The Development of Hydrodynamic and Kinetic Models for the Plasmasphere Refilling Problem Following a Geomagnetic Storm

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THE DEVELOPMENT OF HYDRODYNAMIC AND KINETIC MODELS FOR THE
PLASMASPHERE REFILLING PROBLEM FOLLOWING A GEOMAGNETIC
STORM

by

Kausik Chatterjee

A dissertation submitted in partial fulfillment
of the requirements for the degree
of
DOCTOR OF PHILOSOPHY
in
Physics

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ABSTRACT

The Development of Hydrodynamic and Kinetic Models for the Plasmasphere Refilling Problem Following a Geomagnetic Storm

by

Kausik Chatterjee, Doctor of Philosophy

Utah State University, 2018

Major Professor: Dr. Robert Schunk
Department: Physics

The refilling of the plasmasphere following a geomagnetic storm remains one of the longstanding problems in the area of ionosphere-magnetosphere coupling. The objective of this dissertation is the development of a hydrodynamic model and a kinetic model for the solution of the plasmasphere refilling problem and the comparison of the obtained results. The hydrodynamic model has been developed using the flux-corrected transport method, a numerical method that is extremely well suited to handling problems with shocks and discontinuities. The hydrodynamic model currently includes three ion species ($H^+, \text{He}^+, O^+$) and two neutral species ($H, O$). The kinetic model on the other hand has been developed using the particle-in-cell method, a method that can be well adapted to plasma modeling problems in regimes where the plasma transport equations are not valid. The kinetic model currently includes one ion ($H^+$) species. The comparison of results obtained from the hydrodynamic model and kinetic model will be the principle focus of this dissertation and will be a significant contribution in the area of space plasma
modeling for closed magnetic field line problems. The ultimate objective of this research is the development of a 3D multi-ion hybrid model for the plasmasphere refilling problem; a model that will couple results obtained with the help of the hydrodynamic model at lower altitudes as inputs to the kinetic model at higher altitudes. With additional development, this hybrid model could also be applied to the study of other complex space plasma coupling problems in closed flux tube geometries.
The Development of Hydrodynamic and Kinetic Models for the Plasmasphere Refilling Problem Following a Geomagnetic Storm

Kausik Chatterjee

The objective of this dissertation is the development of computer simulation-based models for the modeling of upper ionosphere, starting from the first principles. The models were validated by exact analytical benchmarks and are seen to be consistent with experimentally obtained results. This area of research has significant implications in the area of global communication. In addition, these models would lead to a better understanding of the physical processes taking place in the upper ionosphere.
To LeenaBina
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To Bitu: “Yes, I should call more often.”

To my mom: “Thanks for putting up with your very, unreligious son for all those years.”

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To LeenaBina: “The girl who can estimate the area of a circle without an ‘equation’ can do anything. It has been a very difficult last couple of years. But I want you to know that some days I can get up from bed only because of you.”

Kausik Chatterjee
CONTENTS

ABSTRACT ............................................................................................................................................... iii
PUBLIC ABSTRACT .......................................................................................................................... v
DEDICATION ....................................................................................................................................... vi
ACKNOWLEDGMENTS ..................................................................................................................... vii
LIST OF TABLES ................................................................................................................................... ix
LIST OF FIGURES .............................................................................................................................. x

CHAPTER

I. INTRODUCTION ............................................................................................................................... 1
II. SINGLE STREAM HYDRODYNAMIC FORMULATION ............................................................... 10
III. TWO-STREAM HYDRODYNAMIC FORMULATION ............................................................... 29
IV. PARTICLE-IN-CELL BASED SOLUTION METHODOLOGY ............................................ 44
V. SUMMARY AND FUTURE WORK ............................................................................................ 77

REFERENCES ........................................................................................................................................ 83
CURRICULUM VITAE .......................................................................................................................... 90
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Coulomb Collision Mean Free Paths after 20 Minutes</td>
<td>34</td>
</tr>
<tr>
<td>3.2</td>
<td>Coulomb Collision Mean Free Paths after 2 Hours</td>
<td>36</td>
</tr>
<tr>
<td>3.3</td>
<td>Coulomb Collision Mean Free Paths after 15 Hours</td>
<td>39</td>
</tr>
<tr>
<td>3.4</td>
<td>Coulomb Collision Mean Free Paths after 40 hours</td>
<td>42</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>The Neutral Atmosphere – Ionosphere – Plasmasphere</td>
</tr>
<tr>
<td>1.2</td>
<td>A depleted flux tube: The ionosphere is shown in hatched lines</td>
</tr>
<tr>
<td>1.3</td>
<td>Comparison of single stream (top) and two-stream (bottom) models</td>
</tr>
<tr>
<td>2.1</td>
<td>Square wave propagation problem without flux-correction</td>
</tr>
<tr>
<td>2.2</td>
<td>Square wave propagation problem with flux-correction</td>
</tr>
<tr>
<td>2.3a</td>
<td>Concentration profile showing plasma escaping into vacuum</td>
</tr>
<tr>
<td>2.3b</td>
<td>Velocity profile showing plasma escaping into vacuum</td>
</tr>
<tr>
<td>2.4a</td>
<td>Hydrogen ion concentration profile after 10 minutes</td>
</tr>
<tr>
<td>2.4b</td>
<td>Hydrogen ion velocity profile after 10 minutes</td>
</tr>
<tr>
<td>2.5a</td>
<td>Hydrogen ion concentration profile after 30 minutes</td>
</tr>
<tr>
<td>2.5b</td>
<td>Hydrogen ion velocity profile after 30 minutes</td>
</tr>
<tr>
<td>2.6a</td>
<td>Hydrogen ion concentration profile after 1 hour</td>
</tr>
<tr>
<td>2.6b</td>
<td>Hydrogen ion velocity profile after 1 hour</td>
</tr>
<tr>
<td>2.7a</td>
<td>Hydrogen ion concentration profile after 2 hours</td>
</tr>
<tr>
<td>2.7b</td>
<td>Hydrogen ion velocity profile after 2 hours</td>
</tr>
<tr>
<td>2.8a</td>
<td>Hydrogen ion concentration profile after 15 hours</td>
</tr>
<tr>
<td>2.8b</td>
<td>Hydrogen ion velocity profile after 15 hours</td>
</tr>
<tr>
<td>2.9a</td>
<td>Hydrogen ion concentration profile after 20 hours</td>
</tr>
<tr>
<td>2.9b</td>
<td>Hydrogen ion velocity profile after 20 hours</td>
</tr>
<tr>
<td>3.1a</td>
<td>Hydrogen ion concentration profile after 20 minutes</td>
</tr>
<tr>
<td>3.1b</td>
<td>Hydrogen ion velocity profile after 20 minutes</td>
</tr>
</tbody>
</table>
4.6d Two-dimensional velocity distribution function, 9 hours
4.6e Two-dimensional velocity distribution function, 12 hours
4.6f Two-dimensional velocity distribution function, 15 hours
4.6g Two-dimensional velocity distribution function, 18 hours
4.6h Two-dimensional velocity distribution function, 40 hours
4.7a Parallel velocity distribution function, 30 minutes
4.7b Parallel velocity distribution function, 3 hours
4.7c Parallel velocity distribution function, 6 hours
4.7d Parallel velocity distribution function, 9 hours
4.7e Parallel velocity distribution function, 12 hours
4.7f Parallel velocity distribution function, 15 hours
4.7g Parallel velocity distribution function, 18 hours
4.7h Parallel velocity distribution function, 40 hours
4.8a Perpendicular velocity distribution function, 30 minutes
4.8b Perpendicular velocity distribution function, 3 hours
4.8c Perpendicular velocity distribution function, 6 hours
4.8d Perpendicular velocity distribution function, 9 hours
4.8e Perpendicular velocity distribution function, 12 hours
4.8f Perpendicular velocity distribution function, 15 hours
4.8g Perpendicular velocity distribution function, 18 hours
4.8h Perpendicular velocity distribution function, 40 hours
I. INTRODUCTION

During the 1970s, it became widely accepted that the ionosphere-magnetosphere coupling processes were key to understanding the particle and field behavior in each of these two regions. The plasmasphere is the innermost region of the magnetosphere. It consists of low-energy cold plasma, lying above the ionosphere and shown in Fig. 1.1.

![Fig. 1.1 The Neutral Atmosphere – Ionosphere – Plasmasphere. 4 Earth Radii on the Dawn Side and 6-7 Earth Radii on the Dusk Side](Windows2Universe.Org, 2018)

The plasmapause, a region defined by two orders of magnitude drop in plasma density, is the outer boundary of the plasmasphere [Gringauz, 1963]. It has been traditionally described as a donut-shaped mass of cold, well-behaved plasma, co-rotating with the earth and with the ion motion inside it governed entirely by the geomagnetic field. However, more recent satellite images have painted a more complex picture [Sandel et al., 2003]. EUV images have indicated that the plasmasphere does not always co-rotate with the earth and certain behavior and features, such as shoulders, channels, notches and plasma erosion events, are seen to appear in these images. These observed
features are fundamental to understanding the effect of the inner magnetospheric electric field on large and meso-scale plasma distributions and as a result, physical modeling and numerical studies can lead to a better understanding of the mechanisms through which the solar wind and interplanetary magnetic field couple to the inner magnetosphere.

The plasma density inside the plasmapause can be as high as $10^2$ to $10^3$ cm$^{-3}$ and thus the interaction between particles is thought to be dominated by Coulomb collisions [Carpenter & Park, 1973]. However, the plasma density near the plasmapause falls abruptly by a factor of 100 and collisions no longer play the dominant role in thermalizing the plasma outside the plasmapause. Within this context, a research area of significance is the plasmaspheric refilling problem after magnetic storms and substorms, when the flux tubes linked to the outer layers of the plasmasphere are peeled away. Subsequently, these flux tubes are convected sunward to the magnetopause, where they lose the plasma contained within them. This removal of plasma due to convection from the outer layers of the plasmasphere causes the location of the plasmapause to move to lower magnitude latitudes. As the geomagnetic storm subsides, magnetospheric convection returns to its former state, leaving the plasma density in the outer layers of the plasmasphere at a significantly depleted level. The resultant pressure gradient causes the ionospheric plasma linked by magnetic flux tubes to this region of the plasmasphere to flow upward, which initiates the process of refilling of the outer layers of the plasmasphere, shown in Fig. 1.2.

The refilling process is not very easy to model, but there was a general understanding within the community of space physicists, that the magnetic field
Fig. 1.2 A depleted flux tube: The ionosphere is shown in hatched lines. The ionosphere is embedded in the neutral atmosphere and above the ionosphere is the depleted flux tube immediately after a geomagnetic storm [Rasmussen & Schunk, 1988].

Geometry within the plasmaspheric flux tubes was relatively well-behaved and the particle sources, as well as the boundary and initial conditions, could be specified with reasonable accuracy. At the very least, the plasmaspheric flux tubes appeared to be the only part of the magnetosphere where there was at least some fundamental basis for the understanding and quantification of complete plasma behavior. It was also understood that the insights gained from these studies could be useful in the study of other complex plasma coupling problems in closed flux tube geometries. Examples of such problems include the plasma sheet flux tubes, which are populated by ionospheric plasma flowing upwards, followed by the precipitation of particles from them to produce the diffuse aurora, and certain loop structures on the surface of the sun where plasma exchange between the photosphere and the low corona takes place.

