_{x_j} D_k^h u_{x_i} D_k^h u$$

$$+ u_{x_i} \zeta^2 D_k^h a_{ij} D_k^h u_{x_j} + 2\zeta \zeta_{x_j} u_{x_i} D_k^h a_{ij} D_k^h u \, dx.$$ 

For tractability we further define

$$A_1 := \sum_{i,j=1}^{n} \int_{\beta} a_{ij}(x + he_k) \zeta^2 D_k^h u_{x_i} D_k^h u_{x_j} \, dx$$

and

$$A_2 := \sum_{i,j=1}^{n} \int_{\beta} 2a_{ij}(x + he_k) \zeta \zeta_{x_j} D_k^h u_{x_i} D_k^h u$$

$$+ u_{x_i} \zeta^2 D_k^h a_{ij} D_k^h u_{x_j} + 2\zeta \zeta_{x_j} u_{x_i} D_k^h a_{ij} D_k^h u \, dx.$$
The uniform ellipticity of the pde implies that

\[ \theta \int_{\beta} \zeta^2 \left| D_k^h \nabla u \right|^2 \, dx \leq A_1 \]

and the \( C^1 \) differentiability and boundedness of the highest order coefficients gives that

\[ |A_2| \leq C_1 \int_{\beta} \Theta \zeta \zeta_{x_j} \left| D_k^h \nabla u \right| \left| D_k^h u \right| \, dx + C_2 \int_{\beta} \zeta^2 \left| D_k^h \nabla u \right| \left| \nabla u \right| \, dx + C_3 \int_{\beta} \zeta \zeta_{x_j} \left| D_k^h u \right| \left| \nabla u \right| \, dx. \]

We may simplify this expression and since \( \zeta \in C^\infty_0(\beta) \) both it, and its derivatives, are bounded, hence we can write, for some constant \( C \), that

\[ |A_2| \leq C \int_{\Omega} \zeta \left( \left| D_k^h \nabla u \right| \left( \left| D_k^h u \right| + \left| \nabla u \right| \right) + \left| D_k^h u \right| \left| \nabla u \right| \right) \, dx. \]

Now the integrand is of the form

\[ \zeta \left[ a(b + c) + bc \right]. \]

Applying Cauchy's inequality to \( a(b + c) + bc \) (where \( a, b \) and \( c \) are non-negative) yields

\[
\begin{align*}
    a(b + c) & \leq \frac{1}{2} \left[ \left( a^2 + (b + c)^2 \right) + b^2 + c^2 \right] \\
    & \leq \frac{1}{2} \left( a^2 + 2b^2 + 2bc + 2c^2 \right) \\
    & \leq \frac{1}{2} a^2 + b^2 + bc + c^2 \\
    & \leq \frac{1}{2} a^2 + (b + c)^2.
\end{align*}
\]

The last inequality holds since \( b \) and \( c \) are non-negative so we may write that

\[ |A_2| \leq \epsilon \int_{\beta} \zeta^2 \left| D_k^h \nabla u \right|^2 \, dx + \frac{C}{\epsilon} \int_{\beta} \zeta^2 (b + c)^2 \, dx \]

by Cauchy's inequality with any \( \epsilon > 0 \). Thus

\[ |A_2| \leq \epsilon \int_{\beta} \zeta^2 \left| D_k^h \nabla u \right|^2 \, dx + \frac{C}{\epsilon} \int_{\beta} \zeta^2 \left( \left| D_k^h u \right| + \left| \nabla u \right| \right)^2 \, dx. \]
By Minkowski's inequality applied to the last term we may write that

\[ |A_2| \leq \epsilon \int_{\beta} \zeta^2 |D_k^h \nabla u|^2 \, dx + \frac{C}{\epsilon} \int_{\beta} \zeta^2 \left( |D_k^h u|^2 + |\nabla u|^2 \right) \, dx \]

and note on any set \( \beta' \) such that \( \text{spt}(\zeta) \subset \subset \beta' \subset \subset \beta \) we still have

\[ |A_2| \leq \epsilon \int_{\beta} \zeta^2 |D_k^h \nabla u|^2 \, dx + \frac{C}{\epsilon} \int_{\beta'} |D_k^h u|^2 + |\nabla u|^2 \, dx \]

as \( \zeta \leq 1 \) on \( \beta \). We now choose \( \epsilon = \theta / 2 \) and note that since \( u \in W^{1,2}(\beta) \) the weak partial derivatives exist and the corresponding difference quotients that comprise the gradient satisfy

\[ \int_{\beta'} |D_k^h u|^2 \, dx \leq C \int_{\beta} |\nabla u|^2 \, dx. \]

Hence we obtain

\[ |A_2| \leq \frac{\theta}{2} \int_{\beta} \zeta^2 |D_k^h \nabla u|^2 \, dx + C \int_{\beta} |\nabla u|^2 \, dx. \]

From the definition of the quantities \( A_1 \) and \( A_2 \), above we find that

\[ \frac{\theta}{2} \int_{\beta} \zeta^2 |D_k^h \nabla u|^2 \, dx - C \int_{\beta} |\nabla u|^2 \, dx \leq A = A_1 + A_2 \]

since

\[ \theta \int_{\beta} \zeta^2 |D_k^h \nabla u|^2 \, dx \leq A_1 \]

and

\[ - \left( \frac{\theta}{2} \int_{\beta} \zeta^2 |D_k^h \nabla u|^2 \, dx + C \int_{\beta} |\nabla u|^2 \, dx \right) \leq A_2. \]

We now proceed to estimate the quantity

\[ B = \int_{\beta} gv \, dx = \int_{\beta} - \left( \sum_{i=1}^{n} b_i(x) u_{x_i} + (c(x) + \lambda) u \right) v \, dx \]
defined above. We have

\[ |B| \leq \int_\beta (\nu |\nabla u| + (\nu + \lambda) |u|) |v| \, dx \]
\[ \leq C \int_\beta (|\nabla u| + |u|) |v| \, dx. \]

By either Gilbarg and Trudinger's Lemma 7.23 (page 168) or Evans' Theorem 3(i), section 5.8.2 (page 277) we may write for \( \beta' \subset \subset \beta \)

\[ \int_\beta |v|^2 \, dx = \int_\beta \left| D_k^h (\zeta^2 D_k^h u) \right| \, dx \]
\[ \leq C \int_{\beta'} \left| D_k^h u \right|^2 + \zeta^2 \left| D_k^h \nabla u \right|^2 \, dx \]
\[ \leq C \int_\beta |\nabla u|^2 + \zeta^2 \left| D_k^h \nabla u \right|^2 \, dx. \]

