Geometric Aspects of Second-Order Scalar Hyperbolic Partial Differential Equations in the Plane

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GEOMETRIC ASPECTS OF SECOND-ORDER SCALAR HYPRBOLIC PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE

by

Martin Juráš

A dissertation submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY in

Mathematical Sciences

UTAH STATE UNIVERSITY
Logan, Utah
1997
ABSTRACT

Geometric Aspects of Second-Order Scalar Hyperbolic
Partial Differential Equations in the Plane

by

Martin Juráš, Doctor of Philosophy
Utah State University, 1997

The purpose of this dissertation is to address various geometric aspects of second-order scalar hyperbolic partial differential equations in two independent variables and one dependent variable

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \]  

(1)

We find a characterization of hyperbolic Darboux integrable equations at level \( k \) (1) in terms of the vanishing of the generalized Laplace invariants and provide an invariant characterization of various cases in the Goursat general classification of hyperbolic Darboux integrable equations (1). In particular we give a contact invariant characterization of equations integrable by the methods of general and intermediate integrals. New relative invariants that control the existence of the first integrals of the characteristic Pfaffian systems are found and used to obtain an invariant characterization for the class of \( j \)-Gordon equations. A notion of a hyperbolic Darboux system is introduced and we show by examples that the classical Laplace transformation is just a special case of a diffeomorphism of hyperbolic Darboux systems. We also construct new examples of homomorphisms between certain hyperbolic systems. We characterize Monge-Ampère equations and explicitly exhibit two invariants whose vanishing is a necessary and sufficient condition for the equation to be of the Monge-Ampère type. The solution to the inverse problem of the calculus of variations for hyperbolic equations (1) in terms of the generalized Laplace invariants is presented. We also obtain some partial results on symplectic conservation laws. We characterize, up to contact equivalence, some classical equations using the generalized Laplace invariants. These results contain characterizations of the wave, Liouville, Klein-Gordon, and certain types of Euler-Poisson equations.
ACKNOWLEDGMENTS

Let me express my gratitude to my thesis advisor, Ian Anderson, who expertly guided my first steps into the geometry of partial differential equations. I thank him for his direction, encouragement, careful proofreading of this research, and suggesting many of the problems that were solved in this dissertation.

Thanks to Mark Fels, who carefully read this research and made many valuable suggestions.

I would like to thank my first mentor, Ivan Studnička, for many enlightening discussions about mathematics.

I wish to thank the Department of Mathematics and Statistics at Utah State University for awarding me the summer research grant in 1995.

I wish to thank the Centre de Recherches Mathématiques for its exceptional hospitality and support during the 1994–95 academic year.

On a personal level, my thanks goes to my friends Renate Schaaf, Bill Bynum, Clarence Bynum, Jordan Cahn, Betsy Davis, Florin Catrina, and Charlie Miller. Without them, my life in Logan would be much more difficult.

I thank my mother, Jarmile Jurášové, and my father, Vladimíru Jurášovi, for their financial support throughout my studies.

Special thanks to my wife, Ireně, for her unfailing support, patience, and wonderful cooking.

Martin Juráš
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CHAPTER 1
INTRODUCTION

The subject matter in this dissertation falls under the general category of the geometric theory of differential equations, the foundations of which were laid by Lie and Darboux in the second half of the last century. The theory was extensively developed by Goursat, Janet, Riquier, Vessiot, Cartan, and many others during the years 1890 - 1940. The goal of this theory is to understand the properties of differential equations through the study of their invariants under the suitable groups of transformations, such as point or contact transformations, differential substitutions, and Bäcklund transformations. In the past ten years there has been a renewed interest in the use of geometric methods to study the properties of differential equations. The topics of central interest today include the systematic computation of the conservation laws and (generalized) symmetries, the inverse problem of the calculus of variations, the problem of Darboux integrability, Hamiltonian and bi-Hamiltonian structures, the Painlevé property, Bäcklund transformations, and equivalence problems. The purpose of this dissertation is to address some of these problems for scalar hyperbolic second-order partial differential equations in the plane

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \]  

\( (1.1) \)

The main results of this dissertation can be summarized as follows.

We find the characterization of hyperbolic Darboux integrable equations of the general form (1.1) at level \( k \) in terms of the vanishing of the generalized Laplace invariants. We invariantly characterize various cases in the Goursat general classification of hyperbolic Darboux integrable equations (1.1). In particular we give a contact invariant characterization of equations integrable by the methods of general and intermediate integrals. New relative invariants that control the existence of the first integrals of the characteristic Pfaffian systems are found. A notion of a hyperbolic Darboux system is introduced. Using this definition, we show by examples that the classical Laplace transformation is just a special case of a diffeomorphism between hyperbolic Darboux systems. We characterize Monge-Ampère equations and explicitly exhibit two invariants whose vanishing is a necessary and sufficient condition for the equation to be of the Monge-Ampère type. The solution to the inverse problem of the calculus of variations for hyperbolic equations of type (1.1) in terms of the generalized Laplace invariants is presented. Some partial results on symplectic conservation laws are also obtained. An invariant characterization of the class of \( f \)-Gordon equations is also given. Lastly, we characterize, up to contact equivalence, some classical equations using the generalized Laplace invariants. These
results contain characterizations of the wave, Liouville, Klein-Gordon, and certain types of Euler-Poisson equations.

1.1. Literature Review

Ever since the discovery of calculus, the basic question of how to solve a given differential equation or a system of differential equations has been of interest. Already Newton realized the importance of understanding solutions of differential equations: "Data aequatione quocunque fluentes quanticae involvente fluxiones invente et vice versa," which may be translated as: "It is useful to solve differential equations." Efforts to answer this fundamental problem gave rise to the theory of differential equations along with many other areas of modern mathematics. The majority of the classical equations studied in mathematics have their origin in the study of important physical phenomena and differential geometry. Therefore, the knowledge of explicit solutions is of great interest. Exact solutions may be used for testing numerical methods, their properties may reflect asymptotic behavior of more general types of solutions, or they can be used as models for physical experiments.

In the second half of the 19th century, Sophus Lie developed his symmetry theory of differential equations. Subsequently Drach, Picard and Vessiot succeeded in developing the Galois theory of differential equations. Lie's method is, even today, the most powerful method for obtaining particular solutions for systems of nonlinear differential equations. In 1919, Emily Noether formulated her famous theorem establishing a correspondence between the symmetries of a Lagrangian and conservation laws of the system of Euler-Lagrange equations. Conservation laws play an important role in integrating differential equations. In the case of ODE's, the classical conservation laws become the first integrals.

Questions concerning integrability of differential equations are questions about in which sense can a given system of differential equations be integrated and what are the methods needed to obtain solutions. For example, first-order linear scalar partial differential equations are integrable by the methods of ODE's. More generally we may ask when can we obtain a general solution to an equation or a system of equations using ODE techniques only? This problem was studied by a number of mathematicians in the 19th century and various methods of integration based on these ideas were developed. Among these were methods of Ampère, Boole, Charpit, Darboux, Lagrange, Laplace, and Monge. The most powerful of these is the method of Darboux. The essence of the method of Darboux for the case of scalar second-order hyperbolic equations in the plane is to construct two new equations involving two arbitrary functions of one variable which are in involution with the original equation, that is, the two new equations together with the original one give rise to an integrable
Pfaffian system. Integrating this system yields the general solution to the original equation. We say that equation (1.1) is \textit{Darboux integrable at level} \( k \) if the two new equations in involution exist and are of order \( \leq k \). We say that the equation (1.1) is \textit{semi-Darboux integrable at level} \( k \) if there is one equation of order \( \leq k \) involving one arbitrary function of one variable that is, in involution with the original equation. The modern definition of Darboux integrability for the second-order scalar hyperbolic equations in the plane will be presented in chapter 2 and is due to Bryant, Griffiths, and Hsu [13].

A considerable portion of Goursat’s analysis of the method of Darboux [26] (pp. 133-171) is devoted to a complete classification of the characteristic systems of Darboux integrable equations. A natural problem then is to classify Darboux integrable equations. The first results on this account are due to Lie [30], who proved that up to a contact transformation the wave and Liouville equations

\[ u_{xy} = e^u \quad \text{and} \quad u_{xy} = 0 \]

are the only Darboux integrable equations of type

\[ u_{xy} = f(u). \]

In [27] Goursat classified all \( f \)-Gordon equations, that is, equations of the type

\[ u_{xy} = f(x, y, u, u_x, u_y), \tag{1.2} \]

integrable by the method of Darboux at the level 2. Goursat listed eleven classes of equations Darboux integrable at the level \( \leq 2 \) but he was unable to give explicit solutions to some of the equations. Vessiot in [43] applied his theory of differential systems to improve Goursat’s classification. He also obtained a closed form solution for every equation on Goursat’s list. The methods of Vessiot are more systematic than those of Goursat and generalize readily to other classes of equations. In [39] and [40], Vassiliou successfully applied Vessiot’s ideas to the systems of coupled \( f \)-Gordon equations.

The current activity in this classical subject is motivated, in part, by a general resurgence of interest in geometric methods for the study of differential equations and also, in part, by a desire to classify various types of completely integrable partial differential equations [31]. Bryant, Griffiths, and Hsu in [13] and [14] introduced the notion of a hyperbolic system of class \( s \), which is a generalization of a hyperbolic second-order differential equation in the plane. For example, a second-order scalar hyperbolic Monge-Ampère equation in the plane, that is, an equation of type

\[ E(u_{xx}u_{yy} - u_{xy}^2) + Au_{xx} + 2Bu_{xy} + Cu_{yy} + D = 0 \tag{1.3} \]
where $A, B, C, D, E$ are functions of variables $x, y, u, u_x, u_y$ only, defines a hyperbolic system of class $s = 1$. A partial differential equation (1.1) which is not Monge-Ampère (1.1) is of class $s = 3$.

Systems of first order partial differential equations in two independent and two dependent variables of type

$$u_x = f(x, y, u, v) \quad \text{and} \quad v_y = g(x, y, u, v)$$

form hyperbolic systems of class $s = 0$. Bryant et al. obtain classification of class $s = 0$ types of hyperbolic systems according to the number of classical conservation laws they possess. They also prove an existence theorem for the initial value problem for Darboux integrable systems. The method used throughout their paper is the Cartan's method of equivalence. Gardner and Kamran [24] have also studied the theory of Bäcklund and Laplace transformations, Riemann invariants, and equivalence problem from the viewpoint of Cartan's ideas on characteristics for second-order hyperbolic equations in the plane. They also provide a new proof for the Lie's characterization of the wave equation and establish a smooth existence theorem for the initial value problem. The work of Anderson and Kamran [8], which was the starting point for this dissertation, focuses on the higher degree conservation laws for scalar hyperbolic second-order equations in the plane. The authors introduced a nonlinear analogue of the classical Laplace invariants and proved that the vanishing of these invariants (at a certain order) is a necessary condition for the equation to be Darboux integrable. They were unable to prove the sufficiency of these conditions. Recently, Sokolov and Zhiber [35] proved that the vanishing of the generalized Laplace invariants at a certain order is sufficient for the Darboux integrability of the $f$-Gordon equations (1.2). Their methods, however, do not readily generalize to more general types of scalar second-order equations in the plane; furthermore, the level of Darboux integrability is not precisely stated.

1.2. Dissertation Structure

The main result of this dissertation is obtained in chapter 6.

We obtain a simple characterization of Goursat's classification of the characteristic systems for hyperbolic Darboux integrable equations (1.1) ([26](pp. 133-171)) in terms of the generalized Laplace invariants and two other invariants (whose vanishing is a necessary and sufficient condition for the equation to be of Monge-Ampère type).

We will discuss the significance of this result through some of its immediate consequences.
Theorem 1.1. A scalar second-order hyperbolic differential equation in the plane (1.1) is Darboux integrable at level $k$ if and only if both generalized Laplace invariants at level $k$ vanish.

We remark that even though the above theorem is an easy consequence of our main result, because of its importance, we prove it in chapter 5, immediately after sufficient technical machinery for the proof is developed. After that we continue with the more detailed analysis of the structure equations of the Laplace-adapted coframe and at the end of chapter 6 we prove our main result.

Theorem 1.1 is a full generalization of the classical theorem, which states that a linear equation is integrable by the method of Darboux if and only if both sequences of the classical Laplace invariants terminate [26]. Not only does it generalize the recent result of Sokolov and Zhiber [35], but it is also an improvement over their result even for the $f$-Gordon equations since our result furnishes the invariants of the lowest possible order.

Theorem 1.1 and the results of Anderson and Kamran [B] (namely, the Theorem on normal forms of form-valued conservation laws) combine to establish the following result.

Theorem 1.2. A second-order hyperbolic equation in the plane (1.1) is Darboux semi-integrable if and only if there are nontrivial $(1, s)$ conservation laws for some $s \geq 3$.

This last statement satisfactorily resolves the problem posed by Tsujishita [36], namely, to give a criterion for Darboux integrability of scalar second-order differential equations in the plane in terms of the cohomology of the associated variational bicomplex.

As we have mentioned earlier, the proof of our main result is based on the Laplace-adapted coframe introduced in [8]. We first derive structure equations for this coframe and with these in hand we carry out the derived flag computations for the characteristic Pfaffian systems for equation (1.1). The computations involved are explicit. A number of examples in chapter 6 are given that illustrate how our derived flag computations can be used to find the characteristic invariants from which the equations in involution are constructed. We also classify all Darboux integrable equations at level 2 of the form $u_{xz} = f(u_{yy})$ and obtain a conclusion that differs from that obtained by Forsyth [23](§267). We then apply our results to the nonlinear wave equation

$$u_{xy} + uu_{xx} + f(u_x) = 0$$

(1.4)

to substantially clarify Calogero's analysis of this equation [15].

In chapter 7, we analyze the characteristic systems of scalar hyperbolic equations in the plane with nonvanishing generalized Laplace invariants. This leads to the discovery of new relative invariants...
which we shall explicitly exhibit. We use these invariants for example to prove characterization results about $f$-Gordon equations in chapter 13.

In chapter 8 we define a hyperbolic Darboux system of class $s$ \cite{13} to be a hyperbolic system of class $s$ satisfying the additional conditions that the Pfaffian systems $\{\omega^1, \omega^2\}$ and $\{\omega^3, \omega^4\}$ are integrable. We show, by examples, that all classical or well-known hyperbolic Darboux integrable scalar partial differential equations in the plane known to the author can be represented as hyperbolic Darboux systems. We show by examples that some classical transformations, for instance the Laplace transform, are just special cases of homomorphisms of hyperbolic Darboux systems. We prove that the equation manifold of a hyperbolic Darboux integrable equation becomes, after a certain number of prolongations, a hyperbolic Darboux system. There are hyperbolic Darboux integrable equations that are not classically contact equivalent, but their corresponding hyperbolic Darboux systems are intrinsically equivalent. For example, the Liouville equation $u_{xy} = e^u$ and the equation $u_{xy} = uu_x$ are equivalent as hyperbolic Darboux systems.

The main goal of chapter 9 is to derive formulas for the Monge-Ampère invariants and partial formulas for the first two generalized Laplace invariants. We also derive recursion formulas for the generalized Laplace invariants. These formulas are a generalization of the recursion formulas for the classical Laplace invariants \cite{23} §198.

In chapter 10, we characterize Monge-Ampère equations. First we prove a theorem on normal forms for analytic hyperbolic systems of class $s = 1$. Our proof follows the ideas of Bryant and Griffiths \cite{12} who proved a similar theorem on normal forms for analytic parabolic systems.

**Theorem 1.3.** Every analytic hyperbolic Monge-Ampère system, that is a hyperbolic system of class $1$, is locally equivalent to a Monge-Ampère system generated by a quasi-linear equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + d = 0 \quad (1.5)$$

where $a, b, c, d$ are functions of $x, y, u, u_x, u_y$ only.

The theorem on normal forms for analytic Monge-Ampère equations follows immediately. Every analytic hyperbolic Monge-Ampère equation is contact equivalent to some quasi-linear equation of type (1.5). According to Bryant and Griffiths \cite{12}, this theorem was already known to Lie and is true even without the analyticity assumption.

In the derived flag computations for the characteristic Pfaffian systems for equation (1.1) we explicitly exhibit two new invariants whose vanishing is necessary and sufficient condition for equation
(1.1) to be Monge-Ampère. We also give a new characterization of Monge-Ampère equations in terms of the generalized Laplace invariants.

**Theorem 1.4.** A second-order scalar hyperbolic partial differential equation in the plane (1.1) is Monge-Ampère if and only if the first two generalized Laplace invariants are of order \( \leq 2 \).

Chapter 11 is devoted to the characterization of second-order scalar hyperbolic equations in the plane that admit a general or complete intermediate integral. Classically, a *first-order general intermediate integral* of a second-order equation (1.1) is a first-order equation

\[ J(x, y, u, u_x, u_y) = \varphi(I(x, y, u, u_x, u_y)) \]

involving an arbitrary function \( \varphi \), such that every solution of this equation is a solution of equation (1.1). Similarly, a *first-order complete intermediate integral* of a second-order equation (1.1) is a first-order equation

\[ V(x, y, u, u_x, u_y, a, b) = 0 \]

involving two arbitrary constants \( a \) and \( b \), such that every solution of this equation is a solution of equation (1.1). From our main result and the results of chapter 9, the following characterization is derived.

**Theorem 1.5.** A second-order hyperbolic equation in the plane (1.1) admits a complete intermediate integral if and only if one of the first-level generalized Laplace invariants vanishes. The equation admits a general intermediate integral if and only if, in addition, it is of Monge-Ampère type.

As our next application of the Laplace-adapted coframe, we solve the inverse problem of the calculus of variations for second-order scalar hyperbolic equations in the plane. For quasi-linear equations the problem was solved by Anderson and Duchamp [3]. We use, in an essential way, the structure equations for the Laplace-adapted coframe to easily derive a characterization of equations arising from variational principle in terms of the generalized Laplace invariants. Recall that an equation is called *multiplier variational* if there exists a Lagrangian whose Euler-Lagrange equation determines the equation manifold. The main result of chapter 12 can be stated as follows.

**Theorem 1.6.** A second-order hyperbolic scalar equation in the plane (1.1) is variational if and only if the two first-level generalized Laplace invariants are equal.

We also give an easy proof that for an equation arising from a variational principle there is a one-to-one correspondence between the generalized symmetries of the equation and the \( H^{1,1} \) cohomology
classes of the associated variational bicomplex. Necessary conditions for the existence of certain symplectic conservation laws are found.

In chapter 13 we invariantly characterize $f$-Gordon equations, certain subclasses of $f$-Gordon equations and some classical equations. More than 100 years ago Lie succeeded in characterizing the wave equation $u_{xy} = 0$. He proved that a second-order hyperbolic scalar equation in the plane is contact equivalent to the wave equation if and only if it admits two independent first-order general intermediate integrals. Due to our characterization of intermediate integrals, it is now easy to obtain a contact invariant characterization of the wave equation in terms of the generalized Laplace invariants, namely an equation is contact equivalent to the wave equation if and only if the first two generalized Laplace invariants vanish. As a by-product of our general classification theorems, we obtain a new proof of the Lie's characterization of the wave equation. The results of chapter 13 also include the characterization of the wave equation, Klein-Gordon equation $u_{xy} = u$ and the Liouville equation $u_{xy} = e^u$ up to contact equivalence. For example, we prove the following.

**Theorem 1.7.** A second-order scalar hyperbolic Monge-Ampère equation in the plane is contact equivalent to either the Liouville equation or to the linear equation $s = 2u/(x+y)^2$ if and only if the two first-level generalized Laplace invariants are equal and nonzero and both second-level generalized Laplace invariants vanish.

To distinguish between the Liouville and the linear equation $s = 2u/(x+y)^2$ is a question that cannot be answered in terms of the generalized Laplace invariants and we use one of the relative invariants which arises in the study of $H^{1,2}$ for this purpose. As yet another application of our theory we, prove that the equation (1.4) studied by Calogero [15] is contact equivalent to a linear equation

$$ u_{xy} + a(x,y) u_x + b(x,y) u_y + c(x,y) u = 0, $$

that is integrable by the method of Laplace (see [23]).
CHAPTER 2
PRELIMINARIES

We need to briefly introduce some basic notions and results from the theory of exterior differential systems, jet bundles, contact transformations, and variational bicomplexes.

2.1. Exterior Differential Systems

Let $M$ be a manifold. By $\Omega^k(M)$ we shall denote the ring of all $k$-forms, in particular $\Omega^0(M)$ denotes the ring of all functions on $M$. Let $\Omega^*(M) = \sum_{k=0}^{\infty} \Omega^k(M)$ be the ring of all differential forms on $M$. A set of one forms that at each point $x \in M$ constitute a basis for the cotangent space is called a coframing of $M$.

Definition 2.1. Let $M$ be a manifold and let $\mathcal{I} \subseteq \Omega^*(M)$ be an ideal. If $d\omega \in \mathcal{I}$ for every $\omega \in \mathcal{I}$, that is, if $\mathcal{I}$ is closed under the exterior differential, then we call $\mathcal{I}$ a differential ideal. Let $\mathcal{G} \subseteq \mathcal{I}$ be a set. We say that a differential ideal $\mathcal{I}$ is algebraically generated by the set $\mathcal{G}$ provided

$$\mathcal{I} = \mathcal{G} \wedge \Omega^*(M).$$

We say that the differential ideal $\mathcal{I}$ is generated by the set $\mathcal{G}$ if $\mathcal{I}$ is algebraically generated by the set

$$\mathcal{G} \cup \{ d\omega; \omega \in \mathcal{G} \}.$$

Definition 2.2. An exterior differential system is a pair $(M, \mathcal{I})$, where $M$ is a manifold and $\mathcal{I} \subseteq \Omega^*(M)$ is a differential ideal. An integral manifold of an exterior differential system $(M, \mathcal{I})$ is a pair $(N, \phi)$, where $N$ is a manifold and $\phi$ is a map $\phi : N \to M$ such that $\phi^*(\mathcal{I}) = \{ 0 \}$.

Definition 2.3. Let $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$ be two exterior differential systems. A $C^\infty$ map $\Phi : M_1 \to M_2$ such that $\Phi^*(\mathcal{I}_2) \subseteq \mathcal{I}_1$, is called a homomorphism of exterior differential systems. Two exterior differential systems $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$ are called equivalent if there exists a diffeomorphism $\Phi : M_1 \to M_2$, which is also a homomorphism.

Definition 2.4. An exterior differential system $(M, \mathcal{I})$, where $\mathcal{I}$ is generated by a set of 1-forms, is called a Pfaffian system. The maximal number of linearly independent 1-forms in the Pfaffian system $(M, \mathcal{I})$ at a point $x \in M$ is called the rank of the Pfaffian system $(M, \mathcal{I})$ at the point $x$. If a
Pfaffian system \((M,\mathcal{I})\) has the same rank \(k\) at every point \(x \in M\), then \((M,\mathcal{I})\) is said to be of rank \(k\). Let \(\mathcal{I}^1\) be the set of 1-forms in \(\mathcal{I}\) The Pfaffian system \((M,\mathcal{I})\) is called integrable if

\[
d\omega \equiv 0 \mod \mathcal{I}^1 \quad \text{for all} \quad \omega \in \mathcal{I}^1. \tag{2.1}
\]

The condition (2.1) is called the Frobenius condition.

Note that the Frobenius condition guarantees that if a differential ideal \(\mathcal{I}\) is generated by the set \(\mathcal{I}^1\), then it is also algebraically generated by \(\mathcal{I}^1\).

**Theorem 2.5. (Frobenius).** Let \((M,\mathcal{I})\) be an integrable Pfaffian system of rank \(k\). Let the dimension of \(M\) be \(n\). Then for every point \(x \in M\) there is a neighborhood \(U\) and a coordinate system \(y_1, \ldots, y_n\) on \(U\) such that \(\mathcal{I}/U\) is generated by \(dy_1, \ldots, y_k\).

### 2.2. Infinite Jet Bundles

The theory of finite jet bundles was developed to give a geometric description of the spaces of partial derivatives of maps between two differentiable manifolds.

To begin, we note a function \(f(x^1, \ldots, x^n)\) with \(n\) independent variables has

\[
\alpha_k^n = \binom{n+k-1}{k}
\]

different \(k\)th-order partial derivatives

\[
\frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}.
\]

indexed over all unordered (symmetric) multi-indices \((i_1, \ldots, i_k), 1 \leq i_k \leq n\). Therefore, the number of partial derivatives of the function \(f(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}\) up to order \(k\) (including the 0th order one) is

\[
\alpha_k^n = \binom{n+k-1}{k} = \sum_{i=0}^{k} \binom{n+i-1}{i}.
\]

Now consider the map \(f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m\). It is obvious that the number of partial derivatives of the map \(f\) up to order \(k\) is

\[
\alpha_k^{n,m} = m \binom{n+k}{k}.
\]
Let $M$ be an $n$ dimensional connected base manifold of a fiber bundle $\pi : E \rightarrow M$ with the fiber dimension $m$. Let

$$\pi^k_M : J^k(E) \rightarrow M$$

be the bundle of $k$-jets of local sections of $E$. We call $J^k(E)$ the $kth$-order jet bundle of $E$. For $x \in M$ the fiber $(\pi^k_M)^{-1}(x)$ in $J^k(E)$ consists of equivalence classes, denoted by $j^k(s)(x)$, of local sections $s$ on $E$. If $U_1$ and $U_2$ are two open neighborhoods of $x$ and if $s_1 : U_1 \rightarrow E$ and $s_2 : U_2 \rightarrow E$ are local sections, then $s_1$ and $s_2$ are in the same equivalence class if their partial derivatives to order $k$ agree at $x$. The dimension of $J^k(E)$ is $n + \alpha^n_{k,m}$. There are natural projections

$$\pi^k_s : J^s(E) \rightarrow J^k(E), \quad \text{for } k \leq s$$

that satisfy

$$\pi^k_k = \text{id}_{J^k(E)} \quad \text{and} \quad \pi^k_s = \pi^k_k \circ \pi_s^t \quad \text{for } k \leq s \leq t. \quad (2.2)$$

By definition a differential form $\omega$ on $J^k(E)$ is called a contact form if for every local section $s$ of $E$

$$(j^k(s))^*(\omega) = 0.$$ 

The set of all contact forms on $J^k(E)$ forms a differential ideal called the contact ideal on $J^k(E)$ and is denoted by $C(J^k(E))$. Thus $(J^k(E), C(J^k(E)))$, is an exterior differential system. Alternatively we may define $kth$-order jet bundle to be special type of an exterior differential system.

Locally a $kth$-order jet space $J^k(E)$ is an exterior differential system $(\mathbb{R}^{n+\alpha^n_{k,m}}, C_k)$, where $\mathbb{R}^{n+\alpha^n_{k,m}}$ has coordinates

$$(x^1, u^\alpha, u^{\alpha}_{i_1}, u^{\alpha}_{i_1i_2}, \ldots, u^{\alpha}_{i_1\ldots i_k})$$

indexed over all ordered multi-indices $(i_1, \ldots, i_k)$, where $1 \leq i_1 \leq \cdots \leq i_k \leq n$ and $1 \leq \alpha \leq m$. It is convenient to adopt the following convention. We will identify $u^{\alpha}_{i_1\ldots i_k}$, where $i_1, \ldots, i_k$ is an unordered sequence, with $u_{i_1\ldots i_k}$, where $i_{j_1}, \ldots, i_{j_k}$ is obtained by permuting $i_1, \ldots, i_k$ so that $i_{j_1} \leq \cdots \leq i_{j_k}$. The contact ideal $C_k$ on $J^k(E)$ is generated by contact 1-forms

$$\theta^\alpha = du^\alpha - u^{\alpha}_{i_1} dx_i, \quad \theta_{i_1}^\alpha = du^\alpha_{i_1} - u^{\alpha}_{i_1i_2} dx_i, \quad \theta_{i_1i_2}^\alpha = du^\alpha_{i_1i_2} - u^{\alpha}_{i_1i_2i_3} dx_i, \quad \ldots$$

$$\theta_{i_1\ldots i_k}^\alpha = du^\alpha_{i_1\ldots i_k} - u^{\alpha}_{i_1\ldots i_{k-1}i_{k+1}} dx_i, \quad \theta_{i_1\ldots i_{k-1}}^\alpha = du^\alpha_{i_1\ldots i_{k-1}} - u^{\alpha}_{i_1\ldots i_{k-2}i_{k}} dx_i,$$

where $1 \leq i_1 \leq \cdots \leq i_l \leq n$ and $1 \leq l \leq k$. 
When working on the finite jet bundles, one frequently runs into a difficulty whenever taking total derivatives. This difficulty is overcome by constructing an infinite jet bundle. The equations (2.2) allow us to construct the inverse limit of the sequence of jet bundles \( \{ J^k(E); k = 0, 1, 2, \ldots \} \).

**Definition 2.6.** The inverse limit \( J^\infty(E) \) of a sequence of jet bundles \( \{ J^k(E); k = 0, 1, 2, \ldots \} \) together with the projections

\[
\pi_M^\infty: J^\infty(E) \to M, \quad \text{and} \quad \pi_k^\infty: J^\infty(E) \to J^k(E), \quad \text{for} \quad k = 0, 1, 2, \ldots
\]

is called the *infinite jet space over E*.

Note that \( J^\infty(E) \) has a naturally defined topological structure

\[
\tau = \{ (\pi_k^\infty)^*(O); O \text{ is an open set in } J^k(E), k = 0, 1, \ldots \}.
\]

**Differential forms** on \( J^\infty(E) \) are defined to be the pullbacks by \( \pi_k^\infty \) of differential forms on \( J^k(E) \).

The set of all \( p \)-forms on \( J^\infty(E) \) is denoted by \( \Omega^p(J^\infty(E)) \) and we set

\[
\Omega^*(J^\infty(E)) = \bigcup_{p=0}^{\infty} \Omega^p(J^\infty(E))
\]

to be the set of all differential forms on \( J^\infty(E) \). In particular this implies that a function (a 0-form) \( f \) on \( J^\infty(E) \) is a map that factors through some \( J^k(E) \), that is, there exists a function \( f_0 \) on \( J^k(E) \) such that \( f = f_0 \circ \pi_k^\infty \). The exterior differential on \( J^\infty(E) \) is defined to be the unique operator \( d \) for which, for every \( k = 0, 1, 2, \ldots \), the following diagram commutes.

\[
\begin{array}{ccc}
\Omega^*(J^\infty(E)) & \xrightarrow{d} & \Omega^*(J^\infty(E)) \\
(\pi_k^\infty)^* & \uparrow & (\pi_k^\infty)^* \\
\Omega^*(J^k(E)) & \xrightarrow{d} & \Omega^*(J^k(E))
\end{array}
\]

The set of all *contact forms* on \( J^\infty(E) \) is a differential ideal \( \mathcal{C}(J^\infty(E)) \) called the *contact ideal* on \( J^\infty(E) \) and is obtained as a union of pullbacks by \( \pi_k^\infty \) of the contact ideals on \( J^k(E) \). A *vector field* \( X \) on \( J^\infty(E) \) is defined to be a derivation on the ring of smooth functions on \( J^\infty(E) \). A vector field on \( J^\infty(E) \) is called a *total vector field* if

\[
X \cdot \omega = 0
\]
for every contact 1-form $\omega$. Total vector fields on $J^\infty(E)$ span an $n$-dimensional involutive distribution. A vector field $X$ on $J^\infty(E)$ is $\pi^\infty_M$ vertical if

$$(\pi^\infty_M)_*(X) = 0.$$ 

Locally $J^\infty(E)$ has coordinates

$$(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1i_2}, u^\alpha_{i_1i_2i_3}, \ldots)$$

and

$$(dx^i, \theta^\alpha_i, \theta^\alpha_{i_1}, \theta^\alpha_{i_1i_2}, \theta^\alpha_{i_1i_2i_3}, \ldots)$$

is a coframing of $J^\infty(E)$. The contact ideal $\mathcal{C}(J^\infty(E))$ on $J^\infty(E)$ is generated by

$$\theta^\alpha, \theta^\alpha_i, \theta^\alpha_{i_1}, \theta^\alpha_{i_1i_2}, \theta^\alpha_{i_1i_2i_3}, \ldots.$$ 

In coordinates a vector field $X$ is given by

$$X = A^i \frac{\partial}{\partial x^i} + \sum_{k=0}^{\alpha} B^\alpha_{i_1\ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1\ldots i_k}},$$

where $A^i$ and $B^\alpha_{i_1\ldots i_k}$ are functions on $J^\infty(E)$. The basis for the total vector fields on $J^\infty(E)$ is, in local coordinates, given by

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_{i_1} \frac{\partial}{\partial u^\alpha} + u^\alpha_{i_1i_2} \frac{\partial}{\partial u^\alpha_{i_1}} + \sum_{1 \leq i_1 \leq i_2 \leq n} u^\alpha_{i_1i_2} \frac{\partial}{\partial u^\alpha_{i_1i_2}} + \ldots, \quad \text{for } k = 0, 1, 2, \ldots.$$ 

In local coordinates a vector field $X$ is $\pi^\infty_M$ vertical if

$$X = \sum_{k=0}^{\alpha} B^\alpha_{i_1\ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1\ldots i_k}},$$

where $A^i$ and $B^\alpha_{i_1\ldots i_k}$ are functions on $J^\infty(E)$. Let us end this section by noting that for vector fields on $J^\infty(E)$ we do not have the notion of the flow. But, using Cartan’s formula we can (at least formally) define the Lie derivative of a vector field $X$

$$\mathcal{L}_X \omega = X \omega - d\omega + d(X \omega),$$
where $\omega$ is a differential form on $J^\infty(E)$.

2.3. The Free Variational Bicomplex

To introduce the notion of the variational bicomplex we need to bi-grade the differential forms on $J^\infty(E)$ in the following way.

**Definition 2.7.** A $p$-form $\omega$ on $J^\infty(E)$ is of type $(r, s)$, where $p = r + s$, if at each point $\sigma \in J^\infty(E)$, holds

$$\omega(X_1, \ldots, X_p) = 0$$

provided that either

(i) more than $s$ of the vectors $X_1, \ldots, X_p$ are $\pi_M^\infty$ vertical, or

(ii) more than $r$ of the vectors $X_1, \ldots, X_p$ are total vector fields.

A type $(1, 0)$ form is called a horizontal form.

In the local coordinates, a type $(r, s)$ form is a sum of terms of the form

$$a \, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge \theta^\alpha_{i_1 \ldots i_p} \wedge \cdots \wedge \theta^\alpha_{j_1 \ldots j_p},$$

where $a$ is a function on $J^\infty(E)$. The space of type $(r, s)$ forms on $J^\infty(E)$ is denoted by $\Omega^{r,s}(J^\infty(E))$ and we have the direct sum decomposition

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)). \quad (2.3)$$

We would like to emphasize that this decomposition is not possible on any finite dimensional jet bundle. The decomposition (2.3) is one of the main reasons we prefer to work on an infinite jet bundle $J^\infty(E)$. Since the exterior differential of a contact 1-form $\theta^\alpha_{i_1 \ldots i_k}$ is

$$d\theta^\alpha_{i_1 \ldots i_k} = dx^i \wedge \theta^\alpha_{i_1 \ldots i_k i},$$

then we have

$$d : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r-1,s}(J^\infty(E)) \oplus \Omega^{r,s+1}(J^\infty(E)).$$

Let

$$\pi^{r,s} : \Omega^p(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$$
be the projection map. Define the horizontal and vertical differentials $d_H$ and $d_V$ by

$$d_H = \pi^{r+1,s} \circ d \quad \text{and} \quad d_V = \pi^{r,s+1} \circ d.$$  

Then the exterior differential can be decomposed into a horizontal and vertical part

$$d = d_H + d_V.$$  

From $d^2 = 0$ it immediately follows that

$$d_H^2 = 0, \quad d_H d_V = -d_V d_H, \quad d_V^2 = 0.$$  

It easy to check that both $d_H$ and $d_V$ are anti-derivations. Recall that an operator $\delta$ is an anti-derivation if

$$\delta(\omega^1 \wedge \omega^2) = \delta(\omega^1) \wedge \omega^2 + (-1)^k \omega^1 \wedge \delta(\omega^2), \quad \text{where } \omega^1 \text{ is a } k\text{-form.}$$  

In particular the horizontal and vertical differentials of a $k$th-order function $f(x^i, u^\alpha, u_{i_1}^\alpha, \ldots, u_{i_k}^\alpha)$ are

$$d_H f = (D_i f) dx^i \quad \text{and} \quad d_V f = \sum_{i_1 \leq \ldots \leq i_s \leq n} \frac{\partial f}{\partial u^\alpha_{i_1 \ldots i_s}} \theta^\alpha_{i_1 \ldots i_s},$$

and we also have the following structure equations

$$d_H (dx^i) = 0 \quad \text{and} \quad d_H \theta^\alpha_{i_1 \ldots i_k} = dx^i \wedge \theta^\alpha_{i_1 \ldots i_k}, \quad (2.4)$$

$$d_V (dx^i) = 0 \quad \text{and} \quad d_V \theta^\alpha_{i_1 \ldots i_k} = 0 \quad (2.5)$$

**Definition 2.8.** The *free variational bicomplex* for the infinite jet bundle $J^\infty(E)$ is a double complex $\Omega^*,*(J^\infty(E), d_H, d_V)$ of differential forms on the infinite jet bundle $J^\infty(E)$. 

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega^{0,3} & \longrightarrow & \ldots & \longrightarrow & \Omega^{n,3} \\
\quad & \overset{d_V}{\longrightarrow} & \Omega^{0,2} & \overset{d_H}{\longrightarrow} & \Omega^{1,2} & \overset{d_V}{\longrightarrow} & \ldots & \overset{d_H}{\longrightarrow} & \Omega^{n-1,2} & \overset{d_V}{\longrightarrow} & \Omega^{n,2} \\
\quad & \overset{d_V}{\longrightarrow} & \Omega^{0,1} & \overset{d_H}{\longrightarrow} & \Omega^{1,1} & \overset{d_V}{\longrightarrow} & \ldots & \overset{d_H}{\longrightarrow} & \Omega^{n-1,1} & \overset{d_V}{\longrightarrow} & \Omega^{n,1} \\
\quad & \overset{d_V}{\longrightarrow} & \Omega^{0,0} & \overset{d_H}{\longrightarrow} & \Omega^{1,0} & \overset{d_V}{\longrightarrow} & \ldots & \overset{d_H}{\longrightarrow} & \Omega^{n-1,0} & \overset{d_V}{\longrightarrow} & \Omega^{n,0} \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^{0,0} & \longrightarrow & \ldots & \longrightarrow & \Omega^{n,0} \\
\end{array}
\]
The rows (except for in the last column) and the columns of the free variational bicomplex are locally exact [1].

**Definition 2.9.** Let \( X \) be vector field on \( J^\infty(E) \) and \( \omega \in \Omega^{r,s}(J^\infty(E)) \). We define the **projected Lie derivative** by

\[
X(\omega) = \pi^{r,s}(\mathcal{L}_X \omega).
\]

In particular we have

\[
D_i \theta^\alpha_{i_1 \ldots i_k} = \theta^\alpha_{i_1 \ldots i_k i}.
\]

### 2.4. Contact Transformations

Let \( \pi : E \to M \) and \( \bar{\pi} : \bar{E} \to \bar{M} \) be two fiber bundles. A smooth map \( \Phi : J^s(E) \to J^k(\bar{E}) \) is called a **contact transformation** if it preserves the contact ideal, that is,

\[
\Phi^*(C(J^s(\bar{E}))) \subseteq C(J^k(\bar{E}))
\]

In other words, \( \Phi \) is a contact transformation if it is a homomorphism of differential systems \((J^s(E), C(J^s(E)))\) and \((J^k(\bar{E}), C(J^k(\bar{E}))\). We say a contact transformation is a **point transformation** if it covers a smooth map \( \Phi_0 : J^0(E) \to J^0(\bar{E}) \), that is, the following diagram commutes.

\[
\begin{array}{ccc}
J^k(E) & \xrightarrow{\Phi} & J^k(\bar{E}) \\
\downarrow \pi^k & & \downarrow \bar{\pi}^k \\
J^0(E) & \xrightarrow{\Phi_0} & J^0(\bar{E})
\end{array}
\]

The following theorem is due to Bäcklund [10].

**Theorem 2.10.** Let \( \pi : E \to M \) and \( \bar{\pi} : \bar{E} \to \bar{M} \) be two trivial fiber bundles with the same dimension of the base manifold and the same fiber dimension. If the fiber dimension > 1, then every contact transformation is a point transformation. If the fiber dimension is 1, then every contact transformation covers a smooth map \( \Phi_1 : J^1(E) \to J^1(\bar{E}) \).

Note that the proof of Bäcklund is very complicated and one can hardly say that it is rigorous.

The proofs of the infinitesimal version of this theorem can be found in the books by Anderson [1] and Olver [33].
**Definition 2.11.** A smooth map $\Phi : J^\infty(E) \to J^\infty(\tilde{E})$ is called a *generalized contact transformation* if it preserves the contact ideal, that is,

$$\Phi^*(C(J^\infty(\tilde{E}))) \subseteq C(J^\infty(E)).$$

$\Phi$ is called a *classical contact transformation* if it is a generalized contact transformation and if $\Phi$ covers a smooth map $\Phi_1 : J^1(E) \to J^1(\tilde{E})$, that is, if the following diagram commutes.

$$
\begin{array}{ccc}
J^\infty(E) & \xrightarrow{\Phi} & J^\infty(\tilde{E}) \\
\pi_1^\infty \downarrow & & \downarrow \pi_1^\infty \\
J^1(E) & \xrightarrow{\Phi_1} & J^1(\tilde{E}).
\end{array}
$$

We remark [1] that there exists exactly one contact transformation $\Phi : J^\infty(E) \to J^\infty(\tilde{E})$, that covers a given smooth map $\Phi_0 : J^0(E) \to J^0(\tilde{E})$ and there exists at most one contact transformation $\Phi : J^\infty(E) \to J^\infty(\tilde{E})$ that covers a given smooth map $\Phi_{1,0} : J^1(E) \to J^0(\tilde{E})$.

To state the next theorem we need to introduce the notion of the Poisson bracket. By definition the *Poisson Bracket* of two first-order functions $g(x^i, u, u_i)$ and $h(x^i, u, u_i)$, where $x^i$ are independent variables, $u$ is one dependent variable and $u_i$ are first derivatives of $u$ with respect to $x^i$, is a function

$$[g, h] = \frac{dg}{dx^i} \frac{\partial h}{\partial u_i} - \frac{dh}{dx^i} \frac{\partial g}{\partial u_i},$$

where

$$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} \quad \text{for} \quad 1 \leq i \leq n.$$

The following form of Lie's theorem is due to Mayer [22] §135. Also see Caratheodory [16].

**Theorem 2.12.** Let $\pi : E \to M$ and $\tilde{\pi} : \tilde{E} \to \tilde{M}$ be two trivial fiber bundles with the same dimension $n$ of base manifolds and let the fiber dimensions of both bundles be 1. Let

$$(x^i, u, u_{i_1}, u_{i_1i_2}, \ldots, u_{i_1\ldots i_k})$$

be the coordinates on $J^k(E)$ and

$$(\tilde{x}^i, \tilde{u}, \tilde{u}_{i_1}, \tilde{u}_{i_1i_2}, \ldots, \tilde{u}_{i_1\ldots i_k})$$
be the coordinates on \( J^k(\mathcal{E}) \), \( k \geq 1 \). Consider a map \( \Phi_{1,0} : J^1(\mathcal{E}) \to J^0(\mathcal{E}) \) given by

\[
\bar{x}^i = f^i(x^i, u, u_i), \quad \text{and} \quad \bar{u} = g(x^i, u, u_i) \quad \text{for} \quad j = 1, \ldots n.
\]

There exists a classical contact transformation \( \Phi : J^k(\mathcal{E}) \to J^k(\mathcal{E}) \) which covers \( \Phi_{1,0} \) if and only if

\[
[f^i, f^j] = 0, \quad \text{and} \quad [g, f^i] = 0 \quad \text{for} \quad 1 \leq i, j \leq n.
\]

2.5. The Variational Bicomplex for a Second-Order Scalar PDE in the Plane

Consider a trivial fiber bundle \( \pi : E \to M \), where \( E = \mathbb{R}^3 \) and \( M = \mathbb{R}^2 \). Consider a second-order jet space \( J^2(E) \) with coordinates

\[
(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})
\]

and the contact ideal generated by

\[
\theta = du - u_x dx - u_y dy, \quad \theta_x = du_x - u_{xx} dx - u_{xy} dy, \quad \theta_y = du_y - u_{xy} dx - u_{yy} dy.
\]

Let \( F : J^2(E) \to \mathbb{R} \) be a smooth function. Then the equation

\[
F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \quad (2.7)
\]

satisfying

\[
(\frac{\partial F}{\partial u_{xx}}, \frac{\partial F}{\partial u_{xy}}, \frac{\partial F}{\partial u_{yy}}) \neq 0,
\]

defines a 7-dimensional hyper surface \( \mathcal{R}^2 \) in \( J^2(E) \) called the equation manifold of equation (2.7).

Let \( i_2 : \mathcal{R}^2 \to J^2(E) \) be the natural embedding. The pullback by \( i_2 \) of the contact ideal on \( J^2(E) \)

\[
\mathcal{C}(\mathcal{R}^2) = i_2^*(\mathcal{C}(J^2(E)))
\]

is called the contact ideal on \( \mathcal{R}^2 \).

Next we consider three equations

\[
F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \quad (2.8)
\]

\[
D_x F = \frac{\partial F}{\partial x} + u_x \frac{\partial F}{\partial u} + u_{xx} \frac{\partial F}{\partial u_x} + u_{xy} \frac{\partial F}{\partial u_y} + u_{xxy} \frac{\partial F}{\partial u_{xx}} + u_{xyy} \frac{\partial F}{\partial u_{xy}} + u_{yyy} \frac{\partial F}{\partial u_{yy}} = 0, \quad (2.9)
\]
\[ D_y F = \frac{\partial F}{\partial y} + u_y \frac{\partial F}{\partial u} + u_{xy} \frac{\partial F}{\partial u_x} + u_{yy} \frac{\partial F}{\partial u_y} + u_{xxy} \frac{\partial F}{\partial u_{xx}} + u_{xyy} \frac{\partial F}{\partial u_{xy}} + u_{yyy} \frac{\partial F}{\partial u_{yy}} = 0. \]  

(2.10)

Since

\( \left( \frac{\partial F}{\partial u_{xx}}, \frac{\partial F}{\partial u_{xy}}, \frac{\partial F}{\partial u_{yy}} \right) \neq 0 \)

one can easily see that equations (2.8), (2.9) and (2.10) define a 9-dimensional submanifold \( \mathcal{R}^3 \) in \( J^3(E) \). Consider the embedding \( i_3 : \mathcal{R}^3 \rightarrow J^3(E) \). The pullback by \( i_3 \) of the contact ideal on \( J^3(E) \)

\[ \mathcal{C}(\mathcal{R}^3) = i_3^*(\mathcal{C}(J^3(E))) \]

defines the contact ideal on \( \mathcal{R}^3 \). The manifold \( \mathcal{R}^3 \) together with the contact ideal is called the first prolongation of \( \mathcal{R}^2 \).

More generally the equation (2.7) together with all its differential consequences up to order \( k \) defines a \( (2k+3) \)-dimensional submanifold \( \mathcal{R}^k \) in \( J^k(E) \). Consider the natural embedding \( i_k : \mathcal{R}^k \rightarrow J^k(E) \). The pullback by \( i_k \) of the contact ideal on \( J^k(E) \)

\[ \mathcal{C}(\mathcal{R}^k) = i_k^*(\mathcal{C}(J^k(E))) \]

defines the contact ideal on \( \mathcal{R}^k \). The manifold \( \mathcal{R}^k \) together with the contact ideal is called the \( (k-2) \)-nd prolongation of \( \mathcal{R}^k \).

For convenience denote \( J^0(E) = E \) by \( \mathcal{R}^0 \) and \( J^1(E) \) by \( \mathcal{R}^1 \). As in the case of the free variational bicomplex, we have natural projections

\[ \pi_k^s : \mathcal{R}^s \rightarrow \mathcal{R}^k, \quad \text{for} \quad k \leq s \]

satisfying

\[ \begin{array}{ccc}
\mathcal{R}^s & \xrightarrow{i_s} & J^s(E) \\
\pi_k^s \downarrow & & \pi_k^s \\
\mathcal{R}^k & \xrightarrow{i_k} & J^k(E). 
\end{array} \]

These projections satisfy

\[ \pi_k^k = \text{id}_{\mathcal{R}^k} \quad \text{and} \quad \pi_k^s = \pi_k^t \circ \pi_s^t \quad \text{for} \quad k \leq s \leq t. \]
Definition 2.13. The inverse limit $\mathcal{R}^\infty$ of the sequence of prolonged equation manifolds $\{ \mathcal{R}^k, k = 0, 1, 2, \ldots \}$ together with the projections

$$
\pi_M^\infty : \mathcal{R}^\infty \to M, \quad \text{and} \quad \pi_k^\infty : \mathcal{R}^\infty \to \mathcal{R}^k, \quad \text{for} \quad k = 0, 1, 2, \ldots
$$

is called the infinite prolongation of $\mathcal{R}^2$.