As a result, several studies have been undertaken over the years to model and quantify plasma transport between the ionosphere and the plasmasphere and these studies
have led to the development of ionosphere-plasmasphere coupling models. These models based on the solution of the plasma transport equations, fall within two broad categories. In one of these two categories, the nonlinear inertial terms in the plasma transport equations are neglected and thus low-speed, diffusion dominated flow can be modeled. Included in this category are the Sheffield University Plasmasphere Ionosphere Model (SUPIM) [Bailey, et al., 1997], the Ionosphere-Plasmasphere Model (IPM) [Schunk et al., 2004], and the Field-Line Interhemispheric Plasma (FLIP) model [Young et al., 1980]. The FLIP model has recently been integrated into the Ionosphere Plasmasphere Electrodynamics (IPE) model developed at the National Oceanic and Atmospheric Administration/Space Weather Prediction Center (NOAA/SWPC) to facilitate a better understanding of the connection between terrestrial and space weather.

The second category of models that exists in the literature is the so-called “hydrodynamic model,” where the nonlinear inertial terms are retained in the plasma transport equations. It was introduced by Banks et al. [1971] and has subsequently been worked on by many researchers [Khazanov et al., 1984; Singh et al., 1986; Rasmussen & Schunk, 1988]. These theoretical models, along with experimental observations, predict three basic features:

A. Fast, supersonic outflow of $H^+$ ions from the conjugate ionospheres during the early stages of refilling.

B. Counter-streaming of these $H^+$ ions, also observed during these early hours.

C. A state of diffusive equilibrium that is reached when the two ion streams are thermalized and there is a decrease from the supersonic velocities reached during the early stages.
However, these three basic features notwithstanding, there is an open question as to whether there is shock formation near the equator as suggested by Banks et al. [1971], Khazanov et al. [1984] and Singh et al. [1986] or if the shock formation is a numerical artifact introduced by the single-stream model (one \( H^+ \) species along the entire closed magnetic field line) chosen to represent the plasma under consideration (shown in Fig. 1.3). It has been suggested by Rasmussen & Schunk [1988] that the modeling of the plasma as a two-stream flow (separate \( H^+ \) species) from the conjugate hemispheres remedies the problem of these shocks at the equator because of the interpenetration of the streams and the result is also presented in Fig. 1.3.

A recent and well-developed single-stream hydrodynamic model of the low-latitude ionosphere is SAMI2 developed by Huba & Joyce [2000]. The model uses a time-splitting scheme where the dynamics of the system is first solved along the geomagnetic field line and then perpendicular to the field line. The motion along the field line is described by a set of advection/diffusion equations where the advection terms are solved using an implicit donor cell method [Smith, 1985]; the diffusion terms, on the other hand, are backward biased [Smith, 1985] for stability. The motion of the plasma perpendicular to the geomagnetic field is determined based on the conservation laws for mass and magnetic flux and under the assumption that the plasma mass and magnetic flux and under the assumption that the plasma drift is an \( \mathbf{E} \times \mathbf{B} \) drift perpendicular to the magnetic field. Based on the foundation led by SAMI2, a 3D global ionosphere code SAMI3 has been developed [Krall & Huba, 2013] where cross-field transport is included as an \( \mathbf{E} \times \mathbf{B} \) drift.
Fig. 1.3 Comparison of single stream (top) and two-stream (bottom) models. Top: Formation of equatorial shock obtained from the “single-stream” model at the onset of refilling [Singh et al., 1986]. Bottom: Absence of equatorial shock obtained from the “two-stream” model at the onset of refilling [Rasmussen & Schunk, 1988].

In the past decade, the plasmasphere has been the subject of new and renewed interest, particularly because of the new and exciting data from the IMAGE mission [Sandel et al., 2003; Darrouzet et al., 2009]. Apart from a better characterization of electron and ion densities of the plasmasphere during quiet periods, the complex restructuring of the plasmasphere becomes more visible during magnetic storms [Goldstein et al., 2002]. In addition, the erosion of the plasmasphere allows the radiation belts to move towards the earth and influence plasma waves that scatter radiation belt particles [Millian & Thorne, 2007]. With that in mind, one of the objectives of this dissertation is the development of a hydrodynamic plasmasphere refilling model, based on the well-known flux corrected transport (FCT) method of Boris & Book [1976], which is extremely well-suited to problems with shocks and discontinuities. The FCT method has previously been used in Singh et al. [1986] and Rasmussen & Schunk [1988].
In Singh et al. [1986], the plasma stream of a given species was modeled as one species, while in Rasmussen & Schunk [1988], the plasma stream was modeled as two separate streams originating from the conjugate hemispheres. However, the models developed in both papers were limited to one ion species ($\text{H}^+$). The first half of this dissertation will be the development of a hydrodynamic model based on both the “one-stream” and the “two-stream” approaches.

The hydrodynamic model developed in this dissertation is however predicated on the assumption that the number densities of the ions and neutrals under consideration are significant enough to justify the validity of the transport equations, and hence, is not the right assumption at high altitudes. Additionally, the probability density functions of the ion species must be close to Maxwellian to justify their validity, an assumption that does not hold true in the early to middle stages of the refilling process. Simply put, even if the density functions are Maxwellian at the base altitude, the plasma ‘flowing up’ is non-Maxwellian. As a result, at high altitudes, one needs to move to kinetic modeling of the plasmasphere refilling problem, where the ions and electrons are treated as “particles” moving under the influence of the forces that are included in the physics of the chosen problem. The numerical method chosen for this kinetic modeling is the well-known ‘particle-in-cell’ method [Hockney & Eastwood, 1988; Birdsall & Langdon, 2005]. There are different variants of the particle-in-cell method, but the variant chosen for our dissertation is the “Macroscopic Particle-in-cell” (MPIC), in which electrons are treated as a neutralizing fluid, but the ions are described kinetically.

In space physics applications, the MPIC method has been applied to both “open-line” problems such as the polar wind problem, and “closed-line” problems such as the
plasmasphere refilling problem which is the problem of our interest. Early works in the application of the MPIC method to the polar wind problem were carried out by many scientists including Wilson et al. [1990], Ho et al. [1992] and Horwitz et al. [1994]. Significant research [Barakat et al., 1998; Demars et al., 1999] in the field have been carried out by my doctoral supervisor Prof. R. W. Schunk and his research group at Utah State University, with Prof. A. R. Barakat being the lead scientist in these endeavors. These research efforts have culminated in the development of a three-dimensional global model [Barakat & Schunk, 2006] for the polar wind problem.

On the other hand, the MPIC method has been applied to the plasmasphere refilling problem [Lin et al., 1991; Lin et al., 1992, Wilson et al., 1992]. Recently, a MPIC-based model [Wang et al., 2015] at high altitudes has been coupled to the Field Line Interhemispheric Plasma (Flip) model [Richards & Torr, 1990; Richards et al., 2000] at low altitudes.

Within this context, this dissertation accomplishes its two objectives:

A. The development of an FCT-based hydrodynamic refilling model for the plasmasphere refilling problem along the “$L = 4$” line.

B. The development of a MPIC-based kinetic model for the plasmasphere refilling problem along the “$L = 4$” line.

The ultimate objective of this research is the hybridization of the FCT-based fluid model at low altitudes with a MPIC-based kinetic model at high altitudes. With the FCT-based model, it is possible to model the chemical processes taking place in the ionosphere where the number densities of ions and neutrals are too high for particle simulation. The outputs of the FCT-based model would then serve as inputs to the
MPIC-based kinetic model at high altitudes and that hybrid model would be a significant contribution to the field of space physics.
II. SINGLE STREAM HYDRODYNAMIC FORMULATION

In this chapter, the plasmasphere refilling problem is modeled as a single ion species ($H^+$ ions) along with electrons and refilling along a 1D flux tube, without considering the curvature of the tube and neglecting the effects from Coulomb collisions. The time-dependent continuity and momentum equations are given below:

\[
\frac{\partial n_{i,e}}{\partial t} + \frac{\partial}{\partial x}(n_{i,e} u_{i,e}) = 0,
\]

\[
\frac{\partial n_{i,e} u_{i,e}}{\partial t} + \frac{\partial n_{i,e} u_{i,e}^2}{\partial x} = -\frac{1}{m_{i,e}} \frac{\partial p_{i,e}}{\partial x} + \frac{e}{m_{i,e}} E - n_{i,e} g(x),
\]  

where $t$ is time, $x$ is the spatial coordinate, $n_{i,e}$ is the ion/electron concentration, $u_{i,e}$ is ion/electron velocity, $m_{i,e}$ is the ion/electron mass, $p_{i,e} = n_{i,e} kT$ is the partial ion/electron pressure, $T$ is the constant temperature along the flux tube, $k$ is the Boltzmann constant, $E$ is the electric field and $g(x)$ is the spatially-dependent gravitational force. Imposing quasi-neutrality, gives rise to $n_i = n_e$ and neglecting the electron mass in the electron momentum equation gives rise to an expression for the electric field given by $E = -(kT/en_i)(\partial n_i/\partial x)$. The substitution of this electric field in the ion momentum equation gives rise to the set of equations given by

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0
\]

\[
\frac{\partial n_i u_i}{\partial t} + \frac{\partial n_i u_i^2}{\partial x} = -\frac{2kT}{m_i} \frac{\partial n_i}{\partial x} - n_{i,e} g(x)
\]

and this set of equations is solved using the FCT method. The details of the method are presented below, as well as the details of the scheme used in this work, adapted from
The system of equations in Eq. (2.2) can be rewritten as

\[
\frac{\partial \mathbf{n}}{\partial t} + \frac{\partial \mathbf{P}}{\partial x} + \mathbf{Q} = 0,
\]

\[
\mathbf{n} = \begin{Bmatrix} n_i \\ n_i u_i \end{Bmatrix}, \quad \mathbf{P} = \begin{Bmatrix} n_i u_i \\ n_i u_i^2 + \frac{2kT}{m_i} n_i \end{Bmatrix}, \quad \mathbf{Q} = \begin{Bmatrix} 1 \\ n_i g(x) \end{Bmatrix}.
\] (2.3)

From a numerical standpoint, the difficulty lies in the fact that immediately after a geomagnetic storm, there is a sharp discontinuity at the ionosphere-plasmasphere boundary. Had it not been for this discontinuity, a second-order scheme such as the Lax-Wendroff [Hoffmann & Chiang, 2000] would have been sufficient, where the numerical method itself does not introduce any diffusion in the problem. The fundamental philosophy behind flux-correction is that “diffusion” is artificially introduced at spatial points where shocks and discontinuities are present. To formulate a flux-correction based solution, a solution based on the Lax-Wendroff scheme denoted as \( \mathbf{n}_i^{LW} \), is first introduced, where the subscript \( i \) represents the spatial index and the superscript \( k \) represents the time index. The Lax-Wendroff scheme is a two-step scheme given by

\[
\mathbf{n}_i^{k+1/2} = \frac{1}{2} \left( \mathbf{n}_i^k + \mathbf{n}_{i+1}^k \right) - \frac{\Delta t}{2\Delta x} \left( \mathbf{p}_i^k - \mathbf{p}_{i+1}^k \right) - \frac{\Delta t}{2} \left( \mathbf{Q}_{i+1}^k - \mathbf{Q}_i^k \right)
\]

\[
\mathbf{n}_i^{k,LW} = \mathbf{n}_i^k - \frac{\Delta t}{\Delta x} \left( \mathbf{p}_i^{k+1/2} - \mathbf{p}_{i+1}^{k+1/2} \right) - \Delta t \left( \mathbf{Q}_{i+1}^{k+1/2} - \mathbf{Q}_i^{k+1/2} \right)
\] (2.4)