Applying Cauchy's inequality with \( \epsilon \) once again gives us that

\[ |B| \leq C \int_\beta (|\nabla u| + |u|) |v| \, dx \]
\[ \leq \epsilon \int_\beta \zeta^2 \left| D_k^h \nabla u \right|^2 \, dx + \frac{C}{\epsilon} \int_\beta |\nabla u|^2 + u^2 \, dx. \]

Setting \( \epsilon = \theta/4 \) provides us with the inequality (recalling that \( A = B \))

\[ |A| = |B| \leq \frac{\theta}{4} \int_\beta \zeta^2 \left| D_k^h \nabla u \right|^2 \, dx + C \int_\beta |\nabla u|^2 + u^2 \, dx. \]

Combining our results then, shows that

\[ \int_{\beta'} \left| D_k^h \nabla u \right|^2 \, dx \leq \int_\beta \zeta^2 \left| D_k^h \nabla u \right|^2 \, dx \leq C \int_\beta |\nabla u|^2 + u^2 \, dx \]

for sufficiently small \( |h| \) and \( k = 1, 2, \ldots, n - 1 \). We have shown that

\[ (\nabla u)_{z_k} \in L^2 (\beta') \]
for \( k = 1, 2, \ldots, n - 1 \) so it remains to show that

\[
(\nabla u)_{x_n} \in L^2 (\beta')
\]

for us to be able to conclude that \( u \in W^{2,2} (\beta') \). In general, we have the estimate, by the arguments given above, for \( i, j \) not both equal to \( n \)

\[
\|u_{x_i x_j}\|_{L^2(\beta')} \leq C \|u\|_{L^2(\beta')}.
\]

Since \( u \in W^{2,2}_{\text{loc}} (\Omega) \) and solves the given partial differential equation a.e. in \( \Omega \), (see Evans' remarks following Theorem 1, in section 6.3.1), and because the differential operator \( L \) may be written in non-divergence form we can write

\[
a_{n,n}(x) u_{x_n x_n} = - \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} - \sum_{i=1}^{n} b_i(x) u_{x_i} - (c(x) + \lambda) u.
\]

Invoking the uniform ellipticity condition,

\[
\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 > 0
\]

for all \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \), we have that \( a_{n,n}(x) \geq \theta > 0 \) throughout \( \beta \). We now can conclude that

\[
|u_{x_n x_n}| \leq C \left( \sum_{i,j=1}^{n} |u_{x_i x_j}| + |\nabla u| + |u| \right)
\]

and that \( u \in W^{1,2}_0 (\beta') \cap W^{2,2} (\beta') \).

We continue with Evans' argument and relax the assumption that the point \( x_0 \in \partial \Omega \) is actually \( B(1,0) \cap \mathbb{R}_+^n \). We consider an arbitrary ball of radius \( r > 0 \) centered at \( x_0 \). As Evans does, we may assume that after a possible relabelling of the coordinate axes that

\[
\Omega \cap B(r, x_0) = \{ x \in B(r, x_0) | x_n > \gamma(x_1, x_2, \ldots, x_{n-1}) \}
\]
where the function $\gamma : \mathbb{R}^n \to \mathbb{R}$ is such that $\gamma \in C^2$. We utilize a diffeomorphism and make the variable change

$$y = \Phi(x)$$
$$x = \Psi(y).$$

We now choose $\eta$ so small that the ball $B(\eta, x_0)$ lies inside the ball $\Phi(B(r, x_0))$. We make the following definitions:

$$U = B(\eta, 0) \cap \{y_n > 0\}$$

and

$$V = B(\frac{\eta}{2}, 0) \cap \{y_n > 0\}.$$  

We also define the function

$$w(y) = w(\Psi(y))$$

for all $y \in U$. It is clear by the differentiability of the diffeomorphism that $w \in W^{1,2}(U)$ and that $w = 0$ on $\partial U \cap \{y_n = 0\}$ since the Jacobian of the transformation is non-zero. We now make the claim that $w$ solves the following partial differential equation

$$\begin{cases} 
Mw = 0 \text{ in } U \\
w = 0 \text{ on } \partial U 
\end{cases}$$

where

$$Mw = \sum_{i,j=1}^{n} (\alpha_{i,j}(y) \omega_{x_i} x_j) + \sum_{i=1}^{n} \beta_i(y) \omega_{x_i} + \sigma(y) w,$$

$$\alpha_k(y) = \sum_{i,j=1}^{n} a_{ij}(\Psi(y))(\Phi_k(\Psi(y)))_{x_i} (\Phi_i(\Psi(y)))_{x_j},$$

$$\beta_k(y) = \sum_{i=1}^{n} b_i(\Psi(y))(\Phi_k(\Psi(y)))_{x_i},$$

and

$$\sigma(y) = (c(\Psi(y)) + \lambda).$$
These equalities hold for index values (e.g., \(i, k\)) \(1, 2, \ldots, n\) and \(y \in U\). If \(v(y) \in W^{1,2}_0(U)\) and we define \(\omega(x) = v(\Phi(x))\) then, by the properties of the diffeomorphism, we can construct the (local) inverse transformation to that just given to conclude that \(w\) is a weak solution of

\[
\begin{cases}
    Mw = 0 \text{ in } U \\
    w = 0 \text{ on } \partial U
\end{cases}
\]

By easy calculations it can be verified that the both the differentiability of the coefficients and a (likely different) uniform ellipticity bound of the coefficient matrix is preserved by the diffeomorphism. We can apply the argument for the special case given earlier, to see that \(w \in W^{2,2}(U \cap B(\frac{1}{2},0))\), hence \(u \in W^{2,2}(\Omega \cap B(r/2,x_0))\).

We note that the arguments above clearly hold for the particular function \(g = 0\) and for future use that for any \(q > 0\), \(g \in L^q(\Omega)\).