Due to the inverse limit construction there exists a uniquely defined map $i_\infty : \mathcal{R}^\infty \to J^\infty(E)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{R}^\infty & \xrightarrow{i_\infty} & J^\infty(E) \\
\pi_k^\infty \downarrow & & \downarrow \pi_k^\infty \\
\mathcal{R}^k & \xrightarrow{i_k} & J^k(E)
\end{array}
$$

$\mathcal{R}^\infty$ inherits the topological structure of $J^\infty(E)$. Differential forms on $\mathcal{R}^\infty$ are defined to be the pullbacks by $i_\infty$ of differential forms on $J^\infty(E)$. The set of all $p$-forms on $\mathcal{R}^\infty$ is denoted by $\Omega^p(\mathcal{R}^\infty)$ and

$$
\Omega^*(\mathcal{R}^\infty) = \bigcup_{p=0}^{\infty} \Omega^p(\mathcal{R}^\infty)
$$

is the set of all forms on $\mathcal{R}^\infty$. The contact ideal on $\mathcal{R}^\infty$ that is, defined to be the pullback by $i_\infty$ of the contact ideal $\mathcal{C}(J^\infty(E))$ on $J^\infty(E)$. Elements of the contact ideal are called contact forms. The pair

$$
\mathcal{R} = (\mathcal{R}^\infty, \mathcal{C}(\mathcal{R}^\infty))
$$

will be referred to as a second-order partial differential equation in the plane.

A vector field $X$ on $\mathcal{R}^\infty$ is defined to be a derivation on the ring of smooth functions on $\mathcal{R}^\infty$. A vector field $X$ is called a total vector field if

$$
X \cdot \omega = 0
$$

for every contact 1-form $\omega$. Total vector fields on $\mathcal{R}^\infty$ span a 2-dimensional involutive distribution.

Let $\mathcal{R}^\infty \subset J^\infty(E)$ and $\tilde{\mathcal{R}} \subset J^\infty(\tilde{E})$ be two scalar second-order partial differential equations in the plane. The map $\Phi : \mathcal{R}^\infty \to \tilde{\mathcal{R}}^\infty$ is called a generalized contact transformation if it preserves the contact ideal, that is,

$$
\Phi^*(\mathcal{C}(\tilde{\mathcal{R}}^\infty)) \subseteq \mathcal{C}(\mathcal{R}^\infty).
$$
A generalized contact transformation is called a classical contact transformation if it covers a map \( \Phi_1 : \mathcal{R}^1 \to \mathcal{R}^1 \). Note that every contact transformation \( \Phi' : J^k(E) \to J^k(\tilde{E}) \) induces a contact transformation \( \Phi : \mathcal{R}^k \to \mathcal{R}^k \). Such contact transformations \( \Phi \) are called external. There are contact transformations \( \Phi : \mathcal{R}^k \to \mathcal{R}^k \) that do not arise from the contact transformations on the underlying jet spaces. Such transformations are called internal contact transformations. Several new examples of internal transformations will be given in chapter 8.

**Definition 2.14.** The variational bicomplex \( \Omega^{r,s}(\mathcal{R}^\infty, d_H, d_V) \) for the second-order partial differential equation in the plane \( \mathcal{R} \) is a pullback by \( i_0 \) of the free variational bicomplex \( \Omega^{r,s}(J^\infty(E), d_H, d_V) \) to \( \mathcal{R}^\infty \).

\[
\begin{align*}
0 & \longrightarrow \Omega^0,0(\mathcal{R}^\infty) \xrightarrow{d_H} \Omega^1,0(\mathcal{R}^\infty) \xrightarrow{d_H} \Omega^2,0(\mathcal{R}^\infty) \\
& \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \\
0 & \longrightarrow \Omega^0,1(\mathcal{R}^\infty) \xrightarrow{d_H} \Omega^1,1(\mathcal{R}^\infty) \xrightarrow{d_H} \Omega^2,1(\mathcal{R}^\infty) \\
& \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \\
0 & \longrightarrow \mathbb{R} \longrightarrow \Omega^0,0(\mathcal{R}^\infty) \xrightarrow{d_H} \Omega^1,0(\mathcal{R}^\infty) \xrightarrow{d_H} \Omega^2,0(\mathcal{R}^\infty).
\end{align*}
\]

We remark that while columns of the variational bicomplex on \( \mathcal{R}^\infty \) are exact, the rows will not be exact in general. We define the cohomology classes \( H^{r,s}(\mathcal{R}^\infty) \) of the variational bicomplex by

\[
H^{r,s}(\mathcal{R}^\infty) = \left\{ \omega \in \Omega^{r,s}(\mathcal{R}^\infty); \ d\omega = 0 \right\}.
\]

The dimensions of \( H^{r,s}(\mathcal{R}^\infty) \) are invariant under the generalized contact transformations (see Anderson [1] for details.) Various interpretations for the cohomology of the variational bicomplex have already been found [1], [5], [6], [8], [9], [36]. It is well-known that the \( H^{1,0}(\mathcal{R}^\infty) \) classes are in one-to-one correspondence with the nontrivial conservation laws. In this dissertation we have found characterization of Darboux integrability in terms of the cohomology of the variational bicomplex. It is very likely that more interpretations will be found in the near future. See, for example, an interesting paper of Anderson and Fels [4].

As in the case of the jet spaces, we define the Lie derivative \( \mathcal{L}_X \omega \) of a vector field \( X \) by
\[ \mathcal{L}_X \omega = X \lrcorner \ d\omega + d(X \lrcorner \ \omega), \]

where \( \omega \) is a differential form on \( \mathcal{R}^\infty \). We define the *projected Lie derivative* \( X(\omega) \) by

\[ X(\omega) = \pi^r,s(\mathcal{L}_X \omega). \]

**Definition 2.15.** Let \( \mathcal{R} \) and \( \tilde{\mathcal{R}} \) be two second-order scalar partial differential equations in the plane. Let \( \Phi : \mathcal{R}^\infty \rightarrow \tilde{\mathcal{R}}^\infty \) be a map between the two infinitely prolonged equation manifolds. For \( \omega \in \Omega^{r,s}(\mathcal{R}^\infty) \) we define the projected pullback \( \Phi^\# : \Omega^{r,s}(\mathcal{R}^\infty) \rightarrow \Omega^{r,s}(\tilde{\mathcal{R}}^\infty) \) by

\[ \Phi^\#(\omega) = \pi^r,s(\Phi^*(\omega)). \]

If \( \Phi \) and \( \Psi \) are generalized contact transformations, then

\[ (\Psi \circ \Phi)^\# = \Phi^\# \circ \Psi^\#. \]

The above result can be found in [1].

We will now make the construction of the variational bicomplex for \( \mathcal{R} \) explicit. Assume that the equation (2.7) is given by

\[ u_{xx} + f(x, y, u, u_x, u_y, u_{xy}, u_{yy}) = 0. \]

The natural coordinates on \( \mathcal{R}^\infty \) are

\[ (x, y, u, u_x, u_y, u_{xy}, u_{yy}, u_{yxx}, u_{yyy}, \ldots, u_{y^k-1}, u_{y^k}, \ldots) \]

and the basis for the contact ideal on \( \mathcal{R}^\infty \) is

\[ \{ \theta, \theta_x, \theta_y, \theta_{xy}, \theta_{yy}, \ldots, \theta_{y^k-1}, \theta_{y^k}, \ldots \}, \]

where

\[ \theta = du - u_x \ dx - u_y \ dy \]

\[ \theta_{y^k-1} = du_{y^k-1} + (D_{y^k} f) \ dx - u_{y^k} \ dy \]

\[ \theta_{y^k} = du_{y^k} - u_{y^k+1} \ dx - u_{y^{k+1}} \ dy. \]
Every total vector field can be written as

$$X = aD_x + bD_y,$$

where $a$ and $b$ are functions on $\mathcal{R}^\infty$ and the vector fields $D_x$ and $D_y$ are

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial x} + u_{yy} \frac{\partial}{\partial u_y} + u_{yy} \frac{\partial}{\partial u_{yy}} + \cdots,$$

$$D_z = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} - f \frac{\partial}{\partial x} + u_{zy} \frac{\partial}{\partial u_y} - D_y f \frac{\partial}{\partial u_{zy}} + u_{zy} \frac{\partial}{\partial u_{zy}} + \cdots.$$

For a $k$th-order function $f(x, y, u, u_x, u_y, u_{xy}, u_{yy}, \ldots, u_{xy}^{k-1}, u_{y}^{k})$ on $\mathcal{R}^\infty$ we have the horizontal and vertical differentials

$$d_H f = (D_x f) \, dx + (D_y f) \, dy$$

and

$$d_V f = \frac{\partial}{\partial u} \theta + \frac{\partial}{\partial x} \theta_x + \frac{\partial}{\partial u_y} \theta_y + \frac{\partial}{\partial u_{xy}} \theta_{xy} + \cdots + \frac{\partial}{\partial u_{xy}^{k-1}} \theta_{xy}^{k-1} + \frac{\partial}{\partial u_y^{k}} \theta_y^{k}.$$

The structure equations for the horizontal and vertical differentials remain the same as in (2.4) and (2.5) except that now

$$d_H \theta_{xy}^{k-1} = d_V (D_y^{k-1} f) \, dx - \theta_{xy}^{k} \, dy.$$

For more detailed study of the variational bicomplex, the reader is referred to [1] and [8].
Consider the trivial bundle $E = \{ \pi : E \to M \}$, with local coordinates $\pi : (x, y, u) \to (x, y)$. Let $\mathcal{R} = (\mathcal{R}^\infty, \mathcal{C}(\mathcal{R}^\infty))$ be a second-order equation given by

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \quad (3.1)$$

With $\mathcal{R}$ we associate the characteristic equation

$$\frac{\partial F}{\partial u_{xx}} \lambda^2 - \frac{\partial F}{\partial u_{xy}} \lambda \mu + \frac{\partial F}{\partial u_{yy}} \mu^2 = 0. \quad (3.2)$$

and the discriminant

$$\Delta = \left( \frac{\partial F}{\partial u_{xy}} \right)^2 - 4 \frac{\partial F}{\partial u_{xx}} \frac{\partial F}{\partial u_{yy}}. \quad (3.3)$$

Let $\sigma$ be a point in $\mathcal{R}^2$. We say that the equation $\mathcal{R}$ is hyperbolic parabolic or elliptic type at the point $\sigma$ if $\Delta(\sigma) > 0$, $\Delta(\sigma) = 0$, or $\Delta(\sigma) < 0$, respectively. We say that the equation (3.1) is hyperbolic parabolic or elliptic type if it is hyperbolic parabolic or elliptic type at every point of the equation manifold $\mathcal{R}$.

In this dissertation we will deal only with hyperbolic equations, that is, with equations satisfying

$$\left( \frac{\partial F}{\partial u_{xy}} \right)^2 - 4 \frac{\partial F}{\partial u_{xx}} \frac{\partial F}{\partial u_{yy}} > 0. \quad (3.4)$$

at every point of the equation manifold $\mathcal{R}$.

If $\mathcal{R}$ is hyperbolic there are two non-proportional real roots $(\mu, \lambda) = (m_x, m_y)$ and $(\mu, \lambda) = (n_x, n_y)$ of the characteristic equation (3.2). The total vector fields

$$X = m_x D_x + m_y D_y, \quad \text{and} \quad Y = n_x D_x + n_y D_y. \quad (3.5)$$

are called the characteristic vector fields and they form a basis for the space of total vector fields on the infinitely prolonged equation manifold $\mathcal{R}^\infty$ of (3.1). The reader should be warned here that in [8] the authors allow the functions $m_x, m_y, n_x, n_y$ to be of any order. Throughout this dissertation we will assume that these functions are of order $\leq 2$. Let $T$ be a total vector field on $\mathcal{R}^\infty$. A function $f$ on $\mathcal{R}^\infty$ is called $T$ invariant if $T(f) = 0$.

We now turn to the definition of Darboux integrability.
Definition 3.1. Let $\mathcal{R}$ be a second-order scalar hyperbolic partial differential equation in the plane with characteristic vector fields $X$ and $Y$. $\mathcal{R}$ is called Darboux integrable at levels $k$, and $l$, where $k, l \geq 1$ if there are two functionally independent $X$ invariant functions of order $\leq k$, and two functionally independent $Y$ invariant functions of order $\leq l$ on $\mathcal{R}^\infty$, or if there are two functionally independent $X$ invariant functions of order $\leq l$, and two functionally independent $Y$ invariant functions of order $\leq k$ on $\mathcal{R}^\infty$. We say that $\mathcal{R}$ is Darboux integrable at level $k$, $k \geq 1$, if there are two functionally independent $X$ invariant functions of order $\leq k$, and two functionally independent $Y$ invariant functions of order $\leq k$ on $\mathcal{R}^\infty$; that is, if there are functions $I$, $\bar{I}$, $J$, $\bar{J}$ on $\mathcal{R}^\infty$ of order $\leq k$, such that

$$X(I) = X(\bar{I}) = 0 \quad \text{and} \quad Y(J) = Y(\bar{J}) = 0$$

and

$$dI \wedge d\bar{I} \neq 0 \quad \text{and} \quad dJ \wedge d\bar{J} \neq 0.$$  

If there are two functionally independent $X$ invariant functions or two functionally independent $Y$ invariant functions on $\mathcal{R}^\infty$ of order $\leq k$, we say that the equation $\mathcal{R}$ is semi-Darboux integrable at level $k$.

It may not be immediately apparent that this definition is just a more precise version of what we have described Darboux integrability to be in the introduction. At the heart of the above definition is the observation that for arbitrary functions $\varphi$ and $\psi$, the system of equations (3.1), its differential consequences, and the equations

$$\bar{I} = \varphi(I) \quad \text{and} \quad \bar{J} = \psi(J)$$

give rise to an integrable Pfaffian system whose integral manifolds describe the general solution to (3.1).

Example 1. Consider the linear hyperbolic equation

$$u_{xy} = \frac{u_y}{x+y} \quad \text{(3.6)}$$

with the characteristic vector fields $X = D_x$, $Y = D_y$. The $X$ invariant functions are

$$I = y \quad \text{and} \quad \bar{I} = \frac{u_y}{x+y}$$
and the $Y$ invariant functions are

$$J = x \quad \text{and} \quad \tilde{J} = u_{xz}.$$ 

The three equations

$$u_{xy} = \frac{u_y}{x+y},$$

$$u_{xx} = \varphi''(x),$$

$$\frac{u_y}{x+y} = \psi''(y),$$

give rise to the Pfaffian system generated by

$$\theta_1 = du - u_x dx - (x+y)\psi''(y) dy,$$

$$\theta_2 = du - \varphi''(x) dx - \psi''(y) dy.$$

it is easy to check that the integrability conditions are satisfied. From $\theta_2 = 0$ we obtain

$$u_x = \varphi'(x) + \psi'(y).$$

Substituting for $u_x$ into $\theta_1 = 0$ we find

$$u = \varphi(x) - \psi(y) + (x+y)\psi'(y),$$

which is the general solution to (3.6).

**Example 2.** Consider the hyperbolic equation

$$u_{xy} = u_x e^u \quad (3.7)$$

with the characteristic vector fields $X = D_x, Y = D_y$. The $X$ invariant functions are

$$I = y \quad \text{and} \quad \tilde{I} = u_y - e^u$$

and the $Y$ invariant functions are

$$J = x \quad \text{and} \quad \tilde{J} = \frac{u_{xx}}{u_x} - u_x.$$
The three equations

\[
\begin{align*}
 u_{xy} &= u_x e^u, \\
 \frac{u_{xx}}{u_x} - u_x &= f(x), \\
 u_y - e^u &= g(y)
\end{align*}
\]

give rise to the Pfaffian system generated by

\[
\begin{align*}
 \theta_1 &= du - u_x dx - (e^u + g(y)) dy, \\
 \theta_2 &= du_x - (u_x^2 + f(x) u_x) dx - u_x e^u dy.
\end{align*}
\]

One can again easily see that \{ \theta_1, \theta_2 \} is integrable. From the equation

\[
\theta_2 - u_x \theta_1 = 0,
\]

we deduce

\[
\frac{du_x}{u_x} = du - f(x) dx - g(y) dy
\]

and hence

\[
u_x = e^{u - F(x) - G(y)},
\]

where \(F'(x) = f(x)\) and \(G'(x) = g(x)\). For convenience denote

\[
e^{-F(x)} = \varphi'(x) \quad \text{and} \quad e^{-G(x)} = \xi'(x).
\]

Substituting into \(\theta_1 = 0\), we get

\[
du - \varphi'(x) \xi'(y) e^u dx - \left( e^u - \frac{\xi''(y)}{\xi'(y)} \right) dy = 0.
\]

Dividing the last equation by \(\xi'(y) e^u\), one arrives at

\[
\frac{1}{\xi'(y) e^u} du + \frac{\xi''(y)}{[\xi'(y)]^2 e^u} dy - \varphi'(x) dx - \frac{1}{\xi'(y)} dy = 0.
\]

Integrating we obtain

\[
-\frac{1}{\xi'(y) e^u} \varphi(x) + \int \frac{dy}{\xi'(y)}.
\]
Denote

\[ \psi(y) = \int \frac{dy}{\xi'(y)}, \]

to arrive to the general solution of (3.7), namely

\[ e^u = -\frac{\psi'(y)}{\varphi(x) + \psi(y)}. \]

**Example 3.** The most well-known example of a Darboux integrable, nonlinear equation is the Liouville equation

\[ u_{xy} = e^u. \]

Here \( X = D_x, Y = D_y \). The \( X \) and \( Y \) invariant functions are

\[ I = y, \quad \tilde{I} = u_{yy} - \frac{u_y}{2}, \quad \text{and} \quad J = x, \quad \tilde{J} = u_{xx} - \frac{u_x}{2}. \]

The Pfaffian system for the equations

\[ u_{xy} = e^u, \]
\[ u_{xx} - \frac{u_x^2}{2} = \varphi(x), \]
\[ u_{yy} - \frac{u_y^2}{2} = \psi(y), \]

is generated by

\[ \theta_1 = du - u_x dx - u_y dy, \]
\[ \theta_2 = du_x - \left( \frac{u_x^2}{2} + \varphi(x) \right) dx - e^u dy, \]
\[ \theta_3 = du_y - e^u dx - \left( \frac{u_y^2}{2} + \psi(y) \right) dy. \]

It is a simple matter to check that this Pfaffian system is integrable and its integration leads to the well known solution

\[ e^u = \frac{2U'V'}{(U + V)^2}, \]
where $U = U(x)$ and $V = V(y)$ and

$$\varphi = \frac{U'''}{U'} - \frac{3}{2} \frac{U''}{(U')^2} \quad \text{and} \quad \psi = \frac{V'''}{V'} - \frac{3}{2} \frac{V''}{(V')^2}.$$ 

Example 4. As our next example consider an equation studied by Goursat [27]

$$u_{xy} + \frac{2\sqrt{u_x u_y}}{x+y} = 0,$$  

(3.8)

with the characteristic vector fields $X = D_x$ and $Y = D_y$. The $X$ invariant functions are

$$I = y \quad \text{and} \quad \tilde{I} = \frac{u_{yy}}{2\sqrt{u_y}} + \frac{\sqrt{u_y}}{x+y}$$

and the $Y$ invariant functions are

$$J = x \quad \text{and} \quad \tilde{J} = \frac{u_{xx}}{2\sqrt{u_x}} + \frac{\sqrt{u_x}}{x+y}.$$ 

The three equations

$$u_{xy} + \frac{2\sqrt{u_x u_y}}{x+y} = 0,$$

$$\frac{u_{xx}}{2\sqrt{u_x}} + \frac{\sqrt{u_x}}{x+y} = f(x),$$

$$\frac{u_{yy}}{2\sqrt{u_y}} + \frac{\sqrt{u_y}}{x+y} = g(y),$$

give rise to the Pfaffian system generated by

$$\theta_1 = du - u_x dx - u_y dy,$$

$$\theta_2 = du_x - \left( \frac{2u_x}{x+y} - 2\sqrt{u_x f(x)} \right) dx + \frac{2\sqrt{u_x u_y}}{x+y} dy,$$

$$\theta_3 = du_y + \frac{2\sqrt{u_x u_y}}{x+y} dx + \left( \frac{2u_y}{x+y} - 2\sqrt{u_y g(y)} \right) dy.$$ 

It is easy to check that the system $\{\theta_1, \theta_2, \theta_3\}$ is integrable. Vessiot [43], page 60, integrates this Pfaffian system and arrives at the general solution of the equation (3.8),

$$x = \varphi''(\alpha), \quad y = \psi''(\beta) \quad \text{and}$$
\[ u = \frac{1}{\psi''(\alpha) + \psi''(\beta)} [(\alpha - \beta)^2 \psi'' \psi'' + 2(\alpha - \beta)(\psi' \psi' - \psi' \psi'') - (\psi' + \psi')^2] + 2(\psi + \psi), \]

where \( \varphi(\alpha) \) is an arbitrary function of \( \alpha \) and \( \psi(\beta) \) is an arbitrary function of \( \beta \).

**Example 5.** As our next example consider the hyperbolic equation

\[ u_{xy} = uu_x. \tag{3.9} \]

The characteristic vector fields are \( X = D_x \) and \( Y = D_y \). The functions

\[ I = y \quad \text{and} \quad \bar{I} = u_y - \frac{u^2}{2} \]

are \( X \) invariant, and the functions

\[ J = x \quad \text{and} \quad \bar{J} = \frac{3u_{xx}^2 - 2u_{xxx}u_x}{2u_x^2} \]

are \( Y \) invariant. The three equations

\[ u_{xy} = uu_x, \]

\[ \frac{3u_{xx}^2 - 2u_{xxx}u_x}{2u_x^2} = \varphi(x), \]

\[ u_y - \frac{u^2}{2} = \psi(y), \]

give rise to the Pfaffian system generated by

\[ \theta_1 = du - u_x dx - \left( \frac{u^2}{2} + \psi(y) \right) dy, \]

\[ \theta_2 = du_x - u_{xx} dx + uu_x dy, \]

\[ \theta_3 = du_{xx} - \left( \frac{3u_{xx}^2}{2u_x} - u_x \varphi(x) \right) dx - \left( u_x^2 + uu_{xx} \right) dy. \]

Goursat, [26] page 134, integrates the above Pfaffian system to obtain the general solution of the equation (3.9),

\[ u = \frac{g''(y)}{g'(y)} - \frac{2g'(y)}{f(x) + g(y)}. \]
Example 6. As our last example consider the hyperbolic non-Monge-Ampère equation

$$u_{xx}u_{xy} - u_x = 0.$$  \hspace{1cm} (3.10)

We choose the characteristic vector fields to be

$$X = u_{xy}D_x + \frac{u_x}{u_{xy}}D_y \quad \text{and} \quad Y = D_x.$$

There are three $X$ invariant second-order functions, namely

$$I_1 = u_{xy} - x, \quad I_2 = y - \frac{u_x}{u_{xy}}, \quad \text{and} \quad I_3 = \frac{u_{xy}^2}{u_x}.$$

and there are two $Y$ invariant functions

$$J_1 = y, \quad \text{and} \quad J_2 = u_{yyy} + \frac{u_{xy}^2u_{xyy}}{u_x}.$$

The three equations

$$u_{xx}u_{xy} = u_x,$$

$$\frac{u_{xy}^2}{u_x} = f(u_{xy} - x),$$

$$u_{yyy} + \frac{u_{xy}^2u_{xyy}}{u_x} = g(y)$$

give rise to a Pfaffian system which is integrated by Goursat [27], page 136. The general solution of the equation (3.10) is given by

$$x = v\varphi(\alpha) - \alpha \quad \text{and} \quad u = 2\int v^2(v\varphi'(\alpha) - 1)\,d\alpha + \psi(y)$$

where $\varphi$ is an arbitrary function of $\alpha$, $\psi$ is an arbitrary function of $y$ and $v$ is a function of $y$ and $\alpha$ defined by

$$v = y\varphi(\alpha) - \varphi(\alpha)\int \frac{d\alpha}{\varphi^2(\alpha)}.$$

Note that $\alpha = u_{xy} - x$.

Many other examples of linear and nonlinear Darboux integrable hyperbolic equations in the plane can be found in the classical works of Goursat [26] and Forsyth [23]. The following theorem, which is proved in chapter 7, explains why Darboux integrable equations can be integrated by the methods of ODE’s.
Theorem 3.2. Let \( \mathcal{R} \) be a scalar hyperbolic second-order partial differential equation in the plane which is Darboux integrable at level \( k+1, k \geq 1 \). Then there exist a coframing of \( \mathcal{R}^{k+1} \)

\[
(\theta^1, \ldots, \theta^{2k+1}; \omega^1, \omega^2, \omega^3, \omega^4)
\]

so that the contact ideal \( \mathcal{C}_k \) is generated algebraically by the forms

\[
\theta^1, \ldots, \theta^{2k+1}; \omega^1 \land \omega^2, \omega^3 \land \omega^4,
\]

and the Pfaffian systems \( \{ \omega^1, \omega^2 \} \) and \( \{ \omega^3, \omega^4 \} \) are integrable.

Using the language of chapter 6, \((\mathcal{R}^{k+1}, \mathcal{C}_k)\) is a hyperbolic Darboux system. Note that the 1-forms \( \omega_1, \omega_2 \) and \( \omega_3, \omega_4 \) can be chosen to be the differentials of \( X \) and \( Y \) invariant functions.

Notice that if \( I, \tilde{I} \) be the two functionally independent first integrals of \( \{ \omega^1, \omega^2 \} \) and \( J, \tilde{J} \) be the two functionally independent first integrals of \( \{ \omega^3, \omega^4 \} \), then on the submanifolds of \( \mathcal{R}^k \) given by the equations

\[
\tilde{I} = \phi(I) \quad \text{and} \quad \tilde{J} = \psi(J),
\]

for any functions \( \phi \) and \( \psi \), the contact ideal \( \mathcal{C}_k \) is integrable.

To motivate the definition of the generalized Laplace, invariants we consider a linear hyperbolic equation

\[
u_{xy} + a(x, y) u_x + b(x, y) u_y + c(x, y) u = 0. \quad (3.11)
\]

Notice that the point transformation

\[
\vec{x} = \varphi(x), \quad \vec{y} = \psi(y), \quad \text{and} \quad \vec{u} = \xi(x, y) u, \quad (3.12)
\]

does not change the form of (3.11). We write the equation (3.11) in the form

\[
\frac{\partial}{\partial x}(u_y + au) + b(u_y + au) - h_0u = 0, \quad (3.13)
\]

where

\[
h_0 = \frac{\partial a}{\partial x} + ab - c. \quad (3.14)
\]

If \( h_0 = 0 \), then the equation (3.11) can be integrated by quadratures. If \( h_0 \neq 0 \), we proceed as follows. Let

\[
\bar{u} = u_y + au. \quad (3.15)
\]
From (3.13) we deduce that

\[ u = \frac{1}{h_0} (\ddot{u}_x + b\ddot{u}). \]  

(3.16)

Substituting (3.16) into (3.15), we obtain

\[ \ddot{u} = \frac{\partial}{\partial y} \left( \frac{1}{h_0} (\ddot{u}_x + b\ddot{u}) \right) + a(\ddot{u}_x + b\ddot{u}), \]

that is,

\[ \ddot{u}_{xy} + a_1(x,y) \ddot{u}_x + b_1(x,y) \ddot{u}_y + c_1(x,y) \ddot{u} = 0, \]

(3.17)

for some functions \( a_1, b_1 \) and \( c_1 \). We then write the equation (3.17) in the form

\[ \frac{\partial}{\partial x} (\ddot{u}_y + a_1 \ddot{u}) + b_1 (\ddot{u}_y + a_1 \ddot{u}) - h_1 \ddot{u} = 0, \]

where

\[ h_1 = \frac{\partial a_1}{\partial x} + a_1 b_1 - c_1. \]

(3.18)

If \( h_1 = 0 \), then the equation (3.17) can be solved by quadratures and, using (3.16), we immediately arrive the general solution to (3.11). If \( h_1 \neq 0 \), we can repeat the above process to define \( h_2 \), and so on. Thus we obtain a sequence of functions \( h_0, h_1, h_2, \ldots \) If this sequence terminates, that is, if \( h_p = 0 \) is the last term of the sequence, then the equation (3.11) can be integrated by quadratures.

Likewise we can write (3.11) in the form

\[ \frac{\partial}{\partial y} (u_x + bu) + a(u_x + bu) - k_0 u = 0, \]

(3.19)

where

\[ k_0 = \frac{\partial b}{\partial y} + ab - c. \]

(3.20)

If \( k_0 = 0 \), then the equation (3.11) can be integrated by quadratures. If \( k_0 \neq 0 \), we proceed similarly as above to construct \( k_1 \) using the transformation

\[ \ddot{u} = u_x + bu \]

(3.21)

instead of the transformation (3.15). Thus we obtain another sequence \( k_0, k_1, k_2, \ldots \) If this sequence terminates, that is, if \( k_q = 0 \) is the last term of the sequence, then the equation (3.11) can be integrated by quadratures.
Summarizing, we conclude that the sequence of functions \( h_0, h_1, h_2, \ldots \) is obtained by successive applications of the Laplace transformation

\[
\ddot{u} = u_y + au,
\]  

(3.22)

and the sequence \( k_0, k_1, k_2, \ldots \) is obtained by successive applications of the Laplace transformation

\[
\ddot{u} = u_x + bu.
\]  

(3.23)

If one is challenged by the problem of integrating a linear equation of type (3.11), one can consider successive applications of either of the transformations (3.22) or (3.23). An easy computation ([23], page 49) shows that these transformations are, in a way, inverses to each other; that is, if we start with the equation (3.11) and apply the transformation (3.22) to get (3.17), then applying the transformation (3.23) to (3.17) yields an equation that is contact equivalent to the equation (3.11) with which we started. Thus the two sequences \( h_0, h_1, h_2, \ldots \), and \( k_0, k_1, k_2, \ldots \), are the only relevant ones to be considered. The functions \( h_i \) and \( k_j \) are invariant under the point transformations of type (3.12), and are called the Laplace invariants (see [23], page 45). If one of the sequences \( h_0, h_1, h_2, \ldots \), or \( k_0, k_1, k_2, \ldots \), terminates, we say the equation (3.11) is integrable by the method of Laplace.

**Example 7.** Consider the linear hyperbolic equation

\[
\frac{u_{xy}}{2u} - \frac{2u}{(x+y)^2} = 0.
\]  

(3.24)

Using the Laplace transform \( \ddot{u} = p \), we transform the equation (3.24) into the equation

\[
\ddot{u}_{xy} + \frac{2\ddot{u}_y}{x+y} - \frac{2\ddot{u}}{(x+y)^2} = 0.
\]  

(3.25)

Using the Laplace transform \( \ddot{u} = \ddot{q} \), we transform the equation (3.25) into the equation

\[
\ddot{u}_{xy} + \frac{2\ddot{u}_x}{x+y} + \frac{2\ddot{u}_y}{x+y} = 0.
\]  

(3.26)

Notice that equation (3.26) can be transformed into (3.24) by \( \ddot{u} = \frac{u}{(x+y)^2} \).

Due to Goursat we have the following result.
Theorem 3.3. A linear, hyperbolic equation (3.11) is Darboux integrable if and only if both sequences of the Laplace Invariants are finite, that is, there are integers $p$ and $q$ such that $h_p = k_q = 0$.

One of the main results of this dissertation is a generalization of this theorem to second-order nonlinear hyperbolic equations in the plane. This will be done in chapter 5.

The purpose of the following chapter is to briefly introduce the Laplace-adapted coframe for a hyperbolic equation and the generalized Laplace invariants, first introduced in [8], and also to obtain sufficient information about the structure equations for the Laplace-adapted coframe.
CHAPTER 4
THE LAPLACE ADAPTED COFRAME

Let $\mathcal{R}$ be a second-order hyperbolic partial differential equation in the plane given by

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \quad (4.1)$$

satisfying

$$\left( \frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_y}, \frac{\partial F}{\partial u_{yy}} \right) \neq 0.$$

Recall that the two characteristic total vector fields

$$X = m_x D_x + m_y D_y \quad \text{and} \quad Y = n_x D_x + n_y D_y. \quad (4.2)$$

are defined by (3.5) and we write the Lie bracket of these total vector fields as

$$[X, Y] = PX + QY, \quad (4.3)$$

where

$$P = \frac{1}{\delta}[m_y X(n_x) - n_x X(m_y) + n_x Y(m_y) - n_y Y(m_x)] \quad (4.4)$$

and

$$Q = \frac{1}{\delta}[m_x X(n_y) - m_y X(n_x) + m_y Y(m_x) - m_x Y(m_y)]. \quad (4.5)$$

Here $\delta$ is defined by

$$\delta = m_x n_y - m_y n_x. \quad (4.6)$$

We denote the horizontal forms dual to $X$ and $Y$ by $\sigma$ and $\tau$, that is

$$\sigma(X) = 1, \quad \sigma(Y) = 0, \quad \tau(X) = 0, \quad \tau(Y) = 1.$$

These forms are given explicitly by

$$\sigma = \frac{1}{\delta}(n_y dx - n_x dy) \quad \text{and} \quad \tau = \frac{1}{\delta}(-m_y dx + m_x dy). \quad (4.7)$$

On the infinitely prolonged equation manifold $\mathcal{R}^\infty$ the contact forms

$$\theta = du - u_x dx - u_y dy, \quad \theta_x = du_x - u_{xx} dx - u_{xy} dy, \quad \theta_y = du_y - u_{xy} dx - u_{yy} dy, \ldots$$
are not independent but are related by the *universal linearization*

\[
\frac{\partial F}{\partial u_{xx}} \theta_{xx} + \frac{\partial F}{\partial u_{xy}} \theta_{xy} + \frac{\partial F}{\partial u_{yx}} \theta_{yx} + \frac{\partial F}{\partial u_{yy}} \theta_{yy} + \frac{\partial F}{\partial u_x} \theta_x + \frac{\partial F}{\partial u_y} \theta_y + \frac{\partial F}{\partial u} \theta = 0.
\]

We let \( \Theta = \varrho \theta \), where \( \varrho \) is any smooth function of order \( \leq 1 \) on \( \mathcal{R}^\infty \). We should warn the reader here that in [8] the authors allow the function \( \varrho \) to be of any order. We rewrite this equation, using the characteristic vector fields (4.2) as

\[
XY(\Theta) + AX(\Theta) + BY(\Theta) + C \Theta = 0 \tag{4.8}
\]

and, equivalently, as

\[
YX(\Theta) + DX(\Theta) + EY(\Theta) + G \Theta = 0, \tag{4.9}
\]

where

\[
A = A_0 - \frac{X(\rho)}{\rho}, \quad B = B_0 - \frac{X(\rho)}{\rho}, \tag{4.10a}
\]

\[
C = C_0 - \frac{XY(\rho)}{\rho} - A_0 \frac{X(\rho)}{\rho} - B_0 \frac{Y(\rho)}{\rho} + 2 \frac{X(\rho)Y(\rho)}{\rho^2} \tag{4.10b}
\]

and

\[
D = A + P, \quad E = B + Q \quad \text{and} \quad G = C, \tag{4.11}
\]

and where

\[
A_0 = \frac{1}{\delta} [(\kappa \frac{\partial F}{\partial u_x} - X(n_x))n_y - (\kappa \frac{\partial F}{\partial u_y} - X(n_y))n_x], \tag{4.12a}
\]

\[
B_0 = \frac{1}{\delta} [- (\kappa \frac{\partial F}{\partial u_x} - X(n_x))m_y + (\kappa \frac{\partial F}{\partial u_y} - X(n_y))m_x], \tag{4.12b}
\]

\[
C_0 = \kappa \frac{\partial F}{\partial u}. \tag{4.12c}
\]

Here \( \kappa \) is defined by the equation

\[
(m_x \lambda - m_y \mu)(n_x \lambda - n_y \mu) = \kappa \left( \frac{\partial F}{\partial u_{xx}} \lambda^2 - \frac{\partial F}{\partial u_{xy}} \lambda \mu + \frac{\partial F}{\partial u_{yy}} \mu^2 \right) = 0.
\]

Observe that formally the equation (4.8) is the same as (3.11.) The Laplace transformation and the generalized Laplace invariants are a nonlinear analogue of similar notions for linear equations. The
Laplace-adapted coframe on $\mathcal{R}^\infty$ is constructed by successive applications of the generalized Laplace transform to (4.8) and (4.9) [8]. We define a contact 1-form $\eta_1$ by

$$\eta_1 = Y(\Theta) + A\Theta,$$  \hspace{1cm} (4.13)

and we set

$$H_0 = X(A) + AB - C,$$  \hspace{1cm} (4.14)

If $H_0 \neq 0$, we apply the projected Lie derivative of $Y$ to the equation (4.8) to arrive at the equation

$$XY(\eta_1) + A_1 X(\eta_1) + B_1 Y(\eta_1) + C_1 \eta_1 = 0,$$  \hspace{1cm} (4.15)

for some functions $A_1, B_1, \text{ and } C_1$. We observe that equation (4.15) is formally the same as equation (4.8) and therefore we may repeat the process. We define

$$\eta_2 = Y(\eta_1) + A_1 \eta_1,$$

and we set

$$H_1 = X(A_1) + A_1 B_1 - C_1.$$  \hspace{1cm} (4.16)

If $H_1 \neq 0$, we apply $Y$ to the equation (4.15) to arrive at the equation

$$XY(\eta_2) + A_2 X(\eta_2) + B_2 Y(\eta_2) + C_2 \eta_2 = 0,$$  \hspace{1cm} (4.17)

for some functions $A_2, B_2, \text{ and } C_2$. We will now define the forms $\eta_i$ by induction. Assume that $H_0 \neq 0, H_1 \neq 0, \ldots, H_{i-1} \neq 0$, and that $\eta_i$ satisfies the identity

$$XY(\eta_i) + A_i X(\eta_i) + B_i Y(\eta_i) + C_i \eta_i = 0.$$  \hspace{1cm} (4.18)

We define

$$\eta_{i+1} = Y(\eta_i) + A_i \eta_i,$$  \hspace{1cm} (4.19)

and set

$$H_i = X(A_i) + A_i B_i - C_i.$$  \hspace{1cm} (4.20)

This process continues until $H_p = 0$, for some $p = 0, 1, 2, \ldots$, in which case we define inductively

$$\eta_{p+i+1} = Y(\eta_{p+i}) \quad \text{for all } i \geq 1.$$  \hspace{1cm} (4.21)
This completes the construction of one half of the Laplace-adapted coframe. Similarly, we construct the other half. We define a contact 1-form $\xi_1$ by

$$\xi_1 = X(\Theta) + E \Theta$$

(4.22)

and we set

$$K_0 = Y(E) + ED - G.$$  

(4.23)

If $K_0 \neq 0$, we apply the projected Lie derivative of $X$ to the equation (4.9) to arrive at the equation

$$YX(\xi_1) + D_1X(\xi_1) + E_1Y(\xi_1) + G_1\xi_1 = 0,$$

(4.24)

for some functions $D_1, E_1, G_1$. We observe that equation (4.24) is formally the same as equation (4.9) and therefore we may repeat the process. We define

$$\xi_2 = X(\xi_1) + E_1 \xi_1,$$

and we set

$$K_1 = Y(E_1) + E_1D_1 - G_1.$$  

(4.25)

If $K_1 \neq 0$, we apply $X$ to the equation (4.24) to arrive at the equation

$$YX(\xi_2) + D_2X(\xi_2) + E_2Y(\xi_2) + G_2\xi_2 = 0,$$

(4.26)

for some functions $D_2, E_2, G_2$. We will now define the forms $\xi_j$ by induction. Assume that $K_0 \neq 0, K_1 \neq 0, \ldots, K_{j-1} \neq 0$, and that $\xi_j$ satisfies the identity

$$YX(\xi_j) + D_jX(\xi_j) + E_jY(\xi_j) + G_j\xi_j = 0$$

(4.27)

We define

$$\xi_{j+1} = X(\xi_j) + E_j \xi_j$$

(4.28)

and set

$$K_j = Y(E_j) + D_jE_j - G_j.$$  

(4.29)

This process continues until $K_q = 0$, for some $q = 0, 1, 2, \ldots$, in which case we define inductively

$$\xi_{q+j+1} = X(\xi_{q+j}), \quad \text{for all } j \geq 1.$$  

(4.30)
The forms \( \{ \sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \ldots \} \) define a coframe on \( \mathcal{R}^\infty \) called the \textit{Laplace-adapted coframe}.

The recursion formulas for \( A_i, B_i, C_i, \ldots, G_i \), as computed in [8], are

\[
A_i = A_{i-1} - Y(\ln H_{i-1}) - P, \quad \text{for } i = 1, \ldots, p, \quad (4.31a)
\]

\[
B_i = B_{i-1} - Q, \quad \text{for } i = 1, 2, 3, \ldots, \quad (4.31b)
\]

\[
C_i = C_{i-1} - X(A_{i-1}) - B_{i-1}Y(\ln H_{i-1}) + Y(B_{i-1}), \quad \text{for } i = 1, \ldots p, \quad (4.31c)
\]

\[
E_j = E_{j-1} - X(\ln K_{j-1}) + Q, \quad \text{for } j = 1, \ldots q, \quad (4.32a)
\]

\[
D_j = D_{j-1} + P, \quad \text{for } j = 1, 2, 3, \ldots, \quad (4.32b)
\]

\[
G_j = G_{j-1} - Y(E_{j-1}) - D_{j-1}X(\ln K_{j-1}) + X(D_{j-1}), \quad \text{for } i = 1, \ldots q. \quad (4.32c)
\]

We now give the transformation rules for the Laplace-adapted coframe. Let \( \mathcal{R}'^\infty \) be another second-order scalar hyperbolic equation in the plane, let \( X' \) and \( Y' \) be the characteristic vector fields and \( \{ \sigma', \tau', \Theta', \eta_1', \xi_1', \eta_2', \xi_2', \ldots \} \) be the Laplace-adapted coframe for \( \mathcal{R}'^\infty \). Let \( \Phi : \mathcal{R}^\infty \to \mathcal{R}'^\infty \) be a classical invertible contact transformation, thus

\[
\Phi'(X') = mX, \quad \Phi'(Y') = nY, \quad \text{and} \quad \Phi'(\Theta') = l\Theta, \quad (4.33)
\]

for some second-order functions \( m, n \) and a first-order function \( l \). Then the generalized Laplace invariants are related by

\[
\Phi'(H_i') = mnH_i \quad \text{and} \quad \Phi'(K_i') = mnK_i, \quad (4.34)
\]

and the Laplace-adapted coframes are related by

\[
\Phi'(\eta_i') = n^i l \eta_i \quad \text{for} \quad 1 \leq i \leq p + 1, \quad (4.35)
\]

\[
\Phi'(\eta_i') \equiv n^i l \eta_i \mod \{ \eta_{p+1}, \ldots, \eta_{i-1} \} \quad \text{for} \quad p + 2 \leq i, \quad (4.36)
\]

and

\[
\Phi'(\xi_j') = m^j l \xi_j \quad \text{for} \quad 1 \leq j \leq q + 1, \quad (4.37)
\]

\[
\Phi'(\xi_j') \equiv m^j l \xi_j \mod \{ \xi_{q+1}, \ldots, \xi_{j-1} \} \quad \text{for} \quad q + 2 \leq j, \quad (4.38)
\]

and
\[ \Phi^*(\sigma') = \frac{1}{m}\sigma \mod \{\Theta, \xi_1, \eta_1\} \quad \text{and} \quad \Phi^*(\tau') = \frac{1}{n}\tau \mod \{\Theta, \xi_1, \eta_1\} \]  

Note that under the projected pullback \( \Phi^\# \) the forms \( \sigma \) and \( \tau \) transform according to

\[ \Phi^\#(\sigma') = \frac{1}{m}\sigma \quad \text{and} \quad \Phi^\#(\tau') = \frac{1}{n}\tau. \]  

For convenience we recall the transformation formulas for \( P, Q, A, B, C, D, E, \) and \( G \) from \([8]\).

\[ \Phi^*(Q') = m(Q + X(ln n)) \quad \text{and} \quad \Phi^*(P') = n(P - Y(ln m)), \]  

(4.41a)

\[ \Phi^*(A') = n(A - Y(ln l)) \quad \text{and} \quad \Phi^*(B') = m(B - X(ln nl)), \]  

(4.41b)

\[ \Phi^*(D') = n(D - Y(ln ml)) \quad \text{and} \quad \Phi^*(E') = m(E - X(ln l)), \]  

(4.41c)

\[ \Phi^*(C') = mn(C - AX(ln l) - BY(ln l) + X(ln n)Y(ln l) - XY(ln l)) \]  

(4.41d)

\[ \Phi^*(G') = mn(G - EY(ln l) - DX(ln l) + Y(ln m)X(ln l) - XY(ln l)) \]  

(4.41e)

Recall that the exterior derivative \( d \) on \( \mathcal{R}^\infty \) splits into two components

\[ d = d_H + d_V, \]  

(4.42)

where

\[ d_H \omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega). \]  

(4.43)

The next proposition ([8]) gives the \( d_H \) structure equations for the Laplace-adapted coframe.

**Proposition 4.1.** Suppose that \( H_p = 0 \) and \( K_q = 0 \). The \( d_H \) structure equations for the Laplace-adapted coframe for the hyperbolic equation \( \mathcal{R}^\infty \) are given by

\[ d_H \sigma = -P \sigma \wedge \tau, \quad d_H \tau = -Q \sigma \wedge \tau, \]  

(4.44a)

\[ d_H(\Theta) = \sigma \wedge (\xi_1 - E \Theta) + \tau \wedge (\eta_1 - A \Theta), \]  

(4.44b)

and

\[ d_H \eta_1 = \sigma \wedge (-B \eta_1 + H_0 \Theta) + \tau \wedge (\eta_2 - A_1 \eta_1), \]  

(4.45a)

\[ d_H \eta_i = \sigma \wedge (-B_{i-1} \eta_i + H_{i-1} \eta_{i-1}) + \tau \wedge (\eta_{i+1} - A_i \eta_i) \quad 2 \leq i \leq p, \]  

(4.45b)

\[ d_H \eta_{p+1} = \sigma \wedge (-B_p \eta_{p+1}) + \tau \wedge \eta_{p+2}, \]  

(4.45c)

\[ d_H \eta_{p+i} = \sigma \wedge \nu_{p+i} + \tau \wedge \eta_{p+i+1} \quad i \geq 2. \]  

(4.45d)
In equation (4.45d), $\nu_{p+i}$ is a contact one form such that

$$\nu_{p+i} \equiv [(i - 1)Q - B] \eta_{p+i} \mod \{ \eta_{p+1}, \ldots, \eta_{p+i-1} \}. \quad (4.46)$$

$$d_H \xi_i = \tau \wedge (-D \xi_i + K_0 \Theta) + \sigma \wedge (\xi_2 - E_1 \xi_1), \quad (4.47a)$$

$$d_H \xi_i = \tau \wedge (-D_{j-1} \xi_j + K_{j-1} \xi_{j-1}) + \sigma \wedge (\xi_{j+1} - E_j \xi_j) \quad \text{for } 2 \leq j \leq q, \quad (4.47b)$$

$$d_H \xi_{q+1} = \tau \wedge (-D_q \xi_{q+1}) + \sigma \wedge \xi_{q+2}, \quad (4.47c)$$

$$d_H \xi_{q+j} = \tau \wedge \mu_{q+j} + \sigma \wedge \xi_{q+j+1} \quad j \geq 2. \quad (4.47d)$$

In equation (4.47d), $\mu_{q+j}$ is a contact one form such that

$$\mu_{q+j} \equiv [-(j - 1)P - D_q] \xi_{q+j} \mod \{ \xi_{q+1}, \ldots, \xi_{q+j-1} \}. \quad (4.48)$$

If $H_i \neq 0$ or $K_j \neq 0$ for all $i$ or $j$, then the structure equations (4.45b) or (4.47b) remain valid for all $i \geq 2$ or $j \geq 2$.

Proof. First recall a well-known formula

$$d\omega(X_1, X_2) = X_1[\omega(X_2)] - X_2[\omega(X_1)] - \omega([X_1, X_2]), \quad (4.49)$$

that holds for any 1-form $\omega$ and any two vector fields $X_1$ and $X_2$. Since $d\nu \sigma$ is a form of type $(1,1)$, we have that $(d\nu \sigma)(X, Y) = 0$. Consequently applying formula (4.49)

$$(d_H \sigma)(X, Y) = (d\sigma)(X, Y) = -\sigma([X, Y]) = -P$$

and this implies that

$$d_H \sigma = -P \sigma \wedge \tau.$$ 

A similar calculation shows that

$$d_H \tau = -Q \sigma \wedge \tau.$$ 

The remaining structure equations are obtained by induction using the formula (4.43).

