

CANONICAL COORDINATES ON LIE GROUPS AND THE BAKER CAMPBELL
HAUSDORFF FORMULA

by

Nicholas Graner

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Approved:

Mark Fels, Ph.D.
Major Professor

Charles Torre, Ph.D.
Committee Member

Ian Anderson, Ph.D.
Committee Member

Mark R. McLellan, Ph.D.
Vice President for Research and
Dean of the School for Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah

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ABSTRACT

Canonical Coordinates on Lie Groups and the Baker Campbell Hausdorff Formula

by

Nicholas Graner, Master of Science

Utah State University, 2018

Major Professor: Mark Fels
Department: Mathematics and Statistics

Lie's third theorem states that for any finite dimensional Lie algebra \mathfrak{g} over the real numbers, there is a simply connected Lie group G which has \mathfrak{g} as its Lie algebra. The main part of this document is concerned with the question of determining the group G , in Lie's third theorem, and its multiplication function when just \mathfrak{g} is given.

The Baker-Campbell-Hausdorff formula (BCH) is implemented in the Maple Differential Geometry package for nilpotent Lie groups where the group multiplication is determined from the Lie algebra using the BCH formula. The BCH formula is also used to give a closed form formula for a local basis for the left invariant vector fields of a Lie group in canonical coordinates of the 1st kind. It is shown how the BCH formula can be used to locally give the group multiplication in canonical coordinates of the 2nd kind. A closed form formula for a local basis for the left invariant vector fields of a solvable Lie group is given in coordinates of the 2nd kind. For nilpotent Lie groups these calculations are implemented in the Maple Differential Geometry package along with the calculation of local left invariant 1-forms for general Lie groups in coordinates of the 2nd kind. Examples of the calculations are given.

(70 pages)

PUBLIC ABSTRACT

Canonical Coordinates on Lie Groups and the Baker Campbell Hausdorff Formula

Nicholas Graner

Lie Groups occur in math and physics as representations of continuous symmetries and are often described in terms of their Lie Algebra. This thesis is concerned with finding a concrete description of a Lie group given its associated Lie algebra. Several calculations toward this end are developed and then implemented in the Maple Differential Geometry package. Examples of the calculations are given.

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CHAPTER 1

INTRODUCTION

Lie's third theorem states that for any finite dimensional Lie algebra \mathfrak{g} over the real numbers, there is a simply connected Lie group G which has \mathfrak{g} as its Lie algebra. The main part of this document is concerned with the question of determining the group G , in Lie's third theorem, and its multiplication function when just \mathfrak{g} is given.

1.1

Chapter Summary

Chapter 2 provides some background regarding Lie algebras and Lie groups and defines some basic terminology that will be used in the document.

Chapter 3 defines canonical coordinates of the 1st kind by use of the exponential map. For any Lie group these coordinates are defined in a neighborhood of the identity. For some simply connected groups, including those with \mathfrak{g} nilpotent, these coordinates are global making the group diffeomorphic to \mathbb{R}^n .

In Section 2 of Chapter 3 we introduce the Baker Campbell Hausdorff formula (BCH) which provides a local formula for the group multiplication in coordinates of the 1st kind. For simply connected Lie groups with nilpotent \mathfrak{g} this will produce the multiplication map in coordinates for the entire group. Examples are given in Section 3 and Maple code implementation is in Appendix A.

In Section 4 of Chapter 3 the BCH formula is used to determine a closed form formula for a basis for the left invariant vector fields of a Lie group in coordinates of the 1st kind on an open set about the identity. For nilpotent \mathfrak{g} these vector fields are defined on all of \mathbb{R}^n . Examples are given in Section 5 and Maple code implementation is in Appendix A.

Chapter 4 defines canonical coordinates of the 2nd kind. For any Lie group these coordinates are defined in a neighborhood of the identity. For simply connected groups with solvable \mathfrak{g} these coordinates are global making the group diffeomorphic to \mathbb{R}^n .

In Section 2 of Chapter 4 we show how the BCH formula can be used to give the group multiplication in coordinates of the 2nd kind for simply connected Lie groups with nilpotent \mathfrak{g} . Examples are given and Maple code implementation is in Appendix A.

In Section 3 of Chapter 4 we modify the basis for left invariant 1-forms given in reference [1] to be in coordinates of the 2nd kind. These 1-forms are defined in a neighborhood of the identity for any Lie group. Examples are given and Maple code implementation is in Appendix A.

In Section 4 of Chapter 4 we show that for Lie groups with solvable \mathfrak{g} the basis of 1-forms given in reference [5] are the same as those computed in coordinates of the 2nd kind using the formula developed in the prior section. We then go on to show that the dual to this basis is the same as one I developed by use of the BCH formula. For simply connected Lie groups with nilpotent \mathfrak{g} these vector fields are defined on all of \mathbb{R}^n . Examples are given in Section 5 and the Maple code implementation is in Appendix A.

1.2

Literature Review and Summary of Results

Eugene Dynkin gave an explicit formulation of the Baker Campbell Hausdorff Formula (BCH) for expressing the multiplication in canonical coordinates of the 1st kind. (see [4] 1.7). In this document we implement the calculation of this formula in the Maple Differential Geometry package for nilpotent Lie groups.

In this document the BCH formula is used to derive a closed form formula for a basis for the left invariant vector fields of a Lie group on a neighborhood of the identity in canonical coordinates of the 1st kind. In developing this formula a result from reference [6] is used to write the sum of some of the coefficients from the BCH formula in terms of Bernoulli numbers. For nilpotent Lie groups these vector fields are implemented in Maple code.

In this document we use the BCH formula to derive a formula for the group multiplication for a nilpotent Lie group in coordinates of the 2nd kind. This is implemented in Maple code.

Reference [1] gives a formula for a basis for the left invariant 1-forms in coordinates, which are similar to coordinates of the 2nd kind, on a neighborhood of the identity for any Lie group. In this document we modify this formula so that it is given in coordinates of the 2nd kind. This formula is implemented in Maple code.

In this document we show that for solvable Lie groups the formula for the left invariant 1-forms given in [1] are, when modified to be in coordinates of the 2nd kind, the same as the left invariant 1-forms given in [5].

In this document a formula is given for a basis for the left invariant vector fields of a solvable Lie group on a neighborhood of the identity. This formula was developed by use of the BCH formula and results from reference [6]. This formula is shown to give the same left invariant vector fields as given in [5].

CHAPTER 2
LIE GROUPS AND LIE ALGEBRAS

First we will introduce Lie groups and Lie algebras and state several important definitions and theorems that we will utilize.

Definition. Lie group - Let G be a group with multiplication μ . The group G is a Lie group if it has the following properties:

- G is a smooth manifold
- $\mu : G \times G \rightarrow G$ is a smooth map
- the map $i : G \rightarrow G$ by $i(a) = a^{-1}$ is a smooth map.

Definition. Lie algebra - A Lie algebra \mathfrak{g} is a vector space with a multiplication $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, having the properties:

- Bilinearity: $[a\mathbf{x} + b\mathbf{y}, \mathbf{z}] = a[\mathbf{x}, \mathbf{z}] + b[\mathbf{y}, \mathbf{z}]$, $[\mathbf{z}, a\mathbf{x} + b\mathbf{y}] = a[\mathbf{z}, \mathbf{x}] + b[\mathbf{z}, \mathbf{y}]$
- Skew Symmetry: $[\mathbf{x}, \mathbf{y}] = -[\mathbf{y}, \mathbf{x}] \Rightarrow [\mathbf{x}, \mathbf{x}] = 0$
- Jacobi Identity: $[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] + [\mathbf{z}, [\mathbf{x}, \mathbf{y}]] + [\mathbf{y}, [\mathbf{z}, \mathbf{x}]] = 0$

where \mathbf{x} , \mathbf{y} , and \mathbf{z} are vectors and $a, b \in \mathbb{R}$.

Given $X \in \mathfrak{g}$, the **Adjoint Endomorphism**, $\text{ad}X : \mathfrak{g} \rightarrow \mathfrak{g}$ is given by $\text{ad}X(Z) = [X, Z]$.

The **Derived Series** of a Lie algebra is a sequence of subalgebras $\mathfrak{g}_k \subset \mathfrak{g}$ where $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}_k] \subset \mathfrak{g}_k$, making each \mathfrak{g}_{k+1} an ideal in \mathfrak{g}_k (note that each \mathfrak{g}_k is also an ideal in \mathfrak{g} (see [8] Exercise 5.1.9)). A Lie algebra is **Solvable** if there is a k such that the term $\mathfrak{g}_k = 0$ in the derived series.

The **Lower Central Series** of a Lie algebra is a sequence of subalgebras $\mathfrak{g}_k \subset \mathfrak{g}$ where $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}] \subset \mathfrak{g}_k$ making each \mathfrak{g}_{k+1} an ideal in \mathfrak{g}_k (note that each \mathfrak{g}_k is also an ideal in \mathfrak{g} based on the Jacobi identity). A Lie algebra is **nilpotent** if there is a k such that the term $\mathfrak{g}_k = 0$ in the lower central series. A nilpotent Lie algebra is said to be **k-step** nilpotent if $\mathfrak{g}_k = 0$ and $\mathfrak{g}_{k-1} \neq 0$ in the lower central series.

We now describe the standard way to associate a Lie algebra with a Lie group.

Definition. Left Invariant Vector Field - Let G be a Lie group. Let $L_g : G \rightarrow G$ be left multiplication by g given by $L_g(h) = gh$, $g, h \in G$. A vector field X on G is a left invariant vector field if $(L_{g*}X)_h = X_{gh}$.

Let \mathfrak{X} be the vector space of vector fields on a Lie group G . Let $X, Y \in \mathfrak{X}$ and let f be a function, $f : G \rightarrow \mathbb{R}$. Define a multiplication operation (i.e. **bracket**), $[X, Y](f) = X(Y(f)) - Y(X(f))$. This makes \mathfrak{X} an infinite dimensional Lie algebra. Let $\mathfrak{g} \subset \mathfrak{X}$ be the set of left invariant vector fields on G .

Theorem 1. *The set \mathfrak{g} of left invariant vector fields on G with bracket, $[,]$ is a finite dimensional Lie algebra with $\dim \mathfrak{g} = \dim G$. Additionally if $X_e \in T_e G$ there is a unique left invariant vector field X on G with the prescribed tangent vector. (See [3] Proposition 7.1)*

In particular $\mathfrak{g} \subset \mathfrak{X}$ is a finite dimensional subalgebra that we call the Lie algebra for G . Since $X \in \mathfrak{g}$ is uniquely determined by X_e , the Lie algebra can be thought of as either the left invariant fields or as the tangent space at the identity since these are in 1-1 correspondence and given two left invariant vector fields, X and Y , we have $[X, Y]_e = [X_e, Y_e]$.

A standard theorem is the converse to Theorem 1. There are a number of forms of this but one such is the following.

Theorem 2. Lie's Third Theorem - *Given any Lie algebra \mathfrak{g} there is a simply connected Lie group G having \mathfrak{g} as its Lie algebra. (See [4] 1.14.3) In fact the simply connected group is unique up to diffeomorphism (See [10] Theorem 3.28).*

2.1

Heisenberg Example:

The Heisenberg group H is the manifold \mathbb{R}^3 with the multiplication:

$$(x^1, x^2, x^3) * (y^1, y^2, y^3) = (x^1 + y^1, x^2 + y^2 + x^1 y^3, x^3 + y^3). \quad (2.1)$$

This multiplication gives a smooth mapping from $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ making \mathbb{R}^3 into a Lie group (albeit a different Lie group than the familiar additive Abelian group). The Heisenberg Lie algebra is often defined as the vector space \mathbb{R}^3 with the Lie bracket:

$$[(x^1, x^2, x^3), (y^1, y^2, y^3)] = (0, x^1 y^3 - x^3 y^1, 0). \quad (2.2)$$

In terms of the standard basis for \mathbb{R}^3 , $\{e_1, e_2, e_3\}$, equation (2.2) gives non-zero products

$$[e_1, e_3] = e_2, [e_3, e_1] = -e_2. \quad (2.3)$$

It is easy to check that $[\cdot, \cdot]$ satisfies the conditions for making \mathbb{R}^3 into a Lie algebra. In terms of the standard basis for \mathbb{R}^3 equation (2.2) gives:

$$[x^i e_i, y^j e_j] = (x^1 y^3 - x^3 y^1) e_2. \quad (2.4)$$

Note that the Heisenberg Lie algebra \mathfrak{h} is 2-step nilpotent since $[\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]] = 0$.

We now compute the Lie algebra \mathfrak{g} of left invariant vector fields for H and show that it is isomorphic to the one described above in equation (2.4). Let $L_g : H \rightarrow H$ by $L_g(x) = gx$, $g = (g^1, g^2, g^3)$, $x = (x^1, x^2, x^3) \in H$. Specifically we have

$$L_g(x^1, x^2, x^3) = (g^1 + x^1, g^2 + x^2 + g^1 x^3, g^3 + x^3).$$

A basis for the tangent space at the identity is given by $\partial_{x^1}, \partial_{x^2}, \partial_{x^3}$. From these we can create a basis for left invariant vector fields by pushing them forward with L_g . We have $L_{g*}(\partial_{x^1}) = \partial_{x^1}$, $L_{g*}(\partial_{x^2}) = \partial_{x^2}$ and $L_{g*}(\partial_{x^3}) = \partial_{x^3} + g^1 \partial_{x^2}$ giving a basis

$$\{X_1 = \partial_{x^1}, X_2 = \partial_{x^2}, X_3 = \partial_{x^3} + x^1 \partial_{x^2}\}$$

for the left invariant vector fields on H .

For $b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$ and $X, Y \in \mathfrak{g}$ we can write

$$\begin{aligned} X &= c^1 X_1 + c^2 X_2 + c^3 X_3 = c^1 \partial_{x^1} + (c^2 + c^3 x^1) \partial_{x^2} + c^3 \partial_{x^3} \\ Y &= b^1 X_1 + b^2 X_2 + b^3 X_3 = b^1 \partial_{x^1} + (b^2 + b^3 x^1) \partial_{x^2} + b^3 \partial_{x^3} \end{aligned} \quad (2.5)$$

with Lie bracket

$$[X, Y] = (c^1 b^3 - b^1 c^3) X_2. \quad (2.6)$$

Since the Lie algebra of left invariant vector fields on the Heisenberg group and the Heisenberg Lie algebra are both 3 dimensional vector spaces we can define a vector space isomorphism ψ between them by $\psi(X_i) = e_i$. Comparing equations (2.4) and (2.6) we see that $[\psi(X), \psi(Y)] = \psi([X, Y])$ making ψ a Lie algebra isomorphism.

CHAPTER 3
CANONICAL COORDINATES OF THE 1ST KIND

In this section we will define the exponential map and show how it can be used to define a coordinate chart about the identity of a Lie group, referred to as canonical coordinates of the 1st kind. We will state the Baker Campbell Hausdorff formula and show how it can be used to give the group multiplication for groups with nilpotent Lie algebras, knowing only the Lie algebra for the group. We will then express the left invariant vector fields of the group in these coordinates and show how they can be calculated algebraically in certain cases.

3.1

The Exponential and Definition of the Coordinates of the 1st Kind

We first define the exponential map which is a generalization of the matrix exponential and then show how this can be used to define a coordinate chart about the identity of a Lie group.

