A Thermal Energy Storage Tank Model for Solar Heating

Robert Arthur Pate

Utah State University

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MODEL FOR SOLAR HEATING

by

Robert Arthur Pate

A dissertation submitted in partial fulfillment of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Engineering
(Mechanical Engineering)

Approved:

UTAH STATE UNIVERSITY
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NOMENCLATURE

A = cross-sectional area of tank

$C_2$ = experimental entrainment coefficient (page 35)

c_p = specific heat

D = diameter of inlet nozzle or port

H = convection coefficient

K = kinematic momentum

k = thermal conductivity

$\lambda$ = height of tank

$m_c$ = mass flow rate in column, $m_c(x,t)$

$m_e$ = entrained mass flow rate, $m_e(x,t)$

$m_{\text{in}}$ = mass flow rate entering top of tank, $m_{\text{in}}(t)$

$m_o$ = mass flow rate in column at $x = 0$

$m_T$ = mass flow rate in tank

$m_u$ = mass flow rate from tank for use, $m_u(t)$

P = perimeter

Q = volumetric flow rate

T = temperature in storage tank, $T(x,t)$

$T_c$ = temperature of column, $T_c(x,t)$

$T_{\text{in}}$ = temperature of inlet water

$T_\infty$ = ambient temperature

u = velocity in x-direction

V = volume of tank

x = distance from top of tank

$\alpha$ = thermal diffusivity
\( \delta = \text{critical depth from top of tank} \)

\( \theta = (T - T_\infty) \)

\( \nu = \text{kinematic viscosity} \)

\( \xi = \text{dimensionless distance from top of tank, } x/\lambda \)

\( \rho = \text{density} \)

\( \sigma = \text{experimental constant defined on page 33} \)

\( \tau = \text{dimensionless time defined on page 29} \)

\( T_{\text{return}} = \text{temperature of return or make-up water} \)

\( U_o = \text{overall heat transfer coefficient for top surface of storage tank} \)

\( U_\lambda = \text{overall heat transfer coefficient for the bottom surface of storage tank} \)

\( J = \text{total momentum} \)

\( \delta' = \text{dimensionless critical depth from top of tank} \)
ABSTRACT

A Thermal Energy Storage Tank Model for Solar Heating

by

Robert Arthur Pate, Doctor of Philosophy

Utah State University, 1977

Major Professor: Dr. Warren F. Phillips
Department: Mechanical Engineering

The results of a combined theoretical and experimental study of the kinetics of a hot-liquid energy-storage tank are presented. A physical model is developed which accurately describes the thermal stratification behavior in a storage tank. The governing differential equations are developed for the physical model. A numerical solution to the system of equations is presented. Some existing models were examined and the predicted results of each are discussed. The concepts developed can be used to predict the thermal stratification behavior in a storage tank under most conceivable operating conditions. These conditions include flow configurations at the top and the bottom of the tank which both have inversionary tendencies. Inversionary behavior could conceivably occur in both the top and the bottom regions of the tank during a combined storage and usage mode which might occur at non-peak storage hours. Although the work was done primarily for the utilization of solar energy, the results are not limited to such application. The results are more significant as a contribution to applied fluid dynamics.

(106 pages)
INTRODUCTION

Most of the alternative energy sources which have been studied (wind, tide, geothermal, solar, etc.) are not yet cost-effective. Some are not even energy-efficient. Their research, development, and construction consumes more energy than they will produce in their term of operation. Considerable research is being done on solar energy as an alternative to fossil fuels. In reality we are in error when we regard the sun as an alternative source of energy. Almost all known forms of energy are traceable directly or indirectly to the sun, nuclear energy being a possible, and to some a questionable, exception. Nature has harvested and stored the sun's energy in the form of fossil and organic fuels which we now enjoy. With the stored forms dwindling, we are looking to the sun for more direct and immediately available energy. Halacy [1] said, "Solar energy is free, to be sure. But the taking isn't." We certainly are finding this to be true as we try to harvest the sun's energy directly. Much research is needed in all related areas to make solar energy competitive. In the southwest, solar utilization looks good because the sun shines when it is needed most. For this reason, solar refrigeration seems practical. Unfortunately, when solar energy is needed for space heating in the North, the sun has gone south for the winter. This is perhaps the greatest handicap to solar space heating.

Daniels [2] indicated that the chief limitations to the use of the sun's energy are economic rather than technological. Though what
he says is basically true, further technological advances can do much to remove the economic limitations. With the present technology we are limited economically. The following research was done in an attempt to understand some of the problems associated with thermal energy storage. It is hoped that additional knowledge and understanding will provide the tools necessary to utilize efficiently the sun's energy for space heating and water heating.

There are two main tasks associated with solar energy usage for heating. They are collection and storage. The intermittent nature of the sun and the constant nature of our needs makes storage very important. The storage problem has been somewhat neglected in the literature. The majority of the effort has gone into devices for collecting solar energy. Most of the published work assumes the existence of adequate storage facilities without taking concern for the details. Many buildings have been designed and constructed with hot water energy-storage systems. The major design criterion seems to have been the total heat capacity with little or no consideration given the kinetics of the process. Though the heat capacity may be sufficient, the rate at which energy may be stored or removed may render the system inadequate. For space heating applications, a system designed to operate on a daily cycle with average weather conditions seems to be economically feasible. Lôf [3] said that it is entirely impractical to carry the entire heating load with solar energy. The truth of this can be seen readily. The extra collection and storage equipment necessary to heat a house on extremely cold or overcast days would increase the capitol cost considerably, and the extra equipment would be used only
a few days each year. There is an optimum size system for each
geographical location. The model to be presented will aid in the
design of the optimum sized system.

The concentration of effort up to the present time has not
been on the storage aspect of solar utilization. Most models to
date have been quite crude. W.A. Beckman and J.A. Duffie, with
various associates, have proposed some simple models for describ­
ing the kinetics of a storage tank. One system proposed by Duffie,
et al [4] models the storage tank as being an insulated cylinder
of uniform temperature. Their justification is as follows:

"Since thermal stratification of the stor­
age tank results in improved performance over an
unstratified system and the degree of stratifica­
tion in the actual system is unknown, the tank
was modeled as being at a uniform temperature."

This model has two virtues; first, it is simple and easy to imple-
ment; and second, it gives conservative results. In another pro-
posed model, Duffie, et al [5] took into account part of the thermal
stratification behavior. They discretized the tank into stratified
finite regions, each having a uniform temperature. The incoming
water seeks the element having a density nearest its own. While
the first model was conservative, this second model is just the
opposite. Density differences result from temperature differences.
The temperature of the incoming water changes as it slips through
elements of different temperatures. This change in temperature
(and consequent change in density) is neglected, and actual re-
sults can differ significantly from predicted results. The
temperature of the cool water at the bottom of the tank must be
known in order to predict the performance of a solar collector. Duffie's completely stratified model predicts that cool water will sink further than actual tests indicate. For example, water at 75°F does not sink to the 75° level, but rather mixes with the water above and may sink only to the 110° level. This mixing partially destroys the stratification in the tank; and as a result, though no energy is lost, the temperature differential is less and the energy in the top of the tank is less available (lower grade energy). Consequently, much more time is required to store the energy and also to extract it for use. Consider, for example, a tank that is 140°F at the top and 70°F at the bottom with a linear temperature profile in-between. Water at 70°F would return to the collector panel to be heated (see Figure 1 for an outline of the basic system). Suppose the tank were completely mixed. Water at the average temperature of 105°F would leave the bottom of the tank and return to the solar panel to be heated. The efficiency of the solar panel would be greatly reduced and would decrease as the water in the tank approached its maximum temperature. The results given by the actual, partially-stratified model would fall somewhere between the results of Duffie's completely mixed and completely stratified models. His completely stratified model predicts that the water leaving the bottom of the tank will be cooler than it actually is and that the water at the top of the tank will be hotter (its energy would be of higher grade) than it actually is.

Duffie and Beckman have avoided the experimental part of the storage research which is necessary to accurately describe the
Figure 1. Schematic of solar utilization system.
stratification in the tank. The purpose of the present study is to understand stratification behavior and to formulate a dynamic model that will predict accurately such behavior. This model should bring us one step closer to the feasible utilization of solar energy.

The following is a dynamic model which will predict the stratification in a hot-liquid energy-storage tank under all conditions encountered in the storage and usage cycles. Although the work was performed primarily for utilization of solar energy, the results are in no way restricted to solar application.
CONCEPTION OF MODEL

When possible, it is desirable to construct an apparatus and observe the phenomenon in question prior to formulating a model of the phenomenon. Because we are interested in describing the kinetics of the liquid in a hot-liquid energy-storage tank, and because the apparatus can be constructed readily, a transparent storage tank apparatus was assembled and instrumented with thermocouples (see Figure 2). The assembly consists of a plexiglass tank, a variable displacement precision pump, and the necessary plumbing to control the water flow rates and temperatures. Figures 3, 4, and 5 show what happens as dyed water at about 50°F is introduced into the tank having a uniform temperature of 125°F. Notice that as the dyed water sinks, it has considerable turbulence. It was observed that not only the dyed water but also a considerable amount of clear water in the immediate vicinity of the dyed water sank. The amount of clear water that was entrained appeared to be much greater than the amount of dyed water entering the tank. Continuity indicates that the clear water adjacent to the sinking column must be rising to replace the water that was entrained.

An initial temperature profile was established in the tank as shown in Figure 6. Figures 7 through 13 show a sequence of what occurred in the tank when a constant mass flow rate of dyed water at 50°F was introduced. The initial temperature profile used for the photographs in Figures 7 through 13 was approximately linear. The water at the top was about 130°F and at the bottom was about 60°F. Figure 8 shows the entrainment. Figures 9 through 11 show the sinking column leveling out at a position having a temperature of about 100°F. Duffie's completely stratified
Figure 2. Visual storage assembly
Figure 3. Cold dyed water entering hot tank.
Figure 4. Cold dyed water entering hot tank
Figure 5. Cold dyed water entering hot tank
Figure 6. Initial temperature profile in the storage tank.
Figure 7. A stratification sequence. (Time = 2 sec)
Figure 8. A stratification sequence. (Time = 4 sec)
Figure 9. A stratification sequence. (Time = 30 sec)
Figure 10. A strafication sequence. (Time = 1 min)
Figure 11. A strafication sequence. (Time = 1.6 min)
Figure 12. A stratification sequence. (Time = 2.5 min)
Figure 13. A strafication sequence. (Time = 4 min)
model predicts that the 50°F inlet water would sink to the bottom without interacting with the water in the tank. The very definite band seen in Figure 10 begins to widen in both the upward and downward directions. The clear water above and below the band indicates that no mixing is taking place, but rather these regions are slowly being displaced.