In the flux-corrected scheme developed in this work, a diffusive flux is generated after the first step using

\[
\mathbf{p}_{i+1/2}^{k,D} = \nu_{i+1/2} \left( \mathbf{n}_{i+1}^k - \mathbf{n}_i^k \right),
\] (2.5)

while an anti-diffusive flux is generated after the second step and defined as follows

\[
\mathbf{p}_{i+1/2}^{k,AD} = \mu_{i+1/2} \left( \mathbf{n}_{i+1}^{k,LW} - \mathbf{n}_i^{k,LW} \right).
\] (2.6)

There can be many choices for the diffusion and anti-diffusion coefficients, but a very
widely used choice [Kuzmin et al., 2012] is given below:

\[ v_{j+1/2} = \alpha_0 + \alpha_1 \left( u_{i+1/2} \frac{\Delta t}{\Delta x} \right)^2, \]

\[ \mu_{j+1/2} = \alpha_0 + \alpha_2 \left( u_{i+1/2} \frac{\Delta t}{\Delta x} \right)^2, \]  

(2.7)

with \( \alpha_0 = 1/6, \alpha_1 = 1/3, \alpha_2 = -1/6 \). The solution with diffusion introduced into it can now be computed using

\[ n_{i}^{k,D} = n_{i}^{k,LIW} + n_{i+1/2}^{k,D} - n_{i-1/2}^{k,D}, \]  

(2.8)

and the variation in the diffusive solution between successive grid points is computed as

\[ \Delta n_{i+1/2}^{D} = n_{i+1}^{D} - n_{i}^{D}. \]  

(2.9)

As mentioned before, the fundamental premise of flux-correction is that diffusion is not required at points in space where the solution is continuous and smooth. As a result, the anti-diffusive flux is modified by comparing the variation in the diffusive solution given by Eq. (2.9) with the anti-diffusive flux \( P_{i+1/2}^{k,AD} \) given by Eq. (2.6):

\[ P_{i+1/2}^{k,FCT} = \sigma_{i+1/2} \max \left[ 0, \min \left[ A, B, C \right] \right] \]

\[ A = \sigma_{i+1/2} \Delta n_{i-1/2}^{k,D}, \quad B = \left| P_{i+1/2}^{k,AD} \right|, \quad C = \sigma_{i-1/2} \left| \Delta n_{i+3/2}^{k,D} \right|, \]  

(2.10)

where \( \sigma_{i+1/2} = \text{sgn} P_{i+1/2}^{k,AD} \). Finally, the modified anti-diffusive flux is applied to the solution and it mitigates the effects of unnecessary diffusion and the flux-corrected solution for Eq. (2.3) is given by

\[ n_{i}^{k+1} = n_{i}^{k,D} + P_{i+1/2}^{k,FCT} - P_{i-1/2}^{k,FCT}. \]  

(2.11)

The results for three benchmark applications are given below:
A. The problem of the propagation of a wave with a constant velocity in a 1D problem domain is considered:

\[
\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}, \quad a > 0, \quad u(x,0) = f(x), \quad u(x,t) = f(x - at) ,
\]

(2.12)

where \( u \) is the propagating quantity of interest, \( x \) is the position coordinate, \( t \) is time, \( a \) is the wave velocity and \( f(x) \) is any given initial condition. However, if the initial condition is a function with spatial discontinuities, such as the square wave, a second-order scheme such as Lax-Wendroff given in Eq. (2.4) produces spatial oscillations as shown in Fig. 2.1.

This problem of spatial oscillations is easily mitigated using the flux-corrected scheme described in Eq. (2.3) – Eq. (2.11), simplified to accommodate the special case of constant velocity transport and the result obtained with the use of flux-correction is given in Fig. 2.2. As expected, flux-correction was able to remedy the problem of oscillation at the edges, but at the expense of the broadening of the solution resulting from the introduction of diffusion, which begs the question if the flux-corrected method can provide the required level of accuracy for the plasmasphere refilling problem. With that in mind, a problem similar in spirit to the refilling problem for which an analytical solution exists is chosen as Benchmark Problem B.
Fig. 2.1 Square wave propagation without flux-correction. A square wave with constant amplitude propagating through a 1D problem domain: Lax-Wendroff scheme without flux-correction. The initial square wave is given in green, the square wave after 1 second is given in red. The fluctuations on the edges arise from the discontinuity in the initial square wave at the edges and the use of a second order algorithm.

Fig. 2.2 Square wave propagation with flux-correction. A square wave with constant amplitude propagating through 1D problem domain: Lax-Wendroff scheme with flux-correction. The initial square wave is given in green, the square wave after 1 second is given in red. The fluctuations on the edges observed in Fig. 2.1 are eliminated by use of flux-correction.
B. The problem in Eq. (2.1) is simplified further by ignoring the contribution from the gravitational force and the result is isothermal plasma expanding into vacuum. The specifics of the problem and solution to this problem is given below in Eq. (2.13). In this problem a constant, sub-sonic drift velocity is imposed on the plasma, and is a generalization of the *self-similar* solution in Schunk & Nagy [2009] where the initial velocity of the plasma was zero:

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial P}{\partial x} &= 0, \quad n = \left\{ \begin{array}{c} n_0 u \\ \frac{P}{m + nu^2} \end{array} \right\}, \\
x &\leq 0: n(x,0) = n_0, u(x,0) = u_0 \\
x &> 0: n(x,0) = 0, u(x,0) = 0 \\
x &\leq (u_0 - u_{th})t: n(x,t) = n_0, u(x,t) = u_0 \\
x &> (u_0 - u_{th})t: n(x,t) = \frac{n_0}{\frac{n_0 u_0}{u_{th}} + 1}, u(x,t) = u_{th}(\xi + 1)
\end{align*}
\]

Fig. 2.3 Concentration profile showing plasma escaping into vacuum. Other than the concentration gradient, the plasma has an initial drift velocity. The initial concentration is shown in green, the analytical solution (after a period of time) is shown in red and the numerical solution (after the same period of time) is shown in blue.
where \( u_{th} = \sqrt{kT/m} \) is the ion-acoustic speed and \( \xi = (x/tu_{th}) \) is known as the self-similar parameter. The solution is characterized by a “rarefaction” wave moving back into the plasma layer with the ion-acoustic speed, and it can be observed from Figs. 2.3a and 2.3b, that there is excellent agreement between numerical solutions and analytical results. The self-similar solution captures the essence of the refilling problem at the onset, when the concentration of the ions is small inside the plasmasphere, while the gradients are big, thus making the sum of the pressure gradient and electric field term in Eq. (2.1) significantly greater than the gravitational force term.

C. In this application example, plasmasphere refilling after a geomagnetic storm is modeled as a single-stream isothermal flow of \( H^+ \) ions, governed by mass and momentum conservation equations. The standard collision-dominated energy
conservation equation is not rigorously valid in the plasmasphere and our eventual objective is to replace the constant temperature in our model with a spatially-varying temperature profile from an empirical model [Titheridge, 1998], which has been seen to produce results more consistent with experiments when it was integrated into the IPM [Schunk et al., 2004]. In this problem, the position coordinate in Eq. (2.2) lies within the range \(0 \leq x \leq X\), where \(X = 58000\) km is the length of the “\(L = 4\)’ magnetic field line. It is assumed that plasma expands into the simulation domain at both ends \((x = 0\) and \(x = X\)). It is also assumed that gravity opposes the inward plasma expansion, so gravity points to the right on one side of the equator and to the left on the other side of the equator. A constant temperature of 3560 K is assumed for both electrons and ions, which corresponds to a thermal velocity \((u_{th})\) of 5.4 km/s. A constant gravitational force is assumed over the entire field line, which is the average of the gravitational force at the equator and at either extremity of the field line:

\[
g(x) = \begin{cases} 
- g', & 0 \leq x < 0.5X \\
0, & x = 0.5X \\
 g', & 0.5X < x \leq X 
\end{cases} \tag{2.14}
\]

where \(g' = 2.45 \text{ km/s}^2\). The boundary conditions imposed for the concentration and velocity of \(H^+\) ions are given by:

\[
n_i(0, t) = n_o, n_i(X, t) = n_o, u_i(0, t) = 2 \text{ km/s}, u_i(X, t) = -2 \text{ km/s} \tag{2.15}
\]

where \(n_o\) is the concentration at the extremities of the flux tube. The constant gravitational force given in Eq. (2.14) and the boundary conditions given by Eq. (2.15) give rise to a steady-state solution given by:
\[ n_i^s = n_0 e^{-\frac{x}{2u_0^2}}, \quad 0 < x < 0.5X \]
\[ n_i^s = n_0 e^{\frac{(x-X)}{2u_0^2}}, \quad 0.5X < x < X \]  \hspace{1cm} (2.16)
\[ u_i^s = 0, \quad 0 < x < X \]

which is obtained by setting \( \partial n_i / \partial t, \partial u_i / \partial t \) and \( \partial u_i / \partial x \) equal to zero in Eq. (2.2). The initial conditions on \( n_i \) and \( u_i \) are assumed to be

\[
\begin{align*}
0 < x < X \\
n_i(x,0) &= 0.00 \ln n_0 . \\
u_i(x,0) &= 0
\end{align*}
\]  \hspace{1cm} (2.17)

In general, the imposition of boundary conditions in numerical models [Oran & Boris, 2001] can be implemented in three different ways:

A. The unknown variables are formulated as linear superpositions of expansion functions where the boundary conditions are satisfied automatically. The method of “flux-correction” cannot be applied systematically to algorithms that use expansions.

B. The second approach is to develop separate finite-difference formulas for boundary cell values in addition to finite-difference formulas developed for points in the interior of the problem domain. This approach can be easy or difficult depending on the complexity of the problem and/or the boundary conditions.

C. The third approach is to develop extrapolations from ghost or guard cells that extends outside the computational domain outside the domain boundary. The cells on the domain boundaries are treated as interior cells and of the three methods, this is the easiest and most flexible.
In our problem of interest, the boundary conditions of interest in Eq. (2.15) are imposed on the external *guard* cells. At small time scales, at the onset of refilling, the concentration gradient term in Eq. (2.2) dominates the gravitational force term. As a result, the concentration and velocity profiles after 10 minutes, shown in Figs. 2.4a and 2.4b, are qualitatively similar to the solution shown in Figs. 2.3a and 2.3b, and the maximum refilling velocity reaches a value greater than 25 km/s, consistent with the values reported in literature [Singh et al., 1986; Rasmussen and Schunk, 1988].

![Graph](image)

**Fig. 2.4a** Hydrogen ion concentration profile after 10 minutes. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube.

The physical picture is that of plasma flowing in from the hemispheres with supersonic velocities being attained at points accessible to the inflowing plasma (Fig 2.4b). On the other hand, at points far from the boundaries, not accessible to the inflowing plasma, the plasma concentration is almost equal to the initial concentration and the gravitational force governs the velocity profile.
Fig. 2.4b Hydrogen ion velocity profile after 10 minutes. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube.

It must be noted that the normalized concentration at each boundary point is less than unity, with the normalized concentration on the guard cell being held to unity. At $t = 0$, there is a sharp discontinuity in the profile (between the two domain boundaries and the two guard cells on each side) as shown in Eq. (2.17) and the drift velocity boundary condition given in Eq. (2.15) is imposed on the guard cells as well. As mentioned before, the discontinuity in profile between the guard cell and the domain boundary is handled with the help of flux-correction where diffusion is added at the expense of slightly diminished accuracy.