Theorem 3. *Let G be a Lie group with identity e and let \mathfrak{g} be the Lie algebra for G . For $X \in \mathfrak{g}$ there exists a unique curve $\gamma : \mathbb{R} \rightarrow G$ satisfying*

1. $\gamma(0) = e$
2. $\gamma_* \frac{d}{dt} = X|_{\gamma(t)}$
3. $\gamma(t + s) = \gamma(t)\gamma(s)$.

The curve γ is the unique maximal integral curve of X through e . Furthermore item (3) tells us that γ is a 1-parameter subgroup of G . (see [7] Corollary 1.5)

We use these unique integral curves to define the exponential map.

Definition. Exponential Map - Let G be a Lie group with associated Lie algebra, \mathfrak{g} and identity e . Let X be a left invariant vector field on G . Let γ be the integral curve of X through e given by Theorem 3. Define the exponential map to be $\exp : \mathfrak{g} \rightarrow G$ by $\exp(X) = \gamma(1)$.

Corollary 4. For each $X \in \mathfrak{g}$. The mapping $\phi : \mathbb{R} \rightarrow G$ by $\phi(t) = \exp(tX)$ is the unique 1-parameter subgroup of G with $\phi(0) = e$ and $\phi_* \frac{d}{dt}|_{t=0} = X|_e$.

The following theorems lay out a few fundamental properties of the exponential map.

Theorem 5. There exists an open set $U \subset G$, $e \in U$ and an open set $V \subset \mathfrak{g}$, $0 \in V$ where $\exp : V \rightarrow U$ is a diffeomorphism of V onto U . (see [9] 2.10.1)

Theorem 6. $\exp(0) = e$.

Proof. Let $\gamma : \mathbb{R} \rightarrow G$ by $\gamma(t) = e$. Let $X = 0 \in \mathfrak{g}$, then γ is the unique integral curve of X through e . Therefore $\exp(0) = \gamma(1) = e$.

□

We now use the exponential map to define a coordinate chart about e in G . Let $\{e_i\}_{1 \leq i \leq m}$ be a basis for \mathfrak{g} and define a mapping $\lambda : \mathbb{R}^m \rightarrow \mathfrak{g}$ by

$$\lambda(x^1, \dots, x^m) = x^i e_i.$$

The function λ defines a topology and a differential structure on \mathfrak{g} by defining $U \subset \mathfrak{g}$ is open iff $\lambda^{-1}(U)$ is open. The pair $(\mathbb{R}^m, \lambda^{-1})$ gives a chart on all of \mathfrak{g} .

Combining the functions \exp and λ we get a function $\psi_1 = (\exp \circ \lambda) : \mathbb{R}^m \rightarrow G$ by

$$\psi_1(x^1, \dots, x^m) = \exp(x^i e_i). \quad (3.1)$$

Theorem 7. There is an open set $U \subset G$ and $W \subset \mathbb{R}^m$, $0 \in W$ with $\psi_1 : W \rightarrow U$ being a diffeomorphism making (U, ψ_1^{-1}) a coordinate chart on G with $\psi_1(0) = e$.

Proof. By Theorem 5 there are open sets $U \subset G$ and $V \subset \mathfrak{g}$, $0 \in V$ with $\exp(V) = U$ a diffeomorphism. Let $W = \lambda^{-1}(V)$. If we restrict ψ_1 to W we get $\psi_1 : W \rightarrow U$ being a diffeomorphism. We also have $\psi_1(0) = \exp(\lambda(0)) = e$.

□

Definition. The local coordinates (U, ψ_1^{-1}) on G are called **Canonical Coordinates of the 1st Kind**.

The following theorem shows that coordinate charts constructed in this way are (essentially) unique up to a change of basis.

Theorem 8. *Let G be a Lie group with Lie algebra \mathfrak{g} . Let $\{e_i\}_{1 \leq i \leq m}$ and $\{f_i\}_{1 \leq i \leq m}$ be two basis for \mathfrak{g} . Let A be the change of basis matrix with $A_i^j f_j = e_i$. By Theorem 7 there is a coordinate chart (U_1, ψ_1^{-1}) on G with $e \in U_1$ where ψ_1 is defined in equation (3.1). Similarly let (V_1, ϕ_1^{-1}) be another coordinate chart on G with $e \in V_1$ where ϕ_1 is defined in the same way as ψ_1 but with $\lambda(x^1, \dots, x^m) = x^i f_i$, using the basis $\{f_i\}$, so that $\phi_1(x^1, \dots, x^m) = \exp(x^i f_i)$. Let $W_1 = \psi_1^{-1}(U_1 \cap V_1)$ and $Z_1 = \phi_1^{-1}(U_1 \cap V_1)$. Let $(x^1, \dots, x^m) \in W_1$. Then the change of coordinates function, $\rho = \phi_1^{-1} \circ \psi_1 : W_1 \rightarrow Z_1$ is given by $\rho(x^1, \dots, x^m)^j = A_i^j x^i$.*

Proof. Let $(y^1, \dots, y^m) = \rho(x^1, \dots, x^m)$. We have $\psi_1(x^1, \dots, x^m) = \phi_1(y^1, \dots, y^m) \Rightarrow \exp(x^i e_i) = \exp(y^j f_j)$. Since \exp is a diffeomorphism on $U_1 \cap V_1$ we have $x^i e_i = y^j f_j \Rightarrow x^i A_i^j f_j = y^j f_j \Rightarrow x^i A_i^j = y^j$, the desired result. □

A well studied problem in Lie theory is when $\exp : \mathfrak{g} \rightarrow G$ is a diffeomorphism. The answer to this is described as follows. A Lie algebra \mathfrak{g} is called **exponential** if for some (and therefore each) Cartan subalgebra \mathfrak{h} of \mathfrak{g} and each root $\lambda : \mathfrak{h}_{\mathbb{C}} \rightarrow \mathbb{C}$ we have $\lambda(\mathfrak{h}) \cap 2\pi i\mathbb{Z} \setminus \{0\} = \emptyset$ (see [11] for the following theorem).

Theorem 9. *When G is simply connected the following are equivalent:*

1. *the exponential map is a diffeomorphism*
2. *the exponential map is injective*
3. *\mathfrak{g} is exponential.*

This theorem tells us that for an exponential Lie algebra, \mathfrak{g} , there is an associated simply connected Lie group, G having \mathfrak{g} as its Lie algebra and G is diffeomorphic to \mathbb{R}^m , with the exponential map giving us coordinates.

When \mathfrak{g} is a nilpotent Lie algebra it is itself a Cartan subalgebra with no non-zero roots λ and is therefore exponential. This gives the following theorem.

Theorem 10. *Let \mathfrak{g} be a nilpotent Lie algebra of dimension n and G its associated simply connected Lie group. The function $\exp : \mathfrak{g} \rightarrow G$ is a diffeomorphism. Furthermore given any basis $\{e_i\}_{1 \leq i \leq n}$ for \mathfrak{g} the function $\psi_1 : \mathbb{R}^n \rightarrow G$, defined above, defines coordinates of the 1st kind. Any other coordinates of the 1st kind are related by a linear change of variables. (See [9] Theorem 3.6.2 for the first part and Theorem 8 for the second.)*

3.1.1

Heisenberg Example

We show how the exponential map gives coordinates of the 1st kind for the Heisenberg group, H , continued from 2.1. Let $X = c^1 \partial_{x^1} + (c^2 + c^3 x^1) \partial_{x^2} + c^3 \partial_{x^3}$, which is a generic left invariant vector field on H as given in equation (2.5).

Let $\gamma(t) : \mathbb{R} \rightarrow H$. We want $\frac{d}{dt}\gamma(t) = (c^1, c^2 + c^3 x^1, c^3)$ and $\gamma(0) = (0, 0, 0)$ so that γ is the integral curve for the vector field X passing through the identity.

Integrating gives $\gamma(t) = (c^1 t, c^2 t + \frac{1}{2} c^3 c^1 t^2, c^3 t)$. Since γ is the unique integral curve for X passing through the identity we have

$$\exp(X) = \gamma(1) = (x^1 = c^1, x^2 = c^2 + \frac{1}{2} c^3 c^1, x^3 = c^3) \quad (3.2)$$

giving us a mapping $\psi_1 : \mathbb{R}^3 \rightarrow H$ by $\psi_1(c^1, c^2, c^3) = \exp(X)$ and defining a coordinate chart (H, ψ_1^{-1}) on all of H in coordinates of the 1st kind.

3.2

Baker Campbell Hausdorff Formula

Let G be a Lie group with Lie algebra \mathfrak{g} , multiplication μ and identity e . According to Theorem 5 there is an open set $V \subset \mathfrak{g}$ and an open set $U \subset G$, $e \in U$, where $\exp : V \rightarrow U$ is a diffeomorphism. Let $\log : U \rightarrow V$ be \exp^{-1} . Let $W = \mu^{-1}(U) \subset G \times G$. There is an open set $U_1 \subset U \subset G$ where $U_1 \times U_1 \subset W$. Let $V_1 = \log(U_1)$. For $X, Y \in V_1$ we can define:

$$Z = \log(\exp X \exp Y). \quad (3.3)$$

The BCH formula provides a power series representation of the right side of equation (3.3) written out explicitly in terms of the Lie bracket on \mathfrak{g} , and is the following:

Theorem 11. *There is an open neighborhood V of $0 \in \mathfrak{g}$ where for $X, Y \in V$ the series:*

$$Z = X + \sum_{\substack{k, m \geq 0 \\ r_i + s_i > 0}} \frac{(-1)^k}{(k+1)(1 + \sum_{i=1}^k s_i)} \frac{(\text{ad}X)^{r_1}}{r_1!} \frac{(\text{ad}Y)^{s_1}}{s_1!} \dots \frac{(\text{ad}X)^{r_k}}{r_k!} \frac{(\text{ad}Y)^{s_k}}{s_k!} \frac{(\text{ad}X)^m}{m!} Y \quad (3.4)$$

converges and satisfies $Z = \log(\exp X \exp Y)$, where $\text{ad}X, \text{ad}Y : \mathfrak{g} \rightarrow \mathfrak{g}$ are the endomorphisms $\text{ad}X(Z) = [X, Z]$, $\text{ad}Y(Z) = [Y, Z]$. (See [8] Proposition 9.2.32)

Two alternative ways of writing the BCH formula are:

$$Z = Y + X + \sum_{\substack{k>0 \\ r_i+s_i>0}} \frac{(-1)^k}{(k+1)(1+\sum_{i=1}^k r_i)} \frac{(\text{ad}X)^{r_1}}{r_1!} \frac{(\text{ad}Y)^{s_1}}{s_1!} \dots \frac{(\text{ad}X)^{r_k}}{r_k!} \frac{(\text{ad}Y)^{s_k}}{s_k!} X$$

(See [4] formula 1.7.3) and

$$Z = X + \sum_{\substack{k>0 \\ r_i+s_i>0}} \frac{(-1)^k}{k(\sum_{i=1}^k r_i + s_i)} \frac{(\text{ad}X)^{r_1}}{r_1!} \frac{(\text{ad}Y)^{s_1}}{s_1!} \dots \frac{(\text{ad}X)^{r_k}}{r_k!} \frac{(\text{ad}Y)^{s_k}}{s_k!} W$$

where W is either X or Y . (See Encyclopedia of Mathematics - Campbell-Hausdorff formula)

In order to implement (3.4) in Maple we need to write the sum as an iterated sum. To do so, we first observe that the sum consists of polynomials in the non-commutative endomorphisms, $\text{ad}X$ and $\text{ad}Y$. We use this observation to write (3.4) as an iterated sum where one summation is over the degree of the polynomial. With this in mind we introduce some notation. Let $s_n = 1$ and let

$$\begin{aligned} & [X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n} = 1] \\ & = (\text{ad}X)^{r_1} (\text{ad}Y)^{s_1} \dots (\text{ad}X)^{r_{n-1}} (\text{ad}Y)^{s_{n-1}} (\text{ad}X)^{r_n} (Y) \\ & = [X, [X, \dots [X, [Y, [Y, \dots [Y, \dots [X, [X, \dots [X, Y]] \dots]], \end{aligned} \tag{3.5}$$

where there are $r_1 X$'s followed by $s_1 Y$'s up to $r_n X$'s followed by $s_n Y$'s and $[Y^1] = Y$.

For example the non-zero terms in equation (3.4) of degree $d = 3$ are then

$$\begin{aligned} & \frac{1}{2}[X^2Y], -\frac{1}{2}[XY^0XY], -\frac{1}{4}[X^2Y^0X^0Y], \\ & \frac{1}{3}[X, Y^0, XY^0X^0Y], -\frac{1}{4}[X^0YXY] \text{ and } \frac{1}{6}[X^0, Y, XY^0X^0Y], \end{aligned} \tag{3.6}$$

which in Lie brackets are

$$\begin{aligned} & \frac{1}{2}[X, [X, Y]], -\frac{1}{2}[X, [X, Y]], -\frac{1}{4}[X, [X, Y]], \\ & \frac{1}{3}[X, [X, Y]], -\frac{1}{4}[Y, [X, Y]] \text{ and } \frac{1}{6}[Y, [X, Y]]. \end{aligned}$$

Let (t_m, \dots, t_1) be a list of natural numbers with $t_1 > 0$. Let

$$W(t_m, \dots, t_1) = \begin{cases} [X^{t_m} Y^{t_{m-1}} X^{t_{m-2}} Y^{t_{m-3}} \dots X^{t_2} Y^{t_1}], & m \text{ even} \\ [Y^{t_m} X^{t_{m-1}} Y^{t_{m-2}} X^{t_{m-3}} \dots X^{t_2} Y^{t_1}], & m \text{ odd.} \end{cases} \quad (3.7)$$

For any list of natural numbers $(r_1, s_1, r_2, s_2, \dots, r_n, s_n)$ with $s_n > 0$ there is a unique list of positive integers (t_m, \dots, t_1) so that $W(t_m, \dots, t_1)$ and

$W(r_1, s_1, r_2, s_2, \dots, r_n, s_n)$ represent the same non-commutative polynomial in $\text{ad}X$ and $\text{ad}Y$. To see this take the ordered list of symbols $\langle r_1, s_1, r_2, s_2, \dots, r_n, s_n \rangle$ and remove any which are equal to zero. In the remaining list if an r_i is adjacent to an r_j , let $r_i = r_i + r_j$ and remove r_j from the list. Do this for the s_i terms also and continue until we have an alternating list $\langle \dots r_i, s_j, r_k, \dots \rangle$ with no terms equal to zero. Relabel these symbols $\langle t_m, \dots, t_1 \rangle$ giving $W(t_m, \dots, t_1)$, representing the same polynomial as $W(r_1, s_1, r_2, s_2, \dots, r_n, s_n)$. Let $d = \sum_1^m t_i$, the degree of the polynomial then (t_m, \dots, t_1) is a **composition** of d , meaning $d = \sum_1^m t_i$ where each t_i is a positive integer.