The photographs shown in Figures 3 through 13 show very definite two-dimensional flow patterns. To formulate a model, it would be helpful to know if the variations in temperature are also two-dimensional. The thermocouples were placed horizontally to establish the existence of horizontal temperature differences. Tests indicated that in the clear portion of the tank, outside the sinking column, the total temperature variation across the width of the tank was less than one-degree Fahrenheit. Additional tests also indicated that at the position where the dyed water leveled out into a band, the total temperature variation across the entire width of the tank was also less than one-degree. There did appear to be some mixing in this band. The water in the clear portions of the tank appeared very calm and free of any mixing. The water in the clear part of the tank did not appear to be able to support the horizontal density gradients that would be caused by horizontal temperature gradients. From the trends that were observed, a physical model was outlined to predict the behavior of the liquid.
DESCRIPTION OF MODEL

The proposed model includes an adiabatic free jet or plume entering the top of the tank. As the water jet enters, it mixes; and the flow becomes turbulent. The friction developed on the periphery of the emerging jet carries with it some of the surrounding fluid which was originally at rest. The free jet is assumed to be adiabatic because the mechanisms of conduction heat transfer are so slow. The temperature of the jet does change, however, but the temperature change results from the energy brought to the jet by the entrained mass and not by conduction from the surrounding fluid.

The storage tank is assumed to have only one significant spatial coordinate. The actual behavior is three-dimensional (in the special case of Figure 2, two-dimensional), but only temperature differences in the x coordinate direction will be considered. As the tests discussed in the previous section indicate, no horizontal temperature gradients of significance were found except in the region of the inlet jet. The storage tank is modeled as two separate parts. One part is the inlet jet, and the other part is the remainder of the tank. The two parts will be referred to as the column or jet and the tank. The vertical depth to which the inlet water sinks before leveling out will be referred to as δ. So the inlet water mixes with some of the tank water and sinks to a depth δ. Below δ the water moves downward at a rate equal to the mass flow rate entering the tank. Above δ the water moves upward at a rate equal to the amount of water that has been entrained in the jet, so that continuity is satisfied at all levels. At δ the entire jet is assumed to enter the
tank so that below \( \delta \) the jet does not exist. The depth \( \delta \) is the depth at which the mixed jet has the same temperature as the adjacent region in the tank. Figure 14 show what is happening. If the temperature profile in the tank were linear as shown, then the jet or column temperature, \( T_c \), would approach the tank temperature, \( T \), at the depth \( \delta \). At \( \delta \) the jet would enter the tank and displace the water below \( \delta \) by an amount \( m_{in} \); and the water above \( \delta \) would be displaced by an amount \( m_e \), which is the mass that was entrained in the jet before it reached \( \delta \) (see Figure 15).
Figure 14. Temperature profiles in the column and the tank.
Figure 15. Relative displacements of temperature profiles.
FORMULATION OF EQUATIONS

A differential formulation will give the desired partial differential equations. Consider first the column (see Figure 16). A mass balance yields

\[
\begin{align*}
\text{mass entering the top} & = m_c \\
\text{mass leaving the bottom} & = (m_c + \frac{\partial m_c}{\partial x} \, dx)
\end{align*}
\]

Considering the conservation of mass, the additional mass leaving the bottom must have been convected or entrained from the tank. Thus,

\[
\text{mass convected from tank} = \frac{\partial m_c}{\partial x} \, dx
\]

An energy balance yields

\[
\begin{align*}
\text{energy convected in the top} & = c_p \, m_c \, T_c \\
\text{energy convected in the side} & = c_p \, T \, \frac{\partial m_c}{\partial x} \, dx \\
\text{energy convected out the bottom} & = c_p \left[ m_c \, T_c + \frac{\partial}{\partial x} \left( m_c \, T_c \right) \, dx \right]
\end{align*}
\]

Equating terms and simplifying yields the following non-linear partial differential equation:

\[
T \, \frac{\partial m_c}{\partial x} = \frac{\partial}{\partial x} \left( m_c \, T_c \right)
\]  

(1)

Note that the thermal inertia of the column has been neglected. It is assumed that the column is quasi-steady.
Figure 16. Differential formulation for the column.
A similar analysis of the tank portion of the differential element yields its governing equation (see Figure 17). The mass balance yields

\[ \text{mass entering the top} = m_T \]
\[
\text{mass leaving the bottom} = (m_T + \frac{\partial m_T}{\partial x} \, dx)
\]
\[
\text{mass leaving the side} = -\frac{\partial m_T}{\partial x} \, dx
\]

Vertical conduction is allowed in the tank and, though it is hoped the tank would be well insulated, a convection loss to the surroundings is accounted for. An energy balance yields

\[ \text{energy conducted in the top} = -kA \frac{\partial T}{\partial x} \]
\[
\text{energy conducted out the bottom} = -kA \left( \frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2} \right) \, dx
\]
\[
\text{energy convected in the top} = c_p \, m_T \, T
\]
\[
\text{energy convected out the bottom} = c_p \left[ m_T \, T + \frac{\partial}{\partial x} (m_T \, T) \, dx \right]
\]
\[
\text{energy convected out the side} = -c_p \, T \, \frac{\partial m_T}{\partial x} \, dx
\]
\[
\text{energy lost by convection to the surroundings} = H \rho A \, (T - T_\infty) \, dx
\]
\[
\text{energy within the element} = \rho A \, c_p \, T \, dx
\]

Equating the net energy entering the control volume to the time rate of change of the energy within the control volume and simplifying yields the following non-linear partial differential equation:
Figure 17. Differential formulation for the tank.
If the second term is expanded, the equation can be reduced still further. The result is

\[ kA \frac{\partial^2 T}{\partial x^2} - \frac{c}{\rho} \frac{\partial}{\partial x} (m_T \frac{\partial T}{\partial x}) + \frac{c}{\rho} T \frac{\partial m_T}{\partial x} - HP (T - T_\infty) = \rho A \frac{\partial T}{\partial t} \]

Equation (2) is valid both above and below \( \delta \). At \( \delta \) the tank temperature is equal to the column temperature, i.e., \( T = T_c \).

Equations (1) and (2) can be put in a dimensionless form by making some variable changes. Define a new space variable such that

\[ \xi = \frac{x}{L} \]

where \( L \) is the total height of the storage tank. Define a dimensionless time \( \tau \) such that

\[ \tau = \frac{m_0}{\rho V} t \]

where \( m_0 \) is the mass flow rate entering the top of the storage tank from the solar panel and \( V \) is the total tank volume. Define a new temperature \( \theta \) such that

\[ \theta = (T - T_\infty) \]

where \( T_\infty \) is the temperature of the immediate surroundings to which the insulated storage tank is exposed. Note that the new temperature is not dimensionless. To non-dimensionalize the temperature in the tank a good reference temperature is needed. The application will provide the best reference temperature. Since \( \theta \) will appear in each term in equation (2) non-dimensionalizing \( \theta \) is not critical. Substituting the above variable changes into equation (2) yields the following equation:
Additional work will be done before equation (1) is non-dimensionalized.

Both equations (1) and equation (2) contain mass flow rates which have been ignored to this point. Both $\dot{m}_T$ and $\dot{m}_c$ are functions of $x$ and for variable inlet mass flow rates they are also functions of time. The mechanics of the jet next will be discussed. With the jet behavior known, all needed mass flow rates can be obtained.
DESCRIPTION OF JET

The water which enters the top of the storage tank has some momentum associated with it. In addition, the entering jet may have a body force on it due to the relative density difference between the water in the column and the water in the tank. As a result, velocity profiles are established between the incoming jet and the initially static water in the tank. The friction developed on the periphery of the emerging jet carries with it some of the surrounding fluid which was originally at rest. A pattern of streamlines similar to those shown in Figure 18 would be developed. The influence of friction causes the jet to spread outwards in the downstream direction. There are two main frictionally caused processes which account for the spreading of the jet. First, friction tends to slow the jet down; thus, continuity considerations require that for an incompressible fluid, the cross-sectional area must increase. Second, friction entrains some originally stationary fluid into the jet. The question to be answered is how much mass is entrained in the jet? This is one of the classical fluid dynamics problems. Some aspect of free jets, plumes, or entrainment has been discussed in almost every volume of the Journal of Fluid Mechanics from the time of the journal's inception to the present [10], [15], [16], [17]. Earlier in this century such men as Herman Schlichting [6], W.G. Bickley [7], and W. Tollmien [8] solved parts of the problem. After imposing several simplifying restrictions, Bickley showed that the governing equations for a two-dimensional jet are integrable in closed form. As he states, exact solutions of Prandtl's boundary layer equations are rare. Years later Albertson, et al [9] described
Figure 18. Streamlines and velocity profiles for jet.
the behavior by approximating profiles with Gaussian distributions. Gaussian profiles are still used by some authors. Much work has been done on entrainment in buoyant plumes, but solutions become much more complex when the surrounding medium has temperature stratification rather than uniformity (Morton, et al [18], Fan [19]). For simplification, the present work neglects the effect of temperature differences on entrainment rates. In 1961, Ricou and Spalding developed an ingenious method for measuring actual entrainment rates. Most of the recent work has been purely experimental.

Early in the progression, H. Schlichting [11] had compiled solutions of one sort or another for the circular and two-dimensional free jets for both laminar and turbulent flow. These may be found in his latest edition of "Boundary-Layer Theory". For the laminar case he indicates that the volumetric flux varies with the distance $x$ as follows:

Two-dimensional jet \[ Q = 3.3019 \left( \frac{K}{\nu} x \right)^3 \]

Circular jet \[ Q = 8 \pi \nu x \]

where $K$ is the kinematic momentum and $\nu$ is the kinematic viscosity. For the turbulent case his velocity profile can be integrated to give

Two-dimensional\[ Q = \left[ \frac{3Kx}{\sigma} \right]^2 \]

where $\sigma$ is an experimental constant found by Reichardt [12] to have 7.67 as its numerical value. Schlichting derives the flow rate for the circular turbulent free jet to be

\[ Q = .404 \sqrt{Kx} \]

See Appendix A for a complete derivation of the above four flow rates.
One should note that at $x = 0$, all of the above expression predict that the jet has no mass flow rate. This ($x = 0$) is a line of singularities for the solution. Bickley states that the occurrence of such a line is a common feature of the solutions of boundary layer equations and can be traced to the omission of the term $\partial^2 u / \partial x^2$ from the equations of motion. He also states that the omission of such a term is equivalent to the neglect of conduction or diffusion against the stream. He indicates, however, that the solution is adequate for moderate and large values of $x$.

Schlichting [11], when discussing the simplifications made, states that it is assumed for simplicity that the slit (for the two-dimensional jet) is infinitely small; but in order to retain a finite momentum as well as a zero volume flow, it is necessary to assume an infinite fluid velocity in the slit. Abramovich [13] discusses the initial region, or the region near $x = 0$, where the slug flow is going through a transition and becoming fully developed boundary layer flow; but unfortunately he does no more than discuss it.

