5-1987

Variational Principles of Fluid Mechanics and Electromagnetism: Imposition and Neglect of the Lin Constraint

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VARIATIONAL PRINCIPLES OF FLUID MECHANICS AND ELECTROMAGNETISM: IMPOSITION AND NEGLECT OF THE LIN CONSTRAINT

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree
of
DOCTOR OF PHILOSOPHY
in
Physics

Approved:

UTAH STATE UNIVERSITY
Logan, Utah

1987

ACKNOWLEDGEMENTS

I am deeply indebted to a number of individuals who have provided invaluable assistance during the preparation of this dissertation. Firstly, I express my utmost gratitude to my major professor, Dr. Farrell Edwards, for consistently going beyond the call of duty in providing both encouragement and assistance during the entire course of my graduate training, and in particular during the final preparation of this manuscript.

I also thank my committee members for their encouragement and constructive criticism. In particular, I thank Dr. Robert Schunk for providing not only his computer facilities for the production of the final manuscript, but also the assistance of his staff. Appreciation is extended to Sherry Thompson for doing an excellent job of typing a difficult document and to Jean Selzer for providing invaluable assistance in the preparation of the text.

I am forever indebted to my parents, Ross and Maurine Allen, for their constant encouragement and support in all my worthwhile pursuits. I particularly thank my mother for typing the original draft of this manuscript and my father for demonstrating through example the rewards of an advanced education.

Finally, to my wife, Tana Jo, I express my love and affection for her constant encouragement and uncomplaining attitude during the long course of my education.

Ross R. Allen Jr.

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ABSTRACT

Variational Principles in Fluid Mechanics

and Electromagnetism:

Imposition and Neglect of the Lin Constraint

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Utah State University, 1987

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Variational principles in classical fluid mechanics and electromagnetism have sprinkled the literature since the eighteenth century. Even so, no adequate variational principle in the Eulerian description of matter was had until 1968 when an Eulerian variational principle was introduced which reproduces Euler's equation of fluid dynamics. Although it successfully produces the appropriate equation of motion for a perfect fluid, the variational principle requires imposition of a constraint which was not fully understood at the time the variational principle was introduced. That constraint is the Lin constraint. The Lin constraint has subsequently been utilized by a number of authors who have sought to develop Eulerian variational principles in both fluid mechanics and electromagnetics (or plasmadynamics). However, few have sought to fully understand the constraint.

This dissertation first reviews the work of earlier authors concerning the development of variational principles in both the Eulerian and Lagrangian nomenclatures. In the process, it is shown rigorously whether or not the Euler-Lagrange equations which result from the variational principles are equivalent to the generally accepted equations of motion. In particular, it is shown in the case of several Eulerian variational principles that imposition of the Lin constraint results in Euler-Lagrange equations which are equivalent to the generally accepted equations of motion. On

the other hand, it is shown that neglect of the Lin constraint results in Euler-Lagrange equations restrictive of the generally accepted equations of motion.

In an effort to improve the physical motivation behind introduction of the Lin constraint a new variational constraint is developed based on the concept of surface forces within a fluid. The new constraint has the advantage of producing Euler-Lagrange equations which are globally correct whereas the Lin constraint itself allows only local equivalence to the standard classical equations of fluid motion.

Several additional items of interest regarding variational principles are presented. It is shown that a quantity often referred to as "the canonical momentum" of a charged fluid is not always a constant of the motion of the fluid. This corrects an error which has previously appeared in the literature. In addition, it is demonstrated that there does not exist an unconstrained Eulerian variational principle giving rise to the generally accepted equations of motion for both a perfect fluid and a cold, electromagnetic fluid.

(172 pages)

I. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

A. Motivation and plan

1. Motivation

The development of variational principles[1] in classical fluid mechanics and electromagnetism has spanned nearly two centuries. Since Lagrange introduced the first such variational principle for fluid mechanics in the eighteenth century, there has been a rather slow but steady parallel progression of variational principles in both the Lagrangian and Eulerian descriptions of matter. In the Lagrangian description of matter a fluid is referenced to a coordinate system comoving with the fluid. Variational principles in this description specify "fluid element labels" as the independent variables and all field quantities such as mass density, velocity, pressure and temperature are written as functions of those fluid element labels. Variational principles in this description usually require the development of special variational techniques.

In the Eulerian description of matter, a fluid is described in reference to a coordinate system fixed in space. Variational principles founded in this description express the field quantities as functions of space and time. The variations of the field quantities in Eulerian variational principles may be performed according to well-known general techniques. Even so, Eulerian variational principles have progressed somewhat more slowly than their Lagrangian counterparts due to a lack of understanding concerning the need for certain types of variational constraints, in particular, the Lin[2] constraint.

As indicated above, Lagrange introduced a fluid mechanical variational principle in the Lagrangian description in the eighteenth century. Taub[3] introduced a general relativistic version of a Lagrangian description variational principle in 1954. Hawking and Ellis[4] and Schutz and Sorkin[5] subsequently have generalized the variational approach and cast it into a more elegant form (in 1973 and 1977, respectively).

On the other hand, Clebsch[6] proposed the first fluid mechanical variational principle in the Eulerian description in 1859. His approach was limited to isentropic, incompressible fluids. Bateman[7] (1929) and Lamb[8] (1932) independently generalized the Clebsch variational principle to account for the motion of compressible, isentropic fluids. However, it was not until 1968 that Seliger and Whitham[9] successfully produced an Eulerian variational principle which gives rise to completely general equations of motion for a perfect fluid.[10] Even so, their variational principle was based on the "mysterious"[9] Lin constraint, and hence they were unable to adequately explain the reason behind its success. In 1970, Schutz[11] was able to generalize the variational principle of Seliger and Whitham to general relativistic notation.

Since the introduction of the Eulerian variational principle of Seliger and Whitham, a number of authors have contributed to the understanding of the Lin constraint, most notably Schutz and Sorkin,[5] Edwards,[12,13] Henyey,[14] Putterman,[15] Taylor,[16] Allen, Edwards, and Clifton,[17] and Ito,[18] some of whom were perhaps unaware of their contribution at the time. Even so, there has not been a fully exhaustive study of this apparently important variational constraint.

2. *Intent*

The purpose of this dissertation is to

(a) Provide an overview of variational principles in fluid mechanics and electromagnetism;

(b) Demonstrate rigorously the equivalence (or lack of equivalence) of the equations obtained from the variational principles with the equations of motion generally ascribed to the appropriate fluid system;

(c) Clarify a number of misunderstandings in the literature concerning the equations that result from a particular variational principle and correct an erroneous conclusion concerning the constancy of the "canonical momentum";

(d) Contribute to the understanding of the Lin constraint through demonstration of its necessity and through development of an alternative variational constraint which has two important new features; (i) the new version is globally correct whereas the former one is only locally valid; (ii) there is a clear physical basis for the new form whereas the previous one is not physically well motivated.

3. Plan

The remainder of this chapter consists of a review of mathematical concepts and machinery which will be utilized extensively throughout the remainder of the work. These include the general notion of a variational principle, a derivation of the Euler-Lagrange equations and a proof and application of Noether's theorem. Also, two theorems of importance, one due to Schutz and Sorkin[5] the other a standard theorem of differential geometry known as Darboux's theorem, are presented.

Chapter II, "Variational Principles in Fluid Mechanics", includes a presentation of several perfect fluid variational principles. By perfect fluid is meant a neutral, inviscid classical fluid which satisfies Euler's equations of fluid dynamics. Variational principles in both the Lagrangian and Eulerian nomenclatures are presented in both non-relativistic and relativistic formats. In each case the resultant Euler-Lagrange equations are shown to be equivalent to Euler's equations.

Chapter III, "Variational Principles in Electromagnetism", presents a similar array of variational principles as Chapter II, but is concerned with charged, zero-temperature, inviscid fluids. In addition, Chapter III includes discussions of complete versus incomplete variational principles in electromagnetics and of field quantity variable transformations.

In Chapter IV, "Neglect of the Lin Constraint", the variational principles in Chapters II and III are modified slightly through the neglect of the Lin constraint. In this chapter it is directly demonstrated that the Lin constraint is a valid constraint, both mathematically and physically, by showing how the resultant Euler-Lagrange

equations are restrictive of the generally accepted equations of fluid motion.

Chapter V, "Physical Interpretations of the Lin Constraint", reiterates the traditional interpretation of the Lin constraint along with its inherent weaknesses. Subsequently, a new variational constraint is developed which is somewhat similar in appearance and function to the Lin constraint. The constraint helps elucidate the interpretation traditionally attached to the Lin constraint and has the advantage of giving rise to Euler-Lagrange equations which are globally equivalent to the generally accepted Maxwell-Lorentz force set of equations. Imposition of the Lin constraint itself achieves only local equivalence to this set of equations. Next, an appeal is made to the Theorem of Schutz and Sorkin as it is demonstrated that there does not exist an unconstrained variational principle in the usual field quantities of fluid mechanics or electromagnetics which yields unrestrictive equations of motion.

B. Notation and conventions

A relativistic viewpoint will be used throughout much of what follows. Often, however, a more general mathematical framework will be developed and then specialized to cases of physical interest. Also, frequent use will be made of non-relativistic Newtonian mechanics. The metric tensor will be taken to have signature -2 and Greek indices will run from 0 to 3 while Latin indices will run from 1 to 3, except as specifically stated. The Einstein summation convention will be observed throughout unless otherwise specifically stated; i.e., identical upper (contravariant) and lower (covariant) indices in a single term indicate an implied sum over all possible values of that index.

Partial derivatives will occasionally be signified by a comma followed by an index designating the variable of differentiation, while covariant derivatives will be signified by a semi-colon followed by the index of differentiation. Unless otherwise specified, all functions are assumed smooth enough that the derivatives which appear exist and are continuous. SI units will be used.

C. Review of the calculus of variations

All subsequent sections are concerned with the physics of continuous fluid systems, both neutral and charged or electromagnetic fluids. Many of the results, then, should be applicable to neutral fluids and to plasmas in the continuum limit. Since a large number of the results are derived through the methods and nomenclature of the calculus of variations, we digress at this point to a brief review of those items in the calculus of variations most relevant to the future developments contained herein.[19]

The calculus of variations has long been a useful tool for the study of physical problems because it leads readily to the discovery of conserved quantities and in addition serves as an elegant packaging for most theories. However, it should be pointed out that the calculus of variations is not the usual starting point for the development of a physical theory. A theory is usually couched in the framework of the calculus of variations following its initial conception. Nevertheless, after the theory has been formulated in the language of the calculus of variations, a great deal of additional physical insight is often obtained.

1. General variational principle on a fixed region — The Euler-Lagrange equations.

In the calculus of variations one begins with the *fundamental integral* or *action* I where

$$I(Q_j) = \int_R L\left(x^1, ..., x^n, Q_j, Q_{j,k}\right) d^n x. \tag{1.1}$$

I may be thought of as a real-valued function on a normed set of functions Q_j, and as such is often referred to as a *functional*, i.e., a real valued function of functions. L is a scalar or scalar density known as the *Lagrangian* or *Lagrangian density* and is in general a function of the *independent variables* or *coordinates* $x^1, ..., x^n$, i.e.,

those variables over which the integration is performed, and the *dependent variables* or *field quantities* $Q_j, j = 1, ..., m$, which are supposed to be functions of the independent variables. L also depends on the first derivatives of the field quantities with respect to the independent variables. More generally L may be considered as a function of still higher order derivatives of field quantities, but we will not need this more general framework in what follows, hence we do not consider it here. L is assumed to be twice continuously differentiable as a function of each of its arguments.

The field quantities are considered to be C^2 in all their variables; that is, we shall take the *space of admissible functions* to be the set of all twice continuously differentiable functions which map $\mathbf{R}^n$ into $\mathbf{R}$. The *variational principle* consists in "varying" the field quantities throughout the space of admissible functions and determining those values of the field quantities for which the fundamental integral $I, (1.1)$, is *stationary.* This is made more precise in the following paragraph.

Consider the arbitrary transformation, or *variation*, of the field quantities

$$\hat{Q}_j = \hat{Q}_j\left(x^1, ..., x^n; \varepsilon\right); \hat{Q}_j\left(x^1, ..., x^n; 0\right) = Q_j\left(x^1, ..., x^n\right) \tag{1.2}$$

which is parameterized by the scalar ε. Assuming that $\hat{Q}_j$ is continuously differentiable in ε, the transformation (1.2) may be rewritten as

$$\hat{Q}_j\left(x^1, ..., x^n; \varepsilon\right) = Q_j\left(x^1, ..., x^n\right) + \varepsilon\xi_j\left(x^1, ..., x^n\right) + \varepsilon 0(\varepsilon) \tag{1.3}$$

where $\xi_j(x^1, ..., x^n) \equiv \frac{d\hat{Q}_j}{d\varepsilon}_{|\varepsilon=0}$ and $0(\varepsilon) \to 0$ as $\varepsilon \to 0$ according to Taylor's theorem. The requirement that $\hat{Q}_j$ be in $C^2(\mathbf{R}^n)$ for for all $\varepsilon \geq 0$ imposes the condition that ξ_j be in $C(\mathbf{R}^n)$ for each j. The *variation* δI of the fundamental integral is defined to be the principle linear part in ε of $I(\hat{Q}_j) - I(Q_j)$. I is said to be *stationary* at those values of the field quantities Q_j for which δI vanishes with respect to arbitrary variations of the field quantities $\hat{Q}_j$ in the space of admissible functions.

The transformation (1.2) does not involve a transformation of the coordinates and hence is a variation in a fixed region of coordinate space.

To compute δI we first compute the principle linear part of $L(x^1, ..., x^n, \hat{Q}_j, \hat{Q}_{j,k}) - L(x^1, ..., x^n, Q_j, Q_{j,k}) \equiv \hat{L} - L$ in ε using Taylor's theorem:

$$\hat{L} - L = L + \varepsilon \sum_{j=1}^{m} \frac{\partial L}{\partial Q_j} \frac{d\hat{Q}_j}{d\varepsilon}|_{\varepsilon=0} + \varepsilon \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{\partial L}{\partial Q_{j,k}} \frac{dQ_{j,k}}{d\varepsilon}|_{\varepsilon=0} + \varepsilon 0(\varepsilon) - L. \quad (1.4)$$

Using (1.3) we find that $\frac{d\hat{Q}_j}{d\varepsilon}|_{\varepsilon=0} = \xi_j$ and $\frac{d\hat{Q}_{j,k}}{d\varepsilon}|_{\varepsilon=0} = \xi_{j,k}$ so that from (1.4) the principle linear part in ε of $\hat{L} - L$ is $\varepsilon \sum_{j=1}^{m} \left(\frac{\partial L}{\partial Q_j} \xi_j + \sum_{k=1}^{n} \frac{\partial L}{\partial Q_{j,k}} \xi_{j,k} \right)$. Since $I(\hat{Q}_j) - I(Q_j) = \int_R (\hat{L} - L) d^n x$ we then see that

$$\delta I = \varepsilon \int_R \sum_{j=1}^{m} \left(\frac{\partial L}{\partial Q_j} \xi_j + \sum_{k=1}^{n} \frac{\partial L}{\partial Q_{j,k}} \right) d^n x. \quad (1.5)$$

The last term under the integral may be expressed as

$$\sum_{k=1}^{n} \frac{\partial L}{\partial Q_{j,k}} \xi_{j,k} = \sum_{k=1}^{n} \frac{d}{dx^k} \left(\frac{\partial L}{\partial Q_{j,k}} \xi_j \right) - \sum_{k=1}^{n} \xi_j \frac{d}{dx^k} \left(\frac{\partial L}{\partial Q_{j,k}} \right). \quad (1.6)$$

Substituting (1.6) into (1.5) and using the generalized divergence theorem[20] we obtain

$$\delta I = \varepsilon \sum_{j=1}^{m} \int_R \left(\frac{\partial L}{\partial Q_j} - \sum_{k=1}^{n} \frac{d}{dx^k} \left(\frac{\partial L}{\partial Q_{j,k}} \right) \right) \xi_j d^n x + \varepsilon \sum_{j=1}^{m} \oint_\Gamma \xi_j \sum_{k=1}^{n} \frac{\partial L}{\partial Q_{j,k}} n_k d\sigma \quad (1.7)$$

where Γ is the bounding surface of the region of integration R, n_k are the components of the outward normal to the surface and $d\sigma$ is the incremental surface area.

The functions ξ_j in the transformation (1.3) are only restricted by the requirement that they be twice continuously differentiable since in order for I, Equation (1.1) to be stationary with respect to arbitrary variations in the field quantities from the space of admissible functions it must in particular be stationary with respect to

variations of the form (1.3) with ξ_j an arbitrary C^2 function. Conversely, requiring stationarity of I with respect to transformations of the form (1.3) with ξ_j an arbitrary C^2 function is sufficient for the stationarity of I with respect to arbitrary variations of the field quantities in the space of admissible functions. We consider first the restrictions imposed by the particular class of transformations defined by $\xi_j = 0$ for $j \neq i$ and ξ_i an arbitrary C^2 function of its variables which vanishes on Γ, the boundary of R. Requiring $\delta I = 0$ under this particular class of transformations (1.7) gives

$$\int_R \left(\frac{\partial L}{\partial Q_i} = \sum_{k=1}^{n} \frac{d}{dx^k} \left(\frac{\partial L}{\partial Q_{i,k}} \right) \right) \xi_i d^n x = 0, \; i = 1, ..., m \tag{1.8}$$

Since ξ_i is arbitrary and C^1 inside of R (1.8) implies by the fundamental theorem of the calculus of variations[21] that

$$E_i(L) \equiv \frac{\partial L}{\partial Q_i} + \sum_{k-1}^{n} \frac{d}{dx^k} \frac{\partial L}{\partial Q_{i,k}} = 0, \; i = 1, ..., m. \tag{1.9}$$

These equations, (1.9), are referred to as the *Euler-Lagrange equations* and will be frequently utilized throughout the following sections and chapters. Any field quantities which make the action I stationary must satisfy the Euler-Lagrange equations, (1.9).

We now consider a wider class of variations but still require I to be stationary, that is $\delta I = 0$. Since (1.9) constitute necessary conditions for the stationarity of I (1.7) reduces to, with $\delta I = 0$,

$$\sum_{j=1}^{m} \oint_\Gamma \xi_j \sum_{k=1}^{n} \frac{\partial L}{\partial Q_{j,k}} n^k d\sigma = 0. \tag{1.10}$$

Let $\xi_j = 0$ for $j \neq i$ and ξ_i be arbitrary on Γ, then (1.10) requires

$$\sum_{k=1}^{n} \frac{\partial L}{\partial Q_{i,k}} n^k \mid_\Gamma = 0, \; i = 1, ..., m. \tag{1.11}$$

Equations (1.11) constitute boundary conditions which necessarily must be satisfied by the field quantities Q_j in order that I be stationary. Conversely, if Equations (1.9) and (1.11) are satisfied, then by (1.7) δI vanishes whence I is stationary.

In the majority of what follows we will assume that no spatial boundary exists and that the field quantities and their derivatives vanish sufficiently fast at spatial infinity. In addition, the time interval of integration will be taken as arbitrary. Under such circumstances the boundary conditions (1.11) are essentially irrelevant in the sense that they are satisfied identically. Hence, in this special case of interest the satisfaction of the Euler-Lagrange equations (1.9) for all times is sufficient for the stationarity of I. We will therefore be concerned mainly with the Euler-Lagrange equations and will assume the boundary conditions (1.11) to be satisfied *a priori*.

2. General variational principle on a variable region—Noether's theorem

Consider now the transformation

$$\hat{x}^i = \hat{x}^i(x^k, Q_j, Q_{j,k}; \varepsilon^h), \tag{1.12a}$$

$$\hat{Q}_j = \hat{Q}_j(x^k, Q_j, Q_{j,k}; \varepsilon^h), \quad i,k = 1,...,n, \;\; j = 1,...,m, \;\; h = 1,...,r, \tag{1.12b}$$

$$\hat{x}^i(x^k, Q_j, Q_{j,k}; 0) = x^i \text{ and } \hat{Q}_j(x^k, Q_j, Q_{j,k}; 0) = Q_j, \tag{1.12c}$$

which is assumed invertible in the sense that by substituting $Q_j = Q_j(x^k)$ and $Q_{j,i} = Q_{j,i}(x^k)$ into Equations (1.12c) one may solve for x^i in terms of $\hat{x}^k$ for all parameter values ε^h. This assumption allows for the expression of Equations (1.12b) as $\hat{Q}_j = \hat{Q}_j(\hat{x}^k)$ and hence the transformation (1.12) carries the surfaces $Q_j = Q_j(x^k)$ into the surfaces $\hat{Q}_j = \hat{Q}_j(\hat{x}^k)$. In addition, we assume that (1.12a,b) are

twice continuously differentiable with respect to each of the variables $x^k, Q_j, Q_{j,k}, \varepsilon^h$. Obviously, Equations (1.12) constitute a much more general class of transformations than does Equation (1.2) in that not only are the field quantities varied but so are the coordinates. Also, in (1.12) the transformations are allowed to depend on r parameters rather than just one as in (1.2).

The fundamental integral I, Equation (1.1), is said to be *invariant* under the transformation (1.12) if $I\left(\hat{Q}_j(\hat{x}^k)\right) = I\left(Q_j(x^k)\right)$, that is,

$$\int_{\hat{R}} L\left(\hat{x}^k, \hat{Q}_j, \hat{Q}_{j,k}\right) d^n\hat{x} = \int_R L\left(x^k, Q_j, Q_{j,k}\right) d^n x, \tag{1.13}$$

where $\hat{R}$ is the image of R under Equation (1.12a). As will be shown, the invariance of I under the transformation (1.12) implies an invariance or symmetry in the Lagrangian (density) L which in turn implies the existence of a conserved quantity. This is the essence of Noether's theorem.

The invariance of L is seen easily by using (1.12a) to write

$$\int_{\hat{R}} L\left(\hat{x}^i, \hat{Q}_j, \hat{Q}_{j,l}\right) d^n\hat{x} = \int_R L\left(\hat{x}^i(x^k), \hat{Q}_j(\hat{x}^i(x^k)), \hat{Q}_{j,l}(\hat{x}^i(x^k))\right) J d^n x \tag{1.14}$$

where $J = det(\partial\hat{x}^i/\partial x^k)$ is the Jacobian determinant of the transformation. Using the invariance of I, Equation (1.13), we conclude that the right hand member of (1.13) equals the right side of (1.14). If this is assumed true on an arbitrary region R of $\mathbf{R}^n$ it follows that

$$L\left(\hat{x}^i(x^k), \hat{Q}_j\left(\hat{x}^i(x^k)\right), \hat{Q}_{j,l}\left(\hat{x}^i(x^k)\right)\right) J = L\left(x^k, Q_j(x^k), Q_{j,l}(x^k)\right). \tag{1.15}$$

Equation (1.15) will be taken as the definition of the *invariance* of L.