For example the terms in equation (3.4) of degree $d = 3$, listed in (3.6) (in the same order) are then

$$\frac{1}{2}W(2, 1), -\frac{1}{2}W(2, 1), -\frac{1}{4}W(2, 1), \frac{1}{3}W(2, 1), -\frac{1}{4}W(1, 1, 1), \frac{1}{6}W(1, 1, 1).$$

Let (a_{n_1}, \dots, a_1) and (b_{n_2}, \dots, b_1) be ordered lists of natural numbers. Define an equivalence relation \sim so that $(a_{n_1}, \dots, a_1) \sim (b_{n_2}, \dots, b_1)$ iff there is a sequence of positive integers (t_m, \dots, t_1) such that $W(a_{n_1}, \dots, a_1)$, $W(b_{n_2}, \dots, b_1)$ and $W(t_m, \dots, t_1)$ represent the same non-commutative polynomial in $\text{ad}X$ and $\text{ad}Y$. Let

$$c(t_m, \dots, t_1) = \sum_{\substack{r_i + s_i > 0 \\ (r_1, s_1, r_2, s_2, \dots, r_n, s_n) \sim (t_m, \dots, t_1)}} \frac{(-1)^{n-1}}{n(\sum_{i=1}^n s_i) \cdot \prod_{i=1}^n r_i! s_i!}. \quad (3.8)$$

For example, for the terms in equation (3.4) of degree $d = 3$ we have

$$c(2, 1) = \frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} = \frac{1}{12} \text{ and} \\ c(1, 1, 1) = -\frac{1}{4} - \frac{1}{6} = -\frac{1}{12}.$$

We can now write equation (3.4) as an iterated sum over the degree of the polynomial with the inner sum being over all compositions of d where d is the degree of the polynomial.

Theorem 12. *With W as defined in formula (3.7) and c as defined in formula (3.8), Z from Theorem 11 is given by*

$$Z = X + \sum_{d=1}^{\infty} \sum_{\substack{\text{compositions} \\ (t_1, \dots, t_m) \\ \text{of } d \\ \text{with } t_1=1}} c(t_m, \dots, t_1) W(t_m, \dots, t_1). \quad (3.9)$$

Equation (3.9) with all terms up to degree $d = 3$ written explicitly is:

$$\begin{aligned} Z &= X + c(1)W(1) + c(1, 1)W(1, 1) + c(1, 2)W(1, 2) + c(1, 1, 1)W(1, 1, 1) + \dots \\ &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]) + \dots \end{aligned}$$

Proof. Starting with equation (3.4), using the notation from (3.5) and letting $n = k + 1$ we get

$$\begin{aligned} Z &= X + \sum_{\substack{k, m \geq 0 \\ r_i + s_i > 0}} \frac{(-1)^k}{(k+1)(1 + \sum_{i=1}^k s_i)} \frac{(\text{ad}X)^{r_1}}{r_1!} \frac{(\text{ad}Y)^{s_1}}{s_1!} \dots \frac{(\text{ad}X)^{r_k}}{r_k!} \frac{(\text{ad}Y)^{s_k}}{s_k!} \frac{(\text{ad}X)^m}{m!} Y \\ &= X + \sum_{\substack{k, m \geq 0 \\ r_i + s_i > 0}} \frac{(-1)^k}{(k+1)(1 + \sum_{i=1}^k s_i)} \frac{[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_k} Y^{s_k} X^m Y^1]}{\prod_{i=1}^m r_i! s_i!} \\ &= X + \sum_{\substack{n > 0 \\ r_i + s_i > 0}} \frac{(-1)^{n-1}}{n(\sum_{i=1}^n s_i) \cdot \prod_{i=1}^n r_i! s_i!} [X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n=1}]. \end{aligned} \quad (3.10)$$

Let $I = (r_1, s_1, \dots, r_n, s_n)$ and let

$$B_I = \frac{(-1)^{n-1}}{n(\sum_{i=1}^n s_i) \cdot \prod_{i=1}^n r_i! s_i!}. \quad (3.11)$$

This allows equation (3.10) to be rewritten as:

$$Z = \log(\exp X \exp Y) = X + \sum_{\substack{n > 0 \\ r_i + s_i > 0}} B_I [X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n=1}]. \quad (3.12)$$

Let $d = \sum_{i=1}^n r_i + s_i$, where we think of d as the degree of the polynomial term $[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n=1}]$ in equation (3.12) in terms of the non commutative variables X and Y . By the definition of multiple

sums, equation (3.12) can be written

$$\begin{aligned}
Z &= X + \lim_{k,l \rightarrow \infty} \sum_{n=1}^k \sum_{\substack{d=n \\ r_i+s_i > 0}}^l B_I[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n} = 1] \\
&= X + \lim_{k,l \rightarrow \infty} \sum_{\substack{d=1 \\ r_i+s_i > 0}}^l \sum_{n=1}^{\min\{k,d\}} B_I[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n} = 1] \\
&= X + \lim_{l \rightarrow \infty} \sum_{\substack{d=1 \\ r_i+s_i > 0}}^l \sum_{n=1}^d B_I[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n} = 1] \\
&= X + \sum_{d=1}^{\infty} \sum_{\substack{n=1 \\ r_i+s_i > 0}}^d B_I[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n} = 1], \tag{3.13}
\end{aligned}$$

noting that $n > d$ is impossible based on the restriction that $r_i + s_i > 0$ and the fact that terms where $k = d$ must go to the same limit as the full sequence of terms.

Using the notation (3.11) we also have

$$c(t_m, \dots, t_1) = \sum_{\substack{r_i+s_i > 0 \text{ and} \\ (r_1, s_1, r_2, s_2, \dots, r_n, s_n) \sim (t_m, \dots, t_1)}} B_I$$

allowing us to rewrite equation (3.13) as

$$\begin{aligned}
Z &= X + \sum_{d=1}^{\infty} \sum_{\substack{n=1 \\ r_i+s_i > 0}}^d B_I[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n} = 1] \\
&= X + \sum_{d=1}^{\infty} \sum_{\substack{\text{compositions} \\ (t_1, \dots, t_m) \\ \text{of } d \\ \text{with } t_1=1 \\ r_i+s_i > 0 \text{ and} \\ (r_1, s_1, r_2, s_2, \dots, r_n, s_n) \sim (t_m, \dots, t_1)}} B_I W(t_m, \dots, t_1) \\
&= X + \sum_{d=1}^{\infty} \sum_{\substack{\text{compositions} \\ (t_1, \dots, t_m) \\ \text{of } d \\ \text{with } t_1=1}} c(t_m, \dots, t_1) W(t_m, \dots, t_1).
\end{aligned}$$

establishing the theorem. □

3.3

BCH - The Nilpotent Case

One important corollary of Theorem 12 occurs when \mathfrak{g} is a nilpotent Lie algebra.

Corollary 13. *Let \mathfrak{g} be k -step nilpotent, then $W(t_m, \dots, t_1) = 0$ when $d \geq k$ and*

$$Z = X + \sum_{d=1}^{k-1} \sum_{\substack{\text{compositions} \\ (t_1, \dots, t_m) \\ \text{of } d \\ \text{with } t_1=1}} c(t_m, \dots, t_1) W(t_m, \dots, t_1).$$

Furthermore if we write Z, X and Y in a basis $\{e_i\}_{1 \leq i \leq m}$ for \mathfrak{g} so that $Z = z^i e_i$, $Y = y^i e_i$ and $X = x^i e_i$ then

$$z^i = x^i + \left(\sum_{d=1}^{k-1} \sum_{\substack{\text{compositions} \\ (t_1, \dots, t_m) \\ \text{of } d \\ \text{with } t_1=1}} c(t_m, \dots, t_1) W(t_m, \dots, t_1) \right)^i. \quad (3.14)$$

Let G be a simply connected Lie group associated with \mathfrak{g} then equation (3.14) gives a global representation of the group multiplication in coordinates of the 1st kind.

Formula (3.14) of Corollary 13 is implemented in Appendix A by the code 4.5.2 to produce the multiplication function for a simply connected nilpotent Lie group with Lie algebra \mathfrak{g} .

3.3.1

Examples 3.3.1

Example 3.3.1.1 The Heisenberg Lie algebra defined in equation (2.2) is 1-step Nilpotent. On account of that, formula (3.4) gives us

$$Z = X + Y + \frac{1}{2}[X, Y].$$

Using the standard basis $\{e_1, e_2, e_3\}$, the Heisenberg Lie algebra is given by the equations in (2.3).

Formula (3.14) then gives:

$$z_1 = x^1 + y^1, z_2 = x^2 + y^2 + \frac{1}{2}(x^1 y^3 - x^3 y^1), z_3 = x^3 + y^3.$$

Example 3.3.1.2 In 4.5.8 of Appendix B a more complicated example computing formula (3.14) is shown.

3.4

Left Invariant Vector Fields in Canonical Coordinates of the 1st Kind

Let G be a Lie group with Lie algebra \mathfrak{g} , multiplication μ and identity e . We will express a basis for \mathfrak{g} in coordinates of the 1st kind and show how they can be calculated by use of the BCH formula with the ultimate goal of computing the vector fields in Maple for \mathfrak{g} nilpotent.

Theorem 14. *Let G be a Lie group and $\{e_i\}_{1 \leq i \leq m}$ be a basis for the left invariant vector fields on G . Let (U, ψ_1^{-1}) be a coordinate chart in coordinates of the 1st kind as defined in Theorem 7. There is an open set $W \subset \psi_1^{-1}(U)$, $0 \in W$ where for $x = (x^1, \dots, x^m) \in W$, $\hat{X}_k|_x = \psi_{1*}^{-1} e_k|_{\psi_1(x)}$ is given by*

$$\begin{bmatrix} \hat{X}_1|_{(x^1, \dots, x^m)} \\ \vdots \\ \hat{X}_m|_{(x^1, \dots, x^m)} \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n ([\mathbf{ad} \ x^i e_i]^T)^n \begin{bmatrix} \partial_{x^1} \\ \vdots \\ \partial_{x^m} \end{bmatrix} \quad (3.15)$$

where B_n is the n 'th Bernoulli number in the standard notation ($B_1 = -\frac{1}{2}$) and $[\mathbf{ad} \ x^i e_i]$ is the matrix representation of $\mathbf{ad} \ x^i e_i$ in the given basis.

Note that the exponential generating function for the Bernoulli numbers is

$$\frac{t}{e^t - t^0} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n \quad (3.16)$$

allowing us to write formula (3.15) as

$$\begin{bmatrix} \hat{X}_1|_{(x^1, \dots, x^m)} \\ \vdots \\ \hat{X}_m|_{(x^1, \dots, x^m)} \end{bmatrix} = \frac{[\mathbf{ad} \ x^i e_i]^T}{I - e^{-[\mathbf{ad} \ x^i e_i]^T}} \begin{bmatrix} \partial_{x^1} \\ \vdots \\ \partial_{x^m} \end{bmatrix} \quad (3.17)$$

where $t = -[\mathbf{ad} \ x^i e_i]^T$ in equation (3.16). Formula (3.17) allows us to compute the vector fields in some cases where the summation in equation (3.15) is infinite. We give an example of this in 3.4.1. Function (3.16) has been well studied, for example the radius of convergence for (3.16) is 2π . Reference [4], in Lemma 1.8.2, gives a formula for right invariant vector fields which is similar to formula (3.17).

Proof. By Theorem (11) there is an opens set $V \subset \mathfrak{g}$ with \exp being a diffeomorphism from $V \rightarrow U = \exp(V)$. Let $x = x^i e_i$, $y = (y^1, \dots, y^m)$, and $z = (z^1, \dots, z^m)$ with $x, y \in W$. Let $X = x^i e_i$,

$Y = y^i e_i$ and $Z = z^i e_i = \log(\exp X \exp Y)$. Let $\mu_1 : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}^m$, by

$$\mu_1(x, y) = z = \psi_1^{-1}(\psi_1(x)\psi_1(y)), \quad (3.18)$$

which is simply the group multiplication expressed in coordinates of the 1st kind. We claim

$$\hat{X}_k|_x = \partial_{y^k} \mu_1(x, y)^a \partial_{x^a}|_{y=0}.$$

To show this let $F_x : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$F_x(y) = \mu_1(x, y) \quad (3.19)$$

giving

$$F_{x*}|_{y=0}(\partial_{y^k}) = \partial_{y^k} \mu_1(x, y)^a \partial_{x^a}|_{y=0}.$$

Since $F_x(y) = \mu_1(x, y) = \psi_1^{-1}(\psi_1(x)\psi_1(y))$ we have

$$\begin{aligned} F_{x*} &= \psi_{1*}^{-1} \circ L_{\psi_1(x)*} \circ \psi_{1*} \\ \Rightarrow F_{x*}|_{y=0}(\partial_{y^k}) &= \psi_{1*}^{-1} \circ L_{\psi_1(x)*} e_k|_e = \psi_{1*}^{-1} e_k|_{\psi_1(x)} = \hat{X}_k|_x \\ \Rightarrow \hat{X}_k|_x &= F_{x*}|_{y=0}(\partial_{y^k}) = \partial_{y^k} \mu_1(x, y)^a \partial_{x^a}|_{y=0}. \end{aligned}$$

We now use equation (3.9) to explicitly calculate $\hat{X}_k|_x$. Since we are going to set $y = 0$ we only need the terms in the formula with exactly one Y . First we note that terms in equation (3.9) with exactly one Y are of the form $c(n, 1)W(n, 1)$. Let K be the sum of terms with more than one Y . Formula (3.9) becomes

$$Z = X + Y + \sum_{n=1}^{\infty} c(n, 1)W(n, 1) + K. \quad (3.20)$$

Let c_{ai}^j be the structure constants for \mathfrak{g} in the basis $\{e_i\}$ so that

$$[\text{ad } x]_k^a = x^i c_{ik}^a.$$

Let i_a, p_a be distinct indices for each a each taking on values from $1, \dots, m$. Let $b_n = c(n, 1)$. we have

$$W(n, 1) = [X, [X, \dots [X, Y] \dots]] = x^{i_1} \dots x^{i_n} y^{i_{n+1}} c_{i_1 p_1}^i c_{i_2 p_2}^{p_1} \dots c_{i_n i_{n+1}}^{p_{n-1}} e_i$$

and formula (3.20) becomes

$$z^i e_i = x^i e_i + y^i e_i + \sum_{n=1}^{\infty} b_n x^{i_1} \cdots x^{i_n} y^{i_{n+1}} c_{i_1 p_1}^i c_{i_2 p_2}^{p_1} \cdots c_{i_n i_{n+1}}^{p_{n-1}} e_i + K. \quad (3.21)$$

We can express the coefficients b_n in terms of Bernoulli numbers by the following Lemma.