In the case of thermal energy storage, a finite flow rate and a finite velocity exist at the inlet. The singularity problem encountered in Schlichting's [11] equations must be resolved. Squire [14] did some interesting work related to the flow configuration in question. He shows that the flow may be regarded as that resulting from the application of a force applied at a point in a viscous fluid which is at rest at infinity. This tends to give support to the thought that an adequate solution to the storage problem might be obtained by merely adding a finite inlet mass flow to Schlichting's [11] results.

In 1971, B.J. Hill [15] did some very helpful work. Using a novel method, which is an adaptation of the porous-wall technique used by Ricou
and Spalding [10], Hill measured the local entrainment rate in the initial region of an axisymmetric turbulent jet. The mass flow rate of the jet was observed to increase linearly with axial distance according to the relation,

$$\frac{D}{m_0} \left( \frac{\rho_0}{\rho_s} \right)^{1/2} \frac{1}{c} \frac{dm}{dx} = C_2$$

(4)

where $m_0$ is the mass flow rate of the jet at the nozzle, $\rho_s$ is the density of the fluid at the nozzle, and $C_2$ is an experimentally determined constant. Hill plots the value of $C_2$ as a function of $x/D$. Figure 19 shows the functional relationship which he found. In the fully developed region, $C_2$ was found to be .32, where fully developed flow begins at about 13 diameters from the nozzle. Notice that even down to five diameters the entrainment coefficient is still near .30.

Figure 19. Variation of entrainment coefficient with axial distance.
If $c_2$ is assumed to be constant at .32, and the density ratio assumed to be unity, then equation (4) can be integrated to give the mass flow rate in the column as

$$\dot{m}_c = \dot{m}_o \left( .32 \frac{x}{D} + 1 \right) \quad (5)$$

where the boundary condition $\dot{m}_c = \dot{m}_o$ at $x = 0$ has been imposed. Schlichting's [11] comparable solution is

$$Q = .404 \sqrt{x}$$

which can be changed to a mass flow rate as follows:

$$\dot{m}_c = \rho Q = .404 \left( \frac{2}{\sqrt{\pi}} \right) \frac{m_o}{D} x \quad (6)$$

where $K$ was replaced by $\frac{2}{\pi} \frac{m_o}{\rho D}$. $K$ can be determined from its definition

$$K = \frac{J}{\rho} = \int_{-\infty}^{+\infty} u^2 dy = \text{constant}$$

$J$ being the total momentum in the $x$-direction. Equation (6) can be simplified to the form

$$\dot{m}_c = \dot{m}_o \left( .46 \frac{x}{D} \right) \quad (7)$$

Hill's solution is

$$\dot{m}_c = \dot{m}_o \left( .32 \frac{x}{D} + 1 \right) \quad (5)$$

Equation (5) is quite similar to equation (7). One difference is the experimental constant (.32 as compared to .46), not a tremendous difference. The significant difference is the value of $m_c$ at $x = 0$. This means that if the inlet mass flow rate $m_o$ were added to Schlichting's solution, it would be of exactly the same form as Hill's solution. In light of the above discussion, equation (4) will be used as the differential equation governing the rate at which mass is laterally entrained into the
circular jet. Similar results can be obtained for the two-dimensional jet by using equation A-29 to determine the mass entrainment rate.

A note at this point is necessary regarding equation (4) as it applies to the solar energy storage problem. Implicit in equation (4) is the fact that the inlet mass is introduced into the tank in the axial direction. This is not the optimum direction for maximum stratification. During most of the storage cycle the water entering the top of the tank is hotter than any liquid in the tank. Under this condition it is very important (for maximum efficiency of the tank and collector panel) that no mixing take place. An axial jet would oppose the natural stratification tendencies (depending on its inlet velocity) and impinge on the cooler layers some distance from the top, causing end entrainment and mixing similar to that described by Baines [16]. It is desirable to introduce the hot water from the solar panel in the horizontal direction at a very low velocity. In the afternoon when the radiation from the sun is decreasing, or when the sun is temporarily hidden by clouds, the water entering the top of the tank may be hotter than the water leaving the tank (returning to the solar panel) but cooler than the water at the top of the tank. Under this condition it is desirable to have the cooler water sink to its appropriate level with as little interaction as possible. This would indicate that the water should be introduced at the top in an axial direction. It is better, however, to sacrifice efficiency at the non-peak hours that occur in the late afternoon than to do so in the morning and early afternoon. Thus, it is best to introduce the inlet water horizontally at a very low velocity. Because the momentum source implied in equation (4) is entering the tank vertically and it is desired
that the momentum source enter horizontally, equation (4) must be modified or some justification must be given for its continued use. To modify the equation would require the momentum source to be replaced by an equivalent buoyant force. To promote stratification, it is desired to maintain moderate velocities. At moderate velocities it has been observed that the horizontal inlet jet maintains its integrity (because of the laminar nature of the flow) as well as its constant cross-sectional area as it bends vertically downward under the influence of the buoyant forces. This 90° turn occurs very rapidly. After the bend has been made, the flow becomes turbulent. In the laminar region of the bend, the velocity appears to be quite constant. For the above reasons, it is quite satisfactory to assume that the turbulent jet begins after the 90° turn has been completed and to apply equation (4) to the jet from that point on. For higher velocities this assumption breaks down, but it should be remembered that high velocities can cause complete mixing in the tank and should be avoided in practical applications.
COMPLETE FORMULATION

The complete set of differential equations which govern the model are as follows:

\[ \alpha \frac{\partial^2 T}{\partial x^2} - \left( \frac{m_T}{\rho A} \right) \frac{\partial T}{\partial x} - \left( \frac{\text{HP}}{\rho A c_p} \right) (T - T_\infty) = \frac{\partial T}{\partial t} \]  \hspace{1cm} (8)

\[ \frac{T}{\partial x} = \frac{\partial}{\partial x} \left( \frac{m_c T_c}{c} \right) \]  \hspace{1cm} (9)

\[ \frac{\dot{m}_c}{\partial x} = c_2 \frac{m_o}{D} \]  \hspace{1cm} Circular jet \hspace{1cm} (10-a)

\[ \frac{\dot{m}_c}{\partial x} = \frac{\rho}{2} \left[ \frac{3k}{40A_x} \right]^2 = \frac{m_o}{D} \left[ \frac{3k}{40A_x} \right]^2 \]  \hspace{1cm} Two-dimensional jet \hspace{1cm} (10-b)

\[ \dot{m}_u = \dot{m}_o - \dot{m}_c - \dot{m}_u \]  \hspace{1cm} (11)

where \( \dot{m}_u \) is the hot water usage rate. It should be noted that \( \dot{m}_c \) is equal to zero below \( \delta \), where \( \delta \) is the value of \( x \) where \( T(x, t) = T_c(x, t) \). Some useful boundary conditions are

\[ \dot{m}_c = \dot{m}_o \text{ at } x = 0 \]

\[ T_c = T_{in} \text{ at } x = 0 \]

\[ (\dot{m}_c - \dot{m}_o) (T_{in} - T_r) + \frac{U A}{c_p} (T_t - T_\infty) - \frac{k A}{c_p} \frac{\partial T}{\partial x} = 0 \text{ at } x = 0 \]

\[ \dot{m}_u (T_{return} - T_{in}) + \frac{U y A}{c_p} (T_{return} - T_\infty) + \frac{k A}{c_p} \frac{\partial T}{\partial x} = 0 \text{ at } x = L \]

Of course, one must also know the initial temperature profile in the tank, \( T(x, 0) \). The above system of differential equations can be simplified, though in the present form they are readily programmable. The
simplifications are similar for both the circular jet and the two-dimension­

al, but since the applicability of the two-dimensional jet is quite

limited, only the circular jet development will be considered further.

Substitution of equation (10-a) and its integrated form, equation

(5), into equation (9) yields

\[
\frac{T_c}{x} + \left(\frac{C_2}{C_2' x + D}\right) (T_c - T) = 0
\]

The \( \dot{m}_T \) in equation (8) is not a particularly meaningful term. It repre­
sents the actual entrained flow rate (as a function of x) within the

tank and can be replaced by boundary conditions. Above \( x = \delta \), \( \dot{m}_T \) can

be replaced by \( (\dot{m}_o - \dot{m}_u - \dot{m}_c) \); and below \( \delta \), it can be replaced by

\( (\dot{m}_o - \dot{m}_u) \). Thus, above \( \delta \), equation (8) becomes

\[
\alpha \frac{\partial^2 T}{\partial x^2} + \left[ \frac{\dot{m}_u + \dot{m}_o}{\rho A} \left( \frac{C_2 x}{D} \right) \right] \frac{\partial T}{\partial x} - \left( \frac{HP}{\rho A c} \right) (T - T_\infty) = \frac{\partial T}{\partial t}
\]

and below \( \delta \), equation (8) becomes

\[
\alpha \frac{\partial^2 T}{\partial x^2} + \left( \frac{\dot{m}_u - \dot{m}_o}{\rho A} \right) \frac{\partial T}{\partial x} - \left( \frac{HP}{\rho A c} \right) (T - T_\infty) = \frac{\partial T}{\partial t}
\]

At this point it is not helpful to non-dimensionalize the tempera­
tures because an appropriate reference temperature is lacking. Such a

temperature would come from the particular application. If the storage

tank is used with a solar panel, an excellent reference temperature might

be \( T_{P \text{max}} \), which is the maximum steady-state solar-panel temperature that

would be reached if the transport fluid in the panel were stopped and

all incoming energy were forced to be re-radiated back to the environ­

ment.

The complete, partially non-dimensional formulation follows:

For \( \xi < \delta' \),
\[
\left( \frac{\alpha \rho V}{m_0 \xi^2} \right) \frac{2 T}{2} + \left( \frac{\dot{m}_u}{m_0} + \frac{\dot{m}_d}{m_0} \right) \left( \frac{C_2 \xi}{D} \right) \frac{\partial T}{\partial \xi} - \left( \frac{\frac{H P}{c c_p m_o}}{} \right) (T - T_\infty) = \frac{\partial T}{\partial T} \tag{15}
\]

and

\[
\frac{\partial T}{\partial \xi} + \left( \frac{C_2}{c_2 \xi + D} \right) (T - T_\infty) = 0 \tag{16}
\]

For \( \xi > \delta' \)

\[
\left( \frac{\partial \rho V}{m_0 \xi^2} \right) \frac{\partial^2 T}{\partial \xi^2} + \left( \frac{\dot{m}_u}{m_0} \right) \frac{\partial T}{\partial \xi} - \left( \frac{H P}{c c_p m_o} \right) (T - T_\infty) = \frac{\partial T}{\partial T} \tag{17}
\]

The boundary conditions are

\[
(\dot{m}_c - \dot{m}_o) (T_{in} - T) + \frac{U A}{c_p} (T_t - T_\infty) - \frac{k A}{c_p \lambda} \frac{\partial T}{\partial \xi} = 0 \text{ at } x = 0
\]

\[
\dot{m}_u (T - T_{return}) + \frac{U A}{c_p} (T_t - T_\infty) + \frac{k A}{c_p \lambda} \frac{\partial T}{\partial \xi} = 0 \text{ at } \xi = 1
\]

\[T_c = T_{in} \text{ at } \xi = 0\]

The initial temperature profile \( T (\xi, 0) \) must also be known.