That I is invariant implies the existence of a conserved density will now be shown. Note firstly that since $I(\hat{Q}_j) - I(Q_j)$ vanishes due to the invariance of I under the

transformation (1.12) the principle linear part $\varepsilon^h, \delta I^h$, must in particular vanish for each h because of the independence of the ε^h's. The vanishing of $\delta I^h, h = 1, ...r$, gives rise to r conserved densities in the following way.

First, expand $\hat{x}^i$ and $\hat{Q}_j$ in a Taylor series about $\varepsilon^h = 0$ to get

$$\hat{x}^i = x^i + \sum_{h=1}^{r} \varepsilon^h \phi_h^i + ||\left(\varepsilon^1, ..., \varepsilon^r\right)|| 0 \left(\varepsilon^1, ..., \varepsilon^r\right) \tag{1.16a}$$

$$\hat{Q}_j = Q_j + \sum_{h=1}^{r} \varepsilon^h \psi_{jh} + ||\left(\varepsilon^1, ..., \varepsilon^r\right)|| 0 \left(\varepsilon^1, ..., \varepsilon^r\right) \tag{1.16b}$$

where

$$\phi_h^i \left(x^k, Q_j, Q_{j,k}\right) \equiv \frac{\partial \hat{x}^i}{\partial \varepsilon^h} \left(x^k, Q_j, Q_{j,k}; \varepsilon^h\right) |_{\varepsilon^1 = ... = \varepsilon^r = 0}$$

and

$$\psi_{jh} \left(x^k, Q_j, Q_{j,k}\right) \equiv \frac{\partial \hat{Q}_j}{\partial \varepsilon^h} |_{\varepsilon^1 = ... = \varepsilon^r = 0}$$

and $0\left(\varepsilon^1, ..., \varepsilon^r\right) \to 0$ as $||\left(\varepsilon^1, ..., \varepsilon^r\right)|| \to 0$. Next, compute $\partial \hat{x}^i / \partial x^k$ using (1.16a) to find

$$\partial \hat{x}^i / \partial x^k = \delta_k^i + \sum_{h=1}^{r} \varepsilon^h \phi_{h,k}^i + ||\left(\varepsilon^1, ..., \varepsilon^r\right)|| 0 \left(\varepsilon^1, ..., \varepsilon^r\right). \tag{1.17}$$

whence

$$J = det\left(\partial \hat{x}^i / \partial x^k\right) = 1 + \sum_{k=1}^{n} \sum_{h=1}^{r} \varepsilon^h \phi_{h,k}^k + ||\left(\varepsilon^1, ..., \varepsilon^r\right)|| 0 \left(\varepsilon^1, ..., \varepsilon^r\right). \tag{1.18}$$

we may also compute $\hat{Q}_{j,k} \equiv \partial \hat{Q}_j / \partial \hat{x}^k$ with the aid of (1.16) and (1.17) by first recognizing the fact that

$$\sum_{i=1}^{n} \left(\partial x^l / \partial \hat{x}^i\right) \left(\partial \hat{x}^i / \partial x^k\right) = \delta_k^l, \tag{1.19}$$

then utilizing this fact in (1.17) to obtain

$$\delta^l_k = \partial x^l/\partial \hat{x}^k + \sum_{h=1}^{r}\sum_{i=1}^{n} \varepsilon^h \phi^i_{h,k} \partial x^l/\partial \hat{x}^i + ||\left(\varepsilon^1, ..., \varepsilon^r\right)||0\left(\varepsilon^1, ..., \varepsilon^r\right). \quad (1.20)$$

It is apparent from expression (1.20) that to zeroth order in $\varepsilon^h, \partial x^l/\partial \hat{x}^i = \delta^l_i$ so that (1.20) may be rewritten as

$$\partial x^l/\partial \hat{x}^k = \delta^l_k - \sum_{h=1}^{r} \varepsilon^h \phi^l_{h,k} + ||\left(\varepsilon^1, ..., \varepsilon^r\right)||0\left(\varepsilon^1, ..., \varepsilon^r\right). \quad (1.21)$$

Now, $\partial \hat{Q}_j/\partial x^i = \sum_{l=1}^{n} \left(\partial \hat{Q}_j/\partial \hat{x}^l\right)\left(\partial \hat{x}^l/\partial x^i\right)$ whence

$$\partial \hat{Q}_j/\partial \hat{x}^k = \sum_{i=1}^{n} \left(\partial \hat{Q}_j/\partial x^i\right)\left(\partial x^i/\partial \hat{x}^k\right) \quad (1.22)$$

by (1.19). The first factor in the right member of (1.22) is easily computed from (1.16b) to be

$$\partial \hat{Q}_j/\partial x^i = Q_{j,i} + \sum_{h=1}^{r} \varepsilon^h \psi_{jh,i} + ||\left(\varepsilon^1, ..., \varepsilon^r\right)||0\left(\varepsilon^1, ..., \varepsilon^r\right). \quad (1.23)$$

Substitution of (1.21) and (1.23) into (1.22) then gives

$$\hat{Q}_{j,k} = Q_{j,k} + \sum_{h=1}^{r} \varepsilon^h \left(\psi_{jh,k} - \sum_{i=1}^{n} Q_{j,i}\phi^i_{h,k}\right) + ||\left(\varepsilon^1, ..., \varepsilon^r\right)||0\left(\varepsilon^1, ..., \varepsilon^r\right). \quad (1.24)$$

Having obtained Equations (1.16), (1.18) and (1.24) we are in a position to calculate δI^h, but for convenience we first define $\hat{L}J$ to be the left member of Equation (1.15). The argument of the integral δI^h is most easily found by computing $\partial(\hat{L}J)/\partial\varepsilon^h|_{\varepsilon^1=...=\varepsilon^r=0}$. We will use the shorthand notation $\varepsilon = 0$ to mean $\varepsilon^1 = ... = \varepsilon^r = 0$.

$$\frac{\partial\left(\hat{L}J\right)}{\partial\varepsilon^h}|_{\varepsilon=0} = \left(\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}\frac{\partial\hat{x}^i}{\partial\varepsilon^h}|_{\varepsilon=0} + \sum_{j=1}^{m}\frac{\partial L}{\partial Q_j}\frac{\partial\hat{Q}_j}{\partial\varepsilon^h}|_{\varepsilon=0}\right.$$
$$\left.+\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial L}{\partial Q_{j,k}}\frac{\partial\hat{Q}_{j,k}}{\partial\varepsilon^h}|_{\varepsilon=0}\right)J|_{\varepsilon=0} + \hat{L}|_{\varepsilon=0}\frac{\partial J_h}{\partial\varepsilon}|_{\varepsilon=0}$$

whence

$$\frac{\partial(\hat{L}J)}{\partial\varepsilon^h}|_{\varepsilon=0} = \left(\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}\phi_h^i + \sum_{j=1}^{m}\frac{\partial L}{\partial Q_j}\psi_{jh} + \sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial L}{\partial Q_{j,k}}\left(\psi_{jh,k} - \sum_{i=1}^{n}Q_{j,i}\phi_{h,k}^i\right)\right)$$
$$+ L\sum_{k=1}^{n}\phi_{h,k}^k. \tag{1.25}$$

where (1.12c), (1.16), (1.18), and (1.24) have been utilized.

We define

$$\eta_{jh} = \psi_{jh} - \sum_{i=1}^{n}Q_{j,i}\phi_h^i \tag{1.26}$$

and compute

$$\frac{d}{dx}k\left(\frac{\partial L}{\partial Q_{j,k}}\eta_{jh}\right) = \frac{d}{dx}k\left(\frac{\partial L}{\partial Q_{j,k}}\right)\eta_{jh} + \frac{\partial L}{\partial Q_{j,k}}\left(\psi_{jh,k} - \sum_{i=1}^{n}Q_{j,i}\phi_{h,k}^i - \sum_{i=1}^{n}Q_{j,ik}\phi_h^i\right)$$

so that

$$\frac{\partial L}{\partial Q_{j,k}}\left(\psi_{jh,k} - \sum_{i=1}^{n}Q_{j,i}\phi_{h,k}^i\right) = \frac{d}{dx^k}\left(\frac{\partial L}{\partial Q_{j,k}}\eta_{jh}\right) - \frac{d}{dx^k}\left(\frac{\partial L}{\partial Q_{j,k}}\right)\eta_{jh}$$
$$+\sum_{i=1}^{n}\frac{\partial L}{\partial Q_{j,k}}Q_{j,ik}\phi_h^i \tag{1.27}$$

Also,

$$\frac{d}{dx^i}\left(L\phi_h^i\right) = \frac{\partial L}{\partial x^i}\phi_h^i + \sum_{j=1}^{m}\frac{\partial L}{\partial Q_j}Q_{j,i}\phi_h^i + \sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial L}{\partial Q_{j,k}}Q_{j,ki}\phi_h^i + L\phi_{h,i}^i$$

so that

$$\frac{\partial L}{\partial x^i}\phi_h^i = \frac{d}{dx^i}\left(L\phi_h^i\right) - \sum_{j=1}^{m}\frac{\partial L}{\partial Q_j}Q_{j,i}\phi_h^i - \sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial L}{\partial Q_{j,k}}Q_{j,ik}\phi_h^i - L\phi_{h,i}^i. \tag{1.28}$$

Substitution of (1.27) and (1.28) into (1.25) yields, after noting that $\partial L/\partial \varepsilon^h = 0$,

$$\begin{aligned}\partial\left(\hat{L}J - L\right)/\partial\varepsilon^h|_{\varepsilon=0} &= \sum_{i=1}^{n}\frac{d}{dx^i}\left(L\phi_h^i + \sum_{j=1}^{m}\frac{\partial L}{\partial Q_{j,i}}\eta_{jh}\right)\\ &+\sum_{j=1}^{m}\left(\frac{\partial L}{\partial Q_j} - \sum_{k=1}^{m}\frac{d}{dx^k}\left(\frac{\partial L}{\partial Q_{j,k}}\right)\right)\eta_{jh}\end{aligned} \tag{1.29}$$

after using the definition for η_{jh}, Equation (1.26), and combining terms. δI^h may be obtained directly from (1.29) by integrating over R and multiplying by ε^h.

If I is invariant under the transformation (1.12) and if the region of integration R is assumed to be completely arbitrary then by Equation (1.15) we conclude that (1.29) must vanish identically for each h since the right member of (1.15) is independent of ε^h for each h. The vanishing of (1.29) may also be deduced by requiring $\delta I^h = 0$ on an arbitrary R for each h. Along extremals, i.e., solutions to the Euler-Lagrange equations, (1.9), the last term of (1.29) vanishes identically whence (1.29) combined with (1.15) gives in this important case

$$\sum_{i=1}^{n}\frac{d}{dx^i}\left(L\phi_h^i + \sum_{j=1}^{m}\frac{\partial L}{\partial Q_{j,i}}\eta_{jh}\right) = 0,\ h = 1, ..., r. \tag{1.30}$$

This proves *Noether's Theorem*: If I is invariant under the transformation (1.12) for an arbitrary region of integration, then along extremal surfaces (1.30) is satisfied.

Special Cases: Equations (1.30) give rise to r conserved quantities, one for each of the r equations, and hence may themselves be viewed as *conservation laws.* To illustrate the manner in which an equation of type (1.30) may give rise to a conserved

quantity consider the well-known equation of charge continuity, $\partial\rho/\partial t + \nabla \cdot (\rho \mathbf{v}) = 0$, where ρ is the fluid charge density and $\mathbf{v}$ the velocity field of the fluid. This expression may be cast into relativistic four-vector notation by first defining the current density four-vector, j^σ by $j^\sigma = \rho_o u^\sigma, \sigma = 0,1,2,3$, where $u^\sigma = \gamma(c, \mathrm{v})$ is the velocity four-vector, c is the speed of light, $\gamma = 1/(1 - v^2/c^2)^{1/2}$, and $\rho_o = \rho/\gamma$ is the rest charge density, i.e., the charge density as measured in a coordinate system which is instantaneously at rest at each point in the fluid. Next, define the position vector $x^\sigma = (ct, x, y, z)$ where t is time and x, y and z are the usual coordinates in Euclidean three-space, i.e., x, y and z are spacial coordinates in a cartesian coordinate system. Finally, let $\partial_\sigma \equiv \frac{\partial}{\partial x^\sigma}$, then the equation of charge continuity may be written as $\partial_\sigma j^\sigma = 0$ (implied summation on σ) which is obviously of the form (1.30).

Returning to the original form of the equation we integrate both sides of the equation over all space to obtain

$$\frac{d}{dt}\int \rho d^3x + \int \nabla \cdot (\rho \mathrm{v})\, d^3x = 0. \tag{1.31}$$

the second term of this equation may be converted to an integral over the surface of the volume of integration, which is at spatial infinity, according to the divergence theorem. Assuming that ρ and $\mathbf{v}$ vanish at spatial infinity for all times t and recognizing that $\int \rho d^3x = Q$, the total charge of the system, Equation (1.31) gives $dQ/dt = 0$ which expresses the fact that the total charge of the system does not change with time, i.e., the total charge of the system is a conserved quantity. In an exactly analogous way, if the fundamental integral I, Equation (1.1), is expressed as an integral over space-time and if all field and transformation quantities are assumed to vanish at spatial infinity, then Equations (1.30) imply that

$$\frac{d}{dt}\int \left(L\phi_h^o + \sum_{j=1}^{m} \frac{\partial L}{\partial Q_{j,o}} \eta_{jh} \right) d^3x = 0, \;\; h = 1, ..., r. \tag{1.32}$$

Hence, the quantities $\int \left(L\phi_h^o + \sum_{j=1}^m \frac{\partial L}{\partial Q_{j,o}} \eta_{jh} \right) d^3x, h = 1, ..., r$, are conserved.

We now consider the invariance of I under some important special classes of transformations (1.12) and the consequent conservation laws (1.30). Firstly, note that in the case in which the field quantities are varied but the coordinates are not, that is $\hat{x}^i = x^i$ and hence $\phi_h^i \equiv 0$, and in which $r = 1$, the conservation law (1.30) implies Equation (1.10) by taking the integral of Equation (1.30) over a volume R of $\mathbf{R}^n$ and converting it to a surface integral. Recall that the boundary conditions (1.11) of the previous subsection were deduced directly from Equation (1.10). Note, however, that the condition (1.30) is stronger than (1.10), a fact that arises due to the assumption leading to Equation (1.30) that the region of integration R is completely arbitrary whereas in the previous subsection R is assumed to be fixed.

Consider now the case in which the Lagrangian density L does not depend explicitly on a coordinate, say $x^k, L = L(x^1, ..., x^{k-1}, x^{k+1}, ..., x^n, Q_j, Q_{j,k})$. In this case L is said to be cyclic in the variable x^k. Then I is invariant under a transformation of the form

$$\begin{aligned} \hat{x}^i &= x^i, \ i \neq k; \\ \hat{x}^k(x^k, \varepsilon) &= x^k + \varepsilon; \\ \hat{Q}_j &= Q_j, \ j = 1, ..., m. \end{aligned}$$

Hence, we may take $r = 1, \phi_1^i = \delta_k^i$ and $\psi_{j1} = 0$ where ϕ_h^i and ψ_{jh} are defined in (1.16). Using the definition of η_{jh} in Equation (1.26) we then conclude from (1.30) that

$$\sum_{i=1}^n \frac{d}{dx^i} \left(L\delta_k^i - \sum_{j=1}^m \frac{\partial L}{\partial Q_{j,i}} Q_{j,k} \right) = 0. \tag{1.33}$$

The quantity

$$T_{ck}^i \equiv \sum_{j=1}^m \frac{\partial L}{\partial Q_{j,i}} Q_{j,k} - L\delta_k^i \tag{1.34}$$

is often referred to as the canonical stress-energy tensor (density)[22] because of its form in the theories of elasticity and electromagnetism. In terms of definition (1.34), Equation (1.33) reads simply $\sum_{i=1}^{n} \frac{d}{dx^i} T^i_{ck} = 0$.

The result (1.33) may be obtained from a much more direct, but less general technique by noting that

$$dL/dx^k = \partial L/\partial x^k + \sum_{j=1}^{m} \frac{\partial L}{\partial Q_j} Q_{j,k} + \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\partial L}{\partial Q_{j,i}} Q_{j,ik}. \tag{1.35}$$

If L is cyclic in x^k, then $\partial L/\partial x^k = 0$. If, in addition, the Euler-Lagrange equations are satisfied, then $\partial L/\partial Q_j = \sum_{i=1}^{n} \frac{d}{dx^i}(\partial L/\partial Q_{j,i})$. Substitution of these two expressions into (1.35) yields

$$\begin{aligned} \frac{dL}{dx^k} &= \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{d}{dx^i} \left(\frac{\partial L}{\partial Q_{j,i}} \right) Q_{j,k} + \frac{\partial L}{\partial Q_{j,i}} \frac{d}{dx^i} \left(Q_{j,k} \right) \right) \\ &= \sum_{i=1}^{n} \frac{d}{dx^i} \left(\frac{\partial L}{\partial Q_{j,k}} \right) \end{aligned} \tag{1.36}$$

By substituting $dL/dx^k = \sum_{i=1}^{n} \frac{d}{dx^i} \left(L\delta^i_k \right)$ into (1.36) and collecting terms we then recover (1.33).

If L describes a mechanical system and if L is cyclic in the time coordinate, then (1.33) expresses the conservation of the total mechanical energy. If L is cyclic in a cartesian coordinate, then (1.33) implies the conservation of the corresponding total momentum. If L is cyclic in an angular coordinate, then (1.33) expresses the conservation of the corresponding total angular momentum.

Finally, when I is invariant under the transformation (1.12), but the region R of integration is not necessarily assumed completely arbitrary we obtain a more general result of interest. By combining (1.13) and (1.14) and using (1.29), which is a general result independent of any assumption about the region of integration R, we find that in this more general case

$$\int_R \sum_{i=1}^{n} \frac{d}{dx^i} \left(L\phi_h^i + \sum_{j=1}^{m} \frac{\partial L}{\partial Q_{j,i}} \eta_{jh} \right) d^n x + \int_R \sum_{j=1}^{m} E_j(L) \eta_{jh} d^n x = 0, \ h = 1, ..., r \tag{1.37}$$

(recall that $E_j(L) = 0$ is the j^{th} Euler-Lagrange equation). The first term may be converted to an integral over the boundary of R, Γ, by utilizing the generalized divergence theorem whence (1.37) becomes

$$\int_R \sum_{j=1}^{m} E_j(L) \eta_{jh} d^n x = \oint_\Gamma \sum_{i=1}^{n} T_h^i d\sigma_i, \ h = 1, ..., r, \tag{1.38}$$

where $d\sigma_i$ is a surface element and $T_h^i \equiv L\phi_h^i - \sum_{j=1}^{m} \frac{\partial L}{\partial Q_{j,i}} \eta_{jh}$. Hence, we have the *generalized Noether's theorem*: If I is invariant under the transformation (1.12), then (1.38) must be satisfied. Along extremals the left member of (1.38) vanishes and (1.38) becomes simply

$$\oint_\Gamma \sum_{i=1}^{n} T_h^i d\sigma_i = 0, \ h = 1, ..., r. \tag{1.39}$$

Equations (1.39) may also be called conservation laws, but in a more general sense than Equations (1.33). In fact, Equations (1.39) lead to the same conserved quantities indicated in Equations (1.32) under less stringent assumptions than the complete arbitrariness of R. To see this, let R be a space-time "cylinder" with cylindrical edges at spatial infinity where field quantities and transformations alike are assumed to vanish. Let the ends of the cylinder be at hypersurfaces of constant time. Then, Equations (1.39) reduce to $\int T_h^o(t_2) d^3x - \int T_h^0(t_1) d^3x = 0$ where t_2 and t_1 are the constant values of time t at the two ends of the cylinder, $T_h^o(t)$ represents the evaluation of T_h^o at t, and the integral extends over all space. We now assume that this expression holds for any two hypersurfaces of constant time, then in particular, $\lim_{t_1 \to 0} \frac{\int T_h^o(t+t_1) d^3x - \int T_h^0(t) d^3x}{t_1} = 0$, which implies (1.32).

3. Variational constraints

The notion of constraining the variations of a variational principle will be relied on heavily in the following sections and chapters and hence is briefly reviewed here.[23] The necessity of applying a constraint arises when the field quantities to be varied are not completely independent, i.e., there exists a relationship between the field quantities which must be satisfied by all variations of those quantities. This relationship may be expressible as a set of ***equations of constraint***

$$f^p\left(x^k, Q_j\right) = 0,\ p = 1, ..., d, \tag{1.40}$$

in which case the constraints which must be applied to the variations of the field quantities Q_j are called ***integrable*** or ***holonomic.*** If the relationship is not expressible in this form the constraints are ***nonholonomic.*** For example, if the relationship is expressible as a set of inequalities rather than equalities or as a set of equalities involving the derivatives of the field quantities $Q_{j,k}$ then the constraints are nonholonomic. Holonomic constraints may be discussed in complete generality whereas nonholonomic constraints have yet to be formulated in a general context, and hence each nonholonomic constraint must be considered individually.

A set of holonomic constraints may be incorporated into a variational principle by first finding ***generalized coordinates*** $q_s, s = 1, ..., m - d$, such that the q_s are entirely independent,

$$Q_j = Q_j\left(x^k, q_s\right), j = 1, ..., m, \tag{1.41}$$

and (1.40) together with (1.41) may be inverted to find q_s in terms of x^k and Q_j. One then expresses L as a function of x^k, q_s, and $q_{s,k}$ and varies the q_s's independently to recover the Euler-Lagrange equations (1.9) in terms of the q_s's rather than the Q_j's. Alternatively, one may add the quantity $\sum_{p=1}^{d} \lambda_p f^p\left(x^k, Q_j\right)$ to $L\left(x^k, Q_j, Q_{j,k}\right)$ and obtain the fundamental integral

$$I\left(Q_j, \lambda_p\right) = \int \left(L\left(x^k, Q_j, Q_{j,k}\right) + \sum_{p=1}^{d} \lambda_p f^p\left(x^k, Q_j\right) \right) d^n x \tag{1.42}$$

in which the Q_j's and λ_p's are to be varied independently. It is easily shown that by varying $\lambda_p, p = 1, ..., d$, independently one obtains the equations of constraint (1.40). The first method yields just $m - d$ equations in the $m - d$ generalized coordinates q_s while the latter method, Equation (1.42), yields $m + d$ equations in the m field quantities Q_j and the d *Lagrange multipliers* λ_p. Although in practice the first method may be more easily solved due to the fewer number of equations involved, the latter method gives more information, information which may have physical interest.