In [8], the problem of classifying the higher degree contact-valued conservation laws for the equation $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ was studied. It was proved that if the two sequences of Laplace invariants $H_i$ and $K_j$ do not terminate, then this equation admits no such conserved forms.
This result follows, without much difficulty, from a normal form theorem for such conservation laws and from the $d_H$ structure equations. But it is easy to see that Darboux integrable equations always admit infinitely many higher degree form-valued conservation laws and therefore we concluded that the vanishing of the generalized Laplace invariants is necessary for Darboux integrability. To prove that the vanishing of the Laplace invariants is sufficient, one must establish the existence of the pairs of $X$ and $Y$ invariant functions and to this end one must have more information about $d$ structure equations for the Laplace-adapted coframe.

Since we already have the $d_H$ structure equations, we turn to the computation of the $d_V$ structure equations. We say that a contact form $\alpha$ is of adapted order $k$ if it belongs to the span of the contact forms $\Theta, \eta_i,$ and $\xi_i, 1 \leq i \leq k$. If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are 1 forms on $\mathcal{R}^\infty$, then $\Omega^*(\alpha_1, \alpha_2, \ldots, \alpha_k)$ denotes the exterior algebra generated over $C^\infty(\mathcal{R}^\infty)$ (the ring of all $C^\infty$ functions on $\mathcal{R}^\infty$) by these forms. We begin with the structure equations for the horizontal forms $\sigma$ and $\tau$.

**Proposition 4.2.** The $d_V$ structure equations for the horizontal forms $\sigma$ and $\tau$ are

$$
d_V\sigma = \sigma \wedge \mu_1 + \tau \wedge \alpha \quad \text{and} \quad d_V\tau = \sigma \wedge \beta + \tau \wedge \mu_2, \tag{4.50}
$$

where $\alpha, \beta, \mu_1, \mu_2$ are contact 1-forms. The adapted order of $\alpha$ and $\beta$ is $\leq 2$. Moreover, the following relations hold:

$$
d_VP = X(\alpha) - Y(\mu_1) + P\mu_2 - Q\alpha, \quad d_VQ = X(\mu_2) - Y(\beta) + Q\mu_1 - P\beta, \tag{4.51}
$$

and

$$
d_V\beta = \beta \wedge (\mu_2 - \mu_1), \quad d_V\mu_2 = \alpha \wedge \beta = -d_V\mu_1, \quad d_V\alpha = \alpha \wedge (\mu_1 - \mu_2). \tag{4.52}
$$

**Proof.** The existence of contact 1 forms $\alpha, \beta, \mu_1, \mu_2$ satisfying (4.50) is immediate from the definition of $d_V$ and the fact that $\sigma$ and $\tau$ are horizontal forms. That the forms $\alpha$ and $\beta$ have adapted order 2 now follows from the explicit formulas for $\sigma$ and $\tau$ in terms of the characteristic roots $(m_x, m_y)$ and $(n_x, n_y)$ and the fact that these roots may be chosen to be second order functions on $\mathcal{R}^\infty$.

Equations (4.51) follow from the integrability conditions arising from (4.44a) and (4.50). To derive (4.52) we apply $d_V$ to (4.50).

**Remark.** 4.3 There exists a coframe such that

$$
\mu_1 = -\alpha \quad \text{and} \quad \mu_2 = -\beta.
$$

To prove this, first note that locally we may choose coordinates so that the equation manifold is given by

$$
u_{xx} + f(x, y, u, u_x, u_y, u_{xy}, u_{yy}) = 0. \tag{4.53}$$
This can be done always since, if \( \partial F/\partial u_{xx} = 0 \), then either interchanging \( x \) and \( y \) or using the point transformation

\[
x' = x + y, \quad y' = x - y, \quad u' = u,
\]
equation (4.1) transforms into an equation of the form (4.53). We choose the characteristic vector fields to be

\[
X = D_x + mD_y \quad \text{and} \quad Y = D_x + nD_y,
\]
where \( m \) and \( n \) are the two distinct, real, second-order functions that satisfy the characteristic equation

\[
\lambda^2 - \frac{\partial f}{\partial u_{xy}} \lambda + \frac{\partial f}{\partial u_{yy}} = 0.
\]

We choose \( \Theta = \theta = du - u_x dx - u_y dy. \) Then

\[
\sigma = \frac{1}{n-m} (ndx - dy) \quad \text{and} \quad \tau = \frac{1}{n-m} (-mdx + dy)
\]
and thus we find

\[
d_V \sigma = \sigma \wedge \frac{1}{n-m} d_V m + \tau \wedge \frac{1}{n-m} d_V n \quad \text{and} \quad d_V \tau = \sigma \wedge \frac{1}{n-m} d_V n + \tau \wedge \frac{1}{n-m} d_V m
\]
and so

\[
\alpha = \frac{1}{n-m} d_V n \quad \text{and} \quad \beta = \frac{1}{n-m} d_V m.
\]

The structure equations (4.50), (4.51) and (4.52) simplify to

\[
d_V \sigma = -\sigma \wedge \beta + \tau \wedge \alpha, \quad d_V \tau = \sigma \wedge \beta - \tau \wedge \alpha
\]
and

\[
d_V P = X(\alpha) + Y(\beta) - (P + Q)\alpha, \quad d_V Q = -X(\alpha) - Y(\beta) - (Q + P)\beta
\]
and

\[
d_V \alpha = \beta \wedge \alpha = -d_V \beta.
\]
Proposition 4.4. The coefficient of $\xi_2$ in $\alpha$ and the coefficient of $\eta_2$ in $\beta$ are relative contact invariants, denoted by $M_\sigma$ and $M_\tau$. Under the classical contact transformation $\Phi: \mathcal{R}^\infty \to \mathcal{R'}^\infty$ using (4.33), $M_\sigma$ and $M_\tau$ are related by

$$
\Phi^*(M'_\sigma) = \frac{n}{m^2 l} M_\sigma \quad \text{and} \quad \Phi^*(M'_\tau) = \frac{m}{n^2 l} M_\tau.
$$

(4.62)

Proof. Let as usual $\{\sigma', \tau', \Theta', \xi'_1, \eta'_1, \ldots\}$ be the Laplace coframe for $\mathcal{R'}^\infty$. Note that $M_\sigma$ is the coefficient of $\tau \wedge \xi_2$ in $d\sigma$ and $M'_\sigma$ is the coefficient of $\tau' \wedge \xi'_2$ in $d\sigma'$. Using the $d_V$ and $d_H$ structure equations for the Laplace-adapted coframe and equations (4.39) and (4.50), we obtain

$$
\Phi^*(d\sigma') = \Phi^*(d_H \sigma + d_V \sigma) = (\Phi^*(-P\sigma \wedge \tau + \sigma \wedge \mu_1 + \tau \wedge \alpha))
$$

$$
\equiv \frac{m^2 l}{n} \Phi^*(M'_\sigma) \tau \wedge \xi_2 \mod \{\sigma, \Theta, \xi_1, \eta_1, \eta_2\},
$$

and

$$
d(\Phi^*(\sigma')) = d\left(\frac{1}{m} - \sigma + a\Theta + b\xi_1 + c\eta_1\right) \equiv \frac{1}{m} M_\sigma \tau \wedge \xi_2 \mod \{\sigma, \Theta, \xi_1, \eta_1, \eta_2\}.
$$

Matching the coefficients of the two last equations, we arrive at the equation (4.62) for $M_\sigma$. A similar proof works for $M_\tau$.

We remark that in chapter 10 we shall prove that the vanishing of $M_\sigma$ and $M_\tau$ are necessary and sufficient conditions for the equation to be of Monge-Ampère type.

Since the Laplace-adapted coframe is defined inductively in terms of the characteristic vector fields, we shall need the commutation rules for $X$, $Y$ and $d_V$.

Proposition 4.5. Let $\alpha$, $\beta$, $\mu_1$ and $\mu_2$ be given by (4.50). Then for any form $\omega$ on $\mathcal{R}^\infty$,

$$
d_V[X(\omega)] - X(d_V \omega) = \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega)
$$

(4.63)

and

$$
d_V[Y(\omega)] - Y(d_V \omega) = \alpha \wedge X(\omega) + \mu_2 \wedge Y(\omega).
$$

(4.64)

Proof. On the one hand we have, by (4.43), that

$$
d_H[d_V(\omega)] = \sigma \wedge X(d_V \omega) + \tau \wedge Y(d_V \omega)
$$

(4.65)
while, on the other hand, from (4.50), we compute that
\[
dv[d_H(\omega)] = dv[\sigma \wedge X(\omega) + \tau \wedge Y(\omega)]
\]
\[
= \sigma \wedge \{\mu_1 \wedge X(\omega) - dv[X(\omega)] + \beta \wedge Y(\omega)\}
\]
\[
+ \tau \wedge \{\alpha \wedge X(\omega) + \mu_2 \wedge Y(\omega) - dv[Y(\omega)]\}.
\]

(4.66)

A comparison of (4.65) and (4.66) leads to (4.63) and (4.64).

Our first approximation to the $dv$ structure equations follows from Proposition 4.5 by induction.

**Proposition 4.6.** The Laplace-adapted coframe satisfies the following congruences:

\[
dv/\Theta \equiv 0 \mod \{\Theta\};
\]

(4.67)

\[
dv/\eta_i \equiv 0 \mod \{\xi_1, \Theta, \eta_1, \ldots, \eta_i\} \quad i \geq 1; \quad \text{and}
\]

(4.68)

\[
dv/\xi_i \equiv 0 \mod \{\eta_1, \Theta, \xi_1, \ldots, \xi_i\} \quad i \geq 1.
\]

(4.69)

**Proof.** Equation (4.67) and equations (4.68) and (4.69), for $i = 1$, are established from (4.13), (4.22), and Proposition 4.5. We prove (4.68) for $i > 1$ by induction. Note that the $d_H$ structure equations given in Proposition 4.1 imply that $X(\eta_j) \equiv 0, \mod \{\Theta, \eta_1, \eta_2, \ldots, \eta_j\}$ and that $Y(\eta_j) \equiv 0, \mod \{\eta_1, \eta_2, \ldots, \eta_{j+1}\}$ for all $j \geq 1$. We also note that $Y(\Theta) = \eta_1 - A\Theta$ and $Y(\xi_1) = -D\xi_1 + K_0\Theta$. Assume that (4.68) is true for all $i \leq j$. Then it is a simple matter to check that $Y(dv/\eta_j) \equiv 0, \mod \{\xi_1, \Theta, \eta_1, \ldots, \eta_{j+1}\}$. Then, for $j \leq p$, where $H_p = 0$, we use (4.19) and (4.64) to compute

\[
dv/\eta_{j+1} = dv(Y(\eta_j) + A_j\eta_j)
\]

\[
= \alpha \wedge X(\eta_j) + \mu_2 \wedge Y(\eta_j) + dv(A_j) \wedge \eta_j + A_jdv(\eta_j)
\]

\[
\equiv 0 \mod \{\xi_1, \Theta, \eta_1, \ldots, \eta_{j+1}\}.
\]

For $j \geq p + 1$, these same computations, but with $A_j = 0$, remain valid. Equation (4.69) is similarly established.

The contact forms that are invariant or relative invariant under the projected Lie derivative of the characteristic vector fields play a distinguished role in the subsequent theory. Let $T$ be a total vector field. We say that a form $\omega \in \Omega^*(\Theta, \eta_1, \xi_1, \ldots)$ is a relative $X$ invariant contact form if $X(\omega) = \lambda \omega$. We say that the form $\omega$ is $X$ invariant contact form if $X(\omega) = 0$. 

Proposition 4.7. Let \( \mathcal{R} \) and \( \mathcal{R}' \) be two second-order scalar hyperbolic partial differential equations in the plane. Let \( \Phi : \mathcal{R}^\infty \rightarrow \mathcal{R}'^\infty \) be an invertible classical contact transformation with

\[
\Phi^*(X') = mX, \quad \Phi^*(Y') = mY \quad \text{and} \quad \Phi^*(\Theta') = l\Theta.
\]

Let \( \omega' \in \Omega^*(\Theta', \eta'_1, \xi'_1, \ldots) \) be a contact form on \( \mathcal{R}'^\infty \) and let \( \omega = \Phi^*(\omega') \). Then \( \omega' \) is \( X' \) invariant if and only if \( \omega \) is \( X \) invariant, that is

\[
X'(\omega') = 0 \quad \text{if and only if} \quad X(\omega) = 0.
\]

Proof. Since \( \Phi \) is invertible we need to prove the theorem only one direction. If \( X'(\omega') = 0 \), then using the structure equations for the Laplace-adapted coframe we obtain

\[
d\omega = dH\omega' + dV\omega' = \tau' \wedge Y'(\omega') + dV\omega' \in \Omega^*(\tau', \Theta', \eta'_1, \xi'_1, \ldots).
\]

Due to the transformation formulas (4.36) - (4.40), we have

\[
d\omega = d\Phi^*(\omega') = \Phi^*(d\omega') \in \Phi^*(\Omega^1(\tau', \Theta', \eta'_1, \xi'_1, \ldots)) = \Omega^*(\tau, \Theta, \eta_1, \xi_1, \ldots)
\]

Since

\[
d\omega = dH\omega + dV\omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega) + dV\omega,
\]

then

\[
X(\omega) = 0.
\]

In [8] it is shown that if \( \omega \in \Omega^*(\Theta, \eta_1, \xi_1, \ldots) \) is a relative \( X \) invariant contact form and \( H_\rho = 0 \), then \( \omega \) is in the exterior algebra \( \Omega^*(\eta_{\rho+1}, \eta_{\rho+2}, \ldots) \). The following generalization of this result will enable use to refine the crude structure equations of Proposition 4.6 just to the degree necessary to prove that the vanishing of the Laplace invariants is the only obstruction to Darboux integrability.

Proposition 4.8. Let \( \rho \) be a nonnegative integer, let \( \omega \in \Omega^*(\Theta, \eta_1, \xi_1, \ldots) \). Suppose \( H_\rho = 0 \) and

\[
X(\omega) \equiv \lambda \omega \mod \{ \eta_{\rho+1}, \eta_{\rho+2}, \ldots, \eta_{\rho+\ell} \}. \quad (4.70)
\]

Then \( \omega \) decomposes uniquely into a sum

\[
\omega = \omega_1 + \omega_2, \quad (4.71)
\]
where \( \omega_1 \equiv 0 \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \} \) and \( \omega_2 \in \Omega^*(\eta_{p+l+1}, \eta_{p+l+2}, \ldots) \).

Similarly, suppose \( K_q = 0 \) and

\[
Y(\omega) \equiv \lambda \omega \quad \mod \{ \xi_{q+1}, \xi_{q+2}, \ldots, \xi_{q+l} \}.
\] (4.72)

Then \( \omega \) decomposes uniquely into a sum

\[
\omega = \omega_1 + \omega_2,
\] (4.73)

where \( \omega_1 \equiv 0 \mod \{ \xi_{q+1}, \xi_{q+2}, \ldots, \xi_{q+l} \} \) and \( \omega_2 \in \Omega^*(\xi_{q+l+1}, \xi_{q+l+2}, \ldots) \).

Proof. We prove only the first part of the statement. The proof of the second part is analogous. We begin with the observation that if a form

\[
\omega_1 \equiv 0 \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}
\] (4.74)

then it is always the case that

\[
X(\omega_1) \equiv \lambda \omega_1 \quad \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}.
\] (4.75)

Now decompose \( \omega \) uniquely into the form \( \omega = \omega_1 + \omega_2 \), where \( \omega_1 \) satisfies (4.74) and

\[
\omega_2 \in \Omega^*(\xi_k, \xi_{k-1}, \ldots, \xi_1, \Theta, \eta_1, \ldots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \ldots).
\]

Write

\[
\omega_2 = \xi_k \land \gamma + \epsilon,
\] (4.76)

where \( \gamma, \epsilon \in \Omega^*(\xi_k, \ldots, \xi_1, \Theta, \eta_1, \ldots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \ldots) \). Then by equations (4.70) and (4.75) it follows that

\[
X(\omega_2) \equiv \lambda \omega_2 \quad \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}.
\]

We compute, using the \( d_H \) structure equations and (4.76)

\[
X(\omega_2) \equiv \xi_{k+1} \land \gamma + \delta \quad \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \},
\]

where \( \delta \in \Omega^*(\xi_{k-1}, \ldots, \Theta, \eta_1, \ldots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \ldots) \). From the congruence (4.70) we now deduce that \( \gamma \equiv 0 \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \} \) and therefore \( \gamma = 0 \). This proves that

\[
\omega_2 \in \Omega^*(\xi_{k-1}, \xi_{k-2}, \ldots, \xi_1, \Theta, \eta_1, \ldots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \ldots).
\]
We can repeat this argument to establish that
\[ \omega_2 \in \Omega^*(\eta_1, \ldots, \eta_p, \eta_{p+1}, \eta_{p+2}, \ldots). \]

This proves the theorem for \( p = 0 \).

Assume \( p \geq 1 \) and now write \( \omega_2 = \gamma + \epsilon \), where \( \gamma, \epsilon \in \Omega^*(\eta_2, \ldots, \eta_p, \eta_{p+1}, \eta_{p+2}, \ldots) \).

This time we compute
\[ X(\omega_2) \equiv H_0 \delta \gamma + \delta \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_p \}, \]
where \( \delta \in \Omega^*(\eta_1, \ldots, \eta_p, \eta_{p+1}, \eta_{p+2}, \ldots) \). Since \( H_0 \neq 0 \), we can conclude from this congruence that \( \gamma = 0 \) and hence \( \omega_2 \in \Omega^*(\eta_2, \ldots, \eta_p, \eta_{p+1}, \eta_{p+2}, \ldots) \). We can repeat this argument until \( H_p = 0 \).

We combine Propositions 4.6 and 4.8 to arrive at the following structure equations.

**Theorem 4.9.** If \( H_p = 0 \), then there is a unique form
\[ \Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \ldots, \eta_p) \] (4.77)
such that for some contact 1-form \( \tilde{\eta} \),
\[ d_{\Upsilon} \eta_{p+1} = \eta_{p+2} \Upsilon + \eta_{p+1} \tilde{\eta} \] (4.78)

The form \( \Upsilon \) satisfies
\[ X(\Upsilon) \equiv -Q \Upsilon + \beta \mod \{ \eta_{p+1}, \eta_{p+2} \} \] (4.79)
and
\[ d_{\Upsilon} \Upsilon = \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \mod \{ \eta_{p+1}, \eta_{p+2} \}. \] (4.80)

The forms \( \eta_{p+i} \), \( i \geq 1 \), satisfy the \( d_{\Upsilon} \) structure equations
\[ d_{\Upsilon} \eta_{p+i} \equiv \eta_{p+i+1} \wedge \Upsilon \mod \{ \eta_{p+1}, \ldots, \eta_{p+i} \}. \] (4.81)

Similarly, if \( K_q = 0 \), then there is a unique form
\[ \Xi \in \Omega^1(\eta_1, \Theta, \xi_1, \ldots, \xi_q) \] (4.82)
such that for some contact 1-form \( \tilde{\xi} \).
The form $\Xi$ satisfies
\[ Y(\Xi) \equiv P\Xi + \alpha \mod \{ \xi_{q+1}, \xi_{q+2} \} \quad (4.84) \]
and
\[ d_V \Xi \equiv \Xi \wedge [\mu_1 - X(\Xi)] \mod \{ \xi_{q+1}, \xi_{q+2} \}. \quad (4.85) \]

The forms $\xi_{q+i}$, $i \geq 1$, satisfy the $d_V$ structure equations
\[ d_V \xi_{q+i} \equiv \xi_{q+i+1} \wedge \Xi \mod \{ \xi_{q+1}, \ldots, \xi_{q+i} \}. \quad (4.86) \]

Proof. As usual we shall prove only the first part of the statement. The proof of the second part is similar. Since
\[
X(d_V \eta_{p+1}) = d_V(X(\eta_{p+1})) - \mu_1 \wedge X(\eta_{p+1}) - \beta \wedge Y(\eta_{p+1})
\]
\[ = d_V(-B_p \eta_{p+1}) - \mu_1 \wedge (-B_p \eta_{p+1}) - \beta \wedge \eta_{p+2} \]
\[ = -B_p d_V \eta_{p+1} \mod \{ \eta_{p+1}, \eta_{p+2} \}, \]
we can deduce from Proposition 4.8 that
\[ d_V \eta_{p+1} = \eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \tilde{\eta} + \omega, \]
where $\omega \in \Omega^2(\eta_{p+3}, \eta_{p+4}, \ldots)$. But then, by Proposition 4.6, we must have $\omega = 0$ and $\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \ldots, \eta_{p+1})$. Any terms in $\Upsilon$ involving $\eta_{p+1}$ can be absorbed into $\tilde{\eta}$. This proves (4.78). The uniqueness of $\Upsilon$ is immediate.

From (4.78) and the $d_H$ structure equations, we compute
\[
d_H(d_V \eta_{p+1}) \equiv (d_H \eta_{p+2}) \wedge \Upsilon - \eta_{p+2} \wedge d_H \Upsilon + d_H \eta_{p+1} \wedge \tilde{\eta} \mod \{ \eta_{p+1} \}
\]
\[ \equiv \sigma \wedge [(Q - B_p) \eta_{p+2} \wedge \Upsilon + \eta_{p+2} \wedge X(\Upsilon)]
\]
\[ + \tau \wedge [\eta_{p+2} \wedge Y(\Upsilon) + \eta_{p+3} \wedge \Upsilon + \eta_{p+2} \wedge \tilde{\eta}] \mod \{ \eta_{p+1} \}
\]
while from (4.45c) and (4.78) we obtain
\[
d_V(d_H \eta_{p+1}) = d_V(-B_p \sigma \wedge \eta_{p+1} + \tau \wedge \eta_{p+2})
\]
\[ \equiv \sigma \wedge [B_p \eta_{p+2} \wedge \Upsilon + \beta \wedge \eta_{p+2}] + \tau \wedge [\mu_2 \wedge \eta_{p+2} - d_V \eta_{p+2}] \mod \{ \eta_{p+1} \}. \]
The comparison of these two equations implies that

\[ \eta_{p+2} \wedge [QT + X(\Upsilon) - \beta] \equiv 0 \mod \{ \eta_{p+1} \} \]

and (4.79) follows. To prove (4.80), we simply take \( d_V \) of (4.78)

\[
0 = d_V (d_V \eta_{p+1}) = d_V (\eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \tilde{\eta})
\]

\[
= d_V \eta_{p+2} \wedge \Upsilon - \eta_{p+2} \wedge d_V \Upsilon + d_V \eta_{p+1} \wedge \tilde{\eta} - \eta_{p+1} \wedge d_V \tilde{\eta}.
\]

Using again (4.78), we obtain

\[
0 = d_V \eta_{p+2} \wedge \Upsilon - \eta_{p+2} \wedge d_V \Upsilon + (\eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \tilde{\eta}) \wedge \tilde{\eta} - \eta_{p+1} \wedge d_V \tilde{\eta}
\]

\[
= d_V \eta_{p+2} \wedge \Upsilon - \eta_{p+2} \wedge d_V \Upsilon + \eta_{p+2} \wedge \Upsilon \wedge \tilde{\eta} - \eta_{p+1} \wedge d_V \tilde{\eta}.
\]

Now use the fact that

\[
d_V \eta_{p+2} = d_V [Y(\eta_{p+1})]
\]

\[
= Y[d_V (\eta_{p+1})] + \alpha \wedge X(\eta_{p+1}) + \mu_2 \wedge Y(\eta_{p+1})
\]

\[
\equiv Y(\eta_{p+1} \wedge \Upsilon + \eta_{p+1} \wedge \tilde{\eta}) + \mu_2 \wedge \eta_{p+1} \mod \{ \eta_{p+1} \}
\]

\[
\equiv \eta_{p+3} \wedge \Upsilon + \eta_{p+2} \wedge Y(\Upsilon) + \eta_{p+2} \wedge \tilde{\eta} + \mu_2 \wedge \eta_{p+2} \mod \{ \eta_{p+1} \}
\]

to conclude that

\[
\eta_{p+2} \wedge [(Y(\Upsilon) - \mu_2) \wedge \Upsilon - d_V \Upsilon] \equiv 0 \mod \{ \eta_{p+1} \}.
\]

From this last congruence (4.80) follows.

We now prove (4.81). First notice that for \( i = 1 \) the equation (4.81) becomes the equation (4.78).

We now proceed by induction. Assume

\[
d_V \eta_{p+i} = \eta_{p+i+1} \wedge \Upsilon + \sum_{j=1}^{i} \eta_{p+j} \wedge \omega_j
\]

(4.87)
for some contact forms $\omega_j$. Using the $d_H$ structure equations and (4.87), we conclude

$$d_H d_V \eta_{p+i} = d_H [\eta_{p+i+1} \wedge \Upsilon + \sum_{j=1}^{i} \eta_{p+j} \wedge \omega_j]$$

$$= \sigma \wedge X(\eta_{p+i+1}) \wedge \Upsilon + \tau \wedge \eta_{p+i+2} \wedge \Upsilon - \eta_{p+i+1} \wedge d_H \Upsilon$$

$$+ \sum_{j=1}^{i} [\sigma \wedge X(\eta_{p+j}) \wedge \omega_j + \tau \wedge \eta_{p+j+1} \wedge \omega_j - \eta_{p+j} \wedge d_H \omega_j]$$

$$\equiv \tau \wedge \eta_{p+i+2} \wedge \Upsilon \mod \{ \eta_{p+1}, \ldots, \eta_{p+i+1} \}.$$

On the other hand,

$$d_V d_H \eta_{p+i} = d_V (\sigma \wedge X(\eta_{p+i}) + \tau \wedge \eta_{p+i+1})$$

$$= d_V \sigma \wedge X(\eta_{p+i}) - \sigma \wedge d_V \eta_{p+i} + d_V \tau \wedge \eta_{p+i+1} - \tau \wedge d_V \eta_{p+i+1}$$

$$= d_V \sigma \wedge X(\eta_{p+i}) - \sigma \wedge (\eta_{p+i+1} \wedge \Upsilon + \sum_{j=1}^{i} \eta_{p+j} \wedge \omega_j) + d_V \tau \wedge \eta_{p+i+1} - \tau \wedge d_V \eta_{p+i+1}$$

$$\equiv -\tau \wedge d_V \eta_{p+i+1} \mod \{ \eta_{p+1}, \ldots, \eta_{p+i+1} \}.$$

Adding these last two congruences, we get

$$\tau \wedge (\eta_{p+i+2} \wedge \Upsilon - d_V \eta_{p+i+1}) \equiv 0 \mod \{ \eta_{p+1}, \ldots, \eta_{p+i+1} \}$$

and so (4.81) follows. This ends the induction.

**Theorem 4.10.** If $H_p = 0$, then form

$$\hat{\tau} = \tau - \Upsilon$$

satisfies

$$d\eta_{p+1} = \hat{\tau} \wedge \eta_{p+2} + \eta_{p+1} \wedge \tilde{\eta}_0 \quad (4.88)$$

for some 1-form $\tilde{\eta}_0$. Under the classical contact transformation $\Phi : R^\infty \to R^\infty$ with (4.33) the form $\hat{\tau}$ transforms according to
\[ \hat{\gamma}' = \frac{1}{m} \hat{\gamma} \]  

(4.89)

provided \( p \geq 1 \).

Similarly, if \( K_q = 0 \), then the form

\[ \hat{\sigma} = \sigma - \Xi \]

satisfies

\[ d\xi_{q+1} = \hat{\sigma} \wedge \xi_{q+2} + \xi_{q+1} \wedge \hat{\xi}_0 \]  

(4.90)

for some 1-form \( \hat{\xi}_0 \). Under the classical contact transformation \( \Phi : \mathcal{R}^\infty \to \mathcal{R}'^\infty \) with (4.33) the form \( \hat{\sigma} \) transforms according to

\[ \hat{\sigma}' = \frac{1}{m} \hat{\sigma} \]  

(4.91)

provided \( q \geq 1 \).

Proof. Again we prove only one half of the statement for \( \hat{\tau} \). Recall from (4.45c) that

\[ d_H \eta_{p+1} = \sigma \wedge (-B_p \eta_{p+1}) + \tau \wedge \eta_{p+2}. \]

Using (4.78), we have

\[ d\eta'_{p+1} = d_V \eta_{p+1} + d_H \eta_{p+1} = (\tau - \hat{\tau}) \wedge \eta_{p+2} + \eta_{p+1} \wedge (B_p \sigma + \hat{\eta}) \]

for some contact 1-form \( \hat{\eta} \), and (4.88) follows.

To prove the transformation formulas we use (4.36), (4.37), and (4.38), to compute

\[ \Phi^*(d\eta'_{p+1}) = \Phi^*(\hat{\tau}' \wedge \eta'_{p+2} + \eta'_{p+1} \wedge \hat{\eta}') = \Phi^*(\hat{\tau}) \wedge n^{p+2} l_H \eta_{p+2} + n^{p+1} l_H \eta_{p+1} \wedge \Phi^*(\hat{\eta}). \]

According to (4.35) and (4.38) we deduce

\[ \Phi^*(\hat{\tau}) \in \Omega^1(\tau, \xi_1, \Theta, \eta_1, \ldots, \eta_p). \]

Equation (4.35) now yields

\[ d(\Phi^*(\eta'_{p+1})) = d(n^{p+1} l_H \eta_{p+1}) = n^{p+1} l_H \tau \wedge \eta_{p+2} + \eta_{p+1} \wedge (\hat{\eta} - d(n^{p+1})). \]
From the comparison of the last two equations the transformation rule (4.89) follows.

REMARK 4.11. Write

\[ \alpha = d_2 \eta_2 + d_1 \eta_1 + c_0 \Theta + c_1 \xi_1 + M_\sigma \xi_2 \]  
and

\[ \beta = b_2 \xi_2 + b_1 \xi_1 + c_0 \Theta + c_1 \eta_1 + M_\tau \eta_2, \]

as in Propositions 4.2 and 4.4. In chapter 9 we prove that \( b_2 = d_2 \). The contact form \( \Upsilon \) is given explicitly by

\[ \Upsilon = b_2 \xi_1 + \tilde{F}_0 \Theta + \sum_{i=1}^{p} \tilde{F}_i \eta_i, \]  
where

\[ \tilde{F}_0 = -X(b_2) + (E_1 - Q)b_2 + b_1, \quad \tilde{F}_{i+1} = -\frac{1}{H_i}(X(\tilde{F}_i) - B_i \tilde{F}_i - c_i) \quad \text{for} \quad i \geq 0 \]

and where \( c_2 = M_\tau \) and \( c_i = 0 \) for \( i \geq 3 \).

Similarly, \( \Xi \) is given explicitly by

\[ \Xi = d_2 \eta_1 + \tilde{G}_0 \Theta + \sum_{j=1}^{q} \tilde{G}_j \xi_j, \]  
where

\[ \tilde{G}_0 = -X(d_2) + (A_1 + P)D - 2 + d_1, \quad \tilde{G}_{j+1} = -\frac{1}{K_j}(Y(\tilde{G}_j) - D_j \tilde{G}_j - e_j) \quad \text{for} \quad j \geq 0 \]

and where \( c_2 = M_\sigma \) and \( e_j = 0 \) for \( j \geq 3 \). The proof of this will be provided in chapter 7.
CHAPTER 5
CRITERIA FOR DARBOUX INTEGRABILITY

If \( I \) and \( J \) are functions of order \( k \) on \( \mathcal{R}^\infty \) such that \( X(I) = 0 \) and \( Y(J) = 0 \), then \( dI \in \mathcal{C}_k(X) \) and \( dJ \in \mathcal{C}_k(Y) \), where

\[
\mathcal{C}_k(X) = \Omega^1(\tau, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k)
\]

and

\[
\mathcal{C}_k(Y) = \Omega^1(\sigma, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k).
\]

We call \( \mathcal{C}_k(X) \) and \( \mathcal{C}_k(Y) \) the \( k \)th-order characteristic Pfaffian systems associated to the second-order hyperbolic equation \( \mathcal{R}^\infty \). The original partial differential equation (4.1) is therefore Darboux integrable (cf. definition 3.1) if for sufficiently large \( k \) the characteristic Pfaffian systems \( \mathcal{C}_k(X) \) and \( \mathcal{C}_k(Y) \) each contain a completely integrable subsystem of dimension \( \geq 2 \). The equation (4.1) is Darboux semi-integrable if either one of the characteristic Pfaffian systems contains an integrable subsystem of dimension \( \geq 2 \).

Theorem 5.1. The Pfaffian systems

\[
\Omega^1(\bar{\tau}, \eta_{p+1}, \ldots, \eta_{p+i}) \quad \text{and} \quad \Omega^1(\bar{\sigma}, \xi_{q+1}, \ldots, \xi_{q+i})
\]

are integrable for \( i \geq 2 \) if \( H_p = 0 \) and \( K_q = 0 \), respectively.

Proof. We will prove the theorem only for \( \Omega^1(\bar{\tau}, \eta_{p+1}, \ldots, \eta_{p+i}) \). The proof for \( \Omega^1(\bar{\sigma}, \xi_{q+1}, \ldots, \xi_{q+i}) \) is similar. This theorem follows directly from the structure equations for the Laplace-adapted coframe and Theorem 4.9.

\[
d(\bar{\tau}) = d(\tau - \Upsilon) = d_H \tau + d_V \tau - d_H \Upsilon - d_V \Upsilon
\]

\[
= -Q\sigma \land \tau + \sigma \land \beta + \tau \land \mu_2 - \sigma \land X(\Upsilon) - \tau \land Y(\Upsilon) - d_V \Upsilon
\]

\[
\equiv -Q\sigma \land \Upsilon + \sigma \land \beta + \Upsilon \mu_2 - \sigma \land X(\Upsilon) - \Upsilon \land Y(\Upsilon) - d_V \Upsilon \quad \text{mod} \{\bar{\tau}\}
\]

\[
\equiv -\sigma \land [X(\Upsilon) + Q\Upsilon - \beta] + \Upsilon \land [\mu_2 - Y(\Upsilon)] - d_V \Upsilon \quad \text{mod} \{\bar{\tau}\}
\]

\[
\equiv 0 \quad \text{mod} \; \Omega^1(\bar{\tau}, \eta_{p+1}, \eta_{p+2}).
\]
By (4.45) and (4.81) for $i \geq 1$

$$d(\eta_{p+i}) = d_H \eta_{p+i} + d_V \eta_{p+i}$$

$$\equiv \sigma \wedge X(\eta_{p+i}) + \tau \wedge \eta_{p+i+1} + \eta_{p+i+1} \wedge \Upsilon \mod \{\eta_{p+1}, \ldots, \eta_{p+i}\}$$

$$\equiv (\tau - \Upsilon) \wedge \eta_{p+i+1} \mod \{\eta_{p+1}, \ldots, \eta_{p+i}\}$$

$$\equiv \hat{\tau} \wedge \eta_{p+i+1} \mod \{\eta_{p+1}, \ldots, \eta_{p+i}\}$$

$$\equiv 0 \mod \Omega^1(\hat{\tau}, \eta_{p+1}, \ldots, \eta_{p+i}).$$

The proof for $\Omega^1(\hat{\tau}, \xi_{q+1}, \ldots, \xi_{q+i})$ is similar.

The results of [8] and Theorem 5.1 combine to establish the following corollary.

**Corollary 5.2.** A second-order hyperbolic scalar equation in the plane,

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

is Darboux integrable if and only if for some integers $p \geq 0$ and $q \geq 0$, the generalized Laplace invariants $H_p$ and $K_q$ vanish. In fact, there are always three independent $X$ invariant functions of order at most $p + 2$ and three independent $Y$ invariant functions of order at most $q + 2$.

As we mentioned in the introduction, Sokolov and Zhiber [35] proved that the vanishing of the Laplace invariants $H_p$ and $K_q$ is sufficient for the Darboux integrability of the equation

$$u_{xy} = f(x, y, u, u_x, u_y). \quad (5.1)$$

However, their method does not appear to furnish the invariants of lowest possible order so that even for equations of the type (5.1), Corollary 5.2 is an improvement over their results.

We now recall that the space $H^{1,s}(\mathcal{R}^\infty)$ of type $(1, s)$ conservation laws for the equation $F = 0$ is the space of $d_H$ closed $(1 + s)$-forms of the type

$$\omega = M \wedge \sigma + N \wedge \tau,$$

where $M$ and $N$ are contact $s$ forms in $\Omega^*(\Theta, \eta_1, \xi_1, \ldots)$, modulo $d_H$ exact forms of this type. In [8] it was shown that if the Laplace invariants $H_p$ and $K_q$ never vanish, then $H^{1,s}(\mathcal{R}^\infty) = 0$ for all $s \geq 3$. It was also shown that Darboux semi-integrable equations possess infinitely many type $(1, s)$ conservation laws for all $s \geq 0$. 
Corollary 5.3. A second-order hyperbolic scalar equation in the plane

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \]

is Darboux semi-integrable if and only if \( H_1, s(\mathcal{R}^\infty) \neq 0 \) for some \( s \geq 3 \).

This last corollary satisfactorily resolves the problem posed by Tsujishita [36], namely, to give a criteria for Darboux integrability of scalar second-order equations in the plane in terms of the cohomology of the associated variational bicomplex.
CHAPTER 6
GOURSAT'S CLASSIFICATION

A considerable portion of Goursat's analysis of the method of Darboux [26] (pp. 133-171) is devoted to a complete classification of the functionally independent $X$ and $Y$ invariant functions at any given order. It is a simple matter for us to refine the congruences (4.79) and (4.80) and thereby re-derive Goursat's results. Moreover, we are able to give explicit formulas for the maximal rank, completely integral subsystems $C_k^{(\infty)}(X)$ and $C_k^{(\infty)}(Y)$ which, as a practical matter, may be quite useful in the implementation of Darboux's method.

To this end we recall that the form $\beta$, defined by (4.50), takes the form (4.93). The function $M_\gamma$ is a relative invariant under contact transforms and will play a central role in what follows. As a first step we sharpen the equation (4.79).

**Proposition 6.1.** Let $H_p = 0$.

(i) If $p \geq 2$, then $X(\mathbf{\gamma}) + QT - \beta = 0$.

(ii) If $p = 1$, then $X(\mathbf{\gamma}) + QT - \beta + M_\gamma \eta_2 = 0$.

(iii) If $p = 0$, then $X(\mathbf{\gamma}) + QT - \beta + c_1 \eta_1 + M_\gamma \eta_2 = 0$.

Similarly let $K_q = 0$.

(iv) If $q \geq 2$, then $Y(\mathbf{\xi}) - P \mathbf{\xi} - \alpha = 0$.

(v) If $q = 1$, then $Y(\mathbf{\xi}) - P \mathbf{\xi} - \alpha + M_\sigma \xi_2 = 0$.

(vi) If $q = 0$, then $Y(\mathbf{\xi}) - P \mathbf{\xi} - \alpha + c_1 \xi_1 + M_\sigma \xi_2 = 0$.

**Proof.** We prove only the first part of the theorem. From (4.77) and (4.79), we have that

$$\mathbf{\gamma} \in \Omega^1(\xi_1, \Theta, \eta_1, \eta_2, \ldots, \eta_p)$$

and

$$X(\mathbf{\gamma}) + QT - \beta \equiv 0 \mod \{\eta_{p+1}, \eta_{p+2}\}.$$  \hspace{1cm} (6.1)

Since

$$X(\eta_i) \equiv 0 \mod \{\Theta, \eta_1, \ldots, \eta_i\},$$
it follows that for \( p \geq 2 \) there are no terms involving \( \eta_{p+1} \) and \( \eta_{p+2} \) on the left-hand side of the congruence (6.1) and hence this congruence must be an identity.

For \( p = 1 \), write \( \Upsilon = g_1 \xi_1 + f_0 \theta + f_1 \eta_1 \) and

\[
X(\Upsilon) + Q \Upsilon - \beta = k_2 \eta_2 + k_3 \eta_3.
\]

Then we use the \( d_H \) structure equations and (4.93) to compute

\[
X(\Upsilon) + Q \Upsilon - \beta = (g_1 - b_2)\xi_2 + (X(g_1) - (E_1 - Q)g_1 + f_0 - b_1)\xi_1 + (X(f_0 - Bf_0 + H_0f_1 - c_0)\Theta
\]

\[
+ (X(f_1 - B_1f_1 - c_1)\eta_1 - c_2 \eta_2 = k_2 \eta_2 + k_3 \eta_3
\]

and match coefficients to deduce that \( k_2 = -M_\tau \) and \( k_3 = 0 \). The proof of (iii) is similar, starting with \( \Upsilon = g_1 \xi_1 + f_0 \theta \).

Next we use Proposition 6.1 to improve the congruences (4.80).

**Proposition 6.2.** Let \( H_p = 0 \).

(i) If \( p \geq 2 \), then \( d_v \Upsilon = \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \).

(ii) If \( p = 1 \), then \( d_v \Upsilon \equiv \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \mod \{ \eta_2 \} \).

(iii) If \( p = 1 \) and \( M_\tau = 0 \), then \( d_v \Upsilon = \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \).

(iv) If \( p = 0 \), then \( d_v \Upsilon \equiv \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \mod \{ \eta_1, \eta_2 \} \).

(v) If \( p = 0 \) and \( M_\tau = 0 \), then \( d_v \Upsilon \equiv \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \mod \{ \eta_1 \} \).

Similarly let \( H_q = 0 \).

(vi) If \( q \geq 2 \), then \( d_v \Xi = \Xi \wedge [\mu_1 - X(\Xi)] \).

(vii) If \( q = 1 \), then \( d_v \Xi \equiv \Xi \wedge [\mu_1 - X(\Xi)] \mod \{ \xi_2 \} \).

(viii) If \( q = 1 \) and \( M_\sigma = 0 \), then \( d_v \Xi = \Xi \wedge [\mu_1 - X(\Xi)] \).

(ix) If \( q = 0 \), then \( d_v \Xi \equiv \Xi \wedge [\mu_1 - X(\Xi)] \mod \{ \xi_1, \xi_2 \} \).

(x) If \( q = 0 \) and \( M_\sigma = 0 \), then \( d_v \Xi \equiv \Xi \wedge [\mu_1 - X(\Xi)] \mod \{ \xi_1 \} \).
Proof. Again we prove only the first part of the statement. We know, from the previous theorem, that
\[ X(\Upsilon) = -Q\Upsilon + \beta + \delta, \quad \text{where} \quad \delta = k_1\eta_1 + k_2\eta_2 \] (6.2)
and where \( k_1 = k_2 = 0 \) if \( p \geq 2 \) and so on. From the Theorem 4.9 we know that
\[ d_\nu \Upsilon = \Upsilon \wedge [\mu_2 - Y(\Upsilon)] + \lambda_{p+1} \wedge \eta_{p+1} + \lambda_{p+2} \wedge \eta_{p+2}. \] (6.3)
Since \( \Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \ldots, \eta_p) \), it follows immediately from Proposition 4.6 that
\[ d_\nu \Upsilon \equiv 0 \mod \{ \xi_1, \Theta, \eta_1, \ldots, \eta_{p'} \}, \]
where \( p' = \max\{1, p\} \), and therefore
\[ \lambda_{p+1}, \lambda_{p+2} \in \Omega^1(\xi_1, \Theta, \eta_1, \ldots, \eta_{p'}). \] (6.4)
In the following computations we will frequently use Propositions 4.2 and 4.5, equation (4.3), and Theorem 4.9.
Substituting from (6.2) and using Proposition 4.2 we obtain
\[ d_\nu X(\Upsilon) = d_\nu (-Q\Upsilon + \beta + \delta) \]
\[ = [X(\mu_2) - Y(\beta) + Q\mu_1 - P\beta] \wedge \Upsilon - Q[\Upsilon \wedge (\mu_2 - Y(\Upsilon)) + \lambda_{p+1} \wedge \eta_{p+1} + \lambda_{p+2} \wedge \eta_{p+2}] \]
\[ + \beta \wedge (\mu_2 - \Upsilon) + d_\nu \delta. \] (6.5)
Substituting from (6.3), we obtain
\[ X(d_\nu \Upsilon) = X(\Upsilon \wedge (\mu_2 - Y(\Upsilon)) + \lambda_{p+1} \wedge \eta_{p+1} + \lambda_{p+2} \wedge \eta_{p+2}) \]
\[ = X(\Upsilon) \wedge (\mu_2 - Y(\Upsilon)) + \Upsilon \wedge (X(\mu_2) - XY(\Upsilon)) \]
\[ + X(\lambda_{p+1} \wedge \eta_{p+1} - \lambda_{p+1} \wedge B_p\eta_{p+1} + X(\lambda_{p+2}) \wedge \eta_{p+2} \]
\[ + \lambda_{p+2} \wedge [(Q - B_p)\eta_{p+2} + \{\} \wedge \eta_{p+1}]. \] (6.6)
Next compute using (4.3) and (6.2)
\[ XY(\Upsilon) = YX(\Upsilon) + PX(\Upsilon) + QY(\Upsilon) \]
\[ = Y(X(\Upsilon) + Q\Upsilon) - Y(Q)\Upsilon + PX(\Upsilon) \]
\[ = Y(\beta) + Y(\delta) - Y(Q)\Upsilon + PX(\Upsilon). \] (6.7)
Substituting equation (6.7) into (6.6), we get
\[
X(d_Y Y) = X(Y) \wedge (\mu_2 - Y(Y)) + \mu_1 \wedge [X(\mu_2) - X(\beta) - X(\delta) - PX(Y)]
\]
\[
+ X(\lambda_{p+1}) \wedge \eta_{p+1} - \lambda_{p+1} \wedge B_p \eta_{p+1} + X(\lambda_{p+2}) \wedge \eta_{p+2}
\]
\[
+ \lambda_{p+2} \wedge [(Q - B_p) \eta_{p+2} + \{\ast\} \eta_{p+1}].
\] (6.8)

We substitute (6.5) and (6.8) into the commutator identity (4.63) and use (6.2) to conclude
\[
0 = -d_Y X(Y) + X(d_Y Y) + \mu_1 \wedge X(Y) + \beta \wedge Y(Y)
\]
\[
= \eta_{p+2} \wedge [X(\lambda_{p+2}) + (2Q - B_p) \lambda_{p+2}]
\]
\[
+ \eta_{p+1} \wedge [X(\lambda_{p+1}) + (Q - B_p) \lambda_{p+1} + \lambda_{p+2} \{\ast\}] + \Delta,
\] (6.9)

where \( \Delta \) denotes the terms in (4.63) which arise from the term \( \delta \) in (6.2), that is,
\[
\Delta = -d_Y \delta + \delta \wedge [\mu_2 - Y(Y)] - \mu_1 \wedge [Y(\delta) + P\delta] - [QY + \delta] \wedge \mu_1 - \mu_1 \wedge \delta - d_Y Q - X(\mu_2) + Y(\beta) \wedge Y.
\]

By Proposition 4.2, this last expression simplifies to
\[
\Delta = d_Y (\delta) - \delta \wedge [\mu_2 - Y(Y)] - \mu_1 \wedge [Y(\delta) + P\delta] - \mu_1 \wedge \delta
\] (6.10)

Suppose that \( p \geq 2 \). Then, by Proposition 6.1, \( \delta = 0 \), \( \Delta = 0 \) and (6.9) therefore implies that
\[
X(\lambda_{p+2}) + (2Q - B_p) \lambda_{p+2} \equiv 0 \mod \{\eta_{p+1}, \eta_{p+2}\}.
\]

We combine Proposition 4.8 with (6.4) to deduce that \( \lambda_{p+2} = 0 \). We return to (6.9) and repeat this argument to deduce that \( \lambda_{p+1} = 0 \).

The same computations apply to prove (iii) since in this case \( \delta = 0 \). In case (ii), we have that \( \delta = M_\gamma \eta_2 \) and it is a simple matter to show that \( \Delta \equiv 0 \mod \{\eta_2\} \) and we can proceed as before.

For \( p = 0 \) and \( M_\gamma = 0 \) we have that \( \delta = c_1 \eta_1 \) and now it happens that \( \Delta \equiv 0 \mod \{\eta_1\} \). We proceed as before to deduce that \( \lambda_2 \equiv 0 \mod \{\eta_1\} \).

Goursat’s results can be summarized as follows.
Theorem 6.3. Let $H_p = 0$, and let $l$ be the adapted order of $\tilde{\tau}$. Let $k \geq 1$.

(i) If $p \geq 2$, then
   (a) $C_k^{(\infty)}(X) = \Omega^1(\tilde{\tau}, \eta_{p+1}, \ldots, \eta_k)$ for $k \geq p$,
   (b) $C_k^{(\infty)}(X) = \Omega^1(\tilde{\tau})$ if $l \leq k \leq p$,
   (c) $C_k^{(\infty)}(X) = \{0\}$ if $k < l$.

(ii) If $p = 1$, then
   (a) $C_k^{(\infty)}(X) = \Omega^1(\tilde{\tau}, \eta_2, \ldots, \eta_k)$ for $k \geq 2$,
   (b) $C_k^{(\infty)}(X) = \Omega^1(\tilde{\tau})$ if $M_r = 0$, and
   (c) $C_k^{(\infty)}(X) = \{0\}$ if $M_r \neq 0$.