In general, the system of equations is non-linear and no Eigenvalue solution is possible. Under certain very restricted conditions a series solution might be possible but the analytic boundary conditions that would be required greatly limit the applicability of the solution. For maximum flexibility a numerical solution is preferred. The numerical scheme used for experimental correlation is contained in Appendix B.
RESULTS

A forward difference scheme was programmed in FORTRAN on a Burroughs 6700 computer using the Burroughs command and edit language. The programs used to obtain the theoretical results for correlation with the experimental data are listed in Appendix B. The results of this study will be presented in four sections: Test I, Test II, Test III, and Conclusion. The first three sections will compare experimental data with the predictions of the present model and previous models for various configurations. Both a circular tank (see Figure 20) and a two-dimensional tank (see Figure 2) are used in the comparison so that the effects of both circular and two-dimensional jets can be examined.

On the figures that follow, the results of the completely mixed model discussed by J.A. Duffie and W.A. Beckman are labeled "mixed." The results of their completely stratified model are labeled "stratified." The results of the present model are labeled "partially stratified". All experimentally determined data points are shown as small circles.
Figure 20. Circular storage tank assembly.
TEST I

A two-dimensional tank was used for Test I. The tank was 12-inches high, 9-inches wide, and 2½ inches deep. Water at 50°F was introduced into the storage tank. The mass flow rate \( m_0 \) was maintained constant at 8.76 lbm/hour. The initial temperature profile in the tank is shown in Figure 21. The temperature fluctuations were monitored and recorded on a twelve channel recorder. The initial conditions and the boundary conditions from the experiment were used as input for the numerical solution scheme. Figure 22 shows how the present theoretical model predicted that the temperature profile in the tank would change with time. For this particular numerical solution, a convection coefficient for the uninsulated vertical walls was assumed to be 2 Btu/hr ft\(^2\)°F. The energy that was stored in the walls was neglected. Figures 23 through 28 show how the theoretical results compare with the actual data. Notice that the correct trends are present.

Additional numerical solutions were obtained neglecting the energy lost to the environment. While the temperature error was found to be very slight for short periods of time, for longer periods of time the accumulative error was significant. The error caused by neglecting environmental losses for the configuration and time shown in Figure 28 is about 5°F. The assumed convective coefficient of 2 Btu/hr ft\(^2\)°F was chosen slightly high to account partially for the evaporation occurring on the continually damp tank surface. But, as additional tests indicated, the solution was not very sensitive to the convective coefficient.
Figure 21. Initial temperature profile in tank for Test I.
Figure 22. Predicted temperature variation with time for Test I.
Figure 23. Comparison between predicted results and experimental data for Test I (time = 2 minutes).
Figure 24. Comparison between predicted results and experimental data for Test I (time - 4 minutes).
Figure 25. Comparison between predicted results and experimental data for Test I (time - 6 minutes).
Figure 26. Comparison between predicted results and experimental data for Test I (time = 8 minutes).
Figure 27. Comparison between predicted results and experimental data for Test I (time = 10 minutes).
Figure 28. Comparison between predicted results and experimental data for Test I (time = 12 minutes).
TEST II

A circular tank having a circular inlet port was used for Test II. The tank was 10½ inches high and 8½ inches in diameter. Water at 52°F was introduced into the storage tank. The mass flow rate $m_o$ was maintained at 22 lbm/hour. The initial temperature profile in the tank is shown in Figure 29. The temperature fluctuations were again recorded. Compatible with the geometry and flow conditions, a numerical solution was generated. Figure 30 shows how the present model predicted that the temperature profile in the tank would change with time. The circular tank was not insulated, so again a convection coefficient of 2 Btu/hr ft$^2$°F was assumed. The environmental temperature was 70°F. The energy stored in the walls was neglected. Figures 31 through 36 show how the theoretical results compare with the actual data. Again the correlation is very good. There is some deviation near the jet.
Figure 29. Initial temperature profile in tank for Test II.
Figure 30. Predicted temperature variation with time for Test III.
Figure 31. Comparison between predicted results and experimental data for Test II (time = 2 minutes).
Figure 32. Comparison between predicted results and experimental data for Test II (time = 4 minutes).
Figure 33. Comparison between predicted results and experimental data for Test II (time = 6 minutes).
Figure 34. Comparison between predicted results and experimental data for Test II (time = 8 minutes).
Figure 35. Comparison between predicted results and experimental data for Test II (time = 10 minutes).
Figure 36. Comparison between predicted results and experimental data for Test II (time = 12 minutes).
TEST III

The two-dimensional tank shown in Figure 2 was used for Test III. The mass flow rate was maintained constant at 13.2 lbm/hour, but the inlet temperature was allowed to vary in an approximately sinusoidal manner. The tank initially had a uniform temperature of 55°F. The inlet temperature was changed from 55°F to 140°F and then back to 55°F as shown in Figure 37. The temperature fluctuations were monitored and recorded. The initial conditions and the boundary conditions for the experiment were used as input for the numerical solution. The side walls of the tank were assumed to be at the same temperature as the adjacent water, and the energy that was stored in the walls was accounted for. A free convection coefficient for the uninsulated vertical walls was assumed to be 2 Btu/hour ft²/F. Figures 38 through 46 show how the theoretical results compare with the actual data. Notice that the correct trends occur in the results of the present model. The results of the present model coincide with the results of Duffie's completely stratified model as long as the inlet water is hotter than the water at the top of the tank. As the inlet temperature begins to decrease, the present model still follows the proper trends, but the completely stratified model does not. Figures 43 through 46 show the difference.

Additional numerical solutions were obtained assuming a linear temperature profile in the side walls rather than a constant profile. The results were about a one-degree increase in temperature at each point.
Figure 37. Inlet temperature variation.
Figure 38. Comparison between predicted results and experimental data for Test III (time = 2 minutes).
Figure 39. Comparison between predicted results and experimental data for Test III (time = 4 minutes).
Figure 40. Comparison between predicted results and experimental data for Test III (time = 6 minutes).
Figure 41. Comparison between predicted results and experimental data for test III (time = 8 minutes).
Figure 42. Comparison between predicted results and experimental data for Test III (time = 10 minutes).
Figure 43. Comparison between predicted results and experimental data for Test III (time = 12 minutes).
Figure 44. Comparison between predicted results and experimental data for test III (time = 14 minutes).
Figure 45. Comparison between predicted results and experimental data for test III (time = 16 minutes).
Figure 46. Comparison between predicted results and experimental data for test III (time = 18 minutes).
CONCLUSIONS

The purpose of the present work was to improve the modeling capabilities associated with a hot-liquid energy storage tank. The model is not intended to give conservative results, but rather it is intended to be a modeling improvement from a best-estimate viewpoint. Those who have written most extensively on solar energy usage, and specifically thermal energy storage, recognize that the existing models are quite simple and can not adequately describe the mixing phenomenon that occurs in a storage tank. The results of two existing models have been presented and compared with experimental data. The completely mixed model was observed to give conservative results. There was always more energy in the tank than the mixed model predicted. The results of the completely stratified model were found to be inadequate as the inlet water temperature decreased (see Figure 46). As the inlet water temperature decreases below the temperature at the top of the tank, an inversion must take place which cannot be described adequately as a complete slip as proposed in the completely stratified model. The cooler inlet water will not sink or slip to the level of corresponding density, but rather it will interact with the surrounding water, with energy and momentum being transferred. Writers are aware of certain deficiencies; but for lack of anything of equivalent simplicity which better describes the phenomena, they choose to use either the completely mixed or the completely stratified model.

With the aid of a little theory of free-jet entrainment, the present model is able to adequately describe the tank mixing that occurs,
without significantly increasing the complexity beyond that of the completely stratified model. The proposed partially stratified model degenerates to the completely stratified model if the entrainment term in the differential equation is set to zero. As can be seen from all of the preceding test results, the addition of the entrainment term greatly improves the results.

All of the preceding model results show, when compared with test data, that the proposed partially stratified model describes the proper trends. There are also, however, slight deviations from test data in all cases. There are many reasons for these deviations, none of which are felt to be of great consequence. The only two phenomena which have caused concern are first, the results of both the completely stratified model and the partially stratified model appear to have slightly higher temperatures than the actual test data show; and second, in the region where the sinking column levels out in the tank, the test data show a steeper temperature gradient than either of the stratified models predicts. There are two things which can account for the slightly higher temperatures. First, the mass flow rate of the test system was determined using a volume flow rate in conjunction with water property tables rather than using an actual mass flow rate measurement. The hot water from the source used in the experiment had very small air bubbles suspended in it. This gave the hot water a milky, opaque appearance. A deviation would result because the water density obtained from the thermodynamic tables used to convert the measured volume flow rate to a mass flow rate did not account for the suspended air bubbles. Some of these bubbles can be seen in Figures 3 through 13 as they adhered to the tank wall surface. The model solution assumes pure water. Thus,
the energy equation of the model would predict higher temperatures than the test data. The energy per volume of water is much greater than the energy per volume of air. The second and probably most significant cause for the higher model temperatures is that there was energy lost from the water to the containment and surroundings which was only partially accounted for in the models. The evaporation and convection from the continually damp outer surfaces of the tank and plumbing and also certain related losses were too involved and of too little significance to receive more than a crude treatment. The losses were compounded by the geometry of the tank apparatus. The shape that was chosen for observation and photographing had a very poor surface-area-to-volume ratio; and thus, the energy losses through the large uninsulated surface area were much greater than would normally be expected.

The second phenomenon which received scrutiny was the steepness of the temperature gradient in the area where the slipping column leveled out. It was first thought that the model might be sensitive to thermal conductivity and thus, cause too much smoothing of the gradients that had appeared sharp in the test results. Additional computer solutions indicated that thermal conductivity was insignificant for the length of time used in the tests. The slight deviation in slope may result from one of the assumptions made in the model. The column was assumed to enter the tank at a horizontal plane. As can be seen in Figures 8 through 10, the column does not enter as a plane having no vertical thickness, but as a band having a finite thickness. Figure 9 shows how the dyed water dips deeper on the left than it does at the center of the tank. This momentum overshoot causes additional mixing as well as lateral entrainment. As can be seen from Curve 2 of Figure
22, lateral entrainment has the effect of steepening the vertical temperature gradient. Additional entrainment below ($\delta$), which has been neglected, may account for the deviation.

There are some areas in which the interested researcher can verify and improve the model. The author feels that a larger well-insulated tank with a better area-to-volume ratio and better plumbing would produce experimental data that would compare even more exactly with the proposed partially stratified model predictions, and that additional study of the entrainment process would be helpful. Entrainment rates, both above and below ($\delta$), could be examined in more detail. As the entrainment process is explained and improved, the model results can be made to approach the actual solution more closely.