Now, since the point \(x_0 \in \partial \Omega\) was arbitrary, it is clear that

\[u \in W^{1,2}_0(\Omega^1) \cap W^{2,2}(\Omega^1)\,.
\]

In fact for each \(i = 1, 2, \ldots\), we have that

\[u^i \in W^{1,2}_0(\Omega^1) \cap W^{2,2}(\Omega^1)\,.
\]

We now want to apply the Sobolev Embedding Theorem (Theorem 4.12, pages 85 and 86, of Adams and Fournier) to the elements of the sequence of functions \(u^i\). We state a somewhat abbreviated version of their theorem here. Noteworthy is the statement of Adams and Fournier on page 84, [AF], that domains which are of class \(C^2\) satisfy the strong local Lipshitz condition and, by implication, the interior cone and segment conditions.

**Theorem 70 (Sobolev Embedding for Strong Local Lipschitz Domains)** Let \(\Omega\) be a domain in \(\mathbb{R}^n\) satisfying the strong Lipschitz condition (and hence the interior cone condition) and let \(k \geq 1\) and \(1 \leq p < \infty\). We have three cases.
Case (i): If $kp > n$ then

$$W^{k,p} (\Omega) \subset C^{0,\alpha} (\Omega)$$

for $\alpha \leq k - \frac{n}{p}$.

Case (ii): If $kp = n$ then

$$W^{k,p} (\Omega) \subset L^q (\Omega)$$

for any $p \leq q < \infty$.

Case (iii): If $kp < n$ then

$$W^{k,p} (\Omega) \subset L^q (\Omega)$$

for any

$$p \leq q \leq p^* = \frac{np}{n-kp} < \infty.$$

Since $k = p = 2$, any of the three cases above, may hold. If the spacial dimension, $n$, of $\Omega$ satisfies $n < 4$ then we have, by the embedding theorem, that the sequence $\{u_i\}_{i=1}^\infty$ has a convergent subsequence in $C^{0,2-n/2} (\bar{\Omega}^1)$ hence

$$u \in C^{0,2-n/2} (\bar{\Omega}^1).$$

If $n = 4$ then Case(ii) holds or if $n > 4$ then Case(iii) holds. In either event we will need a theorem from Gilbarg and Trudinger to first conclude that

$$u \in W^{1,2}_0 (\Omega^1) \cap W^{2,2} (\Omega^1) \cap W^{2,q} (\Omega^1)$$

for some $q$ chosen and fixed such that $q > n/2$. We state the following theorem taken from Gilbarg and Trudinger’s ninth chapter on strong solutions of elliptic partial differential equations. Since, as noted in the introductory material, a domain that has a $C^2$ boundary is also of class $C^{1,1}$ the following theorem may be applied to the situation where $g = \phi = 0$.

**Theorem 71 (9.15, Gilbarg and Trudinger)** Let $\Omega$ be a $C^{1,1} (\Omega)$ domain in $\mathbb{R}^n$, and
let the operator $L$ be strictly elliptic in $\Omega$ with coefficients $a_{ij} \in C^0(\Omega)$, $b_i, c \in L^\infty(\Omega)$ with $i, j = 1, 2, \ldots, n$ and $c \geq 0$. Then if $g \in L^p(\Omega)$ and $u - \phi \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, the Dirichlet problem

\[
\begin{cases}
Lu = g \text{ in } \Omega \\
u = \phi \text{ on } \partial\Omega
\end{cases}
\]

has a unique solution $u \in W^{2,p}(\Omega)$.

Clearly, $u - \phi \in W_0^{1,2}(\Omega)$ and using the theorem we find that each function $u^i \in W^{2,q}(\Omega^1)$ where $q > n/2$ and we may apply the Sobolev embedding theorem to conclude that

\[u \in C^{0,2-n/q}(\Omega^1)\]

Since $u \equiv 0$ on $\Omega^1 \setminus \Omega^i$ for all $i$, we have by continuity that $u \equiv 0$ on $\partial\Omega_0$. This implies that $u \in W_0^{1,2}(\Omega_0)$ (see Kesavan, Theorem 2.2.6, page 61 for details).

A.2 Upper Continuity Proof for Domains with Continuous Boundaries That are a Union of a Finite Number of $C^2$ Pieces in $\mathbb{R}^n$

We will provide a regularity based upper continuity proof for more general domains than those that are of class $C^2$. To clarify the properties of the domains in a decreasing sequence to which the results will apply we make the following definition.

Definition 72 (Sufficiently Regular Decreasing Sequence of Domains) Let $\{\Omega^i\}_{i=1}^\infty$ be a decreasing sequence of bounded and connected domains in $\mathbb{R}^n$ where

\[
\lim_{i \to \infty} \Omega^i = \Omega^0 = \Omega
\]

and for each $i = 0, 1, 2, \ldots$, $\Omega^i$ is sufficiently smooth. That is

(i) $\partial\Omega^i$ is of class $C^{0,1}$ locally and
(ii) $\partial\Omega^i = \bigcup_{j=1}^{N_i} U^i_j$ where each boundary portion $U^i_j$ is of class $C^{2,0}$

where we assume that each $N_i$ is finite.
We want to make some observations before we embark on the task of proving any theorems. Some simple examples of domains that satisfy the conditions given above are rectangles and other regular polygons in \( \mathbb{R}^2 \). In \( \mathbb{R}^3 \), shoe box and cheese shaped domains are examples as are "football" shaped domains. Note that the condition (iii), above, implies that the segment condition holds for each \( \Omega^i \). Since the segment condition holds (as a consequence of (i), above – the strong local Lipschitz condition) this implies that the sets \( U_j^i \) are all of dimension \( n - 1 \). Because the domains satisfy the segment condition, they may have corners, but not cusps. In general, for each \( i = 0, 1, 2, ..., \) there may be a set of points \( X^i \) at which the boundary \( \partial \Omega^i \) is of class, \( C^{0,1} \) but not of class \( C^2 \). We also note that because each \( \Omega^i \) satisfies the segment property the set \( X^i \) must have a set of measure zero with respect to the spacial dimension \( n \) since it too, is at most \( n - 1 \) dimensional.

We want to prove a suitable generalization of the main theorem proved in the previous section. The proof will follow that given above in several respects. First we will show that "near the nice points", \( x \in \bar{\Omega}^i \), (i.e., \( x \notin X^i \)), the functions \( u^i \) are of class \( W^{2,2} \) for some appropriate value of \( q \). We will then show that in fact the functions \( u^i \) satisfy

\[
u^i \in C^{0,2-n/q} \left( \Omega^i \right)
\]

where \( \bar{\Omega}^i \subset \Omega^i \) and \( \mu \left( \Omega^i \setminus \bar{\Omega}^i \right) \) may be taken to be arbitrarily small. We will state the theorem that we want to prove.