The inflowing plasma reaches the equator approximately 30 minutes after the beginning of refilling and the plasma velocity in the equatorial region is zero in the single stream model and a shock is formed with Fig. 2.5a showing a high ion concentration at the equator. The velocity profile provided in Fig. 2.5b is also worth studying. Supersonic ion velocities are observed outside of the shock region, while inside the shock region, the
velocity profile is governed by gravity. The gravitational force acts down on both sides of the equator, with zero velocity being obtained at the equator. As a result, the plasmasphere refills behind the shock front and as refilling continues, there could be

outflowing from the boundary at certain points of time. The inflowing or outflowing nature of the profile can be ascertained from the drift velocity values at the domain boundaries. Throughout the entire simulation time, subsonic drift velocities were observed at domain boundaries. As mentioned before, the shock front moves away from the equator, and the plasmasphere refills behind the shock front. The results after 1 hour are shown in Figs. 2.6a and 2.6b.
Fig. 2.5b Hydrogen ion velocity profile after 30 minutes. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube.

Fig. 2.6a Hydrogen ion concentration profile after 1 hour. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube. The y-axis extends from 0.05 to 1.
Fig. 2.6b Hydrogen ion velocity profile after 1 hour. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube.

The ion velocities inside the “shock region” is gravity dominated and close to zero near the equator with supersonic velocities produced outside of the shock region. The shock front reaches the end-points of the flux tube in approximately 2 hours (Figs. 2.7a and 2.7b). We again note that the normalized concentration of unity is imposed on the guard cell and as a result the concentration at the base altitude is higher than that at all points on the flux tube at all times. Also seen in the concentration profiles are regions of elevated concentrations and regions of depletions, consistent with the refilling from “behind the shock front.”
Fig. 2.7a Hydrogen ion concentration profile after 2 hours. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube. The y-axis extends from 0.05 to 1.

Fig. 2.7b Hydrogen ion velocity profile after 2 hours. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube.
Fig. 2.8a Hydrogen ion concentration profile after 15 hours. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube. The y-axis extends from 0.3 to 1.

The shock fronts described above get reflected at the boundaries and travels back and forth along the flux tube. As refilling continues, the drift velocity transitions from supersonic to subsonic. At 15 hours (Figs. 2.8a and 2.8b), the drift velocities at the boundaries are slightly negative, indicating that at certain points of times, there could be some transport of ions from the plasmasphere to the ionosphere consistent with refilling occurring “behind the shock front.” However, the net transport of ions over the entire refilling period is overwhelmingly from the ionosphere to the plasmasphere, as is expected.

Finally, as can be seen from Figs. 2.9a and 2.9b, in approximately 20 hours, the concentration profile matches the steady-state concentration profile, while the ion
Fig. 2.8b Hydrogen ion velocity profile after 15 hours. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube.

Velocities stay at values approximately two orders of magnitude below the thermal velocity. This refilling time is consistent with the “22-hours refilling time” reported in Singh et al., [1986] for the one-stream model using the FCT method. It should also be borne in mind that the model itself is extremely simplistic because of the assumed linear nature of the field line as opposed to its dipolar geometry, the constant magnitude of the gravitational force term assumed as opposed to its spatially-varying nature, and the lack of a diurnal variation of the ion concentration and velocities at the ionosphere-plasmasphere boundary.

Summarizing, a “single-stream” hydrodynamic model for the plasmasphere
Fig. 2.9a Hydrogen ion concentration profile after 20 hours. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube. The y-axis extends from 0.3 to 1.

Fig. 2.9b Hydrogen ion velocity profile after 20 hours. The concentration normalized to the concentration at the base altitude is imposed on the guard cells. The drift velocity normalized to the hydrogen thermal velocity is imposed on the guard cells. The domain length is normalized to the length of the flux tube.

refilling problem was developed using the FCT method. The model was first validated against exact analytical benchmarks and then applied to the situation of a single ion
species ($H^+$) modeled as a single-stream flow and within half an hour, shock formation was observed at the equatorial region. The refilling process was completed within twenty hours, which was consistent with the refilling period reported in literature [Singh et al., 1986]. In the next chapter, we will report the development of a hydrodynamic refilling model using a two-stream formulation.
III. TWO-STREAM HYDRODYNAMIC FORMULATION

In this chapter, a multi-ion, multi-neutral hydrodynamic model is developed for the plasmasphere refilling problem, where each ion species is formulated as two separate ion streams, with one stream originating from the northern hemisphere and the other originating from the southern hemisphere.

We begin with the time-dependent continuity and momentum equations for a given ion species indicated by the suffix $i$ [Rasmussen & Schunk, 1988; Schunk & Nagy, 2009], but we re-write the equations with the terms related to the cross-section of flux tube on the right-hand side of the equations:

$$\frac{\partial n_i}{\partial t} + \frac{1}{A} \frac{\partial}{\partial s}(An_i u_i) = 0 \quad (3.1a)$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial s}(n_i u_i) = -n_i u_i \frac{1}{A} \frac{\partial A}{\partial s} \quad (3.1b)$$

$$\frac{\partial (n_i u_i^2)}{\partial t} + \frac{1}{A} \frac{\partial}{\partial x}(An_i u_i^2) = - \frac{1}{m_i} \frac{\partial p_i}{\partial s} + \frac{e}{m_i} E(s) - n_i g(s) + \sum_j n_{ij} \nu_{ij} (u_j - u_i) \varphi_{ij} \quad (3.1c)$$

$$\frac{\partial (n_i u_i^2)}{\partial t} + \frac{\partial n_i u_i^2}{\partial x} = - (n_i u_i^2) \frac{1}{A} \frac{\partial A}{\partial s} - \frac{1}{m_i} \frac{\partial p_i}{\partial s} + \frac{e}{m_i} E(s) - n_i g(s) + \sum_j n_{ij} \nu_{ij} (u_j - u_i) \varphi_{ij} \quad (3.1d)$$

where $t$ is time, $s$ is the spatial coordinate along the magnetic field line, $n_i$ is the ion concentration with each species being represented by two streams, $u_i$ is the ion drift velocity, $A$ is the cross section of the flux tube [Schunk & Nagy, 2009], $m_i$ is the ion mass, $p_i = n_i kT$ is the partial ion pressure, $T$ is the constant temperature along the flux tube, $k$ is the Boltzmann constant, $E$ is the electric field, $g(s)$ is the spatially-dependent


gravitational force along the field line, $\nu_j$ is the momentum transfer collision frequency and $\varphi_j$ is a velocity-dependent correction factor [Rasmussen & Schunk, 1988; Schunk & Nagy, 2009], where $j$ represents any other ion or neutral species. From Eqs. (3.1b) and Eq. (3.1d), the curvature of the magnetic field line is accommodated by the two “area-dependent” source terms in the continuity and momentum conservation equations. The spatially-dependent gravitational force at a given point on the field line defined by $(h, \theta)$ (where $h$ is the altitude from the earth’s surface, $\theta$ is the angle between the radial vector and the magnetic axis, $R$ is the radius of the earth) is given by:

$$g(s) = 9.8 \left( \frac{h}{R+h} \right)^2 \frac{2\cos(\theta)}{(1+3\cos^2(\theta))^{1/2}} ms^{-2}. \quad (3.2)$$

The expression given above is obtained by taking a vector dot product of the “acceleration due to gravity vector” (which is directed radially) and a unit vector along the magnetic field line given in Schunk & Nagy [2009]. Further, imposing quasi-neutrality gives rise to $\sum n_i = n_e$, and neglecting the electron mass in the electron momentum equation gives rise to an expression for the electric field given by

$$E = -\frac{kT_e}{en_e} \frac{\partial n_e}{\partial x}. \quad (3.3)$$

which is used in the plasma transport equation.

In our model, each ion species is modeled by two streams; one originating in the northern hemisphere and the other in the southern hemisphere. In the previous chapter, a single-stream, single species model was validated by analytical benchmarks [Chatterjee & Schunk, 2016] including that of a very “simplified” plasmasphere, where the flux line
was approximated as a straight line, by ignoring its curvature. The refilling results are now presented for the “L=4” magnetic field line and these results include three ions (H⁺, O⁺, He⁺) and two neutral (O, H) species [Chatterjee & Schunk, 2017]. It includes both long-range (ion-ion) and short-range (ion-neutral) collisions [Schunk & Nagy, 2009]. For ion-neutral collisions, both resonant and non-resonant collisions [Schunk & Nagy, 2009] are included in the model. The following ion concentrations are assumed at a base altitude of 500 km: \(N(O^+) = 2 \times 10^3 \text{ cm}^{-3}, \ N(H^+) = 10^3 \text{ cm}^{-3}\) and \(N(He^+) = 0.2 \text{ cm}^{-3}\).

An initial concentration profile spatially declining from the base altitude along the field line \([\cos \theta / \cos \theta_0]^4\) given in Rasmussen & Schunk [1988] is used, where \(\theta\) is the co-latitudinal angle along the flux line and \(\theta_0\) is the co-latitudinal angle at the base altitude. The drift velocity for the initial concentration profile is assumed to be zero. Typical values of the neutral concentration(s) were obtained from the MSIS empirical model [Schunk & Nagy, 2009] of terrestrial neutral parameters. The ion temperature was assumed to be 3560 K and the neutral temperature was assumed to be 1463 K, again typical values reported in Schunk & Nagy [2009].

In Fig. 3.1a, we present the hydrogen concentration 20 minutes after the onset of refilling, normalized to the oxygen concentration at the boundaries. The plasmasphere begins to fill up and we observe interpenetration of North and South streams, but no shocks are observed in the middle, consistent with Rasmussen & Schunk [1988]. At the end-points reached by either stream a discontinuity is observed, which is expected. A lot less obvious is a bump in each stream at the domain boundary it is headed to, which indicates coupling between the streams from the electric field term in the transport equations. As it will be observed, with the passage of time these “discontinuities” and
Fig. 3.1a Hydrogen ion concentration profile after 20 minutes. Stream 1 is from the northern hemisphere and stream 2 is from the southern hemisphere. The concentration is normalized to the oxygen ion concentration at the base altitude.

“bumps” will move back and forth along the flux tube. The origin of these structures is electrostatic in nature. The electric field seen by each stream is affected by the concentration gradient in five other streams.

The Coulomb collision mean free paths are estimated at representative latitude points at various stages of refilling and presented in Tables 3.1 to 3.4. They are obtained through estimating the Coulomb collision frequencies between a particular species (designated below by the suffix \(s\)) and a particular target (designated by the target \(t\)), adding the collision frequencies together for different targets and then dividing the thermal velocity of the species by the total collision frequency. The collision frequency between a given species and a target is given by [Schunk & Nagy, 2009]:

\[
\vartheta_{st} = \frac{16}{3} \frac{n_s m_t}{m_s + m_t} \Omega_{st}^{(1,1)}
\]

(3.4)

where \(m\) represents the mass with the appropriate suffix, \(n\) represents the concentration
with the appropriate suffix and $\Omega^{1,1}_{st}$ is a *Chapman Cowling Collision Integral* given in *Chapman & Cowling* [1970].

It will be seen that the Coulomb collision mean free paths are significantly larger than the length scale of these electric field induced discontinuities at all points of time during the refilling process. As the result, the Coulomb collisions cannot play a thermalizing role and the hydrodynamic model which assumes a Maxwellian distribution at a given temperature is itself not valid. This, in turn presents the rationale for the need of a kinetic model, which will be presented in Chapter 4.

Fig. 3.1b Hydrogen ion velocity profile after 20 minutes. Normalized to the thermal velocity of hydrogen atoms at 3560 K. Velocities from the northern to the southern hemisphere are taken to be positive.