Many people find it feasible to sacrifice some degree of accuracy for a large measure of simplicity and convenience. There is a degenerate form of the proposed model which may be very useful for such application. It is obtained by assuming that the partially stratified model has an infinite entrainment rate which would correspond to complete mixing in the region where entrainment takes place; that is, above ($\delta$). Previously ($\delta$) was defined as the elevation where the side column temperature is equal to the adjacent tank temperature. The definition of the new ($\delta$) is similar but must be modified slightly because the side column temperature and the adjacent mixed tank temperature are the same. The new ($\delta$) would be defined as the elevation at which the temperature of the completely mixed region is equal to the temperature of the completely stratified region below it. Assuming that the solid line in Figure 47 represents the temperature profile obtained from using actual entrainment rates, the dashed line would be the temperature
both above and below (δ), could be examined in more detail. As the entrainment process is explained and improved, the model results can be made to approach the actual solution more closely.

Many people find it feasible to sacrifice some degree of accuracy for a large measure of simplicity and convenience. There is a degenerate form of the proposed model which may be very useful for such application. It is obtained by assuming that the partially stratified model has an infinite entrainment rate which would correspond to complete mixing in the region where entrainment takes place; that is, above (δ). Previously (δ) was defined as the elevation where the side column temperature is equal to the adjacent tank temperature. The definition of the new (δ) is similar but must be modified slightly because the side column temperature and the adjacent mixed tank temperature are the same. The new (δ) would be defined as the elevation at which the temperature of the completely mixed region is equal to the temperature of the completely stratified region below it. Assuming that the solid line in Figure 47 represents the temperature profile obtained from using actual entrainment rates, the dashed line would be the temperature profile obtained by using the degenerate form of the model. Note that the degenerate model is slightly less accurate and gives more conservative results.

The author recognizes that while some applications require extreme accuracy, others, less technical, require only reasonable numbers. The proposed partially stratified model and its degenerate form are significant contributions to the variable requirements.
Figure 47. Temperature profile modification for infinite entrainment.
LITERATURE CITED


APPENDICES
APPENDIX A

The following four formulations are as outlined by H. Schlichting in his text, "Boundary Layer Theory".

Laminar two-dimensional jet

We shall adopt a system of coordinates with its origin in the slit and with its axis of abscissae coinciding with the jet axis. The jet spreads outward in the downstream direction owing to the influence of friction, whereas its velocity in the centre decreases in the same direction. For the sake of simplicity we shall assume that the slit is infinitely small; but in order to retain a finite volume of flow as well as a finite momentum, it is necessary to assume an infinite fluid velocity in the slit. The pressure gradient \( \frac{dp}{dx} \) in the \( x \)-direction can hence be neglected because the constant pressure in the surrounding fluid impresses itself on the jet. Consequently, the total momentum in the \( x \)-direction, denoted by \( J \), must remain constant and independent of the distance \( x \) from the orifice. Hence

\[
J = \rho \int_{-\infty}^{+\infty} u^2 dy = \text{const.} \quad \text{(A-1)}
\]

It is possible to make a suitable assumption regarding the velocity distribution if it is considered that the velocity profiles \( u(x,y) \), just as in the case of a flat plate at zero incidence, are most probably similar, because the problem as a whole possesses no characteristic linear dimension. We shall assume, therefore, that the velocity \( u \) is a function of
y/b, where b is the width of the jet, suitably defined. We shall also assume that b is proportional to \( x^p \). Accordingly, we can write the stream function in the form

\[
\psi \sim x^p \left( \frac{y}{b} \right) = x^p f \left( \frac{y}{x^q} \right)
\]

The two unknown exponents \( p \) and \( q \) will be determined from the following conditions:

1. The flux of momentum in the \( x \)-direction is independent of \( x \), according to equation (A-1).
2. The acceleration terms and the friction term in equation (A-2) are of the same order of magnitude.

This gives two equations for \( p \) and \( q \):

\[
2p - q = 0 \quad \text{and} \quad 2p - 1 = p - 3q,
\]

and hence,

\[
p = \frac{1}{3} ; \quad q = \frac{2}{3}
\]

Consequently, the assumptions for the independent variable and for the stream function can be written as

\[
\eta = \frac{1}{3} \left( \frac{y}{x^{1/2}} \right)^{2/3} ; \quad \psi = \left( \frac{y}{x^{1/2}} \right)^{1/3} \frac{1}{x^{1/3}} f(\eta)
\]

if suitable constant factors are included. Therefore, the velocity components are given by the following expressions:
\[ u = \frac{1}{3x^{1/3}} f' (\eta); \quad (A-3) \]

\[ v = -\frac{1}{3} \sqrt[3]{2} x^{-2/3} (f - 2\eta f') \]

Introducing these values into the differential equation (A-2), and equating the pressure term to zero, we obtain the following differential equation for the stream function \( f (\eta) \):

\[ f'^2 + f f'' + f''' = 0 \quad (A-4) \]

with the boundary conditions \( v = 0 \) and \( \partial u/\partial y = 0 \) at \( y = 0 \), and \( u = 0 \) at \( y = \infty \). Thus,

\[ \eta = 0 : f = 0, f'' = 0; \quad \eta = \infty : f' = 0 \quad (A-5) \]

The solution of equation (A-4) is unexpectedly simple. Integrating once we have

\[ f f' + f'' = 0 \]

The constant of integration is zero because of the boundary conditions at \( \eta = 0 \), and the resulting differential equation of the second order could be integrated immediately if the first term contained the factor 2. This can be achieved by the following transformation:

\[ \xi = \alpha \eta; \quad f = 2 \alpha F (\xi), \]

where \( \alpha \) is a free constant to be determined later. Thus, the above equation transforms into

\[ F'' + 2 F F' = 0 \quad (A-6) \]
and the dash now denotes differentiation with respect to $\xi$. The boundary conditions are

$$\xi = 0 : F = 0 ; \xi = \infty : F' = 0, \quad (A-7)$$

and the equation can be integrated once more to give

$$F' + F^2 = 1, \quad (A-8)$$

where the constant of integration was made equal to 1. This follows if we put $F'(0) = 1$, which is permissible without loss of generality because of the free constant $\alpha$ in the relation between $f$ and $F$. Equation (A-8) is a differential equation of Riccati's type and can be integrated in closed terms. We obtain

$$\xi = \int_0^F \frac{dF}{1 - F^2} = \frac{1}{2} \ln \frac{1 + F}{1 - F} = \tanh^{-1} F$$

Inverting this equation we obtain

$$F = \tanh \xi = \frac{1 - \exp(-2\xi)}{1 + \exp(-2\xi)} \quad (A-9)$$

Since, further, $dF/d\xi = 1 - \tanh^2 \xi$, the velocity distribution can be deduced from equation (A-3) and is

$$u = \frac{2}{3} \alpha^2 x^{-1/3} (1 - \tanh^2 \xi) \quad (A-10)$$

It now remains to determine the constant $\alpha$, and this can be done with the aid of condition (A-1) which states that the momentum in the $x$-direction is constant. Combining equations (A-10) and (A-1) we obtain

$$J = \frac{8}{3} \rho \alpha^3 \nu^{1/2} \int_0^\infty (1 - \tanh^2 \xi)^2 d\xi = \frac{16}{9} \rho \alpha^3 \nu^{1/2} \quad (A-11)$$
We shall assume that the flux of momentum, J, for the jet is given. It is proportional to the excess in pressure with which the jet leaves the slit. Introducing the kinematic momentum \( J/\rho = K \), we have from equation (A-11)

\[
\alpha = 0.8255 \left( \frac{K}{\nu^{1/2}} \right)^{1/3}
\]

Hence, for the velocity distribution

\[
u = 0.4543 \left( \frac{K}{\nu \chi} \right)^{1/3} (1 - \tanh^2 \xi), \tag{A-12}
\]

\[
u = 0.5503 \left( \frac{K}{\nu \chi} \right)^{1/3} \left[ 2 \xi (1 - \tanh^2 \xi) - \tanh \xi \right], \tag{A-12}
\]

\[
\xi = 0.2752 \left( \frac{K}{\nu^2} \right)^{1/3} \frac{\chi}{2/3}
\]

The transverse velocity at the boundary of the jet is

\[
v_{\infty} = - 0.550 \left( \frac{K}{\nu \chi} \right)^{1/3} \tag{A-13}
\]

and the volume-rate of discharge per unit height of slit becomes

\[
Q = \int_{-\infty}^{+\infty} u \, dy, \quad \text{or} \quad Q = 3.3019 (K \chi)^{1/3} \tag{A-14}
\]

The volume-rate of discharge increases in the downstream direction because fluid particles are carried away with the jet, owing to friction on its boundaries. It also increases with increasing momentum.

Laminar circular jet

The pressure can here be regarded constant, as in the two-dimensional case. The system of coordinates will be selected with its x-axis in
the axis of the jet, the radial distance being denoted by \( y \). The axial and radial velocity components will be denoted by \( u \) and \( v \), respectively. Owing to the assumption of a constant pressure, the flux of momentum in the direction of \( x \) is constant once more:

\[
J = 2\pi \rho \int_{0}^{\infty} u^2 y \, dy = \text{const.} \quad (A-15)
\]

In the adopted system of coordinates the equation of motion in the direction of \( x \), under the usual boundary-layer simplifications, together with the equation of motion, can be written as

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{v}{y} \frac{\partial u}{\partial y} &= \nu \frac{1}{y} \frac{\partial}{\partial y} \left( y \frac{\partial u}{\partial y} \right), \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{y} &= 0.
\end{align*}
\]

and the boundary conditions are

\[
\begin{align*}
y = 0 : v &= 0; \quad \frac{\partial u}{\partial y} = 0; \\
y = \infty : u &= 0.
\end{align*} \quad (A-17)
\]

As before, the velocity profiles \( u(x,y) \) can be assumed similar. The width of the jet will be taken to be proportional to \( x^n \), it being further assumed that

\[
\psi \sim x^p F(\eta) \text{ with } \eta = \frac{y}{x^n}
\]

In order to determine the exponents \( p \) and \( n \) we can use the same two conditions as in the two-dimensional case. First, the momentum from equation (A-15) must be independent of \( x \); and second, the inertia and frictional terms in equation (A-16a) must be of the same order of magnitude. Hence,
Thus, the following two equations for $p$ and $n$ result:

$$2p - 4n + 2n = 0; 2p - 4n - 1 = p - 4n,$$

so that $p = n = 1$. Consequently, we may now put

$$\psi = v \times F (\eta) \text{ and } \eta = \frac{y}{x}$$

from which it follows that the velocity components are

$$u = \frac{v}{x} \frac{F'}{\eta}; \quad v = \frac{v}{x} \left( F' - \frac{F}{\eta} \right) \quad (A-18)$$