**Theorem 73** Let \( \{ \Omega^i \}_{i=1}^{\infty} \) be a sufficiently regular decreasing sequence of domains in \( \mathbb{R}^n \) as defined above, decreasing to \( \Omega_0 = \Omega^0 \). Furthermore, let \( u^i \) be the first eigenfunction for the problem

\[
\begin{aligned}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega^i \\
\quad u &= 0 \quad \text{on} \quad \partial \Omega^i
\end{aligned}
\]

and \( u^i \to u \) as \( i \to \infty \) in \( W^{1,2}_0 (\Omega^1) \cap W^{2,2}_{\text{loc}} (\Omega^1) \) where

\[
Lu = \sum_{i,j=1}^{n} (a_{ij} (x) u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i (x) u_{x_i} + c(x) u,
\]
\(a_{ij}(x) \in C^1(\bar{\Omega}), b_i(x), c(x) \in L^\infty(\Omega)\) and \(a_{ij} = a_{ji}\) for all appropriate \(i, j = 1, 2, \ldots, n\). As usual, we assume that \(L\) admits a variational formulation. As we have done previously, assume that the operator is strictly elliptic and the coefficient functions are bounded, say

\[
\Theta \geq \sum_{i,j=1}^{n} |a_{ij}(x)|^2
\]

and

\[
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta |\xi|^2
\]

for all \(x \in \bar{\Omega}^1\) and \(\xi \in \mathbb{R}^n\) with

\[
\sum_{i=1}^{n} |b_i(x)| + |c(x)| \leq \nu < \infty.
\]

for all \(x \in \bar{\Omega}^1\). Additionally, assume that for all \(i > 0\), \(u^i \equiv 0\) on \(\Omega^i \setminus \Omega_0\). Then there exists \(\bar{u}\) such that

\[
\|\bar{u}\|_{W^{1,2}(\Omega)} = \|u\|_{W^{1,2}(\Omega)}
\]

and \(\bar{u} \in W^{1,2}_0(\Omega_0) \cap C^{0,\alpha}(\bar{\Omega}^1)\).

**Proof.** Let \(\{\Omega^i\}_{i=1}^{\infty}\) be a sufficiently regular decreasing sequence of domains in \(\mathbb{R}^n\) where

\[
\lim_{i \to \infty} \Omega^i = \Omega^0 = \Omega_0.
\]

We will need a lemma similar to that used in the proof given in the section above.

**Lemma 74** Let \(\Omega = \Omega^i\) for some fixed positive integer \(i\), let \(r > 0\) and \(x \in T \subset U_j^i\) for some fixed \(j\) where \(T\) is a boundary portion of \(\partial \Omega\) such that

\[
T = B(r, x) \cap \partial \Omega \neq \emptyset
\]

but

\(T\) is of class \(C^2\).
Then for any $0 < \rho < \frac{r}{2}$ and $u$ satisfying (for $L$ as above)

\[
\begin{aligned}
Lu + \lambda u &= 0 \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned}
\]

$u \in W_{0}^{1,2}(\Omega \cap B(\rho, x)) \cap W^{2,2}(\Omega \cap B(\rho, x))$.

**Proof.** Let the hypotheses for the lemma hold, the proof is identical to the proof of the lemma given above in the previous section. Since every boundary portion is of class $C^{2}$ the lemma will apply to every boundary portion $U_{j}$ of each domain $\Omega^{i}$. ■

We now claim that there exists, for each value of $i$, a subdomain $\tilde{\Omega}^{i} \subset \Omega^{i}$ with the following properties:

1. $\partial\Omega^{i} \cap \partial\tilde{\Omega}^{i} \neq \emptyset$;
2. $\tilde{\Omega}^{i} \in C^{2}$;
3. $\mu(\Omega \setminus \tilde{\Omega}) \leq \epsilon$.

for any $\epsilon > 0$. We will now state and prove a lemma that substantiates the claim.

**Lemma 75** Let $\Omega$ be an $n$ dimensional domain that is sufficiently smooth. That is

1. $\partial\Omega$ is of class $C^{0,1}$ locally and
2. $\partial\Omega = \bigcup_{j=1}^{N} U_{j}$ where each boundary portion $U_{j}$ is of class $C^{2,0}$

where we assume that $N$ is finite. Furthermore let

\[
X = \{ x \in \partial\Omega \mid x \text{ is not of class } C^{2}\}.
\]

Then for any $\epsilon > 0$ there exists a subdomain $\tilde{\Omega} \subset \Omega$ with the following properties:

1. $\partial\Omega \cap \partial\tilde{\Omega} \neq \emptyset$;
2. $\tilde{\Omega} \in C^{2}$;
3. $\mu(\Omega \setminus \tilde{\Omega}) \leq \epsilon$;
4. $X \cap \overline{\Omega} = \emptyset$. 

Proof. Let the hypotheses of the lemma hold. We assume that $\partial \Omega$ is not of class $C^2$ everywhere or there is nothing to prove. Now, since $\Omega$ has the segment property we know that $\Omega$ lies on one side of $\partial \Omega$. Let us make the following definitions:

$$\partial \Omega_{2\delta} = \{ x \in \bar{\Omega} | \text{dist}(x, \partial \Omega) < 2\delta \} ,$$

$$\Omega_{\delta} = \{ x \in \bar{\Omega} | \text{dist}(x, X) < \delta \}$$

and

$$X_{\delta} = \{ x \in \mathbb{R}^n | \text{dist}(x, X) < \delta \} .$$

We assume that $\delta$ is sufficiently small here. It is clear that $\partial \Omega_{2\delta} \supset \Omega_{\delta}$ and $\Omega_{\delta} \setminus X \cap \partial \Omega \neq \emptyset$, as well. Because $\partial \Omega$ is in fact of class $C^2$ in the set $X_{\delta} \setminus X_{\delta/2}$ where $X_{\delta/2}$ is defined in a fashion similar to $X_{\delta}$, it is possible to find a $C^2$ connection that "bypasses" the set $X$ but lies inside $X_{\delta} \setminus \tilde{X}_{\delta/2} \cap \bar{\Omega}$ and hence in $\Omega$. This is a consequence of the facts that the domain lies on one side of the boundary and that a $C^{0,1}$ connection (i.e., the set $X$) already exists. We are just finding a smoother connection inside $\Omega$ but nearby the set $X$. Also the complications are minimized because there are only a finite number of "joints" between the finite number of $C^2$ boundary portions. There may be a finite number of such connections ($X$ is not necessarily a connected set); we denote these $K_1, K_2, \ldots K_m$. We assume that the sets $K_j$ extend all the way to the boundary of $X_{\delta}$; clearly for all $j = 1..m$, $K_j \subset \bar{\Omega}$. We now consider the set of points

$$\partial \bar{\Omega} = \{ x \in \partial \Omega \setminus \tilde{X}_{\delta} \cup_{j=1}^m K_j \} .$$