Because nonholonomic constraints cannot be discussed in as general a context as holonomic constraints we will defer a more in-depth examination of them until particular cases of interest arise in what follows. In particular, we will examine the nonholonomic Lin constrain.

4. Theorem of Schutz and Sorkin

We now state and prove in a slightly generalized form a theorem originally due to Schutz and Sorkin[5] (their Theorem 2.2) which is important for determining the necessity of constraining the variations in certain physically important variational principles. We begin by assuming that I is invariant under the transformation (1.12) and in accordance with the generalized Noether's theorem conclude that (1.38) must be satisfied. We next vary only the field quantities in (1.38) with an arbitrary transformation of the type (1.3), then equate the principle linear parts in ε of each member of Equation (1.38). Denoting with a δ the principle linear part in ε of each quantity we obtain

$$\int_R \sum_{j=1}^{m} (\delta E_j(L))\, \eta_{jh} d^n x + \int_R \sum_{j=1}^{m} E_j(L)\, (\delta \eta_{jh})\, d^n x = \delta \oint_\Gamma \sum_{i=1}^{n} T_h^i d\sigma_i, h = 1, ..., r. \tag{1.43}$$

Finally, denote by $P_h(\Im, H)$ the quantity

$$P_h(\Im, H) = \int_H \sum_{i=1}^{n} T_h^i d\sigma_i \tag{1.44}$$

where $\Im$ represents the transformation (1.12) and H is an $n-1$-dimensional hypersurface in $\mathbf{R}^n$. We are now in a position to state the theorem.

THEOREM 3.1: Suppose I, Equation (1.1), is invariant under the transformation $\Im$, (1.12), on an arbitrary region of integration and that $P_h(\Im, H)$ is defined by (1.44). Then any two of the following imply the third (there are actually fairly weak additional assumptions required for (b)+(c) implies (a) and (a)+(c) implies (b)):

(a) The field quantities Q_j are invariant under the transformation (1.12); i.e., $\eta_{jh} = 0$ for $j = 1, ..., m$ and $h = 1, ..., r$.

(b) The field quantities Q_j are solutions of the Euler-Lagrange equations: $E_j(L) = 0, j = 1, ..., m$.

(c) For any asymptotically regular hypersurface[24] H of dimension $n-1$, $P_h(\Im, H)$ is an extremum against all variations of the field quantities Q_j of compact support.[25]

Proof. (i) Assume (a) and (b) are satisfied, then the left member of Equation (1.43) vanishes and we obtain $\delta \oint_\Gamma \sum_{i=1}^{n} T_h^i d\sigma_i = 0$. Let H be an asymptotically regular hypersurface and vary the field quantities arbitrarily over a compact region of $\mathbf{R}^n$, i.e., let ψ_j in Equation (1.3) be an arbitrary C^2 real-valued function, defined on the interior of a compact region $K \subseteq \mathbf{R}^n$, which vanishes on the boundary of and exterior to K. Choose a region $R \subseteq \mathbf{R}^n$ which is bounded by H on the one

hand and by a hypersurface H' on the other where $H \cap K = \emptyset$ (see Figure 1), and let Γ be the boundary of R. Then, $\delta \oint_\Gamma \sum_{i=1}^n T_h^i d\sigma_i = \delta \oint_{K\cap H} \sum_{i=1}^n T_h^i d\sigma_i$ and by letting K vary arbitrarily we conclude that

$$P_h(\Im, H) = 0, \tag{1.45}$$

where H is asymptotically regular and δ here means arbitrary compact-support variations of the field quantities Q_j. Hence, (c) is satisfied according to (1.45).

(*ii*) Assume that (a) and (c) are satisfied. Let $R \subseteq \mathbf{R}^n$ be arbitrary and choose H and H' as in Figure 1. Then,

$$\delta \int_\Gamma \sum_{i=1}^n T_h^i d\sigma_i = \delta \int_H \sum_{i=1}^n T_h^i d\sigma_i - \delta \int_{H'} \sum_{i=1}^n T_h^i d\sigma_i$$

which vanishes by (c). Hence, Equation (1.43) reduces in this case to

$$\int_R \sum_{j=1}^m E_j(L)\,(\delta\eta_{jh})\,d^n x = 0.$$

Since the region of integration is arbitrary we conclude that the integrand must vanish identically. Under the *additional assumption* that the variation of transformation (1.12) preserves the arbitrariness of the variation of the field quantities we then conclude that the Euler-Lagrange equations, (1.9), must be satisfied by the field quantities; i.e., (b) must be satisfied.

(*iii*) Assume (b) and (c) true, then according to the argument in part (*ii*) the right member of (1.43) vanishes as does the second term on the left, whence

$$\int_R \sum_{j=1}^m \left[\delta E_j(L)\right] \eta_{jh} d^n x = 0.$$

Since the region of integration R is arbitrary the integrand must vanish identically. Under the *additional assumption* that the variation of the Euler-Lagrange equations

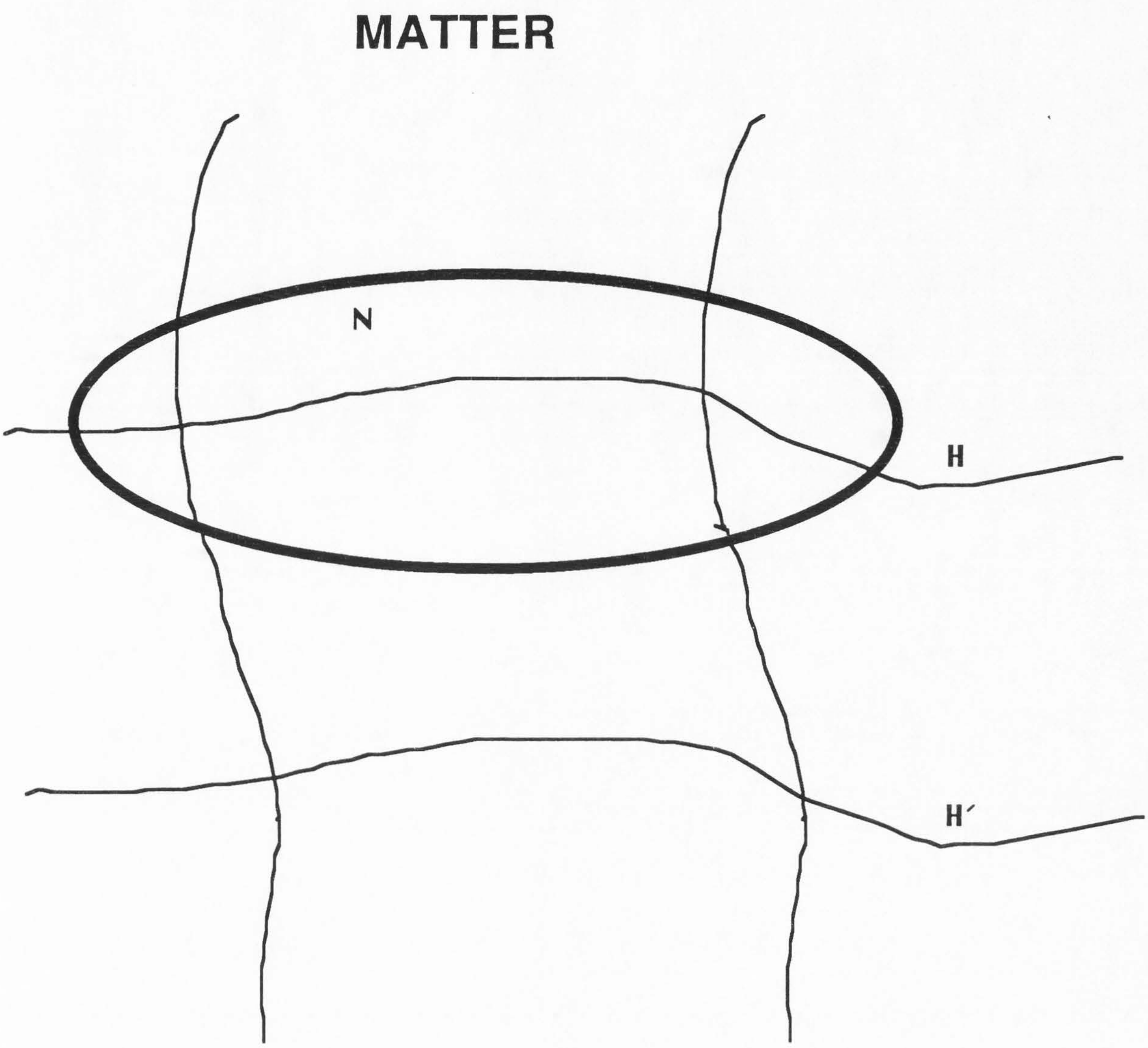

Figure 1: Spacetime diagram of region including the spacelike hypersurfaces H and H'.

The matter between the hypersurfaces is confined to a region of compact support, its boundary indicated by the timelike lines. The perturbation vanishes outside of N.

preserves the generality of the variation of the field quantities we conclude that $\eta_{jh} = 0$ for all j and h so that (a) must be satisfied.

The additional assumptions made in parts (*ii*) and (*iii*) are satisfied in almost all cases of physical interest. Also, (a)+(b) $\Rightarrow$ (c) will be used in what follows almost to the exclusion of the other two cases (recall that (a)+(b) $\Rightarrow$ (c) requires no additional assumption), hence the additional assumptions are not examined in any greater detail here.

To illustrate the utility of Theorem 3.1 we first assume that I is an integral over space-time and that L does not depend explicitly on time t. Then I is invariant under the transformation

$$\hat{t} = t + \varepsilon,\ \hat{x} = x,\ \hat{y} = y,\ \hat{z} = z,\ \hat{Q}_j = Q_j, j = 1, ..., m.$$

Hence, $\phi_1^\beta = \delta_o^\beta$ and $\psi_{j1} = 0$. In this case the stationarity of the field quantities, $\eta_{j1} = 0$, implies $\partial Q_j/\partial t = 0, j = 1, ..., m$, that is, none of the field quantities depend explicitly on the time. From the definition of T_h^i we find that $T_1^\beta = \sum_{j=1}^m \frac{\partial L}{\partial Q_{j,\beta}} Q_{j,o} - L\delta_o^\beta$. On a spacelike hypersurface H , the quantity $P(\Im, H)$ assumes the form

$$P(\Im, H) = \int_V \left(\sum_{j=1}^m \frac{\partial L}{\partial Q_{j,o}} Q_{j,o} - L \right) d^3x. \tag{1.46}$$

Equation (1.46) is usually associated with the total mechanical plus field energy of a system. In this special case Theorem 3.1 states that if the field quantities are time-independent and satisfy the equations of motion, then the total system energy is an extremum against all variations of the field quantities of compact support.

One particular application of this theorem was given by Schutz and Sorkin (their Corollary 2.3) which we now state as:

COROLLARY 3.2: There is no unconstrained variational principle for Maxwell's equations in which the field quantities are the electric and magnetic fields, **E** and **B** (or equivalently, the electromagnetic field tensor $F^{\alpha\beta}$).

Proof. The energy of stationary electromagnetic fields is determined experimentally to be $\int \left(\varepsilon_o E^2 + B^2/\mu_o\right)/2 \; d^3x$ which is only an extremum against all variations of compact support in $\mathbf{E}$ and $\mathbf{B}$ when $\mathbf{E} = \mathbf{B} = 0$. Since the expression holds true even for non-zero, time-independent $\mathbf{E}$ and $\mathbf{B}$ we conclude from Theorem 3.1 that there cannot be an unconstrained variational principle in $\mathbf{E}$ and $\mathbf{B}$ leading to Maxwell's equations.

As is well known, there are two ways to obtain Maxwell's equation from a variational principle, both of which avoid the free variation of the fields $\mathbf{E}$ and $\mathbf{B}$. In the first method one constrains the variations directly by varying $\mathbf{E}$ and $\mathbf{B}$ (alternatively $F_{\alpha\beta}$) with the side conditions $\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \cdot \mathbf{B} = 0$ (alternatively, $F_{\alpha\beta,\nu} + F_{\nu\alpha,\beta} + F_{\beta\nu,\alpha} = 0$). In the second method, one introduces the electromagnetic potential A^σ and obtains an unconstrained variational principle in A^σ.

Theorem 3.1 will be of further use in later sections as we discuss variational principles in fluid mechanics and electromagnetics. In particular, it will motivate a discussion of the Lin constraint.

D. Darboux's and Pfaff's theorems

To conclude the introduction we state without proof a theorem which has wide application in problems of theoretical fluid mechanics and thermodynamics, and which will be used frequently in the following chapters. The theorem is known as Darboux's theorem[26] and is usually stated in the nomenclature of differential forms. An important corollary of Darboux's theorem is known as Pfaff's theorem[27] and will also be stated.

DARBOUX'S THEOREM: Let α be a one-form. Then $\alpha \wedge (d\alpha)^n = 0$ and $(d\alpha)^n \neq 0$, where $(d\alpha)^n$ represents the n-fold Grassman (or wedge) product of the exterior derivative of $\alpha, d\alpha$, is a necessary and sufficient condition for the local existence of zero-forms g_i and $h_i, i = 1, ..., n$, such that $\alpha = \sum_{i=1}^{n} g_i dh_i$ about points at which α does not vanish. Also, $(d\alpha)^{n+1} = 0$ and $\alpha \wedge (d\alpha)^n \neq 0$ and is a necessary and

sufficient condition for the local existence of zero forms $f, g_i, h_i, i = 1, ..., n$, such that $\alpha = df + \sum_{i=1}^{n} g_i dh_i$.

Remark. Taking $n = 0$ in the latter case, i.e., $d\alpha = 0$, we see that the theorem reduces to the Poincaré Lemma[26] and its converse for one-forms, that is, about points at which $\alpha \neq 0, d\alpha = 0$ if and only if there is a locally defined 0-form such that $\alpha = df$.

Darboux's theorem has as an immediate consequence the following corollary, often referred to as Pfaff's Theorem.

COROLLARY (PFAFF'S THEOREM): Let $\alpha = \sum_{i=1}^{n} f_i(x^j)dx^i$ be a one-form defined on an open set of an N-dimensional manifold. Then about each point p in the domain of α there exists an open set U containing p and C^∞ functions $f, g_i, h_i : U \to \mathbf{R}$ such that

$$\begin{aligned} \alpha &= \sum_{i=1}^{N/2} g_i dh_i \qquad \text{if } N \text{ is even} \\ &= df + \sum_{i=1}^{(N-1)/2} g_i dh_i \qquad \text{if } N \text{ is odd,} \end{aligned}$$

so long as $\alpha(p) \neq 0$ and $(d\alpha)^{N/2}(p) \neq 0$ (N even) or $\alpha \wedge (d\alpha)^{(N-1)/2}(p) \neq 0$ (N odd).

Proof. Assume N is even. Then $\alpha \wedge (d\alpha)^{N/2}$ is an $N+1$-form on an N-dimensional manifold and hence must vanish since, if it is expressed in terms of the basis one-forms $dx^i, i = 1, ..., N$, we find $\alpha \wedge (d\alpha)^{N/2} = \sum_{i=1}^{n} F_i dx^i \wedge dx^1 \wedge ... \wedge dx^N$ after rearranging and collecting terms. However, $dx^i \wedge dx^i \wedge ... \wedge dx^N = 0$ for for each i whence $\alpha \wedge (d\alpha)^{N/2} = 0$. Applying Darboux's theorem we conclude the local existence of C^∞ functions (zero-forms) $g_j, h_j, j = 1, ..., N/2$, such that $\alpha = \sum_{j=1}^{N/2} g_j dh_j$.

Assume N is odd. Then $(d\alpha)^{(N+1)/2}$ is an $N + 1$ form on an N-dimensional manifold and hence must vanish by the same argument given above. Application of Darboux's theorem then gives the desired result.

The corollary assumes its most useful form when the N-dimensional manifold is taken to be $\mathbf{R}^N$. In this case, if $W(x^j)$ is a C^∞ vector-valued function defined on $\mathbf{R}^N$ we may define a one-form α by $\alpha = \sum_{i=1}^N W_i(x^j)dx^i$. According to the corollary we then conclude the local existence of C^∞ real-valued f, g_k, h_k functions defined on $\mathbf{R}^N$ such that

$$W_i = \begin{cases} \sum_{k=1}^{N/2} g_k \partial h_k / \partial x^i & N \text{ even} \\ \partial f/\partial x^i + \sum_{k=1}^{(N-1)/2} g_k \partial h_k / \partial x^i & N \text{ odd.} \end{cases}$$

Remark. Pfaff's theorem gives a means of representing a one-form or vector field as efficiently as possible in terms of the number of differentials required for its expression.

II. VARIATIONAL PRINCIPLES IN FLUID MECHANICS

There exist two fundamental means whereby one may specify the fields of fluid mechanics. The most common, the Eulerian description, consists in specifying the fields as functions of both space and time. The Lagrangian description, on the other hand, views a fluid as being composed of small fluid elements each of which has particle-like properties. Hence, in the Lagrangian description the fields are specified as functions of time and a fluid element index to which they pertain. Stated in another way, the Eulerian description may be considered as the description of a fluid according to an observer who is fixed relative to some arbitrary reference frame through which the fluid flows whereas the Lagrangian description is the description of the fluid according to an observer comoving with the fluid. Variational principles in continuum mechanics may be formulated in terms of either description as we show below.

A. Variational principles for a perfect fluid–the Lagrangian description

A variational principle leading to Euler's equation[28] for a perfect fluid[10] in the Lagrangian description was presented first by Lagrange himself. This is not too surprising due to the close parallel between Lagrangian and particle formulations. Indeed, the variational principle may be given in terms of Hamilton's principle, that the Lagrangian[29] (density) be the difference in the kinetic energy (density) and potential energy (density). Much of what follows pertaining to non-relativistic fluids is due to Seliger and Whitham[9] or Saarloos,[30] while Taub[3] and Schutz and Sorkin[5] are responsible for much of the general relativistic formulation.

1. Non-relativistic formulation

In non-relativistic ideal fluid mechanics a fluid element may be indexed by its initial position $\mathbf{a} = (a_1, a_2, a_3)$, i.e., $\mathbf{a}$ is the position of the fluid element at some

initial time t_o. Hence the position of that fluid element $\mathbf{q}$ as a function of time may be expressed in terms of the Lagrangian description as $\mathbf{q} = \mathbf{q}(\mathbf{a}, t)$.

Euler's equation, which models the dynamical behavior of a perfect fluid in the absence of external forces, may be expressed in terms of $\mathbf{q}$ as

$$\rho_m \frac{\partial^2 \mathbf{q}}{\partial t^2} + \nabla_{\mathbf{q}} p = 0 \tag{2.1}$$

where ρ_m is the mass per unit volume at time t, p is the pressure, a function of $\mathbf{q}$ and t, and $\nabla_{\mathbf{q}}$ is the gradient with respect to the generalized coordinates[31] q. ρ_m is related to the initial mass density ρ_{om} according to

$$\rho_m(\mathbf{a}, t) J(\mathbf{a}, t) = \rho_{om}(\mathbf{a}) \tag{2.2}$$

where

$$J(\mathbf{a}, t) \equiv det(\partial q_i / \partial a_j) \tag{2.3}$$

is the Jacobian determinant of $\mathbf{q}$ with respect to $\mathbf{a}$. (2.2) is a result of requiring the conservation of mass for each fluid element, as

$$m_{V_o} = \int_V \rho_m(\mathbf{x}, t) d^3x = \int_{V_o} \rho_m(\mathbf{a}, t) J d^3a = \int_{V_o} \rho_{om}(\mathbf{a}) d^3a$$

must be satisfied for arbitrary V_o from which (2.2) readily follows. By requiring that for fixed t the expression $\mathbf{q} = \mathbf{q}(\mathbf{a}, t)$ be invertible for $\mathbf{a}$ in terms of $\mathbf{q}$, $\mathbf{a} = \mathbf{Q}(\mathbf{q}, t)$, we may utilize Cramer's rule to write

$$\left(\frac{\partial Q_i(\mathbf{q}, t)}{\partial q_j} \right) |_{\mathbf{q}=\mathbf{q}(\mathbf{a},t)} = J_{ji}(\mathbf{a}, t) / J(\mathbf{a}, t) \tag{2.4}$$

where $J_{ij}(\mathbf{a}, t)$ is the cofactor of $\frac{\partial q_i}{\partial a_j}$ in the Jacobian matrix $\left(\frac{\partial q_i}{\partial a_j} \right)$. From (2.2), (2.4) and $\partial / \partial q_j = \sum_{j=1}^{3} \left(\partial Q_i / \partial q_j \right) \partial / \partial a_i$ we may express (2.1) as

$$\rho_{om} \partial^2 q_i / \partial t^2 + \sum_{j=1}^{3} J_{ij} \partial p / \partial a_j = 0. \tag{2.5}$$

For adiabatic flow the specific entropy S is constant along a fluid element trajectory,

$$S = S_o(\mathbf{a}).$$

In order to complete the system of equations (2.5) we assume p is a given function of ρ_m and S (and hence of $\mathbf{a}$ and t).