In Fig. 3.1b, the hydrogen ion drift velocity (normalized to its thermal velocity) profile is presented at 20 minutes after the onset of refilling. High supersonic velocities are observed up to the end-points reached by either stream. Beyond these points, the velocity profile is determined largely by gravity. The total plasma velocity averaged over
the contributions of both the streams is also seen to reach supersonic velocities. The concentration “bumps” at domain boundaries in Fig. 3.1a show corresponding velocity “spikes” in Fig. 3.1b. The Coulomb collision mean free paths at representative latitude points are shown in Table 3.1.

<table>
<thead>
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<th>Latitude (Degrees)</th>
<th>Coulomb Collision Mean Free Path (km)</th>
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Table 3.1 Coulomb Collision Mean Free Paths after 20 Minutes.

In Fig. 3.2a, the hydrogen concentration profile 2 hours after the onset of refilling is presented. The discontinuities in the concentration profile move back and forth as the plasmasphere refills. A discontinuity in the concentration profile of any one stream belonging to any one species, drastically changes the E-field and hence bring sharp changes in the concentration profile of any other stream, belonging to any other species.

In Fig. 3.2b, the hydrogen velocity profile at 2 hours after the onset of refilling is presented. The individual stream velocities are still supersonic, but equally important is the fact that spatial gradients in the velocity profile are significant at some locations. Due
to these discontinuities in the concentration profile, the individual stream velocities, as well as the overall velocity exhibit oscillations throughout the refilling process. But the overall trend is towards “sub-sonic” profiles. Again, Coulomb collisions are incapable of smoothening out these bumps and discontinuities (Table 3.2), and the hydrodynamic model itself is not valid.

In Fig. 3.3a, we present the hydrogen concentration profile at 15 hours after the onset of refilling. We see that the total hydrogen concentration has risen significantly, in that the equatorial concentration is one order of magnitude higher than that at 20 minutes. But the discontinuities are still present and “steady-state” has not been reached yet. In Fig. 3.3b, the hydrogen velocity profile at 15 hours after the onset of refilling is presented. The stream velocities and the overall plasma velocities are sub-sonic everywhere and the Coulomb collision mean free paths though smaller in magnitude than

![HYDROGEN ION CONCENTRATION](image)

Fig. 3.2a Hydrogen ion concentration profile after 2 hours. Stream 1 is from the northern hemisphere and stream 2 is from the southern hemisphere. The concentration is normalized to the oxygen ion concentration at the base altitude.
before is large compared to the length scale of these “electric-field induced” discontinuities, as seen in Table 3.3.

Fig. 3.2b Hydrogen ion velocity profile after 2 hours. Normalized to the thermal velocity of hydrogen atoms at 3560 K. Velocities from the northern to the southern hemisphere are taken to be positive.

<table>
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<th>Latitude (Degrees)</th>
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*Table 3.2. Coulomb Collision Mean Free Paths after 2 Hours.*

In Fig. 3.4a, we present the equatorial concentration for all the three ions as a
function of time. We observe that even at 40 hours, the plasmasphere has not completely settled in when it comes to hydrogen, the lightest of the ion species. It also must be borne in mind that our model did not accommodate diurnal ionospheric variations and spatially asymmetric boundary conditions corresponding to solstices. If instead, we included the diurnal variations and simulated, for example, the boundary conditions for summer and winter solstices, we would expect to see an even more dynamic plasmasphere that never saturates in between geomagnetic storms.

In Fig. 3.4b, we present simulation results that demonstrate the presence of heavier ions (helium and oxygen) during the early hours of refilling. The presence of helium has been experimentally observed [Wilford et al., 2003] and our model is consistent with these experimental results in the sense that the helium peak obtained from our model is significantly larger than the oxygen peak.

![Hydrogen ion concentration profile](image)

**Fig. 3.3a** Hydrogen ion concentration profile after 15 hours. Stream 1 is from the northern hemisphere and stream 2 is from the southern hemisphere. The concentration is normalized to the oxygen ion concentration at the base altitude.
Fig. 3.3b Hydrogen ion velocity profile after 15 hours. Normalized to the thermal velocity of hydrogen atoms at 3560 K. Velocities from the northern to the southern hemisphere are taken to be positive.

Fig. 3.4a Ion concentrations at the equatorial crossing altitude of 19000 km. The ion concentrations are normalized to the oxygen ion concentration at the base altitude at the onset of refilling.
Fig. 3.4b Relative concentration of the hydrogen, oxygen and helium ions.

<table>
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</tbody>
</table>

| Table 3.3 Coulomb Collision Mean Free Paths after 15 Hours. |

At a qualitative level, it is easy to understand these refilling results. In our model $O^+$ trails $He^+$, which in turn trails $H^+$. As a result, $H^+$ has the maximum amount of fluctuations and $O^+$ the minimum. The $He^+$ and $O^+$ peaks correspond to the increased concentration level of these ions matching with dips in the $H^+$ level. It also has to be kept
in mind that the introduction of time-dependent boundary conditions at the base altitude will keep the fluctuations alive in time and could enhance the presence of heavier ions even further in time, which along with the inclusion of other minor ion species and neutrals would capture the physics of an even more dynamic plasmasphere.

In Fig. 3.5a, we present the hydrogen ion concentration profile after 40 hours of refilling. “Steady-state” has not been reached yet, but The North and South streams are seen to be heading in the direction of reaching their respective “diffusive equilibrium” states. More importantly, the total hydrogen ion concentration closely matches the analytical solution obtained assuming an infinite flux line [Rishbeth & Garriott, 1969].

---

Fig. 3.5a Hydrogen ion concentration after 40 hours. Stream 1 is from the northern hemisphere and stream 2 is from the southern hemisphere. The concentration is normalized to the oxygen ion concentration at the base altitude.

In Fig. 3.5b, the hydrogen velocity profile after 40 hours of refilling is presented, and it is observed that the stream velocities, the overall velocities and velocity gradients are almost equal to zero, which again is a confirmation of the fact that at 40
hours, the refilling process is close to reaching the steady-state. The Coulomb collision mean free paths are presented in Table 3.4.

Finally in Fig. 3.6a, we show the oxygen concentration profile after 40 hours of refilling and in Fig. 3.6b, we show the helium concentration profile after the same 40 hours. Comparing Fig. 3.5a to Fig. 3.6a, it can be observed that as we go from the plasmasphere base to higher altitudes along the flux tube, there is an $O^+$ to $H^+$ transition. At the plasmasphere base altitude of 500 km, $O^+$ dominates $H^+$ by a factor of 2:1, which changes to a state of $H^+$ dominance at the equatorial plane by orders of magnitude after 40
Table 3.4 Coulomb Collision Mean Free Paths after 40 Hours.

<table>
<thead>
<tr>
<th>Latitude (Degrees)</th>
<th>Coulomb Collision Mean Free Path (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3115</td>
</tr>
<tr>
<td>-20, +20</td>
<td>2997</td>
</tr>
<tr>
<td>-40, +40</td>
<td>2366</td>
</tr>
<tr>
<td>-58, +58</td>
<td>2111</td>
</tr>
</tbody>
</table>

Also as observed from Fig. 3.6a and Fig. 3.6b, $He^+$ dominates $O^+$ for almost the entirety of the flux tube by virtue of its relatively lighter mass.

Fig. 3.6a Oxygen ion concentration profile after 40 hours. Normalized to the thermal velocity of hydrogen atoms at 3560 k. Velocities from the northern to the southern hemisphere are taken to be positive.
Summarizing, a multi-ion, multi-neutral, two-stream hydrodynamic model has been developed for the plasmasphere refilling problem following a geomagnetic storm. The model correctly models high supersonic velocities at the onset of refilling, the presence of heavier ions in the early hours of refilling, the transition from $O^+$ to $H^+$ dominance in the middle stages and correctly estimates the steady-state ion densities after the completion of the refilling process. However, the model itself is predicated upon the assumption of a Maxwellian velocity distribution function at a given temperature, and the estimated Coulomb collision mean free path lengths show that this assumption is invalid. This brings into question the validity of the estimated concentration and velocity profiles during the middle stages of the refilling process. In Chapter 4, we will provide results from a newly-developed “particle-in-cell” based kinetic model, where this assumption was not necessary.
IV. PARTICLE-IN-CELL BASED SOLUTION METHODOLOGY

In Chapter 2 and Chapter 3, the validity of the “flux-corrected transport” based hydrodynamic model was based on the assumption that the number densities of the ions and neutrals under consideration are large enough to justify the validity of the transport equations, an assumption that does not hold true at high altitudes. Additionally, as has been mentioned before, the probability density functions of the ion species must be close to a Maxwellian to justify the validity of the same transport equations. Either one of these assumptions does not hold true in the early to middle stages of the refilling process. From physical reasoning, even if the density functions are close to Maxwellian at the base altitude, only the ions with velocities from the positive half of the distribution function are injected into the plasmasphere and the density functions of the ion species under consideration will deviate from the Maxwellian until steady-state is achieved. As a result, at high altitudes, one needs to move to kinetic modeling of the plasmasphere refilling problem, where the ions are treated as “particles” moving under the influence of the forces that include electric field, gravitational force and collisional effects. The numerical method chosen for this kinetic modeling is the well-known ‘particle-in-cell’ method [Hockney & Eastwood, 1988; Birdsall & Langdon, 2005]. The method itself has many variants and the variant chosen for this dissertation is the “Macroscopic Particle-in-cell” (MPIC) method, in which electrons are treated as a neutralizing fluid, but the ions are described kinetically. The fundamentals of the MPIC method are described below.

At a conceptual level, space plasma can be thought of as a composition of charged and neutral particles, interacting through electric, magnetic, gravitational and collisional
forces. We associate a suffix $i$ with a particle carrying a charge $q_i$, at a position $x_i$, and velocity $v_i$. The particle is acted on by a combination of electric and magnetic forces along with an external force, which for our problems would be a combination of gravitational, magnetic mirror \cite{Chen2016} and collisional forces:

$$F_i = F_{i,ext} + q_i\left[E(x_i) + v_i \times B(x_i)\right]$$

The electric and magnetic fields are produced by the charges of the particles and their motion along with any external charge and current sources. The fields are given as the solution of the Maxwell’s equations. The evolution of the system of particles in space and time is divided into many such “small time steps” given by $\Delta t$, and within each time step, each particle moves only a small distance; after each such step, the fields and the resultant forces are all recomputed. The particle positions and velocities are updated at the end of each time-step. For space plasma problems with large number densities, a Maxwellian velocity distribution is typically assumed in the derivation of the plasma transport equations. On the other hand, at high altitudes the number densities are small, leading to the presence of non-thermalized plasma.