The set $\partial \bar{\Omega}$ is an $n - 1$ dimensional surface in $\mathbb{R}^n$ that is of class $C^2$; it may have many components because $\Omega$ is connected, but not necessarily simply connected. Furthermore, $\partial \bar{\Omega}$ is clearly contained in $\bar{\Omega}$. The intersection $\partial X_{\delta} \cap \Omega$ is not empty; let us define

$$S = \partial X_{\delta} \cap \Omega .$$

Since $\Omega$ is connected and $\delta$ is sufficiently small, the set of points in $\Omega$ bounded by $\partial \bar{\Omega}$ and
connected to $S$ is also connected and an open set. Hence we define this to be the set $\tilde{\Omega}$. Clearly we can find $\delta$ so small that $\mu(X_\delta) < \varepsilon$. However, since $\mu(X_\delta) < \varepsilon$ and $\Omega \setminus \tilde{\Omega} \subset X_\delta$ we have that $\mu(\Omega \setminus \tilde{\Omega}) < \varepsilon$. 

It might prove helpful to the reader to sketch out the various sets defined in the proof above when, say, the domain is a rectangle in $\mathbb{R}^2$ or a parallelepiped in $\mathbb{R}^3$.

Now, for each $i$, we may find, using the lemma above, a subdomain $\tilde{\Omega}^i$ with $\delta_i$ so small that

$$\mu \left( \Omega_i \setminus \tilde{\Omega}^i \right) < \frac{1}{2^i} \varepsilon$$

for some fixed $\varepsilon$ small. We assume without loss of generality that

$$\delta_{i+1} < \frac{1}{2} \delta_i$$

for all $i \geq 1$. It is clear that as $i \to \infty$, $\tilde{\Omega}^i \to \Omega_0$. We now want to consider the partial differential equation for $L$ as defined above:

$$\begin{cases} Lw + \lambda_i^i w = 0 & \text{in } \tilde{\Omega}^i \\ w = u^i & \text{on } \partial\tilde{\Omega}^i \end{cases}$$

where $u^i$ is the first eigenfunction and $\lambda_i^i$ is the first eigenvalue for

$$\begin{cases} Lu + \lambda u = 0 & \text{in } \Omega^i \\ u = 0 & \text{on } \partial\Omega^i \end{cases}$$

Clearly, the function $u^i$ solves the first pde above. While we can only say that the solutions $u^i \in W^{2,2}_{\text{loc}}(\Omega^i)$, we can say that $u^i \in W^{2,2}(\tilde{\Omega}^i)$ because of the lemma above and the method by which $\tilde{\Omega}^i$ was constructed. There is a small issue concerning the extension of $u^i$ to all of $\Omega$. Since $u^i$ is not identically zero on $\partial\tilde{\Omega}^i$, we cannot extend it by zero beyond $\partial\tilde{\Omega}^i$ however, we can extend it with a suitably smooth cutoff function that is zero outside of $\partial\tilde{\Omega}^i+1$. We can do this because the sets $X_{\delta+1}$ and $X_\delta$ as defined above satisfy $X_{\delta+1} \subset X_\delta$ and of course because $\partial\tilde{\Omega}^i, \partial\tilde{\Omega}^i+1 \in C^2$. Carefully note that we are not asserting that the
extended function $u^i$ satisfies the pde

$$\begin{cases}
Lw + \lambda^i w = 0 \quad \text{in } \tilde{\Omega}^i \\
w = u^i \quad \text{on } \partial \tilde{\Omega}^i
\end{cases}$$

outside of $\tilde{\Omega}^i$. We are now in a position to apply the results of Gilbarg and Trudinger (Theorem 9.15) and the Sobolev embedding theorem as we did in the section above. First we choose and fix $q > n/2$. Recall that

$$g = 0 \in L^q \left( \tilde{\Omega}^i \right)$$

and

$$w - u^i \equiv 0 \in W^{1,q} \left( \tilde{\Omega}^i \right)$$

where $w - u^i$ takes the place of $u - \phi$ in the theorem as stated. The conclusion of the theorem of Gilbarg and Trudinger gives us that $u^i \in W^{2,q} \left( \tilde{\Omega}^i \right)$ is the unique weak solution of

$$\begin{cases}
Lw + \lambda^i w = 0 \quad \text{in } \tilde{\Omega}^i \\
w = u^i \quad \text{on } \partial \tilde{\Omega}^i
\end{cases}$$

and the Sobolev Embedding Theorem allows us to conclude that, for each $i = 1, 2, ...$,

$$u^i \in C^{0,2-n/q} \left( \Omega^i \right).$$

We observe now that where $\partial \Omega^i$ and $\partial \tilde{\Omega}^i$ coincide, $u^i \equiv 0$. We can extend $u^i$ as we did above to all of $\Omega^1$, this time continuously, outside of $\Omega^i$ by zero. Thus we claim that for each value of $i$ we have

$$u^i \in W_0^{1,2} \left( \Omega^1 \right) \cap W^{2,q} \left( \Omega^1 \right) \cap C^{0,2-n/q} \left( \tilde{\Omega}^1 \right).$$

Since the space

$$C^{0,2-n/q} \left( \tilde{\Omega}^1 \right)$$
is a Banach space, the Cauchy sequence \( \{u^i\}_{i=1}^{\infty} \) must have a convergent subsequence and we have that

\[
u^i \to \bar{u} \in C^{0,2-n/q}(\bar{\Omega}^1).
\]

Clearly once again, since \( u \equiv 0 \) on \( \Omega^1 \setminus \bar{\Omega}^i \) for all \( i = 1, 2, \ldots \), we have that \( u \in W^{1,2}_0(\Omega_0) \). □

A simple corollary gives us the result for two dimensional rectangles in \( \mathbb{R}^2 \).