A variational principle which results in Equation (2.5) is

$$\delta \int d^3a \int_{t_1}^{t_2} dt L \equiv \delta \int d^3a \int_{t_1}^{t_2} dt \Big[\frac{1}{2}\rho_{om}(\mathbf{a}) \left(\partial \mathbf{q}(\mathbf{a},t)/\partial t\right)^2 - \rho_{om}(\mathbf{a}) u \left(\rho_m(\mathbf{a},t), S_o(\mathbf{a})\right)\Big] = 0 \tag{2.6}$$

For adiabatic flow the specific entropy S is constant along a fluid element where the fluid element positions $\mathbf{q}$ are varied in such a way that the variations vanish at t_1 and t_2 and on the boundary of the spatial volume of integration. The internal energy per unit mass of the fluid, $u(\rho_m, S)$, is given by the fundamental equation of thermodynamics,

$$du = TdS - pd\left(\frac{1}{\rho_m}\right) \tag{2.7}$$

where $T = T(\rho_m, S)$ is the temperature. Maxwell's relations are in this case

$$\begin{aligned} (\partial u/\partial \rho_m)_S &= p(\rho_m, S)/\rho_m^2; \\ (\partial u/\partial S)_{\rho_m} &= T(\rho_m, S). \end{aligned} \tag{2.8}$$

Recalling from (2.2) that ρ_m depends on the derivatives $\partial q_i/\partial a_j$ and noting that the Lagrangian density represented by (2.6) (the quantity in brackets) is cyclic in all the variables q_i, i.e., it only depends on the derivatives of the q_i's, we find that the Euler-Lagrange equations reduce in this case to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial(\partial q_i/\partial t)}\right) + \sum_{j=1}^{3} \frac{d}{da_j}\left(\frac{\partial L}{\partial(\partial q_i/\partial a_j)}\right) = 0.$$

Hence,

$$\frac{\partial}{\partial t}\left(\rho_{om}(\mathbf{a})\partial q_i(\mathbf{a},t)/\partial t\right)+\sum_{j=1}^{3}\frac{\partial}{\partial a_j}\left(\rho_{om}(\mathbf{a})\left(\frac{\partial u}{\partial \rho_m}\right)_S\frac{\partial \rho_m}{\partial(\partial q_i/\partial a_j)}\right)=0. \tag{2.9}$$

From (2.2) it follows that

$$\frac{\partial \rho_m(\mathbf{a},t)}{\partial(\partial q_i/\partial a_j)}=-\frac{\rho_{om}(\mathbf{a})}{J^2(\mathbf{a},t)}\frac{\partial J(\mathbf{a},t)}{\partial(\partial q_i/\partial a_j)}=-\frac{\rho_m^2(\mathbf{a},t)J_{ij}(\mathbf{a},t)}{\rho_{om}(\mathbf{a})}, \tag{2.10}$$

while the definition of the cofactor J_{ij} implies

$$\sum_{j=1}^{3}\partial J_{ij}/\partial a_j=0. \tag{2.11}$$

Substituting the first equation of (2.8) as well as (2.10) and (2.11) into (2.9) we recover the equation of motion (2.5), thus substantiating the claim that the variational principle (2.6) results in (2.5).

2. Relativistic formulation

A general relativistic generalization of the variational principle for a perfect fluid in the Lagrangian description was first given by Taub.[3] Following Taub, we begin with the Eulerian fundamental integral

$$I_{rel}=\int L d^4x=\int\left(R-2K\rho_o\left(c^2+H_o+\frac{1}{2}\lambda g_{\mu\nu}u^\mu u^\nu\right)\right)\sqrt{(-g)}d^4x \tag{2.12}$$

where R is the scalar curvature formed from the metric tensor $g_{\mu\nu}$, K is the Einstein gravitational constant, ρ_o is the fluid rest number density or concentration, u^μ is the four-velocity, λ is a Lagrange multiplier chosen in such a way that

$$g_{\mu\nu}u^\mu u^\nu=1 \tag{2.13}$$

g is the determinant of $g_{\mu\nu}$, and H_o is the Helmholtz free energy[32]:

$$H_o=U_o-T_oS_o, \tag{2.14}$$

where S_o is the entropy, T_o the temperature and U_o the specific internal energy all as measured by an observer momentarily at rest with respect to the fluid. By using the fundamental equation of thermodynamics (2.7) as well as the relation (2.14) we then conclude that (taking ρ_o and T_o as the independent variables)

$$dH_o = \left(p/\rho_o^2\right) d\rho_o - S_o dT_o. \tag{2.15}$$

We next invoke a Lagrangian description by writing for the position four-vector of a fluid element's world line

$$x^\mu = x^\mu(u, v, w, s), \tag{2.16}$$

where u, v and w label a particular world line and s represents the proper time along that world line, whence $u^\mu = \partial x^\mu / \partial s$. By transforming to a new comoving coordinate system given by $\hat{x}^1 = u, \hat{x}^2 = v, \hat{x}^3 = 2$ and $\hat{x}^o = s$ and using the invariance of the fundamental integral I_{rel} under such a transformation we may write (2.12) as

$$\hat{I}_{rel} = \int \left(\hat{R} - 2K\hat{\rho}_o \left(c^2 + \hat{H}_o + \frac{1}{2}\lambda \hat{g}_{\mu\nu}\hat{u}^\mu \hat{u}^\nu\right)\right) \sqrt{(-\hat{g})} du dv dw ds \tag{2.17}$$

where the hatted quantities obey the appropriate coordinate transformation laws, and $\sqrt{-\hat{g}} = \sqrt{-g} det\left(\partial x^\mu / \partial \hat{x}^\nu\right)$.

As with the non-relativistic case, we require mass conservation which in the relativistic case assumes the form

$$\left(\rho_o u^\mu\right);_\mu = \left(\sqrt{-g}\rho_o u^\mu\right),_\mu / \sqrt{-g} = 0. \tag{2.18}$$

Here, this requires that the quantity $\sqrt{-g}\rho_o det\left(\partial x^\mu / \partial \hat{x}^\nu\right)$ be independent of the proper time of each fluid element described, i.e.,

$$\sqrt{-\hat{g}}\hat{\rho}_o = \sqrt{-g}\rho_o det(\partial x^\mu / \partial \hat{x}^\nu) = M(u, v, w) \tag{2.19}$$

where M is a function only of u, v and w and not of s (compare this expression with its non-relativistic analogue (2.2)). For variations δ which conserve fluid element mass, i.e., $\delta M = 0$,we may use (2.19) to write, with, $J \equiv det\,(\partial x^\mu/\partial \hat{x}^\nu)$,

$$\delta(\sqrt{-g}\rho_o J) = (\delta\sqrt{-g})\rho_o J + \sqrt{-g}(\delta\rho_o)J + \sqrt{-g}\rho_o(\delta J) = 0$$

which, upon dividing by $\sqrt{-g}\rho_o J$, implies

$$\frac{\delta\rho_o}{\rho_o} + \frac{\delta\sqrt{-g}}{\sqrt{-g}} + \frac{\delta J}{J} = 0. \tag{2.20a}$$

But

$$\delta\sqrt{-g}/\sqrt{-g} = -\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\partial\sqrt{-g}}{\partial x^\nu}\delta x^\nu\right) = \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\nu}\delta x^\nu \tag{2.21}$$

and

$$\delta J/J = \frac{1}{J}\frac{\partial J}{\partial\left(\partial x^\mu/\partial\hat{x}^\nu\right)}\delta\left(\partial x^\mu/\partial\hat{x}^\nu\right) = \frac{\partial\hat{x}^\nu}{\partial x^\mu}\frac{\partial}{\partial\hat{x}^\nu}\left(\delta x^\mu\right) = \frac{\partial}{\partial x^\mu}\left(\delta x^\mu\right).$$

Substitution of Equations (2.21) into (2.20a) gives

$$\delta\rho_o/\rho_o + \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu} + \left(\delta x^\nu\right)_{;\nu} = 0 \tag{2.20b}$$

where we have used the fact that $(\delta x^\nu)_{;\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}\delta x^\nu\right)$. Since Equation (2.20b) is manifestly covariant it is also satisfied in the hatted system.

In order to compute the variation of $\hat{I}_{rel}$, Equation (2.17), we need the following two important identities. The first identity is obtained by recognizing that $\hat{g}_{\mu\nu}$ varies not only with $\delta g_{\alpha\beta}$ according to $\delta\hat{g}_{\mu\nu} = \delta g_{\sigma\tau}\frac{\partial x^\sigma}{\partial\hat{x}^\mu}\frac{\partial x^\tau}{\partial\hat{x}^\nu}$ but also with the variation $\delta\hat{x}^\mu$.

Using $\hat{g}_{\mu\nu} = g_{\sigma\tau}\frac{\partial x^\sigma}{\partial\hat{x}^\mu}\frac{\partial x^\tau}{\partial\hat{x}^\nu}$ we find

$$\begin{aligned}\delta_x\hat{g}_{\mu\nu} &= g_{\sigma\tau,\alpha}\frac{\partial x^\alpha}{\partial\hat{x}^\beta}\delta\hat{x}^\beta\frac{\partial x^\sigma}{\partial\hat{x}^\mu}\frac{\partial x^\tau}{\partial\hat{x}^\nu} + g_{\sigma\tau}\frac{\partial}{\partial\hat{x}^\mu}\left(\frac{\partial x^\sigma}{\partial\hat{x}^\alpha}\delta\hat{x}^\alpha\right)\frac{\partial x^\tau}{\partial\hat{x}^\nu} + g_{\sigma\tau}\frac{\partial x^\sigma}{\partial\hat{x}^\mu}\frac{\partial}{\partial\hat{x}^\nu}\left(\frac{\partial x^\tau}{\partial\hat{x}^\alpha}\delta\hat{x}^\alpha\right)\\ &= \frac{\partial}{\partial\hat{x}^\beta}\left(\hat{g}_{\mu\nu}\right)\delta\hat{x}^\beta + \hat{g}_{\mu\beta}\frac{\partial}{\partial\hat{x}^\nu}\left(\delta\hat{x}^\beta\right) + \hat{g}_{\beta\nu}\frac{\partial}{\partial x^\mu}\left(\delta\hat{x}^\beta\right).\end{aligned} \tag{2.22}$$

Define $\frac{\partial}{\partial \hat{x}^{\beta}}\hat{f} = \hat{f}_{,\beta}$ and $\delta\hat{x}_{\nu} = \hat{g}_{\mu\nu}\delta\hat{x}^{\mu}$ where $\hat{f}$ is any tensor or scalar quantity in the hatted system and use these definitions in (2.22):

$$\delta_x \hat{g}_{\mu\nu} = \hat{g}^{\alpha\beta}\hat{g}_{\mu\nu,\alpha}\,\delta\hat{x}_{\beta} + \hat{g}_{\mu\alpha}\left(\hat{g}^{\alpha\beta}\delta\hat{x}_{\beta}\right)_{,\nu} + \hat{g}_{\alpha\nu}\left(\hat{g}^{\alpha\beta}\delta\hat{x}_{\beta}\right)_{,\mu} \tag{2.23}$$

whence

$$\begin{aligned}\delta_x \hat{g}_{\mu\nu} &= \left(\delta\hat{x}_{\mu}\right)_{,\nu} + \left(\delta\hat{x}_{\nu}\right)_{,\mu} + \left(\hat{g}^{\alpha\beta} g_{\mu\nu,\alpha} + \hat{g}_{\alpha\nu}\hat{g}^{\alpha\beta}{}_{,\mu} + \hat{g}_{\mu\alpha}\hat{g}^{\alpha\beta}{}_{,\nu}\right)\delta\hat{x}_{\beta} \\ &= \left(\delta\hat{x}_{\mu}\right)_{,\nu} + \left(\delta\hat{x}_{\nu}\right)_{,\mu} - \hat{g}^{\alpha\beta}\left(\hat{g}_{\alpha\nu,\mu} + \hat{g}_{\mu\alpha,\nu} - \hat{g}_{\mu\nu,\alpha}\right)\delta\hat{x}_{\beta}. \end{aligned} \tag{2.23}$$

But, the coefficient of $\delta\hat{x}_{\beta}$ in the last member of (2.23) is just -2 times the Christoffel symbol of the second kind, so that by recalling the definition of the covariant derivative of a covariant vector we see that (2.23) reduces to

$$\delta_x \hat{g}_{\mu\nu} = \left(\delta\hat{x}_{\mu}\right)_{;\nu} + \left(\delta\hat{x}_{\nu}\right)_{;\mu} \tag{2.24}$$

The second important identity involves the calculation of $\delta_x\left(\hat{g}_{\mu\nu}\hat{u}^{\mu}\hat{u}^{\nu}\right)$:

$$\begin{aligned}\delta_x\left(\hat{g}_{\mu\nu}\hat{u}^{\mu}\hat{u}^{\nu}\right) &= \delta_x\left(g_{\mu\nu}u^{\mu}u^{\nu}\right) \\ &= g_{\mu\nu,\sigma}\,\delta x^{\sigma}u^{\mu}u^{\nu} + 2g_{\mu\nu}u^{\mu}\left(\partial\delta x^{\nu}/\partial s\right) \\ &= g_{\mu\nu,\sigma}\,\delta x^{\sigma}u^{\mu}u^{\nu} + 2g_{\mu\nu}u^{\mu}u^{\sigma}\partial(\delta x^{\nu})/\partial x^{\sigma} \\ &= 2g_{\mu\nu}u^{\mu}u^{\sigma}(\delta x^{\nu})_{;\sigma} \\ &= 2\hat{g}_{\mu\nu}\hat{u}^{\mu}\hat{u}^{\sigma}(\delta\hat{x}^{\nu})_{;\sigma}. \end{aligned} \tag{2.25}$$

We are now in a position to perform the variation of vv$\hat{I}_{rel}$, Equation (2.17). Using (2.19) (with $\delta M = 0$), (2.15), (2.24), (2.25) and the fact that

$$\delta\left(\sqrt{-\hat{g}}\hat{R}\right)/\delta\hat{g}_{\mu\nu} = -\sqrt{-\hat{g}}\hat{G}^{\mu\nu} \equiv \sqrt{-\hat{g}}\left(-\hat{R}^{\mu\nu} + \frac{1}{2}\hat{g}^{\mu\nu}\hat{R}\right)$$

where $G^{\mu\nu}$ is the Einstein field tensor and $\delta/\delta Q$ indicates the functional derivative we obtain

$$\delta\hat{I}_{rel} = \int \Big[-\hat{G}^{\mu\nu}\left(\delta\hat{g}_{\mu\nu} + (\delta\hat{x}_\mu)_{;\nu} + (\delta\hat{x}_\nu)_{;\mu}\right) - 2K\hat{\rho}_o\left(\left(\frac{p}{\rho_o}2\delta\hat{\rho}_o - \hat{S}_o\delta\hat{T}_o\right)\right.$$
$$\left. + \frac{1}{2}\lambda\hat{u}^\mu\hat{u}^\nu\delta\hat{g}_{\mu\nu} + \lambda\hat{g}_{\mu\nu}\hat{u}^\mu\hat{u}^\sigma(\delta\hat{x}^\nu)_{;\sigma}\right)\Big]\sqrt{-\hat{g}}dudvdwds. \qquad (2.26)$$

Substitution of (2.20b) (letting all appropriate quantities therein be hatted) into (2.26) then gives after rearranging terms

$$\delta\hat{I} =_{rel} \int [-\left(\hat{G}^{\mu\nu} + K\hat{T}^{\mu\nu}\right)(\delta\hat{g}_{\mu\nu} + 2(\delta\hat{x}_\mu)_{;\nu}) + 2K\hat{\rho}_o\hat{S}_o\delta\hat{T}_o]\sqrt{-g}dudvdwds \qquad (2.27)$$

where

$$\hat{T}^{\mu\nu} \equiv \lambda\hat{\rho}_o\hat{u}^\mu\hat{u}^\nu - p\hat{g}^{\mu\nu}. \qquad (2.28)$$

Integrating by parts and assuming that the variations vanish on the boundary of integration we may rewrite (2.27) in the form

$$\delta\hat{I}_{rel} = \int [-\left(\hat{G}^{\mu\nu} + K\hat{T}^{\mu\nu}\right)\delta\hat{g}_{\mu\nu} + K\hat{T}^{\mu\nu}_{;\nu}\delta\hat{x}_\mu + 2K\hat{\rho}_o\hat{S}_o\delta\hat{T}_o]\sqrt{-\hat{g}}dudvdwds \qquad (2.29)$$

where we have utilized the identity $G^{\mu\nu}{}_{;\nu} = 0$.

Assuming that $\delta\hat{I}_{rel}$ vanishes for arbitrary variations in the independently varied quantities $\delta\hat{g}_{\mu\nu}$ and $\delta\hat{x}_\mu$ we conclude from (2.29) and the fundamental theorem of the calculus of variations that

$$\hat{G}^{\mu\nu} + K\hat{T}^{\mu\nu} = 0 \qquad (2.30)$$

and

$$\hat{T}^{\mu\nu}_{;\nu} = 0. \qquad (2.31)$$

By choosing the value of the Lagrange multiplier λ to be $\lambda = c^2 + U_o + p/\rho_o$ in Equation (2.28), Equations (2.31) become the equations of motion for a general

relativistic perfect fluid. Taub[3] shows from thermodynamic considerations that such a choice is consistent with the velocity-normalization constraint, Equation (2.13).

To see how the four equations in Equation (2.31) reduce to the non-relativistic equations of perfect fluid motion we consider the components parallel and perpendicular to u^v : $u_\mu T^{\mu\nu}_{;\nu}$ and $\left(\delta^\sigma_\mu - u^\sigma u_\mu\right) T^{\mu\nu}_{;\nu}$. With the aid of the definition of $T^{\mu\nu}$, Equation (2.28), velocity normalization, Equation (2.13), the thermodynamic relation Equation (2.7), and continuity, Equation (2.18), it may easily be shown that $u_\mu T^{\mu\nu}_{;\nu} = \rho_o T u^\nu S_{,\nu}$ and $\left(\delta^\sigma_\mu - u^\sigma u_\mu\right) T^{\mu\nu}_{;\nu} = \rho_o \lambda u^\sigma_{;\nu} u^\nu - \left(\delta^\sigma_\mu - u^\sigma u_\mu\right) g^{\mu\nu} P, \nu$. Setting the first of these to zero obviously gives rise to entropy conservation while the second when equated to zero gives the relativistic analogue of Equation (2.1), including gravitational effects.

It is interesting to note that the variation of $\hat{I}_{rel}$ can lead directly to an equation requiring entropy conservation. If we introduce in a somewhat *ad hoc* manner a scalar-valued function[33] $\hat{\alpha}$ and its variation $\delta\hat{\alpha}$ related to $\delta\hat{T}_o$ according to $\delta\hat{T}_o = (\delta\hat{\alpha})_{,\sigma}\, \hat{u}^\sigma$, we may rewrite the last term in brackets in Equation (2.29) as $2K(\hat{\rho}_o\hat{S}_o\hat{u}^\sigma)_{;\sigma}\delta\hat{\alpha}$ after an integration by parts in which we assume that $\delta\hat{\alpha}$ vanishes on the boundary of the volume of integration. If we then assume that $\delta\hat{I}_{rel} = 0$ for arbitrary variations in the independently varied quantities $\delta\hat{g}_{\mu\nu}$, $\delta\hat{x}_\mu$ and $\delta\hat{\alpha}$ we recover (2.30), (2.31) and in addition obtain

$$\left(\hat{\rho}_o\hat{S}_o\hat{u}^\sigma\right)_{;\sigma} = 0$$

which, after using mass conservation, Equation (2.17), reduces to

$$\hat{\rho}_o\hat{u}^\sigma\hat{S}_{o;\sigma} = 0. \tag{2.32}$$

Equation (2.32) states that entropy $\hat{S}_o$ is conserved along fluid element trajectories.

3. Alternate relativistic formulation

A somewhat more elegant means than that due to Taub of formulating a variational principle for the general relativistic equations of perfect fluid motion in the Lagrangian description has been developed by Hawking and Ellis[4] and improved upon by Schutz and Sorkin[5] in what they term "the minimally constrained variational principle" for reasons to be discussed later. Rather than transforming to a particular comoving coordinate system as in Taub's method, Schutz and Sorkin define a Lagragian variation Δ of the field quantities which acts on an Eulerian fundamental integral.

The Lagrangian variation Δ is defined as the variation *following* a fluid element path in terms of a vector field ξ^α known as the Lagrangian displacement vector. ξ^α is the source of the variations of the fluid element paths or particle world lines in that it moves a world line from its unvaried path to its varied path. Since the Euler-Lagrange equations or equations of motion arise when variations vanish at the initial and final times we require that ξ^α vanish at the initial and final times. If δ represents the Eulerian variation, that is, the variation of field quantities at fixed coordinate values, Δ and δ are related according to

$$\Delta = \delta + L_\xi \tag{2.33}$$

where L_ξ is the Lie derivative with respect to ξ^α, the coordinate-independent generalization of the directional derivative.[34]

In the minimally constrained principle strict entropy and particle number (or mass) conservation need not be constrained *per se*. Rather, it is sufficient to constrain

$$\Delta S_o = 0 \quad \text{and} \quad \Delta j_o^\alpha = 0 \tag{2.34}$$

where S_o is again the specific (rest) entropy and $j_o^\alpha \equiv \sqrt{-g}\rho_o u^\alpha$ is the (mass) current density. What Equations (2.34) do require is that if there are entropy and/or particle sources and sinks in the fluid they must be carried along by ξ^α.

These constraints will be referred to as ***preservation*** of entropy and particles. Also, we choose not to constrain four-velocity normalization, Equation (2.13), as part of the variational principle.