Inserting these values onto equation (A-16a), we obtain the following equation for the stream function:

$$\frac{F F'}{\eta^2} - \frac{F'}{\eta} - \frac{F F''}{\eta} = \frac{d}{d \eta} \left( F'' - \frac{F'}{\eta} \right),$$

which can be integrated once to give

$$F F' = F' - \eta F'' \quad (A-19)$$

The boundary conditions are $u = u_m$ and $v = 0$ for $y = 0$. It follows that $F' = 0$ and $F = 0$ for $\eta = 0$. Since $u$ is an even function of $\eta$, $F'/\eta$ must be even, $F'$ odd, and $F$ even. Because of $F (0) = 0$, the constant term in the expansion of $F$ in powers of $\eta$ must vanish. Thus, one constant of integration is determined. The second constant of integration, which can be denoted by $\gamma$, can be evaluated as follows: If $F (\eta)$ is a solution of equation (A-19), then $F (\gamma \eta) = F (\xi)$ is also a solution. A part solution of the differential equation

$$F \frac{dF}{d\xi} = \frac{dF}{d\xi} - \xi \frac{d^2F}{d\xi^2}$$
which satisfies the boundary condition \( \zeta = 0: F = 0, F' = 0 \), is given by

\[
F = \frac{\zeta^2}{1 + \frac{1}{4} \zeta^2}
\]  
(A-20)

Hence, we obtain from equation (A-18),

\[
u = \frac{\nu}{x} \frac{\gamma^2}{x} \frac{1}{\zeta} \frac{dF}{d\zeta} = \frac{\nu}{x} \frac{2\gamma^2}{(1 + \frac{1}{4} \zeta^2)^2}
\]
(A-20)

\[
v = \frac{\nu}{x} \frac{\gamma}{(\frac{dF}{d\zeta} - \frac{F}{\zeta})} = \frac{\nu}{x} \frac{\xi - \frac{1}{4} \xi^3}{(1 + \frac{1}{4} \xi^2)^2}
\]
(A-20)

Here \( \xi = \gamma y/x \), and the constant of integration \( \gamma \) can now be determined from the given value of momentum.

From equation (A-15) we obtain for the momentum of the jet

\[
J = 2\pi \rho \int_0^\infty u^2 y \ dy = \frac{16}{3} \pi \rho \gamma^2 \nu^2
\]

Finally, the above results can be expressed in a form to contain only the kinematic viscosity, \( \nu \), and the kinematic momentum \( K' = J/\rho \). Thus,

\[
u = \frac{3}{8\pi} \frac{K'}{\nu x} \frac{1}{(1 + \frac{1}{4} \zeta^2)^2}
\]
(A-21)

\[
v = \frac{1}{4} \sqrt{\frac{3}{\pi}} \frac{\sqrt{K'}}{x} \frac{\xi - \frac{1}{4} \xi^3}{(1 + \frac{1}{4} \xi^2)^2}
\]
(A-22)

\[
\zeta = \frac{\sqrt{3}}{\sqrt{16\pi}} \frac{\sqrt{K'}}{\nu} \frac{\gamma}{x}
\]
(A-23)

The volume of flow \( Q = 2\pi \int_0^\infty u \ y \ dy \) (volume per second), which increases with the distance from the orifice, owing to the flow from the surroundings, is represented by the simple equation
This equation should be compared with equation (A-14) for the two-dimensional jet. It is seen that, unexpectedly, the volume of flow at a given distance from the orifice is independent of the momentum of the jet, i.e., independent of the excess of pressure under which the jet leaves the orifice. A jet which leaves under a large pressure difference (large velocity) remains narrower than one leaving with a smaller pressure difference (small velocity). The latter carries with it comparatively more stationary fluid in a manner to make the volume of the flow at a given distance from the orifice equal to that in a faster jet, provided that the kinematic viscosity is the same in both cases.

**Turbulent two-dimensional jet**

The rate of increase in the width of the jet is proportional to $x$, $b \sim x$. The centre-line velocity $U$ is proportional to $x^{-1/2}$. The differential equation is

$$u \ \frac{\partial u}{\partial x} + v \ \frac{\partial u}{\partial y} = \epsilon \ \frac{\partial^2 u}{\partial y^2} \quad (A-25)$$

which must be combined with the equation of continuity. The virtual kinematic viscosity is given by

$$\epsilon = x \ \frac{1}{b} \ U,$$

where $U$ denotes the centre-line velocity. Denoting the centre-line velocity and the width of the jet at a fixed characteristic distance $s$ from the orifice by $U_s$ and $b_s$, respectively, we may write

$$U = U_s \ \left( \frac{x}{s} \right)^{-1/2} \ ; \ b = b_s \ \frac{x}{s}$$

Consequently,
Further, we put

\[ \eta = \sigma \frac{\psi}{x} , \]

where \( \sigma \) denotes a free constant. The equation of continuity is integrated by the use of a stream function \( \psi \), which is assumed to be of the form

\[ \psi = \sigma^{-1} U_s \frac{1}{3} x^{1/2} F(\eta) \]

Thus,

\[ u = U_s \left( \frac{x}{s} \right)^{-1/2} \frac{1}{2} F' ; \quad v = \sigma^{-1} U_s \frac{1}{2} x^{-1/2} (\eta F' - \frac{1}{2} F) \]

On substituting into equation (A-25), we obtain the following differential equation for \( F(\eta) \):

\[ \frac{1}{2} F' + \frac{1}{2} F F'' + \frac{\varepsilon_s}{U_s} \sigma^2 F''' = 0 , \]

with the boundary conditions \( F = 0 \) and \( F' = 1 \) at \( \eta = 0 \), and \( F' = 0 \) at \( \eta = \infty \). Since \( \varepsilon_s \) contains the free constant \( x_1 \), we may put

\[ \sigma = \frac{1}{2} \frac{U_s}{\varepsilon_s} \]

This substitution simplifies the proceeding differential equation which can now be integrated twice, whence we obtain

\[ F^2 = F' = 1 \]

This is exactly the same equation as that for the two-dimensional laminar jet, equation (A-8). Its solution is \( F = \tanh \nu \) so that the velocity is

\[ u = U_x (x/s)^{-1/2} (1 - \tanh^2 \eta) . \]

The characteristic velocity can be
expressed in terms of the constant momentum per unit length: \( J = \rho \int_{-\infty}^{+\infty} u^2 \, dy \). Hence, \( J = \frac{4}{3} \rho U_s^2 s/\sigma \). With \( J/\rho = K \) (kinematic momentum), we obtain the final form of the solution:

\[
\begin{align*}
    u &= \sqrt{3} \sqrt{\frac{K_0}{x}} (1 - \tanh^2 \eta), \\
    v &= \frac{\sqrt{3}}{4} \sqrt{\frac{K}{x_0}} \{2\eta (1 - \tanh^2 \eta) - \tanh \eta\}, \\
    \eta &= a \frac{\sqrt{3}}{x}
\end{align*}
\] (A-28) (A-28) (A-28)

The value of the single empirical constant \( a \) was determined experimentally to be 7.67.

The expression for \( u \) can be integrated to give

\[
Q = \left(\frac{3K}{\sigma^2} \frac{1}{x}\right)^{1/2}
\] (A-29)

**Turbulent circular jet**

The width of the jet is proportional to \( x \) and the centre-line velocity \( U \sim x^{-1} \). Thus, the virtual kinematic viscosity becomes

\[
\varepsilon = x_1 b U \sim x^0 = \text{const} = \varepsilon_0
\]

which means that it remains constant over the whole of the jet, as it was in the two-dimensional wake. Consequently, the differential equation for the velocity distribution becomes formally identical with that for the laminar jet, it being only necessary to replace the kinematic viscosity, \( \nu \), of laminar flow by the virtual kinematic viscosity, \( \varepsilon_0 \), of turbulent flow. It is, therefore, possible to carry over the solution for the laminar, circular-jet, equations (A-21) to (A-23). Introducing, once more, the constant kinematic momentum, \( K \), as a measure of the strength of the jet, we obtain
\[ u = \frac{3}{8\pi} \frac{K}{\varepsilon_0} \frac{1}{\left(1 + \frac{1}{4} \eta^2 \right)^2} \quad \text{(A-30)} \]

\[ v = \frac{1}{4} \sqrt{\frac{3}{\pi}} \frac{\sqrt{K}}{x} \frac{\eta \cdot \frac{1}{4} \eta^2}{\left(1 + \frac{1}{4} \eta^2 \right)^2} \quad \text{(A-30)} \]

\[ \eta = \frac{1}{4} \sqrt{\frac{3}{\pi}} \frac{\sqrt{K}}{\varepsilon_0 x} \quad \text{(A-30)} \]

The empirical constant is now equal to \( \sqrt{K/\varepsilon_0} \). According to the measurement performed by H. Reichardt, the width of the jet is given by \( b_{1/2} = 0.0848 \times \). With \( \eta = 1.286 \) at \( u = \frac{1}{2} u_m \) we have \( b_{1/2} = 5.27 \times \varepsilon_0 / K \); and hence,

\[ \frac{\varepsilon_0}{\sqrt{K}} = 0.0161 \]

On the other hand we have

\[ \sqrt{K} = 1.59 b_{1/2} U \]

so that

\[ \varepsilon_0 = 0.0256 b_{1/2} U \]

where, as before, \( b_{1/2} \) denotes half the width at half depth.

The mass of fluid carried at a distance \( x \) from the orifice can be calculated from equation (A-24). Inserting the above value for \( \varepsilon_0 \), we obtain

\[ Q = 0.404 \sqrt{K} x \quad \text{(A-31)} \]
APPENDIX B

NUMERICAL SOLUTION

An explicit forward difference technique was used to obtain numerical solutions. The partials in equation (8) were replaced as follows:

\[ \frac{\partial^2 T}{\partial x^2} \approx \frac{T(I+1,J) - 2T(I,J) + T(I-1,J)}{\Delta x^2} \]

\[ \frac{\partial T}{\partial x} \approx \frac{T(I,J) - T(I-1,J)}{\Delta x} \]

\[ \frac{\partial T}{\partial t} \approx \frac{T(I, J+1) - T(I,J)}{\Delta t} \]

Upon substitution an expression for \( T(I,J+1) \) is obtained.

\[ T(I,J+1) = T(I,J) \left[ 1 - 2 \left( \frac{\Delta t}{\Delta x^2} \right) + m_T \left( \frac{\Delta t}{\alpha \rho \Delta x} \right) - HP \left( \frac{\Delta t}{\alpha \rho c_p} \right) \right] + \]

\[ T(I+1,J) \left[ \left( \frac{\Delta t}{\Delta x^2} \right) - m_T \left( \frac{\Delta t}{\alpha \rho \Delta x} \right) \right] + \]

\[ T(I-1,J) \left( \frac{\Delta t}{\Delta x^2} \right) + T_\infty \left( \frac{HP \Delta t}{\alpha \rho c_p} \right) \]  \hspace{1cm} (A-32)

The \( I \) subscript is for position and the \( J \) subscript is for time.