Lemma 1. *Let $c(n, 1)$ be defined as in (3.8) then*

$$b_n = c(n, 1) = \frac{(-1)^n}{n!} B_n. \quad (3.22)$$

(See [6] Theorem 3)

We can now use (3.21) to write out a formula for \hat{X}_k . We have

$$\mu_1(x, y)^a = z^a = x^a + y^a + \sum_{n=1}^{\infty} b_n x^{i_1} \cdots x^{i_n} y^{i_{n+1}} c_{i_1 p_1}^a c_{i_2 p_2}^{p_1} \cdots c_{i_n i_{n+1}}^{p_{n-1}} + k^a$$

giving

$$\begin{aligned} \hat{X}_k|_x &= \frac{\partial}{\partial y^k} (x^a + y^a + \sum_{n=1}^{\infty} b_n x^{i_1} \cdots x^{i_n} y^{i_{n+1}} c_{i_1 p_1}^a c_{i_2 p_2}^{p_1} \cdots c_{i_n i_{n+1}}^{p_{n-1}}) \Big|_{y=0} \\ &= (\delta_k^a + \sum_{n=1}^{\infty} b_n x^{i_1} \cdots x^{i_n} c_{i_1 p_1}^a c_{i_2 p_2}^{p_1} \cdots c_{i_n i_k}^{p_{n-1}}) \partial_{x^a} \\ &= (\delta_k^a + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} B_n [\text{ad } x]_{p_1}^a [\text{ad } x]_{p_2}^{p_1} \cdots [\text{ad } x]_{i_k}^{p_{n-1}}) \partial_{x^a} \end{aligned}$$

which in matrix notation is (3.15) proving the theorem. □

Corollary 15. *Let \mathfrak{g} be a Lie algebra where formula (3.9) converges for all elements in $\mathfrak{g} \times \mathfrak{g}$ and the map ψ_1 is a diffeomorphism $\mathbb{R}^m \rightarrow \mathbb{G}$ then the vector fields given by equation (3.15) are left invariant vector fields on \mathbb{R}^m with the group multiplication in coordinates of the 1st kind given by $\mu_1 : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ by*

$$\mu_1(x, y) = z = \psi_1^{-1}(\psi_1(x)\psi_1(y)).$$

3.4.1

Example

The following example is a case where we can compute the left invariant vector fields in coordinates of the 1st kind by using equation (3.17) and diagonalization of the adjoint matrix.

Let \mathfrak{g} be a 3 dimensional Lie algebra with basis $\{e_1, e_2, e_3\}$ and structure equations $[e_1, e_2] = 0$, $[e_1, e_3] = e_1$ and $[e_2, e_3] = e_2$. We have

$$[\text{ad } x^i e_i]^T = \begin{bmatrix} -x^3 & 0 & 0 \\ 0 & -x^3 & 0 \\ x^1 & x^2 & 0 \end{bmatrix}.$$

Putting this matrix in Jordan form gives

$$\begin{bmatrix} -x^3 & 0 & 0 \\ 0 & -x^3 & 0 \\ x^1 & x^2 & 0 \end{bmatrix} = \text{QJQ}^{-1}$$

where

$$\text{Q} = \begin{bmatrix} 0 & \frac{x^1 - x^2}{x^1} & -\frac{x^2}{x^1} \\ 0 & 1 & 1 \\ \frac{x^1}{x^3} & -\frac{x^1}{x^3} & 0 \end{bmatrix}$$

and

$$\text{J} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -x^3 & 0 \\ 0 & 0 & -x^3 \end{bmatrix}.$$

Since J is diagonal we can now compute

$$\frac{[\text{ad } x^i e_i]^T}{\mathbb{I} - e^{-[\text{ad } x^i e_i]^T}} = \text{Q} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-x^3}{1 - \exp(x^3)} & 0 \\ 0 & 0 & \frac{-x^3}{1 - \exp(x^3)} \end{bmatrix} \text{Q}^{-1}$$

which by equation (3.17) gives

$$\begin{bmatrix} \hat{X}_1|_{(x^1, x^2, x^3)} \\ \hat{X}_2|_{(x^1, x^2, x^3)} \\ \hat{X}_3|_{(x^1, x^2, x^3)} \end{bmatrix} = \begin{bmatrix} \frac{-x^3}{1-\exp(x^3)} \partial_{x^1} \\ \frac{-x^3}{1-\exp(x^3)} \partial_{x^2} \\ \frac{-x^1(1-\exp(x^3)+x^3)}{-x^3(1-\exp(x^3))} \partial_{x^1} + \frac{-x^2(1-\exp(x^3)+x^3)}{-x^3(1-\exp(x^3))} \partial_{x^2} + \partial_{x^3} \end{bmatrix}.$$

3.5

Left Invariant Vector Fields in Canonical Coordinates of the 1st Kind - Nilpotent Case

One important case where Corollary 15 applies is when \mathfrak{g} is a nilpotent Lie algebra.

Corollary 16. *Let \mathfrak{g} be k -step nilpotent. We have $[\text{ad } x^i e_i]^k = 0$ so that (3.15) has only finitely many terms allowing for exact calculation of the vector fields.*

In the case Corollary 16 applies, formula (3.15) is implemented in Appendix A by the code 4.5.3.

3.5.1

Examples 3.5.1

Example 3.5.1.1 As an example consider the Heisenberg Lie algebra. Using the standard basis $\{e_1, e_2, e_3\}$, the Heisenberg Lie algebra is given by the equations in (2.3). The transpose of the ad matrix is then given by:

$$[\text{ad } x^i e_i]^T = \begin{bmatrix} 0 & -x^3 & 0 \\ 0 & 0 & 0 \\ 0 & x^3 & 0 \end{bmatrix}.$$

Therefore formula (3.15) gives:

$$\begin{aligned}
\begin{bmatrix} \hat{X}_1|_{(x^1, x^2, x^3)} \\ \hat{X}_2|_{(x^1, x^2, x^3)} \\ \hat{X}_3|_{(x^1, x^2, x^3)} \end{bmatrix} &= \sum_{n=0}^1 \frac{(-1)^n}{n!} B_n \begin{bmatrix} 0 & -x^3 & 0 \\ 0 & 0 & 0 \\ 0 & x^1 & 0 \end{bmatrix}^n \begin{bmatrix} \partial_{x^1} \\ \partial_{x^2} \\ \partial_{x^3} \end{bmatrix} \\
&= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & -x^3 & 0 \\ 0 & 0 & 0 \\ 0 & x^1 & 0 \end{bmatrix} \right) \begin{bmatrix} \partial_{x^1} \\ \partial_{x^2} \\ \partial_{x^3} \end{bmatrix} \\
&= \begin{bmatrix} \partial_{x^1} + \frac{1}{2}x^3\partial_{x^2} \\ \partial_{x^2} \\ \partial_{x^3} - \frac{1}{2}x^1\partial_{x^2} \end{bmatrix}.
\end{aligned}$$

For $b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$ we can write generic vector fields

$$\begin{aligned}
X &= c^1 \hat{X}_1 + c^2 \hat{X}_2 + c^3 \hat{X}_3 = c^1 \partial_{x^1} + (c^2 + c^3 x^1) \partial_{x^2} + c^3 \partial_{x^3} \\
Y &= b^1 \hat{X}_1 + b^2 \hat{X}_2 + b^3 \hat{X}_3 = b^1 \partial_{x^1} + (b^2 + b^3 x^1) \partial_{x^2} + b^3 \partial_{x^3}
\end{aligned}$$

with Lie bracket

$$[X, Y] = (c^1 b^3 - b^1 c^3) \hat{X}_2$$

agreeing with the result we saw in equation (2.6).

Example 3.5.1.2 In 4.5.9 of Appendix B a more complicated example computing equation (3.15) is shown.

CHAPTER 4
CANONICAL COORDINATES OF THE 2ND KIND

In the prior section we saw how the exponential map could be used to define a chart on a neighborhood about the identity for any Lie group. In this section we will use a different method of using the exponential map to give a chart about the identity and these are called coordinates of the 2nd kind. When the Lie group is simply connected and solvable this chart will cover the entire group.

4.1

Defining the Coordinates of the 2nd Kind

Let G be a Lie group with Lie algebra \mathfrak{g} , multiplication μ and identity e . Let $\{e_i\}_{1 \leq i \leq m}$ be a basis for \mathfrak{g} . Let ψ_1 be defined as in equation (3.1). Let $\hat{x}^j = (0, \dots, 0, x^j, 0, \dots, 0)$, where x^j is the j th position. We define the mapping $\psi_2 : \mathbb{R}^m \rightarrow G$ by

$$\begin{aligned} \psi_2(x^1, \dots, x^m) &= \exp(x^1 e_1) \exp(x^2 e_2) \cdots \exp(x^m e_m) \\ &= \psi_1(\hat{x}^1) \psi_1(\hat{x}^2) \cdots \psi_1(\hat{x}^m). \end{aligned} \tag{4.1}$$

It should be noted that the function ψ_2 utilizes the group product whereas the function ψ_1 did not. We give an example of this calculation in 4.1.1.

Theorem 17. *There is an open set $W \subset \mathbb{R}^m$, $0 \in W$ and an open set $U \subset G$, $e \in U$ such that $\psi_2 : W \rightarrow U$ is a diffeomorphism, thus making (U, ψ_2^{-1}) a coordinate chart about e in G*

Proof. We have $\mu_*|_e : T_e G \times T_e G \rightarrow T_e G$ with $\mu_*|_e(e_i|_e, 0) = \mu_*(0, e_i|_e) = e_i|_e$. By induction, $\hat{\mu} : G \times \cdots \times G \rightarrow G$, by repeated group multiplication gives $\hat{\mu}_*|_e : T_e G \times \cdots \times T_e G \rightarrow T_e G$ with $\hat{\mu}_*|_e(0, \dots, 0, e_i|_e, 0, \dots, 0) = e_i|_e$.

Let ψ_1 be defined as in equation (3.1). We have

$$\begin{aligned}\psi_2(x^1, \dots, x^m) &= \hat{\mu}(\psi_1(x^1, 0, \dots, 0), \psi_1(0, x^2, 0, \dots, 0), \dots, \psi_1(0, \dots, 0, x^m)) \\ \Rightarrow \psi_{2*}|_0(\partial_{x^i}) &= \hat{\mu}_*|_e(0, \dots, 0, \psi_{1*}(\partial_{x^i}), 0, \dots, 0) \\ &= \hat{\mu}_*|_e(0, \dots, 0, e_i|_e, 0, \dots, 0) = e_i|_e.\end{aligned}$$

This makes $\psi_{2*}|_0$ invertible and therefore by the inverse function theorem there is an open set W , $0 \in W$ with $\psi_2 : W \rightarrow \mathcal{U} = \psi_2(W)$ a diffeomorphism. Since $\psi_2(0) = e$, the theorem then follows. \square

Definition. The coordinates $(\mathcal{U}, \psi_2^{-1})$ are called **Coordinates of the 2nd Kind**.

Definition. Let \mathfrak{g} be a solvable Lie algebra and let $\{e_1, \dots, e_m\}$ is a basis for \mathfrak{g} where $\mathfrak{h}_i = \text{span}\{e_1, \dots, e_i\}$ is a subalgebra of \mathfrak{g} and \mathfrak{h}_i is an ideal in \mathfrak{h}_{i+1} then we call $\{e_1, \dots, e_m\}$ an **adapted basis**.

Theorem 18. *Let \mathfrak{g} be a solvable Lie algebra then there exists an adapted basis for \mathfrak{g} . (See [7] Lemma 2.1)*

Theorem 19. *Let G be a simply connected solvable Lie group with Lie algebra \mathfrak{g} . Let $\{e_1, \dots, e_m\}$ be an adapted basis for \mathfrak{g} , then the map (4.1) is a diffeomorphism of $\mathbb{R}^m \rightarrow G$. (See [9] Theorem 3.18.11)*

The general approach used in the proof of [9] Theorem 3.18.11 is as follows. Let $\mathfrak{b}_i = \text{span}\{e_i\}$, for each i . In [9] Varadarajan shows that \mathfrak{h}_{m-1} and \mathfrak{b}_m are the Lie algebras for closed simply connected subgroups H and B_m of G with $G = HB_m$ and the mapping $(\mathfrak{h}, \mathfrak{b}_m) \rightarrow \mathfrak{h}\mathfrak{b}_m$, $\mathfrak{h} \in H$, $\mathfrak{b}_m \in B_m$ is a diffeomorphism. By induction we get the mapping $(\mathfrak{b}_1, \dots, \mathfrak{b}_m) \rightarrow \mathfrak{b}_1 \cdots \mathfrak{b}_m$, $\mathfrak{b}_i \in B_i$ being a diffeomorphism. Since each B_i is simply connected and 1-dimensional the map $\exp : \mathfrak{t} \rightarrow \exp(\mathfrak{t}e_i)$ is a diffeomorphism of $\mathbb{R} \rightarrow B_i$ giving the desired result.

4.1.1

Heisenberg Example

As an example consider the Heisenberg group, H , defined in equation (2.1). Using the standard basis $\{e_1, e_2, e_3\}$, the Heisenberg Lie algebra is given by the equations in (2.3). With ψ_1 as defined

in equation (3.1) equation (4.1) then gives:

$$\begin{aligned}\psi_2(x^1, x^2, x^3) &= \psi_1(x^1, 0, 0)\psi_1(0, x^2, 0)\psi_1(0, 0, x^3) \\ &= (x^1, x^2 + x^1x^3, x^3).\end{aligned}$$

4.2

Group Multiplication in Coordinates of the 2nd Kind -Nilpotent Case

We will now show how to give the multiplication function for a simply connected Lie group with nilpotent Lie algebra in coordinates of the 2nd kind. We first note an immediate corollary to Theorem 19.

Corollary 20. *Let G be a simply connected Lie group with nilpotent Lie algebra \mathfrak{g} then the map ψ_2 defined in equation (4.1) is a diffeomorphism of $\mathbb{R}^m \rightarrow G$.*

Let G be a simply connected Lie group with nilpotent Lie algebra \mathfrak{g} . Let $\{e_1, \dots, e_m\}$ be a basis for \mathfrak{g} and let ψ_1 be defined as in (3.1). By Theorem 10 ψ_1 is a diffeomorphism from $\mathbb{R}^m \rightarrow G$. Define a function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$\phi(x) = \psi_1^{-1} \circ \psi_2(x) \tag{4.2}$$

for $x \in \mathbb{R}^m$. This function is the change of coordinates function between coordinates of the 2nd kind and coordinates of the 1st kind.

Theorem 21. *Let G be a simply connected Lie group with nilpotent Lie algebra \mathfrak{g} . Let $x, y \in \mathbb{R}^m$. Let ϕ be defined as in equation (4.2). Let μ_1 be defined as in equation (3.18). Let $\mu_2 : \mathbb{R}^m \times \mathbb{R}^m$ by*

$$\mu_2(x, y) = \phi^{-1} \circ \mu_1(\phi(x), \phi(y)) \tag{4.3}$$

then μ_2 gives the group multiplication in coordinates of the 2nd kind.

Proof. Since ψ_1 and ψ_2 are diffeomorphisms ϕ is a diffeomorphism. We have

$$\begin{aligned}\mu_2(x, y) &= \phi^{-1} \circ \mu_1(\phi(x), \phi(y)) \\ &= (\psi_2^{-1} \circ \psi_1) \circ \psi_1^{-1}(\psi_1(\phi(x))\psi_1(\phi(y))) \\ &= \psi_2^{-1}(\psi_2(x)\psi_2(y)).\end{aligned}$$

establishing that μ_2 gives the group multiplication in coordinates of the 2nd kind.

□

We now show that for nilpotent \mathfrak{g} that $\phi(x)$ and then $\mu_2(x, y)$ can be calculated by use of formula (3.14). Let $x = (x^1, \dots, x^m)$. Let $\hat{x}^j = (0, \dots, 0, x^j, 0, \dots, 0)$, where x^j is the j th position. By equations (4.2) and (4.1) we have

$$\phi(x) = \psi_1^{-1} \circ \psi_2(x) = \psi_1^{-1}(\psi_1(\hat{x}^1)\psi_1(\hat{x}^2) \cdots \psi_1(\hat{x}^m)).$$

Formula (3.14) first allows us to compute

$$\mu_1(\hat{x}^1, \hat{x}^2) = \psi_1^{-1}(\psi_1(\hat{x}^1)\psi_1(\hat{x}^2)).$$

By inductive application of formula (3.14) we can compute

$$\phi(x) = \psi_1^{-1}(\psi_1(\hat{x}^1)\psi_1(\hat{x}^2) \cdots \psi_1(\hat{x}^m)). \quad (4.4)$$

Formula (4.4) and the calculation of ϕ^{-1} is implemented in Appendix A by the code 4.5.4.