(iii) If $p = 0$, then
   (a) $C_k^{(\infty)}(X) = \Omega^1(\tilde{\tau}, \eta_1, \ldots, \eta_k)$ for $k \geq 2$,
   (b) $C_k^{(\infty)}(X) = \Omega^1(\tilde{\tau}, \eta_1)$ if $M_r = 0$ and,
   (c) $C_k^{(\infty)}(X) = \{0\}$ if $M_r \neq 0$.

Similarly let $K_q = 0$, and let $l$ be the adapted order of $\tilde{\sigma}$. Let $k \geq 1$.

(iv) If $q \geq 2$, then
   (a) $C_k^{(\infty)}(Y) = \Omega^1(\tilde{\sigma}, \eta_{q+1}, \ldots, \eta_k)$ for $k \geq q$,
   (b) $C_k^{(\infty)}(Y) = \Omega^1(\tilde{\sigma})$ if $l \leq k \leq q$,
   (c) $C_k^{(\infty)}(Y) = \{0\}$ if $k < l$.

(v) If $q = 1$, then
   (a) $C_k^{(\infty)}(Y) = \Omega^1(\tilde{\sigma}, \eta_2, \ldots, \eta_k)$ for $k \geq 2$,
   (b) $C_k^{(\infty)}(Y) = \Omega^1(\tilde{\sigma})$ if $M_\sigma = 0$, and
   (c) $C_k^{(\infty)}(Y) = \{0\}$ if $M_\sigma \neq 0$.

(vi) If $q = 0$, then
   (a) $C_k^{(\infty)}(Y) = \Omega^1(\tilde{\sigma}, \eta_1, \ldots, \eta_k)$ for $k \geq 2$,
   (b) $C_k^{(\infty)}(Y) = \Omega^1(\tilde{\sigma}, \eta_1)$ if $M_\sigma = 0$ and,
   (c) $C_k^{(\infty)}(Y) = \{0\}$ if $M_\sigma \neq 0$. 
Proof. We can use the refined structure equations given in Proposition 6.1 and 6.2 to show that the Pfaffian systems listed in the theorem are all integrable. The theorem then follows immediately from the easily established fact that

\[ \mathcal{G}_k^{(\infty)}(X) \subset \Omega^1(\tilde{\tau}, \eta_{p+1}, \ldots, \eta_k). \]

Goursat proves that once a pair of \( X \) invariant functions are known, a new \( X \) invariant function can always be constructed from the ratio of the total derivatives of the given invariants (see also [8]). In case (i) there is exactly one invariant \( I_1 \) of order \( l \) equal to the adapted order of \( \tilde{\tau} \) and one invariant of order \( p+1 \). In case (ii)b there is one \( X \) invariant of order 1 and one new \( X \) invariant of order 2 (so that this case is similar to case (i)) while in case (ii)c there are no invariants of order 1 but two invariants \( I_2 \) and \( I_2' \) of order 2. In case (iii)b there are two invariants \( i_1 \) and \( i_1' \) of order 1. In case (iii)c there are no invariants of order 1 but there are three invariants \( I_2, I_2' \) and \( I_2'' \) of order two. In the next chapter, we find a relative invariants whose vanishing determines the existence of an \( X \) invariant function of a given order.

Finally, we remark that for second-order scalar elliptic equations in the plane, the characteristic total vector fields are complex. A complex valued Laplace-adapted coframe and complex generalized Laplace invariants were defined in [8]. In this situation, one needs the complex version of then Frobenius theorem [29], (p. 23) to determine the existence of complex valued characteristic invariant functions. But it is not difficult to show that Theorem 6.3, with the obvious modifications, remains valid.

Example 1. For our first example, we consider the equation

\[ u_{xx}u_{xy} = u_x \]

and we show how the results of Theorem 6.3 can be used to find the characteristic invariants. We use the notation \( p = u_x, q = u_y, r = u_{xx}, s = u_{xy} \) and \( t = u_{yy} \) when convenient. We find that \( H_0 = 0, M_x = -2 \frac{q}{p} \) and \( K_2 = 0 \) so that, according to this theorem, there are three second-order \( X \) invariant functions \( I, I', \) and \( I'' \), which are the first integrals of the integrable Pfaffian system \( \{ \tilde{\tau}, \eta_1, \eta_2 \} \), one \( Y \) invariant function \( J \) of order one arising from the integrable 1-form \( \tilde{\sigma} \), and one \( Y \) invariant of order 3, which together with the invariant \( J \) is the first integral of the integrable system.
\{\hat{\eta}, \xi_3\}. We find that

\[ \hat{\tau} = dx - \frac{s^2}{p} dy, \quad \eta_1 = \theta_x = dp - \frac{p}{s} dx - sdy \quad \text{and} \]

\[ \eta_2 = -\frac{p}{s^3} \theta_{xy} + \frac{1}{s} \theta_z = -\frac{p^2 u_{xyy}}{s^4} \hat{\tau} - dy + \frac{1}{s} dp - \frac{p}{s^2} ds \]

and hence

\[ \eta_2 \equiv d(-y + \frac{p}{s}) \mod \hat{\tau}, \quad \eta_1 \equiv -s^2 d\left(\frac{p}{s^2}\right) \mod \{\hat{\tau}, \eta_2\}, \quad \text{and} \]

\[ \hat{\tau} = d(x - s) - \frac{s^3}{p} d\left(\frac{p}{s^2}\right) + \frac{s^2}{p} d(y - \frac{p}{s}). \]

The \(X = D_x + \frac{p}{s^2} D_y\) invariants are thus

\[ I = -y + \frac{p}{s}, \quad I' = \frac{p}{s^2}, \quad I'' = x - s. \]

Likewise, we compute \(\hat{\sigma} = \frac{s^2}{p} dy\) and

\[ \xi_3 = \frac{p^3}{s^6} \theta_{yy} + \frac{p^2}{s^4} \theta_{xxy} + \frac{2u_{xyy}}{s^2} \theta_{xy} - \frac{pu_{xyy}}{s^4} \theta_z \]

\[ \equiv \frac{p^3}{s^6} du_{yy} + \frac{p^2}{s^4} du_{xyy} + \frac{2p^2 u_{xyy}}{s^5} ds - \frac{pu_{xyy}}{s^4} dp \mod \hat{\sigma} \]

\[ \equiv \frac{p^3}{s^6} d\left(\frac{s^2}{p} u_{xyy} + u_{yyy}\right) \mod \hat{\sigma} \]

to arrive at the \(Y = D_x\) invariants \(J = y\) and \(J' = \frac{s^2}{p} u_{xyy} + u_{yyy}. \)

**Example 2.** For our second example, we consider the equation

\[ u_{xy} + uu_{xx} + f(y,p) = 0. \quad (6.11) \]

where, as usual, \(p = u_x\). This equation was first studied in detail by Calogero [15] who proved, by ad hoc methods, that a general solution to (6.11) can be obtained by quadratures. Our general results can be applied to (6.11) to substantially clarify Calogero’s analysis.
The characteristic vector fields for (6.11) are

\[ X = uD_x + D_y \quad \text{and} \quad Y = D_z, \]

the commutator is \([X, Y] = -u_y Y\) and we easily see that the coefficients of the universal linearization (4.8) and (4.9) are \(A = 0, B = f_p, C = u_{xx}\) and \(D = 0, E = f_p - u_x\) and \(G = u_{xx}\). It is a simple matter to directly check that

\[ H_0 = -u_{xx} \quad \text{and} \quad H_1 = 0. \tag{6.12} \]

The following general recursion formula for the generalized Laplace invariants will be derived in chapter 9, Theorem 9.3.

\[ K_n = 2K_{n-1} - K_{n-2} - YX(\log K_{n-1}) - PX(\log K_{n-1}) + Y(Q) - X(P) + 2PQ, \]

where \(n \geq 1\) and \(K_{-1} = H_0\). It follows that

\[ K_n = Y(\alpha_n), \tag{6.13} \]

where the functions \(\alpha_n(y, p)\) are given recursively by

\[ \alpha_0 = f_p - 2u_x, \quad \alpha_1 = 3(f_p - u_x) + \frac{f_{ppp} - f_{ypp}}{f_{yp} - 2}, \quad \text{and} \]

\[ \alpha_n = 2\alpha_{n-1} - \alpha_{n-2} + f_p + \left( \frac{\partial}{\partial p} - \frac{\partial}{\partial y} \right) \log \left( \frac{\partial \alpha_{n-1}}{\partial p} \right). \]

For simplicity, we now assume that \(f = f(p)\). By Theorem 6.3, we are assured that there are two \(X\) invariant functions and that (6.11) is always semi-Darboux integrable. Indeed, we know that \(C_k^{(\infty)}(X) = \Omega^1(\hat{\tau}, \eta_2)\), where

\[ \hat{\tau} = dx - udy + \frac{1}{r}\theta_x = \frac{1}{r}[dp + fdy] \]

and

\[ \eta_2 = \theta_{xx} - \frac{u_{xx}}{r}\theta_x \equiv dr - \frac{1}{f}(p + f')dp \mod \hat{\tau} \]

from which it is a simple matter to arrive at the invariants

\[ I_1 = y + \int \frac{dp}{f} \quad \text{and} \quad I_2 = \frac{r}{f} \exp \left[ -\int \frac{pdp}{f} \right]. \]
if \( f \neq 0 \) and, for \( f = 0 \),
\[
I_1 = p \quad \text{and} \quad I_2 = re^{yp}.
\]

The equation
\[
I_2 = \phi(I_1) \tag{6.14}
\]
is a second-order general intermediate integral for (6.11). Viewed as an ordinary differential equation for \( p \), we can solve (6.14) by quadratures to obtain \( p \) in terms of \( x \) and \( y \), the arbitrary function \( \phi \) and a constant of integration \( \gamma(y) \). The second derivative \( u_{xx} \) is then determined by (6.14) while \( u_{xy} = \frac{dp}{dy} \) and the general solution to (6.11) is therefore
\[
u = -\frac{1}{u_{xx}}[u_{xy} + f(p)]. \tag{6.15}
\]

For example, for the equation
\[
u_{xy} + uu_{xy} - 2ux(ux + a) = 0, \tag{6.16}
\]
where \( a \) is a constant, the invariants are
\[
I_1 = \frac{p + a}{p} e^{2ay} \quad \text{and} \quad I_2 = \frac{u_{xx}}{p^{\sqrt{p/a} + a}}
\]
and with
\[
\phi(z) = -2b \sqrt{\frac{cz + a - c}{ac}} \tag{6.17}
\]
for constants \( b \) and \( c \), equation (6.14) becomes
\[
\frac{dp}{dx} = -2bp \sqrt{\frac{p}{h(y)}} + 1, \quad \text{where} \quad h(y) = \frac{ace^{2ay}}{a - c + ce^{2ay}}.
\]
The integration of (6.17) leads to \( p = -h(y) \text{sech}^2(bx + \gamma(y)) \) in which case (6.15) becomes
\[
u = -\frac{h(y)}{b} \tanh(bx + \gamma(y)) - \frac{\gamma'(y)}{b}.
\]
This solution to (6.16) contains, as a special case, the solution obtained by Calogero. It is perhaps worth noting that it is not possible to find a closed form solution to (6.11) involving two arbitrary
functions since this would imply (6.11) is Darboux integrable. This is not the case since for (6.16) we find that

$$K_n = -(2n^2 + 7n + 6)u_{xx}. \quad (6.18)$$

Nevertheless, it is possible to find functions $f(p)$ for which (6.11) is Darboux integrable at low orders. For example, the equation

$$u_{xy} + uu_{xx} + u_x^2 = 0 \quad (6.19)$$

satisfies $H_1 = 0$ and $K_0 = 0$. The $X$ invariants are now

$$I_1 = y - \frac{1}{p} \quad \text{and} \quad I_2 = \frac{r}{p^3}$$

and the $Y$ invariants are

$$J_1 = y \quad \text{and} \quad J_2 = q + up$$

and hence, in accordance with the method of Darboux, we set

$$r = p^3 \phi_0(y - \frac{1}{p}) = \frac{p^3}{\phi''(\alpha)} \quad \text{and} \quad q = -up + \psi''(\beta),$$

where $\alpha = y - \frac{1}{p}$ and $\beta = y$. Then, in terms of the variables $\alpha, \beta, x,$ and $u$, the differential system

$$du - p\,dx - q\,dy = 0 \quad \text{and} \quad dp - r\,dx - s\,dy = 0$$

becomes

$$du - \frac{dx}{\beta - \alpha} + \left[ \frac{u}{\beta - \alpha} - \psi''(\beta) \right] d\beta = 0 \quad \text{and} \quad \phi''(\alpha)\,d\alpha - \frac{dx}{\beta - \alpha} + \frac{u}{\beta - \alpha}\,d\beta = 0.$$

We therefore deduce that the general solution to (6.19) is given by

$$u = \phi'(\alpha) + \psi'(\beta), \quad x = \beta\phi'(\alpha) - \alpha\psi'(\alpha) + \phi(\alpha) + \psi(\beta), \quad y = \beta.$$  

**Example 3.** For our third example, we classify all hyperbolic equations of the form

$$r = f(t), \quad (6.20)$$

where $f' > 0$, which are Darboux integrable on the second jet bundle. Recall that $r = u_{xx}$ and $t = u_{yy}$. It is convenient to write

$$f'(t) = \frac{1}{g^2(t)}, \quad (6.21)$$

where $g(t) > 0$. 
**Proposition 6.4.** Let $H_0 \neq 0$ and $H_1 = 0$. Then equation (6.20) is contact equivalent to one of the following three equations determined by

\[
\begin{align*}
g(t) &= t^2, \\
\tan(\sqrt{g} - t) &= \sqrt{g}, \quad \text{or} \quad (6.23) \\
\tanh(\sqrt{g} - t) &= \sqrt{g}. \quad (6.24)
\end{align*}
\]

**Proof.** We first note that the constant of integration arising from (6.21) in the determination of $f$ from $g$ can always be absorbed by the change of variables $\bar{u} = u + \lambda x^2$. We remark that (6.22) leads to $3rt^3 + 1 = 0$ which is integrated in Goursat [26] (example IV, p. 130). The computations of Forsyth [23] would seem to indicate, contrary to our conclusions, that (6.22) is the only Darboux integrable equation of the type (6.20)

An easy computation, using Maple, shows that in order for the highest order terms in $H_1$ to vanish, the function $g$ must satisfy

\[
4g^2(g'')^2 - 4g^2g''g' - 4g(g')^2g'' + (g')^4 = 0. \quad (6.25)
\]

Since $g \neq 0$, the only singular solution to this equation is $g' = 0$. But then $g = k$ is a constant and $H_0 = 0$ and this is the case which we have excluded. The general solution to (6.25) for $g = g(t)$ depends upon three arbitrary constants and so $f$ depends upon four arbitrary constants. To prove the theorem, we must show that these constants can be normalized by contact transformations so as to arrive at exactly one of the three equations (6.22), (6.23) or (6.24).

We observe that under the reflection

\[
\begin{align*}
\bar{x} &= y, \\
\bar{y} &= x, \\
\bar{u} &= u
\end{align*}
\]

the equation $r = f(t)$ is transformed into $r = f^{-1}(t)$. Also, under the simple point transformations

\[
\begin{align*}
\bar{x} &= x, \\
\bar{y} &= y, \\
\bar{u} &= u + \frac{1}{2}ay^2
\end{align*}
\]

and

\[
\begin{align*}
\bar{x} &= x, \\
\bar{y} &= b^2y, \\
\bar{u} &= cu
\end{align*}
\]

the equation (6.20) is transformed into $\bar{r} = \bar{f}(\bar{t})$, where

\[
\bar{f}(\bar{t}) = f(\bar{t} - a) \quad \text{and} \quad f(\bar{t}) = cf(\frac{b^4}{c} - \bar{t}),
\]

where $c$ and $f$ are constants.
respectively. Hence the function \( g \) transforms under (6.29) and (6.30) as

\[
\tilde{g}(\tilde{t}) = g(t - \alpha) \quad \text{and} \quad \tilde{g}(\tilde{t}) = \frac{1}{b^2} g(\frac{b^4}{c} \tilde{t}).
\]  

Since the condition \( H_1 = 0 \) is contact invariant, it follows that the equation (6.25) is invariant under the three-parameter solvable group (6.31) and is therefore solvable by quadratures. Applying Lie's method for the solution of such equations, or alternatively, by noting that

\[
\frac{g'}{\sqrt{g}} + 2 \sqrt{g} g'' = 4\alpha \quad \text{and} \quad \sqrt{g} g' - 2g^{3/2}g'' = 4\beta
\]

are first integrals for (6.25), we deduce that every solution to (6.25) is a solution to the first-order equation

\[
g' = 2\alpha \sqrt{g} + \frac{2\beta}{\sqrt{g}}
\]  

for some choice of constants \( \alpha \) and \( \beta \). It is easy to check that this equation implies that \( H_1 = K_1 = 0 \) so that every solution to (6.32) determines a Darboux integrable equation.

Under the change of variables (6.30), the first integrals \( \alpha \) and \( \beta \) transform as

\[
\tilde{\alpha} = \frac{b^3}{c} \alpha \quad \text{and} \quad \tilde{\beta} = \frac{b}{c} \beta.
\]

Thus according to whether (i) \( \alpha \neq 0, \beta = 0 \); (ii) \( \alpha = 0, \beta \neq 0 \); (iii) \( \alpha \beta > 0 \); or (iv) \( \alpha \beta < 0 \), we can use the point transformation (6.30) to transform \( g \) so that (i) \( \alpha = 1, \beta = 0 \); (ii) \( \alpha = 0, \beta = 1 \); (iii) \( \alpha = 1, \beta = 1 \); or (iv) \( \alpha = 1, \beta = -1 \). In case (i) we find that \( g(t) = (t + k)^2 \) and we use (6.29) to transform this to \( g(t) = t^2 \). This gives the equation \( 3rt^3 + 1 = 0 \). In case (ii), we find that \( g(t) = (3t)^{2/3} \), which yields the equation obtained from (6.22) by the reflection (6.26). In case (iii), we find that \( g \) is uniquely determined by (6.23) while (iv) implies that

\[
\sqrt{g} + \frac{1}{2} \log \left| \frac{\sqrt{g} - 1}{\sqrt{g} + 1} \right| = t.
\]

There are two subcases to consider here depending on whether \( 0 < g < 1 \) or \( g > 1 \). In the first case we find the \( g \) satisfies (6.24) while the second case can be reduced to the first, again by the reflection (6.26).
It is not difficult to see that the Lie algebra of contact symmetries for (6.22), (6.23), and (6.24) consists solely of point symmetries. For (6.22), there is a 9-dimensional Lie algebra of contact symmetries while equations (6.23) and (6.24) have equivalent 7-dimensional Lie algebras of contact symmetries ([28] (p. 215)). This shows that (6.22) is not equivalent to either (6.23) or (6.24). We have, as yet, been unable to distinguish (6.23) from (6.24) by contact invariant means.

For the choice of $g(t) = t^2$ (6.22) we get $f(t) = -\frac{3}{t^5}$ and so equation (6.20) becomes

$$3rt^3 + 1 = 0. \quad (6.33)$$

For the equation (6.33), we have $X = D_x + \frac{1}{t^2} D_y$, $Y = D_x - \frac{1}{t^2} D_y$, $H_1 = K_1 = 0$ and $M_x = M_y = \frac{t^3}{4}$, so that, by Theorem 6.3, there are two $X$ functionally independent invariants of order 2. We determine

$$\tau = \frac{1}{2} (st + 1) dx - \frac{t}{2} dq \quad \text{and} \quad \eta_2 = \frac{2}{t^4} \theta_{yy} - \frac{2}{t^2} \theta_{xy} + \frac{2}{t^5} (t^2 u_{yy} - u_{yy}) \theta_y \equiv -\frac{2}{t^2} ds + \frac{2}{t^4} dt \mod \tau$$

$$\equiv -\frac{2}{t^2} d(s + \frac{1}{t}) \mod \tau$$

so that one invariant is $I = s + \frac{1}{t}$. Since $\tau = \frac{t}{2} (dx - dq)$ it follows that the other invariant is $I' = Ix - q$. 
CHAPTER 7
THE FIRST INVARIANTS OF THE CHARACTERISTIC SYSTEMS

In chapter 6, we have analyzed in detail the necessary and sufficient conditions for the characteristic Pfaffian systems of order $k$, $C_k(X)$ and $C_k(Y)$, to have at least rank 2 integrable subsystems. The purpose of this chapter is to find necessary and sufficient conditions for the characteristic Pfaffian systems of order $k$ to have at least rank 1 integrable subsystems, that is, the cases when there exists at least one $X$ or $Y$ invariant function. This process will lead to the discovery of new relative invariants that control the existence of invariant functions. We also discuss the equation $T(f) = 0$, where $T$ is a total vector field.

Notice that if $H_0 \neq 0, H_1 \neq 0, \ldots, H_{k-1} \neq 0$, $k \geq 1$, then the $d$ structure equations immediately imply that $\dim C_k^{(\infty)}(X)$, is 0 or 1. If $\dim C_k^{(\infty)}(X) = 1$, then again using the $d$ structure equations we deduce

$$C_k^{(\infty)}(X) = \Omega^1(\tau - \Sigma),$$

where $\Sigma$ is a contact form of adapted order $\leq k$.

Let $\Sigma$ be a contact form. Using equations (4.42), (4.43), and (4.44a), we easily conclude that the necessary and sufficient conditions for

$$d(\tau - \Sigma) \equiv 0 \mod \{ \tau - \Sigma \}$$

are

$$X(\Sigma) + Q\Sigma - \beta = 0 \quad \text{and} \quad d\nu\Sigma = \Sigma \wedge (\mu_2 - Y(\Sigma)),$$

(7.1)

where $\beta$ and $\mu_2$ are given by (4.50).

Let $T$ be a total vector field. Recall that a form $\omega \in \Omega^*(\Theta, \xi_1, \eta_1, \ldots)$ is called a relative $T$ invariant contact form if $T(\omega) = \lambda \omega$ for some function $\lambda$.

**Lemma 7.1.** Let $\Sigma$ be a contact form on $\mathcal{R}^{\infty}$ such that

$$X(\Sigma) + Q\Sigma - \beta = 0.$$  

(7.2)

Then

$$\omega = d\nu\Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma))$$

is a relative $X$ invariant contact form.
Proof. Using Propositions 4.5 and 4.8, we deduce that

\[ d_V \omega = d_V (X(\Sigma) + Q\Sigma - \beta) = X(d_V \Sigma) + Qd_V \Sigma + \beta \wedge [Y(\Sigma) - \mu_2] + [X(\mu_2) - Y(\beta) - p\beta] \wedge \Sigma + \mu_1 \wedge [X(\Sigma) + Q\Sigma - \beta]. \]

From equation (7.2) we solve for \( \beta \) and substitute into the last equation to conclude

\[ X(d_V \Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma))) + Q \left[ d_V \Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma)) \right] = 0. \]

Let \( \Sigma \) be a contact form of adapted order \( \leq k \) satisfying (7.2). From Proposition 4.8 we deduce that if \( H_p = 0 \) for some \( p \geq k \), then

\[ d_V \Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma)) \in \Omega^2(\eta_{p+1}, \eta_{p+2}, \ldots). \]

From the \( d_V \) structure equations it now easily follows that

\[ d_V \Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma)) \equiv 0 \mod \{ \Theta, \xi_1, \ldots, \xi_k, \eta_k \}. \]

Hence

\[ d_V \Sigma = \Sigma \wedge (\mu_2 - Y(\Sigma)). \]  \hspace{1cm} (7.3)

If \( H_i \neq 0 \) for all \( i = 0, 1, 2, \ldots \), then either from the slight modification of the proof of Proposition 4.8 or by the theorem of Anderson and Kamran [8] it follows that there are no nonzero relative \( X \) invariant contact forms. Thus again we obtain (7.3).

We have proved the following.

**Proposition 7.2.** Let \( \Sigma \) be any contact form on \( R^n \) of adapted order \( \leq k \). Then \( \tau - \Sigma \) is integrable, that is

\[ d(\tau - \Sigma) \equiv 0 \mod \{ \tau - \Sigma \} \]

if and only if

\[ X(\Sigma) + Q\Sigma - \beta = 0. \]

Recall that \( \alpha \) and \( \beta \), defined in Proposition 4.2, are contact forms of adapted order two,

\[ \alpha = d_2 \eta_2 + d_1 \eta_1 + e_0 \Theta + e_1 \xi_1 + M_\tau \xi_2 \] \hspace{1cm} (7.4)

and

\[ \beta = b_2 \xi_2 + b_1 \xi_1 + c_0 \Theta + c_1 \eta_1 + M_\tau \eta_2. \] \hspace{1cm} (7.5)
Let $H_0 \neq 0, H_1 \neq 0, \ldots, H_{k-1} \neq 0$, $k \geq 1$. Define the functions $F_0, \ldots, F_{k+1}$ as follows:

\begin{align}
F_0 &= -X(b_2) + (E_1 - Q)b_2 + b_1, \\
F_1 &= -X(F_0) + BF_0 + c_0, \\
F_2 &= -X\left(\frac{F_1}{H_0}\right) + \frac{B_1 F_1}{H_0} + c_1, \\
F_3 &= -X\left(\frac{F_2}{H_1}\right) + \frac{B_2 F_2}{H_1} + M_r, \\
F_i &= -X\left(\frac{F_{i-1}}{H_{i-2}}\right) + \frac{B_{i-1} F_{i-1}}{H_{i-2}}, \quad \text{for } i = 4, \ldots, k+1.
\end{align}

Similarly, let $K_0 \neq 0, K_1 \neq 0, \ldots, K_{k-1} \neq 0$, $k \geq 1$. Define the functions $G_0, \ldots, G_{k+1}$ as follows:

\begin{align}
G_0 &= -Y(d_2) + (B_1 + P)d_2 + d_1, \\
G_1 &= -Y(G_0) + EG_0 + e_0, \\
G_2 &= -Y\left(\frac{G_1}{K_0}\right) + \frac{E_1 G_1}{K_0} + e_1, \\
G_3 &= -Y\left(\frac{G_2}{K_1}\right) + \frac{E_2 G_2}{K_1} + M_r, \\
G_j &= -Y\left(\frac{G_{j-1}}{K_{j-2}}\right) + \frac{E_{j-1} G_{j-1}}{K_{j-2}}, \quad \text{for } j = 4, \ldots, k+1.
\end{align}

**Lemma 7.3.** Let $H_0 \neq 0, H_1 \neq 0, \ldots, H_{k-1} \neq 0$, $k \geq 2$. Let $\Sigma$ be a contact form of adapted order $\leq k$ on $\mathcal{R}^\infty$. Then

\begin{equation}
X(\Sigma) + Q\Sigma - \beta = 0
\end{equation}

if and only if

\begin{equation}
\Sigma = b_2 \xi_1 + F_0 \Theta + \sum_{i=1}^k \frac{F_i}{H_{i-1}} \eta_i \quad \text{and} \quad F_{k+1} = 0.
\end{equation}

Similarly let $K_0 \neq 0, K_1 \neq 0, \ldots, K_{k-1} \neq 0$, $k \geq 2$. Let $\Sigma$ be a contact form of adapted order $\leq k$. Then

\begin{equation}
Y(\Sigma) - P\Sigma - \alpha = 0
\end{equation}
if and only if

$$
\Sigma = d_2 \eta_1 + G_0 \Theta + \sum_{j=1}^{k} \frac{G_j}{K_{j-1}} \xi_j \quad \text{and} \quad G_{k+1} = 0.
$$

(7.11)

**Proof.** We will prove only the first part of the Lemma. The proof of the second part is analogous.

Let

$$
\Sigma = \sum_{j=1}^{k} g_j \xi_j + f_0 \Theta + \sum_{i=1}^{k} f_i \eta_i.
$$

Using the defining equations for the Laplace-adapted coframe, we compute

$$
\begin{align*}
X(\Sigma) + Q\Sigma - \beta \\
= g_k \xi_{k+1} + \sum_{j=3}^{k} (X(g_j) - (E_j - Q)g_j + g_{j-1}) \xi_j + (X(g_2) - (E_2 - Q)g_2 - b_2 + g_1) \xi_2 \\
+ (X(g_1) - (E_1 - Q)g_1 - b_1 + f_0) \xi_1 + (X(f_0) - B f_0 - c_0 + H_0 f_1) \Theta \\
+ (X(f_1) - B_1 f_1 - c_1 + H_1 f_2) \eta_1 + (X(f_2) - B_2 f_2 - M_2 + H_2 f_3) \eta_2 \\
+ \sum_{i=3}^{k-1} (X(f_i) - B_i f_i + H_i f_{i+1}) \eta_i + (X(f_k) - B_k f_k) \eta_k.
\end{align*}
$$

Imposing condition (7.8) equations (7.9) follow.

Denote

$$
\Upsilon_k = b_2 \xi_1 + F_0 \Theta + \sum_{i=1}^{k} \frac{F_i}{H_{i-1}} \eta_i, \quad \text{and} \quad \Xi_k = d_2 \eta_1 + G_0 \Theta + \sum_{j=1}^{k} \frac{G_j}{K_{j-1}} \xi_j.
$$

(7.12)

From Proposition 6.1 it follows that if $H_p = 0$ for $p \geq 2$, we have $\Upsilon_p = \Upsilon$ and so $\hat{\sigma} = \sigma - \Upsilon_p$. Similarly, if $K_q = 0$ for $q \geq 2$, we have $\Xi_q = \Xi$ and so $\hat{\sigma} = \sigma - \Xi_p$. Recall that $\Upsilon$ and $\Xi$ denote the contact forms defined in Theorem 4.9. Theorem 4.6 and Lemma 7.3 combine to establish the following result.

**Theorem 7.4.** Let $\mathcal{R}$ be a scalar second-order hyperbolic equation in the plane.
(i) There exists an X invariant function of order \( \leq 1 \) if and only if \( M_r = 0 \) and either \( H_0 = 0 \) or \( F_2 = 0 \).

(ii) If \( k \geq 2 \), then there exists an X invariant function of order \( \leq k \) if and only if either \( H_p = 0 \) for some \( p \leq k \) or \( F_{k+1} = 0 \).

(iii) There exists a Y invariant function of order \( \leq 1 \) if and only if \( M_\sigma = 0 \) and either \( K_0 = 0 \) or \( G_2 = 0 \).

(iv) If \( k \geq 2 \), then there exists a Y invariant function of order \( \leq k \) if and only if either \( K_q = 0 \) for some \( q \leq k \) or \( G_{k+1} = 0 \).

Remark. 7.5 Notice that due to Theorem 6.3, if \( H_1 = M_r = 0 \), then there exists a X invariant function of order \( \leq 1 \). Similarly, if \( K_1 = M_\sigma = 0 \), then there exists a Y invariant function of order \( \leq 1 \).

Consider the hyperbolic equation studied by Tsujishita [37]

\[
r = \frac{t^3}{3}
\]

with the characteristic vector fields

\[
X = D_x + tD_y \quad \text{and} \quad Y = D_x - tD_y.
\]

Using Maple, we obtain \( H_0 \neq 0, H_1 \neq 0, H_2 \neq 0, F_2 \neq 0 \) and \( F_3 = 0 \). By the above theorem there exists a second-order X invariant function, namely

\[
I = s - \frac{t^2}{2}.
\]

So far we have dealt only with the equations \( X(I) = 0 \) or \( Y(I) = 0 \). Let us now investigate a more general equation \( T(I) = 0 \), where \( T \) is an arbitrary total vector field.

**Theorem 7.6.** Let \( R \) be a scalar hyperbolic second-order partial differential equation in the plane. Let \( T \) be a total vector field on \( R^\infty \). If there are two functionally independent \( T \) invariant functions on \( R^\infty \), then \( T \) is a characteristic vector field.

**Proof.** Let \( \{ T, W \} \) be a basis for the total vector fields on \( R^\infty \). Let \( \{ \mu, \epsilon \} \) be the dual basis for the horizontal forms, that is,
\[ \mu(T) = 1 \quad \text{and} \quad \mu(W) = 0, \]
\[ \epsilon(T) = 0 \quad \text{and} \quad \epsilon(W) = 1. \]

If \( I \) is a \( T \) invariant function of order \( k \), then
\[
dI = d_H I + d_V I = T(I)\mu + W(I)\epsilon + d_V I
\]
\[
= W(I)\epsilon + d_V I \in \Omega^1(\epsilon, \Theta, \xi_1, \eta_1, \ldots, \xi_k, \eta_k).
\]

Thus there are functionally independent \( T \) invariant functions of order \( \leq k \) if and only if the Pfaffian system
\[
C_k(T) = \Omega^1(\epsilon, \Theta, \xi_1, \eta_1, \ldots, \xi_k, \eta_k)
\]
has an integrable subsystem of rank 2. Let
\[
\epsilon = a\sigma + b\tau \tag{7.13}
\]
and assume that \( a \neq 0, \ b \neq 0 \). Let \( \{ \alpha, \beta \} \subseteq C_k(T) \) be a rank 2 integrable subsystem, where
\[
\alpha = \epsilon + \epsilon \Theta + \sum_{j=1}^{k} a_j \xi_j + \sum_{i=1}^{k} b_i \eta_i \tag{7.14}
\]
and
\[
\beta \equiv p_l \xi_l + q_l \eta_l \mod \{ \Theta, \xi_1, \eta_1, \ldots, \xi_{l-1}, \eta_{l-1} \}, \quad 1 \leq l \leq k
\]
and where \( a_j, b_i, p_l, \) and \( q_l \) are \( C^\infty \) functions on \( \mathcal{R} \). Using the \( d_H \) and \( d_V \) structure equations for the Laplace-adapted coframe (Propositions 4.1 and 4.6), we obtain
\[
d\beta = d_H \beta + d_V \beta \equiv \sigma \wedge X(\beta) + \tau \wedge Y(\beta) \mod \{ \Theta, \xi_1, \eta_1, \ldots, \xi_l, \eta_l \}
\]
\[
\equiv \sigma \wedge p_l \xi_{l+1} + \tau \wedge q_l \eta_{l+1} \mod \{ \Theta, \xi_1, \eta_1, \ldots, \xi_l, \eta_l \}
\]
\[
\equiv \frac{1}{a}(\alpha - b\tau - \sum_{j=1}^{k} a_j \xi_j - \sum_{i=1}^{k} b_i \eta_i) \wedge p_l \xi_{l+1} + \tau \wedge q_l \eta_{l+1} \mod \{ \Theta, \xi_1, \eta_1, \ldots, \xi_l, \eta_l \}
\]
\[
\equiv \tau \wedge (q_l \eta_{l+1} - \frac{b}{a} p_l \xi_{l+1}) - \frac{1}{a} \left( \frac{1}{a} \sum_{j=1+2} a_j \xi_j + \sum_{i=1+1} b_i \eta_i \right) \wedge p_l \xi_{l+1} \mod \{ \alpha, \Theta, \xi_1, \eta_1, \ldots, \xi_l, \eta_l \}.
\]
By assumption

\[ d\beta \equiv 0 \mod \{ \alpha, \Theta, \xi_1, \eta_1, \ldots, \xi_l, \eta_l \}. \]

The comparison of the two last congruences yields \( p_l = q_l = 0 \), that is,

\[ \beta \in \Omega^1(\alpha, \Theta, \xi_1, \eta_1, \ldots, \xi_{l-1}, \eta_{l-1}). \]

Because \( l \), \( 1 \leq l \leq k \) was arbitrary, it follows that \( \beta = c\Theta \). We have from (7.13) and (7.14)

\[ d\beta \equiv \sigma \wedge c\xi_1 + \tau \wedge c\eta_1 \mod \{ \Theta \}. \]

\[ \equiv \frac{1}{a} (\alpha - b\tau - \sum_{j=1}^k a_j \xi_j - \sum_{i=1}^k b_i \eta_i) \wedge c\xi_1 + \tau \wedge c\eta_1 \mod \{ \Theta \}. \]

\[ \equiv c\tau \wedge (\eta_1 - \frac{b}{a} \xi_1) - \frac{c}{a} \left( \sum_{j=1}^k a_j \xi_j + \sum_{i=1}^k b_i \eta_i \right) \wedge \xi_1 \mod \{ \alpha, \Theta \}. \]

By assumption

\[ d\beta \equiv 0 \mod \{ \alpha, \Theta \}. \]

The comparison of the last two congruences yields \( c = 0 \), that is, \( \beta = 0 \).

**Theorem 7.7.** Let \( \mathcal{R} \) be a scalar hyperbolic second-order partial differential equation in the plane. Let \( T = mD_x + nD_y \) be a total vector field on \( \mathcal{R}^\infty \) and let \( m \) and \( n \) be a function of order \( \leq k \). If there is a \( T \) invariant function \( I \) of order \( \geq k \), then \( T \) is a characteristic vector field.

**Proof.** Assume that the equation manifold \( \mathcal{R} \) is given by

\[ r + f(x, y, u, p, q, s, t) = 0 \]

and so a natural choice of coordinates on \( \mathcal{R}^\infty \) is

\[ (x, y, u, p, q, s, t, u_{xyy}, u_{yyy}, \ldots). \]

Let \( I = I(x, y, u, p, q, s, t, \ldots, u_{xy^{l-1}}, u_{y^l}) \) be an \( T \) invariant function of order \( l \geq k \). Then

\[ T(I) = [(n - m \frac{\partial f}{\partial s}) \frac{\partial I}{\partial u_{xy^{l-1}}} + m \frac{\partial I}{\partial u_{y^l}}] u_{xy^{l-1}} = [-m \frac{\partial f}{\partial t} \frac{\partial I}{\partial u_{xy^{l-1}}} + n \frac{\partial I}{\partial u_{y^l}}] u_{y^l} \]

\[ + \{ \text{function of order } \leq k \}. \]
From $T(I) = 0$ we obtain that

\[(n - m \frac{\partial f}{\partial s}) \frac{\partial I}{\partial u_{xy'} - 1} + m \frac{\partial I}{\partial u_{y'}} = 0 \quad \text{and} \quad -m \frac{\partial f}{\partial t} \frac{\partial I}{\partial u_{xy'} - 1} + n \frac{\partial I}{\partial u_{y'}} = 0.\]

Since $I$ is of order $l$, that is,

\[\frac{\partial I}{\partial u_{xy'} - 1} \neq 0 \quad \text{or} \quad \frac{\partial I}{\partial u_{y'}} \neq 0,
\]

we conclude that

\[(n - m \frac{\partial f}{\partial s}) n + m^2 \frac{\partial f}{\partial t} = n^2 - \frac{\partial f}{\partial s} mn + \frac{\partial f}{\partial t} m^2 = 0,
\]

which ends the proof.
CHAPTER 8
HYPERBOLIC DARBOUX SYSTEMS

Bryant, Griffiths, and Hsu [13] introduced the following notion of a hyperbolic system.

**Definition 8.1.** A hyperbolic system of class $s$ is an exterior differential system $(M, \mathcal{I})$ where $M$ is a manifold of dimension $s + 4$ and $\mathcal{I}$ is a differential ideal with the property that every point of $M$ lies in the neighborhood $U$ on which there exists a coframing

$$(\theta^1, \ldots, \theta^s; \omega^1, \omega^2, \omega^3, \omega^4)$$

so that, on $U$, the ideal $\mathcal{I}$ is generated algebraically by

$$\{\theta^1, \ldots, \theta^s; \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4\}.$$

For simplicity we will refer to $\mathcal{I}$ as to a hyperbolic system.

Recall from chapter 2 that an integral manifold of a differential system $(M, \mathcal{I})$ is a pair $(N, \phi)$, where $N$ is a manifold and $\phi$ is a map $\phi : N \to M$ such that $\phi^*(\mathcal{I}) = \{0\}$. Since $\Omega^3(M) = \mathcal{T}^3$, where $\mathcal{T}^3$ denotes the 3-forms in $\mathcal{I}$ then it easily follows that a hyperbolic system $(M, \mathcal{I})$ has at most 2-dimensional integral submanifolds.

In [13] the authors show by examples that many classical or well-known systems of differential equations can be represented as hyperbolic systems. It is proved that a partial prolongation of a hyperbolic system of class $s$ is a hyperbolic system of class $s + 1$. The authors introduce the notion of Darboux integrability for a hyperbolic system and prove that a hyperbolic system $\mathcal{I}$ is semi-Darboux integrable if and only if after certain number of partial prolongations the prolonged system $\mathcal{I}'$ is algebraically generated by

$$\{\theta^1, \ldots, \theta^s; \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4\} \quad (8.1)$$

and at least one of the Pfaffian systems $\{\omega^1, \omega^2\}$ and $\{\omega^3, \omega^4\}$ is integrable. A hyperbolic system $\mathcal{I}$ is Darboux integrable if and only if after certain number of prolongations the prolonged system $\mathcal{I}'$ is algebraically generated by (8.1) and both Pfaffian systems $\{\omega^1, \omega^2\}$ and $\{\omega^3, \omega^4\}$ are integrable.

This motivates the following definition.

**Definition 8.2.** A hyperbolic Darboux system of class $s$ is a hyperbolic system of class $s$ satisfying the additional conditions that the Pfaffian systems $\{\omega^1, \omega^2\}$ and $\{\omega^3, \omega^4\}$ are integrable.
The purpose of this chapter is to prove in detail that if a second-order scalar hyperbolic equation in the plane $\mathcal{R}$ is Darboux integrable at level $k + 1$, then the prolonged equation manifold $\mathcal{R}^{k+1}$ together with the contact ideal $\mathcal{C}_{k+1}$ form a hyperbolic Darboux system. To prove this result we will introduce the notion of the characteristic coframe on $\mathcal{R}^\infty$. We would be much happier if we were able to prove this result using the Laplace-adapted coframe. Unfortunately this is not possible. Indeed, in general if $\mathcal{R}$ is a Darboux integrable equation at level $k + 1$, then $\Theta, \xi_1, \eta_1, \ldots, \xi_k, \eta_k$ are differential forms on $\mathcal{R}^{2k+1}$. For example, consider the equation $\mathcal{R}$ given by

$$3r t^3 + 1 = 0,$$

which we have studied in the last example of chapter 6. As we have seen, this equation is Darboux integrable at level 2. An easy computation gives

$$\eta_1 = \theta_x - \frac{1}{t^2} \theta_y - \frac{u_{xxy} t^2 + u_{yxy}}{t^3} \theta,$$

which is a form on $\mathcal{R}^3$ but not on $\mathcal{R}^2$. We conjecture that for a Monge-Ampère equation that is Darboux integrable at level $k + 1$, the differential forms $\Theta, \xi_1, \eta_1, \ldots, \xi_k, \eta_k$ are forms on $\mathcal{R}^{k+1}$.

We show by examples that the classical Laplace transformation, is just a special case of a homomorphism between hyperbolic Darboux systems. We show that the Liouville equation $u_{xy} = e^u$ and the equation $u_{xy} = uu_x$ are equivalent as hyperbolic Darboux systems. We construct examples of homomorphisms from the Darboux system corresponding to a partial prolongation of the wave equation into the Darboux systems corresponding to the partial prolongations of certain equations of Moutard type (see Forsyth [23] §231).

Denote

$$\alpha_1 = X(\Theta) \quad \text{and} \quad \alpha_{j+1} = X(\alpha_j) \quad \text{for} \ j = 1, 2, 3, \ldots,$$

and

$$\beta_1 = X(\Theta) \quad \text{and} \quad \beta_{i+1} = X(\beta_i) \quad \text{for} \ i = 1, 2, 3, \ldots.$$

The forms $\sigma, \tau, \Theta, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ form a coframe on $\mathcal{R}^\infty$ and the contact ideal $\mathcal{C}_\infty$ on $\mathcal{R}^\infty$ as a differential ideal is generated by the forms $\Theta, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ (see [8]).

**Definition 8.3.** We call the coframe $(\sigma, \tau, \Theta, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots)$ the characteristic coframe on $\mathcal{R}^\infty$.  

Theorem 8.4. The contact ideal $C_{k+1}$ on $\mathcal{R}^{k+1}$ as a differential ideal is generated by the forms $\Theta, \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$.

Proof. To prove the theorem we need to show that $\Theta, \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ are forms on $\mathcal{R}^{k+1}$. One can readily see that $\Theta, \alpha_1, \beta_1$ are forms on $\mathcal{R}^2$. If $\omega$ is a form on $\mathcal{R}^k$, then $X(\omega)$ and $Y(\omega)$ are forms on $\mathcal{R}^{k+1}$ and so the proof follows by induction.

With the exception of the cases $H_0 = 0$ or $H_1 = 0$, it follows from the Theorem 6.3 that for hyperbolic Darboux integrable equations, the form $\hat{\tau}$ is integrable, that is,

$$d\hat{\tau} \wedge \hat{\tau} = 0.$$

Similarly, with the exception of the cases $K_0 = 0$ or $K_1 = 0$, the form $\hat{\sigma}$ is integrable, that is,

$$d\hat{\sigma} \wedge \hat{\sigma} = 0.$$

Definition 8.5. Let $H_0 = 0$ or $H_1 = 0$. Then $\hat{\tau}$ will denote the unique form in $\Omega^1(\tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2)$ such that

(i) $\hat{\tau} - \tau$ is a contact form, and

(ii) $d\hat{\tau} \wedge \hat{\tau} = 0$.

Similarly, let $K_0 = 0$ or $K_1 = 0$. Then $\hat{\sigma}$ will denote the unique form in $\Omega^1(\sigma, \Theta, \xi_1, \eta_1, \xi_2, \eta_2)$ such that

(i) $\hat{\sigma} - \sigma$ is a contact form, and

(ii) $d\hat{\sigma} \wedge \hat{\sigma} = 0$.

Lemma 8.6. Let $\omega$ be a form on $\mathcal{R}^{\infty}$ and let $T$ be a total vector field. Then the projected Lie derivative commutes with the horizontal differential, that is,

$$T(d_H \omega) - d_H T(\omega) = 0.$$

Proof. By the virtue of Cartan formula

$$\mathcal{L}_T \omega = d(T \lrcorner \omega) + T \lrcorner d\omega.$$

Due to linearity, it is sufficient to assume that $\omega$ is of type $(r, s)$. By definition

$$T(\omega) = \pi^{r,s}(\mathcal{L}_T \omega).$$
We obtain
\[ d_H(\mathcal{L}_T \omega) = d_H[d_H(T \omega) + d_V(T \omega) + T \omega] + d_H d_V \omega \]
\[ = d_H d_V(T \omega) + d_H(T d_H \omega) + d_H(T d_V \omega). \]

and
\[ \mathcal{L}_T(d_H \omega) = d_H(T \omega) + d_V(T d_H \omega) + T \omega. \]

Since \( T \omega \) is of type \((r-1,s)\), one can easily deduce that
\[ T(d_H \omega) = d_H(T \omega) = d_H T(\omega). \]

As an immediate consequence of the previous Lemma and Proposition 4.5, we obtain commutation rules for the projected Lie derivatives of the characteristic vector fields and the exterior differential.