The Lagrangian density in this case may be obtained directly from Hamilton's principle, i.e., it may be taken as the difference in the kinetic energy density and the potential energy density. A relativistic quantity which has the property of reducing to this difference in the non-relativistic limit is $-\sqrt{-g}\rho_o(c^2+U_o)$ by arguments outlined in Schutz and Sorkin[5] (page 23), so we adopt this quantity as the Lagrangian density. The fundamental integral I_R then becomes

$$I_R = -\int \sqrt{-g}\rho_o(c^2+U_o)d^4x. \tag{2.35}$$

The reason that we have not included the scalar curvature in the fundamental integral I_R is that at present we are only interested in obtaining the matter equations, Equation (2.31), and not the Einstein equations, Equation (2.30). Hence, in varying I_R we hold the metric tensor fixed, that is we require $\delta g_{\mu\nu}=0$. From the relation between δ and Δ, Equation (2.33), we then conclude

$$\Delta g_{\mu\nu} = L_\xi g_{\mu\nu} = g_{\mu\nu},_\alpha \xi^\alpha + g_{\mu\nu}\xi^\alpha,_\mu + g_{\mu\alpha}\xi^\alpha,_\nu = \xi_{\mu;\nu} + \xi_{\nu;\mu}. \tag{2.36}$$

In order to find the Lagrangian variation of I_R the following identities are needed and are readily calculated:

$$\Delta(-g)^{1/2} = \frac{1}{2}(-g)^{1/2}g^{\mu\nu}\Delta g_{\mu\nu}, \tag{2.37a}$$

$$\Delta\rho_o = \Delta\left(-g_{\mu\nu}j_o^\mu j_o^\nu/g\right)^{1/2} = \frac{1}{2}\rho_o\left(u^\mu u^\nu - g^{\mu\nu}\right)\Delta g_{\mu\nu}, \tag{2.37b}$$

and

$$\Delta U_o = T_o\Delta S_o + (p/\rho_o^2)\Delta\rho_o = (p/2\rho_o)(u^\mu u^\nu - g^{\mu\nu})\Delta g_{\mu\nu}. \tag{2.37c}$$

Substituting Equations (2.37) into the expression $\Delta I_R = \int \Delta[-\sqrt{-g}\rho_o(c^2+U_o)]d^4x$ we obtain

$$\Delta I_R = -\int \Big[\frac{1}{2}\rho_o(c^2+U_o)g^{\mu\nu} + (c^2+U_o)\left(\frac{1}{2}\rho_o(u^\mu u^\nu - g^{\mu\nu})\right)$$
$$+\rho_o\left((p/2\rho_o)(u^\mu u^\nu - g^{\mu\nu})\right)\Big](-g)^{1/2}\Delta g_{\mu\nu}d^4x$$
$$= -\frac{1}{2}\int \sqrt{-g}T^{\mu\nu}\Delta g_{\mu\nu}d^4x = \int \sqrt{-g}T^{\mu\nu}_{;\nu}\xi_\mu d^4x \tag{2.38}$$

where the last equality follows from (2.36) and an integration by parts assuming that ξ_μ vanishes on the boundary of the region of integration. Allowing ξ_μ to assume arbitrary values within the region of interest, which is the same as requiring fluid element paths to be varied arbitrarily, we then conclude from Equation (2.38) that if I_R is stationary with respect to Lagrangian variations ($\Delta I_R = 0$) under the constraints of Equation (2.34), then the matter equations $T^{\mu\nu}_{;\nu} = 0$ are satisfied.

We conclude with one last comment concerning the constraint equations (2.34). The matter equations (2.31) imply the following:

$$u_\mu T^{\mu\nu}_{;\nu} = u_\mu\left(\lambda u^\mu(\rho_o u^\nu)_{;\nu} + \rho_o\lambda_{,\nu}u^\mu u^\nu + \rho_o\lambda u^\mu_{;\nu}u^\nu - p_{,\nu}g^{\mu\nu}\right)$$
$$= \lambda(\rho_o u^\nu)_{;\nu} + \rho_o u^\nu\left(U_{o,\nu} + (p/\rho_o)_{,\nu} - p_{,\nu}/\rho_o\right)$$
$$= \lambda(\rho_o u^\nu)_{;\nu} + \rho_o T_o u^\nu S_{o,\nu} = 0 \tag{2.39}$$

where the second equality follows from the definition of λ and four-velocity normalization, Equation (2.13), and the third from the thermodynamic relation (2.7). Thus, although Equations (2.34) do not require strict conservation of particle number or entropy *per se* the matter equations require a particular relation between particle production and entropy production given by Equation (2.39). In fact, if either particle number or entropy is conserved, then the matter equations imply that the other must also be conserved as can easily be seen from Equation (2.39). The exact form of particle number or entropy production (but not both) must be imposed as an *ad hoc* additional assumption in order to obtain a closed set of equa-

tions from this minimally constrained variational principle; e.g., one might impose no entropy production (entropy conservation) or no particle production.

B. Variational principles for a perfect fluid –the Eulerian description

As the previous section (Section A) demonstrates, developing variational principles in the Lagrangian description often involves the development of specialized and occasionally *ad hoc* techniques. Although it may be argued that the Lagrangian description is based on the actual physics of the problem in that a real fluid is actually composed of particles or fluid elements, in order to avoid some of its inherent complications, such as the need for specialized techniques, it would be well to seek for a more elegant variational principle leading to the equations of motion for a perfect fluid. The development of variational principles in the Eulerian description is motivated by its increased convenience and elegance over the development of those in the Lagrangian description. For example, in Eulerian variational principles one is able to incorporate any necessary holonomic constraint on the variations through the Lagrange multiplier method described in Section C.3 of Chapter I, something not generally possible in Lagrangian variational principles. In fact, nonholonomic constraints can often be incorporated successfully with a Lagrange multiplier-type constraint in Eulerian variational principles, e.g., mass conservation, entropy conservation and the soon-to-be-discussed Lin constraint. It might further be argued that the Eulerian variational principles contain just as much physics as their Lagrangian counterparts in that they both give rise to the same physical predictions; however, the physics is often masked in the Eulerian description and must be closely scrutinized.

Notwithstanding their advantages in fluid mechanics Eulerian variational principles have progressed somewhat more slowly than their Lagrangian counterparts, probably due to what some have termed the "mysterious" nature of the Lin constrain. In fact, Clebsch[6] in 1859 was the first to develop a variational principle in the Eulerian description leading to the equations of motion for a perfect fluid,

but his work was restricted to incompressible isentropic fluids. In 1929, Bateman[7] extended the work of Clebsch to include compressible isentropic fluids (this was also done independently by Lamb[8] in 1932). However, it was not until 1968 that Seliger and Whitham[9] presented a variational principle for a general non-relativistic perfect fluid, i.e., a fluid which is generally compressible and nonisentropic. Two years later, Schutz[11] extended the method of Seliger and Whitham to general relativistic perfect fluids. In this section, we first present the non-relativistic formulation of Seliger and Whitham, then Schutz' relativistic extension.

1. Non-relativistic formulation

In the Eulerian description we are not required to treat the displacement vector of a fluid element as a field quantity; instead, we recognize that the dynamical behavior of a perfect fluid can be modeled completely by specifying the values of the fluid's mass density ρ_m, pressure p, specific entropy S, and velocity field $\mathbf{v}$ all as functions of the spatial and temporal coordinates $\mathbf{x}$ and t. Since in this description we do not concern ourselves with the question of where a particular fluid element (or particle) is at a given time, we lose the close similarity with a system of particles which is maintained by a Lagrangian description.

As is well known, the equations of motion for a non-relativistic perfect fluid in the Eulerian description are

$$\rho_m \frac{D}{Dt}\mathbf{v} + \nabla p = 0, \tag{2.40}$$

$$\partial \rho_m / \partial t + \nabla \cdot (\rho_m \mathbf{v}) = 0, \tag{2.41}$$

and

$$DS/Dt = 0, \tag{2.42}$$

where $D/Dt \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$ is the convective derivative and $p = p(\rho_m, S)$. Equation (2.40) follows its Lagrangian counterpart, Equation (2.1), by defining $\mathbf{v}\left(\mathbf{q}\left(\mathbf{a}, t\right), t\right) =$

$\partial \mathrm{q}(\mathrm{a},t)/\partial t$, Equation (2.41) is equivalent to conservation of mass along fluid element trajectories, and Equation (2.42) requires entropy to be convected with the fluid.

If we rely on Hamilton's principle, that the Lagrangian density be the difference in the kinetic energy density and the potential energy density, as we did in the Lagrangian description, we encounter unforeseen difficulties. To illustrate these difficulties consider the fundamental integral defined by Hamilton's principle

$$I_1 = \int [\frac{1}{2}\rho_m u^2 - p_m u(\rho_m, S)] d^3x dt. \tag{2.43}$$

Free variation of each of the three components of the velocity field v_i, $i = 1,2,3$, gives $v_i = 0$, a great restriction over the expected result. Hence the class of variations is much too large. We must somehow restrict the variations of those field quantities in order to retrieve the more general equations of motion (2.40).

A reasonable first attempt at restricting the variations of the field quantities is to vary I_1 subject to the side conditions that mass and entropy be conserved (Eqs. (2.41) and (2.42)). We incorporate those equations as constraints on the variations of the field quantities by using the Lagrange multiplier technique described in Section C.3 of Chapter I. Before doing so, for convenience (not of necessity) we choose to rewrite (2.42), using (2.41), in the form

$$\partial(\rho_m S)/\partial t + \nabla \cdot (\rho_m S \mathbf{v}) = 0. \tag{2.44}$$

The action I_1 is then transformed to (with λ and μ Lagrange multipliers)

$$I_2 = \int \Big[\frac{1}{2}\rho_m v^2 - \rho_m u(\rho_m, S) + \lambda\left(\partial \rho_m/\partial t + \nabla \cdot (\rho_m \mathbf{v})\right)$$
$$+ \mu\left(\partial(\rho_m S)/\partial t + \nabla \cdot (\rho_m S \mathbf{v})\right)\Big] d^3x dt.$$

The attendant Euler-Lagrange equations include

$$\delta v_i : \quad \rho_m v_i - \rho_m \partial\lambda/\partial x^i - \rho_m S \partial\mu/\partial x^i = 0, \; i = 1,2,3,$$

which simplifies to

$$\mathbf{v} = \nabla\lambda + S\nabla\mu. \tag{2.45}$$

Unfortunately, Equation (2.45) is still too restrictive, for in the isentropic case (S = constant), (2.45) implies that the velocity field $\mathbf{v}$ be curl-free ($\nabla \times \mathbf{v} = 0$). From experiment we know that isentropic perfect fluid flow with $\nabla \times \mathbf{v} \neq 0$ exists; hence, Equation (2.45) is not completely general. It was not until 1958 that this difficulty was resolved, at which time Lin[2] developed a constraint which he imposed on the variations of the field quantities of Equation (2.43). His constraint proved sufficient to recover the completely general equations of perfect fluid motion. Lin argued that although the Lagrangian fluid labels are no longer explicitly needed in the Eulerian description of a fluid, the motion of the fluid should be such that labels $\mathbf{a}$ can always be found for any given fluid element. Since the components of $\mathbf{a}$ are interpreted as the initial coordinates of a fluid element, Lin's requirement may be formulated in terms of Eulerian coordinates as

$$D\mathbf{a}(\mathbf{x},t)/Dt = 0, \tag{2.46}$$

i.e., the initial conditions do not change along the path of a fluid element. This is known as the Lin constraint. By invoking the Lin constrain we are requiring, in effect, that the motion of a given fluid element be traceable backwards in time to its original position.

We incorporate the Lin constraint by constraining the variations of I_1 using all three of Equations (2.41), (2.44) and (2.46). This is the method originally employed by Lin to obtain completely general equations of motion. It has the disadvantage of having more field quantities than are actually necessary and hence is overly complex. Instead of using Lin's method here we choose the alternate approach, due to Schutz and Sorkin,[5] of first applying Pfaff's theorem[26] (see Section D of Chapter I) to simplify the last term of the integrand in Equation (2.40). We first let $\mathbf{B}$ denote the Lagrange multiplier which constrains Eq. (2.46). The Lin constraint term of

the variational integral is then

$$\rho_m \mathbf{B} \cdot D\mathbf{a}/Dt. \tag{2.47}$$

since the term must be converted to a density through multiplication by ρ_m. Define also $j_m^\sigma, \sigma = 0,1,2,3$, by $j_m^\sigma = \rho_m(c_1, v_1, v_2, v_3)$ and $\partial_\sigma \equiv \partial/\partial x^\sigma$ where $x^\sigma = (ct, x, y, z)$. Then equation (2.47) may be written as

$$\rho_m \mathbf{B} \cdot D\mathbf{a}/Dt = j_m^\sigma \mathbf{B} \cdot \partial_\sigma \mathbf{a} \tag{2.48}$$

(implied sum on σ). Since $\mathbf{a}(\mathbf{x}, t)$ represents the initial position of a fluid element one must be able to invert the expression for $\mathbf{a}$ as a function of $\mathbf{x}$ to find $\mathbf{x}$ as a function of $\mathbf{a}$ at each t. This means that all functions of $\mathbf{x}$ and t may also be considered as functions of $\mathbf{a}$ and t. Suppressing the dependence on t and expressing $\mathbf{B} \cdot \partial_\sigma \mathbf{a}$ in differential geometric notation we may write

$$\mathbf{B} \cdot \partial_\sigma \mathbf{a} dx^\sigma = B_1(\mathbf{a}) da_1 + B_2(\mathbf{a}) da_2 + B_3(\mathbf{a}) da_3 = d\alpha + \beta d\gamma,$$

where in the last step we have utilized Pfaff's theorem to introduce functions α, β, γ. Converting back to the notation of Equation (2.48) we then find that

$$\rho_m \mathbf{B} \cdot D\mathbf{a}/Dt = j_m^\sigma \partial_\sigma \alpha + \beta j_m^\sigma \partial_\sigma \gamma = \rho_m (D\alpha/Dt + \beta D_\gamma/Dt). \tag{2.49}$$

In place of varying $\mathbf{a}$ and $\mathbf{B}$ independently of other field quantities we now vary α, β and γ arbitrarily. By introducing α, β and γ we have included the additional condition that the Lagrange multiplier $\mathbf{B}$ depend only on $\mathbf{a}$. Imposition of the Lin constraint allows us to ignore the equation of continuity as an equation of constraint, since the requirement that a fluid element's path be traceable back to its initial position also requires that it maintain its integrity, i.e., that continuity be satisfied.

We now add Equation (2.49) to the fundamental integral I_2 and remove the

equation of continuity constraint term to obtain the fundamental integral

$$I_3 = \int \Big[\frac{1}{2}\rho_m v^2 - p_m u + \mu\Big(\partial(\rho_m S)/\partial t + \nabla\cdot(\rho_m S\mathbf{v})\Big) + \rho_m\Big(D\alpha/Dt + \beta D_\gamma/Dt\Big)\Big] d^3x dt. \tag{2.50}$$

The Euler-Lagrange equations that result from the variation of the field quantities of I_3, Equation (2.50), are as follows:

$$\delta v_i: \quad \rho_m v_i - \rho_m S\partial\mu/\partial x^i + \rho_m \partial\alpha/\partial x^i + \rho_m\beta\alpha_\gamma/\partial x^i = 0, \quad i = 1,2,3, \tag{2.51a}$$

$$\delta\rho_m: \quad \frac{1}{2}v^2 - u - \rho_m\partial u/\partial\rho_m - SD\mu/Dt + D\alpha/Dt + \beta D\gamma/Dt = 0, \tag{2.51b}$$

$$\delta S: \quad -\rho_m\partial u/\partial S - \rho_m D\mu/Dt = 0, \tag{2.51c}$$

$$\delta\beta: \quad \rho_m D\gamma/Dt = 0, \tag{2.51d}$$

$$\delta\gamma: \quad \partial(\rho_m\beta)/\partial t + \nabla\cdot(\rho_m\beta\mathbf{v}) = 0, \tag{2.51e}$$

$$\delta\mu: \quad \partial(\rho_m S)/\partial t + \nabla\cdot(\rho_m S\mathbf{v}) = 0, \tag{2.44}$$

and

$$\delta\alpha: \quad \partial\rho_m/\partial t + \nabla\cdot(\rho_m\mathbf{v}) = 0. \tag{2.41}$$

Note that the second-to-the-last equation is the equation of constraint requiring conservation of entropy, while the last equation is the equation of continuity. we may rewrite these equations as follows:

$$\mathbf{v} = S\nabla\mu - \nabla\alpha - \beta\nabla\gamma, \tag{2.52a}$$

$$D\alpha/Dt = g(\rho_m, S) - \frac{1}{2}v^2, \tag{2.52b}$$

$$D\mu/Dt = -T(\rho_m, S), \tag{2.52c}$$

$$D\gamma/Dt = D\beta/Dt = DS/Dt = O, \tag{2.52d}$$

$$D\rho_m/Dt = -\rho_m \nabla \cdot \mathbf{v}. \tag{2.52e}$$

Equation (2.52a) follows directly from (2.51a). Equations (2.52b,c) follow from the fundamental equation of thermodynamics and from the definition of the specific Gibbs free energy $g(\rho_m, S) = u(\rho_m, S) + p(\rho_m, S)/\rho_m - ST(\rho_m, S)$. The quantity μ is called the *thermasy.*[35] (2.52d) is a result of (2.41), (2.44), (2.51d) and (2.51e), while (2.52e) is a direct consequence of the equation of continuity, Equation (2.41). Equations (2.52a) through (2.52e), or the globally equivalent set (2.41), (2.44), and (2.51a) - (2.51e), constitute a closed set of equations which is locally equivalent to the standard set of equations of perfect fluid motion, Equations (2.40), (2.41) and (2.42), as will be shown.

Before demonstrating the equivalence of the two sets of equations of motion we pause to make three remarks. Firstly, we point out that Equation (2.52a) does not physically restrict the fluid's velocity field as does Equation (2.45). To see this we take the curl of $\mathbf{v}$ to find, from equation (2.52a), that

$$\omega \equiv \nabla \times \mathbf{v} = \nabla S \times \nabla \mu - \nabla \beta \times \nabla \gamma. \tag{2.53}$$

This expresses the fluid's *vorticity* ω as the sum of two contributions: one caused by entropy gradients, the other introduced initially. The quantity $\nabla \beta \times \nabla \gamma$ represents the intersection of the family of surfaces β = constant and γ = constant and

Equations (2.52d) states that these surfaces are convected with the fluid. Note that the initial vorticity is probably itself due to the entropy gradients associated with external or viscous forces.

Secondly, we note that the Lin constraint is somewhat mysterious from a mathematical point of view in that it is not clear that it is a constraint at all. Since in the side condition, Equation (2.46), which is the essence of the Lin constraint, we introduce a new field quantity a which does not appear elsewhere in the Lagrangian density it would seem that the constraint should be vacuous, i.e., constraining Equation (2.46) should lead to equations of motion equivalent to those obtained by ignoring the constraint. The manipulations of the constraint term resulting in the revised constraint term, Equation (2.45), do not seem to alter the impression that the constraint should be vacuous. Nevertheless, use of the constraint leads to a more general set of equations of motion so that it is indeed a valid constraint on the variations. This point will be discussed more thoroughly in the following sections.

Lastly, we consider a simplified form of I_3, Equation (2.50), due to Seliger and Whitham.[9] We begin by integrating the term involving S by parts, assuming that the field quantities vanish on the boundary of the region of integration. The term may then be replaced by $-\rho_m S D_\mu/Dt$. Now, substitute Equations (2.51d) and (2.52b,c) into the resultant expression for I_3 to get

$$I_3 = \int [\frac{1}{2}\rho_m v^2 - \rho_m u - \rho_m S(-T) + \rho_m \left(g - \frac{1}{2}v^2\right)]d^3x dt = \int p d^3x dt.$$

Hence, the Lagrangian density is simply the pressure p. One may begin with $p = p(h, S)$ for the Lagrangian density, where $h \equiv u + p/\rho_m$ is the specific enthalpy, and recover the equations of motion (2.52a)-(2.52e) by first noting the thermodynamic relation

$$dp = \rho_m dh - \rho_m T ds \quad (i.e., \partial p/\partial h = \rho_m \quad \text{and} \quad \partial p/\partial S = -\rho_m T). \tag{2.54}$$

Also, one must express h in terms of potentials and entropy as

$$h = \partial\alpha/\partial t + \beta\partial\gamma/\partial t - S\partial\mu/\partial t - \frac{1}{2}(S\nabla\mu - \nabla\alpha - \beta\nabla\gamma)^2 \tag{2.55}$$

(note that this follows from (2.51b) after using $h = u + p_m\partial u/\partial\rho_m$ and (2.52a) for $\mathbf{v}$), then vary each of the quantities α, β, γ, S and μ independently, and finally define $\mathbf{v}$ according to the "Clebsch representation," Equation (2.52a), in order to simplify the resultant equations of motion.

2. Equivalence of the non-relativistic equations of motion

In order to demonstrate the equivalence of the standard set of equations of motion for a perfect fluid, Equations (2.40)-(2.42), to the potential set, Equations (2.52a)-(2.52e), we must show that one set implies the other when each set is appended with the relations of thermodynamics. We begin by showing that the potential set implies the standard set. Hence, we assume the existence of $\mathbf{v}, S, \mu, \alpha, \beta, \gamma$ and ρ_m such that the potential set is satisfied. Both continuity, Equation (2.41), and entropy conservation, Equation (2.42), are included in the potential set and hence are an immediate consequence of the potential set; thus, we need only show that Equation (2.40) follows from the potential set. To do so, we simply calculate $D\mathbf{v}/Dt$ using the potential representation, Equation (2.52a), for $\mathbf{v}$ and the vector identities

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) \tag{2.56}$$

and

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{2.57}$$

$$D\mathbf{v}/Dt \equiv \partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v} = \partial\mathbf{v}/\partial t + \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v}) \tag{2.58}$$

follows from (2.56). Now, from Equation (2.52a),

$$\partial\mathbf{v}/\partial t = \frac{\partial}{\partial t}(S\nabla\mu - \nabla\alpha - \beta\nabla\gamma)$$

$$= \nabla(S\mu_t - \alpha_t - \beta\gamma_t) + S_t\nabla\mu - \beta_t\nabla\gamma - \mu_t\nabla S + \gamma_t\nabla\beta, \tag{2.59}$$

where, for brevity, we have used a subscript t to denote partial differentiation with respect to t. Also, from (2.52a), (2.53) and (2.57) we compute

$$\begin{aligned}
\mathbf{v}\times(\nabla\times\mathbf{v}) &= (S\nabla\mu - \nabla\alpha - \beta\nabla\gamma)\times(\nabla S\times\nabla\mu - \nabla\beta\times\nabla\gamma)\\
&= S\nabla\mu\times(\nabla S\times\nabla\mu) - S\nabla\mu\times(\nabla\beta\times\nabla\gamma) - \nabla\alpha\times(\nabla S\times\nabla\mu) + \nabla\alpha\times(\nabla\beta\times\nabla\gamma)\\
&\quad - \beta\nabla\gamma\times(\nabla S\times\nabla\mu) + \beta\nabla\gamma\times(\nabla\beta\times\nabla\gamma)\\
&= S(\nabla\mu)^2\nabla S - (S\nabla\mu\cdot\nabla S)\nabla\mu\\
&\quad - (S\nabla\mu\cdot\nabla\gamma)\nabla\beta + (S\nabla\mu\cdot\nabla\beta)\nabla\gamma - (\nabla\alpha\cdot\nabla\mu)\nabla S + (\nabla\alpha\cdot\nabla S)\nabla\mu + (\nabla\alpha\cdot\nabla\gamma)\nabla\beta\\
&\quad - (\nabla\alpha\cdot\nabla\beta)\nabla\gamma - (\beta\nabla\gamma\cdot\nabla\mu)\nabla S + (\beta\nabla\gamma\cdot\nabla S)\nabla\mu + \beta(\nabla\gamma)^2\nabla\beta - (\beta\nabla\gamma\cdot\nabla\beta)\nabla\gamma\\
&= (\mathbf{v}\cdot\nabla\mu)\nabla S - (\mathbf{v}\cdot\nabla S)\nabla\mu - (\mathbf{v}\cdot\nabla\gamma)\nabla\beta + (\mathbf{v}\cdot\nabla\beta)\nabla\gamma
\end{aligned} \tag{2.60}$$

Substitution of Equations (2.59) and (2.60) into (2.58) yields

$$\begin{aligned}
D\mathbf{v}/Dt &= \nabla(S\mu_t - \alpha_t - \beta\gamma_t + \frac{1}{2}v^2) + (DS/Dt)\nabla\mu\\
&\quad - (D\beta/Dt)\nabla\gamma - (D\mu/Dt)\nabla S + (D\gamma/Dt)\nabla\beta
\end{aligned} \tag{2.61}$$

Comparing the first quantity in brackets with Equation (2.55), keeping in mind the velocity representation, Equation (2.52a), we see that it is just the negative of the specific enthalpy h (recall that this expression for h follows from the potential set of equations). Using this fact and Equations (2.52c,d) we may simplify expression (2.61) to the form

$$D\mathbf{v}/Dt = -\nabla h + T\nabla S = -\frac{1}{\rho_m}\nabla p$$

where the last equality follows from the thermodynamic relation (2.54). Hence, (2.40) follows from the potential set.