Since we can compute ϕ^{-1} , μ_1 , and ϕ we can then by the use of formula (4.3) calculate $\mu_2(x, y)$.

Formula (4.3) is implemented in Appendix A by the code 4.5.5.

4.2.1

Heisenberg Example

In 4.5.10 of Appendix B we show how Theorem 21 can be applied to the Heisenberg group.

4.3

Left Invariant 1-Forms in Coordinates of the 2nd Kind

In equation 3.3 of [1] a basis for the left invariant 1-forms is given in coordinates similar to coordinates of the 2nd kind. We modify this formula slightly to determine the basis in actual coordinates of the 2nd kind.

Let G be a Lie group with Lie algebra \mathfrak{g} . Let $\{f_1, \dots, f_n\}$ be a basis for \mathfrak{g} . Let $\mathbf{y} = (y^1, \dots, y^n)$ and define coordinates about the identity in G using

$$\hat{\psi}_2(y^1, \dots, y^n) = \exp(y^n f_n) \exp(y^{n-1} f_{n-1}) \cdots \exp(y^1 f_1). \quad (4.5)$$

The following theorem is given in [1] p. 8.

Theorem 22. *A basis for the left invariant 1-forms, dual to the $\{f_i\}$, at (y^1, \dots, y^n) is given by*

$$\begin{bmatrix} \hat{\omega}^1 \\ \hat{\omega}^2 \\ \hat{\omega}^3 \\ \vdots \\ \hat{\omega}^n \end{bmatrix} = I_n \begin{bmatrix} dy^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + M_2(\mathbf{y}) \begin{bmatrix} 0 \\ dy^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + M_n(\mathbf{y}) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ dy^n \end{bmatrix}, \quad (4.6)$$

where

$$M_k(\mathbf{y}) = \exp(-y^1 [\text{ad } f_1]) \exp(-y^2 [\text{ad } f_2]) \cdots \exp(-y^{k-1} [\text{ad } f_{k-1}]), k > 1. \quad (4.7)$$

We now use this theorem to obtain our desired result.

Theorem 23. *Let $\beta = \{e_1, \dots, e_n\}$ be a basis for \mathfrak{g} . Let $\mathbf{x} = (x^1, \dots, x^n)$ be coordinates of the 2^{nd} kind given by ψ_2 as defined in equation (4.1). A basis for the left invariant 1-forms, dual to the $\{e_i\}$, at (x^1, \dots, x^n) is then given by*

$$\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \vdots \\ \omega^n \end{bmatrix} = I_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ dx^n \end{bmatrix} + N_{n-1}(\mathbf{x}) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ dx^{n-1} \\ 0 \end{bmatrix} + \dots + N_1(\mathbf{x}) \begin{bmatrix} dx^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (4.8)$$

where

$$N_k(\mathbf{x}) = \exp(-x^n [\text{ad } e_n]) \exp(-x^{n-1} [\text{ad } e_{n-1}]) \cdots \exp(-x^{k+1} [\text{ad } e_{k+1}]), k < n. \quad (4.9)$$

Proof. Let Q be the $n \times n$ involutory anti-diagonal matrix consisting of 1's on the anti-diagonal.

Let

$$\begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} = Q \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \text{ and } [f_1 \cdots f_n] = [e_1 \cdots e_n]Q. \quad (4.10)$$

In particular equation (4.10) gives $y_{n+1-k} = x_k$ and $f_{n+1-k} = e_k$. In terms of the basis $\beta' = \{f_1, \dots, f_n\}$, using (4.5), (4.1) and (4.10), we observe that

$$\begin{aligned} \hat{\psi}_2(Qx) &= \exp(x^1 f_n) \exp(x^2 f_{n-1}) \cdots \exp(x^n f_1) \\ &= \exp(x^1 e_1) \exp(x^2 e_2) \cdots \exp(x^n e_n) \\ &= \psi_2(x) \end{aligned}$$

Let $\phi = \hat{\psi}_2 \circ \psi_2^{-1}$ be the change of coordinates given by $y = Qx$. The pullback of 1-forms by the transformation ϕ gives

$$\begin{bmatrix} dx^1 \\ \vdots \\ dx^n \end{bmatrix} = \phi^*(Q \begin{bmatrix} dy^1 \\ \vdots \\ dy^n \end{bmatrix}). \quad (4.11)$$

Let

$$\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \vdots \\ \omega^n \end{bmatrix} = \phi^*(Q \begin{bmatrix} \hat{\omega}^1 \\ \hat{\omega}^2 \\ \hat{\omega}^3 \\ \vdots \\ \hat{\omega}^n \end{bmatrix})$$

where $\hat{\omega}$ are the left invariant 1-forms in the coordinates y given by equation (4.6). Since Q is invertible and constant the ω^i form a basis for the left invariant 1-forms in coordinates of the

second kind. Using equations (4.6) and (4.11) we get

$$\begin{aligned}
\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \vdots \\ \omega^n \end{bmatrix} &= \phi^*(QI_n \begin{bmatrix} dy^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}) + \phi^*(QM_2(y)Q^{-1}Q \begin{bmatrix} 0 \\ dy^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}) + \dots + \phi^*(QM_n(y)Q^{-1}Q \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ dy^n \end{bmatrix}) \\
&= I_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ dx^n \end{bmatrix} + \phi^*(QM_2(y)Q^{-1}) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ dx^{n-1} \\ 0 \end{bmatrix} + \dots + \phi^*(QM_n(y)Q^{-1}) \begin{bmatrix} dx^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (4.12)
\end{aligned}$$

We have from (4.7)

$$\begin{aligned}
QM_{n+1-k}(y)Q^{-1} &= Q \exp(-y^1[\mathbf{ad} f_1])Q^{-1} \dots Q \exp(-y^{n-k}[\mathbf{ad} f_{n-k}])Q^{-1} \\
&= \exp(-y^1 Q[\mathbf{ad} f_1]Q^{-1}) \dots \exp(-y^{n-k} Q[\mathbf{ad} f_{n-k}]Q^{-1}). \quad (4.13)
\end{aligned}$$

Since Q is the change of basis matrix from the basis β' to the basis β we have

$$[\mathbf{ad} e_{n+1-k}] = [\mathbf{ad} f_k]_{\beta} = Q[\mathbf{ad} f_k]_{\beta'} Q^{-1}. \quad (4.14)$$

Using equation (4.14) in equation (4.13) gives

$$QM_{n+1-k}(y)Q^{-1} = \exp(-y^1[\mathbf{ad} e_n]) \dots \exp(-y^{n-k}[\mathbf{ad} e_{k+1}]),$$

so that

$$\phi^*(QM_{n+1-k}(y)Q^{-1}) = \exp(-x^n[\mathbf{ad} e_n]) \dots \exp(-x^{k+1}[\mathbf{ad} e_{k+1}]).$$

We observe that $N_k(x) = \phi^*(QM_{n+1-k}(y)Q^{-1})$, where $N_k(x)$ is defined as in equation (4.9). This demonstrates that equation (4.12) is the same as equation (4.8), thus establishing the theorem. \square

Formula (4.8) of Theorem 23 is implemented in Appendix A by the code 4.5.6.

4.3.1

Example

In 4.5.11 of Appendix B we show an example of equation (4.8) being calculated from $\mathfrak{so}(3)$. Testing indicated that with an Intel I5-6400 processor the left invariant 1-forms were calculated in a few seconds for $\mathfrak{so}(5)$, about 3 minutes for \mathfrak{g}_2 and in about 5 minutes for $\mathfrak{so}(6)$. Testing $\mathfrak{so}(7)$ was halted after about an hour.

4.4

Coordinates of the 2nd Kind - Solvable Case

We will now demonstrate a additional formulas for left invariant 1-forms and left invariant vector fields expressed in coordinates of the 2nd kind.

4.4.1

Left Invariant 1-Forms in Coordinates of the 2nd Kind - Solvable Case

Let G be a Lie group with solvable Lie algebra \mathfrak{g} . The following theorem found on page 10 of [5], gives a basis for the left invariant 1-forms of G .

Theorem 24. *Let $\{e_1, \dots, e_n\}$ be an adapted basis for a solvable Lie algebra \mathfrak{g} . Let $\mathfrak{t}_s = \text{span}\{e_1, \dots, e_{n-s}\}$. Note that this numbering of sub algebras differs from that used when we defined adapted basis in Definition 4.1, having $\mathfrak{t}_0 = \mathfrak{g}$. Let $\text{ad}_s(e_{n-s})$ be the restriction of $\text{ad}(e_{n-s})$ to the subspace \mathfrak{t}_s . Let \hat{A}_s be the $n-s$ by $n-s$ matrix representation of $\text{ad}(e_{n-s})$ in the basis for \mathfrak{t}_s . Let I_s be the $s \times s$ identity matrix and 0_s the $n-s$ by s zero matrix. The following formula gives us a basis, τ_1, \dots, τ_n for the left invariant 1-forms on G , dual to the $\{e_i\}$,*

$$\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} = e^{-\hat{A}_0^T x^n} \begin{bmatrix} e^{-\hat{A}_1^T x^{n-1}} & 0_1 \\ 0_1^T & 1 \end{bmatrix} \dots \begin{bmatrix} e^{-\hat{A}_{n-2}^T x^2} & 0_{n-2} \\ 0_{n-2}^T & I_{n-2} \end{bmatrix} \begin{bmatrix} dx^1 \\ \vdots \\ dx_n \end{bmatrix}. \quad (4.15)$$

We now show that for \mathfrak{g} solvable formula (4.15) matches formula (4.8), thus establishing that the vector fields from Theorem 24 are given in coordinates of the 2nd kind.

Theorem 25. *Assume that \mathfrak{g} is solvable and that $\{e_1, \dots, e_n\}$ is an adapted basis as defined in Definition 4.1 then equations (4.8) and (4.15) are identical.*

Proof. Let t_s , \hat{A}_s , I_s and 0_s be defined as in Theorem 24. Since \mathfrak{g} is solvable, with the choice of basis given in the theorem, the matrix $[\text{ad } e_{n-j}]$ is of the form

$$\begin{bmatrix} \hat{A}_j & * \\ 0_j^T & 0_j \end{bmatrix}$$

and $\exp(-x^{n-j}[\text{ad } e_{n-j}])$ is of the form

$$\begin{bmatrix} e^{\hat{A}_j(-x^{n-j})} & * \\ 0_j^T & I_j \end{bmatrix}, \quad (4.16)$$

where $*$ is an $n-j \times j$ matrix. We are going to show that

$$N_k(x) \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix} = e^{A_0(-x^n)} \begin{bmatrix} e^{A_1(-x^{n-1})} & 0_1 \\ 0_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} e^{A_{n-2}(-x^2)} & 0_{n-2} \\ 0_{n-2}^T & I_{n-2} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix}, \quad (4.17)$$

where $N_k(x)$ is defined in equation (4.9), which will allow us to prove the theorem. We note that based on equation (4.16) we have

$$\exp(-x^{k+1}[\text{ad } e_{k+1}]) \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} e^{\hat{A}_{n-(k+1)}(-x^{k+1})} & 0_{n-(k+1)} \\ 0_{n-(k+1)}^T & I_{n-(k+1)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix}.$$

Assume that

$$\begin{aligned} & \begin{bmatrix} e^{\hat{A}_{r-s}(-x^{n-(r-s)})} & * \\ 0_{r-s}^T & I_{r-s} \end{bmatrix} \cdots \begin{bmatrix} e^{\hat{A}_{r-1}(-x^{n-(r-1)})} & * \\ 0_{r-1}^T & I_{r-1} \end{bmatrix} \begin{bmatrix} e^{\hat{A}_r(-x^{n-r})} & 0 \\ 0_r^T & I_r \end{bmatrix} \\ = & \begin{bmatrix} e^{\hat{A}_{r-s}(-x^{n-(r-s)})} & 0_{r-s} \\ 0_{r-s}^T & I_{r-s} \end{bmatrix} \cdots \begin{bmatrix} e^{\hat{A}_{r-1}(-x^{n-(r-1)})} & 0_{r-1} \\ 0_{r-1}^T & I_{r-1} \end{bmatrix} \begin{bmatrix} e^{\hat{A}_r(-x^{n-r})} & 0 \\ 0_r^T & I_r \end{bmatrix} \end{aligned}$$

for $s < r$. We observe that

$$\begin{aligned} & \begin{bmatrix} e^{\hat{A}_{r-(s+1)}(-x^{n-(r-(s+1))})} & * \\ 0_{r-s}^T & I_{r-(s+1)} \end{bmatrix} \begin{bmatrix} e^{\hat{A}_{r-s}(-x^{n-(r-s)})} & 0 \\ 0_{r-s}^T & I_{r-s} \end{bmatrix} \\ &= \begin{bmatrix} e^{\hat{A}_{r-(s+1)}(-x^{n-(r-s)})} & 0_{r-(s+1)} \\ 0_{r-s}^T & I_{r-s} \end{bmatrix} \begin{bmatrix} e^{\hat{A}_{r-s}(-x^{n-(r-s)})} & 0_{r-s} \\ 0_{r-s}^T & I_{r-s} \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned} & \begin{bmatrix} e^{\hat{A}_{r-(s+1)}(-x^{n-(r-(s+1))})} & * \\ 0_{r-s}^T & I_{r-(s+1)} \end{bmatrix} \cdots \begin{bmatrix} e^{\hat{A}_{r-1}(-x^{n-(r-1)})} & * \\ 0_{r-1}^T & I_{r-1} \end{bmatrix} \begin{bmatrix} e^{\hat{A}_r(-x^{n-r})} & 0 \\ 0_r^T & I_r \end{bmatrix} \\ &= \begin{bmatrix} e^{\hat{A}_{r-(s+1)}(-x^{n-(r-s)})} & 0_{r-(s+1)} \\ 0_{r-s}^T & I_{r-s} \end{bmatrix} \cdots \begin{bmatrix} e^{\hat{A}_{r-1}(-x^{n-(r-1)})} & 0_{r-1} \\ 0_{r-1}^T & I_{r-1} \end{bmatrix} \begin{bmatrix} e^{\hat{A}_r(-x^{n-r})} & 0 \\ 0_r^T & I_r \end{bmatrix}. \end{aligned}$$

We can then conclude by induction that

$$\begin{aligned} & e^{\hat{A}_0(-x^n)} \begin{bmatrix} e^{\hat{A}_1(-x^{n-1})} & * \\ 0_{n-1}^T & I_{n-1} \end{bmatrix} \cdots \begin{bmatrix} e^{\hat{A}_{k+1}(-x^{n-(k+1)})} & 0 \\ 0_{n-1}^T & I_{n-1} \end{bmatrix} \\ &= e^{\hat{A}_0(-x^n)} \begin{bmatrix} e^{\hat{A}_1(-x^{n-1})} & 0 \\ 0_{n-1}^T & I_{n-1} \end{bmatrix} \cdots \begin{bmatrix} e^{\hat{A}_{k+1}(-x^{n-(k+1)})} & 0 \\ 0_{n-1}^T & I_{n-1} \end{bmatrix} \end{aligned}$$

and therefore

$$\mathbf{N}_k(x) \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix} = e^{\hat{A}_0(-x^n)} \begin{bmatrix} e^{\hat{A}_1(-x^{n-1})} & 0_1 \\ 0_1^T & I_1 \end{bmatrix} \cdots \begin{bmatrix} e^{\hat{A}_{n-(k+1)}(-x^{k+1})} & 0_{n-(k+1)} \\ 0_{n-(k+1)}^T & I_{n-(k+1)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix}.$$