Equation (9) can be expanded and modified as follows:

\[ \frac{\partial m_c}{\partial x} = \frac{m_c(I,J) - m_c(I-1,J)}{\Delta x} \]

\[ \frac{\partial T}{\partial x} = \frac{T_c(I+1,J) - T_c(I,J)}{\Delta x} \]
Upon substitution and simplification an expression for $T_c(I+1, J)$ is obtained.

$$T_c(I+1, J) = \{T(I, J) - T_c(I, J)\} \left[ \frac{\dot{m}_c(I, J) - \dot{m}_c(I-1, J)}{\dot{m}_c(I, J)} \right] + T(I, J) \quad (A-33)$$

The tank mass flow rate, $\dot{m}_t$, was replaced by its appropriate function of $\dot{m}_c$. The column mass flow rate was replaced by either equation (5) for the circular case, or $\dot{m}_o$ added to equation (7) for the two-dimensional case. The mass flow rate in equation (A-32) depends upon where the temperature is being determined (above or below $\delta$). The indices in equations (A-32) and (A-33) need to be modified at the first and last tank nodes to avoid "invalid index" problems.

The three programs to follow were used to obtain the solutions plotted in the text. The first program is for the two-dimensional jet case, using a modified form of equation (7) to calculate the appropriate mass flow rates. The second program is for the circular jet case, using equation (5) to calculate the appropriate mass flow rates. In the first two programs only constant inlet conditions are used. The third program allows the inlet temperature to vary. Tabular inlet conditions are used with linear interpolation between points.

The energy stored in the walls is partially accounted for in the third program. The thermal diffusivity is multiplied by a constant defined as follows:

$$\frac{1}{1 - \frac{\rho_s A_s c_{ps}}{\rho A c_p}}$$

where the $s$ subscript refers to the structural components.
The emphasis of the work completed was to develop a model and not perfect a computer program. A stability analysis was not performed. Forward difference numerical solutions are stable only for small enough step sizes. These programs were used strictly as an aid and are not intended to be the goal. Had a sophisticated program been the goal, implicit methods would have been used. Many of the implicit methods are always stable.
$ P E S E T $ F P E E
FILE 1=PROFIL.E UNIT=LISKPACK RECORD=14 BLOCKING=15
FILE 2=FLOTDATA UNIT=LISKPACK RECORD=14 BLOCKING=15
C
C
NN = NUMBER OF NODES
NT = NUMBER OF TIME STEPS
CM = MASS ADDITION TO COLUMN
DELTA = Grid spacing
DELTA = TIME STEP SIZE
TK = THERMAL CONDUCTIVITY OF STORAG LE L IQU I D
CP = SPECIFIC HEAT OF STORAG LE L IQU I D
A = CROSS-SECTIONAL AREA OF STORAG LE TANK
W = LENGTH OF SLIT
S = WIDTH OF SLIT
MU = VISCOSITY IN LBF-SEC/FT SQUA REED
PMC = MASS FLOW RATE COMING DOWN SIDE COLUMNS
PMT = MASS FLOW RATE IN TOP SECTION OF TANK
FM = MASS FLOW RATE ENTERING TANK (LBM/HOUR)
TIN = TEMPERATURE OF INCOMING WATER
XLEN = HEIGHT OF TANK IN FEET
C
DIMENSION T(26,261), TC(26,261), TH(261)

PM(1)=PM+.62545*SQRT((CM**2*DELTA*(1-1)/S))

DPHI(DIFF)=-.00007064*DIFF**2+.001672*DIFF+.532

1 "FITE(6,/) 'INPUT PROFILE DATA SOURCE, 1=FILE, 5=TELETYPE'
READ(5,/) M1
IF( M1, EC, 1) GO TO 7
W FITE(6,/) 'INPUT NN (NUMBER OF NODES)'
7 READ(5,/) XN
IF( M1, EC, 1) GO TO 8
W FITE(6,/) 'INPUT T(L,1), I=1,NN'
8 READ(5,/) (T(L,1),I=1,NN)
PREM 1
2 W FITE(6,/) 'INPUT TK, CP, PHO, A, XLEN, S, W'
READ(5,/) TK, CP, PHO, A, XLEN, S, W
DELTA=XLEN/((NN-1)
3 W FITE(6,/) 'INPUT DELT (IN MINUTES) AND NT (TIME NODES)' READ(5,/) DELT, NT
DELTA=DELT/60.
4 W FITE(6,/) 'INPUT FM AND TIN'
READ(5,/) FM, TIN
5 W FITE(6,/) 'INPUT TAME, H,F'
READ(5,/) TAME, H,F
ALPHA=TK/(PHO*CP)

ATX=ALPHA*DELTA/(DELTA**2)

TFS=DELTA/(A*PHO*DELTA)

HFT=H*F*DELTA/(PHO*A*CP)
DO 160 J = B, N T
TC(I,J) = TIN
DO 20 I = 1, NN
DIFF = TC(I,J) - T(I,J)
DIFF = DP0C(DIFF)
TC(I+1,J) = (TC(I,J) - FC(I+1) + TC(I,J) * FC(I)) / FC(I+1)
DO 20 I = 1, NN
10 CONTINUE
20 CONTINUE
IF(I .GT. I) NCPIT = I - 1
IF(I .EQ. NN) NCPIT = NN
C CRITICAL NODE
C T(NCPIT, J+1) = TC(NCPIT, J)
C TOP OF TANK (ABOVE CRITICAL NODE)
C DO 40 K = 1, NCPIT - 2
I = NCPIT - K
IF(I .LE. 1) GO TO 40
FMTE = FM - FM (I)
TC(I+1,J) = TC(I,J) * ( 1 - 2 * ATX + FMTE * TAPX )
& + T(I+1,J) * ( ATX + TAPX * FMTE ) + T(I-1,J) * ATX + HFT + TAMB
40 CONTINUE
C BOTTOM OF TANK (BELOW CRITICAL NODE)
C DO 50 I = NCPIT + 1, NN - 1
IF(I .GT. NN) GO TO 50
FMTE = FM
TC(I,J) = TC(I,J) * ( 1 - 2 * ATX + FMTE * TAPX - HFT ) + T(I+1,J) * ATX +
& T(I-1,J) * ( ATX + TAPX * FMTE ) + TAMB * HFT
TC(I,J+1) = TC(I,J)
50 CONTINUE
C TOP NODE
C I = 1
FMTE = FM - FM (I)
TC(I,J+1) = TC(I,J) * ( 1 - 2 * ATX + FMTE * TAPX - HFT ) + T(I+1,J) * ( 2 * ATX
& - FMTE * TAPX )
C BOTTOM NODE
C I = NN
TC(I,J+1) = TC(I,J) * ( 1 - 2 * ATX + TAPX * FM - HFT ) + T(I-1,J) * ( 2 * ATX
& + TAPX * FM ) + TAMB * HFT
TC(I,J+1) = TC(I,J)
100 CONTINUE
60 WRITE(6,/) 'PROFILE=1, PROPERTIES=2, DELTAT AND NT=3, FM AND
\ TIN=4,
& TIME=5, PPRINT CURVE=6'
READ(5,/) NFLAG
GO TO (1, 2, 3, 4, 5, 6) NFLAG
6 WRITE(6,/) 'INPUT TIME OF DESIRED CURVE IN MINUTES'
READ(5,/) TIME
NC = 1+TIME/DELT
IF(NC.GT.NT) WRITE(6,/) 'TIME GREATER THAN TEST TIME!
IF(NC.GT.NT) GO TO 6
DO 70 I = 1, NN
70 WRITE(6,/) TC(I, NC), TC(I, NC)
GO TO 60
STOP
END
PROGRAM NO. 2

FILE 1 = PROFILE, UNIT = DISKPACK, RECORD = 14, EOLocking = 15
FILE 2 = FLOTDATA, UNIT = DISKPACK, RECORD = 14, EOLocking = 15

C C
C NN = NUMBER OF NODES
C NT = NUMBER OF TIME STEPS
C CM = MASS ADIABATIC COLUMN
C DELTAX = GRID SPACING
C DELTAT = TIME STEP SIZE
C TK = THERMAL CONDUCTIVITY OF STOPLAGE LIQUID
C CF = SPECIFIC HEAT OF STOPLAGE LIQUID
C A = CROSS-SECTIONAL AREA OF STOPLAGE TANK
C W = LENGTH OF SLIT
C S = WIDTH OF SLIT
C MU = VISCOSITY IN LB-SEC/FT SQUARED
C RMC = MASS FLOW RATE COMING LOW SIDE COLUMN
C RATE = MASS FLOW RATE IN TOP SECTION OF TANK
C FM = MASS FLOW RATE ENTERING TANK (LB/HOUR)
C TIN = TEMPERATURE OF INCOMING WATER
C XLEN = HEIGHT OF TANK IN FEET
C
C DIMENSION T(201, 201), TC(201, 201), TM(201)
RMCM = FM*C/(2*DELTAX*(1-1)/D+1.)
1 WRITE(6, '/') 'INPUT PROFILE DATA SOURCE, 1=FILE, 5=TELETYPE'
READ(5, '/') NI
IF(MI. EQ. 1) GO TO 7
WRITE(6, '/') 'INPUT NN (NUMBER OF NODES)'
7 READ(MI, '/') NN
IF(MI. EQ. 1) GO TO 8
WRITE(6, '/') 'INPUT T(1, 1) I=1, NN'
8 READ(MI, '/') (T(I, 1), I=1, NN)
PEMLD 1
2 WRITE(6, '/') 'INPUT TK, CF, PHO, /XLEN, D'
READ(5, '/') TK, CF, PHO, /XLEN, D
DELTAX = XLEN/(NN-1)
3 WRITE(6, '/') 'INPUT DELT (IN MINUTES) AND NT (TIME NODES)'
READ(5, '/') DELT, NT
DELTAT = DELT/60.
4 WRITE(6, '/') 'INPUT FM AND TIN'
READ(5, '/') FM, TIN
5 WRITE(6, '/') 'INPUT TAME, H=P'
READ(5, '/') TAME, H=P
ALPHA = TK/(PHO*CF)
ATX = ALPHA*DELTAT/(DELTAX**2)
TAP = DELTAT/(A*PHO*DELTAX)
HFT = H*P*DELTAT/(PHO*A*CF)
DO 100 J = 1, NT
TC(1, J) = TIN

100
DO 20 I = 1, NJ
TC(I+1,J) = (TC(I,J) + TC(I,J) * FM(I+1) - FM(I)) / FM(I+1)
IF (TC(I,J) - TC(I,J)) < 10** -30, 30
10 CONTINUE
20 CONTINUE
30 NCPI = 1
IF (I .GT. J) NCPI = I - 1
IF (I .GT. NJ) NCPI = NJ

C CITICAL NODE

C T(NCPI, J+1) = TC(NCPI, J)