We now show that the potential set follows from the standard set. Note firstly that both continuity, Equation (2.52e), and entropy conservation, Equation (2.52d), follow immediately from the standard set. Applying Pfaff's theorem (Section D of

Chapter I) to the one-form $\varepsilon = \sum_{i=1}^{3} \left(S\partial\mu/\partial x^i - v_i\right) dx^i$ where $\mathbf{v}, S$ and ρ_m are assumed to solve Equations (2.40)-(2.42) and μ is a yet unspecified C^∞ function we conclude the local existence of C^∞ real-valued functions α, β and γ such that

$$S\nabla\mu - \mathbf{v} = \nabla\alpha + \beta\nabla\gamma.$$

A simple rearrangement of terms then leads to Equation (2.52a). For $\mathbf{v}$ of this form $D\mathbf{v}/Dt$ takes the form given by Equation (2.61). Using Equation (2.40) we then find that

$$\begin{aligned}(1/\rho_m)\nabla p = -D\mathbf{v}/Dt &= -\nabla(S\mu_t - \alpha_t - \beta\gamma_t + \frac{1}{2}v^2) \\ &- (D\beta/Dt)\nabla\gamma - (D\mu/Dt)\nabla S + (D\gamma/Dt)\nabla\beta. \end{aligned} \tag{2.62}$$

If we now specify μ to be the thermasy, i.e., a solution to Equation (2.52c), and utilize the thermodynamic relationship (2.54), Equation (2.62) may be rewritten as

$$\nabla(h + S\mu_t - \alpha_t - \beta\gamma_t + \frac{1}{2}v^2) + (D\beta/Dt)\nabla\gamma - (D\gamma/Dt)\nabla\beta = 0, \tag{2.63}$$

where h is the specific enthalpy. Define H as

$$H \equiv h + S\mu_t - \alpha_t - \beta\gamma_t + \frac{1}{2}v^2 = g - \frac{1}{2}v^2 - D\alpha/Dt - \beta D\gamma/Dt, \tag{2.64}$$

so as to simplify the expression of (2.63) to the form

$$\nabla H = (D\gamma/Dt)\nabla\beta - (D\beta/Dt)\nabla\gamma. \tag{2.65}$$

This implies that the set of vectors $\nabla_H, \nabla_\beta, \nabla_\gamma$ are linearly dependent and hence that the matrix $(\nabla_H, \nabla_\beta, \nabla_\gamma)$ has vanishing determinant. This matrix is the transpose of the Jacobian matrix of the vector-valued function (H, β, γ) and hence the Jacobian determinant $\frac{\partial(H,\beta,\gamma)}{\partial(x^1,x^2,x^3)}$ must vanish, since matrix transposition does not affect the evaluation of the determinant. We conclude that H, β, γ are functionally dependent, that is, there exists a *functional* $F = F(f_1, f_2, f_3, t)$ such that

$F(H, \beta, \gamma, t) = 0$. We calculate the gradient of F to find

$$(\partial F/\partial H)\nabla H + (\partial F/\partial \beta)\nabla \beta + (\partial F/\partial \gamma)\nabla \gamma = 0. \tag{2.66}$$

Comparing (2.66) with (2.65) we see that we may take $\partial F/\partial H = 1$, and so from $F(H, \beta, \gamma, t) = H + G(\beta, \gamma, t) = 0$ we may write $H = H(\beta, \gamma, t)$. Taking the gradient of H and comparing the result with (2.65) we then conclude that

$$\partial H/\partial \beta = D\gamma/Dt \quad \text{and} \quad \partial H/\partial \gamma = -D\beta/Dt. \tag{2.67}$$

Notice the resemblance of Equations (2.67) to Hamilton's equations.

Now, the potentials α, β and γ are not completely determined by the velocity representation, Equation (2.52a), which we obtained by application of Pfaff's theorem; neither is μ completely determined by the relation $D\mu/Dt = -T$ since an arbitrary function ν with $D\nu/Dt = 0$ may be added to μ without affecting the relation. Hence, there remains some gauge freedom between the potentials which may be exploited in order to obtain the remainder of the potential set of equations (recall that so far we have shown that (2.52a), (2.52c), entropy conservation, continuity, and (2.63) follow from the standard set). By choosing the gauge of μ in such a way that $\nabla S \times \nabla \mu$ vanishes at the initial time t, it can be seen that $\nabla \beta \times \nabla \gamma$ must be responsible for the initial vorticity. Since all subsequent vorticity is caused by entropy gradients we may attribute to $\nabla \beta \times \nabla \gamma$ the interpretation of vortex lines which are convected with the fluid. Hence, we may choose the gauge of β and γ such that $D\beta/Dt = D\gamma/Dt = 0$. Now, from (2.65) or (2.67) we conclude that H is an arbitrary function of time t. This arbitrary function of t may be absorbed into the gauge of α so as to make $H = 0$; hence, there is enough gauge freedom in μ, α, β and γ to allow for the satisfaction of $D\beta/Dt = D\gamma/Dt = 0$, Equation (2.52d), and $H = 0$. From Equation (2.64) it is seen that $H = 0$ together with $D\gamma/Dt = 0$ implies Equation (2.52b). Thus, the potential set follows from the standard set. This concludes the proof of equivalence.

3. Relativistic formulation

A relativistic Eulerian variational principle may be obtained from its Lagrangian description counterpart in much the same way as the non-relativistic Eulerian version was obtained from its Lagrangian counterpart. We begin with essentially the same fundamental integral as was introduced by Taub, Equation (2.12), but we replace $H_o(\rho_o, T_o)$ with $U_o(\rho_o, S_o)$ (i.e., we choose ρ_o and S_o as independent variables rather than ρ_o and T_o). As done previously we let λ be a Lagrange multiplier constraining velocity normalization, Equation (2.13), but we now vary λ explicitly. We then constrain the variations via entropy conservation (in the form $(\rho_o u^\nu S_o)_{;\nu} = 0$), and the Lin constraint, which we include in the Lagrangian density in the form $\rho_o u^\nu(\alpha_{,\nu} + \beta\gamma_{,\nu})$ (compare this form with the very similar non-relativistic version in Equation (2.49)). The fundamental integral, Equation (2.12) is then transformed into

$$I'_{rel} = \int \Big[R - 2K\Big(\rho_o(c^2 + U_o + \frac{1}{2}\lambda(g_{\sigma\nu}u^\sigma u^\nu - 1) \\ - \mu(\rho_o S_o u^\nu)_{;\nu} + \rho_o u^\nu(\alpha_{,\nu} + \beta\gamma_{,\nu})\Big)\Big](-g)^{1/2} d^3x dt,$$

where μ is a Lagrange multiplier which constrains entropy conservation. As with the non-relativistic case we now vary ρ_o and u^ν directly rather than vary each fluid element path. Before actually performing the variations it is advantageous, in fact, necessary, to integrate the term involving μ by parts, requiring the field quantities to vanish on the boundary of integration. The fundamental integral then becomes

$$I''_{rel} = \int \Big[R - 2K\rho_o\Big(c^2 + U_o + \frac{1}{2}\lambda(g_{\sigma\nu}u^\sigma u^\nu - 1) \\ + u^\nu(S_o\mu_{,\nu} + \alpha_{,\nu} + \beta\gamma_{,\nu})\Big)\Big](-g)^{1/2} d^3x dt. \qquad (2.68)$$

The Euler-Lagrange equations that follow from I''_{rel}, Equation (2.68), are as follows:

$$\delta g_{\sigma\nu}: \quad -(-g)^{1/2}\Big[G^{\sigma\nu} + K\rho_o\Big(\lambda u^\sigma u^\nu + g^{\sigma\nu}\Big(c^2 + U_o + \frac{1}{2}\lambda(g_{\eta\delta}u^\eta u^\delta - 1)$$

$$+\,u^{\eta}(S_o\mu,_{\eta}+\alpha,_{\eta}+\beta\gamma,_{\eta}))\Big)\Big] = 0, \tag{2.69a}$$

$$\delta u^{\nu}:\quad \lambda u_{\nu}+S_o\mu,_{\nu}+\alpha,_{\nu}+\beta\gamma,_{\nu}=0, \tag{2.69b}$$

$$\delta\rho_o:\quad c^2+U_o+\rho_o(\partial U_o/\partial\rho_o)_{S_o}+\frac{1}{2}\lambda(g_{\sigma\nu}u^{\sigma}u^{\nu}-1)+u^{\nu}(S_o\mu,_{\nu}+\alpha,_{\nu}+\beta\gamma,_{\nu})=0, \tag{2.69c}$$

$$\delta S_o:\quad (\partial U_o/\partial S_o)_{\rho_o}+u^{\nu}\mu,_{\nu}=0, \tag{2.69d}$$

$$\delta\lambda:\quad g_{\sigma\nu}u^{\sigma}u^{\nu}=1, \tag{2.13}$$

$$\delta\mu:\quad (\rho_o S_o u^{\nu});_{\nu}=0, \tag{2.69e}$$

$$\delta\alpha:\quad (\rho_o u^{\nu});_{\nu}=0, \tag{2.18}$$

$$\delta\beta:\quad \rho_o u^{\nu}\gamma,_{\nu}=0, \tag{2.69f}$$

$$\delta\gamma:\quad (\rho_o\beta u^{\nu});_{\nu}=0. \tag{2.69g}$$

Equation (2.69a) is obtained with the aid of the relations $\delta(\sqrt{-g}R)/\delta g_{\sigma\nu}=\sqrt{-g}G^{\sigma\nu}$ and $\partial\sqrt{-g}/\partial g_{\sigma\nu}=\frac{1}{2}\sqrt{-g}g^{\sigma\nu}$. This set of equations may be reduced to the set of globally equivalent equations (obtained through strictly algebraic manipulations)

$$G^{\sigma\nu}+KT^{\sigma\nu}=0, \tag{2.70a}$$

$$\lambda u_\nu = -(S_o \mu_{,\nu} + \alpha_{,\nu} + \beta \gamma_{,\nu}), \tag{2.70b}$$

$$D\alpha/D\tau = -(c^2 + U_o + p/\rho_o - S_o T_o), \tag{2.70c}$$

$$D\mu/D\tau = -T_o, \tag{2.70d}$$

$$g_{\sigma\nu} u^\sigma u^\nu = 1, \tag{2.13}$$

$$DS_o/D\tau = 0, \tag{2.70e}$$

$$D\rho_o/D\tau = -\rho_o u^\nu{}_{;\nu}, \tag{2.70f}$$

$$D\gamma/D\tau = D\beta/D\tau = 0, \tag{2.70g}$$

where $D/D\tau \equiv u^\nu \partial/\partial x^\nu$ is the relativistic version of the convective derivative, and as before $T^{\sigma\nu} \equiv \rho_o \lambda u^\sigma u^\nu - p g^{\sigma\nu}$ (see Equation (2.28)). Equation (2.70b) follows directly from (2.69b) as does (2.70f) from (2.18) and $D\gamma/D\tau = 0$ from (2.69f). Equation (2.70e) is a result of (2.69e) and (2.18) while $D\beta/D\tau = 0$ results from (2.69g) and (2.18). The thermodynamic relation (2.7) may be used along with (2.69d) to obtain (2.70d) and Equation (2.70c) results from (2.69c) after using (2.7), (2.13), (2.69f), and (2.70d). Lastly, (2.70a) follows from (2.69a) through the use of (2.13), (2.69f), (2.70c) and (2.70d). Note that Taub's expression[3] $\lambda = c^2 + U_o + p/\rho_o$ which he deduced indirectly through thermodynamic arguments follows directly from this

set by multiplying (2.70b) by u^ν, summing over ν and using (2.70c), (2.70d) and (2.70g). This set of equations as expressed in terms of potentials (appended with the usual relations of thermodynamics) is locally equivalent to the standard set comprised of four-velocity normalization, Equation (2.13), continuity, Equation (2.18) (or equivalently Equation (2.70f)), entropy conservation, Equation (2.32) (or equivalently Equation (2.70e)), and Equations (2.30) (or (2.70a)) and (2.31) (the standard set is also appended with the usual relations of thermodynamics).

Before proceeding with the proof of equivalence of the two sets of equations of motion we remark that by substituting the Euler-Lagrange equations (2.13), (2.70c), (2.70d) and (2.70g) into the fundamental integral I''_{rel}, Equation (2.68), the integral is transformed into

$$I'''_{rel} = \int [R + 2Kp] d^3x dt \tag{2.71}$$

so that the Lagrangian density is essentially the sum of the scalar curvature R and the pressure p. Schutz,[11] in his derivation of the equations of motion for a relativistic perfect fluid from a variational principle, begins with the fundamental integral I'''_{rel}, Equation (2.71), then imposes explicity four-velocity normalization, Equation (2.13), and the four-velocity representation, Equation (2.70b), but with $S_o\mu,_\nu$ replaced by $-\mu S_o,_\nu$ and α and β by $-\alpha$ and $-\beta$, respectively. He then varies only the potentials while using the usual thermodynamic relations and thereby obtains Equations (2.70d) through (2.70g). He replaces Equation (2.70c) by $D\alpha/D\tau = -\lambda = -(c^2 + U_o + P/\rho_o)$. His method is exactly analogous to the alternate procedure for non-relativistic perfect fluids due to Seliger and Whitham as discussed previously near the end of Subsection II.B.1. It has the disadvantage that several of the equations of motion must be imposed explicitly, they do not arise naturally from the variational principle. It has the advantage that the fundamental integral is quite simple.

4. Equivalence of the relativistic equations of motion

As was the case with the two sets of non-relativistic equations of motion, the two sets of relativistic equations of motion have several equations in common. Not only are the usual equations of thermodynamics common to both sets, but so are continuity, Equation (2.18) (or (2.70f)), entropy conservation, Equation (2.32) (or (2.70e)), four-velocity normalization, Equation (2.13), and Einstein's field equations, Equation (2.30) (or (2.70a)). The proof of equivalence therefore reduces to the demonstration on the one hand that the potential set implies that the covariant divergence of the energy-momentum tensor $T^{\sigma\nu}$ vanishes and on the other hand that the standard set implies the (local) existence of potentials α, β, γ and μ such that (2.70b), (2.70c), (2.70d) and (2.70g) are satisfied. It is noteworthy that Einstein's field equations, Equation (2.70a), actually imply the vanishing of the covariant divergence of the energy-momentum tensor $T^{\sigma\nu}$ since the covariant divergence of the Einstein tensor $G^{\sigma\nu}$ vanishes according to the Bianchi identities.[36] Due to this fact it is seen that the standard set follows immediately from the potential set without even invoking any of the equations which involve the potentials α, β, γ and μ. Nevertheless, one must yet show that the equations involving these potentials are consistent with the vanishing of the covariant divergence of the energy-momentum tensor $T^{\sigma\nu}$.

To prove the equivalence of the two sets of equations of motion we follow the elegant and precise method of Schutz[11] (his Appendix B) by first introducing the following theorem.

THEOREM 6.1: Let u^ν describe the four-velocity vector field of a one component perfect fluid with scalar pressure p. In addition, let ρ_o be its number density and define λ by $\lambda = -c^2 - U_o - p/\rho_o$ where c is the speed of light and U_o is the fluid's (rest) internal energy satisfying the thermodynamic relation

$$dU_o = T_o dS + (p/\rho_o^2) d\rho_o, \tag{2.72}$$

T_o being the (rest) temperature field and S_o the (rest) entropy field of the fluid. define the fluid's energy-momentum tensor according to

$$T^{\sigma\nu} = \rho_o \lambda u^\sigma u^\nu - p g^{\sigma\nu} \tag{2.28}$$

and define (up to the appropriate gauge freedom) the scalars μ and α by

$$D\mu/D\tau = -T_o \tag{2.70d}$$

and

$$D\alpha/D\tau = -\lambda + S_o T_o. \tag{2.73}$$

Finally, assume continuity and conservation of entropy,

$$(\rho_o u^\nu)_{;\nu} = 0 \tag{2.18}$$

and

$$DS_o/D\tau = 0 \tag{2.70e}$$

are satisfied. Assuming nothing else

$$L_u(\lambda u_\nu + S_o \mu_{,\nu} + \alpha_{,\nu}) = g_{\nu\sigma} T^{\sigma\beta}{}_{;\beta} / \rho_o \tag{2.74}$$

is a mathematical identity so long as all derivatives exist and are continuous and $\rho_o \neq 0$, where L_u denotes the Lie derivative with respect to u^ν.[34]

Proof. Calculate $L_u(\lambda u_\nu + S_o \nu_{,\nu} + \alpha_{,\nu})$ to find

$$L_u(\lambda u_\nu + S_o \mu_{,\nu} + \alpha_{,\nu}) =$$

$$L_u(\lambda u_\nu + S_o \mu_{,\nu} + \alpha_{,\nu})_{,\sigma} u^\sigma + (\lambda u_\sigma + S_o \mu_{,\sigma} + \alpha_{,\sigma}) u^\sigma{}_{,\nu}$$

$$= \lambda_{;\sigma} u_\nu u^\sigma + \lambda(u_{\nu,\sigma} u^\sigma{}_{,\nu}) + (DS_o/D\tau)\mu_{,\nu} + S_o(D\mu/D\tau)_{,\nu} + (D\alpha/D\tau)_{,\mu}$$

$$= \lambda_{;\sigma} u_\nu u^\sigma + \lambda(u_{\nu,\sigma} u^\sigma{}_{,\nu}) - \lambda_{,\nu} + T_o S_{o,\nu} \tag{2.75}$$

where the last equality follows from (2.18), (2.70d) and (2.73). Using the definition of λ and the thermodynamic relation (2.72) it is easily seen that

$$-\lambda,_\nu + T_o S_o,_\nu = -p,_\nu / \rho_o. \tag{2.76}$$

Now, since u^ν is the four-velocity field for a perfect fluid it must satisfy the four-velocity normalization condition Equation (2.13). Hence,

$$(g_{\sigma\beta} u^\sigma u^\beta),_\nu = g_{\sigma\beta},_\nu u^\sigma u^\beta + 2 u_o u^\sigma,_\nu = 0,$$

which implies

$$u_\sigma u^\sigma,_\nu = -\frac{1}{2} g_{\sigma\beta},_\nu u^\sigma u^\beta. \tag{2.77}$$

Equation (2.77) in turn implies that

$$L_u u_\nu = u_{\nu,\sigma} u^\sigma + u_\sigma u^\sigma_{,\nu} = u^\sigma (u_{\nu,\sigma} - \frac{1}{2} g^{\beta\delta} g_{\sigma\beta,\nu} u_\delta) = u^\sigma u_{\nu;\sigma}, \tag{2.78}$$

where the last equality follows from the definition of the covariant derivative in terms of the metric tensor $g_{\sigma\beta}$. Substitution of (2.76) and (2.78) into (2.75) yields

$$L_u(\lambda u_\nu + S_o \mu,_\nu + \alpha,_\nu) = (\lambda u_\nu);_\sigma u^\sigma - p,_\nu / \rho_o. \tag{2.79}$$

We now compute $T^{\sigma\beta};_\beta / \rho_o$ from the definition of the energy-momentum tensor $T^{\sigma\beta}$, Equation (2.28), and continuity, Equation (2.18), to find

$$T^{\sigma\beta};_\beta / \rho_o = (\lambda u^\sigma);_\beta u^\beta - p,_\beta g^{\sigma\beta} / \rho_o,$$

whence

$$g_{\nu\sigma} T^{\sigma\beta};_\beta / \rho_o = (\lambda u_\nu);_\sigma u^\sigma - p,_\nu / \rho_o. \tag{2.80}$$

Comparison of Equations (2.79) and (2.80) leads immediately to the claimed result, Equation (2.74). This concludes the proof of Theorem 6.1.

We now show that the equations involving the potentials α, β, γ and μ in the potential set of equations are consistent with, and in fact imply, the vanishing of the covariant divergence of the energy-momentum tensor $T^{\sigma\beta}$. From the four-velocity representation, Equation (2.70b), and Equations (2.70g) we conclude

$$L_u(\lambda u_\nu + S_o\mu_{,\nu} + \alpha_{,\nu}) = -L_u(\beta\gamma_{,\nu}) = (D\beta/D\tau)\gamma_{,\nu} + \beta(D\gamma/D\tau)_{,\nu} = 0. \quad (2.81)$$

From Theorem 6.1 the left-hand member of Equation (2.74) is equal to $g_{\nu\sigma}T^{\sigma\beta}{}_{;\beta}/\rho_o$. Multiplying this expression by $\rho_o g^{\nu\mu}$, summing over ν and using Equation (2.81) leads to the conclusion that $T^{\mu\beta}{}_{;\beta} = 0$; i.e., the equations involving the potentials α, β, γ and μ imply that the covariant divergence of the energy-momentum tensor $T^{\mu\beta}$ vanishes.