Since \mathfrak{g} is solvable, based on our choice of basis, \hat{A}_{n-k} has zeros in column k and in row k . So

$e^{\hat{A}_{n-k}(-x^k)}$ has zeros in column k except for a 1 in column k , row k . This implies that

$$N_k(x) \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix} = N_k(x) \begin{bmatrix} e^{\hat{A}_{n-k}(-x^k)} & 0_{n-k} \\ 0_{n-k}^T & I_{n-k} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix}.$$

Since each matrix

$$\begin{bmatrix} e^{\hat{A}_{n-j}(-x^j)} & 0_{n-j} \\ 0_{n-j}^T & I_{n-j} \end{bmatrix}, j < k$$

has a k^{th} column with all zeros except for a 1 in row k we see that

$$\begin{bmatrix} e^{A_1(-x^{n-k})} & 0_1 \\ 0_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} e^{A_{n-2}(-x^2)} & 0_{n-2} \\ 0_{n-2}^T & I_{n-2} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix}$$

and therefore

$$N_k(x) \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix} = e^{A_0(-x^n)} \begin{bmatrix} e^{A_1(-x^{n-1})} & 0_1 \\ 0_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} e^{A_{n-2}(-x^2)} & 0_{n-2} \\ 0_{n-2}^T & I_{n-2} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ dx^k \\ \vdots \\ 0 \end{bmatrix}$$

thus establishing equation (4.17). By expanding formula (4.15) out and utilizing equation (4.17) we see that

$$\begin{bmatrix} \tau^1 \\ \vdots \\ \tau^n \end{bmatrix} = I_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ dx^n \end{bmatrix} + N_{n-1}(x) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ dx^{n-1} \\ 0 \end{bmatrix} + \dots + N_1(x) \begin{bmatrix} dx^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix}$$

establishing the theorem. □

4.4.2

Left Invariant Vector Fields in Coordinates of the 2nd Kind - Solvable Case

We now demonstrate that an alternative formula, derived by the use of the BCH formula, yields a basis of left invariant vector fields dual to the basis of left invariant 1-forms shown in Theorem 24.

The following equation found on page 10 of [5], gives a basis for the left invariant vector fields of a solvable Lie group G , which we have established in Theorem 25 to be in coordinates of 2nd kind.

Theorem 26. *Let \mathfrak{g} be a solvable Lie algebra of dimension n . Let $\{X_i\}$ be vector fields on \mathbb{R}^m given by*

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = e^{\hat{A}_s^T x^n} \begin{bmatrix} e^{\hat{A}_1^T x^{n-1}} & 0_1 \\ 0_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} e^{\hat{A}_{n-2}^T x^2} & 0_{n-2} \\ 0_{n-2}^T & I_{n-2} \end{bmatrix} \begin{bmatrix} \partial_{x^1} \\ \vdots \\ \partial_{x_n} \end{bmatrix} \quad (4.18)$$

with \hat{A}_s defined as in Theorem 24, then $\{X_i\}$ form a basis for \mathfrak{g} in coordinates of the 2nd kind.

We will now show that a formula built using BCH methods gives the same left invariant vector fields as those given in equation (4.18). Let G be a Lie group of dimension m with identity e and solvable Lie algebra \mathfrak{g} . The mapping ψ_1 as defined in equation (3.1) and the mapping ψ_2 defined in equation (4.1) gives us coordinate charts about $e \in G$. We need a subset of the intersection of these charts with some specific properties. This is given by the next theorem.

Theorem 27. *Let (U_1, ψ_1^{-1}) and (U_2, ψ_2^{-1}) be charts about $e \in G$ as given by Theorem 11 and Theorem 17, then there is an open ball W about 0 in \mathbb{R}^m with $U = \psi_1(W) \subset U' = U_1 \cap U_2$ with $a^{-1} \in U$ for $a \in U$ and $abc \in U'$ for $a, b, c \in U$.*

Proof. Let $V = \mu^{-1}(\mu^{-1}(U')) \subset G \times G \times G$. We have V open with $abc \in U'$ for $a, b, c \in V$. Let $W' = \psi_1^{-1}(V)$. There is an open ball $W \subset W'$ about 0. Let $U = \psi_1(W)$. Let $a, b, c \in U$, then $a, b, c \in V \Rightarrow abc \in U'$ and there is an $x \in W$ with $a = \psi_1(x)$, which implies $-x \in W \Rightarrow a^{-1} = \psi_1(-x) \in U$, which proves the theorem. □

Let \mathfrak{g} be a Lie algebra and let $\{e_i\}_{1 \leq i \leq m}$ be an adapted basis of \mathfrak{g} as defined in 4.1 with $\mathfrak{h}_j = \text{span}\{e_i\}_{1 \leq i \leq j}$. Let

$$\text{ad}_j(e_i) = \text{ad } e_i|_{\mathfrak{h}_j}, \quad 1 \leq i \leq j \leq m. \quad (4.19)$$

Let 0_j be a j by $m - j$ matrix of zeros. Let I_{m-j} be the $m - j$ identity matrix.

Let

$$A_{x^j} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n([\text{ad}_j x^j e_j]^T)^n & 0_j \\ 0_j^T & I_{m-j} \end{bmatrix} \quad (4.20)$$

where there is no summation over j and the 0 and I matrices are omitted for $j = m$.

Theorem 28. *Let G be a simply connected Lie group with solvable Lie algebra \mathfrak{g} and let $\{e_i\}_{1 \leq i \leq m}$ be an adapted basis for \mathfrak{g} . Let A_{x^j} be defined as in (4.20). Let $[\text{ad } x^i e_i]$ be the matrix representation of $\text{ad } x^i e_i$ in the basis $\{e_i\}$. Let (U_1, ψ_1^{-1}) , (U_2, ψ_2^{-1}) , and U be defined as in Theorem 27. Then for $a \in U$, $x = (x^1, \dots, x^m) = \psi_2^{-1}(a)$, $\hat{X}_k|_x = \psi_{2*}^{-1} e_k|_a$ is given by*

$$\begin{bmatrix} \hat{X}_1|_x \\ \vdots \\ \hat{X}_m|_x \end{bmatrix} = (A_{-x^m}^{-1} A_{x^m}) (A_{-x^{m-1}}^{-1} A_{x^{m-1}}) \cdots (A_{-x^2}^{-1} A_{x^2}) \begin{bmatrix} \partial_{x^1} \\ \vdots \\ \partial_{x^m} \end{bmatrix}. \quad (4.21)$$

Proof. We will prove this theorem by establishing that formula (4.21) gives the same vector fields as given by equation (4.18). □

Formula (4.21) of Theorem 28 is implemented in Appendix A by the code 4.5.7. There is an example shown in 4.5.1.

Theorem 29. *The basis for left invariant vector fields given in formulas (4.18) and (4.21) are identical.*

Proof. To prove the theorem we only need to show that

$$\begin{bmatrix} e^{\hat{A}_{n-j}^T x^j} & 0_j \\ 0_j^T & I_j \end{bmatrix} = A_{-x^j}^{-1} A_{x^j}$$

for $2 \leq j \leq n$. In other words we will show

$$A_{-x^j} \begin{bmatrix} e^{\hat{A}_{n-j}^T x^j} & 0_j \\ 0_j^T & I_j \end{bmatrix} = A_{x^j}. \quad (4.22)$$

We first note that using the notation of formula (4.19) that $[\text{ad}_j(e_j)] = \hat{A}_{n-j}$ in the notation of formula (4.15) so that

$$A_{x^j} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n (\hat{A}_{n-j}^T x^j)^n & 0_j \\ 0_j^T & I_{n-j} \end{bmatrix}.$$

Let $M = \hat{A}_{n-j}^T x^j$. In order to establish (4.22) we must show that

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n (-M)^n \right) e^M = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n M^n.$$

Inserting the power series for exp gives

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n (-M)^n \right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} M^n \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n M^n.$$

Using the Cauchy product of power series formula on the left hand side we then need to show

$$\sum_{n=0}^{\infty} \sum_{i=0}^n \frac{(-1)^i}{i!} B_i (-1)^i \frac{1}{(n-i)!} M^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n M^n$$

which is equivalent to showing

$$\sum_{i=0}^n \frac{B_i}{i!(n-i)!} = \frac{(-1)^n}{n!} B_n.$$

For $n = 0$ we get $1 = 1$ and For $n = 1$ we get $1 - \frac{1}{2} = \frac{1}{2}$. For $n > 1$, $B_n = 0$ for n odd so that $(-1)^n B_n = B_n$, so we only need to show that

$$\sum_{i=0}^{n-1} \frac{B_i}{i!(n-i)!} = 0$$

but this is a known identity for Bernoulli numbers (See Wolfram Math World - Bernoulli Number) thus establishing equation (4.22) and proving the theorem. □

4.5

Left Invariant Vector Fields in Coordinates 2nd Kind - Nilpotent Case

One important case is where Theorem 28 applies is if \mathfrak{g} is a nilpotent Lie algebra.

Corollary 30. *Let \mathfrak{g} be k -step nilpotent. We have $[\text{ad } x^i e_i]^k = 0$ so that (4.21) has only finitely many terms allowing for exact calculation of the vector fields in terms of finite sums of brackets.*

4.5.1

Example

In 4.5.12 of Appendix B we show an example of equation (4.21) of Theorem 28 being calculated.

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APPENDICES

APPENDIX A

Maple Code

4.5.2

Maple Code for Equation (3.14) of Corollary 13

► Initialize Packages.

▼ Procedures

[Input: A Lie algebra.

[Output: The result of equation 3.14 for generic points x and y in coordinates of the 1st kind.

```
Alg > NilpotentBCHmult := proc(alginput)
  local i, X, Y, Z, algbasis, adim, basis;
  if Query(alginput, "Nilpotent") then
    ChangeLieAlgebraTo(alginput);
    adim := DGinformation(alginput, "LieAlgebraDimension");
    basis := DGinfo(alginput, "FrameBaseVectors");
    X := evalDG(0·basis[1]);
    Y := evalDG(0·basis[1]);
    for i from 1 to adim do
      X := evalDG(X + x||i* basis[i]);
      Y := evalDG(Y + y||i* basis[i]);
    end do;
    Z := evalDG(X + Y + BCH(X, Y, Y, 1, 1, 0, 1, 1, 0));
    algbasis := [e1];
    for i from 2 to adim do algbasis := [op(algbasis), e||i] end do;
    GetComponents(Z, algbasis);
  else
    print("Not a nilpotent Lie Algebra");
  end if;
end:
```

The following function is an internal subroutine for NilpotentBCHmult. It is designed to be called recursively. It is not for external use.

```

Alg > BCH := proc(X, Y, rt, sum, prod, leadx, leady, n, ind)
  local lx, lx1, lx2, lx3, lx4, ly, ly1, ly2, ly3, ly4, ln, ln1, ln2;
  ln := n;
  lx1 := 0; lx2 := 0; lx3 := 0; lx4 := 0;
  ly1 := 0; ly2 := 0; ly3 := 0; ly4 := 0;
  lx := LieBracket(X, rt);
  if lx ≠ 0 &mult e1 then
    if leadx=0 then
      ln1 := (-1)^(trunc((n + ind)/2) * (trunc((n + ind)/2) + 1));
      ln2 := (-1)^(trunc((n + ind + 2)/2) * (trunc((n + ind + 2)/2)
+ 1));
      lx1 := evalDG(lx * (1/ln1 + 1/ln2) / (sum) / prod / leady!);
      lx2 := BCH(X, Y, lx, sum, prod * leady!, 1, 0, ln + 1, ind);
      lx3 := BCH(X, Y, lx, sum, prod * leady!, 1, 0, ln + 3, ind);
    else
      ln1 := (-1)^(trunc((n + 1)/2) - 1) * trunc(((n + 1)/2));
      ln2 := (-1)^(trunc((n + 3)/2) - 1) * trunc(((n + 3)/2));
      lx1 := evalDG(lx / ln1 / (sum) / prod / (leadx + 1)!);
      lx2 := BCH(X, Y, lx, sum, prod, leadx + 1, 0, ln, ind);
      lx3 := evalDG(lx / ln2 / (sum) / prod / leadx!);
      lx4 := BCH(X, Y, lx, sum, prod * leadx!, 1, 0, ln + 2, ind);
    end if;
  end if;
  ly := LieBracket(Y, rt);
  #print(6,ln,Y,rt);
  if ly ≠ 0 &mult e1 then
    if leady=0 then
      ln1 := (-1)^(trunc((n + ind)/2) * (trunc((n + ind)/2) + 1));
      ly1 := evalDG(ly / ln1 / (sum + 1) / prod / leadx!);
      ly2 := BCH(X, Y, ly, sum + 1, prod * leadx, 0, 1, ln + 1, ind);
    else
      ln1 := (-1)^(trunc((n + ind + 1)/2) - 1) * trunc(((n + ind
+ 1)/2));
      ln2 := (-1)^(trunc((n + ind + 3)/2) - 1) * trunc(((n + ind
+ 3)/2));
      ly1 := evalDG(ly / ln1 / (sum + 1) / prod / (leady + 1)!);
      ly2 := BCH(X, Y, ly, sum + 1, prod, 0, leady + 1, ln, ind);
      ly3 := evalDG(ly / ln2 / (sum + 1) / prod / leady!);
      ly4 := BCH(X, Y, ly, sum + 1, prod * leady!, 0, 1, ln + 2, ind);
    end if;
  end if;
  evalDG(lx1 + lx2 + lx3 + lx4 + ly1 + ly2 + ly3 + ly4);
end:

```

4.5.3

Maple Code for Equation (3.15) of Theorem 14

```

Input:  A Lie algebra.
        A manifold of the same dimension as the Lie algebra.
Output: The vector fields given by formula (3.15) in the thesis using the default basis for
        the Lie algebra input.
M > NilpotentLivf1stKind := proc(alginput, manifold)
    local i, b, dv, gen_element, A, z, adim, coords, vects, basis;
    if Query(alginput, "Nilpotent") then
        ChangeLieAlgebraTo(alginput);
        adim := DGinformation(alginput, "LieAlgebraDimension");
        basis := DGinformation(alginput, "FrameBaseVectors");
        dv := [ ];
        gen_element := evalDG(0·e1);
        A := IdentityMatrix(adim);
        coords := DGinformation(M, "FrameIndependentVariables");
        vects := DGinformation(M, "FrameBaseVectors");
        for i from 1 to adim do
            gen_element := evalDG(gen_element + coords[i]·basis[i]);
            dv := [op(dv), vects[i]];
        end do;
        for i from 1 to adim do
            A := A +  $\frac{(-1)^i}{i!}$  · bernoulli(i) · (Adjoint(gen_element) + )i;
        end do;
        A.Matrix(dv) + ;
    else
        print("Not a nilpotent Lie Algebra");
    end if;
end:

```

4.5.4

Maple Code for Equation (4.4)

```

Input:  A Lie algebra
        A point in  $R^n$  (where n is the dimension of the Lie algebra) entered as a list
        If first_or_second="second" the function changes coordinates of the 2nd kind to
        coordinates of the first kind
        otherwise the function changes coordinates of the 1st kind to coordinates of the
        second kind.

Output: The result of equation (4.4) in the thesis or its inverse.
> NilpotentCanonicalCoordinateChange := proc (alginput, coordinate, first_or_second)
  local i, X, Y, W, Z, adim, basis, dgsetupvarsx, dgsetupvarsy, rt, sc, ab, ls, liedata, phi, psi,
    Mult, wvars, zvars, subvars, Mult2;
  if Query(alginput, "Nilpotent") then
    ChangeLieAlgebraTo(alginput);
    adim := DGinformation(alginput, "LieAlgebraDimension");
    ls := Series("Lower");
    ab := ls[nops(ls)-1];
    for i from nops(ls) - 1 by -1 to 2 do
      ab := [op(ab), op(ComplementaryBasis(ls[i], ls[i-1]))];
    end do;
    liedata := LieAlgebraData(ab, alginput);
    DGsetup(liedata);
    basis := DGinfo(alginput, "FrameBaseVectors");
    rt := y1·basis[1];
    for i from 2 to adim do
      X := rt;
      Y := y||i·basis[i];
      rt := evalDG(rt + Y + BCH(X, Y, Y, 1, 1, 0, 1, 1, 0));
    end do;
    sc := GetComponents(rt, basis);
    dgsetupvarsx := [ ];
    dgsetupvarsy := [ ];
    wvars := [ ];
    zvars := [ ];
    subvars := [ ];
    for i from 1 to adim do
      dgsetupvarsx := [op(dgsetupvarsx), x||i];
      dgsetupvarsy := [op(dgsetupvarsy), y||i];
      wvars := [op(wvars), w||i];
      zvars := [op(zvars), z||i];
      subvars := [op(subvars), x||i = sc[i]];
    end do;
    DGsetup(dgsetupvarsx, M);
    DGsetup(dgsetupvarsy, N);
    phi := Transformation(N, M, subvars);
    psi := InverseTransformation(phi);
    if first_or_second = "second" then
      ApplyTransformation(phi, coordinate);
    else
      ApplyTransformation(psi, coordinate);
    end if;
  else
    print("Not a nilpotent Lie Algebra");
  end if;
end proc;

```

```
| | end if;  
| | end;
```

4.5.5

Maple Code for Equation (4.3) of Theorem 21

Input: A Lie algebra

Output: The result of equation 4.3 for generic points x and y in coordinates of the 2nd kind

```

Alg > NilpotentBCHmult2 := proc(alginput)
  local i, X, Y, W, Z, adim, basis, dgsetupvarsx, dgsetupvarsy, rt, sc, ab, ls,
    liedata, phi, psi, Mult, wvars, zvars, subvars, Mult2;
  if Query(alginput, "Nilpotent") then
    adim := DGinformation(alginput, "LieAlgebraDimension");
    wvars := [ ];
    zvars := [ ];
    for i from 1 to adim do
      wvars := [op(wvars), w||i];
      zvars := [op(zvars), z||i];
    end do;
    Mult := NilpotentBCHmult(alginput);
    W := NilpotentCanonicalCoordinateChange(alginput, wvars, "second");
    Z := NilpotentCanonicalCoordinateChange(alginput, zvars, "second");
    subvars := [ ];
    for i from 1 to adim do
      subvars := [op(subvars), x||i = W[i]];
      subvars := [op(subvars), y||i = Z[i]];
    end do;
    Mult2 := subs(subvars, Mult);
    Mult2 := NilpotentCanonicalCoordinateChange(alginput, Mult2,
"first");
    subvars := [ ];
    for i from 1 to adim do
      subvars := [op(subvars), w||i = x||i];
      subvars := [op(subvars), z||i = y||i];
    end do;
    Mult2 := subs(subvars, Mult2)
  else
    print("Not a nilpotent Lie Algebra");
  end if;
end:

```

4.5.6

Maple Code for Equation (4.8) of Theorem 23

Input: A Lie algebra.

A manifold of the same dimension as the Lie algebra.

Output: The 1-form fields given by equation (4.8) in the thesis using the default basis for the Lie algebra input.

```
M > CalcLeftInv1Forms := proc(alginput, manifold)
  local i, j, B, V, adim, ls, ab, liedata, forms;
  ChangeLieAlgebraTo(alginput);
  adim := DGinformation(alginput, "LieAlgebraDimension");
  forms := DGinformation(manifold, "FrameBaseForms");
  B := [ ];
  for i from 1 to adim do B := [op(B), 0] end do;
  B := convert(B, Matrix) +;
  for i from 1 to adim do
    V := [ ];
    for j from 1 to adim do
      if j = i then
        V := [op(V), forms[j]];
      else
        V := [op(V), 0];
      end if;
    end do;
    V := convert(V, Matrix) +;
    if i < adim then
      B := evalDG(B + calc_Nk(alginput, manifold, adim, i).V);
    else
      B := evalDG(B + V);
    end if;
  end do;
end;
```

The following function is an internal subroutine for calc_left_forms. It is not for external use.

```
M > calc_Nk := proc(alginput, manifold, adim, cnt)
  local i, Nk, basis, coords;
  basis := DGinformation(alginput, "FrameBaseVectors");
  coords := DGinformation(manifold, "FrameIndependentVariables");
  Nk := AdjointExp((-coords[adim]) · (basis[adim]));
  if adim - 1 > cnt then
    for i from adim - 1 by -1 to 1 + cnt do
      Nk := Multiply(Nk, AdjointExp((-coords[i]) · (basis[i])))
    end do;
  else
    return Nk;
  end if;
end;
```

4.5.7

Maple Code for Equation (4.21) of Theorem 28

```

Input:  A Lie algebra.
        A manifold of the same dimension as the Lie algebra.
Output: The vector fields from equation (4.21) in the thesis using a basis adapted to lower
        central series for the Lie algebra input.
M > NilpotentLivf2ndKind := proc (alginput, manifold)
    local i, dv, A, adim, ls, ab, liedata, vects;
    if Query(alginput, "Nilpotent") then
        ChangeLieAlgebraTo(alginput);
        adim := DGinformation(alginput, "LieAlgebraDimension");
        ls := Series("Lower");
        ab := ls[nops(ls) - 1];
        for i from nops(ls) - 1 by -1 to 2 do
            ab := [op(ab), op(ComplementaryBasis(ls[i], ls[i - 1]))];
        end do;
        liedata := LieAlgebraData(ab, alginput);
        DGsetup(liedata);
        dv := [];
        A := IdentityMatrix(adim);
        vects := DGinformation(manifold, "FrameBaseVectors");
        for i from 1 to adim do
            dv := [op(dv), vects[i]];
        end do;
        for i from adim by -1 to 2 do
            A := A.Admatrix(alginput, manifold, adim, i, -1)-1.Admatrix(alginput,
            manifold, adim, i, 1);
        end do;
        A := A.Matrix(dv)+;
    else
        print("Not a nilpotent Lie Algebra");
    end if;
end:

```


The following function is an internal subroutine for *NilpotentLiyf2ndKind*. It is not for external use.

```

M > Admatrix := proc(alginput, manifold, inidim, adim, s)
  local i, j, b, A, A0, L2, dgsetupvars, coords, basis;
  dgsetupvars := [ ];
  A := IdentityMatrix(adim);
  A0 := IdentityMatrix(inidim, compact=false);
  for i from 1 to adim do
    dgsetupvars := [op(dgsetupvars), e||i];
  end do;
  L2 := LieAlgebraData(dgsetupvars, alginput);
  DGsetup(L2);
  basis := DGinformation(alginput, "FrameBaseVectors");
  coords := DGinformation(manifold, "FrameIndependentVariables");
  for i from 1 to adim do
    A := A +  $\frac{(-1)^i}{i!}$  · bernoulli(i) ·  $\left( \text{Adjoint}(\text{evalDG}(s \cdot \text{coords}[\text{adim}] \cdot \text{basis}[\text{adim}]))^+ \right)^i$ ;
  end do;
  for i from 1 to adim do
    for j from 1 to adim do
      A0[i, j] := A[i, j];
    end do;
  end do;
  A0;
end:

```

The following are functions used for testing.

```

Alg > AdaptedBasis := proc(ls)
    local basis, i;
    basis := ls[nops(ls) - 1];
    for i from nops(ls) - 1 by - 1 to 2
        do basis := [op(basis), op(ComplementaryBasis(ls[i], ls[i - 1]) )];
        end do;
    end:
> Livf_from_mult := proc(mult)
    local adim, i, j, dgsetupvars, subvars, vf, l, L;
    dgsetupvars := [ ];
    subvars := [ ];
    vf := [ ];
    adim := DGinformation(Alg, "LieAlgebraDimension") :
    for i from 1 to adim do
        dgsetupvars := [op(dgsetupvars), x || i];
        subvars := [op(subvars), y || i = 0];
    end do :
    DGsetup(dgsetupvars, M) :
    for i from 1 to adim do
        l := subs(subvars, diff(mult, y || i));
        L := evalDG(l[1]·D_x l);
        for j from 2 to adim do
            L := evalDG(L + l[j]·D_x || j)
        end do;
        vf := [op(vf), L]
    end do;
end:
> CheckAssociativity := proc(mu)
    local i, subvars, a, ans1, ans2;
    subvars := [ ];
    a := [ ];
    for i from 1 to nops(mu) do subvars := [op(subvars), y || i = w || i, x || i = y || i] end do:
    for i from 1 to nops(mu) do a := simplify( [op(a), subs(subvars, mu[i]) ] ) end do:

    subvars := [ ];
    for i from 1 to nops(mu) do subvars := [op(subvars), y || i = a[i]] end do:
    ans1 := simplify(subs(subvars, mu) );
    a := mu;
    subvars := [ ];
    for i from 1 to nops(mu) do subvars := [op(subvars), y || i = w || i] end do:
    a := subs(subvars, a);
    subvars := [ ];
    for i from 1 to nops(mu) do subvars := [op(subvars), x || i = mu[i]] end do:
    ans2 := subs(subvars, a);
    simplify(ans1 - ans2);
end:

```

```

M > CheckLeftInvariant := proc(X, mu)
  local i, subvars, left_mult, yX, ans, X_at_xy;
  subvars := [ ];
  for i from 1 to nops(mu) do subvars := [op(subvars), x||i=mu[i]] end do;
  X_at_xy := subs(subvars, X);
  left_mult := Transformation(N, M, subvars);
  subvars := [ ];
  for i from 1 to nops(mu) do subvars := [op(subvars), D_x||i=D_y||i, x||i=y
||i] end do;
  yX := subs(subvars, X);
  ans := [ ];
  for i from 1 to nops(mu) do
    ans := [op(ans), Pushforward(left_mult, yX[i])]
  end do;
  evalDG(ans - X_at_xy);
end:

```

APPENDIX B

Examples

4.5.8

Example 3.3.1.2**► Synopsis****1. Multiplication in Canonical Coordinates of the 1st Kind - Nilpotent Case.**

We retrieve a Lie algebra from Winternitz.

> `LieAlg := Retrieve("Winternitz", 1, [6, 2], Alg)`
`LieAlg := [e1, e2]=e3, [e1, e3]=e4, [e1, e4]=e5, [e1, e5]=e6, [e1, e6]=0, [e2, e3]=0, [e2, e4]=0, [e2, e5]=0, [e2, e6]=0, [e3, e4]=0, [e3, e5]=0, [e3, e6]=0, [e4, e5]=0, [e4, e6]=0, [e5, e6]=0` **(5.1)**

> `DGsetup(LieAlg) :`

We switch to an adapted basis for convenience in comparing vector fields in coordinates of the 1st kind and coordinates of the second kind.

> `DGsetup(LieAlgebraData(AdaptedBasis(Series("Lower")), Alg)) :`

> `MultiplicationTable()`

<i>Alg</i>	<i>e1</i>	<i>e2</i>	<i>e3</i>	<i>e4</i>	<i>e5</i>	<i>e6</i>
<i>e1</i>	0	0	0	0	0	0
<i>e2</i>	0	0	0	0	<i>e1</i>	0
<i>e3</i>	0	0	0	0	<i>e2</i>	0
<i>e4</i>	0	0	0	0	<i>e3</i>	0
<i>e5</i>	0	- <i>e1</i>	- <i>e2</i>	- <i>e3</i>	0	<i>e4</i>
<i>e6</i>	0	0	0	0	- <i>e4</i>	0

We check to see that our Lie algebra is nilpotent.

> `Query(Alg, "Nilpotent")`
true **(5.3)**

We use the BCH formula to calculate $z=xy$, where z, x , and y are group elements in coordinates of the 1st kind.

> `z := NilpotentBCHmult(Alg)`

$$z := \left[x1 + y1 + \frac{1}{2} x2 y5 - \frac{1}{2} x5 y2 - \frac{1}{24} x4 x5 y5^2 + \frac{1}{24} x5^2 y4 y5 - \frac{1}{180} x5^2 y5^2 y6 \right. \quad (5.4)$$

$$+ \frac{1}{180} x5 x6 y5^3 + \frac{1}{12} x3 y5^2 - \frac{1}{12} x5 y3 y5 - \frac{1}{12} x3 x5 y5 + \frac{1}{12} x5^2 y3$$

$$+ \frac{1}{720} x5^4 y6 - \frac{1}{720} x5^3 x6 y5 + \frac{1}{180} x5^3 y5 y6 - \frac{1}{180} x5^2 x6 y5^2 - \frac{1}{720} x5 y5^3 y6$$

$$+ \frac{1}{720} x6 y5^4, x2 + y2 + \frac{1}{2} x3 y5 - \frac{1}{2} x5 y3 - \frac{1}{24} x5^2 y5 y6 + \frac{1}{24} x5 x6 y5^2$$

$$\left. + \frac{1}{12} x4 y5^2 - \frac{1}{12} x5 y4 y5 - \frac{1}{12} x4 x5 y5 + \frac{1}{12} x5^2 y4, x3 + y3 + \frac{1}{2} x4 y5 \right]$$

$$\begin{aligned}
& -\frac{1}{2} x_5 y_4 + \frac{1}{12} x_5 y_5 y_6 - \frac{1}{12} x_6 y_5^2 - \frac{1}{12} x_5^2 y_6 + \frac{1}{12} x_5 x_6 y_5, x_4 + y_4 \\
& + \frac{1}{2} x_5 y_6 - \frac{1}{2} x_6 y_5, x_5 + y_5, x_6 + y_6 \Big]
\end{aligned}$$

We verify that this is a group multiplicaiton.

> *adim* := *DGinformation*(*Alg*, "LieAlgebraDimension") :

> *identvars* := [] :

> *invvars* := [] :

> **for** *i* **from** 1 **to** *adim* **do**

identvars := [*op*(*identvars*), *y* || *i* = 0];

invvars := [*op*(*invvars*), *y* || *i* = -*x* || *i*];

end do:

Setting *y*=0 in the multiplication gives *x*, verifying that (0,0,0,0,0,0) is the identity.

> *simplify*(*subs*(*identvars*, *z*))

$$[x_1, x_2, x_3, x_4, x_5, x_6] \quad (5.5)$$

Setting *y*=(-*x*₁, -*x*₂, -*x*₄, -*x*₄, -*x*₅, -*x*₆) gives 0 verifying that (-*x*₁, -*x*₂, -*x*₄, -*x*₄, -*x*₅, -*x*₆) is the inverse.