**A.2.1 Upper Continuity Proof for Rectangles in \( \mathbb{R}^2 \)**

The proof of this special case is simply a consequence of the previous theorem.
In order to obtain earlier results, we proved the monotonicity of eigenvalues lemma. Briefly, if imprecisely, it showed that if \( \Omega_i \subset \Omega_j \) then \( \lambda_j \leq \lambda_i \) where \( \lambda_i \) and \( \lambda_j \) are first eigenvalues for some appropriate fixed operator \( L \) for the problems

\[
\begin{align*}
&\begin{cases}
Lu + \lambda u = 0 &\text{in } \Omega_i \\
u = 0 &\text{on } \partial \Omega_i
\end{cases} \\
&\begin{cases}
Lu + \lambda u = 0 &\text{in } \Omega_j \\
u = 0 &\text{on } \partial \Omega_j
\end{cases}
\end{align*}
\]

It may be proved that the inequality \( \lambda_j \leq \lambda_i \) is actually strict. Two versions of the proof are given here.

**B.1 Harnack Inequality for Weak Solutions Version (Gilbarg and Trudinger)**

We state the lemma we wish to prove. We recall the properties of the operator \( L \). We assume that

\[
Lu = \sum_{i,j} D_i (a_{ij}(x) D_j u) + \sum_i (D_i (b_i(x) u) + c_i(x) D_i u) + d(x) u
\]

and that the operator \( L \) satisfies both uniform bound and ellipticity conditions. That is, there exist positive constants, \( \Theta \) and \( \theta \), so that

\[
\Theta \geq \sum_{i,j=1}^n |a_{ij}(x)|^2
\]

and

\[
\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2
\]
for all \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \). Also we let it be understood that

\[
\frac{1}{\theta^2} \sum_{i=1}^{n} |b_i(x)|^2 + |c_i(x)|^2 + \frac{1}{\theta} |d(x)|^2 \leq \nu^2 < \infty.
\]

As elsewhere we will assume that the coefficient functions satisfy the following properties: the functions \( a_{ij}(x) \) are \( C^0(\Omega) \) for \( i, j = 1, 2, \ldots, n \) and \( b_i(x) \) and \( c(x) \) for \( i = 1, 2, \ldots, n \) are \( L^\infty(\Omega) \). Recall also that the corresponding bilinear form \( B \) (on \( \Omega \)) is written:

\[
B_\Omega[u, \phi] = \int_\Omega \sum_{j,k} a_{j,k} D_k u D_j \phi + \sum_j b_j u D_j \phi - \left( \sum_j c_j D_j u + d u \right) \phi \, dx.
\]

**Lemma 76 (Strict Monotonicity of First Eigenvalue)** For the partial differential equations with the operator \( L \), as above and fixed admits a variational formulation and has respective minimizing solutions \( u_i \) and \( u_j \) for the problems

\[
\begin{cases}
Lu + \lambda_i u &= 0 \text{ in } \Omega_i \\
u &= 0 \text{ on } \partial \Omega_i
\end{cases}
\]

and

\[
\begin{cases}
Lu + \lambda_j u &= 0 \text{ in } \Omega_j \\
u &= 0 \text{ on } \partial \Omega_j
\end{cases}
\]

where \( \Omega_i \) and \( \Omega_j \) satisfy \( \Omega_i \subset \Omega_j \), \( \Omega_i \neq \Omega_j \). Then the first eigenvalues \( \lambda_i \) and \( \lambda_j \) satisfy, \( \lambda_i > \lambda_j \).

**Proof.** Let the hypotheses of the lemma hold and \( L \) have the requisite properties. The proof will be a consequence of the Harnack inequality, which in turn is a consequence of the strong maximum principle. We state the Corollary presented in Gilbarg and Trudinger, page 199, \([GT]\), to their version of the Harnack inequality.

**Theorem 77 (Gilbarg and Trudinger Corollary, 8.21)** "Let \( L \) be as above and let
\( u \in W^{1,2}(\Omega) \) satisfy \( u \geq 0 \) in \( \Omega \). Then for any \( \bar{\Omega} \subset \subset \Omega \), we have

\[
\sup_{\bar{\Omega}} u \leq C \inf_{\bar{\Omega}} u
\]

where \( C = C(n, \Theta, \nu, \bar{\Omega}, \Omega) \).

Now suppose that the first eigenfunction \( u_i \geq 0 \) for

\[
\begin{cases}
Lu + \lambda_i u = 0 & \text{in } \Omega_i \\
u = 0 & \text{on } \partial \Omega_i
\end{cases}
\]

extended to all of \( \Omega_j \) satisfies

\[
\begin{cases}
Lu + \lambda_j u = 0 & \text{in } \Omega_j \\
u = 0 & \text{on } \partial \Omega_j
\end{cases}
\]

with \( \lambda_i = \lambda_j \). Take \( \bar{\Omega} \) so that \( \bar{\Omega} \cap \Omega_j \setminus \bar{\Omega}_i \) is of positive measure. Then

\[
\inf_{\bar{\Omega}} u_i = 0
\]

which implies that

\[
\sup_{\bar{\Omega}} u_i = 0,
\]

a contradiction. \( \blacksquare \)

B.2 Regularity Version (Heywood Noussair, and Swanson)

A proof of the strict monotonicity of the first eigenvalues with respect to the domain can be easily had using the theorems of Heywood, Noussair, and Swanson.

**Lemma 78** Let the operator \( L \), and the domains \( \Omega_i \), and \( \Omega_j \) with \( \Omega_i \subset \Omega_j \), \( \Omega_i \neq \Omega_j \) be such that the first eigenfunctions exist and the theorems of Heywood, Noussair, and Swanson
apply to the following:

\[
\begin{cases}
Lu + \lambda_i u = 0 & \text{in } \Omega_i \\
u = 0 & \text{on } \partial \Omega_i
\end{cases}
\]

and

\[
\begin{cases}
Lu + \lambda_j u = 0 & \text{in } \Omega_j \\
u = 0 & \text{on } \partial \Omega_j
\end{cases}
\]

Then the first eigenvalue of the first equation, \(\lambda_i\), and \(\lambda_j\), the first eigenvalue for the second equation, satisfy \(\lambda_i > \lambda_j\).