**Proposition 8.7.** Let \( \alpha, \beta, \mu_1 \) and \( \mu_2 \) be given as in Proposition 4.2. Then for a form \( \omega \) on \( \mathbb{R}^\infty \)
\[ d[X(\omega)] - X(d\omega) = \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega) \]
and
\[ d[Y(\omega)] - Y(d\omega) = \alpha \wedge X(\omega) + \mu_2 \wedge Y(\omega). \]

**Lemma 8.8.** The characteristic coframe on \( \mathbb{R}^\infty \) satisfies the following congruences.
\[ Y(\alpha_j) \equiv 0 \mod \{ \beta_1, \Theta, \alpha_1, \ldots, \alpha_j \} \quad \text{for } j = 1, 2, \ldots, \quad (8.2a) \]
\[ X(\beta_i) \equiv 0 \mod \{ \alpha_1, \Theta, \beta_1, \ldots, \beta_i \} \quad \text{for } i = 1, 2, \ldots. \quad (8.2b) \]

**Proof.** From the universal linearization we deduce
\[ Y(\alpha_1) \equiv 0 \mod \{ \beta_1, \Theta, \alpha_1 \} \quad \text{and} \quad X(\beta_1) \equiv 0 \mod \{ \alpha_1, \Theta, \beta_1 \}. \]

We proceed by induction. Assume that \((8.2a)\) holds for some fixed \( j \), that is,
\[ Y(\alpha_j) = b^1 \beta_1 + a^0 \Theta + \sum_{l=1}^j a^l \alpha_l, \]
and so
\[ Y(\alpha_{j+1}) = YX(\alpha_j) = XY(\alpha_j) - PX(\alpha_j) - QY(\alpha_j) \]
\[ = X(b^1 \beta_1 + a^0 \Theta + \sum_{l=1}^j a^l \alpha_l) - P\alpha_{j+1} - Q(b^1 \beta_1 + a^0 \Theta + \sum_{l=1}^j a^l \alpha_l) \]
\[ \equiv 0 \mod \{ \beta_1, \Theta, \alpha_1, \ldots, \alpha_{j+1} \}. \]

The congruence \((8.2b)\) follows similarly.
Theorem 8.9. The characteristic coframe on $\mathcal{R}^\infty$ satisfies the following structure equations.

$$d\sigma \equiv -P \sigma \wedge \tau \mod \{\Theta, \alpha_1, \beta_1, \alpha_2, \beta_2\},$$  \hspace{1cm} (8.3a)

$$d\tau \equiv -Q \sigma \wedge \tau \mod \{\Theta, \alpha_1, \beta_1, \alpha_2, \beta_2\},$$  \hspace{1cm} (8.3b)

$$d\Theta \equiv \sigma \wedge \alpha_1 + \tau \wedge \beta_1 \mod \{\Theta\},$$ \hspace{1cm} (8.3c)

$$d\alpha_j \equiv \sigma \wedge \alpha_{j+1} \mod \{\beta_1, \Theta, \alpha_1, \ldots, \alpha_j\}, \quad j = 1, 2, \ldots, \hspace{1cm} (8.3d)$$

$$d\beta_i \equiv \tau \wedge \beta_{i+1} \mod \{\alpha_1, \Theta, \beta_1, \ldots, \beta_i\}, \quad i = 1, 2, \ldots. \hspace{1cm} (8.3e)$$

Proof. Congruences (8.3a) and (8.3b) follow immediately from Propositions 4.1 and 4.2. Recall that $\Theta = \rho \theta$ where $\rho$ is a function of at most order one and $\theta = du - u_x dx - u_y dy$. To prove (8.3c) compute

$$d\Theta = d\rho \Theta + d\nu \Theta = \sigma \wedge X(\Theta) + \tau \wedge Y(\Theta) + d\nu (\rho \theta)$$

$$= \sigma \wedge \alpha_1 + \tau \wedge \beta_1 + d\nu (\ln \rho) \Theta.$$ 

To prove (8.3d) we proceed by induction. For $\alpha_0 = \Theta$ the statement is true. Assume that (8.3d) is satisfied for some fixed $j$. By Proposition 8.7 and Lemma 8.8

$$d\alpha_{j+1} = dX(\alpha_j) = X(d\alpha_j) + \mu_1 \wedge X(\alpha_j) + \beta \wedge Y(\alpha_j)$$

$$\equiv X(\sigma \wedge \alpha_{j+1}) \mod \{\beta_1, \Theta, \alpha_1, \ldots, \alpha_{j+1}\}$$

$$\equiv \sigma \wedge \alpha_{j+2} \mod \{\beta_1, \Theta, \alpha_1, \ldots, \alpha_{j+1}\}.$$ 

the proof of (8.3c) is similar.

Theorem 8.10. Let $\mathcal{R}$ be a scalar hyperbolic second-order partial differential equation in the plane which is Darboux integrable at level $k + 1 \geq 2$. Then there are two 1-forms $\pi^1$ and $\pi^2$ on $\mathcal{R}^{k+1}$ such that $(\tilde{\sigma}, \pi^1, \pi^2, \Theta, \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)$ is a coframing of $\mathcal{R}^{k+1}$ so that the following structure equations hold.

$$d\tilde{\sigma} \equiv 0 \mod \{\tilde{\sigma}\} \quad \text{and} \quad d\tilde{\sigma} \equiv 0 \mod \{\tilde{\sigma}\},$$ \hspace{1cm} (8.5a)

$$d\pi^1 \equiv 0 \mod \{\pi^1\} \quad \text{and} \quad d\pi^2 \equiv 0 \mod \{\pi^2\},$$ \hspace{1cm} (8.5b)
and

\[ d\Theta \equiv 0 \mod C_{k+1}, \]  
\[ d\alpha_j \equiv 0 \mod C_{k+1}, \quad j = 1, \ldots, k - 1, \]  
\[ d\beta_i \equiv 0 \mod C_{k+1}, \quad i = 1, \ldots, k - 1, \]  
\[ d\alpha_k \equiv \tilde{\sigma} \wedge \pi^1 \mod C_{k+1}, \]  
\[ d\beta_k \equiv \tilde{\tau} \wedge \pi^2 \mod C_{k+1}, \]  

where \( C_{k+1} \) denotes the contact ideal on \( \mathcal{R}^{k+1} \).

**Proof.** Assume that \( H_p = 0 \) and \( K_q = 0 \). By assumption \( \max\{p, q\} \leq k \).

First we shall prove that the form \( \tilde{\tau} \) is a form on \( \mathcal{R}^{k+1} \). By Theorem 6.3 and Definition 8.5, the equations (8.5a) hold and so there exists a function \( I \) on \( \mathcal{R}^k \) such that

\[ \tilde{\tau} = a dI \]

for some nonvanishing function \( a \) on \( \mathcal{R}^\infty \). Moreover, since

\[ \tilde{\tau} \equiv \tau \equiv \frac{1}{m_x n_y - m_y n_x} (-m_y dx + m_x dy) \mod \{ \alpha_1, \Theta, \beta_1, \ldots, \beta_k \}, \]

and

\[ \tilde{\tau} = a dI = a (d_H I + d_V I) \equiv a (D_x (I) dx + D_y (I) dy) \mod \{ \alpha_1, \Theta, \beta_1, \ldots, \beta_k \} \]

we deduce that \( a \) is of order \( \leq k + 1 \). Hence \( \tilde{\tau} \) is a form on \( \mathcal{R}^{k+1} \). Similarly \( \tilde{\sigma} \) is a form on \( \mathcal{R}^{k+1} \).

By Theorem 6.3 the Pfaffian system

\[ \Omega^1 (\tilde{\tau}, \eta_{p+1} \ldots, \eta_{k+1}) \]

is integrable. Choose a function \( \tilde{I} \) on \( \mathcal{R}^{k+1} \) such that

\[ b d\tilde{I} \equiv \eta_{k+1} \mod \{ \tilde{\tau}, \eta_{p+1} \ldots, \eta_{k+1} \}, \]

for some nonvanishing function \( b \) on \( \mathcal{R}^\infty \). Similarly we choose a function \( \tilde{J} \) on \( \mathcal{R}^{k+1} \) such that

\[ c d\tilde{J} \equiv \xi_{k+1} \mod \{ \tilde{\sigma}, \xi_{q+1} \ldots, \xi_{k+1} \}, \]
for some nonvanishing function \( c \) on \( \mathcal{R}^\infty \). Then on \( \mathcal{R}^\infty \)

\[
\hat{\sigma} \wedge \hat{\tau} \wedge \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_k \wedge \beta_k \wedge d\bar{I} \wedge d\bar{J}
\]

\[
= \sigma \wedge \tau \wedge \xi_1 \wedge \eta_1 \wedge \cdots \wedge \xi_k \wedge \eta_k \wedge \frac{1}{b} \eta_{k+1} \wedge \frac{1}{c} \xi_{k+1} \neq 0.
\]

This proves that \((\bar{\sigma}, d\bar{J}, \bar{\tau}, d\bar{I}, \Theta, \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)\) is a coframing of \( \mathcal{R}^{k+1} \). Congruences (8.7a) - (8.7c) were proven in the previous theorem. From the fact that

\[
\eta_{k+1} \equiv \beta_{k+1} \mod \{ \Theta, \beta_1, \ldots, \beta_k \}
\]

we deduce

\[
d\beta_k \equiv \tau \wedge \beta_{k+1} \mod \{ \alpha, \Theta, \beta_1, \ldots, \beta_k \}
\]

\[
\equiv \bar{\tau} \wedge \eta_{k+1} \mod \{ \alpha, \Theta, \beta_1, \ldots, \beta_k \}
\]

\[
\equiv \bar{\tau} \wedge bd\bar{I} \mod \{ \alpha, \Theta, \beta_1, \ldots, \beta_k \}, \quad (8.9)
\]

on \( \mathcal{R}^\infty \). Since \( d\beta_k \) is a 2-form on \( \mathcal{R}^{k+1} \) it follows that

\[
d\beta_k \equiv \bar{\tau} \wedge bd\bar{I} \mod \{ \alpha, \Theta, \beta_1, \ldots, \beta_k \}
\]

holds on \( \mathcal{R}^{k+1} \). This, in particular, means that \( b \) is a function of order \( \leq k + 1 \). We now choose \( \pi^2 = b \, d\bar{I} \). This ends the proof of (8.7e). A similar proof holds for (8.7d). \( \blacksquare \)

The following Corollary is immediate.

**Corollary 8.11.** Let \( k \geq 1 \) and let \( \mathcal{R} \) be a scalar hyperbolic second-order partial differential equation in the plane which is Darboux integrable at level \( k+1 \). Then there exists a coframing of \( \mathcal{R}^{k+1} \)

\[
(\theta^1, \ldots, \theta^{2k+1}; \omega^1, \omega^2, \omega^3, \omega^4)
\]

so that the contact ideal \( \mathcal{C}_{k+1} \) is generated algebraically by

\[
\{ \theta^1, \ldots, \theta^{2k+1}; \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4, \}
\]

and the Pfaffian systems \( \{ \omega^1, \omega^2 \} \) and \( \{ \omega^3, \omega^4 \} \) are integrable. In other words, \((\mathcal{R}^{k+1}, \mathcal{C}_{k+1})\) is a hyperbolic Darboux system.

Note that we can choose the 1-forms \( \omega^1, \omega^2 \) and \( \omega^3, \omega^4 \) to be the differentials of the \( X \) and \( Y \) invariant functions.
Example 1. The wave equation

\[ s = 0 \]

which is Darboux integrable at level 1, can be represented as a hyperbolic Darboux system of class 1 on a 5-manifold. In local coordinates \((x, y, u, p, q)\), the differential ideal \(\mathcal{I}\) is algebraically generated by \(\theta, \omega^1 \wedge \omega^2, \text{ and } \omega^3 \wedge \omega^4\), where

\[ \theta = du - p \, dx - q \, dy, \quad \omega^1 = dx, \quad \omega^2 = dp, \quad \omega^3 = dy, \quad \omega^4 = dq. \]

We remark that the wave equation may also be represented by a hyperbolic Darboux system \(\mathcal{I}'\) of class 0 on a 4-manifold with local coordinates \((x, y, p, q)\). The differential ideal \(\mathcal{I}'\) is algebraically generated by \(\omega^1 \wedge \omega^2, \text{ and } \omega^3 \wedge \omega^4\). Indeed the integral manifolds \((p, q) : (x, y) \to (x, y, p, q)\) of \(\mathcal{I}'\), satisfying the independence condition \(\omega^1 \wedge \omega^3 = dx \wedge dy \neq 0\) satisfy

\[ \frac{\partial p}{\partial y} = 0 \quad \text{and} \quad \frac{\partial q}{\partial x} = 0. \]

Thus there exists a function \(u\) such that

\[ \frac{\partial u}{\partial x} = p \quad \text{and} \quad \frac{\partial u}{\partial y} = q. \]

Integrating, we obtain the general solution to the wave equation \(u = \varphi(x) + \psi(y)\) where \(\varphi\) and \(\psi\) are arbitrary functions.

Example 2. The linear equation

\[ s = \frac{q}{x + y} \quad (8.10) \]

which is Darboux integrable at levels 1 and 2, can be represented as a hyperbolic Darboux system of class 2 on a 6-manifold. In local coordinates \((x, y, u, p, q, r)\) the differential ideal \(\mathcal{I}\) is algebraically generated by \(\theta, \theta_x, \omega^1 \wedge \omega^2, \text{ and } \omega^3 \wedge \omega^4\), where

\[ \theta = du - p \, dx - q \, dy, \quad \theta_x = dp - r \, dx - \frac{q}{x + y} \, dy \]

and

\[ \omega^1 = dx, \quad \omega^2 = d\tilde{I}, \quad \text{and} \quad \omega^3 = dy, \quad \omega^4 = d\tilde{J} \]

and where

\[ \tilde{I} = \frac{q}{x + y} \quad \text{and} \quad \tilde{J} = r. \]
We remark that the equation (8.10) can be also represented by a hyperbolic Darboux system $\mathcal{I}'$ of class 1 on a 5-manifold with local coordinates $(x, y, p, q, r)$. The differential ideal $\mathcal{I}'$ is algebraically generated by $\theta_x, \omega^1 \wedge \omega^2$ and $\omega^3 \wedge \omega^4$.

**Example 3.** The equation

$$s = pe^u$$

which is Darboux integrable at levels 1 and 2, can be represented as a hyperbolic Darboux system of class 2 on a 6-manifold. In local coordinates $(x, y, u, p, q, r)$ the differential ideal $\mathcal{I}$ is algebraically generated by $\theta, \theta_x, \omega^1 \wedge \omega^2$ and $\omega^3 \wedge \omega^4$, where

$$\theta = du - pdx - qdy, \quad \theta_x = dp - rdx - pe^u dy$$

and

$$\omega^1 = dx, \quad \omega^2 = d\tilde{J} \quad \text{and} \quad \omega^3 = dy, \quad \omega^4 = d\tilde{I}$$

and where

$$\tilde{I} = q - e^u \quad \text{and} \quad \tilde{J} = \frac{r}{p} - p.$$

**Example 4.** The Liouville equation

$$s = e^u,$$

which is Darboux integrable at level 2, can be represented as a hyperbolic Darboux system of class 3 on a 7-manifold. In local coordinates $(x, y, u, p, q, r, t)$ the differential ideal $\mathcal{I}$ is algebraically generated by $\theta, \theta_x, \theta_y, \omega^1 \wedge \omega^2$ and $\omega^3 \wedge \omega^4$ where

$$\theta = du - pdx - qdy, \quad \theta_x = dp - rdx - e^u dy,$$

$$\theta_y = dq - e^u dx - tdy$$

and

$$\omega^1 = dx, \quad \omega^2 = d\tilde{J} \quad \text{and} \quad \omega^3 = dy, \quad \omega^4 = d\tilde{I}$$

and

$$\tilde{I} = t - \frac{q^2}{2} \quad \text{and} \quad \tilde{J} = r - \frac{q^2}{2}.$$
Example 5. The equation

\[ s = up, \]

which is Darboux integrable at levels 1 and 3, can be represented as a hyperbolic Darboux system of class 3 on a 7-manifold. In local coordinates \((x, y, u, p, q, r, u_{xxx})\), the differential ideal \(\mathcal{I}\) is algebraically generated by \(\theta, \theta_x, \theta_{xx}, \omega^1 \wedge \omega^2\) and \(\omega^3 \wedge \omega^4\), where

\[
\theta = du - pdx - qdy,
\]

\[
\theta_x = dp - rdx - updy,
\]

\[
\theta_{xx} = dr - u_{xxx}dx - (p^2 + ur)dy
\]

and

\[
\omega^1 = dx, \quad \omega^2 = d\bar{J} \quad \text{and} \quad \omega^3 = dy, \quad \omega^4 = d\bar{I}
\]

and

\[
\bar{I} = q - \frac{u^2}{2} \quad \text{and} \quad \bar{J} = \frac{3r^2 - 2u_{xxx}p}{2p^2}.
\]

Definition 8.12. Let \((M_1, \mathcal{I}_1)\) and \((M_2, \mathcal{I}_2)\) be two hyperbolic systems. A \(C^\infty\) map \(\Phi : M_1 \to M_2\) is called a homomorphism of hyperbolic systems if it is a homomorphism of exterior differential systems, that is, if \(\Phi^\ast(\mathcal{I}_2) \subseteq \mathcal{I}_1\). Two hyperbolic systems \((M_1, \mathcal{I}_1)\) and \((M_2, \mathcal{I}_2)\) are called equivalent if they are equivalent as exterior differential systems, that is, if there exists a diffeomorphism \(\Phi : M_1 \to M_2\), which is a homomorphism.

Observe that if \((N, \phi)\) is an integral manifold of the hyperbolic system \((M, \mathcal{I})\), that is, \(\phi^\ast(\mathcal{I}) = \{0\}\), and if \(\Phi : M \to \tilde{M}\) is a homomorphism of hyperbolic systems \((M, \mathcal{I})\) and \((\tilde{M}, \tilde{\mathcal{I}})\), then \((N, \Phi \circ \phi)\) is an integral submanifold of \((\tilde{M}, \tilde{\mathcal{I}})\). Indeed,

\[
(\Phi \circ \phi)^\ast(\tilde{\mathcal{I}}) = \phi^\ast(\Phi^\ast(\tilde{\mathcal{I}})) \subseteq \phi^\ast(\mathcal{I}) = \{0\}.
\]

We have seen by examples that a Darboux integrable equation with \(H_p = 0\) and \(K_q = 0\) can be represented as a hyperbolic Darboux system of class \(s = p + q + 1\).

Let \(\mathcal{R}\) and \(\mathcal{R}'\) be two Darboux integrable linear equations, that is, equations of type

\[
s + a(x, y)p + b(x, y)q + c(x, y)u = 0,
\]

which are transformed one into another by a sequence of classical Laplace transformations. If \(H_p = K_q = 0\) and \(H'_{pp'} = K'_{qq'} = 0\), then it is easy to see that

\[
s = p + q + 1 = p' + q' + 1
\]
and so \( \mathcal{R} \) and \( \mathcal{R}' \) can be represented as hyperbolic Darboux systems \( \mathcal{I} \) and \( \mathcal{I}' \) of class \( s \).

We conjecture that if \( \mathcal{R} \) and \( \mathcal{R}' \) be two Darboux integrable linear equations that are transformed one into another by a sequence of classical Laplace transformations, then they can be represented as equivalent hyperbolic Darboux systems.

In the following example we shall construct an invertible homomorphism between the hyperbolic systems associated with two nonequivalent linear differential equations. The construction of the homomorphism is based on the classical Laplace transformation.

**Example 6.** The linear hyperbolic equation

\[
 s - \frac{2u}{(x + y)^2} = 0. \tag{8.11}
\]

is transformed, using the Laplace transformation \( \bar{x} = x, \bar{y} = y \) and \( \bar{u} = p \), into the equation

\[
 \bar{s} + \frac{2q}{\bar{x} + \bar{y}} - \frac{2\bar{u}}{(\bar{x} + \bar{y})^2} = 0. \tag{8.12}
\]

Using the Laplace transformation, we build a diffeomorphism which maps the hyperbolic Darboux system corresponding to the equation (8.11) into the hyperbolic Darboux system corresponding to the equation (8.12). The equation (8.11) is Darboux integrable at level 2, that is, \( H_1 = K_1 = 0 \). The functions

\[
 I_1 = y, \quad I_2 = t + \frac{2q}{x + y},
\]

are the \( D_x \) invariant for (8.11) and

\[
 J_1 = x, \quad J_2 = r + \frac{2p}{x + y},
\]

are the \( D_y \) invariant functions. The equation (8.12) is Darboux integrable at levels 3 and 1, that is, \( H_2 = K_0 = 0 \). The functions

\[
 \bar{I}_1 = \bar{y}, \quad \bar{I}_2 = (\bar{x} + \bar{y})^2 \bar{u}_{gbb} + 6(\bar{x} + \bar{y}) \bar{i} + 6\bar{q}
\]

are the \( D_z \) invariant for (8.12) and

\[
 \bar{J}_1 = \bar{x}, \quad \bar{J}_2 = \bar{p} + \frac{2\bar{u}}{\bar{x} + \bar{y}}
\]
are the $D_y$ invariant functions.

We associate the equation (8.11) with the hyperbolic Darboux system $\mathcal{I}$ on a 7-manifold with coordinates $(x, y, u, p, q, r, t)$. $\mathcal{I}$ is algebraically generated by $\theta, \theta_x, \theta_y, \omega^1 \wedge \omega^2$ and $\omega^3 \wedge \omega^4$, where

$$\theta = du - pdx - qdy,$$

$$\theta_x = dp - rdx - \frac{2u}{(x+y)^2} dy,$$

$$\theta_y = dq - \frac{2u}{(x+y)^2} dx - tdy$$

and

$$\omega^1 = dI_1, \quad \omega^2 = dI_2 \quad \text{and} \quad \omega^3 = dJ_1, \quad \omega^4 = dJ_2,$$

We associate the equation (8.12) with the hyperbolic Darboux system $\mathcal{I}$ on a 7-manifold with coordinates $(\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}, \bar{t}, \bar{u}_{yyy})$. $\mathcal{I}$ is algebraically generated by $\bar{\theta}, \bar{\theta}_y, \bar{\theta}_{yy}, \bar{\omega}^1 \wedge \bar{\omega}^2$ and $\bar{\omega}^3 \wedge \bar{\omega}^4$, where

$$\bar{\theta} = d\bar{u} - \bar{p}d\bar{x} - \bar{q}d\bar{y},$$

$$\bar{\theta}_y = d\bar{q} + \left( \frac{2\bar{q}}{\bar{x} + \bar{y}} - \frac{2\bar{u}}{(\bar{x} + \bar{y})^2} \right) d\bar{x} - \bar{t}dy,$$

$$\bar{\theta}_{yy} = d\bar{t} - \left( \frac{2\bar{t}}{\bar{x} + \bar{y}} - \frac{4\bar{q}}{(\bar{x} + \bar{y})^2} + \frac{4\bar{u}}{(\bar{x} + \bar{y})^3} \right) dx - \bar{u}_{yyy} dy$$

and

$$\bar{\omega}^1 = d\bar{I}_1, \quad \bar{\omega}^2 = d\bar{I}_2 \quad \text{and} \quad \bar{\omega}^3 = d\bar{J}_1, \quad \bar{\omega}^4 = d\bar{J}_2.$$

Consider the map $\Phi : (x, y, u, p, q, r, t) \rightarrow (\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}, \bar{t}, \bar{u}_{yyy})$ defined by

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{u} = p, \quad \bar{p} = r, \quad \bar{q} = \frac{2u}{(x+y)^2},$$

$$\bar{t} = \frac{2q}{(x+y)^2} - \frac{4u}{(x+y)^3} \quad \text{and} \quad \bar{u}_{yyy} = \frac{2t}{(x+y)^2} - \frac{8q}{(x+y)^3} + \frac{12u}{(x+y)^4}.$$ 

One can easily see that $\Phi$ is invertible and since

$$\Phi^*(\bar{\theta}) = \theta_x, \quad \Phi^*(\bar{\theta}_y) = \frac{2}{(x+y)^2} \theta, \quad \Phi^*(\bar{\theta}_{yy}) = \frac{2}{(x+y)^2} \theta_y - \frac{4}{(x+y)^3} \theta.$$
and
\[ \Phi^*(I_2) = 2I_2, \quad \Phi^*(J_2) = J_2, \]
the map \( \Phi \) is a diffeomorphism.

We say that a second-order scalar hyperbolic equation in the plane
\[ F(x, y, u, p, q, r, s, t) = 0 \]
is of Moutard type, see [23] §231, if its general solution can be written in a closed form
\[ u = f(x, y, \varphi(x), \varphi'(x), \ldots, \varphi^{(p)}(x), \psi(y), \psi'(y), \ldots, \psi^{(q)}(y)), \quad (8.13) \]
where \( \varphi \) and \( \psi \) are arbitrary functions. An equation of Moutard type with the general solution (8.13) is Darboux integrable at levels \( p + 1 \) and \( q + 1 \) and so it can be represented as a hyperbolic Darboux system \( \mathcal{I} \) of class \( s = p + q + 1 \). Denote by \( \mathcal{I}^k_w \) the hyperbolic Darboux system of class \( k \) associated with the \( k \)th-prolongation of the wave equation, that is, \( \mathcal{I}^k_w \) is algebraically generated by
\[ \theta = du - px dx - qdy, \quad \theta_x = dp - r dx \quad \theta_y = dq - t dy, \quad \ldots, \]
\[ \ldots, \theta_{x^{p+1}} = du_{x^{p+1}} - u_{x^{p+1}} dx, \quad \theta_{y^{q+1}} = du_{y^{q+1}} - u_{y^{q+1}} dy \]
and
\[ dx \wedge du_{x^{k}}, \quad dy \wedge du_{y^{k}}. \]

We conjecture that if \( \mathcal{I} \) is a hyperbolic Darboux system of class \( s = p + q + 1 \) associated with the equation of Moutard type that admits a general solution (8.13), then there exists a homomorphism from \( \mathcal{I}^k_w \) into \( \mathcal{I} \), for some \( k = 2, 3, \ldots \).

Example 7. The wave equation \( s = 0 \) can be represented as a hyperbolic Darboux system \( \mathcal{I} \) on a 7-manifold with coordinates \((x, y, u, p, q, r, t)\). The differential ideal \( \mathcal{I} \) is algebraically generated by \( \theta_x, \theta_y, dx \wedge dr \), and \( dy \wedge dt \), where
\[ \theta = du - px dx - qdy, \quad \theta_x = dp - r dx \quad \text{and} \quad \theta_y = dq - t dy. \]
The linear equation \( \tilde{s} = \frac{\bar{q}}{\bar{x} + \bar{y}} \) can be represented as a hyperbolic Darboux system on a 6-manifold with coordinates \((\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}, \bar{r})\). The differential ideal \( \tilde{\mathcal{I}} \) is algebraically generated by \( \tilde{\theta}_x, \tilde{\omega}^1 \wedge \tilde{\omega}^2 \), and \( \tilde{\omega}^3 \wedge \tilde{\omega}^4 \), where
\[ \tilde{\theta} = d\bar{u} - p\bar{dx} - q\bar{dy}, \quad \tilde{\theta}_x = d\bar{p} - r\bar{dx} - \frac{q}{\bar{x} + \bar{y}} d\bar{y}. \]
and
\[ \tilde{\omega}^1 = d\tilde{x}, \quad \tilde{\omega}^2 = d\tilde{J}, \quad \tilde{\omega}^3 = d\tilde{y}, \quad \tilde{\omega}^4 = d\tilde{I} \]
and where
\[ \tilde{I} = \frac{\tilde{q}}{\tilde{x} + \tilde{y}} \quad \tilde{J} = \tilde{r}. \]

Consider the map \( \Phi : (x, y, u, p, q, r, t) \to (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{p}, \tilde{q}, \tilde{r}) \) defined by
\[
\begin{align*}
\tilde{x} = x, & \quad \tilde{y} = y, \quad \tilde{u} = u - (x + y)q \\
\tilde{p} = p - q, & \quad \tilde{q} = -(x + y)t, \quad \tilde{r} = r.
\end{align*}
\]
Since
\[
\Phi^*(\tilde{\theta}) = -(x + y)\theta_y + \theta, \quad \Phi^*(\tilde{\theta}_z) = \theta_z - \theta_y, \quad \Phi^*(\tilde{I}) = -t, \quad \Phi^*(\tilde{J}) = r
\]
and
\[
\Phi^*(\tilde{I}) = t, \quad \Phi^*(\tilde{J}) = r,
\]
the map \( \Phi \) is a homomorphism.

**Example 8.** In this example we construct a homomorphism from a hyperbolic Darboux system associated with the second prolongation of the wave equation \( s = 0 \) into a hyperbolic Darboux system associated with the Liouville equation \( \tilde{s} = e^\theta. \) Let \( \mathcal{I} \) be a differential ideal on a 9-manifold with coordinates \((x, y, u, p, q, r, t, u_{xxx}, u_{yyy})\) algebraically generated by \( \theta, \theta_x, \theta_y, \theta_{xx}, \theta_{yy}, dx \wedge du_{xxx} \)
and \(dy \wedge du_{yyy},\) where
\[
\begin{align*}
\theta = du - p dx - q dy, \quad & \theta_x = dp - r dx, \quad \theta_y = dq - t dy,
\theta_{xx} = dr - u_{xxx} dx & \quad \text{and} \quad \theta_{yy} = dt - u_{yyy} dy.
\end{align*}
\]
Let \( \tilde{\mathcal{I}} \) be a differential ideal on a 7-manifold with coordinates \((\tilde{x}, \tilde{y}, \tilde{u}, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{t})\), algebraically generated by \( \tilde{\theta}, \tilde{\theta}_z, \tilde{\theta}_{xx}, \tilde{\omega}^1 \wedge \tilde{\omega}^2 \) and \( \tilde{\omega}^3 \wedge \tilde{\omega}^4,\) where
\[
\begin{align*}
\tilde{\theta} = d\tilde{u} - \tilde{p} d\tilde{x} - \tilde{q} d\tilde{y}, \quad & \tilde{\theta}_z = d\tilde{p} - \tilde{r} d\tilde{x} - e^\theta d\tilde{y},
\tilde{\theta}_y = d\tilde{q} - e^\theta d\tilde{x} - \tilde{t} d\tilde{y} & \quad \text{and} \quad \tilde{\omega}^1 = d\tilde{x}, \quad \tilde{\omega}^2 = d\tilde{J}, \quad \tilde{\omega}^3 = d\tilde{y}, \quad \tilde{\omega}^4 = d\tilde{I}.
\end{align*}
\]
and where

\[ \tilde{I} = \tilde{t} - \frac{q^2}{2}, \quad \tilde{J} = \tilde{r} - \frac{\tilde{p}^2}{2}. \]

Consider the map \( \Phi : (x, y, u, p, q, r, t, u_{xxx}, u_{yyy}) \rightarrow (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{t}) \) defined by

\[ \begin{align*}
\tilde{x} &= x, \quad \tilde{y} = y, \quad \tilde{u} = \ln \frac{2pq}{u^2}, \\
\tilde{p} &= \frac{r}{p} - \frac{2p}{u}, \quad \tilde{q} = \frac{t}{q} - \frac{2q}{u}, \\
\tilde{r} &= \frac{u_{xxx}}{p} - \frac{r^2}{p^2} - \frac{2r}{u} + \frac{2p^2}{u^2}, \\
\tilde{t} &= \frac{u_{yyy}}{q} - \frac{t^2}{q^2} - \frac{2t}{u} + \frac{2q^2}{u^2}.
\end{align*} \]

Since

\[ \Phi^* (\theta) = \frac{1}{p} \theta_x + \frac{1}{q} \theta_y - \frac{2}{u} \theta, \]

\[ \Phi^* (\theta_x) = \frac{1}{p} \theta_{xx} - \left( \frac{r}{p^2} + \frac{2}{u} \right) \theta_x + \frac{2p}{u^2} \theta, \]

\[ \Phi^* (\theta_y) = \frac{1}{q} \theta_{yy} - \left( \frac{t}{q^2} + \frac{2}{u} \right) \theta_y + \frac{2q}{u^2} \theta \]

and

\[ \Phi^* (\omega^2) \equiv du_{xxx} - \frac{3r}{p^2} \theta_{xx} + \frac{3p^2}{p^3} \theta_x \mod \{dx\}, \]

\[ \Phi^* (\omega^4) \equiv du_{yyy} - \frac{3t}{q^2} \theta_{yy} + \frac{3t^2}{q^3} \theta_y \mod \{dy\}, \]

the map \( \Phi \) is a homomorphism of hyperbolic Darboux systems \( \mathcal{I} \) and \( \tilde{\mathcal{I}} \).

We remark that the classical Laplace transformation can be generalized to define a transformation on a larger family of equations. This family in particular includes all equations of Moutard type. This “generalized” Laplace transformation can be used to define invertible homomorphisms between hyperbolic Darboux systems associated with nonequivalent equations. In the following example we will use this “generalized” Laplace transformation to show that the hyperbolic Darboux systems associated with the Liouville equation \( s = e^u \) and the equation \( \tilde{s} = \tilde{u}\tilde{p} \) are equivalent.

**Example 9.** The Liouville equation and the equation \( \tilde{s} = \tilde{u}\tilde{p} \) are two nonlinear equations that are not contact equivalent since they are Darboux integrable at different levels. They are also both of Moutard type since they possess general solutions in closed form. The hyperbolic Darboux system
for the Liouville equation is given as before by a differential ideal $\mathcal{I}$ on a 7-manifold with coordinates $(x, y, u, p, q, r, t)$. $\mathcal{I}$ is algebraically generated by $\theta, \theta_z, \theta_y, \omega^1 \wedge \omega^2$ and $\omega^3 \wedge \omega^4$, where

\[
\begin{align*}
\theta &= du - pdx - qdy, \\
\theta_z &= dp - rdx - e^uy, \\
\theta_y &= dq - e^ydx - tdy
\end{align*}
\]

and

\[
\omega^1 = dx, \quad \omega^2 = dJ \quad \text{and} \quad \omega^3 = dy, \quad \omega^4 = dI,
\]

where

\[
I = t - \frac{q^2}{2} \quad \text{and} \quad J = r - \frac{p^2}{2}.
\]

The equation $\dot{s} = \dot{u}\dot{p}$ is associated with a hyperbolic Darboux system $\hat{\mathcal{I}}$ on a 7-manifold with coordinates $(\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}, \bar{r}, \bar{u}_{zzz})$. $\hat{\mathcal{I}}$ is algebraically generated by $\bar{\theta}, \bar{\theta}_z, \bar{\theta}_{zz}, \bar{\omega}^1 \wedge \bar{\omega}^2$ and $\bar{\omega}^3 \wedge \bar{\omega}^4$, where

\[
\begin{align*}
\bar{\theta} &= du - \bar{p}dx - \bar{q}dy, \\
\bar{\theta}_z &= dp - \bar{r}dx - \bar{u}\bar{p}dy, \\
\bar{\theta}_{zz} &= d\bar{r} - \bar{u}_{zzz}dx - (\bar{p}^2 + \bar{u}\bar{r})dy
\end{align*}
\]

and

\[
\bar{\omega}^1 = d\bar{x}, \quad \bar{\omega}^2 = d\bar{J}, \quad \bar{\omega}^3 = d\bar{y}, \quad \bar{\omega}^4 = d\bar{I}
\]

and

\[
\begin{align*}
\bar{I} &= \bar{q} - \frac{\bar{u}^2}{2}, \\
\bar{J} &= \frac{3\bar{r}^2 - 2\bar{u}_{zzz}\bar{p}}{2\bar{p}^2}.
\end{align*}
\]

Consider the map $\Phi : (x, y, u, p, q, r, t) \to (\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}, \bar{r}, \bar{u}_{zzz})$ defined by

\[
\begin{align*}
x &= \bar{x}, & \bar{y} &= y, & \bar{u} &= q, & \bar{p} &= e^u, & \bar{q} &= t, \\
\bar{r} &= e^up, & \bar{u}_{zzz} &= e^u(p^2 + r).
\end{align*}
\]

One can easily see that $\Phi$ is invertible and since

\[
\begin{align*}
\Phi^*(\bar{\theta}) &= \theta_y, & \Phi^*(\bar{\theta}_z) &= e^u\theta, & \Phi^*(\bar{\theta}_{zz}) &= e^u p\theta + e^u \theta_x
\end{align*}
\]

and

\[
\Phi^*(\bar{I}) = I, \quad \Phi^*(\bar{J}) = -J,
\]

the hyperbolic systems $\mathcal{I}$ and $\hat{\mathcal{I}}$ are equivalent.
CHAPTER 9
FORMULAS

In chapters 10 - 13 of this dissertation, we will characterize Monge-Ampère equations, equations which admit complete or intermediate first integrals, solve the inverse problem ot the calculus of variations and characterize certain classical equations. Many of these characterizations will be obtained in terms of the Monge-Ampère invariants $M_σ, M_τ$ and the first Laplace invariants $H₀, K₀$.

The purpose of this chapter is to derive explicit formulas for these invariants. We also derive the recursion formulas for $H_n$ and $K_n$ that are generalizations of the formulas in [23] (page 52) for the classical Laplace invariants. Let $ℝ$ be a hyperbolic equation given by

$$F(x,y,u,p,q,r,s,t) = 0.$$  

(9.1)

Since

$$dx(X) = m_x, \quad dx(Y) = n_x \quad \text{and} \quad dy(X) = m_y, \quad dy(Y) = n_y$$

we easily conclude that $dx$ and $dy$ are in terms of $σ$ and $τ$ given by

$$dx = m_xσ + n_xτ \quad \text{and} \quad dy = m_yσ + n_yτ.$$  

(9.2)

From the characteristic equation, we easily deduce

$$m_xn_xθ_{xx} + (m_yn_x + m_xn_y)θ_{xy} + m_yn_yθ_{yy} ≡ 0 \mod \{θ, θ_x, θ_y\}.$$  

(9.3)

Recall that

$$Θ = ρθ.$$  

We easily compute

$$ξ_2 ≡ ρ(m_x^2θ_{xx} + 2m_xm_yθ_{xy} + m_y^2θ_{yy}) \mod \{θ, θ_x, θ_y\},$$  

(9.4a)

$$η_2 ≡ ρ(n_x^2θ_{xx} + 2n_xn_yθ_{xy} + n_y^2θ_{yy}) \mod \{θ, θ_x, θ_y\}.$$  

(9.4b)

Let $g(x,y,u,p,q,r,s,t)$ be a second-order function on $ℝ^∞$. Then

$$dVG ≡ \frac{∂g}{∂r}θ_{xx} + \frac{∂g}{∂s}θ_{xy} + \frac{∂g}{∂t}θ_{yy} \mod \{θ, θ_x, θ_y\}.$$  

Our next goal is to find functions $a$ and $b$ such that

$$\frac{∂g}{∂r}θ_{xx} + \frac{∂g}{∂s}θ_{xy} + \frac{∂g}{∂t}θ_{yy} ≡ aξ_2 + bη_2 \mod \{θ, θ_x, θ_y\}.$$  


Let \( a_1, a_2, a_3 \) be functions satisfying

\[
a_1 \varrho \xi_1 + a_3 \varrho \eta_1
\]

\[
\equiv a_1 \varrho (m_x^2 \theta_{xx} + 2m_x m_y \theta_{xy} + m_y^2 \theta_{yy}) + a_2 \varrho (m_x n_x \theta_{xx} + (m_y n_n + m_x n_y) \theta_{xy} + m_y n_y \theta_{yy})
\]

\[+ a_3 \varrho (n_x^2 \theta_{xx} + 2n_x n_y \theta_{xy} + n_y^2 \theta_{yy}) \mod \{ \theta, \theta_x, \theta_y \} \]

\[
\equiv \frac{\partial \varrho}{\partial r} \theta_{xx} + \frac{\partial \varrho}{\partial s} \theta_{xy} + \frac{\partial \varrho}{\partial t} \theta_{yy} \mod \{ \theta, \theta_x, \theta_y \} \]

\[
\equiv d_v \varrho \mod \{ \theta, \theta_x, \theta_y \},
\]

and so we arrive at

\[
\frac{\partial \varrho}{\partial r} = \varrho (a_1 n_x^2 + a_2 m_x n_x + a_3 n_y^2), \tag{9.5a}
\]

\[
\frac{\partial \varrho}{\partial r} = \varrho (2a_1 m_x m_y + a_2 (m_x n_y + m_y n_x) + 2a_3 n_x n_y), \tag{9.5b}
\]

\[
\frac{\partial \varrho}{\partial t} = \varrho (a_1 m_y^2 + a_2 m_y n_y + a_3 n_y^2). \tag{9.5c}
\]

Since,

\[
\text{det} \begin{pmatrix}
m_x^2 & m_x n_x & n_x^2 \\
2m_x m_y & m_x n_y + m_y n_x & 2n_x n_y \\
m_y^2 & m_y n_y & n_y^2
\end{pmatrix} = \delta^3,
\]

where

\[
\delta = m_x n_y - n_x m_y,
\]

we can solve for \( a_1, a_2, a_3 \) from (9.5) to obtain

\[
a_1 = \frac{1}{\theta^2} \left( n_y^2 \frac{\partial \varrho}{\partial r} - n_x n_y \frac{\partial \varrho}{\partial s} + n_x^2 \frac{\partial \varrho}{\partial t} \right),
\]

\[
a_3 = \frac{1}{\theta^2} \left( m_y \frac{\partial \varrho}{\partial r} - m_x m_y \frac{\partial \varrho}{\partial s} + m_x^2 \frac{\partial \varrho}{\partial t} \right)
\]

and so we conclude

\[
d_v \varrho \equiv \frac{1}{\delta^3} \left[ \left( n_y^2 \frac{\partial \varrho}{\partial r} - n_x n_y \frac{\partial \varrho}{\partial s} + n_x^2 \frac{\partial \varrho}{\partial t} \right) \xi_2 + \left( m_y \frac{\partial \varrho}{\partial r} - m_x m_y \frac{\partial \varrho}{\partial s} + m_x^2 \frac{\partial \varrho}{\partial t} \right) \eta_2 \right] \mod \{ \theta, \theta_x, \theta_y \}. \tag{9.6}
\]
Recall that

\[ d_{\gamma} \sigma = \sigma \wedge \mu_1 + \tau \wedge \alpha \quad \text{and} \quad d_{\gamma} \tau = \sigma \wedge \beta + \tau \wedge \mu_2, \]

where \( \alpha \) and \( \beta \) are contact forms of adapted order 2 (see Proposition 4.2),

\[ \alpha = d_{\gamma} \eta_2 + d_{\gamma} \eta_1 + e_0 \theta + e_1 \xi_1 + M_\sigma \xi_2 \]

and

\[ \beta = b_2 \xi_2 + b_1 \xi_1 + c_0 \theta + c_1 \eta_1 + M_\tau \eta_2. \]

Also recall that

\[ \sigma = \frac{1}{\delta} (n_y dx - n_z dy) \quad \text{and} \quad \tau = \frac{1}{\delta} (-m_y dx + m_z dy). \]

Using (9.2), we have

\[ d_{\gamma} \sigma \equiv \frac{1}{\delta} (d_{\gamma} n_y \wedge dx - d_{\gamma} n_z \wedge dy) \mod \sigma \]

\[ \equiv -\frac{1}{\delta} \tau \wedge (n_x d_{\gamma} n_y - n_y d_{\gamma} n_x) \mod \sigma. \]

Similarly

\[ d_{\gamma} \tau \equiv \frac{1}{\delta} \sigma \wedge (m_z d_{\gamma} m_y - m_y d_{\gamma} m_z) \mod \tau. \]

By equation (9.6), it is a simple matter to conclude

\[ M_\sigma = -\frac{1}{\delta \sigma^3} [\left( \frac{\partial n_y}{\partial r} n_x - \frac{\partial n_x}{\partial r} n_y \right) n_y^2 - \left( \frac{\partial m_y}{\partial s} n_x - \frac{\partial m_x}{\partial s} n_y \right) n_x n_y + \left( \frac{\partial m_y}{\partial t} n_x - \frac{\partial n_x}{\partial t} n_y \right) n_z^2], \]

\[ M_\tau = \frac{1}{\delta \sigma^3} [\left( \frac{\partial m_y}{\partial r} m_z - \frac{\partial m_z}{\partial r} m_y \right) m_y^2 - \left( \frac{\partial m_y}{\partial s} m_z - \frac{\partial m_z}{\partial s} m_y \right) m_x m_y + \left( \frac{\partial m_y}{\partial t} m_z - \frac{\partial m_z}{\partial t} m_y \right) m_z^2] \]

and

\[ d_2 = -\frac{1}{\delta \sigma^3} [\left( \frac{\partial n_y}{\partial r} n_x - \frac{\partial n_x}{\partial r} n_y \right) n_y^2 - \left( \frac{\partial m_y}{\partial s} n_x - \frac{\partial m_x}{\partial s} n_y \right) n_x m_y + \left( \frac{\partial n_y}{\partial t} n_x - \frac{\partial n_x}{\partial t} n_y \right) m_z^2], \]

\[ b_2 = \frac{1}{\delta \sigma^3} [\left( \frac{\partial m_y}{\partial r} m_z - \frac{\partial m_z}{\partial r} m_y \right) m_y^2 - \left( \frac{\partial m_y}{\partial s} m_z - \frac{\partial m_z}{\partial s} m_y \right) n_x n_y + \left( \frac{\partial m_y}{\partial t} m_z - \frac{\partial m_z}{\partial t} m_y \right) n_z^2]. \]
Now let \( \mathcal{R}^\infty \) be given by a hyperbolic equation
\[
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\]
We choose the characteristic vector fields to be
\[
X = D_x + mD_y \quad \text{and} \quad Y = D_x + nD_y,
\]
where \( m \) and \( n \) are the two distinct, real second-order functions that satisfy the characteristic equation
\[
\lambda^2 - \frac{\partial f}{\partial s} \lambda + \frac{\partial f}{\partial t} = 0,
\]
that is
\[
m + n = \frac{\partial f}{\partial s} \quad \text{and} \quad mn = \frac{\partial f}{\partial t},
\]
and let
\[
\Theta = \theta = du - pdx - qdy.
\]
The \( d_V \) structure equations for \( \sigma \) and \( \tau \) simplify to
\[
d_V\sigma = -\sigma \wedge \beta + \tau \wedge \alpha \quad \text{and} \quad d_V\tau = \sigma \wedge \beta - \tau \wedge \alpha,
\]
and the structure equations (4.59), (4.60), and (4.61) hold. The formulas (4.10) and (4.12) for the coefficients \( A, \ldots, G \) of the universal linearization simplify to
\[
A = \frac{1}{n-m} (X(n) + n \frac{\partial f}{\partial p} - \frac{\partial f}{\partial q}),
\]
\[
B = -A + \frac{\partial f}{\partial p} \quad \text{and} \quad C = G = \frac{\partial f}{\partial u}
\]
\[
D = A + P, \quad \text{and} \quad E = B + Q
\]
and where
\[
P = -Q = \frac{1}{n-m} (Y(m) - X(n)).
\]
Then the formulas for $M_{\sigma}$, $M_{\tau}$, $b_2$ and $d_2$ simplify to

$$M_{\sigma} = \frac{1}{(n-m)^3} \left( \frac{\partial n}{\partial t} - n \frac{\partial m}{\partial s} \right) \quad \text{and} \quad M_{\tau} = \frac{1}{(n-m)^3} \left( \frac{\partial m}{\partial t} - m \frac{\partial n}{\partial s} \right). \tag{9.16a}$$

$$b_2 = \frac{1}{(n-m)^3} \left( \frac{\partial m}{\partial t} - n \frac{\partial m}{\partial s} \right) \quad \text{and} \quad d_2 = -\frac{1}{(n-m)^3} \left( \frac{\partial n}{\partial t} - m \frac{\partial n}{\partial s} \right). \tag{9.16b}$$

Using (9.10), we compute

$$b_2 - d_2 = \frac{1}{(n-m)^3} \left( \frac{\partial m}{\partial t} - n \frac{\partial m}{\partial s} - \frac{\partial n}{\partial t} + m \frac{\partial n}{\partial s} \right) = \frac{1}{(n-m)^3} \left( \frac{\partial f}{\partial t \partial s} - \frac{\partial f}{\partial s \partial t} \right) = 0.$$

**Proposition 9.1.** Let $R$ be a second-order scalar hyperbolic partial differential equation in the plane. Let $\alpha$ and $\beta$ be given as in Proposition 4.2, namely,

$$\alpha = d_2 \eta_2 + d_1 \eta_1 + c_0 \Theta + c_1 \xi_1 + M_{\sigma} \xi_2$$

and

$$\beta = b_2 \xi_2 + b_1 \xi_1 + c_0 \Theta + c_1 \eta_1 + M_{\tau} \eta_2.$$

Then the coefficients $b_2$ and $d_2$ are equal.

**Proof.** We have just proved that $b_2 - d_2 = 0$ for the special coframe considered above. To prove the proposition, we will show that $b_2 - d_2$ is a relative invariant; namely, we will show that if $\Phi : R^{\infty} \rightarrow R'^{\infty}$ is a classical contact transformation for which (4.33) holds, then the function $b_2 - d_2$ is a relative invariant that transforms according to

$$\Phi^*(b_2' - d_2') = \frac{1}{m n d} (b_2 - d_2). \tag{9.17}$$

If

$$\Phi^*(\sigma') = \frac{1}{m} \sigma + a \Theta + b \xi_1 + c \eta_1, \tag{9.18a}$$

$$\Phi^*(\tau') = \frac{1}{n} \tau + e \Theta + h \xi_1 + g \eta_1, \tag{9.18b}$$

then

$$d\Theta' = d_H \Theta' + d_{\nu} \Theta' \equiv \sigma' \wedge \xi_1' + \tau' \wedge \eta_1' \mod \{\Theta'\}. \tag{9.19}$$
Substituting from equations (9.19), (9.18), and (4.35) - (4.38), one obtains

\[
\Phi^*(d\Theta') \equiv l(\sigma \land \xi_1 + \tau \land \eta_1) + (-cml + hnl)\xi_1 \land \eta_1 \quad \text{mod} \{\Theta\}.
\]

Next compute

\[
d(\Phi^*(\Theta')) = d(l\Theta) \equiv l(\sigma \land \xi_1 + \tau \land \eta_1) \quad \text{mod} \{\Theta\}.
\]

Comparison of the last two congruences yield

\[
cm = hn.
\]

(9.20)

Using the Propositions 4.1, 4.2, and 4.6, we compute

\[
\Phi^*(d\sigma') = \Phi^*(-P\sigma' \land \tau' + \sigma' \land \mu'_1 + \tau \land \alpha')
\]

\[
\equiv \frac{1}{n} \tau \land \Phi^*(\alpha') \quad \text{mod} \{\sigma, \Theta, \xi_1, \eta_1\}
\]

\[
\equiv \frac{1}{n} \tau \land (\Phi^*(d'_2)\eta_2 + \Phi^*(M'_\sigma)\eta_2) \quad \text{mod} \{\sigma, \Theta, \xi_1, \eta_1\}.
\]

On the other hand,

\[
d(\Phi^*\sigma') = d(\frac{1}{m}\sigma + a\Theta + b\xi_1 + c\eta_1)
\]

\[
\equiv \frac{1}{m} \tau \land \alpha + c\tau \land \eta_2 \quad \text{mod} \{\sigma, \Theta, \xi_1, \eta_1\}
\]

\[
\equiv \tau \land \left[\frac{1}{m} M\sigma \xi_2 + \left(\frac{1}{m} d_2 + c\right) \eta_2\right] \quad \text{mod} \{\sigma, \Theta, \xi_1, \eta_1\}.
\]

The comparison of the last two congruences yields

\[
\Phi^*(d'_2) = \frac{1}{mnl}d_2 + \frac{c}{nl}.
\]

(9.21)

Similarly, \(\Phi^*(d\tau') = d(\Phi^*\tau')\) yields

\[
\Phi^*(b'_2) = \frac{1}{mnl}b_2 + \frac{h}{ml}.
\]

(9.22)

Using (9.20) and equations (9.21) and (9.22), we conclude

\[
\Phi^*(b'_2 - d'_2) = \frac{1}{mnl}(b_2 - d_2).
\]
This proves the proposition.