> *subs*(*invvars*, *z*)

$$[0, 0, 0, 0, 0, 0] \quad (5.6)$$

We check the associative property by computing *x*(*yw*)-(xy)*w*.

> *CheckAssociativity*(*z*)

$$[0, 0, 0, 0, 0, 0] \quad (5.7)$$

We compute left invariant vector fields by taking derivatives of the multiplication function.

> *Livf_mult* := *convert*(*Livf_from_mult*(*z*), *Matrix*)⁺

$$\textit{Livf_mult} := \left[\left[\partial_{x_1} \right], \quad (5.8)
\right.$$

$$\left[-\frac{x_5}{2} \partial_{x_1} + \partial_{x_2} \right],$$

$$\left[\frac{x_5^2}{12} \partial_{x_1} - \frac{x_5}{2} \partial_{x_2} + \partial_{x_3} \right],$$

$$\left[\frac{x_5^2}{12} \partial_{x_2} - \frac{x_5}{2} \partial_{x_3} + \partial_{x_4} \right],$$

$$\left[\left(\frac{1}{2} x_2 - \frac{1}{12} x_3 x_5 - \frac{1}{720} x_5^3 x_6 \right) \partial_{x_1} + \left(\frac{x_3}{2} - \frac{x_4 x_5}{12} \right) \partial_{x_2} + \left(\frac{x_4}{2} + \frac{x_5 x_6}{12} \right) \partial_{x_3} \right.$$

$$\left. - \frac{x_6}{2} \partial_{x_4} + \partial_{x_5} \right],$$

$$\left[\frac{x_5^4}{720} \partial_{x_1} - \frac{x_5^2}{12} \partial_{x_3} + \frac{x_5}{2} \partial_{x_4} + \partial_{x_6} \right]]$$

We create a new Lie algebra with these vector fields.

> *L2* := *LieAlgebraData*(*convert*(*Livf_mult*, *list*), *Alg2*)

$$\begin{aligned}
L2 := [e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = 0, [e_1, e_6] = 0, [e_2, e_3] = 0, [e_2, e_4] \\
= 0, [e_2, e_5] = e_1, [e_2, e_6] = 0, [e_3, e_4] = 0, [e_3, e_5] = e_2, [e_3, e_6] = 0, [e_4, e_5] = e_3, \\
[e_4, e_6] = 0, [e_5, e_6] = e_4
\end{aligned} \quad (5.9)$$

> *DGsetup*(*L2*)

$$\textit{Lie algebra: Alg2} \quad (5.10)$$

We see that they give the same multiplication table that we started with.

> *MultiplicationTable*()

<i>Alg2</i>	<i>e1</i>	<i>e2</i>	<i>e3</i>	<i>e4</i>	<i>e5</i>	<i>e6</i>
<i>e1</i>	0	0	0	0	0	0
<i>e2</i>	0	0	0	0	<i>e1</i>	0
<i>e3</i>	0	0	0	0	<i>e2</i>	0
<i>e4</i>	0	0	0	0	<i>e3</i>	0
<i>e5</i>	0	$-e1$	$-e2$	$-e3$	0	<i>e4</i>
<i>e6</i>	0	0	0	0	$-e4$	0

(5.11)

4.5.9

Example 3.5.1.2

2. Left Invariant Vector Fields in Coordinates of the 1st Kind by Algebraic Formula - Nilpotent Case.

We set up a coordinate manifold.

> *dgsetupvars* := []:

> **for** *i* **from** 1 **to** *adim* **do** *dgsetupvars* := [*op*(*dgsetupvars*), *x*||*i*] **end do** :

> *DGsetup*(*dgsetupvars*, *M*) :

We compute left invariant vector fields by formula [1] from the slides.

> *Liv_algebraic* := *NilpotentLivf1stKind*(*Alg*, *M*)

$$\mathit{Liv_algebraic} := \left[\left[\partial_{x1} \right], \right. \quad (7.1)$$

$$\left[-\frac{1}{2} x5 \partial_{x1} + \partial_{x2} \right],$$

$$\left[\frac{1}{12} x5^2 \partial_{x1} - \frac{1}{2} x5 \partial_{x2} + \partial_{x3} \right],$$

$$\left[\frac{1}{12} x5^2 \partial_{x2} - \frac{1}{2} x5 \partial_{x3} + \partial_{x4} \right],$$

$$\left[\left(\frac{1}{2} x2 - \frac{1}{12} x3 x5 - \frac{1}{720} x5^3 x6 \right) \partial_{x1} + \left(\frac{1}{2} x3 - \frac{1}{12} x4 x5 \right) \partial_{x2} + \left(\frac{1}{2} x4 \right. \right.$$

$$\left. \left. + \frac{1}{12} x5 x6 \right) \partial_{x3} - \frac{1}{2} x6 \partial_{x4} + \partial_{x5} \right],$$

$$\left[\frac{1}{720} x5^4 \partial_{x1} - \frac{1}{12} x5^2 \partial_{x3} + \frac{1}{2} x5 \partial_{x4} + \partial_{x6} \right]]$$

We see that these are the same vector fields given by taking derivatives of the multiplication function.

> *evalDG*(*Liv_mult* - *Liv_algebraic*)

$$\begin{bmatrix} 0 \\ 0 \partial_{x1} \\ 0 \partial_{x1} \\ 0 \partial_{x1} \\ 0 \partial_{x1} \\ 0 \partial_{x1} \end{bmatrix}$$

(7.2)

We verify that these vector fields are left invariant.

We set up second coordinate manifold.

> *dgsetupvars* := []:


```

> for i from 1 to adim do dgsetupvars := [op(dgsetupvars), y||i] end do :
> DGsetup(dgsetupvars, N) :
We check left invariance by pushing forward Livf_algebraic evaluated at y to xy and see that it is
the same vector as Liff_algebraic evaluated at xy.
This test is similar to the one illustrated on the online help for GroupActions
[InvariantVectorsAndForms].
> CheckLeftInvariant(convert(Livf_algebraic, list), z)
      [0, 0  $\frac{\partial}{\partial x^1}$ , 0  $\frac{\partial}{\partial x^1}$ , 0  $\frac{\partial}{\partial x^1}$ , 0  $\frac{\partial}{\partial x^1}$ , 0  $\frac{\partial}{\partial x^1}$ ]

```

(7.3)

4.5.10

Example 4.2.1**3. Multiplication in Canonical Coordinates of the 2nd Kind - Nilpotent Case.**

We calculate coordinates of the second kind for a generic point in coordinates of the 1st kind.

```
> coordinate_1st := [ ]:
```

```
> for i from 1 to adim do coordinate_1st := [op(coordinate_1st), x||i] end do:
```

```
> NilpotentCanonicalCoordinateChange(Alg, coordinate_1st, "first")
```

$$\left[\frac{1}{30} x^5 x^4 x^6 - \frac{1}{24} x^5 x^3 x^4 + \frac{1}{6} x^5 x^2 x^3 - \frac{1}{2} x^5 x^2 + x^1, -\frac{1}{8} x^5 x^3 x^6 + \frac{1}{6} x^5 x^2 x^4 - \frac{1}{2} x^3 x^5 \right. \\ \left. + x^2, \frac{1}{3} x^5 x^2 x^6 - \frac{1}{2} x^4 x^5 + x^3, -\frac{1}{2} x^5 x^6 + x^4, x^5, x^6 \right] \quad (9.1)$$

We check that converting back to coordinates of the 1st kind gives the original point we started with.

```
> simplify(NilpotentCanonicalCoordinateChange(Alg, %, "second"))
```

$$[x1, x2, x3, x4, x5, x6] \quad (9.2)$$

We calculate $z=xy$, where z, x , and y are group elements in coordinates of the 2nd kind.

```
> z2 := NilpotentBCHmult2(Alg) :
```

We verify that this is a group multiplication.

Setting $y=0$ in the multiplication gives x , verifying that $(0,0,0,0,0,0)$ is the identity.

```
> simplify(subs(identvars, z2))
```

$$[x1, x2, x3, x4, x5, x6] \quad (9.3)$$

Calculate the Inverse of a generic point by switching to coordinates of the 1st kind and taking the negative of each coordinate then switching back to coordinates of the 2nd kind.

```
> xvars := [ ]:
```

```
> invvars := [ ]:
```

```
> for i from 1 to adim do xvars := [op(xvars), x||i] end do:
```

```
> inverse := NilpotentCanonicalCoordinateChange(Alg, xvars, "second") :
```

```
> for i from 1 to adim do invvars := [op(invvars), -inverse[i]] end do:
```

```
> inverse2 := NilpotentCanonicalCoordinateChange(Alg, invvars, "first") :
```

We check that this is in fact the inverse.

```
> subvars := [ ]:
```

```
> for i from 1 to adim do subvars := [op(subvars), y||i = inverse2[i]] end do:
```

```
> simplify(subs(subvars, z2))
```

$$[0, 0, 0, 0, 0, 0] \quad (9.4)$$

We check the associative property by computing $x(yw)-(xy)w$.

```
> CheckAssociativity(z2)
```

$$[0, 0, 0, 0, 0, 0] \quad (9.5)$$

We compute left invariant vector fields by taking derivatives of the multiplication function.

> $Liv_mult2 := \text{convert}(Liv_from_mult(z2), \text{Matrix})^+$

$$Liv_mult2 := \begin{pmatrix} \partial_{x1} \\ -x5 \partial_{x1} + \partial_{x2} \\ \frac{x5^2}{2} \partial_{x1} - x5 \partial_{x2} + \partial_{x3} \\ -\frac{x5^3}{6} \partial_{x1} + \frac{x5^2}{2} \partial_{x2} - x5 \partial_{x3} + \partial_{x4} \\ \frac{x5^3 x6}{6} \partial_{x1} - \frac{x5^2 x6}{2} \partial_{x2} + x5 x6 \partial_{x3} - x6 \partial_{x4} + \partial_{x5} \\ \partial_{x6} \end{pmatrix} \quad (9.6)$$

We create a new Lie algebra with these vector fields.

> $DGsetup(\text{LieAlgebraData}(\text{convert}(Liv_mult2, \text{list}), Alg3))$

Lie algebra: Alg3

(9.7)

We see that they give the same multiplication table that we started with.

> $\text{MultiplicationTable}()$

Alg3	e1	e2	e3	e4	e5	e6
e1	0	0	0	0	0	0
e2	0	0	0	0	e1	0
e3	0	0	0	0	e2	0
e4	0	0	0	0	e3	0
e5	0	-e1	-e2	-e3	0	e4
e6	0	0	0	0	-e4	0

(9.8)

4.5.11

Example 4.3.1**4. Calculate Left Invariant 1-Forms.**

We retrieve a Lie algebra $\mathfrak{so}(3)$ from Winternitz.

```
> LieAlgF := Retrieve("Winternitz", 1, [3, 9], FormAlg)
      LieAlgF := [e1, e2] = e3, [e1, e3] = -e2, [e2, e3] = e1
```

 (11.1)

```
> DGsetup(LieAlgF) :
```

```
> MultiplicationTable( )
```

FormAlg	$e1$	$e2$	$e3$
$e1$	0	$e3$	$-e2$
$e2$	$-e3$	0	$e1$
$e3$	$e2$	$-e1$	0

 (11.2)

We set up a coordinate manifold.

```
> adimF := DGinformation(FormAlg, "LieAlgebraDimension") :
```

```
> dgsetupvars := [ ] :
```

```
> for i from 1 to adimF do dgsetupvars := [op(dgsetupvars), x||i] end do :
```

```
> DGsetup(dgsetupvars, MF) :
```

We compute left invariant 1-forms by formula [2] from the slides.

```
> omega := CalcLeftInv1Forms(FormAlg, MF)
```

$$\omega := \begin{bmatrix} \cos(x3) \cos(x2) dx1 + \sin(x3) dx2 \\ -\sin(x3) \cos(x2) dx1 + \cos(x3) dx2 \\ \sin(x2) dx1 + dx3 \end{bmatrix}$$
 (11.3)

We calculate the dual vectors to these 1-forms.

```
> omegavars := [ ] :
```

```
> for i from 1 to adimF do omegavars := [op(omegavars), omega[i, 1]] end do :
```

```
> Livf_from_1forms := simplify(convert(DualBasis(omegavars), Matrix) ^ +)
```

$$\text{Livf_from_1forms} := \begin{bmatrix} \frac{\cos(x3)}{\cos(x2)} \partial_{x1} + \sin(x3) \partial_{x2} - \frac{\cos(x3) \sin(x2)}{\cos(x2)} \partial_{x3} \\ -\frac{\sin(x3)}{\cos(x2)} \partial_{x1} + \cos(x3) \partial_{x2} + \frac{\sin(x3) \sin(x2)}{\cos(x2)} \partial_{x3} \\ \partial_{x3} \end{bmatrix}$$
 (11.4)

We create a new Lie algebra with these vector fields.

```
> DGsetup(LieAlgebraData(convert(Livf_from_1forms, list), Alg4))
```

Lie algebra: Alg4

(11.5)

We see that they give the same multiplication table that we started with.

```
> MultiplicationTable( )
```

$\left[\begin{array}{c} | \\ | \\ | \end{array} \right]$

$$\begin{array}{c|ccc} \text{Alg4} & e1 & e2 & e3 \\ \hline e1 & 0 & e3 & -e2 \\ e2 & -e3 & 0 & e1 \\ e3 & e2 & -e1 & 0 \end{array}$$

(11.6)

4.5.12

Example 4.5.1

5. Left Invariant Vector Fields in Coordinates of the 2nd Kind by Algebraic Formula - Nilpotent Case.

We compute left invariant vector fields by formula [3] from the slides.

> `Livf2_algebraic := NilpotentLivf2ndKind(Alg, M)`

$$\text{Livf2_algebraic} := \begin{bmatrix} \partial_{x1} \\ -x5 \partial_{x1} + \partial_{x2} \\ \frac{1}{2} x5^2 \partial_{x1} - x5 \partial_{x2} + \partial_{x3} \\ -\frac{1}{6} x5^3 \partial_{x1} + \frac{1}{2} x5^2 \partial_{x2} - x5 \partial_{x3} + \partial_{x4} \\ \frac{1}{6} x5^3 x6 \partial_{x1} - \frac{1}{2} x5^2 x6 \partial_{x2} + x5 x6 \partial_{x3} - x6 \partial_{x4} + \partial_{x5} \\ \partial_{x6} \end{bmatrix} \quad (13.1)$$

We see that these are the same vector fields given by taking derivatives of the multiplication function in coordinates of the 2nd kind.

> `evalDG(Livf_mult2 - Livf2_algebraic)`

$$\begin{bmatrix} 0 \\ 0 \partial_{x1} \\ 0 \partial_{x1} \\ 0 \partial_{x1} \\ 0 \partial_{x1} \\ 0 \\ 0 \end{bmatrix} \quad (13.2)$$

We check left invariance by pushing forward `Livf_algebraic` evaluated at `y` to `xy` and see that it is the same vector as `Livf_algebraic` evaluated at `xy`.

> `CheckLeftInvariant(convert(Livf2_algebraic, list), z2)`

$$[0, 0 \partial_{x1}, 0 \partial_{x1}, 0 \partial_{x1}, 0 \partial_{x1}, 0 \partial_{x1}] \quad (13.3)$$