The second half of the equivalence proof proceeds as follows. First, define $w_\mu \equiv \lambda u_\nu + S_o\mu_{,\nu} + \alpha_{,\nu}$ where μ and α are defined by (2.70d) and (2.73), and hence exist locally by standard existence theorems for first-order partial differential equations[37] so long as T_o, S_o and λ satisfy sufficient smoothness conditions. Then,

$$w_\nu u^\nu = \lambda + S_o(D\mu/D\tau) + D\alpha/D\tau = 0 \quad \text{and} \quad L_u(w_\nu) = 0 \quad (2.82)$$

according to (2.70d), (2.73), Theorem 6.1 and Equation (2.31) which is part of the standard set. Choose a comoving coordinate system $\tau, u^i, i = 1,2,3$, so that $u^\nu = \delta^\nu_o$. Then the two conditions on w, Equations (2.82), imply that $w_o = 0$ and $w_{i,o} = 0, i = 1,2,3$ so that in this particular coordinate system w_ν has but three components each of which exhibits no dependence on τ. Utilizing Pfaff's theorem (Section D of Chapter I) we may then conclude the local existence of scalar-valued functions β, γ and ϕ such that

$$w_i dy^i = -\beta d\gamma + d\phi$$

where β, γ and ϕ are functions only of $y^i, i = 1,2,3$. As a consequence

$$\beta_{,o} = \delta^\nu_o \beta_{,\nu} = u^\nu\beta_{,\nu} = D\beta/D\tau = D\gamma/D\tau = D\phi/D\tau = 0,$$

where the last few equalities are valid in any coordinate system. This allows us to conclude the (local) existence of scalars β, γ and ϕ such that

$$\lambda u_\nu + S_o \mu_{,\nu} + (\alpha - \phi)_{,\nu} = -\beta \gamma_{,\nu} \tag{2.83}$$

where

$$D\beta/D\tau = D\gamma/D\tau = D\phi/D\tau = 0. \tag{2.84}$$

Recalling that α has some "gauge" freedom in that an arbitrary scalar with vanishing convective derivative may be added to α without changing its defining evolutionary equation, Equation (2.73), we conclude that ϕ in Equation (2.83) may be absorbed into the gauge of α. Equations (2.83), (2.84) and the defining relations for μ and α, Equations (2.70d) and (2.73) thus lead directly to the equations involving the potentials α, β, γ and μ in the potential set of equations. Since the only equations needed to imply the existence of potentials α, β, γ and μ satisfying (2.70d), (2.73), (2.83) and (2.84) are included in the standard set we conclude that the standard set implies the potential set. This concludes the proof of equivalence of the two sets of equations of motion.

C. Conclusion

The multiplicity of variational principles leading to the equations of motion for a perfect fluid has been illustrated by the several principles developed within Sections II.A and II.B. We emphasize here the fact that in all of these variational principles several constraints have been imposed on the variations of the field quantities, e.g., entropy and particle (mass) conservation or preservation and Lagrangian variations or the Lin constraint. It has not as yet been shown that the imposition of such constraints is necessary except that as some of the constraints are not imposed the resultant equations of motion are physically restrictive. The necessity of imposing these constraints is shown more rigorously in Chapters IV and V.

The Lin constraint deserves slightly more attention than entropy and particle (mass) conservation or preservation as the latter seem somewhat more conceptual from a physical standpoint. On the other hand, since a fluid can be equivalently described in terms of the Lagrangian or Eulerian prescriptions it is not immediately clear why Lagrangian variations should be preferred over Eulerian variations and hence why the Lin constraint necessarily must be imposed in Eulerian variational principles. As the Lin constraint was introduced in this section the historical motivation for its original introduction by Lin[2] was given. That is, in reality a fluid is composed of particles and hence the Lagrangian description of a fluid is more accurate from a physical standpoint. Hence, Lin introduces the physically motivated constraint that each fluid element be labeled, as in the Lagrangian prescription, by its initial position. As originally formulated by Lin the constraint has remained mysterious, in particular in its mathematical form, for many years. Nevertheless, it has been used extensively because of the resultant equations of motion which follow after its imposition as opposed to those that result without imposing it. It is the intent of the following sections to unmask some of the mystery that enshrouds the Lin constraint. Before doing so, however, we will examine variational principles which lead to the equations of electromagnetism.

III. VARIATIONAL PRINCIPLES IN ELECTROMAGNETISM

As with variational principles in fluid mechanics, electromagnetic variational principles may be cast in either the Lagrangian or Eulerian description. However, most of the literature pertaining to electromagnetic variational principles maintains a more stringent picture of the particulate nature of a fluid than maintained in fluid mechanics. Usually, an explicity sum over individual fluid particles is contained in electromagnetic Lagrangians and the macroscopic averaging process leading to the Eulerian equations of motion is carried out after the variational principle has given the equations of motion for each fluid particle. Perhaps this is due to the fact that the advent of quantum mechanics was originally more closely linked to charged elementary particles than to neutral fluids, or perhaps it is due to the increased complexity of the equations of electrodynamics over those of fluid mechanics, which makes the more general Lagrangian nomenclature too cumbersome. Whatever the actual case may be, we choose here to use the nomenclature most often selected in the literature for two reasons. First, it is likely most familiar to the reader; and second, we have not used the approach involving explicit particle notation to this point and the approach gives further insight into the Lin constraint. In this section we will only consider zero-entropy (zero-temperature) electromagnetic fluids.

We begin this section by motivating the need for a "complete" variational principle by first reviewing several familiar "incomplete" principles. We then consider several principles in the Lagrangian and mixed Lagrangian/Eulerian descriptions. Finally, variational principles in the strictly Eulerian description are presented. Some of this chapter is based on an unpublished manuscript by Edwards entitled "A Review of Electromagnetism and the Formulation of a New, Classical Action Having Connections With Quantum Mechanics."

A. Incomplete electromagnetic variational principles

By "complete" is meant a variational principle which gives rise to a closed set of equations describing the motion of an electromagnetic fluid. Such a set should

include Maxwell's equations and a force or momentum equation. An incomplete principle would yield a subset of, or restrictions to these equations. For example, Maxwell's equations in a vacuum may be derived from the familiar fundamental integral

$$I_{EM1} = \int \varepsilon_o[\frac{1}{2}c^2B^2 - \frac{1}{2}E^2 + \mathbf{A}\cdot(\partial\mathbf{E}/\partial t - c^2\nabla\times\mathbf{B}) + \phi\nabla\cdot\mathbf{E}]d^3x dt \qquad (3.1)$$

by free variation of the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields and the Lagrange multipliers $\mathbf{A}$ and ϕ (ε_o is the dielectric constant or permittivity of free space and c is the speed of light in vacuum). To see this we compute the Euler-Lagrange equations:

$$\delta\mathbf{A}: \quad \nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t},$$

$$\delta\phi: \quad \nabla\cdot\mathbf{E} = 0,$$

$$\delta\mathbf{B}: \quad \mathbf{B} = \nabla\times\mathbf{A}, \qquad (3.2a)$$

$$\delta\mathbf{E}: \quad \mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}. \qquad (3.2b)$$

The last two equations, Equations (3.2), are the well known expressions of the magnetic and electric fields in terms of the vector $\mathbf{A}$ and scalar ϕ potentials, while the first two equations are the equations of constraint, two of Maxwell's equations in the absence of sources. Equations (3.2) lead immediately to the other two Maxwell's equations

$$\nabla\cdot\mathbf{B} = 0 \qquad (3.3a)$$

and

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}. \qquad (3.3b)$$

This variational principle is incomplete in two respects. Firstly, it does not provide for the possibility that the fields may be created by charges and currents and as such constitutes a restrictive principle. Secondly, it does not give rise to a force or

momentum equation and hence yields only a subset of the complete set of equations of motion.

The first difficulty may be overcome by introduction of the charge density ρ_{oe} and the (charge) current density $\mathbf{j}$, which we express together in four-vector notation (with $\mathbf{j} = \gamma\rho_{oe}\mathbf{v}$) as $j^\sigma = \gamma\rho_{oe}(c, \mathbf{v}) \equiv \rho_{oe}v^\sigma$ where $\gamma \equiv \frac{1}{\sqrt{1-(v^2/c^2)}}$ (ρ_{oe} here represents the "rest" charge density, i.e., the charge density measured by an inertial observer locally at rest with respect to the charge distribution. We shall occasionally use the notation $\rho_e \equiv \gamma\rho_{oe}$). The velocity field $\mathbf{v}$ should be smooth[38] (obtained, perhaps, through macroscopic averaging) and describes the motion of the charge distribution. c is the the vacuum speed of light. For convenience, we also define the skew-symmetric electromagnetic field-strength tensor $F_{\beta\sigma}$ in terms of the electromagnetic four-potential $A_\sigma = (\phi/c, \mathbf{A})$ as

$$F_{\beta\sigma} = \partial_\beta A_\sigma - \partial_\sigma A_\beta \tag{3.4}$$

where ϕ and $\mathbf{A}$ are the familiar scalar and vector potentials, respectively, appearing in Equations (3.2) above. For mathematical simplicity we use A_σ in the majority of our calculations and let Equations (3.2) be defining equations for the fields $\mathbf{E}$ and $\mathbf{B}$. Recall that $\partial_\beta \equiv \partial/\partial x^\beta$ where $x^\beta = (ct, x, y, z)$. Note that

$$\partial_\beta F_{\gamma\sigma} + \partial_\sigma F_{\beta\gamma} + \partial_\gamma F_{\sigma\beta} = 0 \tag{3.5}$$

follows from the definition of the electromagnetic field-strength tensor $F_{\beta\sigma}$, Equation (3.4). Using the definitions of $\mathbf{E}$ and $\mathbf{B}$ in Equations (3.2) it is easily seen that Equation (3.5) is equivalent to the *internal* Maxwell equations, Equations (3.3). This identity becomes more evident by writing $F_{\beta\sigma}$ explicitly in terms of $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ as

$$F_{\beta\sigma} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{3.6}$$

Consider the fundamental integral[39]

$$I_{EM2} = -\int \left\{ F_{\beta\sigma} F^{\beta\sigma}/4\mu_o + A_\sigma j^\sigma \right\} d^3x dt \tag{3.7}$$

(μ_o is the magnetic permeability of free space) where it is assumed that A_σ constitute the field quantities to be varied. All inner products are to be interpreted in terms of the Minkowski metric

$$g_{\beta\sigma} = g^{\beta\sigma} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

e.g., $F_{\beta\sigma}F^{\beta\sigma} = F_{\beta\sigma}g^{\beta\gamma}F_{\gamma\delta}g^{\delta\sigma} = 2(B^2 - E^2/c^2)$ and $A_\sigma j^\sigma = g_{\sigma\beta}A^\beta j^\sigma = \gamma\rho_e(\phi - \mathbf{A}\cdot\mathbf{v})$ where $A^\beta \equiv (\phi/c, \mathbf{A})$. By using $\varepsilon_o\mu_o = c^{-2}$ and the evaluation of $F_{\beta\sigma}F^{\beta\sigma}$ it is easily seen that the first term under the integral sign of I_{EM2}, $-F_{\beta\sigma}F^{\beta\sigma}/4\mu_o$,(see Equation (3.7)) is equal to the negative of the first two terms of I_{EM1}, $-\frac{1}{2}\varepsilon_o(c^2B^2 - E^2)$, (see Equation (3.1)). Hence, the fundamental integrals are essentially equivalent except for the appearance of $A_\sigma j^\sigma$ in I_{EM2}.[40] The term $A_\sigma j^\sigma$ describes a field-current interaction and allows for the creation of the field A_σ from the current density j^σ. This conclusion follows from variation of A_σ from which is obtained the Euler-Lagrange equations

$$\delta A_\sigma : \quad \partial_\beta F^{\beta\sigma} = \mu_o j^\sigma . \tag{3.8}$$

Using the explicit expression for $F_{\beta\sigma}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, Equation (3.6), the Minkowski metric and the definition of j^σ, one may write Equation (3.8) as

$$\nabla \cdot \mathbf{E} = \rho_e/\varepsilon_o$$

and

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_o \mathbf{j}$$

which are the familiar Maxwell equations in the presence of sources. Hence, I_{EM2} is not restrictive of Maxwell's equations as is I_{EM1}, but it does not give rise to a force or momentum equation and hence remains incomplete.

To motivate a complete variational principle we next consider a familiar principle which leads only to a force (momentum) equation for each fluid particle. Then by combining this with a variational principle that yields only Maxwell's equations we may obtain a "complete" principle. To this end, consider the fundamental integral[41]

$$I_F = -\sum_{i=1}^{N} \int \left(q_i v_i^\sigma A_{i\sigma} + m_i c (v_i^\sigma v_{i\sigma})^{1/2} \right) d\tau_i \tag{3.9}$$

where q_i and m_i are the charge and mass respectively of particle i, N is the total number of particles, c is the speed of light and $A_{i\beta}$ represents the electromagnetic four-potential acting at the site of particle i. If $x^\sigma(\tau_i) \equiv x_i^\sigma$ represents the four-position of particle i, parametrized by τ_i, then $v_i^\sigma \equiv \frac{dx_i^\sigma}{d\tau_i} |_{\tau_i}$ and $A_{i\beta} = A_\beta(x_i^\sigma)$. If τ_i is taken to be the proper time of particle i, then v_i^σ represents its four-velocity.

In I_F, Equation (3.9), we vary the trajectory of each particle,[42] which is the same as varying x_i^σ for each i, in order to find an equation of motion for each particle. Before computing the Euler-Lagrange equations for I_F we remark that I_F is simply the relativistic version of Hamilton's principle for a system of charged particles as can easily be seen by finding its non-relativistic approximation I'_F,

$$I'_F = \sum_{i=1}^{N} \int \left(\frac{1}{2} m_i v_i^2 - (m_i c^2 + q_i \phi_i - q_i \mathbf{v}_i \cdot \mathbf{A}_i) \right) dt_i .$$

The first term of I'_F is obviously the (non-relativistic) kinetic energy of particle i, while the terms in parenthesis represent the sum of the mass and electromagnetic potential energies. Hence $I'_F(I_F)$ expresses the sum of integrals of the kinetic less potential energies over all N system particles (Hamilton's principle).

The Euler-Lagrange equations which follow from I_F, Equation (3.9), are

$$\delta x_i^\sigma : \quad \partial(q_i v_i^\beta A_{i\beta})/\partial x_i^\sigma - \frac{d}{d\tau_i} \left(q_i A_{i\sigma} + m_i c v_{i\sigma} / (v_i^\beta v_{i\beta})^{1/2} \right) = 0, \ i = 1, ..., N.$$

Taking τ_i to be the proper time of particle i so that v_i^β is its four-velocity one readily obtains

$$v_i^\beta v_{i\beta} = c^2. \tag{3.10}$$

Also,

$$\frac{dA_{i\sigma}}{d\tau_i} = \frac{\partial A_{i\sigma}}{\partial x_i^\beta} v_i^\beta,$$

so that the Euler-Lagrange equations become

$$m_i \frac{dv_{i\sigma}}{d\tau_i} = q_i v_i^\beta F_{i\sigma\beta}, \; i = 1, ...N, \tag{3.11}$$

where

$$F_{i\sigma\beta} \equiv \partial A_{i\beta}/\partial x_i^\sigma - \partial A_{i\sigma}/\partial x_i^\beta. \tag{3.12}$$

Equation (3.11) gives N Lorentz force equations, one for each particle. $F_{i\sigma\beta}$ may be thought of as the electromagnetic field-strength tensor evaluated at the position of particle i, which interpretation, after using (3.6) with appropriate subscripts i on the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields, leads to the expression of (3.11) as

$$m_i d(\gamma_i \mathbf{v}_i)/dt_i = q_i(\mathbf{E}_i + \mathbf{v}_i \times \mathbf{B}_i) \tag{3.13a}$$

and

$$d(\gamma_i m_i c^2)/dt_i = q_i \mathbf{v}_i \cdot \mathbf{E}_i, \; i = 1, ..., N, \tag{3.13b}$$

where the usual relationships $v_i^\sigma = \gamma_i(c, \mathbf{v}_i), \gamma_i = (1 - v_i^2/c^2)^{-1/2}$ and $dt_i = \gamma_i d\tau_i$ have been utilized. Equation (3.13a) is obviously the relativistic version of the well-known Lorentz force relation. It is interesting to note that Equation (3.13b) follows directly from (3.13a) by taking the scalar product of $\mathbf{v}_i$ with Equation (3.13a) and by expressing γ_i explicitly in terms of v_i^2; hence, (3.13b) is not an independent equation.

Once the fields have been determined, the Lorentz force equation, (3.11), determines the motion of each fluid particle. Of course, as the particles move the fields

are altered and there is no way of determining the induced dynamical behavior of the fields from just the Lorentz force equation; hence the equations of motion and therefore the variational principle from which they follow are incomplete in the case of I_F. Of course, if the fields produced by the particles themselves are negligible in comparison with the externally applied fields, then the Lorentz force equation, (3.11), is adequate for the determination of particle trajectories. Although this may be true in a number of important cases (e.g., magnetically confined plasmas, etc.), it is generally the exception rather than the rule.

Similarly, if Maxwell's equations, (3.5) and (3.8), are taken by themselves, then no information may be obtained concerning the motion of the system particles which might in turn alter the fields. For this reason, I_{EM2}, Equation (3.7), is considered incomplete. Of course, if the motion of the fluid particles is negligible, i.e., if the currents are negligible, or if they (the fluid particles or currents) are constrained to flow in a predetermined pattern, then Maxwell's equations are, by themselves, sufficient. Again, this is the exception rather than the rule (the exceptions are again important and include the rich and well-studied fields for electro- and magnetostatics).

Having motivated the need for a "complete" variational principle we now introduce several complete principles.

B. Complete particulate (Lagrangian) variational principle

A complete variational principle may be obtained from Equation (3.9) by appropriately defining A_i^σ so as to encompass those fields produced by all particles in the fluid except particle i (this avoids the problem of infinite energies associated with self-interactions). Such a definition is provided by Wheeler and Feynman,[43]

$$A_i^\sigma(x_i^\beta) \equiv \frac{\mu_o c}{4\pi} \sum_{k\neq i} q_k \int_{-\infty}^{+\infty} \delta\left[\left(x_i^\beta - x_k^\beta\right)(x_{i\beta} - x_{k\beta})\right] v_k^\sigma d\tau_k, \qquad (3.14)$$

where $\delta()$ denotes the Dirac delta function (or "measure" in the mathematical

nomenclature). Substitution of expression (3.14) into I_F transforms I_F into the direct action fundamental integral of Schwarzschild,[44] Tetrode,[45] and Fokker[46] in which each system particle is assumed to be influenced directly via retarded and advanced "actions" of all other system particles. Since the particle interactions are assumed direct, the Schwarzschild, Tetrode, Fokker action (or fundamental integral) I_{STF} does not require the introduction of the unphysical concept of a field; instead, fields may be defined directly in terms of physical quantities as in (3.14).

It is clear that variations of the particle trajectories in I_{STF} give the Lorentz force equations, (3.11), after defining A_i^σ according to (3.14) (the derivation of (3.11) from I_{STF} is precisely the same as its derivation from I_F since I_{STF} is identical to I_F after defining A_i^σ according to (3.14)). Hence, in order to show that I_{STF} is complete we need only show that Maxwell's equations follow from the defining relation (3.14) and that the defining relation is sufficiently general to encompass all appropriate solutions of Maxwell's equations.

Let us first calculate $\partial A_i^\sigma / \partial x_i^\sigma$ (implied summation over σ, but no implied sum over i),

$$\begin{aligned}\partial A_i^\sigma / \partial x_i^\sigma &= \frac{\mu_o c}{4\pi} \sum_{k \neq i} q_k \int_{-\infty}^{+\infty} \delta' \left[\left(x_i^\beta - x_k^\beta \right) \left(x_{i\beta} - x_{k\beta} \right) \right] \left[2 \left(x_{i\sigma} - x_{x\sigma} \right) \right] v_k^\sigma d\tau_k \\ &= \frac{\mu_o c}{4\pi} \sum_{k \neq i} q_k \int \frac{d\delta}{d\tau_k}(\tau_k) d\tau_k = \frac{\mu_o c}{4\pi} \sum_{k \neq i} q_k \delta(\tau_k)\big|_{-\infty}^{+\infty} = 0.\end{aligned}$$

Hence, A_i^σ as defined in (3.14) is in the *Lorentz gauge.* Next, we employ Dirac's identity[47] to obtain the equality

$$\partial \left\{ \partial \left[\delta \left[(x_i^\sigma - x_k^\sigma)(x_{i\sigma} - x_{k\sigma}) \right] \right] / \partial x_{i\beta} \right\} / \partial x_i^\beta = 4\pi \delta(x_i^\nu - x_k^\nu). \tag{3.15}$$

Using the identity (3.15) we may now compute $\frac{\partial}{\partial x_i^\beta} \frac{\partial}{\partial x_{i\beta}}(A_i^\sigma) = \partial^2 A_i^\sigma / \partial x_i^\beta \partial x_{i\beta}$,

$$\partial^2 A_i^\sigma / \partial x_i^\beta \partial x_{i\beta}(x_i^\nu) = \frac{\mu_o c}{4\pi} \sum_{k \neq i} q_k \int \left[4\pi \delta(x_i^\nu - x_k^\nu) \right] v_k^\sigma d\tau_k = \mu_o j_i^\sigma \tag{3.16}$$

where

$$j_i^\sigma(x_i^\nu) \equiv c \sum_{k \neq i} q_k \int \delta(x_i^\nu - x_k^\nu) v_k^\sigma d\tau_k \tag{3.17}$$

defines the effective four-current density seen by particle i (note its apparent singular nature). Employing the fact that A_i^σ satisfies the Lorentz gauge condition we may subtract $\partial(\partial A_i^\beta / \partial x_{i\sigma})/\partial x_i^\beta = 0$ from the left hand member of Equation (3.16) to obtain

$$\frac{\partial}{\partial x_i^\beta}\left(\partial A_i^\sigma / \partial x_{i\beta} - \partial A_i^\beta / \partial x_{i\sigma}\right) = \mu_o j_i^\sigma .$$

From the previous definition of $F_{i\beta\sigma}$, Equation (3.12), and the definition of the Minkowski metric the quantity in brackets in this last equation is recognized as $F_{i\beta\sigma}$, hence the equation reduces to

$$\partial F_i^{\beta\sigma} / \partial x_i^\beta = \mu_o j_i^\sigma$$

which has the same form as the external Maxwell equations, Equation (3.8) (note that the internal Maxwell equations, Equation (3.5), are satisfied identically by the definition of $F_{i\beta\sigma}$, Equation (3.12)). Hence, definition (3.14) satisfies Maxwell's equations.