**Proof.** Let the hypotheses hold. Then have that \(u_i \in W^{1,2}_0(\Omega_i)\), and \(u_j \in W^{1,2}_0(\Omega_j)\); because \(\Omega_i \subset \Omega_j\) we know that \(u_i \in W^{1,2}_0(\Omega_j)\) (after suitable extension by zero) as well. As before, we also have by regularity arguments and Theorem 6.13 of Gilbarg and Trudinger, that

\[
u_i, u_j \in W^{2,2}_{\text{loc}}(\Omega_j) \cap C^0(\Omega_j) \cap C^{2,\alpha}(\Omega_j)
\]

as well. Therefore assume, for the purposes of obtaining a contradiction, that \(\lambda_j \leq \lambda_i\) with respective corresponding positive solutions \(u_i\) and \(u_j\) but \(\Omega_i \subset \Omega_j\). Now for,

\[
L_i(u) = Lu + \lambda_i u \quad \text{and}
\]

\[
L_j(u) = Lu + \lambda_j u, \quad \text{we have}
\]

\[
g[u] = \int_{\Omega_j} L_i(u) - L_j(u) dx
\]

\[
= \int_{\Omega_j} (\lambda_i - \lambda_j) u dx \leq 0 \quad \text{if and only if}
\]

\[
\lambda_j \geq \lambda_i.
\]

Applying the theorem of Heywood, Noussair, and Swanson, (GHNS), we have that either \(u_j\) is a multiple of \(u_i\) on \(\Omega_j\) or \(u_j\) has a zero in \(\Omega_j\) – both of which are false. ■
APPENDIX C
SOME OBSERVATIONS CONCERNING ESTIMATED EIGENVALUES
FOR PARTIAL DIFFERENTIAL EQUATIONS IN
CERTAIN DOMAINS IN $\mathbb{R}^2$

While working on the results of the first section for this dissertation, we found it expedient to try to find graphical (numerical) approximations to solutions of elliptic partial differential equations of the form

$$\begin{align*}
 Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
 u &= \phi \quad \text{on} \quad \partial \Omega
\end{align*}$$

with

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x)u$$

and where the domain $\Omega$ in $\mathbb{R}^2$ was of the form

$$\Omega = \{(x, y)| x \in (a, b), \ f(x) < y < g(x)\}.$$ 

We made some interesting observations during this exploratory phase of the research, and in this section we will present, rather heuristically, some of these observations. Although our approximation scheme is applicable to domains with inhomogeneous boundary conditions (as well as higher order pdes) we will restrict our interest here to the case where $u = 0$ on $\partial \Omega$.

There exist standard approximation algorithms for partial differential equations that are easily implemented on rectangular domains in $\mathbb{R}^2$. Such a scheme is documented in "Theory and Problems of Partial Differential Equations," by Paul DuChateau and David W. Zachmann, [DZSOL], pages 167-168. As is typical in these algorithms, one chooses nodes inside the rectangle and approximates the values of the derivative terms of the unknown solution to the pde by difference quotients. The known (coefficient and boundary term) functions are evaluated at the nodes and the result is a banded linear system of equations where each
variable corresponds to an approximation to the solution at an individual node. In order to apply the method to our desired class of domains, the use of coordinate transformations for both the various coefficient functions, derivative terms and boundary terms was required. It is easy to see that for $C^2$ or better functions, $f(x)$ and $g(x)$, as well as many less smooth functions, the domain may easily be mapped to the square $(0,1) \times (0,1)$, and that an inverse transformation will exist. The Vessiot, [Vess], package for the well known Maple, [Maple], computer algebra system, written partially by the author, was used to construct the coordinate transformations as needed and a standard approximation algorithm was implemented by using Matlab software, [Matlab]. The numerical results were then transformed back to the original domain and presented graphically using the software.

There are certain issues that arise when using a linear scheme to solve a homogeneous partial differential equation with homogeneous boundary conditions. Typically, the scheme will find the ever present zero solution which, to say the least, is not helpful. To circumvent this problem, since it is well known that the eigenvalue problems have a positive solutions and that any constant multiple of the solution is also a solution, we assigned a non zero value to a particular node and replaced the corresponding variable throughout the system with its assigned value. We give a simple example for illustration.

Consider the system

\[
\begin{bmatrix}
-4 & 1 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 0 \\
0 & 1 & -4 & 1 & 0 \\
0 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 1 & -4 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

It is obvious that the only solution is the trivial solution. If we assign the value $u_3 = 1$ and substitute it into the above equation, we obtain an equivalent reduced order system:
This system typically has a non-zero solution. We of course include \( u_3 = 1 \) as part of the solution vector. Now in constructing numerical approximations to eigenvalue problems we cannot leave the eigenvalue free, but must furnish a guess for the value. We will denote a guess for an eigenvalue \( \lambda \), by \( \lambda^* \).

We will present the outline of an algorithm so that the interested reader might duplicate our results. Suppose we wish to find a linear approximation to the problem

\[
\begin{align*}
Lu + \lambda u &= 0 \quad \text{in} \quad \Omega \\
\quad u &= \phi \quad \text{on} \quad \partial \Omega
\end{align*}
\]

where

\[
\Omega = \{(x, y) | x \in (a, b), \ f(x) < y < g(x)\}
\]

for some suitable \( f \) and \( g \). First, we construct the transformation

\[
\begin{align*}
X(x, y) &= x \\
Y(x, y) &= \frac{y - f(x)}{g(x) - f(x)}
\end{align*}
\]

and its inverse

\[
\begin{align*}
x(X, Y) &= X \\
y(X, Y) &= (g(X) - f(X)) Y + f(X)
\end{align*}
\]

where we set

\[
v(X, Y) = u(x(X, Y), y(X, Y)).
\]

It is clear that the new domain \( \tilde{\Omega} = (a, b) \times (0, 1) \), a rectangle. Second, we prolong the
inverse of the transformation and we attempt to solve

\[
\begin{align*}
\tilde{L}v + \lambda v &= 0 \quad \text{in} \quad \tilde{\Omega} \\
v &= \phi \quad \text{on} \quad \partial\tilde{\Omega}
\end{align*}
\]

where the operator \( \tilde{L} \) is the transformed operator. Third, we choose a (guessed) value \( \lambda^* \) for the eigenvalue, \( \lambda \), and substitute it into the equation. Fourth, we choose a grid of nodes, construct central difference quotients for the boundary value problem in the ordinary way and construct the corresponding linear system of equations. We could attempt to solve this \( n^{th} \) order linear system at this point, but for the case \( \phi = 0 \) we would obtain the trivial solution to the system. To avoid this, our fifth step is to choose a node, say \( (X, Y) = (A, B) \), and set \( v(A, B) = K \). Sixth, we substitute \( K \) into the system of linear equations to obtain a linear system of order \( n - 1 \). Since this system is no longer homogeneous, we may obtain a solution that is non-trivial. The seventh step is to solve the reduced order linear system of equations. Eighth, for each node, \( (X_i, Y_i), i = 1, 2, ..., n \) (including \( (X, Y) = (A, B) \)) we determine

\[ u_i(x, y) = v(x(X_i, Y_i), y(X_i, Y_i)) \]

where the value of \( v \) at \( (X_i, Y_i) \) is given by the solution of the reduced order linear system. Last we plot the ordered triplets \( (x_i, y_i, u_i) \) where

\[ x_i = X_i \quad \text{and} \]
\[ y_i = (g(X_i) - f(X_i)) Y_i + f(X_i). \]