The objective of our particle-in-cell based model is the estimation of the phase space distribution function $f_s(x,v,t)$ for a given species $s$, defined as number density per unit element of the phase space, which in the absence of collisions is defined by the solution of the Vlasov equation \cite{SchunkNagy2009}. We will describe the fundamentals of the method within the context of the Vlasov equation and introduce collisions later in our model.
The Vlasov equation in 1D can be defined as

\[
\frac{\partial f_s}{\partial t} + v \frac{\partial f_s}{\partial x} + a \frac{\partial f_s}{\partial v} = 0, \tag{4.2}
\]

where \(a\) is the acceleration. In our chosen problem of interest, transport occurs along the flux tube and magnetic mirror force \([Chen, 2016]\) plays a role. The Poisson’s equation for the electrostatic scalar potential describes the electric field:

\[
\varepsilon_0 \frac{\partial^2 \varphi}{\partial x^2} = -\rho. \tag{4.3}
\]

The position and time-dependent charge are obtained by integrating over the distribution function in the velocity space and given by:

\[
\rho(x, t) = \sum_s q_s \int f_s (x, v, t) dv \tag{4.4}
\]

In some sense, the PIC method can be regarded as an extension of the well-known finite element method \([Reddy, 2006]\), where the elements themselves are in motion and overlapping with each other. The computational framework of the PIC method is formulated by assuming that the distribution function of each species is given by the superposition of several elements

\[
f_s(x, v, t) = \sum_p f_p(x, v, t). \tag{4.5}
\]

A superparticle represents many physical particles that are near each other in the phase space. The derivation of the equations of motion for the superparticles is based on the understanding that the Vlasov equation is linear in \(f_s\) and the equation satisfied by each superparticle must be the same Vlasov equation. The overall distribution function is
a result of the superposition of the elements and if each element satisfies the Vlasov equation, their superposition will have to satisfy it too. As a result, we can write the Vlasov equation for each superparticle:

$$\frac{\partial f_p}{\partial t} + v \frac{\partial f_p}{\partial x} + a \frac{\partial f_p}{\partial v} = 0$$  \hspace{1cm} (4.6)

The equations of motion for the superparticles are obtained by taking the zeroth order moment of the Vlasov equation, the first order moment in the position coordinate and the second order moment in the velocity coordinate respectively:

$$\frac{dN_p}{dt} = 0, \quad \frac{dx_p}{dt} = v_p, \quad \frac{dv_p}{dy} = a_p.$$  \hspace{1cm} (4.7)

Thus, the PIC method is very intuitive in the sense that the evolution equations of the superparticles have the same structure as the Newton's laws of motion, with the understanding that the forces on a superparticle could be forces that depend on all the other superparticles. One such force that will be handled in this dissertation is the electric field. In most PIC formulations, in the absence of magnetic fields, the electric potential is obtained through the solution of Gauss's law and the electric field is obtained as the derivative of this potential. However, in our problem of interest, the electron temperature is specified, and as a result, an expression for the electric field is obtained in terms of ion densities alone under the assumption of charge neutrality [Schunk & Nagy, 2009] and will be used in our PIC formulation.

The equations given by Eq. (4.7) are the equations of Newtonian mechanics and are simple ordinary differential equations. The most commonly used algorithms for discretizing these algorithms in the PIC literature is the so-called “leap frog algorithm”
which staggers the time levels of the position and velocity by a half time-step. The scheme is given by:

\[ x_{p}^{n+1} = x_{p}^{n} + (\Delta t) v_{p}^{n+\frac{1}{2}}, v_{p}^{n+\frac{1}{2}} = v_{p}^{n+\frac{3}{2}} + (\Delta t)a_{p}(x_{p}^{n}) \]  

(4.8)

where \( a_{p} \) is the acceleration of a superparticle caused by externally specified forces (such as gravitational force in our chosen problem of interest) and forces caused by the collective action of the particles (such as the electric field in our problem of interest). The initial velocity of the first-time cycle is moved by half a time-step using an explicit method:

\[ v_{p}^{1/2} = v_{p}^{0} + (\Delta t)a_{p}(x_{p}^{0}). \]  

(4.9)

We now outline the steps in the PIC algorithm used in this work within a one-dimensional framework:

A. The plasma contains a certain number of superparticles and the one-dimensional problem domain is divided into a certain number of spatial bins or cells.

B. The superparticles are placed in these spatial bins, according to a predetermined probability distribution, as dictated by the physics of the problem.

C. The superparticles are also assumed to follow a known velocity distribution function given from the physics of the problem and are placed in a predetermined number of velocity bins.
D. Under the influence of all the known forces, the superparticles are advanced by one time-step using Eq. (4.9).

E. The particles are then re-arranged in the different spatial bins.

F. Within each spatial bin, the particles are re-arranged in the pre-determined number of velocity bins.

G. After every time step, the velocity distribution of the particles in a given spatial bin, the number of particles in a given spatial bin, and the average velocity of the particles in each bin are available.

H. The forces on the particles are then computed and we go back to Step D.

I. The process completes when a given number of time-steps are completed.

As has been mentioned before the objective of this dissertation is to develop a PIC based kinetic model for the plasmasphere refilling problem following a geomagnetic storm. The refilling results will be provided in the following section. But before doing that, we provide the results of the validation of our PIC model with analytical benchmarks, which are given below:

1. **Benchmark 1**: The first benchmark problem is a problem in the velocity domain with no spatial and time dependence. A charged particle is introduced in a sea of neutrals and is propelled by an external electric field. The goal is to estimate the steady-state velocity distribution function. The framework needed for this problem is extremely simple in the sense that one only needs velocity bins and spatial bins are not needed. The problem is one-dimensional, and the ions and neutrals are particles constrained to move in a line. In the PIC formulation, the motion of an ion is followed for many
collisions, and its velocity is monitored. The following scheme was used [Schunk & Nagy, 2009]:

A. An initial ion velocity was picked and to remove bias the first few hundred collisions were ignored.

B. The mean time to collision with neutrals is a constant and given by \( \tau \). The time interval \( (t_c) \) between consecutive collisions is given by \( t_c = -\tau \ln(r) \), where \( r \) is a uniformly distributed random number between 0 and 1.

C. The ion is followed in the velocity space and its velocity is given by

\[
v_i(t) = v_i(t - t_c) + \frac{qE}{m} t_c
\]

(4.16)

D. At the end of the collision, the ion picks up the velocity of the neutral background, which is picked at random from a Maxwellian distribution function.

E. The process then restarts and is continued for a large number \( (10^6) \) of collisions.

F. The entire velocity space is divided into many bins and the amount of time the particle spends in each bin is proportional to the value of the velocity distribution function at that bin.

In Fig. 4.1a, we provide the results for the case of zero electric field, where the ion assumes the Maxwellian velocity distribution of the neutral background in the steady-state. In Fig. 4.2b, the results are provided for the case of a strong electric field, and the ion distribution function in the steady-state is the well-known drifted Maxwellian
distribution function [Schunk & Nagy, 2009].

Fig. 4.1a Steady-state velocity distribution, zero electric field. The particle-in-cell solution is shown in blue and the analytical solution is shown in red.

Fig. 4.1b Steady-state velocity distribution, finite electric field. The particle-in-cell solution is shown in blue and the analytical solution is shown in red.

2. **Benchmark 2:** For Benchmark Problem 2, we revisit the square wave propagation problem previously solved using the FCT method in Chapter 2. However, the problem is more interesting than the previous problem where the particles had only their drift velocities, given by the solution of the one-
dimensional wave equation. Here, the particles have a given Maxwellian thermal velocity distribution function superimposed on a given drift velocity. In the particle-in-cell formulation, the particles are divided into a certain number of spatial bins within a given region. The particles in a given spatial bin are then arranged into a certain number of velocity bins based on the Maxwellian distribution function. This is a problem with “zero force” and the particles move with the randomly assigned thermal velocity and the constant drift velocity. The algorithmic framework is like the refilling problem, because we need to divide both the physical space and the velocity space into bins. The problem is of course simpler than the refilling problem because of the absence of forces, but the computational framework is the same. The evolution of the square wave is plotted in three limits:

A. Fig. 4.2a shows the result for a problem where a certain number of particles is uniformly distributed in a certain region of space. The particles follow the Maxwellian distribution with finite temperature, but the drift velocity is zero. As expected, after a certain amount of time has elapsed, there will be no net motion of the particles and the particles will diffuse into the adjoining space.

B. Fig. 4.2b shows the result for constant drift velocity and zero thermal velocity. This was the problem that was solved by the FCT method in Chapter 2 (Fig. 2.2), and in Fig. (4.2b), the same result is
reproduced using the “particle in cell” method. Intuitively, the particles have no “thermal motion” and “drifts” into another region of space.

C. Fig. 4.2c shows the result for a finite thermal velocity and a constant drift velocity. Here the finite thermal velocity produces diffusion superimposed on the associated drift. The result is a decrease in the amplitude of the square wave with an increase in the spatial spread.

For all these three cases mentioned above, 10 million particles were used and distributed in 100 spatial and 100 velocity bins. Having established a computational framework, we now present the details of a PIC-based solution methodology for the plasmasphere refilling problem following a geomagnetic storm.

Fig. 4.2a Square wave propagation, zero thermal and constant drift velocity. X-axis: Arbitrary length scale. Y-axis: Number of particles per unit length. In “blue”: The particle density at time zero. In red: The particle density after a certain amount of time has elapsed.
Fig. 4.2b Square wave propagation, finite thermal and zero drift velocity. X-axis: Arbitrary length scale. Y-axis: Number of particles per unit length. In “blue”: The particle density at time zero. In red: The particle density after a certain amount of time has elapsed.

Fig. 4.2c Square wave propagation, finite thermal and constant drift velocity. X-axis: Arbitrary length scale. Y-axis: Number of particles per unit length. In “blue”: The particle density at time zero. In red: The particle density after a certain amount of time has elapsed.

The MPIC method has been applied to both “open-line” problems such as the polar wind problem, and “closed-line” problems such as the plasmasphere refilling problem which is the problem of our interest. Our literature survey shows that early
research in the application of the MPIC method to the polar wind problem were carried out by many scientists including Wilson et al. [1990], Ho et al. [1992] and Horwitz et al. [1994]. Important research [Barakat et al., 1998; Demars et al., 1999] in the field has been carried out by my doctoral supervisor Prof. R. W. Schunk and his research group at Utah State University. These research efforts have resulted in the development of a three-dimensional global model [Barakat & Schunk, 2006].

On the other hand, the MPIC method has been applied to the plasmasphere refilling problem [Lin et al., 1991; Lin et al., 1992, Wilson et al., 1992]. Recently, a MPIC-based model [Wang et al., 2015] has been coupled to the Field Line Interhemispheric Plasma (Flip) model [Richards & Torr., 1990; Richards et al., 2000].

The ultimate objective of this research is the hybridization of the “flux-corrected transport” based fluid model at low altitudes with a kinetic model at high altitudes. With that in mind, in this section, we revisit the physics of the plasmasphere refilling problem along a flux tube. In Chapter 2, the curvature of the field line was neglected, and we studied the problem as a purely one-dimensional problem. In Chapter 3, the curvature of the field line was incorporated as “source terms” in the conservation equations. The kinetic model is however one-dimensional in the position space, while being two-dimensional in the velocity space. The forces that act on the particles are gravitational force, magnetic mirror force [Chen, 2016] and ambipolar electric field [Schunk & Nagy, 2009]. The gravitational force on the computational cells distributed on the dipolar field line was imposed externally [Schunk & Nagy, 2009]. The derivative of the ambipolar electric potential led to an expression for the electric field already provided in Chapter 2 and Chapter 3. The electrons were assumed to be isothermal and following the
Boltzmann distribution with the assumption of charge neutrality between the electrons and the ions. The magnetic mirror force [Chen, 2016] follows from the invariance of the magnetic moment of the gyrating particle. As the gyrating particle moves from a region with weaker magnetic field to a region of stronger magnetic field, its velocity perpendicular to the magnetic field must increase to keep the magnetic moment constant. The energy of the particle also must be constant and the velocity of the particle parallel to the magnetic field must decrease. If the magnetic field reaches values that are high enough, the velocity parallel to the magnetic field must be zero and the particle is “reflected” back into the region of the magnetic field. For our problem of interest, upward flowing thermal ions flowing from the ionosphere because of the pressure gradient become trapped within the flux tube and these trapped ions cause refilling of the plasmasphere.