To obtain the formulas for \( H_0 \) and \( K_0 \) we arrive at

\[
A = -(n - m)^2 M_x X(u_{yy}) + \{ \text{function of order } \leq 2 \},
\]

\[
E = -(n - m)^2 M_y Y(u_{yy}) + \{ \text{function of order } \leq 2 \}.
\]

Using equations (9.10), (9.11), and (9.13), we conclude

\[
H_0 = (n - m)^3 M_x X(u_{yy}) + \{ \text{function of order } \leq 3 \},
\]

\[
K_0 = -(n - m)^3 M_y Y(u_{yy}) + \{ \text{function of order } \leq 3 \}.
\]

To end this chapter, we introduce recursion formulas for the generalized Laplace invariants.

**Lemma 9.2.** For \( n \geq 1 \)

\[
H_{n-1} = Y(B_n + Q) + (A_n + P)(B_n + Q) - C_n,
\]

\[
K_{n-1} = X(D_n - P) + (E_n - Q)(D_n - P) - G_n.
\]

**Proof.** Using the formulas (4.31), we compute

\[
Y(B_n + Q) + (A_n + P)(B_n + Q) - C_n = Y(B_{n-1}) + (A_{n-1} + Y(\ln H_{n-1})) B_{n-1}
\]

\[
-C_n + X(A_{n-1} + B_{n-1} Y(\ln H_{n-1}) - Y(B_{n-1}) = X(A_{n-1} + A_{n-1} B_{n-1} - C_{n-1} = H_{n-1}.
\]

This proves (9.25a). The proof of (9.25b) is similar.

**Theorem 9.3.** The generalized Laplace invariants satisfy the following recursion formulas.

\[
H_{n+1} = 2H_n - H_{n-1} - XY(\ln H_n) + QY(\ln H_n) + Y(Q) + 2PQ - X(P)
\]

and

\[
K_{n+1} = 2K_n - K_{n-1} - YX(\ln K_n) - PX(\ln K_n) + Y(Q) + 2PQ - X(P),
\]

where \( n \geq 0 \) and where \( H_{-1} = K_0 \) and \( K_{-1} = H_0 \).

**Proof.** By definition

\[
H_1 = X(A_1) - A_1 B_1 - C_1.
\]
Using (4.31) and (4.11), we have

\[ H_1 = 2[X(A) + AB - C] - [Y(B + Q) + (B + Q)(A + P)c] + Y(Q) - XY(ln H_0) \]

\[ + QY(ln H_0) + 2PQ - X(P) \]

\[ = 2H_0 - K_0 - XY(ln H_0) + QY(ln H_0) + Y(Q) + 2PQ - X(P). \]

Let \( n \geq 1 \). By definition

\[ H_{n+1} = X(A_{n+1}) + A_{n+1}B_{n+1} - C_{n+1}. \]

Using (4.31), we conclude

\[ H_{n+1} = X(A_n - Y(ln H_n) - P) + (A_n - Y(ln H_n) - P)(B_n - Q) \]

\[ - (C_n - X(A_n) - B_nY(ln H_n) + Y(B_n)) \]

\[ = 2H_n - [Y(B_n + Q) + (A_n + P)(B_n + Q) - C_n] - XY(ln H_n) + QY(ln H_n) \]

\[ + Y(Q) + 2PQ - X(P). \]

Equation (9.26) now follows from equation (9.25a). The proof for (9.27) is similar.

**Theorem 9.4.** Let \( R \) be a second-order scalar hyperbolic equation in the plane with commuting characteristic vector fields \( X \) and \( Y \), that is,

\[ [X, Y] = 0. \]

Then the formulas (9.26) and (9.27) simplify to

\[ H_{n+1} = 2H_n - H_{n-1} - XY(ln H_n), \]

(9.28)

and

\[ K_{n+1} = 2K_n - K_{n-1} - YX(ln K_n), \]

(9.29)

where \( n \geq 0 \) and where \( H_{-1} = K_0 \) and \( K_{-1} = H_0 \).

We remark that Sokolov and Zhiber [35] applied the integrability theory of Toda lattice equations to formulas (9.28) and (9.29) to partially solve the conjecture of Anderson and Kamran [8] on Darboux integrable equations. Namely, they proved that an \( f \)-Gordon equation is Darboux integrable if and only if the two sequences of generalized Laplace invariants \( \{ H_i \} \) and \( \{ K_j \} \) are finite.
CHAPTER 10
MONGE-AMPÈRE EQUATIONS

A Monge-Ampère equation is an equation of the form

$$E(u_{xx}u_{yy} - u_{xy}^2) + Au_{xx} + 2Bu_{xy} + Cu_{yy} + D = 0,$$

(10.1)

where $A, B, C, D, E$ are functions of $x, y, u, u_x, u_y$ only. We assume that equation (10.1) is hyperbolic, thus

$$\Delta = B^2 + ED - AC > 0.$$

To (10.1) we associate a hyperbolic system $\mathcal{I}$ of class 1 on a 5-manifold $M$ with coordinates $(x, y, u, p, q)$. The differential ideal $\mathcal{I}$ is generated by the forms

$$\theta = du - pdx - qdy,$$

(10.2a)

$$\Psi = E dp \wedge dq + A dp \wedge dy + B (dq \wedge dy + dx \wedge dp) + C dx \wedge dq + D dx \wedge dy.$$  

(10.2b)

Denote

$$\Phi = d\theta = dx \wedge dp + dy \wedge dq.$$

It is easy to see that $\mathcal{I}$ is algebraically generated by the forms $\theta, \Psi, \Phi$. Define

$$\Omega^1 = \Phi + \Psi \sqrt{\Delta} \quad \text{and} \quad \Omega^2 = \Phi - \Psi \sqrt{\Delta},$$

and notice that

$$\Omega^1 \wedge \Omega^1 = 0 = \Omega^2 \wedge \Omega^2.$$

From the last equation we deduce that locally there are 1-forms $\omega^1, \omega^2, \omega^3, \omega^4$, such that

$$\Omega^1 = \omega^1 \wedge \omega^2 \quad \text{and} \quad \Omega^2 = \omega^3 \wedge \omega^4.$$

A straightforward computation yields

$$\theta \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 = \theta \wedge \Omega^1 \wedge \Omega^2 = -4 dx \wedge dy \wedge du \wedge dp \wedge dq \neq 0.$$

Thus $\{\theta, \omega^1, \omega^2, \omega^3, \omega^4\}$ is a local coframing of $M$ and the differential ideal $\mathcal{I}$ is locally algebraically generated by $\{\theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4\}$. Hence $\mathcal{I}$ is a hyperbolic system of class 1.
Definition 10.1. A hyperbolic system of class 1 is called a hyperbolic Monge-Ampère system.

We now prove a theorem on normal forms of hyperbolic Monge-Ampère systems. The proof is an easy modification of the proof of the theorem on normal forms for parabolic Monge-Ampère systems of Bryant and Griffiths [12].

Theorem 10.2. Let $\mathcal{I}$ be a hyperbolic analytic Monge-Ampère system on a 5-manifold $M$. Then $\mathcal{I}$ is locally equivalent to the Monge-Ampère system generated by a hyperbolic quasi-linear equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + d = 0,$$

where $a, b, c, d$ are functions of $x, y, u, u_x, u_y$ only.

Proof. Let $p \in M$. Then there is a neighborhood $U \subset M$, $U \cong \mathbb{R}^5$ of $p$ such that there is a coframing $(\theta, \omega^1, \omega^2, \omega^3, \omega^4)$ of $U$ and $\mathcal{I}$ restricted to $U$ is algebraically generated by the forms $\theta, \omega^1 \wedge \omega^2$ and $\omega^3 \wedge \omega^4$. Consider a 7-dimensional manifold $X = U \times \mathbb{R}^2$ with coordinates $(x, y, u, p, q, b_1, b_2)$. Define the following 1-forms on $X$.

$$\eta_0 = \theta, \quad \eta_1 = \omega^1 + b_1 \omega^2, \quad \text{and} \quad \eta_2 = \omega^3 + b_2 \omega^4.$$

Then for $i = 1, \ldots, 4$,

$$d\omega^i = T^i \omega^2 \wedge \omega^4 \mod \{\eta_0, \eta_1, \eta_2\}$$

for some functions $T^i$ on $X$. Thus

$$d\eta^0 \equiv 0 \mod \{\eta_0, \eta_1, \eta_2\},$$

$$d\eta^1 \equiv \beta_1 \wedge \omega^2 \mod \{\eta_0, \eta_1, \eta_2\},$$

$$d\eta^2 \equiv \beta_2 \wedge \omega^4 \mod \{\eta_0, \eta_1, \eta_2\},$$

where

$$\beta_1 = db_1 - (T^1 + b_1 T^2) \omega^4,$$

$$\beta_2 = db_2 + (T^3 + b_1 T^4) \omega^2.$$

Let $\mathcal{J}$ be a differential ideal generated by two 5-forms

$$T_1 = d\eta_1 \wedge \eta_0 \wedge \eta_1 \wedge \eta_2 = \beta_1 \wedge \omega_2 \wedge \eta_0 \wedge \eta_1 \wedge \eta_2,$$

$$T_2 = d\eta_2 \wedge \eta_0 \wedge \eta_1 \wedge \eta_2 = \beta_2 \wedge \omega_4 \wedge \eta_0 \wedge \eta_1 \wedge \eta_2.$$
with the independence condition
\[ \Omega = \theta \wedge \omega^1 \wedge \omega^3 \wedge \omega^2 \wedge \omega^4 = \eta_0 \wedge \eta_1 \wedge \eta_2 \wedge \omega^2 \wedge \omega^4 \neq 0. \]

So \((\eta_0, \eta_1, \eta_2, \omega^2, \omega^4, \beta_1, \beta_2)\) is a coframing of \(X\). Fix a point \(x_0 \in X\). We will compute the Cartan characters \(s_0, \ldots, s_4\) of \(J\) in \(x_0 \in X\). One can immediately see that
\[ s_0 = s_1 = s_2 = s_3 = 0. \]

Since \(J\) is generated by two linearly independent 5-forms, then
\[ s_4 = 2. \]

Let \((e_1, \ldots, e_7)\) be the framing at \(x_0\) dual to \((\eta_0, \eta_1, \eta_2, \omega^2, \omega^4, \beta_1, \beta_2)\). Consider the 5-dimensional integral element \(E_0\) of \(J\) spanned by \(\{e_1, e_2, e_3, e_4 + e_5, e_6\}\). Then
\[ \Omega_{x_0}(E_0) \neq 0. \]

Consider the integral flag
\[ \{0\} \subset \{e_1\} \subset \{e_1, e_2\} \subset \{e_1, e_2, e_3\} \subset \{e_1, e_2, e_3, e_4 + e_5\} \subset \{e_1, e_2, e_3, e_4 + e_5, e_6\}, \]
and compute the polar spaces for the vector spaces \(V\) of this integral flag
\[ H(V) = \{v \in T_x(X) \mid \omega(V, v) = 0 \text{ for all } \omega \in J\}. \]

We have
\[ H(\{0\}) = H(\{e_1\}) = H(\{e_1, e_2\}) = H(\{e_1, e_2, e_3\}) = H(\{e_1, e_2, e_3, e_4 + e_5\}) \approx \mathbb{R}^7. \]

Compute
\[ \Upsilon_1(e_1, e_2, e_3, e_4 + e_5, \sum_{i=1}^7 a^i e_i) = -a^6, \]
\[ \Upsilon_2(e_1, e_2, e_3, e_4 + e_5, \sum_{i=1}^7 a^i e_i) = -a^7. \]

Thus
\[ H(\{e_1, e_2, e_3, e_4 + e_5\}) \approx \mathbb{R}^5. \]
By definition the integral element $E_0$ is ordinary. According to the Cartan-Kähler theorem, there exists a 5-dimensional integral submanifold $N \subset X$ such that $TN/x_0 = E_0$. Since $\Omega/x_0(E_0) \neq 0$, the submanifold $N$ can be locally described as a graph $(c_1, c_2) : U_0 \to R^2$, where $U_0 \subseteq U$ is a neighborhood of $x_0$. Let $\mathcal{K}$ denote the differential ideal generated by

$$
\alpha_0 = \theta, \quad \alpha_1 = \omega^1 + c_1 \omega^2, \quad \text{and} \quad \alpha_2 = \omega^3 + c_2 \omega^4.
$$

It is easy to check that $\mathcal{K}$ is integrable. Moreover, $\mathcal{I} \subseteq \mathcal{K}$. Let $\bar{x}, \bar{y}, \bar{u}$, be three functionally independent first integrals of $\mathcal{K}$. Since $\theta \in \mathcal{I}$, we may assume

$$
\theta \wedge d\bar{x} \wedge d\bar{y} \neq 0
$$

and so there is a nonvanishing function $m$ on $U_0$ such that

$$
\theta = m (d\bar{u} - p dx - \bar{q} dy).
$$

Since $d\theta \wedge d\theta \wedge \theta \neq 0$, it follows that $\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}$ are functionally independent and therefore $(\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q})$ are local coordinates on $U_0$. The ideal $\mathcal{I}$ is algebraically generated by

$$
\theta_0 = d\bar{u} - \bar{p} d\bar{x} - \bar{q} d\bar{y}, \quad d\theta_0 = dx \wedge dp + dy \wedge dq
$$

and the form $\Psi$. Since $\Psi \in \mathcal{I} \subseteq \mathcal{K}$, then

$$
\Psi = e dp \wedge dq + a dp \wedge dy + b_1 dq \wedge dy + b_2 d\bar{x} \wedge dp + c d\bar{x} \wedge dq + d dp \wedge dq \quad \text{mod} \{ \theta \}.
$$

From $\Psi \wedge d\bar{x} \wedge dq \wedge \theta_0 = 0$ we deduce $e = 0$. Consider the form

$$
\Psi_0 = \Psi + \frac{b_1 - b_2}{2} d\theta_0 = a dp \wedge dy + b (dq \wedge dy + dx \wedge dp) + c d\bar{x} \wedge dq + d d\bar{x} \wedge dy,
$$

where

$$
b = \frac{b_1 + b_2}{2}.
$$

The differential ideal $\mathcal{I}$ on $U_0$ is generated by $\theta_0$ and $\Psi_0$. It is now easy to see that the integral manifolds of $\mathcal{I}$ with the independence condition $dx \wedge dy \neq 0$ are in a one-to-one correspondence with the solutions of the quasi-linear equation

$$
a u_{xx} + b_2 u_{xy} + c u_{yy} + d = 0.
$$
**Theorem 10.3.** Every hyperbolic analytic Monge-Ampère equation is contact equivalent to some quasi-linear equation

\[ au_{xx} + 2bu_{xy} + cu_{yy} + d = 0. \]

**Proof.** To (10.1) we associate a hyperbolic system \( \mathcal{I} \) of class 1 on a 5-manifold \( M \) with coordinates \((x, y, u, p, q)\). The differential ideal \( \mathcal{I} \) is generated by the forms

\[ \theta = du - pdx - qdy, \]

\[ \Psi = Edp \wedge dq + Adp \wedge dy + B(dq \wedge dy + dx \wedge dp) + Cdx \wedge dq + Ddx \wedge dy. \]

By the previous theorem there exists a local change of coordinates on an open set \( U \subseteq M \) such that

\[ \phi: (x, y, u, p, q) \rightarrow (\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}) \]

such that

\[ \phi^*(\theta) = m (d\bar{u} - \bar{p} d\bar{x} - \bar{q} d\bar{y}) \]

for some nonvanishing function \( m \) and the differential ideal \( \mathcal{I} = \phi^*(\mathcal{I}) \) is generated by

\[ \bar{\theta} = d\bar{u} - \bar{p} d\bar{x} - \bar{q} d\bar{y}, \quad \text{and} \]

\[ \bar{\Psi} = a \, d\bar{p} \wedge d\bar{q} + b \, (d\bar{q} \wedge d\bar{y} + d\bar{x} \wedge d\bar{p}) + c \, d\bar{x} \wedge d\bar{q} + d \, d\bar{x} \wedge d\bar{y}. \]

Consider the manifold \( U \times \mathbb{R}^3 \) with coordinates \((x, y, u, p, q, r, s, t)\). Denote

\[ \theta_x = dp - r \, dx - s \, dy \quad \text{and} \quad \theta_y = dq - s \, dx - t \, dy. \]

Let \( \bar{r} \), \( \bar{s} \) and \( \bar{t} \) be three functions that are uniquely determined by the following relations

\[ d\bar{p} - \bar{r} \, d\bar{x} - \bar{s} \, d\bar{y} \equiv 0 \mod \{ \theta, \theta_x, \theta_y \} \quad \text{and} \]

\[ d\bar{q} - \bar{s} \, d\bar{x} - \bar{t} \, d\bar{y} \equiv 0 \mod \{ \theta, \theta_x, \theta_y \}. \]

So \((\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t})\) is a coordinate system on \( U \times \mathbb{R}^3 \). We have

\[ [E (rt - s^2) + Ar + 2Bs + Ct + D] \, dx \wedge dy \equiv \Psi \mod \{ \theta, \theta_x, \theta_y \} \]

\[ \equiv a \, d\bar{p} \wedge d\bar{q} + b \, (d\bar{q} \wedge d\bar{y} + d\bar{x} \wedge d\bar{p}) + c \, d\bar{x} \wedge d\bar{q} + d \, d\bar{x} \wedge d\bar{y} \mod \{ \theta, \theta_x, \theta_y \} \]

\[ \equiv (a\bar{r} + 2b\bar{s} + c\bar{t} + d) \, d\bar{x} \wedge d\bar{y} \mod \{ \theta, \theta_x, \theta_y \}. \]
It follows that
\[ [E (rt - s^2) + Ar + 2Bs + Ct + D] = l (ar + 2bs + c\bar{e} + d), \]
for some function \( l = l(x, y, u, p, q, r, s, t) \). Since \( \theta \wedge d\bar{e} \wedge dy \neq 0 \) on \( M \), then
\[ d\bar{e} \wedge dy \wedge \theta \wedge \theta_x \wedge \theta_y = l \, dx \wedge dy \wedge \theta \wedge \theta_x \wedge \theta_y \neq 0, \]
and so the function \( l \) is nonvanishing.

According to Bryant and Griffiths [12], this result was already known to Lie and the theorem is true even without the analyticity of the functions. Our proof requires the analyticity assumption since we use the Cartan-Kähler theorem in an essential way.

As an example consider the Monge-Ampère equation
\[ rt - s^2 + 1 = 0. \quad (10.3) \]
Both first generalized Laplace invariants \( H_0 \) and \( K_0 \) vanish. By the results of chapter 13, the equation is contact equivalent to the wave equation \( s = 0 \). We now explicitly exhibit this contact transformation.

Consider the differential ideal \( \mathcal{I} \) over the space of variables \((x, y, u, p, q)\) generated by the forms
\[ \theta = du - p \, dx - q \, dy \quad \text{and} \quad \Psi = dp \wedge dq + dx \wedge dy. \]

Let
\[ \Phi = d\theta = dx \wedge dp + dy \wedge dq \]
and
\[ \Omega^1 = \Phi + \Psi = (dp + dq) \wedge (dq - dx), \]
\[ \Omega^2 = \Phi - \Psi = (dx + dq) \wedge (dp - dy). \]

Let
\[ \eta_0 = \theta = du - p \, dx - q \, dy \]
\[ \eta_1 = dp + dq + b_1 (dq - dx), \]
\[ \eta_2 = dx + dq + b_2 (dp - dy). \]

We can easily see that for \( b_1 = b_2 = 0 \) the forms
vanish. Thus the Pfaffian system $K$ generated by
\[ \alpha_0 = du - p\,dx - q\,dy, \quad \alpha_1 = dp + dy \quad \text{and} \quad \alpha_2 = dx + dq \]
is integrable. The functions
\[ \ddot{x} = x + q, \quad \ddot{y} = y + p, \quad \text{and} \quad \ddot{u} = u + pq \]
are three independent first integrals of $K$. The condition
\[ \theta \wedge d\ddot{x} \wedge d\ddot{y} \neq 0 \]
is satisfied. We easily compute
\[ \ddot{p} = p \quad \text{and} \quad \ddot{q} = q. \]
In the bared coordinates,
\[ \Phi = -(d\ddot{x} \wedge d\ddot{p} + d\ddot{q} \wedge d\ddot{y}) + d\ddot{x} \wedge d\ddot{y}. \]
We conclude that the contact transformation above transforms equation (10.3) into the equation
\[ 2\ddot{s} - 1 = 0. \quad (10.4) \]
The point transformation
\[ x = \ddot{x}, \quad y = \ddot{y} \quad \text{and} \quad u = \ddot{u} + \frac{1}{2} \ddot{x} \ddot{y} \]
transforms equation (10.4) into the wave equation.

Note that since the coefficients $m$ and $n$ in (4.33) are of order $\leq 2$, then, according to the transformation formulas (4.34), the condition that $H_0$, or $K_0$ is of order $\leq 2$ is invariant under classical contact transformations.

**Theorem 10.4.** Let $\mathcal{R}$ be a second-order scalar hyperbolic partial differential equation in the plane.
The following conditions are equivalent:

(i) $H_0$ and $K_0$ are of order $\leq 2$.
(ii) $M_\sigma = M_r = 0$.
(iii) $\mathcal{R}$ is a Monge-Ampère equation.

Proof. (i)$\Rightarrow$(ii). Follows immediately from (9.24). (ii)$\Rightarrow$(iii). Let the equation manifold $\mathcal{R}$ be given by the equation
\[ \eta_1 = d\eta_1 \wedge \eta_0 \wedge \eta_1 \wedge \eta_2 \quad \text{and} \quad \eta_2 = d\eta_2 \wedge \eta_0 \wedge \eta_1 \wedge \eta_2 \]
\[ r + f(x, y, u, p, q, s, t) = 0. \]

We choose the characteristic vector fields (9.8), where \( m \) and \( n \) satisfy (9.10). Denote
\[
\Delta = (n - m)^2 = \left( \frac{\partial f}{\partial s} \right)^2 - 4 \frac{\partial f}{\partial t}. \tag{10.5}
\]

Using (9.16) from \( M_s = M_t = 0 \) follows
\[
\frac{1}{(n - m)^2} \left( \frac{\partial n}{\partial t} - n \frac{\partial n}{\partial s} \right) = 0 \quad \text{and} \quad \frac{1}{(n - m)^3} \left( \frac{\partial m}{\partial t} - m \frac{\partial m}{\partial s} \right) = 0.
\]

Adding and subtracting these equations, we get
\[
\frac{1}{2} \frac{\partial}{\partial s} (n^2 + m^2) - \frac{\partial}{\partial t} (n + m) = 0 \quad \text{and} \quad \frac{1}{2} \frac{\partial}{\partial s} (-n^2 + m^2) - \frac{\partial}{\partial t} (-n + m) = 0. \tag{10.6}
\]

Using (9.10), we obtain
\[
n^2 + m^2 = \left( \frac{\partial f}{\partial s} \right)^2 - 2 \frac{\partial f}{\partial t} \quad \text{and} \quad -n^2 + m^2 = \frac{\partial f}{\partial s} (m - n).
\]

Upon substitution into (10.6) and a simplifying, we obtain
\[
\frac{\partial \Delta}{\partial s} = 0 \quad \text{and} \quad \frac{\partial f}{\partial t} \frac{\partial f}{\partial s} - \frac{\partial f}{\partial t} \frac{\partial f}{\partial \partial s} = 0, \tag{10.7}
\]

Next, we take derivatives of (10.7) and (10.8) with respect to \( s \) and \( t \). Eliminating \( \frac{\partial f}{\partial t \partial t} \frac{\partial f}{\partial \partial t} \partial \partial s \), and \( \frac{\partial f}{\partial \partial t \partial \partial s} \), from these four equations, we obtain
\[
\frac{\partial f}{\partial s \partial s \partial s} = \frac{1}{2 \Delta} \frac{\partial \Delta}{\partial s} \frac{\partial f}{\partial s \partial s}.
\]

From (10.7) follows
\[
\frac{\partial f}{\partial s \partial s \partial s} = 0.
\]
Hence $f = as^2 + bs + c$, where $a, b, c$ are arbitrary functions of $x, y, u, p, q, t$. Substituting into (10.7), we get
\[
\frac{\partial a}{\partial t} = a^2 \quad \text{and} \quad \frac{\partial b}{\partial t} = ab.
\]
Integrating, we obtain
\[
a = \frac{1}{A-t}, \quad b = \frac{2B}{A-t} \quad \text{or} \quad a = 0, \quad b = 2B,
\]
where $A, B$ are arbitrary functions of $x, y, u, p, q$. We have
\[
f = \frac{s^2 + 2Bs}{A-t} + c \quad \text{or} \quad f = 2Bs + c,
\]
Substituting (10.9) into (10.8) yields
\[
\frac{2}{A-t} \frac{\partial c}{\partial t} = \frac{\partial c}{\partial t} \frac{\partial c}{\partial t} \quad \text{or} \quad \frac{\partial c}{\partial t} = 0.
\]
Integrating, we obtain
\[
c = \frac{Ct + D}{A-t} \quad \text{or} \quad c = Ct + D,
\]
where $C, D$ are arbitrary functions of $x, y, u, p, q$. Substituting into (10.9), we get
\[
f = \frac{s^2 + 2Bs + Ct + D}{A-t} \quad \text{or} \quad f = 2Bs + Ct + D,
\]
which proves the claim.

(iii)$\Rightarrow$(i). An easy computation using Maple proves the claim. We present a proof for quasi-linear equations
\[
r + (m + n)s + mnt + c = 0,
\]
where $m, n, c$ are functions of $x, y, u, p, q$. Let
\[
X = D_x + mD_y, \quad \text{and} \quad Y = D_x + mD_y,
\]
be the characteristic vector fields. Using (9.11), we get
\[ A = \frac{1}{n-m} \left( \frac{\partial m}{\partial q} - n \frac{\partial m}{\partial p} \right) Y(q) + \{ \text{function of order } \leq 1 \}. \]

Hence

\[ H_0 = \frac{1}{n-m} \left( \frac{\partial m}{\partial q} - n \frac{\partial m}{\partial p} \right) XY(q) + \{ \text{function of order } \leq 2 \}. \]

But it is easy to see that \( XY(q) \) is of order \( \leq 2 \), and so \( H_0 \) is of order \( \leq 2 \). The proof that \( K_0 \) is of order \( \leq 2 \) is similar.

Since the Monge-Ampère invariants \( M_\tau \) and \( M_\tau \) are relative invariants under contact transformations, we have established the following corollary.

**Corollary 10.5.** The class of Monge-Ampère equations is invariant under the group of classical contact transformations.
CHAPTER 11
INTERMEDIATE INTEGRALS

The purpose of this chapter is to address the classical problem of the existence of general and complete intermediate integrals for the second-order scalar hyperbolic equations in the plane. We derive necessary and sufficient conditions for an equation to admit a general or a complete intermediate integral in terms of the generalized Laplace invariants.

Roughly speaking, a general intermediate integral of a second-order equation

\[ F(x, y, u, p, q, r, s, t) = 0 \]  

is a first-order equation

\[ J(x, y, u, p, q) = \varphi(I(x, y, u, p, q)) \]  

involving an arbitrary function \( \varphi \), such that every solution of the equation \( (11.2) \) is a solution of the equation \( (11.1) \). Taking the first total derivatives of \( (11.2) \) gives

\[ D_x J = \varphi'(I)D_x I \quad \text{and} \quad D_y J = \varphi'(I)D_y I, \]

and eliminating \( \varphi' \) from these two equations we obtain the equation

\[ D_x I D_y J - D_y I D_x J = 0. \]  

**Lemma 11.1.** Let \( I \) and \( J \) be first-order functions. A function \( f(x, y) \) is a solution to the Monge-Ampère equation

\[ D_x I D_y J - D_y I D_x J = 0, \]  

if and only if it is a solution to the first-order equation

\[ \psi(I, J) = 0, \]  

for some function \( \psi = \psi(u, v) \), where \( \partial \psi / \partial u \neq 0 \) or \( \partial \psi / \partial v \neq 0 \).

**Proof.** That the equation \( (11.4) \) is Monge-Ampère is immediate.

Let \( f(x, y) \) be a solution to the equation \( (11.4) \). Denote by

\[ \psi(I, J) = 0, \]
\[ G(x, y) = I(x, y, f(x, y), \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)), \]

\[ H(x, y) = J(x, y, f(x, y), \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)), \]

Equation (11.4) yields

\[ \frac{\partial G}{\partial x} \frac{\partial F}{\partial y} - \frac{\partial G}{\partial y} \frac{\partial F}{\partial x} = 0, \]

that is, \( G \) and \( H \) are functionally dependent. Thus there is a function \( \psi(u, v) \), with \( \partial \psi/\partial u \neq 0 \) or \( \partial \psi/\partial v \neq 0 \), such that

\[ \psi(G(x, y), H(x, y)) = 0. \]

Hence for this \( \psi \) the function \( f \) is a solution of the equation (11.5).

Conversely, taking the total derivatives of equation (11.5), we obtain

\[ \frac{\partial \psi}{\partial u}(I, J)D_xI + \frac{\partial \psi}{\partial v}(I, J)D_yJ = 0, \]

\[ \frac{\partial \psi}{\partial u}(I, J)D_yI + \frac{\partial \psi}{\partial v}(I, J)D_xJ = 0. \]

Since \( \partial \psi/\partial u \neq 0 \) or \( \partial \psi/\partial v \neq 0 \), then we have (11.4).

**Definition 11.2.** We say that a second-order scalar partial differential equation in the plane \( \mathcal{R} \) admits a general intermediate integral

\[ J = \varphi(I), \]

if it is contact equivalent to the equation \( \mathcal{R}' \) given by

\[ F(x, y, u, p, q, r, s, t) = D_xID_yJ - D_yID_xJ = 0 \quad (11.6) \]

for some functionally independent functions \( I \) and \( J \) of order \( \leq 1 \).

For instance, the equation

\[ \frac{1+q}{p} = \varphi(x+u) \]
is a general intermediate integral of the equation

\[ q(q + 1)r - (1 + p + q + 2pq)s + p(p + 1)t = 0. \]

Let \( R \) be a scalar second-order equation in the plane that admits a general intermediate integral. We may assume that \( R \) is given by equation (11.6) for some functions \( I \) and \( J \) of order \( \leq 1 \), which as we pointed out is a Monge-Ampère equation. Next observe that \( I \) and \( J \) are functionally independent. Indeed if not, then there would exist a function \( \varphi \) such that \( J = \varphi(I) \) and so the left side of equation (11.6) would vanish identically, which would be a contradiction. Compute

\[
\frac{\partial F}{\partial r} = \frac{\partial I}{\partial p} D_y J - \frac{\partial J}{\partial p} D_y I,
\]

\[
\frac{\partial F}{\partial s} = \frac{\partial I}{\partial q} D_y J + \frac{\partial J}{\partial p} D_z I - \frac{\partial I}{\partial p} D_z J - \frac{\partial J}{\partial q} D_y I,
\]

\[
\frac{\partial F}{\partial t} = \frac{\partial J}{\partial q} D_z I - \frac{\partial I}{\partial q} D_z J.
\]

Thus

\[
\frac{\partial F}{\partial r} (-D_z I)^2 - \frac{\partial F}{\partial s} (-D_z I)(D_y I) + \frac{\partial F}{\partial t} (D_y I)^2 = 0
\]

on the equation manifold \( R \), and so we conclude that the total vector field

\[ Z = D_y ID_z - D_z ID_y \tag{11.7} \]

is a characteristic vector field on \( R \). It is immediate that

\[ Z(I) = Z(J) = 0. \tag{11.8} \]

Note that if \( Z \) is given by (11.7), then from (11.8) and Theorem 7.6 immediately follows that \( Z \) is a characteristic vector field. Also notice that the discriminant of (11.6) is given by

\[
\left( \frac{\partial F}{\partial s} \right)^2 - 4 \frac{\partial F}{\partial r} \frac{\partial F}{\partial t} = \left( \frac{\partial I}{\partial q} D_y J - \frac{\partial J}{\partial p} D_z I - \frac{\partial I}{\partial p} D_z J - \frac{\partial J}{\partial q} D_y I \right)^2.
\]

We have proved the following.
Proposition 11.3. A second-order scalar hyperbolic partial differential equation in the plane $\mathcal{R}$ that admits a general intermediate integral $J = \varphi(I)$ is a Monge-Ampère equation and $I$, $J$ are both either $X$ invariant or $Y$ invariant functions.

We will now prove the converse of this proposition.

Proposition 11.4. Let $\mathcal{R}$ be a second-order hyperbolic equation in the plane that admits two functionally independent functions $I$ and $J$ of order $\leq 1$ that are both $X$ invariant or $Y$ invariant, that is

$$X(I) = X(J) = 0 \quad \text{or} \quad Y(I) = Y(J) = 0.$$  

Then the equation

$$J = \varphi(I)$$

is a general intermediate integral of $\mathcal{R}$.

Proof. Without restrictions assume $X(I) = X(J) = 0$ and let

$$X = m_x D_x + m_y D_y \quad \text{and} \quad Y = n_x D_x + n_y D_y$$

as usual. Then

$$D_x ID_y J - D_y ID_x J = \frac{1}{m_x n_y - m_y n_x} [X(I) Y(J) - Y(I) X(J)]$$

on $\mathcal{R}^2$ and so

$$D_x ID_y J - D_y ID_x J = 0 \quad (11.9)$$

on $\mathcal{R}^2$. Since $I$ and $J$ are functionally independent, the equation (11.9) defines a 7-dimensional manifold $\mathcal{R}'^2$ such that $\mathcal{R}^2 \subset \mathcal{R}'^2$. Since we are working locally, $\mathcal{R}^2 = \mathcal{R}'^2$ and the theorem follows.

Summarizing the last two propositions we get

Theorem 11.5. A second-order hyperbolic partial differential equation in the plane $\mathcal{R}$ admits a general intermediate integral

$$J = \varphi(I)$$

if and only if the functions $I$ and $J$ satisfy either
From Theorem 6.3 we obtain the following characterization.

**Corollary 11.6.** A second-order hyperbolic partial differential equation in the plane \( \mathcal{R} \) admits a general intermediate integral if and only if \( H_0 = M_r = 0 \) or \( K_0 = M_s = 0 \).

Roughly speaking, a complete intermediate integral of a second-order equation

\[
F(x, y, u, p, q, r, s, t) = 0
\]

is a first-order equation

\[
V(x, y, u, p, q, a, b) = 0
\]

involving two arbitrary constants \( a \) and \( b \), such that any solution of (11.11) is a solution of the equation (11.10).

**Definition 11.7.** We say that a scalar second-order partial differential equation in the plane \( \mathcal{R} \) admits a complete intermediate integral

\[
V(x, y, u, p, q, a) = b
\]

involving two arbitrary constants \( a \) and \( b \), if

\[
da \wedge dV \wedge d(\frac{\partial V}{\partial a}) \neq 0
\]

and if eliminating the constant \( a \) from the equations

\[
D_x V = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial u} p + \frac{\partial V}{\partial p} r + \frac{\partial V}{\partial q} s = 0
\]

\[
D_y V = \frac{\partial V}{\partial y} + \frac{\partial V}{\partial u} q + \frac{\partial V}{\partial p} s + \frac{\partial V}{\partial q} t = 0
\]

yields an equation which equation manifold \( \mathcal{R}' \) is contact equivalent to \( \mathcal{R} \).

Note that if the equation \( \mathcal{R} \) admits a general intermediate integral \( J = \varphi(I) \), then it admits a complete intermediate integral, namely

\[
J = aI + b.
\]
The rest of this chapter is devoted to the characterization of equations that admit a complete intermediate integral. We first prove that if a scalar second-order hyperbolic partial differential equation in the plane \( \mathcal{R} \) admits a complete intermediate integral, then there are three functionally independent functions of order \( \leq 2 \) that are all either \( X \) invariant or \( Y \) invariant. Next we prove the converse. Namely, if there are three \( X \) functionally independent functions \( I, J, \) and \( K \) of order \( \leq 2 \), we prove that we can always choose these so that the relation

\[
K = \frac{Y(J)}{Y(I)}
\]

is satisfied. Next we show that there exists a function \( V(x, y, u, p, q, a) \) such that

\[
J = V(x, y, u, p, q, I).
\]

We then show that the equation

\[
V(x, y, u, p, q, a) = b
\]

is a complete intermediate integral for \( \mathcal{R} \).

**Lemma 11.8.** Let \( \mathcal{R} \) be a scalar second-order hyperbolic partial differential equation in the plane. Let \( I \) and \( J \) be two \( X \) invariant functions such that \( Y(I) \neq 0 \). Then

\[
K = \frac{Y(J)}{Y(I)}
\]

is an \( X \) invariant function.

**Proof.** Since \( X(I) = X(J) = 0 \), then

\[
d_H I = Y(I)\tau \quad \text{and} \quad d_H J = Y(J)\tau.
\]

Hence

\[
d_H J = \frac{Y(J)}{Y(I)}d_H I = Kd_H I.
\]

By Lemma 8.6 the horizontal differential commutes with the projected Lie derivative and so we compute that the right side of (11.13) is

\[
X(Kd_H I) = X(K)d_H I + KX(d_H I) = X(K)Y(I)\tau + Kd_H (X(I)) = X(K)Y(I)\tau
\]

and the left side of (11.13) is
Thus

\[ X(d_H J) = d_H X(J) = 0. \]

Thus

\[ X(K)Y(I)\tau = 0. \]

since \( Y(I) \neq 0 \) it follows \( X(K) = 0. \)

Lemma 11.9. Let \( \mathcal{R} \) be a scalar second-order hyperbolic partial differential equation in the plane. Let \( I \) and \( J \) be two \( X \) invariant second-order functions on \( \mathcal{R}^\infty \) and let

\[ V = V(x, y, u, p, q, \alpha) \]

be a function such that

\[ J = V(x, y, u, p, q, I). \] \hspace{1cm} (11.14)

Let

\[ K = \frac{Y(J)}{Y(I)}. \] \hspace{1cm} (11.15)

The following statements are equivalent.

(i) \( K \) is a function of order \( \leq 2. \)

(ii) \( [Y(V)](x, y, u, p, q, I) = 0, \) where

\[ Y(V) = \frac{\partial V}{\partial x} Y(x) + \frac{\partial V}{\partial y} Y(y) + \frac{\partial V}{\partial u} Y(u) + \frac{\partial V}{\partial p} Y(p) + \frac{\partial V}{\partial q} Y(q). \] \hspace{1cm} (11.16)

If one of the conditions (i), (ii) is satisfied, then

\[ K = \frac{\partial V}{\partial \alpha}(x, y, u, p, q, I). \] \hspace{1cm} (11.17)

Proof. Assume that the equation manifold is given by equation

\[ r + f(x, y, u, p, q, s, t) = 0. \] \hspace{1cm} (11.18)

The natural coordinates on \( \mathcal{R}^\infty \) are

\[ (x, y, u, p, q, s, t, u_{xyy}, u_{yyy}, u_{xyyy}, u_{yyyy}, \ldots). \] \hspace{1cm} (11.19)
We choose the characteristic vector fields to be

\[ X = D_x + mD_y \quad \text{and} \quad Y = D_x + nD_y, \tag{11.20} \]

where \( m, n \) are the two distinct roots of the characteristic equation

\[ \lambda^2 - \frac{\partial f}{\partial s} \lambda + \frac{\partial f}{\partial t} = 0. \tag{11.21} \]

From \( X(I) = X(J) = 0 \), we have

\[ D_x I = -mD_y I \quad \text{and} \quad D_x J = -mD_y J \]

and so

\[ Y(I) = (n - m) \left( \frac{\partial I}{\partial s} u_{xyy} + \frac{\partial I}{\partial t} u_{yyy} \right) + \{\text{function of order } \leq 2\}, \tag{11.22a} \]

\[ Y(J) = (n - m) \left( \frac{\partial J}{\partial s} u_{xyy} + \frac{\partial J}{\partial t} u_{yyy} \right) + \{\text{function of order } \leq 2\}. \tag{11.22b} \]

(i) \( \Rightarrow \) (ii). Assume that \( K \) is of order \( \leq 2 \). From (11.22) we deduce

\[ K = \frac{Y(J)}{Y(I)} = \frac{\frac{\partial J}{\partial s} u_{xyy} + \frac{\partial J}{\partial t} u_{yyy} + \{\text{function of order } \leq 2\}}{\frac{\partial I}{\partial s} u_{xyy} + \frac{\partial I}{\partial t} u_{yyy} + \{\text{function of order } \leq 2\}}. \]

Hence

\[ \frac{\partial J}{\partial s} = K \frac{\partial I}{\partial s} \quad \text{and} \quad \frac{\partial J}{\partial t} = K \frac{\partial I}{\partial t}. \tag{11.23} \]

Since \( I \) is of second order, either \( \partial I/\partial s \neq 0 \) or \( \partial I/\partial t \neq 0 \). First assume \( \partial I/\partial s \neq 0 \). From the first equation in (11.23) and from the definition of \( K \), we obtain

\[ \frac{\partial J}{\partial s} Y(I) = \frac{\partial I}{\partial s} Y(J). \]

Substituting for \( J \) from (11.14) into the above formula yields

\[ \frac{\partial V}{\partial a} \frac{\partial I}{\partial s} Y(I) = \frac{\partial I}{\partial s} \left( \frac{\partial V}{\partial x} Y(x) + \frac{\partial V}{\partial y} Y(y) + \frac{\partial V}{\partial u} Y(u) + \frac{\partial V}{\partial p} Y(p) + \frac{\partial V}{\partial q} Y(q) + \frac{\partial V}{\partial a} Y(I) \right), \]

evaluated at \((x, y, u, p, q, I)\). From here (ii) follows immediately. In the case \( \partial I/\partial t \) the proof is similar.
(ii) $\Rightarrow$ (i). Applying $Y$ to equation (11.14) we obtain

$$Y(J) = [Y(V)](x, y, u, p, q, I) + \frac{\partial V}{\partial a}(x, y, u, p, q, I) Y(I).$$

Hence, by the assumption,

$$Y(J) = \frac{\partial V}{\partial a}(x, y, u, p, q, I) Y(I).$$

Thus

$$K = \frac{Y(J)}{Y(I)} = \frac{\partial V}{\partial a}(x, y, u, p, q, I).$$

is of second order. This also proves the formula (11.17).

Assume that the scalar second-order hyperbolic partial differential equation in the plane $\mathcal{R}$ admits a complete intermediate integral

$$V(x, y, u, p, q, a) = b. \quad (11.24)$$

Since $a, V, \partial V/\partial a$ are functionally independent, then $D_x(\partial V/\partial a) \neq 0$ or $D_y(\partial V/\partial a) \neq 0$.

(Otherwise $\partial V/\partial a = \varphi(a)$ for some function $\varphi$). Hence

$$\frac{\partial}{\partial a}(D_x V) = D_x \left(\frac{\partial V}{\partial a}\right) \neq 0 \quad \text{or} \quad \frac{\partial}{\partial a}(D_y V) = D_y \left(\frac{\partial V}{\partial a}\right) \neq 0.$$

Therefore, we can solve from one of the equations

$$D_x V = 0 \quad \text{or} \quad D_y V = 0 \quad (11.25)$$

for $a$ (recall that $D_x V$ and $D_y V$ are given by the equations (11.12)) to obtain

$$a = I(x, y, u, p, q, r, s, t). \quad (11.26)$$

Note that $I$ is a second-order function, that is, depends explicitly on $r, s$ or $t$. Substituting (11.26) into the other equation in (11.25), we obtain the equation $\mathcal{R}$, that is,

$$[D_x V](x, y, u, p, q, I) = 0 \quad \text{and} \quad [D_y V](x, y, u, p, q, I) = 0, \quad (11.27)$$

on $\mathcal{R}$, and substituting (11.25) into (11.24) we get

$$b = V(x, y, u, p, q, I) = J(x, y, u, p, q, r, s, t). \quad (11.28)$$
Taking the total differentials of (11.28), we have

\[ D_x J = D_x \left(V(x, y, u, p, q, I)\right) \quad \text{and} \quad D_y J = D_y \left(V(x, y, u, p, q, I)\right). \quad (11.29) \]

From (11.27) and (11.29), we conclude that on \( \mathcal{R}^\infty \),

\[ D_x J = \frac{\partial V}{\partial a}(x, y, u, p, q, I)D_x I \quad \text{and} \quad D_y J = \frac{\partial V}{\partial a}(x, y, u, p, q, I)D_y I. \]

Multiplying the first equation by \( D_y I \) and the second by \( D_x I \) and subtracting, we obtain

\[ D_x ID_y J - D_y ID_x J = 0 \quad (11.30) \]

on \( \mathcal{R}^\infty \). Consider the vector field

\[ X = D_y ID_x - D_x ID_y \]

on \( \mathcal{R}^\infty \). By (11.30)

\[ X(I) = X(J) = 0. \]

Since \( a \) and \( V \) are functionally independent and \( I \) is of second order, then \( J = V(x, y, u, p, q, I) \) and \( I \) are functionally independent on \( \mathcal{R} \). By Theorem 7.6 \( X \) is a characteristic vector field of the equation \( \mathcal{R} \). Let

\[ Y = n_x D_x + n_y D_y \]

be a characteristic vector field of equation \( \mathcal{R} \) that is, non-proportional to \( X \). Then from (11.27) we deduce

\[ [Y(V)](x, y, u, p, q, I) = n_x [D_x V](x, y, u, p, q, I) + n_y [D_y V](x, y, u, p, q, I) = 0. \]

The condition \((ii)\) of Lemma 11.9 is satisfied and so it follows

\[ K = \frac{Y(J)}{Y(I)} \]

is of order \( \leq 2 \). By Lemma 11.8, \( K \) is \( X \) invariant and again by Lemma 11.9 we conclude that

\[ K = \frac{\partial V}{\partial a}(x, y, u, p, q, I). \]

Since \( a, V, \partial V/\partial a \) are functionally independent and \( I \) is of second order, then

\[ I, \quad J = V(x, y, u, p, q, I), \quad \text{and} \quad K = \frac{\partial V}{\partial a}(x, y, u, p, q, I) \]

are functionally independent on \( \mathcal{R}^\infty \). We have proved the following theorem.
Theorem 11.10. Let $\mathcal{R}$ be a scalar second-order hyperbolic partial differential equation in the plane. If $\mathcal{R}$ admits a complete intermediate integral

$$V(x, y, u, p, q, a) = b,$$

then there are three functionally independent functions on $\mathcal{R}$ of order $\leq 2$.

To prove the converse, we start with the following definition found in Olver [33] (page 138).

Definition 11.11. Let $\mathcal{R}^\infty$ be an infinitely prolonged equation manifold of the equation (11.1). Let $I$ and $J$ be two $k$th-order functions on $\mathcal{R}^\infty$. We say that functions $I$ and $J$ are strictly independent if $I$, $J$ and all the coordinate functions on $\mathcal{R}^\infty$ up to order $(k - 1)$ are functionally independent. We say that functions $I$, and $J$ are weakly dependent if the functions $I$ and $J$ are not strictly independent.

Lemma 11.12. Let $\mathcal{R}$ be a scalar second-order hyperbolic partial differential equation in the plane. Let $I$ and $J$ be two $X$ invariant second-order functions. Then $I$ and $J$ are weakly dependent.