We discovered that the graphical representation of the solution typically provides insight into whether or not the guess, \( \lambda^* \), is too high, too low or pretty close. Subject somewhat to the fineness of the grid mesh, we made the following observations.

(i) The number of the sign changes in the graph indicates the order of the next highest eigenvalue.

(ii) If the guess, \( \lambda^* \) is too small, the graph at the assigned node will exhibit a sharp point.
(iii) If the guess, $\lambda^*$ is too large, the graph at the assigned node will exhibit a sharp point and an "extra" sign change.

(iv) If the guess, $\lambda^*$ is close, the graph will appear smooth throughout and at the assigned node.

The behavior is best captured by some simple examples. We will consider the following partial differential equation

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega = (0, \pi) \times (0, \pi) \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

A list of first and second eigenfunctions and their corresponding eigenvalues is provided below:

(i) $u_1 = \sin x \sin y$, $\lambda_1 = 2$

(ii) $u_{2x} = \sin 2x \sin y$, $\lambda_{2x} = 5$

(iii) $u_{2y} = \sin x \sin 2y$, $\lambda_{2y} = 5$

Below we present a graphical approximation to the solution of

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega = (0, \pi) \times (0, \pi) \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

where we used the value $\lambda^* = 0$ as a guess for $\lambda$; we used a $23 \times 23$ (internal) grid and assigned a value of $u(11\pi/24, 11\pi/24) = 1$.

Figure 9. Low First Eigenvalue Estimate
Noteworthy is the single signed graph and the sharp spike at the assigned value indicating that $\lambda^* < \lambda_1$. Below, in Figure 10, we have modified our guess, solving the given system with $\lambda^* = 3$. The assigned value parameters are as above. We now see a two signed solution indicating that $\lambda^* > \lambda_1$.

![Figure 10. High First Eigenvalue Estimate](image1)

Improving on our most recent guesses we choose $\lambda^* = 1$, and obtain a graph that we would expect. This is shown below in Figure 11.

![Figure 11. Correct First Eigenvalue Estimate](image2)

Since we know that the second eigenvalue for our pde is 5, but that it is non-simple, we obtain a solution that is smooth looking, but is somewhat unexpected. It turns out that the solution that has been found is actually a linear combination of the two second
eigenfunctions. Specifically, it is the solution,

\[ u = u_{2x} + u_{2y} \]
\[ = \sin 2x \sin y + \sin x \sin 2y \]
\[ = 2 \cos x \sin x \sin y + 2 \sin x \cos y \sin y. \]

This may be verified by noting that the solution is zero on the line \( y = \pi - x \). This solution was found because the assigned node lies on the line \( y = x \). A graph is shown in Figure 12, below.

![Figure 12. Linear Combination of Second Eigenfunctions](image)

When the assigned node is changed to \((\pi/2, \pi/4)\) we obtain a graph that corresponds to the eigenfunction \( u_{2y} = \sin x \sin 2y \). Again \( \lambda^* = \lambda_2 = 5 \) and graph is displayed in Figure 13:

![Figure 13. \( u_{2y} = \sin x \sin 2y \)](image)

We now make a guess at the third eigenvalue, \( \lambda_3 = 8 \), by choosing \( \lambda^* = 7 \). We see now
that there are three distinct humps or sign changes indicating that $\lambda_2 < \lambda^* < \lambda_3$. We have returned to the original node assignment for this last graph, Figure 14.

![Figure 14. Eigenvalue Estimate $\lambda^*$ where $\lambda_2 < \lambda^* < \lambda_3$](image)

We will now speculate somewhat on what we have shown above. We suspect that instead of solving the given partial differential equation (in the first three cases) on the given domain we are actually approximating, by a standard linear technique, the solution to

$$
\begin{align*}
\Delta u + \lambda^2 u &= 0 \quad \text{in} \quad \tilde{\Omega} = (0, \pi) \times (0, \pi) \setminus \left(\frac{11\pi}{24}, \frac{11\pi}{24}\right) \\
\quad x &\in (0, \pi), \quad y = 0 \\
\quad x &\in (0, \pi), \quad y = \pi \\
\quad x &= 0, \quad y \in (0, \pi) \\
\quad x &= \pi, \quad y \in (0, \pi) \\
\quad u &= 1 \quad \text{at} \quad (x, y) = (\frac{11\pi}{24}, \frac{11\pi}{24})
\end{align*}
$$

That is, we are finding an approximation to a solution of the original equation on a punctured domain. Interestingly under finer and finer (and yet, fairly coarse, in reality) meshes, the solution appears to be smooth on the domain, although not on its closure. The boundary properties of the domain are rather abysmal – the boundary is not connected and satisfies neither a cone condition nor the segment property! Yet it seems that the solution is smooth! Of course, when the guessed eigenvalue is an actual eigenvalue of the original problem, the eigenfunction is a smooth solution to the problem with the punctured domain.

Regarding the sign change properties that were observed, it seems reasonable that the
guessed value for the eigenvalue has an effect on the eigenvalue for the linear system. Since the display of the graph is a record of the value of the components of the system, all positive values probably correspond to a linear system with eigenvalues of one sign. When the graph exhibited displays a sign change, it indicates that there are two signs among the eigenvalues. This can easily be seen to be the case in orthogonal system. It is also clear that there cannot be more sign changes than nodes in the graph - a factor for determining approximations to higher eigenvalues. Perhaps details on this are known and can be found in the literature.

As a last observation, we note that approximations were found on domains of the more general form mentioned above and which do not satisfy the segment condition. It is this observation that prompted some of the comments presented in the epilogue to the Heywood, Noussair, and Swanson section.
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References: Available upon request.