Fig. 1.2 presents the partially depleted “$L = 4$” flux tube immediately following a geomagnetic storm. The hatched area represents the part of the ionosphere that is a reservoir of singly charged hydrogen ions. Above the hatched region is the depleted plasmasphere, which is the domain of simulation. The base altitude is assumed to be 500 km and the thermal velocity of the hydrogen ions is assumed to be 3560 K. At the base of each one of the conjugate ionospheres, the ions injected into the flux tube are assumed to have velocity distributions consistent with the half of a Maxwellian that leads to upward flow:

$$f(v_{pa}, v_{pr}) = A_0 \exp \left[ -\frac{v_{pa}^2 + v_{pr}^2}{2v_T^2} \right], v_{pa} \geq 0, A_0 = \frac{2n_0}{(2\pi^2v_T^2)^{3/2}}, v_T^2 = \frac{kT_0}{m}$$ (4.10a)

$$f(v_{pa}, v_{pr}) = 0, v_{pa} \leq 0$$ (4.10b)

Above $v_{pa}$ is the component of ion velocity that is parallel to the flux tube, $v_{pr}$ is the
component of the ion velocity that is perpendicular to the flux tube and \( n_o \) is the density of hydrogen ions at the base altitude taken to be equal to 2000 cm\(^{-3} \). Electron temperature is assumed to be equal to the thermal velocity of the injected ions (3560 K) and held constant throughout the simulation. At the onset of simulation, the flux tube is not considered to be perfectly empty and an initial concentration profile (given in Chapter 3) is assumed.

The model includes the effects of small-angle Coulomb collision through the methodology suggested by Takizuka & Abe [1977]. In this formulation [Barghouti, 1994], a pair of hydrogen ions within a given spatial cell is chosen at random. The Coulomb collision does not change the absolute value of the relative velocity \( U \) but causes its direction to change.

A hydrogen ion makes many small angle scatterings in a time period \( \Delta t \). The deflection in the relative velocity \( U \) is computed from the accumulation of these many small angle scatterings. The variable \( \delta = \tan(\theta/2) \) is chosen at random from a Gaussian distribution with a mean of zero, and the variance of the distribution is given by

\[
VAR = \frac{2\pi e^4 n_i \lambda}{m_H u^3} \Delta t \quad (4.11)
\]

where \( n_i \) is the Hydrogen ion density, \( m_H \) is the hydrogen mass, \( e \) is the electronic charge and \( \lambda \) is the Coulomb logarithm [Takizuka & Abe, 1977]. The change in the velocity of each element of the ion pair is given by

\[
v_{i1}' = v_{i1} + \frac{\Delta u}{2}, \quad v_{i2}' = v_{i2} - \frac{\Delta u}{2}
\]

where the velocities after collision are denoted by primes. Eq. (4.12) indicates conservation of momentum and energy for the colliding pair and as a result for the entire population. In this dissertation, the simulation results for 1 million and 10 million
superparticles are provided. Reduced statistical error has been achieved for the case of 10 million particles, thus demonstrating the validity of the developed model.

In Figs. 4.3, 4.4 and 4.5 respectively, the number density, parallel temperature and perpendicular temperature are presented as a function of latitude. These results are presented from the early hours of refilling up until forty hours, and there is no significant change in the density and temperature values after that time. From the density plots, we observe that the refilling is smooth with no signature of any shock waves.

To address this issue in detail, we summarize the origins of shock waves in the hydrodynamic plasma. As early as the early 70s, Banks et al. [1971] hypothesized that two supersonic ion streams originating at the base altitudes of the conjugate ionospheres as a result of strong density gradients, meet each other at the magnetic equator. At this point, a pair of shock fronts is triggered, and these shock fronts travel down to the base of the flux tube. Relatively dense warm plasma fills up the space between these shock fronts. As a result, it can be noted that the primary mechanism for refilling during the early hours of refilling is the initiation and downward propagation of these pair of shock waves. Singh et al. [1986] analyzed this refilling problem with a hydrodynamic model, that assumed the plasma to be isothermal as well as isotropic. In Chapter 2 of this dissertation, a one-stream model similar to Singh et al. [1986] has been developed which produces refilling of the plasmasphere behind the shock front during the early hours of the refilling.

However, several researchers [Singh et al., 1986; Rasmussen & Schunk, 1988; Singh, 1988; Singh, 1990] have observed that from a physical standpoint, two supersonic streams would be interpenetrating at the equator and would not produce a shock front.
However, in a “single-stream” hydrodynamic model, a shock wave is created where the two supersonic streams meet, because the velocity of the “one stream flow” is forced to go to zero at that point. Thus, the shock wave at the equator is believed to be a numerical artifact by most scientists and researchers in the field.

The problem of “shock formation” at the equator has been remedied through the development of “two-stream” hydrodynamic models [Singh, 1988; Rasmussen & Schunk, 1988; Singh & Torr, 1990]. These models do not form shock fronts at the equator where the two streams meet. Instead, they are formed at the base altitude, where the ions pile up. This “shock formation” at the boundaries can also be observed from the results presented in Chapter 3 of this dissertation. The time taken for these shocks to dissipate is a little bit higher than in the single stream model and as a result, the refilling times in two-stream models are a bit higher than in the single-stream model, as seen from the literature referenced above, as well as the two models developed in Chapter 2 and Chapter 3 of this dissertation.

Within the context of the refilling results from hydrodynamic models, we turn our attention to the refilling results shown in Fig. 4.3. The concentration plots from 30 minutes to 40 hours clearly show that refilling is always taking place from the two conjugate ionospheres and never from the middle. The parallel temperature in the equatorial region during the early hours of refilling is very high, reaching a value of 30000 K after 30 minutes. However, this temperature has been referred to as a “pseudo temperature” in literature [Wilson et al., 1992] which is a result of the fact, that “one”
Fig. 4.3a Normalized concentration, 1 million simulation particles. The concentration is normalized to 1000 particles per cubic centimeter. “$L = 4$” flux tube and snapshots at selected times during the refilling.

Fig. 4.3b: Normalized concentration, 10 million simulation particles. The concentration is normalized to 1000 particles per cubic centimeter. “$L = 4$” flux tube and snapshots at selected times during the refilling.

temperature is used to describe “two” counter streaming beams. After about 18 hours, this temperature comes down to a value of around 3560 K, and the two counter streaming beams coalesce and morph into one beam with a Maxwellian distribution. During the early hours of refilling, the perpendicular temperature decreases with altitude consistent with the inverse of magnetic field drop-off related to the conservation of magnetic dipole
moment of particles [Chen, 2016]. With the passage of time, the perpendicular temperature increases to values near 3560 K, with energy being provided from the flow of the two counter streaming beams in the parallel direction.

The time evolution of the parallel and perpendicular temperature contains some interesting physics. From Figs. 4.4 and 4.5, it can be seen that the parallel and perpendicular temperatures evolve to almost equal values in between 18 hours and 40 hours. It can also be noted that the parallel temperature remains above the perpendicular temperature for the entire simulation time and both temperatures remain above 3560 K in the neighborhood of the magnetic equator. This can be attributed to the fact that as the ions flow to the equator, they gain kinetic energy from the ambipolar electric field; then because of collisions, parallel energy is converted to thermal energy.

![Parallel temperature graph](image)

Fig. 4.4a Parallel temperature, 1 million simulation particles. “L = 4” flux tube and snapshots at selected times during the refilling process.
Fig. 4.4b Parallel temperature, 10 million simulation particles. “$L = 4$” flux tube and snapshots at selected times during the refilling process.

Fig. 4.5a Perpendicular temperature, 1 million simulation particles. “$L = 4$” flux tube and snapshots at selected times during the refilling process.
The value of the ambipolar electric field depends on the increase in potential from the equator to the base of the flux tube, which depends on the difference in densities between these two points in space. This can be attributed to the assumption of Boltzmann-distributed isothermal electrons and also to the fact that the gravitational potential difference is significantly smaller than the electric potential difference. It can also be noted that centrifugal force stemming from co-rotation would make this potential energy difference even larger. However, as time progresses, the equatorial density increases which leads to a decrease in the potential drop.

The parallel temperature is always higher than the perpendicular temperature. This is because of the fact that at the onset of refilling, ions gain kinetic energy in the parallel direction because of the pressure gradient and the ambipolar electric field. The parallel kinetic energy then gets slowly converted to perpendicular energy. As a result, the parallel temperature decreases, and perpendicular temperature increases throughout refilling; in the final stages of simulation, the parallel temperature was seen to be just
slightly higher than the perpendicular temperature.

In Fig. 4.6, several velocity space plots are provided at the magnetic equator for the same times as in Figs. (4.3 – 4.5). These plots are produced through putting particles in velocity bins based on their parallel and perpendicular velocities near the equatorial region. The earliest reported plots were at 30 minutes when the velocity space plots corresponded to that of two streams whose velocity distributions peaked around 25 km/s. At 30 minutes, the parallel temperature was 30000 K and the perpendicular temperature was about 100 K. Subsequently to that time, the parallel temperature decreased, and the perpendicular temperature increased. As refilling continued, the spread of the perpendicular velocity distribution and perpendicular temperature increased. Around 15 hours, the two streams morph into one, and we see “one distribution” that is close to a Maxwellian.

![Fig. 4.6a Two-dimensional velocity distribution function, 30 minutes. There is a four orders of magnitude difference between regions of highest magnitude (yellow) and regions of smallest magnitude (deep blue).](image)
Fig. 4.6b Two-dimensional velocity distribution function, 3 hours. There is a four orders of magnitude difference between regions of highest magnitude (yellow) and regions of smallest magnitude (deep blue).

Fig. 4.6c Two-dimensional velocity distribution function, 6 hours. There is a four orders of magnitude difference between regions of highest magnitude (yellow) and regions of smallest magnitude (deep blue).
Fig. 4.6d Two-dimensional velocity distribution function, 9 hours. There is a four orders of magnitude difference between regions of highest magnitude (yellow) and regions of smallest magnitude (deep blue).

Fig. 4.6e Two-dimensional velocity distribution function, 12 hours. There is a four orders of magnitude difference between regions of highest magnitude (yellow) and regions of smallest magnitude (deep blue).
Fig. 4.6f Two-dimensional velocity distribution function, 15 hours. There is a four orders of magnitude difference between regions of highest magnitude (yellow) and regions of smallest magnitude (deep blue).

Fig. 4.6g Two-dimensional velocity distribution function, 18 hours. There is a four orders of magnitude difference between regions of highest magnitude (yellow) and regions of smallest magnitude (deep blue).
Fig. 4.6h Two-dimensional velocity distribution function, 40 hours. There is a four orders of magnitude difference between regions of highest magnitude (yellow) and regions of smallest magnitude (deep blue).

In Figs. 4.7a-h, we provide the line plots for the parallel velocity distribution at the magnetic equator for the same times as before, which are obtained by integrating the two-dimensional velocity distributions of Fig. 4.6a-h with respect to the perpendicular velocity variable. The parallel velocity distributions also corresponded to the physical picture of two streams merging into one.

On the other hand, in Figs. 4.8a-h, we present a similar set of line plots for the perpendicular velocity distribution, obtained by integrating the velocity distributions of Fig. 4.6a-h with respect to the parallel velocity. At 30 minutes, the distribution function had a very narrow spread, that gradually transformed into a Maxwellian over the refilling process. The distributions in Figs. 4.7a-h and Fig. 4.8a-h are normalized to an area of unity under the curve and in each frame, we place a Maxwellian at 3560 K for comparison.

From Figs. 4.7a-h and 4.8a-h, a very interesting observation can be made. It can be seen, that there is consistently more “noise” at the peak of the distribution functions.
Fig. 4.7a Parallel velocity distribution function, 30 minutes. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.

Fig. 4.7b Parallel velocity distribution function, 3 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.
Fig. 4.7c Parallel velocity distribution function, 6 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.