C TOP OF TANK (ABOVE CRITICAL NODE)

C DO 40 K = 1, NCPI - 2
I = NCPI - K
IF (I .LE. I) GO TO 40
FMT = FM - FM(I)
T(I, J+1) = T(I, J) * (1 - 2 * ATX + HFT + FMT * TAFX) & + T(I, J) * (ATX + TAFX * FMT) + T(I - 1, J) * ATX + HFT + TME
40 CONTINUE

C BOTTOM OF TANK (BELOW CRITICAL NODE)

C DO 50 I = 1, NCPI + 1, NJ - 1
IF (I .GE. NJ) GO TO 50
FMT = FM
T(I, J+1) = T(I, J) * (1 - 2 * ATX + FMT * TAFX - HFT) + T(I + 1, J) * ATX & + T(I - 1, J) * (ATX + TAFX * FMT) + TME * HFT
T(C(I, J+1) = T(I, J+1)
50 CONTINUE

C TOP NODE

C I = 1
FMT = FM - FM(I)
T(I, J+1) = T(I, J) * (1 - 2 * ATX + FMT * TAFX - HFT) + T(I + 1, J) * (2 * ATX & + FMT * TAFX)

C BOTTOM NODE

C I = NJ
T(C(I, J+1) = T(I, J+1)
100 CONTINUE

60 WRITE(6,') 'PROFILE = 1, PROPERTIES = 2, DELTAT AND NT = 3, FM AN
\ TIN = 4
& H AND TAME = 5, PPRINT CURVE = 6'
READ(5,/) NFLAG
GO TO (1, 2, 3, 4, 5, 6) NFLAG
6 WRITE(6,/) 'INPUT TIME OF DESIRED CURVE IN MINUTES'
READ(5,/) TIME
N = 1 + TIME/DELT
IF(NC.GT.NT) WRITE(6,/) 'TIME GREATER THAN TEST TIME'
IF(NC.GT.NT) GO TO 6
DO 70 I = 1, NN
70 WRITE(6,/) T(I, NC), TC(I, NC)
GO TO 60
STOP
END
PROGRAM NO. 3

100 $ RESET FREE

102 FILE 1=PROFILE UNIT=DISKPACK RECORD=14 BLOCKING=15
104 FILE 2=PLOTDATA UNIT=DISKPACK RECORD=14 BLOCKING=15
106 C
108 C
110 C NN = NUMBER OF NODES
112 C NT = NUMBER OF TIME STEPS
114 C DELTAX = X GRID SPACING
116 C DELTAT = TIME STEP SIZE
118 C TK = THERMAL CONDUCTIVITY OF STORAGE LIQUID
120 C CP = SPECIFIC HEAT OF STORAGE LIQUID
122 C A = CROSS-SECTIONAL AREA OF STORAGE TANK
124 C W = LENGTH OF SLIT
126 C S = WIDTH OF SLIT
128 C D = DIAMETER OF INLET PORT
130 C NCASE = 1 FOR ENTRAINMENT OR 0 FOR COMPLETELY STRATIFIED
132 C RM = MASS FLOW RATE COMING DOWN SIDE COLUMN
134 C RMT = MASS FLOW RATE IN TOP SECTION OF TANK
136 C RM = MASS FLOW RATE ENTERING TANK (LIM/HOUR)
138 C TIN(K) = TEMPERATURE OF INCOMING WATER
140 C TTIN(K) = TIME AT WHICH TIN(K) IS GIVEN IN MINUTES
142 C XLEN = HEIGHT OF TANK IN FEET
144 C
146 C
148 DIMENSION TC(50, 402), TC(50, 402), TIN(50), TTIN(50), TMIX(402),
& TOP(402)
150 RMCI(I) = RM+NCASE*6254*SORT(ABS(RM**2*DELTAX*(I-1)/(S*W))
154 WRITE(6, ' ' ) 'INPUT PROFILE DATA SOURCE I=FILE S=TELETYPE'
156 READ(5, ) MI
162 IF(MI .EQ. 1) GO TO 7
158 WRITE(6, ' ' ) 'INPUT NN (NUMBER OF NODES)' 
160 READ(MI, ) NN
154 IF(MI .EQ. 1) GO TO 7
162 READ(MI, ) NN
166 WRITE(6, ' ' ) 'INPUT T(K, I) I=1, NN'
168 READ(MI, ) (TC(I, I), I=1, NN)
170 REWIND 1
172 WRITE(6, ' ' ) 'INPUT NTIN (NUMBER OF TEMP POINTS)'
174 READ(5, ) NTIN
176 WRITE(6, ' ' ) 'INPUT TTIN(K) (MIN) AND TIN(K)'
178 DO 9 K=1, NTIN
180 READ(5, ) TTIN(K), TIN(K)
182 TTIN(K)=TTIN(K)/60.
184 9 CONTINUE
186 2 WRITE(6, ' ' ) 'INPUT TK, CP, RHO, A, XLEN, S, W'
188 READ(5, ) TK, CP, RHO, A, XLEN, S, W
190 DELTAX=XLEN/(NN-1)
192 3 WRITE(6, ' ' ) 'INPUT DELT (IN MINUTES) AND NT (TIME NODES)'
194 READ(5, ) DELT, NT
196 DELTAT=DELT/60.
198 4 WRITE(6, ' ' ) 'INPUT RM AND NCASE (1 FOR ENTRAINMENT OR 0 FOR\ 
\ NON-ENTRAINMENT)'
READ(5,/) FM,N CASE
5 WRITE(6,/) 'INPUT TAM,H,P'
READ(5,/) TAM,H,P
ALPHA= TP/(RH0*CP)* 1. 23
ATX= ALPHA*DEL TAT/(DEL TAX** 2)
TAFP= DEL TAT/(A*RH0*DEL TAX)
HPT= H*P*DEL TAT/(RH0*A*CP)
TMIX(J)= TIN(J)
TOP(J)= TIN(J)
15 DO 100 J= 1,NT
220 TIME= DEL TAT*(J- 1)
222 DO 11 KE= 1,NTIN- 1
224 IF(TIME.GE.TTIN(K).AND. TIME.LE.TTIN(K+ 1)) KE= K
226 11 CONTINUE
228 TO= (TIN(KE+ 1)- TIN(KE))*(TIME- TTIN(KE))/( TTIN(KE+ 1)-
& TTIN(KE))= TIN(KE)
232 TOP(J+ 1)= TOP(J)* (1. - FM*CP*DEL TAT/* 6- H*DEL TAT* 347) +
& TO*(FM*CP*DEL TAT/* 6)+ TAMB*(H*DEL TAT* 347)
236 TC(J+ 1)= TOP(J+ 1)
238 24 CONTINUE
240 TMIX(J+ 1)= (DEL TAT/ FM/(RH0*AXLEN))*(- TMIX(J)+ TC(J,J)) + TMIX(J)
242 DO 20 I= 1,NN
244 TC(I,J+ 1)= (TC(I,J)* (FM(I,J+ 1)- FM(I,J)) + TC(I,J)* FM(I))/FM(I)
246 IF(TC(I,J)- TC(I,J)) LT 0.1 K= 10,30,30
248 CONTINUE
250 20 CONTINUE
252 30 NCRIT= I
254 IF(I.GT. 1) NCRIT= I- 1
256 IF(I.GT.NN) NCRIT= NN
258 C
260 C CRITICAL NODE
262 C
264 IF(NCRIT.GE. 1) GO TO 35
266 GO TO 36
268 35 TC(J+ 1)= TC(J,J)* (1. - (FM(I,J)* TAFP+C*H/(RH0*CP*DEL TAT))* 2.) +
&TC(NCRIT,J)* (FM(C(NCRIT)* TAFP* 2.) + TAMF* 2.* 0/H/(RH0*CP*DEL TAT))
272 GO TO 37
274 36 TNCRIT(J+ 1)= TNCRIT(J)* (1. - FM(C(NCRIT)* TAFP)) + TC(NCRIT,J)
276 &FM(C(NCRIT)* TAFP)
278 37 CONTINUE
280 C
282 C TOP OF TANK (ABOVE CRITICAL NODE)
284 C
286 IF(NCRIT.GE. 1) GO TO 45
288 DO 40 K= 1,NCRIT- 2
290 I=NCRIT-K
292 IF(I.LE. 1) GO TO 40
294 FMTE= FM- FM(I)
296 TC(I,J+ 1)= TC(I,J)* (1. - 2.* ATX- HPT+ FMTE* TAFP)
298 & T(I+ 1,J)* (ATX- TAFP*FMTE)+ T(I- 1,J)* ATX+ HPT* TAMF
300 40 CONTINUE
302 C
304 C BOTTOM OF TANK (BELOW CRITICAL NODE)
306 C
308 45 CONTINUE
310 DO 50 I=NCRIT+1,NN-1
312 IF(I*GE*NN) GO TO 50
314 FMTF=FM
316 T(I,J+1)=T(I,J)*((1-2.*ATX*FMTF*FARX*HPT)+(1+1/J)*ATX+
318 &T(I-1,J)*((2.*ATX+FMTF*FARX)+TM*B*HPT)+(1+A/(DEL*P)))
320 TC(I,J+1)=T(I,J+1)
322 50 CONTINUE
324 C
326 C TOP NODE
328 C
330 IF(NC*EO*1) GO TO 98
332 I=1
334 FMTF=FM-RMC(I)
336 T(I,J+1)=T(I,J)*((1-2.*ATX+FMTE*FARX*HPT*(1+A/(DEL*P)))))
338 &T(I-1,J)*((2.*ATX*FARX*FM)+TM*B*HPT*(1+A/(DEL*P)))
340 C
342 C BOTTOM NODE
344 C
346 98 CONTINUE
348 I=NN
350 T(I,J+1)=T(I,J)*((1-2.*ATX-TARX*RM-HPT*(1+A/(DEL*P))))
352 &T(I-1,J)*((2.*ATX+TARX*RM)+TM*B*HPT*(1+A/(DEL*P)))
354 TC(I,J+1)=T(I,J+1)
356 100 CONTINUE
358 60 WRITE(6,/) 'PROFILE=1, PROPERTIES=2, DELTAT AND NT=3, FM AND \ 
\ TIN=4, & H AND TAM=5, PRINT CURVE=6'
360 N=READ(5,/) NFLAG
362 GO TO (1,2,3,4,5,6) NFLAG
366 6 WRITE(6,/) 'INPUT TIME OF DESIRED CURVE IN MINUTES'
368 N=READ(5,/) NTIME
370 NC=1+NTIME/DEL
372 IF(NC*GT*NT) WRITE(6,/) 'TIME GREATER THAN TEST TIME'
374 IF(NC*GT*NT) GO TO 6
376 DO 70 I=1,NN
378 70 WRITE(6,/) T(I,NC), TC(I,NC)
380 WRITE(6,/) TMIX(NT)
382 GO TO 60
384 STOP
386 END
VITA

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