Proof. Without the loss of generality assume that the equation manifold is given by (11.18). The natural coordinates on $\mathcal{R}^\infty$ are given by (11.19). We start with an observation that two $k$th-order functions $g_1, g_2$ on $\mathcal{R}^\infty$ are weakly dependent if and only if

$$\frac{\partial g_1}{\partial u_{x y}^{k-1}} - \frac{\partial g_2}{\partial u_{y y}^{k-1}} = 0.$$  

We choose the characteristic vector fields (11.20). Compute

$$Y(I) = au_{x y} + bu_{y y} + \{\text{function of order } \leq 2\}, \quad (11.31a)$$

$$Y(J) = cu_{x y} + cu_{y y} + \{\text{function of order } \leq 2\}, \quad (11.31b)$$

where

$$a = \left( -\frac{\partial f}{\partial s} + n \right) \frac{\partial I}{\partial s} + \frac{\partial I}{\partial t} \quad \text{and} \quad b = -\frac{\partial f}{\partial t} \frac{\partial I}{\partial s} + n \frac{\partial I}{\partial t},$$

$$c = \left( -\frac{\partial f}{\partial s} + n \right) \frac{\partial J}{\partial s} + \frac{\partial J}{\partial t} \quad \text{and} \quad e = -\frac{\partial f}{\partial t} \frac{\partial J}{\partial s} + n \frac{\partial J}{\partial t}.$$  

Thus

$$ae - bc = \left( \frac{\partial I}{\partial s} \frac{\partial J}{\partial t} - \frac{\partial I}{\partial t} \frac{\partial J}{\partial s} \right) \left( n^2 - \frac{\partial f}{\partial s} n + \frac{\partial f}{\partial t} \right) = 0, \quad (11.32)$$

where $a$, $b$, $c$, $d$, and $e$ are the characteristic vector fields.
by (11.21). From $X(I) = X(J) = 0$ we get

$$D_x I = -m D_y I \quad \text{and} \quad D_x J = -m D_y J.$$  \hfill (11.33)

Using (11.33), we again compute $Y(I)$ and $Y(J)$ to conclude that these are given by (11.31), where this time

$$a = (n - m) \frac{\partial I}{\partial s} \quad \text{and} \quad b = (n - m) \frac{\partial I}{\partial t},$$

$$c = (n - m) \frac{\partial J}{\partial s} \quad \text{and} \quad e = (n - m) \frac{\partial J}{\partial t}.$$  

So

$$ae - bc = (n - m)^2 \left( \frac{\partial I}{\partial s} \frac{\partial J}{\partial t} - \frac{\partial I}{\partial t} \frac{\partial J}{\partial s} \right).$$  \hfill (11.34)

Comparing the equations (11.32) and (11.34), we obtain

$$\frac{\partial I}{\partial s} \frac{\partial J}{\partial t} - \frac{\partial I}{\partial t} \frac{\partial J}{\partial s} = 0,$$

and so the functions $I$ and $J$ are weakly dependent.

Lemma 11.13. Let $\mathcal{R}$ be a scalar second-order hyperbolic partial differential equation in the plane. Let $X$ and $Y$ be the characteristic vector fields and let $I, I_1$ be two functionally independent $X$ invariant second-order functions on $\mathcal{R}^\infty$. Let

$$I_2 = \frac{Y(I_1)}{Y(I)}$$

be of order $\leq 2$. Then $I, I_1$ and $I_2$ are functionally independent $X$ invariant functions.

Proof. Let $(x, y, u, p, q, I, J)$, for some function $J$, be coordinates on $\mathcal{R}^2$. With no loss of generality assume that

$$dx \wedge dy \wedge du \wedge dp \wedge dq \wedge dr \wedge ds \neq 0$$

on $\mathcal{R}$. By Lemma 11.8, $I_2$ is an invariant function. By Lemma 11.12 $I$ and $I_1$ are weakly dependent, that is, there exists a function $V$ such that

$$I_1 = V(x, y, u, p, q, I).$$  \hfill (11.35)
By lemma 11.9

\[ I_2 = \frac{\partial I_1}{\partial I} \]

and

\[ Y(x) \frac{\partial I_1}{\partial x} + Y(y) \frac{\partial I_1}{\partial y} + Y(u) \frac{\partial I_1}{\partial u} + Y(p) \frac{\partial I_1}{\partial p} + Y(q) \frac{\partial I_1}{\partial q} = 0. \]  

(11.36)

Applying \( X \) to the equation (11.35) we obtain

\[ X(x) \frac{\partial I_1}{\partial x} + X(y) \frac{\partial I_1}{\partial y} + X(u) \frac{\partial I_1}{\partial u} + X(p) \frac{\partial I_1}{\partial p} + X(q) \frac{\partial I_1}{\partial q} = 0. \]  

(11.37)

From equations (11.36) and (11.37) we easily conclude

\[ \frac{\partial I_1}{\partial x} + p \frac{\partial I_1}{\partial u} + r \frac{\partial I_1}{\partial p} + s \frac{\partial I_1}{\partial q} = 0, \]  

(11.38a)

\[ \frac{\partial I_1}{\partial y} + q \frac{\partial I_1}{\partial u} + s \frac{\partial I_1}{\partial p} + t \frac{\partial I_1}{\partial q} = 0. \]  

(11.38b)

If

\[ \frac{\partial I_1}{\partial u} = \frac{\partial I_1}{\partial p} = \frac{\partial I_1}{\partial q} = 0, \]

then from (11.38) we deduce

\[ \frac{\partial I_1}{\partial x} = \frac{\partial I_1}{\partial y} = 0 \]

and so \( I \) and \( I_1 \) are functionally dependent. Hence either

\[ \frac{\partial I_1}{\partial u} \neq 0 \quad \text{or} \quad \frac{\partial I_1}{\partial p} \neq 0 \quad \text{or} \quad \frac{\partial I_1}{\partial q} \neq 0. \]

Assume for instance that

\[ \frac{\partial I_1}{\partial p} \neq 0. \]

Assume that \( I, I_1, \) and \( I_2 \) are functionally dependent, that is,

\[ dI \wedge dI_1 \wedge dI_2 = 0. \]  

(11.39)
Equation (11.39) yields

\[
\frac{\partial I_1}{\partial x} \frac{\partial I_1}{\partial I_0 p} - \frac{\partial I_1}{\partial p} \frac{\partial I_1}{\partial I_0 x} = 0 \quad \text{and} \quad (11.40a)
\]

\[
\frac{\partial I_1}{\partial p} \frac{\partial I_1}{\partial I_0 q} - \frac{\partial I_1}{\partial q} \frac{\partial I_1}{\partial I_0 p} = 0. \quad (11.40b)
\]

Solving for \( \partial I_1/\partial x \) from (11.38a) and substituting into (11.40a) and using (11.40b), we obtain

\[
\frac{\partial I_1}{\partial p} \left[ \frac{\partial r}{\partial I} \frac{\partial I_1}{\partial p} + \frac{\partial s}{\partial I} \frac{\partial I_1}{\partial q} \right] = 0.
\]

By assumption \( \partial I_1/\partial p \neq 0 \), and so we conclude

\[
\frac{\partial r}{\partial I} \frac{\partial I_1}{\partial p} + \frac{\partial s}{\partial I} \frac{\partial I_1}{\partial q} = 0. \quad (11.41)
\]

If, instead of \( \partial I_1/\partial p \neq 0 \), we assume \( \partial I_1/\partial u \neq 0 \) or \( \partial I_1/\partial q \neq 0 \), then we again obtain the equation (11.41). Differentiating (11.38a) with respect to \( J \) gives

\[
\frac{\partial r}{\partial J} \frac{\partial I_1}{\partial p} + \frac{\partial s}{\partial J} \frac{\partial I_1}{\partial q} = 0, \quad (11.42)
\]

because

\[
\frac{\partial I_1}{\partial J} = \frac{\partial}{\partial J} [V(x, y, u, p, g, I)] = 0.
\]

Since

\[
\left[ \frac{\partial r}{\partial J} \frac{\partial s}{\partial I} - \frac{\partial r}{\partial I} \frac{\partial s}{\partial J} \right] dx \wedge dy \wedge du \wedge dp \wedge dq \wedge dI \wedge dJ = dx \wedge dy \wedge du \wedge dp \wedge dq \wedge dr \wedge ds \neq 0,
\]

we conclude from (11.41) and (11.42)

\[
\frac{\partial I_1}{\partial p} = \frac{\partial I_1}{\partial q} = 0.
\]

Substituting into (11.38), we deduce

\[
\frac{\partial I_1}{\partial x} = \frac{\partial I_1}{\partial y} = \frac{\partial I_1}{\partial u} = 0.
\]

So \( I \) and \( I_1 \) are functionally dependent. This is a contradiction and so \( I, I_1 \) and \( I_2 \) are functionally independent.
Lemma 11.14. Let $\mathcal{R}$ be a scalar second-order hyperbolic partial differential equation in the plane that admits three functionally independent $X$ invariant functions of order 2 on $\mathcal{R}^\infty$. Then there are functionally independent $X$ invariant functions $I$, $I_1$, and $I_2$ on $\mathcal{R}^\infty$ of order $\leq 2$, such that

$$I_2 = \frac{Y(I_1)}{Y(I)}.$$ 

Proof. Let $I$, $I_1$, and $I_2$ be three, second-order, functionally independent $X$ invariant functions. Let $(x, y, u, p, q, I, J)$, for some function $J$, be coordinates on $\mathcal{R}^2$. By Lemma 11.12, $I$, $I_1$ and $I_2$ are weakly dependent, that is, there are functions $V$ and $W$ such that

$$I_1 = V(x, y, u, p, q, I) \quad \text{and} \quad I_2 = W(x, y, u, p, q, I).$$

Assume that

$$I_3 = \frac{Y(I_1)}{Y(I)} \quad \text{and} \quad I_4 = \frac{Y(I_2)}{Y(I)}$$

are third-order functions. We readily obtain

$$Y(I_1) = f_1 + \frac{\partial I_1}{\partial I} Y(I) \quad \text{and} \quad Y(I_2) = f_2 + \frac{\partial I_2}{\partial I} Y(I),$$

where $f_1$ and $f_2$ are functions of order $\leq 2$. Since $I_3$ is of third order $f_1 \neq 0$. An easy computation yields

$$I_4 = \frac{Y(I_2)}{Y(I)} = f_3 I_3 + f_4,$$

where

$$f_3 = \frac{f_2}{f_1} \quad \text{and} \quad f_4 = \frac{\partial I_2}{\partial I} - \frac{f_2}{f_1} \frac{\partial I_1}{\partial I}$$

are functions of order $\leq 2$. By Lemma 11.11 $I_4$ is an $X$ invariant function. By Theorem 6.3

$$I_4 = f_3 I_3 + f_4 = f(I, I_1, I_2, I_3)$$

for some function $f$. Hence

$$f_4 = f(I, I_1, I_2, 0)$$
and it now easily follows that $f_3 = g(I, I_1, I_2)$ for some function $g(x_1, x_2, x_3)$. Since $I_4$ is of order 3, then $g \neq 0$. Let $\alpha = \alpha(x_1, x_2, x_3)$ satisfies the equation

$$\frac{\partial \alpha}{\partial x_2} + g \frac{\partial \alpha}{\partial x_3} = 0.$$  \hspace{1cm} (11.45)

Denote

$$I_5 = \alpha(I, I_1, I_2).$$

Using equations (11.43), (11.44), and (11.45), we conclude

$$Y(I_5) = \left[ \frac{\partial \alpha}{\partial x_1}(I, I_1, I_2) + f_4 \frac{\partial \alpha}{\partial x_2}(I, I_1, I_2) \right] Y(I)$$

and so

$$I_6 = \frac{Y(I_5)}{Y(I)}$$

is an $X$ invariant function of order $\leq 2$. Since $g \neq 0$, then from (11.45) we deduce

$$\frac{\partial \alpha}{\partial x_2} \neq 0.$$ 

Therefore, $I$ and $I_5$ are functionally independent. By Lemma 11.13 we conclude that $I$, $I_5$, and $I_6$ are functionally independent $X$ invariant functions of order $\leq 2$.

**Theorem 11.15.** Let $R$ be a scalar second-order hyperbolic partial differential equation in the plane. If there are three functionally independent functions of order $\leq 2$ on $R^\infty$ which all all either $X$ invariant or $Y$ invariant, then $R$ admits a complete intermediate integral.

**Proof.** Without restrictions assume that $I_1$, $I_2$ and $I_3$ are three functionally independent $X$ invariant functions of order $\leq 2$. By Theorem 6.3, $H_0 = 0$. If one of the functions $I$, $I_1$, or $I_2$ is of order $\leq 1$, then $M_7 = 0$ and so $R$ admits a general intermediate integral and therefore also a complete intermediate integral. Assume now that $I_1$, $I_2$, and $I_3$ are second-order functions. By the previous lemma there are three functionally independent $X$ invariant functions $I$, $J$, and $K$ on $R^\infty$ of order $\leq 2$, such that

$$K = \frac{Y(J)}{Y(I)}.$$ 

By Lemma 11.12, $I$ and $J$ are weakly dependent, that is, there exists a function
\[ V = V(x, y, u, p, q, a), \]
such that
\[ J = V(x, y, u, p, q, I). \] \hspace{1cm} (11.46)

From Lemma 11.9 we deduce
\[ K = \frac{\partial V}{\partial a}(x, y, u, p, q, I) \] \hspace{1cm} (11.47)
and that
\[
\frac{\partial V}{\partial x} Y(x) + \frac{\partial V}{\partial y} Y(y) + \frac{\partial V}{\partial u} Y(u) + \frac{\partial V}{\partial p} Y(p) + \frac{\partial V}{\partial q} Y(q) = 0. \] \hspace{1cm} (11.48)

Applying \( X \) to the equation (11.46) and taking into account that \( X(I) = X(J) = 0 \), we obtain
\[
\frac{\partial V}{\partial x} X(x) + \frac{\partial V}{\partial y} X(y) + \frac{\partial V}{\partial u} X(u) + \frac{\partial V}{\partial p} X(p) + \frac{\partial V}{\partial q} X(q) = 0. \] \hspace{1cm} (11.49)

From (11.48) and (11.49), we get
\[
[D_x V](x, y, u, p, q, I) = 0 \quad \text{and} \quad [D_y V](x, y, u, p, q, I) = 0
\]
on \( R^\infty \). Since \( I, J, \) and \( K \) are functionally independent, then from (11.46) and (11.47) we conclude that \( a, V, \) and \( \partial V/\partial a \) are as well functionally independent. This proves that \( V(x, y, u, p, q, a) = b \) is a complete intermediate integral of \( R \).

Theorems 11.10 and 11.15 combine to establish

**Theorem 11.16.** Let \( R \) be a scalar second-order hyperbolic partial differential equation in the plane. Then \( R \) admits a general intermediate integral if and only if there are three functionally independent \( X \) invariant functions of order \( \leq 2 \) on \( R^\infty \).

From Theorem 6.3 we have the following characterization result.

**Theorem 11.17.** Let \( R \) be a scalar second-order hyperbolic partial differential equation in the plane. Then \( R \) admits a complete intermediate integral if and only if \( H_0 = 0 \) or \( K_0 = 0 \).

**Example 1.** The Liouville equation
\[ s = e^u, \]
admits neither general nor complete intermediate integrals since $H_0 = K_0 = e^u$.

**Example 2.** Consider the Monge-Ampère equation

$$p^2 q(s^2 - rt) + 2pqyr + (p^2y + qx)s + xpt - xy = 0,$$

(11.50)

with the characteristic vector fields

$$X = p(2y - qt)D_x + (p^2s + x)D_y,$$

$$Y = q(2y - pt)D_x + p(qs + y)D_y.$$  

Since

$$I_1 = pq - \frac{1}{2}y^2,$$

$$I_2 = p^2q - \frac{1}{2}x^2$$

are $X$ invariant functions, then (11.50) admits a general intermediate integral $I_2 = \varphi(I_1)$, that is,

$$p^2q - \frac{1}{2}x^2 = \varphi(pq - \frac{1}{2}y^2),$$

where $\varphi$ is an arbitrary function of one variable.

**Example 3.** As our next example, consider the equation

$$rs - p = 0.$$  

(11.51)

We choose the characteristic vector fields to be

$$X = D_x + \frac{p}{s^2}D_y$$

and

$$Y = D_x.$$  

It is easy to check that $H_0 = 0$ and $K_0, K_1 \neq 0, K_2 = 0$ and $M_\sigma = 0, M_\tau \neq 0$. Therefore, by Corollary 11.6 and Theorem 11.17, the equation (11.51) admits a complete intermediate integral but not a general intermediate integral. The $X$ invariant second-order functions are

$$I_1 = s - x,$$

$$I_2 = y - \frac{p}{s},$$

$$I_3 = \frac{s^2}{p}.$$  

(11.52)

Notice that

$$\frac{Y(I_1)}{Y(I_2)} = I_3.$$  

On the equation manifold of (11.51), the $X$ invariant functions $I_1$ and $I_2$ satisfy the relation

$$(I_1 + x)(y - I_2) - p = 0.$$  

(11.53)
The equation
\[(a + x)(y - b) - p = 0,\]
where \(a, b\) are arbitrary constants, is a complete intermediate integral of the equation (11.51).

**Example 4.** The equation
\[(sq - tp)^2 = (rt - s^2)(sp - rq)\]
has a complete intermediate integral
\[u + ap + a^2q + b = 0,\]
and the equation
\[\left( x - \frac{pt - qr}{rt - s^2} \right)^2 + \left( y - \frac{qr - ps}{rt - s^2} \right)^2 = 1\]
has a complete intermediate integral
\[px + qy - 2u = p \cos a + q \sin a + b.\]

For more examples of intermediate integrals, the reader is referred to Forsyth [23] or Goursat [26].

A classical problem in partial differential equations is to find methods that construct a general solution, that is, a solution involving arbitrary functions, from the knowledge of a solution that involves arbitrary constants and find criteria when these methods work. These problems were intensively studied by Lagrange, Bour, Charpit, and Ismchenetsky. The most well-known of these methods is the method of variations of constants developed by Lagrange and successfully applied to first-order equations. In the case of partial differential equation of first order, we can always construct a general integral, provided a complete integral is known. In the case of partial differential equations of second order, there are obstructions that will sometimes not allow to construct a general integral from the complete integral. See Forsyth [23], chapter XIX, for detailed discussion of these topics.

§272 In [23] (§272) Forsyth presents the problem of constructing a general intermediate integral from a complete intermediate integral. The following result immediately follows from theorems 11.6 and 11.17, and formulas (9.24).
**Theorem 11.18.** Let \( R \) be a scalar second-order hyperbolic partial differential equation in the plane that admits a complete intermediate integral. Then \( R \) admits a general intermediate integral if and only if \( M_r = M_\sigma = 0 \).

The characterization of Monge-Ampère equations obtained in chapter 11 together with the theorem 11.18 establishes the following fact.

**Proposition 11.19.** Let \( R \) be a scalar second-order hyperbolic partial differential equation in the plane that admits a complete intermediate integral. Then \( R \) admits a general intermediate integral if and only if \( R \) is Monge-Ampère.
CHAPTER 12
THE INVERSE PROBLEM

To solve the inverse problem of the calculus of variations in the case of second-order scalar partial differential equation in the plane

\[ F(x, y, u, p, q, r, s, t) = 0 \]  \hspace{1cm} (12.1)

is to characterize those equations for which there exists a nonzero first-order function \( m(x, y, u, p, q) \), called the \textit{variational multiplier} and a second-order Lagrangian \( \lambda = L(x, y, u, p, q, r, s, t) \, dx \wedge dy \), such that

\[ m(x, y, u, p, q)F(x, y, u, p, q, r, s, t) \, dx \wedge dy = \mathcal{E}(L(x, y, u, p, q, r, s, t)), \]  \hspace{1cm} (12.2)

where \( \mathcal{E}(\lambda) \) is the \textit{Euler-Lagrange form}

\[ \mathcal{E}(\lambda) = \left[ \frac{\partial L}{\partial u} - D_x(\frac{\partial L}{\partial p}) - D_y(\frac{\partial L}{\partial q}) + D_{xx}(\frac{\partial L}{\partial r}) + D_{xy}(\frac{\partial L}{\partial s}) + D_{yy}(\frac{\partial L}{\partial t}) \right] \, \theta \wedge dx \wedge dy. \]  \hspace{1cm} (12.3)

The formulation of the multiplier problem is easily extended to higher order equations or to systems of equations, in which case the multiplier becomes a matrix. The interest in the inverse problem of the calculus of variations was originated by Darboux \[18\] (§604, 605). Darboux showed that a scalar second-order ordinary differential equation always arises from the variational principle. Indeed, if \( u'' = F(x, u, u') \) is a second-order ODE the Helmholtz conditions for the operator

\[ T = m(u'' - F) \]

reduce to a single first-order partial differential equation for the multiplier \( m = m(x, u, u') \). Such an equation has a general solution involving an arbitrary function of two variables. We conclude that a second-order scalar ordinary differential equation always admits infinitely many linearly independent Lagrangians (over the field of real numbers). Almost half a century later, Douglas published his acclaimed paper in which he studied a system of two second-order ordinary differential equations. Douglas applied the Helmholtz conditions to the system of two second-order operators and arrived at an overdetermined system of partial differential equations with the entries of the multiplier matrix.
as unknowns. He analyzed the degree of generality of the solution space for this system using Janet-Riquer theory and, unlike the scalar case, discovered obstructions for the existence of a multiplier. He arrived at the conclusion that some systems do not admit multiplier, some admit a unique multiplier, some admit finitely many linearly independent multipliers, and some admit infinitely many multipliers. Anderson and Duchamp [3] solved the inverse problem of the calculus of variations for second-order scalar quasi-linear equations (in arbitrary independent variables). They explicitly constructed certain one-form \( \Sigma \) and showed that a second-order scalar quasi-linear hyperbolic or elliptic equation is multiplier variational if and only if \( d_H \Sigma = 0 \). We remark here that for a second-order scalar quasi-linear hyperbolic equation in the plane, the form \( \Sigma \) pulled back on the equation manifold is essentially the form \( E\sigma + A\tau \) studied in [8] (the precise relation will be given later in the text). Anderson and Thompson [9] studied the inverse problem of the calculus of variations for systems of ordinary differential equations. The authors rederived the necessary and sufficient conditions of Douglas for second-order equations and extended them to systems of higher order using the methods of the variational bicomplex and then used the Cartan-Kähler theory to determine the degree of generality of the Lagrangians. They showed that the existence of a multiplier was in direct correspondence with the existence of special cohomology classes arising in the variational bicomplex associated with the system of equations. They also proved that a system of ordinary differential equations of order higher than two will admit only finitely many linearly independent multipliers. The case of a scalar fourth-order ordinary differential equation was studied by M. Fels [21]. The author used the formulation of the inverse problem given by Anderson and Thompson and wrote down an invariant coframe for the problem obtained through the Cartan’s equivalence method. He found two relative invariants whose vanishing is a necessary and sufficient condition for the fourth-order scalar ODE to arise from a variational principle and also proved that the multiplier, if it exists, is unique.

In our solution of the multiplier problem for second-order scalar hyperbolic equations in the plane we first show, in the spirit of Anderson-Thompson [9], that the existence of the multiplier is equivalent to the existence of a certain cohomology class in the variational bicomplex associated with the equation. We also show that it is equivalent to the \( d_H \) exactness of the form \( E\sigma + A\tau \), which was studied in the last chapter of [7]. Next we show that the multiplier is unique up to a multiplication by a constant, provided the multiplier exists. Using the structure equations for the Laplace-adapted coframe, we uncover a single relative invariant whose vanishing is a necessary and sufficient condition for the existence of a multiplier. Namely, we prove that a second-order scalar
hyperbolic equation in the plane arises from the variational principle if and only if the first two
Laplace invariants $H_0$ and $K_0$ are equal.

We start with the following proposition.

**Proposition 12.1.** If a second-order scalar partial differential equation in the plane admits a vari­
ational multiplier, then it is Monge-Ampère.

*Proof. Assume the equation*

$$F(x,y,u,p,q,r,s,t) = 0$$

admits a variational multiplier $m = m(x,y,u,p,q)$. The Helmholtz conditions for $T = mF$ are

$$\frac{\partial T}{\partial p} = D_x\left(\frac{\partial T}{\partial r}\right) + \frac{1}{2} D_y\left(\frac{\partial T}{\partial s}\right), \quad (12.4a)$$

$$\frac{\partial T}{\partial q} = \frac{1}{2} D_x\left(\frac{\partial T}{\partial s}\right) + D_y\left(\frac{\partial T}{\partial t}\right). \quad (12.4b)$$

Expanding these equations, we get

$$\frac{\partial T}{\partial r\partial r} u_{xxx} + 3 \frac{\partial T}{\partial r\partial s} u_{xxy} + \left(\frac{\partial T}{\partial r\partial t} + \frac{1}{2} \frac{\partial T}{\partial s\partial s}\right) u_{xxy} + \frac{1}{2} \frac{\partial T}{\partial s\partial t} u_{yy} + \{\text{function of order } \leq 2 \} = 0$$

and

$$\frac{1}{2} \frac{\partial T}{\partial r\partial r} u_{xxx} + \left(\frac{\partial T}{\partial r\partial t} + \frac{1}{2} \frac{\partial T}{\partial s\partial s}\right) u_{xxy} + 3 \frac{\partial T}{\partial r\partial t} u_{xxy} + \frac{\partial T}{\partial t\partial t} u_{yy} + \{\text{function of order } \leq 2 \} = 0.$$
From (12.5) follows

\[ F = crt + ar + ct + g, \]  

(12.8)

where the functions \( a, c, e, g \) are functions of \( x, y, u, p, q, s \) only. From (12.6) we obtain

\[ \frac{\partial e}{\partial s} = \frac{\partial a}{\partial s} = \frac{\partial c}{\partial s} = 0. \]

From (12.7) we conclude

\[ e = -\frac{1}{2} \frac{\partial g}{\partial s s}, \]

that is,

\[ g = -es^2 + bs + f, \]

where \( b, f \) are functions of \( x, y, u, p, q, s \) only. Substituting for \( g \) into (12.8) yields that \( T \) is of Monge-Ampère type and hence so is \( F \).

A generalization of the above result can be found in [1], sec 4.c.

Let

\[ T = m(x, y, u, p, q) [e(s^2 - rt) + ar + 2bs + ct + d], \]  

(12.9)

where \( a, b, c, d, e \) are functions of \( x, y, u, p, q, \) be a Monge-Ampère operator with the unknown first-order function \( m \). We use the Helmholtz conditions (12.4) for \( T \) to derive a system of equations for \( m \). Expanding the Helmholtz conditions for \( T \) and the comparing the coefficients in the second order variables, we arrive at four independent conditions, which when written in the matrix form, are

\[
\begin{pmatrix}
-e & 0 & c & -b \\
0 & e & b & a \\
a & b & 0 & -d \\
\end{pmatrix} \begin{pmatrix}
\frac{\partial m}{\partial x} \\
\frac{\partial m}{\partial y} \\
\frac{\partial m}{\partial p} \\
\frac{\partial m}{\partial q} \\
\end{pmatrix} = m \begin{pmatrix}
\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - \frac{\partial c}{\partial p} \\
\frac{\partial a}{\partial y} + \frac{\partial b}{\partial q} - \frac{\partial c}{\partial p} \\
\frac{\partial a}{\partial p} + \frac{\partial b}{\partial q} - \frac{\partial c}{\partial p} \\
\frac{\partial a}{\partial q} + \frac{\partial b}{\partial p} - \frac{\partial c}{\partial p} \\
\end{pmatrix} \]  

(12.10)

where

\[
\frac{d}{dx} = \frac{\partial}{\partial x} + p \frac{\partial}{\partial u} \quad \text{and} \quad \frac{d}{dy} = \frac{\partial}{\partial y} + q \frac{\partial}{\partial u}. \]  

(12.11)

Notice that the determinant of the matrix on the left side of the equation (12.10) is \( \Delta^2 \) where
\[ \Delta = b^2 - ac + ed \]  \hspace{1cm} (12.12)

Assuming that the equation is hyperbolic yields

\[ \Delta > 0. \]

and so the matrix

\[
\begin{pmatrix}
-e & 0 & c & -b \\
0 & e & b & a \\
b & c & 0 & -d \\
a & b & -d & 0
\end{pmatrix}
\]

has an inverse, namely

\[
\frac{1}{\Delta} \begin{pmatrix}
-d & 0 & b & -c \\
0 & d & -a & -b \\
-a & b & 0 & -e \\
-b & c & -e & 0
\end{pmatrix}. \hspace{1cm} (12.13)
\]

Denoting

\[ M = \ln m, \]

we can rewrite the system (12.10) as a system of equations for \( M \)

\[
\begin{pmatrix}
\frac{dM}{dx} \\
\frac{dM}{dy} \\
\frac{\partial M}{\partial p} \\
\frac{\partial M}{\partial q}
\end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix}
-d & 0 & b & -c \\
0 & d & -a & -b \\
-a & b & 0 & -e \\
-b & c & -e & 0
\end{pmatrix} \begin{pmatrix}
\frac{\partial \Theta}{\partial x} + \frac{\partial \Theta}{\partial q} - \frac{\partial \Theta}{\partial p} \\
\frac{\partial \Theta}{\partial y} + \frac{\partial \Theta}{\partial q} - \frac{\partial \Theta}{\partial p} \\
\frac{\partial \Theta}{\partial p} - \frac{\partial \Theta}{\partial q} + \frac{\partial \Theta}{\partial p} \\
\frac{\partial \Theta}{\partial q} - \frac{\partial \Theta}{\partial q} + \frac{\partial \Theta}{\partial p}
\end{pmatrix}. \hspace{1cm} (12.14)
\]

We will now interpret these equations in terms of the existence of a special \( H^{1,2}(\mathcal{R}^\infty) \) cohomology class that can be represented by a nonvanishing form having the structure

\[ a \Theta \wedge (\sigma \wedge \xi_1 - \tau \wedge \eta_1), \hspace{1cm} (12.15) \]

for some \( a \in C^\infty(\mathcal{R}^\infty) \). It is easy to see that the form of type (12.15) cannot be \( d_H \) exact unless \( a = 0 \). Denote

\[ \omega_0 = \Theta \wedge (\sigma \wedge \xi_1 - \tau \wedge \eta_1). \hspace{1cm} (12.16) \]

Using the \( d_H \) structure equations, we compute
where

\[ d_H \omega_0 = 2 \omega_0 \wedge \lambda_0 \]  \hspace{1cm} (12.17)

\[ \lambda_0 = E \sigma + A \tau. \]  \hspace{1cm} (12.18)

Let \( \omega = a \omega_0 \) and \( a \neq 0 \). The condition

\[ d_H \omega = 0 \]

yields

\[ a (d_H (\ln a) - 2 \lambda_0) \wedge \omega_0 = 0, \]

which implies

\[ \lambda_0 = \frac{1}{2} d_H (\ln a). \]  \hspace{1cm} (12.19)

That is,

\[ X(\ln a) = 2E \quad \text{and} \quad Y(\ln a) = 2A. \]  \hspace{1cm} (12.20)

Note that we have just recovered the form \( \lambda_0 \), which Anderson and Kamran introduced in [8].

From the equation (12.19) it is clear that the necessary condition for the existence of an \( H^{1,2}(\mathcal{R}^\infty) \) cohomology class with a representative of type (12.15) is that \( \lambda_0 \) be \( d_H \) exact. But one can easily see that this is also a sufficient condition. If \( \lambda \) is \( d_H \) exact, that is, if equation (12.19) holds for some \( a \in C^\infty(\mathcal{R}^\infty) \), then

\[ d_H (a \omega_0) = d_H a \wedge \omega_0 + ad_H \omega_0 = d_H a \wedge \omega_0 - a d_H (\ln a) \wedge \omega_0 = 0. \]

The existence of an \( H^{1,2}(\mathcal{R}^\infty) \) cohomology class represented by a form of type (12.15) is a contact invariant condition. To see this, use the transformation laws for the Laplace-adapted coframe. Under the projected pullback by a contact transformation \( \Phi : \mathcal{R} \to \mathcal{R}' \) (4.33), the form \( \omega_0 \) transforms according to

\[ \Phi^\# (\omega'_0) = \Phi^\# (\Theta' \wedge (\sigma' \wedge \xi'_1 - \tau' \wedge \eta'_1)) = l^2 \omega_0. \]

This also proves that \( \lambda_0 = E \sigma + A \tau \) being \( d_H \) exact, is a contact invariant condition. Another way to see this is to investigate the transformation law of \( \lambda_0 \) under the projected pullback by a classical contact transformation (4.33)
\[ \Phi^\#(\lambda_0) = \Phi^\#(E' \sigma' + A' \tau') = E \sigma + A \tau - d_H(\ln l) = \lambda_0 - d_H(\ln l). \] (12.21)

We now choose the characteristic vector fields and the coframe as in the Remark 4.3, with \( m \) and \( n \) being second-order functions. Assuming that the equation is Monge-Ampère, formulas (9.23) yield that \( A \) and \( E \) are of second-order. Thus \( \lambda_0 \) is \( d_H \) exact if there exists a function \( g \) such that

\[ X(g) = E \quad \text{and} \quad Y(g) = A. \] (12.22)

Since the functions \( A \) and \( E \) are of order \( \leq 2 \), it is easy to conclude that \( g \) must be of order \( \leq 1 \).

Expanding these equations and equating the coefficients of the second-order variables, we obtain

\[
\begin{pmatrix}
\frac{dg}{dx} \\
\frac{dg}{dy} \\
\frac{dg}{\partial \theta} \\
\frac{dg}{\partial \varphi}
\end{pmatrix} = \frac{1}{2\Delta} \begin{pmatrix}
-d & 0 & b & -c \\
0 & d & -a & -b \\
-a & b & 0 & -e \\
-b & c & -e & 0
\end{pmatrix} \begin{pmatrix}
\frac{dc}{dx} + \frac{\partial b}{\partial q} - \frac{\partial c}{\partial p} \\
\frac{dc}{dy} + \frac{\partial a}{\partial q} - \frac{\partial c}{\partial p} \\
\frac{db}{dx} - \frac{\partial d}{\partial q} + \frac{\partial b}{\partial p} \\
\frac{db}{dy} - \frac{\partial d}{\partial q} + \frac{\partial b}{\partial p}
\end{pmatrix} + \frac{1}{4} \begin{pmatrix}
\frac{d}{dx}(\ln \Delta) \\
\frac{d}{dy}(\ln \Delta) \\
\frac{d}{\partial \theta}(\ln \Delta) \\
\frac{d}{\partial \varphi}(\ln \Delta)
\end{pmatrix},
\] (12.23)

where \( \Delta \) is given by (12.12). Let

\[ \tilde{g} = 2g - \frac{1}{2} \ln \Delta. \] (12.24)

Then the system (12.23) can be written as

\[
\begin{pmatrix}
\frac{d\tilde{g}}{dx} \\
\frac{d\tilde{g}}{dy} \\
\frac{d\tilde{g}}{\partial \theta} \\
\frac{d\tilde{g}}{\partial \varphi}
\end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix}
-d & 0 & b & -c \\
0 & d & -a & -b \\
-a & b & 0 & -e \\
-b & c & -e & 0
\end{pmatrix} \begin{pmatrix}
\frac{dc}{dx} + \frac{\partial b}{\partial q} - \frac{\partial c}{\partial p} \\
\frac{dc}{dy} + \frac{\partial a}{\partial q} - \frac{\partial c}{\partial p} \\
\frac{db}{dx} - \frac{\partial d}{\partial q} + \frac{\partial b}{\partial p} \\
\frac{db}{dy} - \frac{\partial d}{\partial q} + \frac{\partial b}{\partial p}
\end{pmatrix}.
\] (12.25)

Since the systems (12.14) and (12.25) are identical, we conclude that a Monge-Ampère equation is multiplier variational if and only if the form \( \lambda_0 = E \sigma + A \tau \) is \( d_H \) exact. The Helmholtz conditions for the multiplier can be rewritten as

\[ X \left( \frac{1}{2} \ln m + \frac{1}{4} \ln \Delta \right) = E \quad \text{and} \quad Y \left( \frac{1}{2} \ln m + \frac{1}{4} \ln \Delta \right) = A \] (12.26)

From these equations it is apparent that the multiplier is unique up to a multiplication by a constant.

Summarizing, we obtain the following.

**Theorem 12.2.** The following three statements are equivalent.

(i) A hyperbolic equation \( R \) admits a variational multiplier.
(ii) The form \( \lambda_0 = E\sigma + A\tau \) is \( d_H \) exact.

(iii) There is a nontrivial cohomology class in \( H^{1,2}(\mathcal{R}^\infty) \) that can be represented by a form of type

\[ a \Theta \wedge (\sigma \wedge \xi_1 - \tau \wedge \eta_1), \]

for some \( a \in C^\infty(\mathcal{R}^\infty) \).

Moreover, if the multiplier exist, then it is uniquely determined up to a multiplication by a constant.

Note also that (ii) or (iii) imply that (i) is a contact invariant condition. We remark that the form \( \Sigma \) constructed by Anderson and Duchamp [3] in the case of a second-order scalar quasi-linear hyperbolic equation in the plane becomes

\[ \Sigma = 2(E\sigma + A\tau) - \frac{1}{2}d_H(\ln \Delta) \]

when pulled back to the equation manifold \( \mathcal{R}^\infty \).

Since

\[ d_H(\lambda_0) = d_H(E\sigma + A\tau) = [X(A) - Y(E) - PE - QA] \sigma \wedge \tau = (H_0 - K_0) \sigma \wedge \tau, \]

we have proved the following theorem.

**Proposition 12.3.** If a hyperbolic equation \( \mathcal{R} \) admits a variational multiplier, then it is Monge-Ampère and \( H_0 = K_0 \).

To prove the converse we will need the following two lemmas.

**Lemma 12.4.** Let \( \mathcal{R} \) be a scalar hyperbolic equation \( \mathcal{R} \) in the plane and let \( \omega \in \Omega^{0,s}(\mathcal{R}^\infty) \). If \( d_H \omega = 0 \), then \( \omega = 0 \) for \( s \geq 1 \).

**Proof.** Let \( X \) and \( Y \) be the characteristic vector fields for \( \mathcal{R} \). Since

\[ d_H \omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega) \]

then

\[ X(\omega) = 0 \quad \text{and} \quad Y(\omega) = 0 \]

so that \( \omega \) is both \( X \) and \( Y \) invariant. By Proposition 4.8 \( \omega = 0 \).

We will need the full structure equations for \( \sigma, \theta, \tau, \xi_1, \), and \( \eta_1 \). Using the \( d_H \) structure equations and Propositions 4.3 and 4.5, we obtain the following.
Lemma 12.5. The forms \( \sigma, \theta = du - p \, x \, d - q \, dy, \tau, \xi_1, \) and \( \eta_1 \) satisfy the following structure equations.

\[
\begin{align*}
d\sigma &= -P \, \sigma \wedge \tau + \sigma \wedge \mu_1 + \tau \wedge \alpha, \\
d\tau &= -Q \, \sigma \wedge \tau + \sigma \wedge \beta + \tau \wedge \mu_2 \\
d\theta &= d_H \theta = \sigma \wedge (\xi_1 - E \, \theta) + \tau \wedge (\eta_1 - A \, \theta), \\
d\xi_1 &= \tau \wedge (-D \, \xi_1 + K_0 \, \theta) + \sigma \wedge (\xi_2 - E_1 \, \xi_1) + \mu_1 \wedge \xi_1 + \beta \wedge \eta_1 \\
&\quad + (d_v \, E - E \, \mu_1 - A \, \beta) \wedge \theta, \\
d\eta_1 &= \sigma \wedge (-B \, \eta_1 + H_0 \, \theta) + \tau \wedge (\eta_2 - A_1 \, \eta_1) + \alpha \wedge \xi_1 + \mu_2 \wedge \eta_1 \\
&\quad + (d_v \, A - A \, \mu_2 - E \, \alpha) \wedge \theta.
\end{align*}
\]

(12.27a) (12.27b) (12.27c) (12.27d)

Theorem 12.6. A hyperbolic analytic equation \( R \) with \( H_0 = K_0 \) is Monge-Ampère and admits a variational multiplier.

Proof. Assume

\( H_0 = K_0. \)

From formulas (9.24) we deduce that \( M_\sigma = M_\tau = 0. \) The equation \( R \) is Monge-Ampère and so by formulas (9.23), \( A \) and \( E \) are functions of order \( \leq 2. \) Thus

\[
d_v (E \sigma + A \tau) = \sigma \wedge (A \beta + E \mu_1 - d_v \, E) + \tau \wedge (E \alpha + A \mu_2 - d_v \, A)
\]

\[
= \sigma \wedge (a_1 \, \theta + a_2 \, \xi_1 + a_3 \, \eta_1 + a_4 \, \xi_2 + a_5 \, \eta_2) + \tau \wedge (a_6 \, \theta + a_7 \, \xi_1 + a_8 \, \eta_1 + a_9 \, \xi_2 + a_{10} \, \eta_2)
\]

for some functions \( a_1, \ldots, a_{10}. \) Using the \( d_H \) structure equations, we obtain

\[
d_H (a_4 \, \xi_1 + a_{10} \, \eta_1) \equiv \sigma \wedge a_4 \, \xi_2 + \tau \wedge a_{10} \, \eta_2 \quad \text{mod } \{ \theta, \xi_1, \eta_1 \}
\]

and so

\[
d_v (E \sigma + A \tau) - d_H (a_4 \, \xi_1 + a_{10} \, \eta_1) = \sigma \wedge (c_1 \, \theta + c_2 \, \xi_1 + c_3 \, \eta_1 + c_4 \, \eta_2) + \tau \wedge (c_5 \, \theta + c_6 \, \xi_1 + c_7 \, \eta_1 + c_8 \, \xi_2)
\]

(12.28)

for some functions \( c_1, \ldots, c_8. \) Taking the horizontal differential of the left side of the equation (12.28) yields
\[ d_H[d_V(E\sigma + A\tau) - d_H(a_4\xi_1 + a_{10}\eta_1)] = -d_V d_H(E\sigma + A\tau) - d_H(d_H(a_4\xi_1 + a_{10}\eta_1)) = -d_V[H_0 - K_0] \sigma \wedge \tau = 0. \]

Hence taking the horizontal differential of the equation (12.28), one gets
\[ \sigma \wedge \tau \wedge (c_4\eta_3 - c_8\xi_3) \equiv 0 \quad \text{mod} \{ \theta, \xi_1, \eta_1, \xi_2, \eta_2 \} \]
and so
\[ c_4 = c_8 = 0. \]

With this in mind, we again take the horizontal differential of the equation (12.28) to obtain
\[ \sigma \wedge \tau \wedge (c_3\eta_2 - c_6\xi_2) \equiv 0 \quad \text{mod} \{ \theta, \xi_1, \eta_1 \} \]
and so
\[ c_3 = c_6 = 0. \]

We conclude that
\[ d_V(E\sigma + A\tau) - d_H(a_4\xi_1 + a_{10}\eta_1) = \sigma \wedge (c_1\theta + c_2\xi_1) + \tau \wedge (c_5\theta + c_7\eta_1). \quad (12.29) \]

Recall that \( \omega_0 \) is given by the equation (12.16), namely,
\[ \omega_0 = \theta \wedge (\sigma \wedge \xi_1 - \tau \wedge \eta_1). \quad (12.30) \]

Using the equations (12.27) and Proposition 9.1, we deduce
\[ d\omega_0 = 2 \omega_0 \wedge \lambda + 2\theta \wedge (\sigma \wedge b_2\xi_2 \wedge \eta_1 - \tau \wedge d_2\eta_2 \wedge \xi_1), \quad (12.31) \]
where
\[ \lambda = E\sigma + A\tau + b_1\xi_1 + d_1\eta_1. \quad (12.32) \]

By Theorem 10.3, \( \mathcal{R} \) is given by a quasi-linear equation and so, by (9.16), \( b_2 = d_2 = 0 \). Taking the exterior differential of (12.31) yields
\[ d\lambda \wedge \omega_0 = 0. \quad (12.33) \]

Substituting from (12.30), (12.32), and (12.29) into (12.33), we obtain
\[ (c_2 - c_7) \sigma \wedge \tau \wedge \theta \wedge \xi_1 \wedge \eta_1 \wedge \cdots = 0, \]
and consequently
\[ c_2 = c_1. \] (12.34)

Consider
\[ \lambda = \lambda + c_2 \theta = E \sigma + A \tau - a_4 \xi_1 - a_{10} \eta_1 - c_2 \theta. \]

Using equations (12.29) and (12.34), we compute
\[ d(V(E \sigma + A \tau) - d_H(a_4 \xi_1 + a_{10} \eta_1 + c_2 \theta) = (c_9 \sigma + c_{10} \tau) \wedge \theta, \]

for some functions \( c_9 \) and \( c_{10} \). Taking the horizontal differential of the last equation, we deduce that
\[ \sigma \wedge \tau \wedge (c_9 \eta_1 - c_{10} \xi_1) \equiv 0 \mod \{ \theta \} \]

and so
\[ c_9 = c_{10} = 0. \]

Finally, we conclude
\[ d(V(E \sigma + A \tau) - d_H(a_4 \xi_1 + a_{10} \eta_1 + c_2 \theta) = 0. \] (12.35)

We already knew that
\[ d_H(E \sigma + A \tau) = (H_0 - K_0) \sigma \wedge \tau. \] (12.36)

Using (12.35), we obtain
\[ d_H d_V(a_4 \xi_1 + a_{10} \eta_1 + c_2 \theta) = -d_V d_H(a_4 \xi_1 + a_{10} \eta_1 + c_2 \theta) = d_V d_V(E \sigma + A \tau) = 0. \]

By lemma 12.4
\[ d_V(a_4 \xi_1 + a_{10} \eta_1 + c_2 \theta) = 0. \] (12.37)

From (12.35), (12.36), and (12.37) follows
\[ d(\lambda) = d_H(E \sigma + A \tau) + d_V(E \sigma + A \tau) - d_H(a_4 \xi_1 + a_{10} \eta_1 + c_2 \theta) - d_V(a_4 \xi_1 + a_{10} \eta_1 + c_2 \theta) = 0. \]
By Poincaré Lemma there exists a function \( g \in C^\infty(\mathcal{R}^\infty) \) such that

\[
dg = \dot{\lambda},
\]

but this implies

\[
d_Hg = E\sigma + A\tau.
\]

The statement now follows from Theorem 12.2.

We have the following result.

**Corollary 12.7.** A second-order scalar hyperbolic equation \( \mathcal{R} \) in the plane admits a variational multiplier if and only if \( H_0 = K_0 \). Moreover, if \( d_H(g) = E\sigma + A\tau \), then the variational multiplier \( m(x, y, u, p, q) \) is given by

\[
m = \frac{e^{2g}}{\sqrt{\Delta}}.
\]

Note that the second statement follows immediately from (12.26).

We now develop another criterion for the equation to be variational in terms of a normal form of the universal linearization.

Let \( \mathcal{R}^\infty \) be the infinitely prolonged equation manifold of a second-order hyperbolic scalar partial differential equation in the plane

\[
F(x, y, u, p, q, r, s, t) = 0.
\]

Consider a total differential operator \( \mathcal{F} : \Omega^{0, s}(\mathcal{R}^\infty) \rightarrow \Omega^{0, s}(\mathcal{R}^\infty) \). The adjoint operator \( \mathcal{F}^* \) is, by definition, the unique total differential operator

\[
\mathcal{F} : \Omega^{0, s}(\mathcal{R}^\infty) \rightarrow \Omega^{0, s}(\mathcal{R}^\infty),
\]

such that for every \( \rho \in \Omega^{0, s}(\mathcal{R}^\infty) \) and every \( \omega \in \Omega^{0, s'}(\mathcal{R}^\infty) \)

\[
[p \wedge F(\omega) - \mathcal{F}^*(\omega) \wedge \rho] \wedge \sigma \wedge \tau = d_H\gamma
\]

for some \( \gamma \in \Omega^{1, s+s'}(\mathcal{R}^\infty) \).

The following proposition was proved in [7].
Proposition 12.8. For a second-order scalar hyperbolic equation $\mathcal{R}$ in the plane with characteristic vector fields $X$ and $Y$ we consider the second-order total differential operator

$$\mathcal{F}(\omega) = XY(\omega) + AX(\omega) + BY(\omega) + C\omega.$$  \hfill (12.38)

The adjoint of the operator (12.38) is a second-order total differential operator

$$\mathcal{F}^*(\omega) = XY(\omega) + A^* X(\omega) + B^* Y(\omega) + C^* \omega,$$  \hfill (12.39)

where

$$A^* = -A, \quad B^* = -B - 2Q,$$

and

$$C^* = C - X(A) - Y(E) - AB - E(A - P).$$

We call the operator $\mathcal{F}$ self-adjoint if $\mathcal{F} = \mathcal{F}^*$. It is immediate that the necessary and sufficient conditions for the operator (12.38) to be self-adjoint are

$$A = 0 \quad \text{and} \quad E = 0. \quad \hfill (12.40)$$

Conditions (12.40) imply that $H_0 = K_0$, that is, if $\mathcal{F}$ is self-adjoint, then the equation $\mathcal{R}$ is multiplier variational.

Theorem 12.9. Let $\mathcal{R}$ be a second-order scalar hyperbolic equation in the plane. Then $\mathcal{R}$ is multiplier variational if and only if there exist Laplace-adapted coframe $(\sigma', \tau', \Theta', \xi_1', \eta_1', \ldots)$ on $\mathcal{R}^\infty$ such that the operator

$$\mathcal{F}'(\omega) = X'Y'(\omega) + A'X'(\omega) + B'Y'(\omega) + C'\omega.$$  

is self-adjoint.