Fig. 4.7d Parallel velocity distribution function, 9 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.
Fig. 4.7e Parallel velocity distribution function, 12 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.

Fig. 4.7f Parallel velocity distribution function, 15 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.
Fig. 4.7g Parallel velocity distribution function, 18 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.

Fig. 4.7h Parallel velocity distribution function, 40 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.
Fig. 4.8a Perpendicular velocity distribution function, 30 minutes. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.

Fig. 4.8b Perpendicular velocity distribution function, 3 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.
Fig. 4.8c Perpendicular velocity distribution function, 6 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.

Fig. 4.8d Perpendicular velocity distribution function, 9 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.
Fig. 4.8e Perpendicular velocity distribution function, 12 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.

Fig. 4.8f Perpendicular velocity distribution function, 15 hours. Red: Numerical simulation results. Blue: Maxwellian at 3560 K for comparison.
This particular observation is “unusual” because there are more particles at the “middle” than at the “tail.” It is my belief that these plots are indicative of additional, interesting physics that I will explore in my future research endeavors. In Chapter 5, I will conclude this dissertation with a summary and a description of my future research plans.
V. SUMMARY AND FUTURE WORK

Summarizing, the objective of this dissertation was the development of a hydrodynamic model and a kinetic model for the plasmasphere refilling problem and the comparison of the obtained results. The hydrodynamic model was based on the well-known “flux-corrected transport method,” a numerical method very well-suited to problems with shocks and discontinuities. The method was validated with three important analytical benchmarks and excellent agreement was obtained between exact, analytical solutions and numerical results. These problems were chosen to build the computational framework required for the plasmasphere refilling problem and we will summarize the validation benchmarks briefly.

First, the FCT method was applied to the square wave propagation in Benchmark Problem A of Chapter 2. This is a very important benchmark problem because it captures wave propagation with spatial discontinuities at the edges. These spatial discontinuities [Hoffmann & Chiang, 2000] cause oscillations in higher-order finite-difference based algorithms (Fig. 2.1) which are mitigated through flux-correction (Fig. 2.2).

Next, the FCT method was applied to the problem of isothermal plasma expanding into vacuum (Benchmark Problem B) of Chapter 2. This problem is similar to the plasmasphere refilling problem in its early stages, in the absence of gravity and collisional effects. The model correctly captures the “rarefaction wave” (Fig. 2.3a) and supersonic ion drift velocities (Fig. 2.3b).

In the last and final benchmark problem [Benchmark Problem C, Chapter 1], the FCT method was applied to a simplified 1D refilling problem with only the hydrogen ion
and a constant gravitational force, where the plasma was modeled as “one-stream.” The FCT method correctly predicts the high supersonic speeds that are observed at the onset of refilling Fig. 2.4b and a shock at the equator approximately 30 minutes after the onset of refilling (Figs. 2.5a-b). It has been stated in literature [Singh et al., 1986; Rasmussen & Schunk, 1988; Singh, 1988; Singh, 1990] that this shock is a numerical artifact originating from the fact that the plasma is modeled as “one-stream” even though there are two separate plasma streams originating from the two hemispheres. After the shock formation, the plasmasphere is seen to refill behind the shock front and the nature of the flow transitions from supersonic to subsonic. (Figs. 2.6 to 2.9). Finally, as seen from Fig. 2.10a and Fig. 2.10b, steady-state is reached in around 20 hours with zero drift velocity (as expected) and after the successful completion of these validation examples, the FCT-based model was applied to a multi-ion refilling problem in Chapter 3.

In Chapter 3, the hydrodynamic model was applied to the refilling problem with three ions (\(H^+\), \(He^+\) and \(O^+\)) and two neutrals (\(H\) and \(O\)). Each ion species was formulated as two separate ion streams, with one stream originating from the northern hemisphere and the other originating from the southern hemisphere, and the problem of shock formation originating from the “one stream flow” was remedied, as seen from Fig. (3.1a). From Fig. (3.1b), high supersonic velocities at the onset of refilling can be observed, with drift velocity profile consistent with two “North-bound” and “South-bound” interpenetrating ion streams.

The inclusion of multiple ion species in the model however introduces additional complexities that were not present in a “one-species” model. As the plasmasphere refills, the discontinuities in the profile move back and forth and present the picture of a
plasmasphere that is more “dynamic” than a plasmasphere with one ion. This is because the discontinuity in the concentration profile of any one stream belonging to any one species, drastically changes the electric field, and hence, can bring very sharp changes in the concentration profile of any other stream, belonging to any other species. This can be seen from Fig. 3.2a, which presents the concentration profile of hydrogen ions, two hours after the onset of refilling.

In Fig. (3.3a-b), the hydrogen concentration and velocity profile at 15 hours after the onset of refilling are presented. Even though the stream velocities and the overall plasma velocities are sub-sonic everywhere, there are high velocity gradients at a significantly larger number of locations. The spatial gradient of the velocity presents itself in the momentum equation leading to flow with shocks and discontinuities even after 15 hours of refilling.

In Fig. 3.4a, the equatorial concentration for all the three ions as a function of time are presented. It can be seen that even 40 hours after the onset of refilling, the plasmasphere has not reached steady-state. Besides that, the results presented in this dissertation did not accommodate diurnal variations at the base altitude and spatially asymmetric “solstice-like” boundary conditions. With these factors taken into consideration, we expect to see a plasmasphere that would not saturate between geomagnetic storms.

The simulation results in Fig. 3.4b demonstrate the relatively increased presence of heavier ions (helium and oxygen) during the early hours of refilling. The presence of helium has been experimentally observed [Wilford et al., 2003] and can be described as a
success of our hydrodynamic model. The introduction of other heavy ions such as $NO^+$ would present an even more dynamic plasmasphere.

In Fig. 3.5a, the hydrogen concentration profile after 40 hours of refilling was presented. The North and South streams are identical, which has to be true because of the Equinox-type boundary conditions. In addition, the total ion concentration matches the exact analytical solution [Rishbeth & Garriott, 1969], which again can be described as a validation of the model itself, considering the fact that the analytical solution was derived in the dipolar geometry. From Fig. 3.5b, it can be seen that the stream velocities, the overall velocities and velocity gradients are almost equal to zero, which again is a confirmation of the fact that “steady-state” has been reached.

Finally in Fig. 3.6a and and Fig. 3.6b, the oxygen and helium concentration after 40 hours of refilling are presented and an $O^+$ to $H^+$ transition after the completion of the refilling process was observed. At 40 hours, $O^+$ is dominated by $He^+$ for almost the entirety of the flux tube by virtue of its relatively lighter mass.

Summarizing, the following important results have been obtained from our multi-ion hydrodynamic refilling model and they are stated below:

A. Fast, supersonic outflow of ions in the early hours of refilling.
B. Counter-streaming of ions in the early to middle stages of the refilling process.
C. Diffusiive equilibrium in the steady-state.
D. The presence of heavy ions in the early stages of refilling.
E. Transition from $O^+$ to $H^+$ dominance after the completion of the refilling process.

As has been mentioned before, the hydrodynamic model is restricted to altitudes where the number densities justify the validity of the transport equations. There is also the assumption of Maxwellian density function, which is questionable in the early hours of refilling with supersonic ion speeds. With that fact in consideration, a kinetic model is developed for the same refilling problem with only the hydrogen ion. In Chapter 4, the results from the kinetic model are presented at $T = 3560$ K for only the hydrogen ion.

The kinetic model, one-dimensional in space and two-dimensional in velocity, does not exhibit the shocks exhibited by the one-stream and two-stream models in Chapter 2 and Chapter 3. The physical picture corresponds to the plasmasphere always refilling from the conjugate ionospheres and never from the middle. Refilling happens through the ions that are injected from the ionosphere but are trapped in the flux tube because of the magnetic mirror forces. In this model, the ions “glide” past each other and shocks are not formed.

A comparative study of the hydrodynamic models (in Chapter 2 and Chapter 3) show agreement in certain aspects and disagreement in others. Both the models show high drift velocities at the onset of refilling which settled down to “zero drift” at steady-state. In the hydrodynamic model, the parallel ion temperature was held to 3560 K, but the kinetic model produced parallel and perpendicular ion temperatures as a function of latitude at all hours of refilling. The one- and two-dimensional velocity distribution functions also match the expected physics. As a result, we can say that the dissertation
accomplished its objective of independently developing a hydrodynamic and a kinetic model for the refilling problem.

The application of the hydrodynamic and kinetic models to different application examples would form the basis of future work. As for the hydrodynamic model, the future objectives include its use in the simulation of the following application examples:

A. Summer and Winter solstic type boundary conditions.

B. Inclusion of diurnal variations in the base altitude.

C. Inclusion of additional ion species in the models.

The kinetic model is currently a single species model and needs significantly more development to be applicable to the same application examples. An important milestone for this research would be the coupling of results from the hydrodynamic model at low altitudes as boundary conditions to the kinetic model, which would then be applied to problems at higher altitudes. We believe that such a hybrid model would be a significant contribution to the area of ionosphere-magnetosphere coupling research.
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CURRICULUM VITAE

Kausik Chatterjee

(October 2018)

CAREER OBJECTIVE

Career in research and development in the area of space physics.

EDUCATION

Master of Technology, Nuclear Engineering, Indian Institute of Technology (05/96) GPA: 9.6 (10.0 = A). PhD in Physics, Utah State University, Logan, Utah (expected 12/18). GPA: 3.96 (4.0 = A).

APPOINTMENTS

- 12/11 – present, Utah State University, Logan, UT. Job responsibilities:
  a) Research Scientist, Collaborative research projects with Air Force Research Laboratory at Kirtland Air Force Base as a part of Space Dynamics Laboratory, which is a part of Utah State University Research Foundation
  b) UTAH/NASA Space Grant Consortium Fellow

- 09/05 – 12/11, Cooper Union for the Advancement of Science and Art, New York, NY, Lecturer, Department of Electrical and Computer Engineering. Job responsibilities: Teaching and research

- 09/02-08/05 – California State University, Fresno, CA, Lecturer, Department of Electrical and Computer Engineering. Job responsibilities: Teaching and research

- 08/96– 07/02– LSI Logic, San Jose, CA, Research Scientist. Job responsibilities:
  a) Monte Carlo modeling of electromagnetic propagation scattering in complex IC interconnect structures at high frequencies.
  b) Modeling of amorphous silicon neutron detectors.
RESEARCH INTERESTS

- The research problem I am currently working on is also the subject of my doctoral dissertation and it involves the refilling of the plasmasphere following a geomagnetic storm. The specific research objectives involve the development of hydrodynamic and kinetic models for the solution of the plasmasphere refilling problem and the comparison of the obtained results. At its essence, the hydrodynamic model involves the solution of the plasma transport equations through the use of the flux-corrected transport method, a numerical method that is extremely well suited to handling problems with shocks and discontinuities. The kinetic model on the other hand has been developed using the particle-in-cell method, a method that can be well adapted to plasma modeling problems in regimes where the plasma transport equations are not valid. The methodologies developed are very general and would be applicable to plasma modeling problems in other application areas.

- Modeling of the communication breakdown problem through the modeling of the ionized layer around a space vehicle at the time of re-entry. The numerical method used was the Monte Carlo method.

- Modeling of amorphous silicon neutron detectors using the Monte Carlo method.

- Modeling of electromagnetic propagation/scattering in problems where the inputs and system parameters varied in the statistical sense. The numerical method used was the Method of Moments.

- Monte Carlo modeling of electromagnetic propagation scattering in complex IC interconnect structures at high frequencies.