Proof. Let $(\sigma, \tau, \Theta, \xi_1, \eta_1, \ldots)$ be a Laplace-adapted coframe on $\mathcal{R}^\infty$. Assume that $\mathcal{R}$ is multiplier variational. Then $H_0 = K_0$. This condition is equivalent to the existence of a function $l$ such that

$$Y(\ln l) = A \quad \text{and} \quad X(\ln l) = E.$$  

The equation $\mathcal{R}$ is Monge-Ampère and so by formulas (9.23) $A$ and $E$ are functions of order $\leq 2$ and therefore $l$ is of order $\leq 1$. Set

$$\Theta' = l\Theta.$$
Now consider the Laplace-adapted coframe \((\sigma, \tau, \Theta', \xi_1', \eta_1', \ldots)\) Using the formulas (4.41) we conclude that

\[ A' = E' = 0. \]

and so the operator \(F'\) is self-adjoint.

The converse is immediate.

**Corollary 12.10.** Let \(\mathcal{R}\) be a second-order scalar hyperbolic equation in the plane. If \(\mathcal{R}\) is multiplier variational, then the universal linearization of \(\mathcal{R}\) can be written as

\[
XY(\Theta) - QY(\Theta) + C\Theta = 0 \quad \text{and} \quad YX(\Theta) + PX(\Theta) + C\Theta = 0.
\]

**Proof.** By the previous theorem, we know that there is a Laplace-adapted coframe \((\sigma, \tau, \Theta, \xi - 1, \eta_1, \ldots)\) such that \(A = E = 0\). Hence the universal linearization of \(\mathcal{R}\) can be written as

\[
XY(\Theta) + BY(\Theta) + C\Theta = 0 \quad \text{and} \quad YX(\Theta) + DX(\Theta) + C\Theta = 0.
\]

Since

\[
XY(\Theta) - YX(\Theta) = PX(\Theta) + QY(\Theta),
\]

it is easy to conclude that

\[
B = -Q \quad \text{and} \quad D = P.
\]

We remark here that a function \(\rho\) on \(\mathcal{R}^\infty\) is a characteristic of a generalized symmetry for the equation \(\mathcal{R}\) if and only if

\[ F(\rho) = 0, \]

where \(F\) is given by the equation (12.38). Next recall the structure theorem for the type \((1, s)\), \(s \geq 1\), conservation laws of Anderson and Kamran [8].

**Theorem 12.11.** Let \(\mathcal{R}\) be a second-order scalar hyperbolic equation in the plane. Let \(s \geq 1\) and let \(\omega \in \Omega^{1,s}(\mathcal{R}^\infty)\) be a \(d_H\) closed form. Then there are contact forms

\[
\rho \in \Omega^{0,s-1}(\mathcal{R}^\infty) \quad \text{and} \quad \gamma \in \Omega^{0,s}(\mathcal{R}^\infty)
\]

such that \(\omega\) is given by

\[ \omega = \psi(\rho) + d_H\gamma \]
where the map
\[ \psi : \Omega^{0,s-1}(\mathcal{R}^\infty) \to \Omega^{1,s}(\mathcal{R}^\infty) \]
is defined by
\[ \psi(\omega) = \frac{1}{2} \sigma[\Theta \wedge (X(\omega) - E\omega) - \xi_1 \wedge \omega] + \frac{1}{2} \tau[\Theta \wedge (-Y(\omega) + A\omega) + \eta_1 \wedge \omega] \]
and \( \rho \) satisfies the adjoint equation
\[ \mathcal{F}^*(\rho) = X Y(\rho) + A^* X(\rho) + B^* Y(\rho) + C^* \rho. \]

When \( s = 1 \), the function \( \rho \) and the type \((0,1)\) form \( \gamma \) are unique.

From Theorem 12.9 and 12.11 we easily deduce the following theorem.

**Theorem 12.12.** Let \( \mathcal{R} \) be a second-order scalar hyperbolic equation in the plane, which is multiplier variational. Then there is a one-to-one correspondence between the generalized symmetries of \( \mathcal{R} \) and the nontrivial cohomology classes \( H^{1,1}(\mathcal{R}^\infty) \).

Let \( \mathcal{R} \) be a second-order scalar hyperbolic equation in the plane given by
\[ F(x,y,u,p,q,r,s,t) = 0. \]
After solving the variational multiplier problem, a natural question to ask is whether there is a second-order linear differential operator \( \mathcal{L} = aD_x + bD_y + c \), where \( a, b, \) and \( c \) are functions of order \( \leq 3 \) such that
\[ \mathcal{L}(F) \theta \wedge dx \wedge dy = \mathcal{E}(\lambda), \tag{12.41} \]
for some Lagrangian \( \lambda \). Here \( \mathcal{E}(\lambda) \) is the Euler-Lagrange form. Using the geometric version of the first variational formula
\[ d_V \lambda = \mathcal{E}(\lambda) + d_H \eta \]
for some form \( \eta \), and the "snake lemma" (see [1] or [9] for details), we deduce that if there is a second-order linear differential operator \( \mathcal{L} = aD_x + bD_y + c \), where \( a, b, \) and \( c \) are functions of order \( \leq 3 \) such that (12.41) is satisfied, then there is a \( d_H \) closed form \( \omega \in \Omega^{1,2}(\mathcal{R}^\infty) \) having the structure
\[ \omega = a^1(\sigma \wedge \xi_1 \wedge \xi_2 - K_0 \tau \wedge \xi_1 \wedge \Theta) + a^2(\sigma \wedge \xi_1 \wedge \Theta - \tau \wedge \eta_1 \wedge \Theta) \]
\[ + a^3(-H_0 \sigma \wedge \eta_1 \wedge \Theta + \tau \wedge \eta_1 \wedge \eta^2) \]
Since $d_H \omega = 0$, the coefficients $a_1, a_2,$ and $a_3$ satisfy

\[ Y(a^1) = (D_1 + A) a^1, \quad X(a^3) = (B_1 + E) a^3 \] (12.42)

and

\[ Y(a^2) - 2Aa^2 = -X(K_0 a^1) + (E_1 + E + Q) K_0 a^1, \] (12.43a)

\[ -X(a^2) + 2Ea^2 = -Y(H_0 a^3) + (A_1 + A - P) H_0 a^3. \] (12.43b)

Assuming $H_0 \neq 0$ and $K_0 \neq 0$, we conclude that the necessary conditions for the existence of a nontrivial form $\omega$ satisfying (12.42) and (12.43) are

\[ J_1 = J_2 = J_3 = J_4 = J_5 = J_6 = 0, \]

where

\[ J_1 = Y(\alpha_{11}) + 3P\alpha_{11} - X(\alpha_9) + (Q + \alpha_{14})\alpha_9 + \alpha_2 I_3 + \alpha_3 \alpha_13 + \alpha_5 I_4, \] (12.44a)

\[ J_2 = Y(\alpha_{12}) - (2a - P)\alpha_{12} + \alpha_6 I_4, \] (12.44b)

\[ J_3 = K_1 + \left(6 \frac{H_0}{K_0} - 7\right) H_1 - 5(H_0 + K_0), \] (12.44c)

\[ J_4 = Y(\alpha_{13}) + 2P\alpha_{13} - X(\alpha_{10}) + (Q + \alpha_{14})\alpha_{10} - K_0 I_5 + \alpha_7 I_4 - \alpha_9, \] (12.44d)

\[ J_5 = Y(I_4) + (2(A - P) - 3Y(\ln H_0)) I_4 + \alpha_{12}, \] (12.44e)

\[ J_6 = 3(K_1 - H_1 + K_0 - H_0) - YX(\ln \frac{K_0}{H_0}) - PX(\ln \frac{K_0}{H_0}). \] (12.44f)

and where

\[ I_1 = \frac{2}{H_0}(H_0 - K_0), \quad I_2 = -\frac{K_0}{H_0}, \quad I_3 = X(\ln \frac{H_0}{K_0}), \quad I_4 = 3\frac{H_0}{K_0}(H_1 - K_0), \]
\[\begin{aligned}
\alpha_1 &= -2H_0(A - P - Y(\ln H_0)), \\
\alpha_2 &= 2K_0(E + Q - X(\ln K_0)), \\
\alpha_3 &= 2(H_0 + X(P) - AE - PQ + C), \\
\alpha_4 &= 2(K_0 - Y(Q) - AE - PQ + C), \\
\alpha_5 &= \frac{1}{H_0}(X(\alpha_2) - (2E + Q)\alpha_2), \\
\alpha_6 &= \frac{1}{H_0}((2A - P)\alpha_1 - Y(\alpha_1)), \\
\alpha_7 &= \frac{K_0}{H_0}(4E + 3Q - 3X(\ln K_0)), \\
\alpha_8 &= 4A - 3P - 3Y(\ln H_0), \\
\alpha_9 &= X(\alpha_3) - Q\alpha_3, \\
\alpha_{10} &= 4H_0 + 3X(P) - 3PQ - 4AE + 4C, \\
\alpha_{11} &= \frac{1}{I_2}(2E\alpha_5 - X(\alpha_5)), \\
\alpha_{12} &= \frac{1}{I_2}(Y(\alpha_4) + (P - \alpha_3)\alpha_4 - X(\alpha_6) + 2Q\alpha_6 - I_1\alpha_1), \\
\alpha_{13} &= \frac{1}{I_2}((2B + Q)\alpha_7 - X(\alpha_7) + Q\alpha_3 - \alpha_5), \\
\alpha_{14} &= 6E + 3Q - 3X(\ln K_0) - X(\ln \frac{K_0}{H_0}).
\end{aligned}\]

Let \(\Phi : \mathcal{R}^\infty \to \mathcal{R}'^\infty\) be a classical invertible contact transformation, thus

\[\Phi^*(X') = mX, \quad \Phi^*(Y') = nY, \quad \text{and} \quad \Phi^*(\Theta') = l\Theta,\]

for some second-order functions \(m, n\) and a first-order function \(l\). Then the transformation laws for \(J_1, \ldots, J_6\) are

\[\Phi^*(J'_1) = m^3nJ_1, \quad \Phi^*(J'_2) = mn^2J_2, \quad \Phi^*(J'_3) = mnJ_3,\]
\[ \Phi^* (J'_4) = m^2_n J_4, \quad \Phi^* (J'_5) = mn^2 J_1, \quad \Phi^* (J'_6) = mn J_6. \]
CHAPTER 13
CLASSIFICATION RESULTS

A detailed discussion of equations with $H_0 = 0$ or $K_0 = 0$, that is, equations that admit a complete intermediate integral, was given in chapter 11. We will therefore mainly focus on the cases where $H_0 \neq 0$ and $K_0 \neq 0$. We invariantly characterize $f$-Gordon equations and some of their subclasses. We prove structure theorems for $f$-Gordon equations which satisfy the relation $aH_0 = bK_0$ where $a$ and $b$ are constants. Results of this chapter include characterizations of wave, Liouville, Klein-Gordon, and Euler-Poisson equations.

Recall that a second-order scalar hyperbolic partial differential equation in the plane is called $f$-Gordon if it can be written in the form

$$s + f(x, y, u, p, q) = 0. \quad (13.1)$$

**Lemma 13.1.** Let $\mathcal{R}$ be a second-order scalar hyperbolic partial differential equation in the plane. Let $I$ and $J$ be two non-constant functions of order $\leq 1$ such that

$$X(I) = 0 \quad \text{and} \quad Y(J) = 0.$$

Then the Poisson bracket

$$[I, J] = 0.$$

**Proof.** By definition (2.6) the Poisson bracket is

$$[I, J] = \frac{dI}{dx} \frac{\partial J}{\partial p} - \frac{dJ}{dx} \frac{\partial I}{\partial p} + \frac{dI}{dy} \frac{\partial J}{\partial q} - \frac{dJ}{dy} \frac{\partial I}{\partial q}, \quad (13.2)$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p \frac{\partial}{\partial u} \quad \text{and} \quad \frac{d}{dy} = \frac{\partial}{\partial y} + q \frac{\partial}{\partial u}.$$

By definition we have

$$D_x = \frac{1}{\delta} (n_y X - m_y Y) \quad \text{and} \quad D_y = \frac{1}{\delta} (m_x Y - n_x X). \quad (13.3)$$

where

$$\delta = m_x n_y - m_y n_x.$$
Using (13.3) we deduce

\[
[I, J] = \frac{dI}{dx} \frac{\partial J}{\partial p} - \frac{dJ}{dx} \frac{\partial I}{\partial p} + \frac{dI}{dy} \frac{\partial J}{\partial q} - \frac{dJ}{dy} \frac{\partial I}{\partial q} 
\]

\[
= D_x I \frac{\partial J}{\partial p} - D_x J \frac{\partial I}{\partial p} + D_y I \frac{\partial J}{\partial q} - D_y J \frac{\partial I}{\partial q} 
\]

\[
= \frac{1}{\delta} \left[ X(J)(n_x \frac{\partial I}{\partial q} - n_y \frac{\partial I}{\partial p}) + Y(I)(m_x \frac{\partial J}{\partial q} - m_y \frac{\partial J}{\partial p}) \right]. 
\]

By Theorem 7.4

\[ M_\sigma = M_\tau = 0 \]

and so by Theorem 10.4 \( \mathcal{R} \) is a Monge-Ampère equation. We need to show that for Monge-Ampère equations

\[ n_x \frac{\partial I}{\partial q} - n_y \frac{\partial I}{\partial p} = 0 \quad \text{and} \quad m_x \frac{\partial J}{\partial q} - m_y \frac{\partial J}{\partial p} = 0. \]

We will prove the above equations are satisfied for quasi-linear equations. The computations for generic Monge-Ampère equations are similar, but more complicated. Assume that \( \mathcal{R} \) is given by a quasi-linear equation

\[ r + (m + n)s + mn t + f = 0, \quad (13.4) \]

where \( m, n, \) and \( f \) are functions of \( x, y, u, p, q \) only and since \( \mathcal{R} \) is hyperbolic \( m \neq n \). Notice that now we have \( m_x = 1, m_y = m, n_x = 1 \) and \( n_y = n \). Expanding \( X(I) = 0 \), we obtain

\[ \frac{\partial I}{\partial p} r + \left( \frac{\partial I}{\partial q} + m \frac{\partial I}{\partial p} \right) s + m \frac{\partial J}{\partial q} + \frac{dI}{dx} + m \frac{dI}{dy} = 0. \quad (13.5) \]

Solving for \( r \) from equation (13.4) and substituting into (13.5), we easily conclude

\[ \frac{\partial I}{\partial q} - n \frac{\partial I}{\partial p} = 0. \]

Similarly from \( Y(I) = 0 \) we deduce

\[ \frac{\partial J}{\partial q} - m \frac{\partial J}{\partial p} = 0. \]
Lemma 13.2. Let \( \varphi = \varphi(x, y, u, u_x, u_y) \) and \( \psi = \psi(x, y, u, u_x, u_y) \) be two independent functions of order \( \leq 1 \) such that

\[
[\varphi, \psi] = 0.
\]

Then there exists a function \( \phi = \phi(x, y, u, u_x, u_y) \) such that

\[
[\varphi, \phi] = 0 \quad \text{and} \quad [\psi, \phi] = 0.
\]

Proof. Define two vector fields \( A, B \) on \( J^1(E) \) by

\[
A(f) = [f, \varphi] \quad \text{and} \quad B(f) = [f, \psi],
\]

for an arbitrary first-order function \( f \). In coordinates

\[
A = \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial y} + \left( p \frac{\partial \varphi}{\partial p} + q \frac{\partial \varphi}{\partial q} \right) \frac{\partial}{\partial u} - \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial p} - \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial q},
\]

\[
B = \frac{\partial \psi}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial y} + \left( p \frac{\partial \psi}{\partial p} + q \frac{\partial \psi}{\partial q} \right) \frac{\partial}{\partial u} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial p} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial q}.
\]

A straightforward but lengthy computation, using \([\varphi, \psi] = 0\), shows

\[
[A, B] = \frac{\partial \varphi}{\partial u} B - \frac{\partial \psi}{\partial u} A.
\]

By Frobenius theorem there are three functionally independent functions of order \( \leq 1 \) that solve the system of first-order partial differential equations

\[
A(f) = 0 \quad \text{and} \quad B(f) = 0. \quad (13.6)
\]

Obviously \( \varphi \) and \( \psi \) are two functionally independent solutions of (13.6). Denote by \( \phi(x, y, u, u_x, u_y) \) a solution of (13.6) such that \( \varphi, \psi, \) and \( \phi \) are functionally independent. This ends the proof of the statement.

In the view of Theorem 2.12 we easily deduce that we are free to specify either \( \bar{x}(x, y, u, u_x, u_y) \) or \( \bar{y}(x, y, u, u_x, u_y) \) or \( \bar{u}(x, y, u, u_x, u_y) \) and we can always find the remaining two functions to construct a contact transformation.
Theorem 13.3. A second-order scalar hyperbolic partial differential equation in the plane $\mathcal{R}$ is contact equivalent to an $f$-Gordon equation if and only if there exists at least one $X$ invariant non-constant function of order $\leq 1$ and if there exists at least one $Y$ invariant non-constant function of order $\leq 1$, that is, if

$$\dim(C_1^{(\infty)}(X)) \geq 1 \quad \text{and} \quad \dim(C_1^{(\infty)}(Y)) \geq 1.$$  \hspace{1cm} (13.7)

Proof. Let $\mathcal{R}$ an $f$-Gordon equation (13.1). Then the characteristic vector fields are $D_x, D_y$, and so $y$ is a $D_x$ invariant function and $x$ is a $D_y$ invariant function.

To prove the converse, assume that the condition (13.7) is satisfied. Let $I$ and $J$ be the first-order non-constant first integrals of $C_1(X)$ and $C_1(Y)$, that is,

$$X(I) = 0 \quad \text{and} \quad X(J) = 0.$$  

By Lemma 13.1 it follows

$$[I, J] = 0.$$  

$I$ and $J$ are functionally independent. Indeed, if $J = \varphi(I)$ for some function $\varphi$, then from $X(I) = 0$ and $X(J) = 0$ we immediately conclude that $I$ is a constant, which is a contradiction. Thus using Theorem 2.12 and Lemma 13.2, we conclude that there exists a function $K$ of order $\leq 1$ such that

$$x = J, \quad y = I, \quad u = K$$

defines a classical contact transformation $\Phi : \mathcal{R}^\infty \to \mathcal{R}^\infty$ for which

$$x = J, \quad y = I \quad \text{and} \quad u = K.$$  

By Proposition 4.7

$$X(y) = 0 \quad \text{and} \quad Y(x) = 0.$$  

and consequently

$$X = aD_x \quad \text{and} \quad Y = bD_y,$$

for some nonvanishing functions $a$ and $b$. So in the new coordinates $\mathcal{R}$ be given by

$$F(x, y, u, p, q, r, s, t) = 0.$$  \hspace{1cm} (13.8)
From the characteristic equation
\[
\frac{\partial F}{\partial r} \lambda^2 - \frac{\partial F}{\partial s} \lambda \mu + \frac{\partial F}{\partial t} \mu^2 = 0
\]
we conclude
\[
\frac{\partial F}{\partial r} = \frac{\partial F}{\partial t} = 0.
\]
hence \(\partial F/\partial s \neq 0\), and so, due to implicit function theorem, we can resolve the equation (13.8) for \(s\) and conclude that \(R'\) is an equation manifold of
\[
s + f(x, y, u, p, q) = 0,
\]
for some function \(f\).

The problem of characterizing \(f\)-Gordon equations is now transformed to the problem of characterizing the dimensions of the maximal integrable subsystems of the characteristic Pfaffian systems \(C_1(X)\) and \(C_1(Y)\). But we have already done this in chapters 6 and 7. From Theorem 7.4 immediately follows our next result.

**Theorem 13.4.** Let \(R\) be a scalar second-order hyperbolic equation in the plane. Then \(R\) is contact equivalent to \(f\)-Gordon equation if and only if \(M_\sigma = M_\tau = 0\) and the following two conditions are satisfied.

(i) \(H_0 = 0\) or \(F_2 = 0\) and
(ii) \(K_0 = 0\) or \(G_2 = 0\).

**Remark 13.5** From the Remark 7.5 we conclude that if \(H_1 = M_\tau = 0\) and \(K_1 = M_\sigma = 0\), then \(R\) is contact equivalent to \(f\)-Gordon equation.

The characterization of the wave equation follows.

**Theorem 13.6.** A second-order scalar hyperbolic partial differential equation in the plane \(R\) is contact equivalent to the wave equation \(s = 0\) if and only if \(H_0 = K_0 = 0\).

**Proof.** It is immediate that for the wave equation \(H_0 = K_0 = 0\).

We now prove the converse. Assume \(H_0 = K_0 = 0\). By formulas (9.24) \(M_\sigma = M_\tau = 0\) and so by 13.4 the equation is contact equivalent to the \(f\)-Gordon equation
\[
s + f(x, y, u, p, q) = 0.
\]
We have
\[ H_0 = \frac{\partial f}{\partial p} r + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} p - \frac{\partial f}{\partial q} f + \frac{\partial f}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial f}{\partial u} = 0, \]
\[ K_0 = \frac{\partial f}{\partial q} t + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} q - \frac{\partial f}{\partial p} f + \frac{\partial f}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial f}{\partial u} = 0. \]

From the last two equations follows
\[ \frac{\partial f}{\partial p} = 0 \quad \text{and} \quad \frac{\partial f}{\partial q} = 0, \quad (13.10) \]
that is, the equation (13.9) is of the form
\[ s + g(x, y, u)pq + a(x, y, u)p + b(x, y, u)q + c(x, y, u) = 0. \quad (13.11) \]

Consider the point transformation
\[ \bar{x} = x, \quad \bar{y} = y, \quad \text{and} \quad \bar{u} = \bar{u}(x, y, u), \quad (13.12) \]
where \(\bar{u}\) is determined by a first-order ODE
\[ \frac{\partial \bar{u}}{\partial u} = e^G \quad \text{and where} \ G \ satisfies \ \frac{\partial G}{\partial u} = g. \]

The equation (13.11) is transformed into equation of the form
\[ s + a(x, y, u)p + b(x, y, u)q + c(x, y, u) = 0. \quad (13.13) \]

We now compute
\[ H_0 = -\frac{\partial b}{\partial u} q + \frac{\partial a}{\partial x} - \frac{\partial c}{\partial u} + ab = 0 \quad \text{and} \quad K_0 = -\frac{\partial a}{\partial u} p + \frac{\partial b}{\partial y} - \frac{\partial c}{\partial u} + ab = 0 \]
to conclude
\[ \frac{\partial a}{\partial u} = \frac{\partial b}{\partial u} = 0 \quad \text{and} \quad \frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}. \]

These conditions imply the existence of the function \(\varphi = \varphi(x, y)\), such that
\[ \frac{\partial \varphi}{\partial y} = a \quad \text{and} \quad \frac{\partial \varphi}{\partial x} = b. \]
Under the point transformation

\[ \bar{x} = x, \quad \bar{y} = y, \quad \text{and} \quad \bar{u} = e^{\sigma}u, \]

the equation (13.13) is transformed into an equation of the form

\[ s + c(x, y, u) = 0. \]  \hspace{1cm} (13.14)

Since

\[ \frac{\partial c}{\partial u} = -H_0 = 0, \]

then \( c = c(x, y) \). The point transformation

\[ \bar{x} = x, \quad \bar{y} = y, \quad \text{and} \quad \bar{u} = u + C, \]

where \( C = C(x, y) \) is a function satisfying the differential equation

\[ \frac{\partial C}{\partial x \partial y} = c, \]

carries (13.14) into the wave equation.

From the last Corollary and from Theorem 6.3 easily follows Lie's Theorem.

**Corollary 13.7.** A second-order scalar hyperbolic partial differential equation in the plane that is, contact equivalent to the wave equation \( s = 0 \) if and only if there are two independent \( X \) invariant functions and two independent \( Y \) invariant functions of order at most one.

**Theorem 13.8.** Let \( R \) be contact equivalent to an \( f \)-Gordon equation with \( H_0 \neq 0 \) and \( K_0 \neq 0 \). Then \( R \) is contact equivalent to an equation of type

\[ s + a(x, y, u)p + b(x, y, u)q + c(x, y, u) = 0, \]  \hspace{1cm} (13.15)

if and only if \( \frac{K_0}{H_0} \) is of order \( \leq 1 \).

**Proof.** Note that according to the transformation rules (4.34) the function \( K_0/H_0 \) is an absolute invariant for the group of the classical contact transformations.

An easy computation shows that \( K_0/H_0 \) for an equation of type (13.15) is a function of order \( \leq 1 \).
We now prove the converse. Considering the \( f \)-Gordon equation

\[ s + f(x, y, u, p, q) = 0 \]  

(13.16)

we arrive at

\[
H_0 = \frac{\partial f}{\partial p} r + \frac{\partial f}{\partial x} q + \frac{\partial f}{\partial u} p - \frac{\partial f}{\partial p} f + \frac{\partial f}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial f}{\partial u},
\]

(13.17)

\[
K_0 = \frac{\partial f}{\partial q} t + \frac{\partial f}{\partial y} q + \frac{\partial f}{\partial u} q - \frac{\partial f}{\partial p} f + \frac{\partial f}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial f}{\partial u}.
\]

(13.18)

Using \( H_0/K_0 \) is of order \( \leq 1 \), we deduce

\[
\frac{\partial f}{\partial p} = 0 \quad \text{and} \quad \frac{\partial f}{\partial q} = 0,
\]

and consequently (13.16) is of the form

\[ s + f(x, y, u, p, q) = g(x, y, u)pq + a(x, y, u)p + b(x, y, u)q + c(x, y, u) = 0. \]  

(13.19)

Consider the point transformation

\[
\bar{x} = x, \quad \bar{y} = y, \quad \text{and} \quad \bar{u} = \bar{u}(x, y, u),
\]

(13.20)

where \( \bar{u} \) is determined by a first-order ordinary differential equation

\[
\frac{\partial \bar{u}}{\partial u} = e^G \quad \text{and where} \quad G \text{ satisfies} \quad \frac{\partial G}{\partial u} = g.
\]

(13.21)

The contact transformation (13.20), (13.21) transforms the equation (13.16) into the equation of the type (13.15).

We will now prove structure theorems for \( f \)-Gordon equations that satisfy the relation \( aH_0 = bK_0 \) where \( a \) and \( b \) are constants. Notice that if a second-order hyperbolic equation in the plane \( \mathcal{R} \) satisfies \( aH_0 = bK_0 \), then the formulas (9.24) and the results of chapter 10 easily imply that \( \mathcal{R} \) is Monge-Ampère. We will start with the case when \( H_0 = K_0 \).

**Theorem 13.9.** Let \( \mathcal{R} \) be contact equivalent to an \( f \)-Gordon equation. Then \( \mathcal{R} \) is contact equivalent to an equation of type

\[ s + c(x, y, u) = 0, \]  

(13.22)
if and only if \( H_0 = K_0 \).

**Proof.** Assume that the equation is of type (13.22). Then \( H_0 = K_0 \) follows by an easy computation.

We now prove the converse. Let \( \mathcal{R} \) be an \( f \)-Gordon equation with \( H_0 = K_0 \).

If \( H_0 = K_0 = 0 \), then the equation is contact equivalent to the wave equation and the statement follows.

Assume \( H_0 \neq 0 \). Then by the previous theorem \( \mathcal{R} \) is given by

\[
s + a(x, y, u)p + b(x, y, u)q + c(x, y, u) = 0, 
\]

for some functions \( a, b, \) and \( c \). Computing \( H_0, \) and \( K_0 \) we obtain

\[
H_0 = -\frac{\partial b}{\partial u}q + \frac{\partial a}{\partial x} - \frac{\partial c}{\partial y} + ab = -\frac{\partial a}{\partial u}p + \frac{\partial b}{\partial y} - \frac{\partial c}{\partial u} + ab = K_0, 
\]

which implies

\[
\frac{\partial a}{\partial u} = \frac{\partial b}{\partial u} = 0 \quad \text{and} \quad \frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}. 
\]

These conditions yield the existence of the function \( \varphi = \varphi(x, y) \), such that

\[
\frac{\partial \varphi}{\partial y} = a \quad \text{and} \quad \frac{\partial \varphi}{\partial x} = b. 
\]

Under the point transformation

\[
\bar{x} = x, \quad \bar{y} = y, \quad \text{and} \quad \bar{u} = e^\varphi u, 
\]

the equation (13.23) becomes of type (13.22).

Theorems 13.9 and Corollary 12.7 yield the classification result for \( f \)-Gordon variational equations.

**Theorem 13.10.** Let \( \mathcal{R} \) be contact equivalent to an \( f \)-Gordon equation. Then the following statements are equivalent.

(i) \( \mathcal{R} \) is multiplier variational.

(ii) \( \mathcal{R} \) is contact equivalent to an equation of type \( s + c(x, y, u) = 0 \).

(iii) \( H_0 = K_0 \).
Theorem 13.11. Let \( \mathcal{R} \) be contact equivalent to an \( f \)-Gordon equation. If \( \mathcal{R} \) satisfies the condition \( H_0 = kK_0 \), for some real number \( k \), \( k \neq 1 \), then \( \mathcal{R} \) is contact equivalent to the linear equation of the form

\[
s + a(x, y)p + b(x, y)q + c(x, y)u = 0, \tag{13.24}
\]

where

\[
c = \frac{1}{1-k} \left[ \frac{\partial a}{\partial x} - k \frac{\partial b}{\partial y} \right] + ab. \tag{13.25}
\]

Proof. Let \( H_0 = kK_0 \). If \( k = 0 \), then \( \mathcal{R} \) is contact equivalent to the wave equation. If \( k \neq 0 \), then by Theorem 13.8 \( \mathcal{R} \) is contact equivalent to the equation of the form

\[
s + a(x, y, u)p + b(x, y, u)q + c(x, y, u) = 0.
\]

Computing \( H_0 \) and \( K_0 \), we obtain

\[
H_0 = -\frac{\partial b}{\partial u} q + \frac{\partial a}{\partial x} - \frac{\partial c}{\partial u} + ab = k \left[ -\frac{\partial a}{\partial u} p + \frac{\partial b}{\partial y} - \frac{\partial c}{\partial u} + ab \right] = kK_0,
\]

from which we deduce

\[
\frac{\partial a}{\partial u} = \frac{\partial b}{\partial u} = 0,
\]

that is,

\[
a = a(x, y), \quad b = b(x, y)
\]

and

\[
\frac{\partial a}{\partial x} - k \frac{\partial b}{\partial y} = (1-k) \left[ \frac{\partial c}{\partial u} - ab \right].
\]

Differentiating the last equation with respect to \( u \), we obtain

\[
(k-1) \frac{\partial c}{\partial u} = 0.
\]

Since \( k \neq 1 \) then \( c = gu + h \) for some functions \( g = g(x, y) \) and \( h = h(x, y) \). Hence \( \mathcal{R} \) can be written in the form

\[
s + a(x, y)p + b(x, y)q + g(x, y)u + h(x, y) = 0. \tag{13.26}
\]

Let \( \varphi = \varphi(x, y) \) be any solution to (13.26). Then the transformation \( \bar{u} = u - \varphi(x, y) \) carries the equation (13.26) into a linear equation of the form (13.24). The equation (13.25) follows immediately from the condition \( H_0 = kK_0 \).

Theorem 13.11 and the classification theorem for linear equations [34] (page 112), combine to establish the following result.
Theorem 13.12. Let $\mathcal{R}$ be contact equivalent to an $f$-Gordon equation. If

$$\frac{K_0}{H_0} = a \quad \text{and} \quad \frac{H_1}{H_0} = b,$$

where $a, b$ are constants and $c = a + b - 2 \neq 0$, then $\mathcal{R}$ is contact equivalent to the Euler-Poisson equation

$$s + \frac{2}{c(x+y)} p + \frac{2a}{c(x+y)} q + \frac{4a}{c^2(x+y)^2} u = 0.$$

Next we characterize the Klein-Gordon and Liouville equations.

Theorem 13.13. Let $\mathcal{R}$ be contact equivalent to an $f$-Gordon equation. Then $\mathcal{R}$ is contact equivalent to the Klein-Gordon equations if and only if the following conditions are satisfied.

(i) $H_0 = K_0 \neq 0$ and

(ii) $H_0 = H_1$.

Proof. Let $\mathcal{R}$ be Klein-Gordon. Then the conditions (i) and (ii) are satisfied.

Conversely, let (i) and (ii) be satisfied for an $f$-Gordon equation $\mathcal{R}$. Using (i), we conclude from Theorem 13.9 that $\mathcal{R}$ is given by

$$s + c(x,y,u) = 0.$$

It is immediate that $H_0 = -\frac{\partial c}{\partial u}$. Using Theorem 9.4, we have

$$H_1 = 2H_0 - K_0 - D_xD_y(\ln H_0).$$

From the highest order coefficients of $H_0 = K_0$, we deduce

$$\frac{\partial c}{\partial u} \frac{\partial c}{\partial u} \frac{\partial c}{\partial u} - \left(\frac{\partial c}{\partial u}\right)^2 = 0.$$

Integrating this equation, we obtain

$$c = e^{\frac{1}{a}ax+b} + g, \quad \text{where} \quad a \neq 0,$$

or

$$c = e^{au} + b,$$
where \( a, b, g \) are arbitrary functions of \( x \) and \( y \) only. Note that \( c = c(x, y) \) is also a solution to (13.29) but it yields \( H_0 = K_0 = 0 \), which contradicts (i). First assume that \( c \) is given by (13.29). Then

\[
H_0 = K_0 = e^{au + b}
\]

and so using (13.27) we arrive at

\[
H_1 = -\frac{\partial a}{\partial y} p - \frac{\partial a}{\partial x} q - \frac{\partial a}{\partial x \partial y} u + a g - \frac{\partial b}{\partial x \partial y}.
\]

From \( H_1 = H_0 \) we obtain

\[
\frac{\partial a}{\partial x} = 0 \quad \text{and} \quad \frac{\partial a}{\partial y} = 0,
\]

and so \( a \) is a constant. Consequently,

\[
a g - \frac{\partial b}{\partial x \partial y} = H_1 = H_0 = e^{au + b}.
\]

Differentiating the last equation with respect to \( u \) yields \( a = 0 \), which contradicts the assumption.

Next assume that \( c \) is given by (13.30). Then

\[
H_0 = e^{a}
\]

and so using (13.27) we obtain

\[
H_1 = e^{a} - \frac{\partial a}{\partial x \partial y}.
\]

from \( H_1 = H_0 \) we conclude \( \frac{\partial a}{\partial x \partial y} = 0 \), that is,

\[
a = \xi(x) + \eta(y),
\]

where \( \xi \) and \( \eta \) are arbitrary functions of its variables. Denote

\[
f(x) = e^{\xi(x)}, \quad \text{and} \quad g(y) = e^{\eta(x)}.
\]

We conclude that \( \mathcal{R} \) is given by the equation

\[
s + f(x)g(y)u + b(x, y) = 0. \quad (13.31)
\]
Consider a point transformation

\[ \tilde{x} = F(x), \quad \tilde{y} = G(y), \quad \text{and} \quad \tilde{u} = u, \]

where \( F'(x) = f(x) \) and \( G'(y) = g(y) \). The equation (13.31) is transformed into

\[ \tilde{s} + \tilde{u} + \tilde{b}(\tilde{x}, \tilde{y}) = 0. \quad (13.32) \]

Finally, consider a point transformation

\[ x = -\tilde{x}, \quad y = \tilde{y}, \quad \text{and} \quad u = \tilde{u} - \varphi(\tilde{x}, \tilde{y}), \]

where \( \varphi \) satisfies the differential equation (13.32) This transformation carries (13.32) into \( s = u \). \( \blacksquare \)

**Theorem 13.14.** A second-order scalar hyperbolic partial differential equation in the plane \( \mathcal{R} \) is contact equivalent to the Liouville equation \( s = e^u \) or to the linear equation \( s = \frac{2u}{(x+y)^2} \) if and only if the following conditions are satisfied.

(i) \( M_\sigma = M_\tau = 0 \) and

(ii) \( H_0 = K_0 \neq 0 \) and

(iii) \( H_1 = 0 \).

**Proof.** If \( \mathcal{R} \) is the Liouville equation or the linear equation, the verification of the conditions (i)-(iii) is straightforward.

We now prove the converse. Let \( \mathcal{R} \) be the equation manifold and let (i), (ii), and (iii) be satisfied. Using Theorem 9.4, we easily deduce that \( K_1 = 0 \) and so from Remark 13.5 we conclude that \( \mathcal{R} \) is contact equivalent to an \( f \)-Gordon equation. From Theorem 13.9 \( \mathcal{R} \) is given by

\[ s + c(x, y, u) = 0. \quad (13.33) \]

From the highest coefficients in \( H_1 = 0 \) we conclude that equation (13.28) holds and therefore the function \( c \) is given by (13.29) or (13.30), namely,

\[ c = \pm \frac{1}{a} e^{au} + b, \quad \text{where} \quad a \neq 0, \quad (13.34) \]

or

\[ c = \pm e^a u + b, \quad (13.35) \]
where \(a, b, g\) are arbitrary functions of \(x\) and \(y\) only. Assuming first that \(c\) is given by (13.34), we follow the steps of the proof of Theorem 13.13 to conclude that the function \(a\) is a constant and functions \(b\) and \(g\) are related by the equation

\[
\frac{\partial b}{\partial x \partial y} - ag = 0.
\]

Thus, the equation (13.33) is

\[
s + \frac{1}{a} e^{au+b} + \frac{1}{a} \frac{\partial b}{\partial x \partial y} = 0
\]

(13.36)

The point transformation

\[
\bar{x} = x, \quad \bar{y} = y, \quad \text{and} \quad \bar{u} = au + b
\]

transforms (13.36) into

\[
\bar{s} + e^u = 0 \quad \text{or} \quad \bar{s} - e^u = 0.
\]

The last two equations are obviously equivalent.

Secondly, assume that the function \(c\) is given by (13.35). Without the loss of generality we assume that \(c = e^u + b\). If not, make the transformation \(\bar{x} = -x, \ \bar{y} = y, \ \bar{u} = u\). Hence \(R\) is given by

\[
s + e^u + b = 0.
\]

(13.37)

Imposing \((iii)\), we deduce

\[
\frac{\partial a}{\partial x \partial y} = e^u.
\]

and so

\[
\frac{e^a}{\left(\phi(x) + \psi(y)\right)^2} = \frac{2\phi(x)\psi(y)}{(\phi(x) + \psi(y))^2},
\]

where \(\phi\) and \(\psi\) are arbitrary functions. Consider the point transformation

\[
\bar{x} = \phi(x), \quad \bar{y} = \psi(y), \quad \text{and} \quad \bar{u} = u.
\]

In the new coordinates \(R\) is given by the equation

\[
\bar{s} + \frac{2\bar{u}}{(\bar{x} + \bar{y})^2} + b(\bar{x}, \bar{y}) = 0,
\]

(13.38)
where \( \tilde{b}(\tilde{x}, \tilde{y}) = b(\phi^{-1}(\tilde{x}), \psi^{-1}(\tilde{y})) \). Finally, a point transformation

\[
\begin{align*}
x &= -\tilde{x}, \\
y &= \tilde{y}, \\
u &= \tilde{u} + \varphi(\tilde{x}, \tilde{y}),
\end{align*}
\]

where \( \varphi \) satisfies a differential equation (13.38) carries the equation (13.38) into the equation

\[
s = \frac{2u}{(x + y)^2}
\]

It is obvious that we cannot distinguish between the Liouville and the linear equations \( s = \frac{2u}{(x + y)^2} \) in terms of the generalized Laplace invariants. Luckily, we can use the relative invariant \( J_1 \) (12.44a) for this purpose. By a straightforward computation using Maple, we obtain the following result.

**Proposition 13.15.** For the Liouville equation \( s = e^{u} \) we have

\[
J_1 = 2e^{u}(r + p^2) - 2u_{xxx},
\]

and for the linear equation \( s = \frac{2u}{(x + y)^2} \) we have

\[
J_1 = 0.
\]

We now proceed with another characterization theorem.

**Theorem 13.16.** Let \( \mathcal{R} \) be contact equivalent to an \( f \)-Gordon equation. If \( H_0 \neq 0, K_0 \neq 0, X(K_0/H_0) \neq 0, \) and \( Y(H_0/K_0) \neq 0, \) then \( \mathcal{R} \) is contact equivalent to an equation of type

\[
s + a(x, y)p + b(x, y)q + c(x, y, u) = 0,
\]

if and only if the following conditions are satisfied.

(i) The order of \( \frac{H_0}{K_0} \) is \( \leq 1 \) and

(ii) the order of \( L = \frac{1}{H_0} X(\frac{K_0}{H_0})Y(\frac{H_0}{K_0}) \) is \( \leq 1 \).

**Proof.** Note that \( L \) is an absolute invariant under the action of the group of classical contact transformations.

Assume that \( \mathcal{R} \) is contact equivalent to the equation of form (13.39). To verify that the conditions (i) and (ii) are satisfied is a straightforward computation.
Conversely, assume that (i) and (ii) are satisfied. By Theorem 13.8 we may assume that \( R \) is given by (13.15), namely,

\[
s + a(x, y, u)p + b(x, y, u)q + c(x, y, u) = 0.
\]

We will prove that \( \frac{\partial a}{\partial u} = \frac{\partial b}{\partial u} = 0 \). The first generalized Laplace invariants \( H_0 \) and \( K_0 \) are

\[
H_0 = -\frac{\partial b}{\partial u}q + \frac{\partial a}{\partial x} - \frac{\partial c}{\partial u} + ab \quad \text{and} \quad K_0 = -\frac{\partial a}{\partial u}p + \frac{\partial b}{\partial y} - \frac{\partial c}{\partial u} + ab.
\]

An easy computation yields

\[
X\left(\frac{K_0}{H_0}\right) = D_x \left(\frac{K_0}{H_0}\right) = -r \frac{\partial a}{H_0 \partial u} + \alpha_1 \neq 0, \quad (13.40)
\]

\[
Y\left(\frac{H_0}{K_0}\right) = D_y \left(\frac{H_0}{K_0}\right) = -t \frac{\partial b}{K_0 \partial u} + \alpha_2 \neq 0, \quad (13.41)
\]

where \( \alpha_1, \alpha_2 \) are functions of order \( \leq 1 \). Hence

\[
L = \frac{1}{H_0^2 K_0} \frac{\partial a}{\partial u} \frac{\partial b}{\partial u} - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial u} + \alpha_1 \alpha_2. \quad (13.42)
\]

If \( \alpha_1 = \alpha_2 = 0 \), then from the last equation follows \( \frac{\partial a}{\partial u} = 0 \) or \( \frac{\partial b}{\partial u} = 0 \), which yields a contradiction with (13.40) or (13.41). Consequently, either \( \alpha_1 \) or \( \alpha_2 \) does not vanish. Say \( \alpha_1 \neq 0 \). Then from (13.42) follows \( \frac{\partial b}{\partial u} = 0 \). By (13.41) \( \alpha_2 \neq 0 \) and so using (13.42) follows \( \frac{\partial a}{\partial u} = 0 \). If \( \alpha_2 \neq 0 \), then a similar argument shows that \( \frac{\partial a}{\partial u} = 0 \) and \( \frac{\partial b}{\partial u} = 0 \).

We now turn our attention once more to the equation of Calogero [15], namely,

\[
s + ur + f(p) = 0, \quad f(p) \neq 0, \quad (13.43)
\]

which we studied in Example 2 in chapter 6. The characteristic vector fields are

\[
X = uD_x + D_y, \quad \text{and} \quad Y = D_x.
\]

If we denote
\[ f(p) = \frac{1}{g''(p)}, \]

then

\[ I_1 = y + g'(p), \quad \text{and} \quad I_2 = rg''(p)e^{pg'(p)-g(p)}, \]

are \( X \) invariant functions and

\[ J_1 = y \]

is \( Y \) invariant function. Thus, by Theorem 13.3 the equation (13.43) is contact equivalent to an \( f \)-Gordon equation. Recall that

\[ H_0 = -r, \quad H_1 = 0, \quad \text{and} \quad K_0 = r(f''(p) - 2). \]

One can readily verify that for a nonlinear function \( f(p) \), the conditions of Theorem 13.16 are satisfied and so (13.43) is contact equivalent to the equation of type (13.39). Namely, the contact transformation

\[ \begin{align*}
\bar{x} &= I_1, \quad \bar{y} = J_1, \quad \bar{u} = px - u, \quad \bar{p} = xF(p), \quad \bar{q} = -q - xf(p), \\
\bar{r} &= f(p)(xf'(p) + \frac{f(p)}{r}), \quad \bar{s} = -f(p)(\frac{s}{r} + xf'(p) + \frac{f(p)}{r}), \\
\bar{t} &= -t + f(p)(\frac{s}{r} + xf'(p) + \frac{f(p)}{r})(1 + \frac{s}{f(p)}) - xf'(p)s.
\end{align*} \]

transforms the equation (13.43) into the linear equation

\[ \bar{s} + a(\bar{x}, \bar{y})\bar{p} + b(\bar{x}, \bar{y})\bar{u} = 0, \quad (13.44) \]

where the functions \( a \) and \( b \) are given by

\[ a = f'(h(\bar{x} - \bar{y})) - h(\bar{x} - \bar{y}), \quad \text{and} \quad b = h(\bar{x} - \bar{y}) \]

and where \( h = (g')^{-1} \). Since \( H_1 = 0 \) the equation (13.44) is integrable by the method of Laplace (we refer the reader to the introduction to this dissertation or better see Forsyth [23] chapter XII for details).

Finally consider a special case of the equation (13.43), namely,

\[ s + ur - 2p(p + c) = 0 \quad (13.45) \]
where \( c \) is a constant. We have

\[
H_0 = -r, \quad H_1 = 0, \quad \text{and} \quad K_0 = -6r.
\]

Therefore,

\[
\frac{K_0}{H_0} = 6 \quad \text{and} \quad \frac{H_1}{H_0} = 0
\]

and so by Theorem 13.12 the equation (13.45) is contact equivalent to the Euler-poisson equation

\[
s + \frac{p}{2(x + y)} + \frac{3q}{x + y} + \frac{3u}{2(x + y)^2} = 0.
\]
CHAPTER 14
FUTURE RESEARCH

The following is a brief outline for possible future research in related areas.

1. Use the generalized Laplace invariants to classify some special types of Darboux integrable equations. For example, we have classified the equations of type \( u_{xx} = f(u_{yy}) \) that are Darboux integrable at level 2. Classification theorems for \( f \)-Gordon equations satisfying the condition \( aH_0 = bK_0 \), where \( a \) and \( b \) are arbitrary constants, are known. Very little is known about the classification of non-Monge-Ampère Darboux integrable equations or about the classification of nonlinear Darboux integrable equations that are integrable at level \( > 2 \).

2. Apply the ideas of Vessiot [42] [43] to other classes of equations. For the case of coupled \( f \)-Gordon equations, this was done by Vassiliou [39] and [40].

3. Determine how the cohomology of the variational bicomplex associated with third-order scalar differential equation or a system of two second-order equations in the plane reflects the property of being Darboux integrable. We have studied this problem for scalar hyperbolic differential equations in the plane. A particularly interesting situation is encountered in the case of a scalar parabolic differential equation in the plane. Cartan [17] showed that such an equation is Darboux integrable if and only if the Goursat’s invariant vanishes. Anderson and Kamran [8], [6] proved that the Goursat’s invariant vanishes if and only if there are infinitely many conservation laws of type (1, s) for every \( s \geq 0 \).

4. Compute the cohomology group \( H^{1,2} \) of the variational bicomplex for the scalar hyperbolic second-order differential equation in the plane. This problem is closely related to the generalized inverse problem of the calculus of variations as formulated by Anderson and Fels [5]. In our preliminary work on this problem we have uncovered new geometric invariants. Using the structure theorem for higher order conservation laws [8], we have computed \( H^{1,2} \) for certain equations. For example, \( H^{1,2} \) for the Liouville equation is infinite-dimensional and \( H^{1,2} \) for the sine-Gordon equation \( u_{xy} = \sin u \) is 1-dimensional.

5. Compute the cohomology groups \( H^{1,1} \) and \( H^{1,0} \) of the variational bicomplex for the scalar hyperbolic Darboux integrable second-order differential equation in the plane. In the case of a variational equation, the \( H^{1,1} \) cohomology group is in a one-to-one correspondence with the generalized symmetries of the equation and, due to Emily Noether theorem, there is a one-to-one
correspondence between the group of generalized variational symmetries and the cohomology group $H^{1,0}$.

6. We introduced a notion of a hyperbolic Darboux system as a special case of a hyperbolic system defined in [13]. By examples we showed that the Laplace transform is just a special case of a homomorphism of corresponding hyperbolic Darboux systems. We show that as hyperbolic Darboux systems the Liouville equation and the equation $u_{xy} = uu_{x}$ are diffeomorphic. We conjecture that Moutard equations are exactly those equations for which there is a surjective homomorphism from a hyperbolic Darboux system associated with some partial prolongation of the wave equation onto the hyperbolic Darboux system associated with the given equation. Bryant, Griffiths, and Hsu in [13] and [14] classified hyperbolic systems of class $s = 0$. Classification of hyperbolic Darboux systems of classes $s = 1, 2, \text{ and } 3$ would lead to some new classification results concerning Darboux integrable equations at level 2.
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