Conversely, if there exist fields at the location of fluid particle i satisfying Maxwell's equations, those fields must be a result of a current density four-vector of the form given in Equation (3.17). Let Equation (3.14) define the quantity A_i^σ and Equation (3.12) define $F_{i\beta\sigma}$, then this tensor defines fields which are consistent with Maxwell's equations and the Lorentz force equation, Equation (3.11). We thereby conclude that there is always enough gauge freedom in A_i^σ so as to define any physically realizable electromagnetic field tensor $F_{i\beta\sigma}$ by Equation (3.12) where A_i^σ is as defined in (3.14). Therefore, I_{STF} is complete.

We next consider a traditional fundamental integral in mixed Lagrangian/Eulerian notation which yields a complete set of dynamical equations.

C. Complete Lagrangian/Eulerian variational principles

Another method of expanding the incomplete variational principles of Section III.A so as to obtain a complete principle is to combine directly a principle yielding only Maxwell's equations with one yielding only the Lorentz force relations. The result may take one of two forms, both of which mix particulate/Lagrangian notation with Eulerian notation. The two forms are actually one and the same, and differ only in the form chosen to display the fundamental integral.

At the outset we define the fluid's total current density four-vector as

$$j^\sigma(x^\nu) \equiv c \sum_{i=1}^{N} q_i \int_{-\infty}^{+\infty} \delta(x^\nu - x_i^\nu) v_i^\sigma d\tau_i. \tag{3.18}$$

This definition differs from the definition of the effective current density four-vector seen by particle i, Equation (3.17), only in the respects that (i) it is evaluated at a general space-time position x^ν rather than just the space-time position of particle i, x_i^ν, and (ii) the sum over particles does not exclude any fluid particle, in particular particle i is not excluded. From definition (3.18) follows

$$\begin{aligned}\int A_\sigma(x^\nu) j^\sigma(x^\nu) d^3x dt &= \sum_{i=1}^{N} q_i \int_{-\infty}^{+\infty} v_i^\sigma \left[\int A_\sigma(x^\nu)\delta(x^\nu - x_i^\nu) d^4x \right] d\tau_i \\ &= \sum_{i=1}^{N} \int_{-\infty}^{+\infty} q_i v_i^\sigma A_{i\sigma} d\tau_i\end{aligned}$$

where, as is usual, we write d^4x for $dx^o d^3x$. Hence, with this definition of the current density four-vector, which is the obvious one for a system of N charged particles, the second term of the fundamental integral I_{EM2}, Equation (3.7), corresponds precisely with the first term of I_F, Equation (3.9). This information suggests that a complete variational principle may be obtained by either adding the first term of I_{EM2} to I_F or by adding the last term of I_F to I_{EM2} and in both cases varying A^σ

and x_i^σ while taking (3.18) into account. Hence, we take

$$I_{C1} = -\int \left[F_{\beta\sigma}F^{\beta\sigma}/4\mu_o\right] d^3x dt - \sum_{i=1}^{N} \int \left[q_i v_i^\sigma A_{i\sigma} + m_i c(v_i^\sigma v_{i\sigma})^{1/2}\right] d\tau_i. \quad (3.19a)$$

and

$$I_{C2} = -\int \left[F_{\beta\sigma}F^{\beta\sigma}/4\mu_o + A_\sigma j^\sigma\right] d^3x dt - \sum_{i=1}^{N} \int m_i c(v_i^\sigma v_{i\sigma})^{1/2} d\tau_i, \quad (3.19b)$$

and reiterate that with definition (3.18) I_{C1} and I_{C2} are one and the same.

The Euler-Lagrange equations which follow from I_{C1}, Equation (3.19a), and/or, I_{C2} Equation (3.19b) are most easily obtained by varying x_i^σ in I_{C1}, then A^σ in I_{C2} whereupon one obtains the Lorentz force relations, Equation (3.11), in the first case and Maxwell's equations, (3.5) and (3.8), in the latter. We remark that such an approach depends strongly on the definition of the current density four-vector, Equation (3.18).

Although I_{C1} and I_{C2} are complete the equations which follow from them are not entirely consistent. This conclusion follows from the fact that A^σ, computed from the external Maxwell equations, Equation (3.8), includes a contribution from each of the fluid's N particles and hence $A_i^\sigma(x_i^\beta) \equiv A^\sigma(x^\beta)\mid_{x^\beta = x_i^\beta}$ must include a contribution from particle i, i.e., a self-interaction term which acts on particle i through the Lorentz force equation, Equation (3.11). Such interactions, of course, are unphysical and are explicitly excluded from the direct-action definition of A_i^σ provided by Wheeler and Feynman, Equation (3.14). We note, however, that in the continuum approximation of the current density four-vector j^σ, perhaps obtained through a macroscopic averaging process, the self-interaction problem is no longer an issue as an infinitesimal fluid element may only interact infinitesimally and hence negligibly with itself. Hence, one might be tempted to view the current density four-vector appearing in the external Maxwell equations, (3.8), as having been macroscopically averaged over the particle paths suggested by the Lorentz force

relations, Equation (3.11). This viewpoint, however, introduces a logical inconsistency in that Maxwell's equations and the Lorentz force relations are coupled not only through their dependence on the particle trajectories, but also through their dependence on the electromagnetic four-potential A^σ. The inconsistency, then, is that Maxwell's equations include no self-interactions while the Lorentz force relation retains self-interactions.

The "strongly-coupled" nature of the two sets of equations (Maxwell and Lorentz force) leads us to conjecture that physical and logical self-consistency can only be maintained through variational principles written entirely in terms of either particulate (Lagrangian) or Eulerian notation. I_{STF}, introduced in Section III.B is seen to be completely self-consistent (no self-interactions or mixed notations are involved) and is written entirely in particulate (Lagrangian) notation. On the other hand I_{C1} and I_{C2}, Equations (3.19), yield equations which either include self-interactions or are written in mixed Lagrangian/Eulerian notation and as such are not logically self-consistent. In the following sections we examine complete variational principles in the Eulerian description which should also be physically and logically self-consistent.

D. Complete Eulerian variational principles

One method of obtaining a complete variational principle in the Eulerian description consists in performing smoothing operations on the Euler-Lagrange equations obtained from either a complete particulate (or Lagrangian) fundamental integral (such as I_{STF}) or a complete Lagrangian/Eulerian fundamental integral. Stated another way, we may begin with a Lagrangian, or part Lagrangian, fundamental integral, vary the field quantities to obtain the Euler-Lagrange equations, then subsequently transform the Euler-Lagrange equations to their equivalent Eulerian form. By so doing one obtains the Eulerian form of Maxwell's equations, Equations (3.5) and (3.8), and

$$\rho_{om} v^\beta \partial_\beta v_\sigma = \rho_{oe} v^\beta F_{\sigma\beta}, \tag{3.20}$$

which is the Lorentz force relation in Eulerian notation. Equation (3.20) is easily seen to follow from the Lagrangian form of the Lorentz force relations, Equation (3.11), by using the identity $d/d\tau_i = v_i^\beta \partial/\partial x_i^\beta$. In the case that the charge (mass) current density, j^β (j_m^β), may be written as the product of the four-velocity field v^β and the rest charge (mass) density ρ_{oe} (ρ_{om}),[48] the Eulerian Lorentz force relation, Equation (3.20), may be simplified to

$$j_m^\beta \partial v_\sigma = j^\beta F_{\sigma\beta}. \tag{3.21}$$

A second method consists in performing the smoothing process first so as to obtain a fundamental integral expressed entirely in terms of the Eulerian description, then subsequently varying the appropriate Eulerian field quantities so as to obtain the Eulerian equations of motion. Such a method may be motivated through the performance of a mathematical transformation prompted by the definition of the total charge current density j^σ, Equation (3.18). We shall consider first the fundamental integral I_{C1} and/or I_{C2}, Equations (3.19), then the fundamental integral I_{STF} under this mathematical transformation in the special case of a one-species fluid, $m_i = m$ and $q_i = q$ for $i = 1, ..., N$. In each case we shall introduce the Lin constraint in order to obtain a fundamental integral which leads to unrestricted equations of motion.

1. Transformation of I_{C1} and I_{C2}

As noted previously, I_{C1} and I_{C2} are equivalent. Since we desire to transform these integrals to Eulerian notation we choose to transform I_{C2} since it contains but one Lagrangian term, namely its last term. Given the positions x_i^β and the velocities v_i^β of all N particles i in the fluid, one may define a smooth (in fact, infinitely differentiable) velocity field $v^\beta(x^\nu)$ such that $v^\beta \mid_{x_i^\nu(\tau_i)} = v_i^\beta(\tau_i)$ for each particle i and any value of the parameter τ_i. That is, this may be done as long as no two particles ever occupy the same space-time position; and of course, no two particles may occupy the same space-time position, assuming they are of the same

charge and interact electromagnetically, without imparting infinite energy to the system. There is enough freedom in the definition of v^β to allow $v^\beta v_\beta = c^2$ for all x^ν. Such a definition of v^β is not guaranteed to be unique, but is guaranteed to exist. With such a four-velocity field v^β in hand we may use the definition of j^σ, Equation (3.18), to write

$$j^\sigma(x^\nu) = \rho_{oe}(x^\nu)v^\sigma(x^\nu) \tag{3.22}$$

where we define the rest charge density ρ_{oe} by

$$\rho_{oe}(x^\nu) = c\sum_{i=1}^{N} q_i \int \delta(x^\nu - x_i^\nu)d\tau_i, \tag{3.23}$$

Consider now a single species fluid, $m_i = m$ and $q_i = q > O$ for all $i = 1,...,N$. For this case we evaluate the quantity

$$\begin{aligned}(mc/q)\int (j^\sigma j_\sigma)^{1/2} d^3x dt &= (mc/q)\int \rho_{oe}(v^\sigma v_\sigma)^{1/2} d^3x dt \\ &= (mc^2/q)\sum_{i=1}^{N} q \int\int \delta(x^\nu - x_i^\nu)d\tau_i (v^\sigma v_\sigma)^{1/2} d^3x dt \\ &= mc\sum_{i=1}^{N}\int (v_i^\sigma v_{i\sigma})^{1/2} d\tau_i. \end{aligned} \tag{3.24}$$

The first equality follows from Equation (3.22), the second from Equation (3.23), and the last from a change in the order of integration and the well-known properties of the δ- function. Note that the last member of Equation (3.24) is equal to the last term in I_{C2} when all fluid particles have the same mass m and charge q. Hence, for a single species fluid, I_{C2} (and hence I_{C1}) is equal to

$$I_{C3} = -\int \left[F_{\beta\sigma}F^{\beta\sigma}/4\mu_o + A_\sigma j^\sigma + (mc/q)(j_\sigma j^\sigma)^{1/2}\right] d^3x dt. \tag{3.25}$$

The quantities to be varied in I_{C3} include the electromagnetic four-potential A^β and the particle trajectories, x_i^β, $i = 1,...,N$, which no longer appear explicitly. Note, however, that both ρ_{oe} and v^σ depend implicitly on x_i^β.

2. Variational principle with A^σ as an independent parameter

We now present entirely Eulerian variational principles which yield a complete set of Euler-Lagrange equations (equivalent to Maxwell's equations plus the Lorentz force relation).

Recall from the previous chapter on perfect fluids that when a fundamental integral is considered in the Eulerian rather than Lagrangian description, the variation of ρ_{om} and **v** or u^σ must be constrained according to the Lin constraint in order to recover the equations of motion for a perfect fluid in their complete generality. This is consistent with our findings of the previous subsection, in which a mathematical transformation related the charge current density to the quantity to be varied, the particle trajectory x_i^β. This suggests that j^σ should be varied, but that the variation should be performed consistent with the transformation relating j^σ to x_i^β. We thereby surmise that the variation of $j^\sigma = \rho_{oe} v^\sigma$ should be constrained. Examining Equation (2.54) and assuming a complete parallelism between neutral fluid mechanics and electromagnetic fluid mechanics suggests that the Lin constraint should appear in the fundamental integral in the form $j^\sigma(\partial_\sigma \alpha + \beta \partial_\sigma \gamma)$. Hence, we consider the fundamental integral

$$I_{C4} = I_{C3} - \int j^\sigma \left(\partial_\sigma \alpha + \beta \partial_\sigma \gamma\right) d^3x dt, \tag{3.26}$$

where $A^\sigma, j^\sigma, \alpha, \beta$ and γ are to be independently varied and I_{C3} is as given in Equation (3.25).

The Euler-Lagrange equations that follow from the variation of the field quantities in I_{C4} are

$$\delta A_\sigma : \quad \partial_\beta F^{\beta\sigma} = \mu_o j^\sigma, \tag{3.8}$$

$$\delta j^\sigma : \quad A_\sigma + (mc/q) j_\sigma / (j_\eta j^\eta)^{1/2} + \partial_\sigma \alpha + \beta \partial_\sigma \gamma = 0, \tag{3.27a}$$

$$\delta \alpha : \quad \partial_\sigma j^\sigma = 0, \tag{3.27b}$$

$$\delta\beta: \quad j^{\sigma}\partial_{\sigma}\gamma = 0, \tag{3.27c}$$

and

$$\delta\gamma: \quad \partial_{\sigma}(\beta j^{\sigma}) = j^{\sigma}\partial_{\sigma}\beta = 0. \tag{3.27d}$$

It is necessary to show the equivalence of this set of "potential" equations with the Eulerian set of Maxwell-Lorentz force equations discussed previously. As will be seen, the equivalence is local rather than global. We append to these potential equations the expressions $j^{\sigma} = \rho_{oe}v^{\sigma}$, Equation (3.22), and $v_{\sigma}v^{\sigma} = c^2$, since the former is used in the transformation which results in I_{C3} and the latter is a relativistic necessity. Of course, these two expressions may be constrained by the Lagrange multiplier method to be a part of the variational principle. However, one may easily demonstrate that both Lagrangian multipliers used to constrain these two expressions of necessity must vanish and hence use of the multipliers does not mandate any essential change in the potential set of equations listed above other than the addition of the two appended expressions.

Before proceeding with the demonstration of equivalence we make one remark concerning the fundamental integral I_{C4}, Equation (3.26), and the Euler-Lagrange equations which result therefrom. The equation of charge continuity, Equation (3.27b), does not need to follow directly from the variational principle since it is guaranteed from the skew-symmetry of $F_{\beta\sigma}$ (and hence of $F^{\beta\sigma}$) and the external Maxwell equations, Equation (3.8). In order to obtain a more "minimally constrained" variational principle which does not give rise redundantly to charge continuity we may delete the term involving α from I_{C4}. In essence, what this amounts to is the absorption of $\partial_{\sigma}\alpha$ into the guage of A_{σ}. The equations of motion which result from the variation of the field quantities $A_{\sigma}, j^{\sigma}, \beta$ and γ in the more minimally constrained principle are identical to the corresponding Euler-Lagrange equations for I_{C4} given above, except that in the δj^{σ} equation the $\partial_{\sigma}\alpha$ term of Equation (3.27a) is deleted, i.e.,

$$\delta j^{\sigma}: \quad A_{\sigma} + (mc/q)j_{\sigma}/(j_{\eta}j^{\eta})^{1/2} + \beta\partial_{\sigma}\gamma = 0. \tag{3.28}$$

Making use of the other equations of motion, Equation (3.28) can be seen to restrict the gauge of A_σ according to

$$v^\sigma A_\sigma + mc^2/q = 0. \tag{3.29}$$

Evidently, this is an unfamiliar guage condition, for an electromagnetic four-potential which satisfies Equation (3.29) does not generally satisfy either the Lorentz or Coulomb gauge conditions. Hence, the price that is paid for the use of this more minimally constrained variational principle is an unfamiliar electromagnetic gauge condition on A_σ. Nevertheless, this restriction may be removed *after* the variations have been performed by subsequently performing an arbitrary gauge transformation on A_σ through adding $\partial_\sigma \alpha$ to the left member of Equation (3.28) and thereby recovering the more general Equation (3.27a). The end result is a set of equations, Equations (3.8) and (3.27), (globally) equivalent to those obtained from I_{C4}.

3. Equivalence of I_{C1} and I_{C2} with I_{C5}

We now demonstrate the (local) equivalence of the potential equations with the usual Eulerian Maxwell-Lorentz force set of equations. We reiterate that the potential set of equations consists of the definition of $F_{\beta\sigma}$, Equation (3.4), the external Maxwell equations, Equation (3.8), the expression of j^σ as the product of ρ_{oe} and v^σ, Equation (3.22), four-velocity "normalization", $v_\sigma v^\sigma = c^2$ and Equations (3.27). The Eulerian Maxwell-Lorentz force set of equations consist of the definition of $F_{\beta\sigma}$, Equation (3.4), the external Maxwell equations, Equation (3.8), the expression of j^σ as the product of ρ_{oe} and v^σ, Equation (3.22), four-velocity "normalization", $v_\sigma v^\sigma = c^2$, the Eulerian Lorentz force relation, Equation (3.20) or (3.21), and $\rho_{om}/\rho_{oe} = m/q$ which follows from the fact that we are considering a single-species fluid. The constant rest mass density to rest charge density ratio is also a part of the potential set, but since ρ_{om} does not appear anywhere in that set we choose not to complicate the potential set with this additional relation. In the

Maxwell-Lorentz force set we use it only to reduce the Lorentz force relation to

$$mv^\eta \partial_\eta v_\sigma = qv^\eta F_{\sigma\eta}, \tag{3.30}$$

which will be referred to as "the Lorentz force relation" throughout the remainder of the proof of equivalence.

Because of the duplication of a number of equations between the two equation sets the proof of equivalence may be reduced to the demonstration that (i) if the fields and potentials satisfy the potential set of equations, then the fields must satisfy the Lorentz force relation, Equation (3.30), and (ii) for every set of fields which satisfy the Maxwell-Lorentz force set of equations there exist potentials α, β and γ such that Equations (3.27) are satisfied. That part (i) holds is readily demonstrated through straightforward computations in the following manner. Using $j^\sigma = \rho_{oe} v^\sigma$ and $v_\sigma v^\sigma = c^2$, Equation (3.27a) may be put in the form

$$A_\sigma + (m/q)v_\sigma + \partial_\sigma \alpha + \beta \partial_\sigma \gamma = 0. \tag{3.31}$$

Next, use Equation (3.31) to compute

$$v^\eta \left[\partial_\eta (A_\sigma + (m/q)v_\sigma) - \partial_\sigma (A_\eta + (m/q)v_\eta)\right] = v^\eta (\partial_\eta \beta \partial_\sigma \gamma - \partial_\sigma \beta \partial_\eta \gamma) = 0, \tag{3.32}$$

where the last equality follows from (3.27c) and (3.27d). By rearranging terms and using the definition of $F_{\sigma\eta}$, Equation (3.4), and $v_\eta \partial_\sigma v^\eta = \frac{1}{2}\partial_\sigma (v_\eta v^\eta) = 0$, Equation (3.32) is seen to give rise to the Lorentz force relation, Equation (3.30). Hence, part (i) holds.

Part (ii) is seen to be satisfied by first assuming the existence of fields $j^\sigma = \rho_{oe} v^\sigma$ and A^σ (consequently $F_{\sigma\eta}$) which satisfy the Maxwell-Lorentz force set of equations. It is clear that charge continuity, Equation (3.27b), follows immediately from the skew-symmetry of $F_{\beta\sigma}$ and the external Maxwell equations (3.8), so we need only show the existence of an α, β and γ such that (3.27a,c,d) are satisfied. Now, define

a scalar-valued function α such that

$$D\alpha/D\tau \equiv v^\sigma \partial_\sigma \alpha = -v^\sigma A_\sigma - mc^2/q, \tag{3.33}$$

whose local existence is guaranteed as long as v^σ and A_σ are sufficiently well-behaved, i.e., sufficiently smooth (note that α has enough gauge freedom that one may add any function f with $Df/D\tau = 0$ to α without disrupting its defining relation, Equation (3.33)). With such a definition for α it is clear that $A_\sigma + (m/q)v_\sigma + \partial_\sigma \alpha$ is orthogonal to v^σ, that is

$$v^\sigma (A_\sigma + (m/q)v_\sigma + \partial_\sigma \alpha) = 0. \tag{3.34}$$

We next compute the Lie derivative

$$\begin{aligned} L_v(A_\sigma + (m/q)v_\sigma + \partial_\sigma \alpha) &\equiv v^\eta \partial_\eta (A_\sigma + (m/q)v_\sigma + \partial_\sigma \alpha) \\ &\quad + (A_\eta + (m/q)v_\eta + \partial_\eta \alpha)\partial_\sigma v^\eta \\ &= v^\eta \left(\partial_\eta (A_\sigma + (m/q)v_\sigma) - \partial_\sigma (A_\eta + (m/q)v_\eta)\right) \\ &= v^\eta F_{\eta\sigma} + (m/q)v^\eta \partial_\eta v_\sigma \\ &= 0. \end{aligned} \tag{3.35}$$

The second equality follows from the orthogonality of v^σ and $A_\sigma + (m/q)v_\sigma + \partial_\sigma \alpha$, and from the fact that $\partial_\eta \partial_\sigma \alpha = \partial_\sigma \partial_\eta \alpha$. The third equality follows from the definition of $F_{\eta\sigma}$, Equation (3.4), and the fact that $v^\eta \partial_\sigma v_\eta = \frac{1}{2}\partial_\sigma (v^\eta v_\eta) = 0$, while the last equality follows from the fact that the fields satisfy the Lorentz force relation, Equation (3.30).

Equalities (3.34) and (3.35) imply that in the comoving coordinate system $(\tau, y^i, i = 1, 2, 3)$ where $v^\sigma = c\delta^\sigma_o$, both $w_o = 0$ and $\partial_o w_i = O, i = 1, 2, 3$, where $w_\sigma \equiv A_\sigma + \frac{m}{q} v_\sigma + \partial_\sigma \alpha$. It is clear, then, that in this particular coordinate system w_σ has but three components $w_i, i = 1, 2, 3$, all of which depend only on the spatial coordinates $y^i, i = 1, 2, 3$. Invoking the corollary to Pfaff's theorem contained in Section I.D, it is clear that (locally) there exist functions ϕ, β and γ depending only

on the y^i such that $w_i = \partial\phi/\partial y^i - \beta\partial\gamma/\partial y^i$, or

$$\omega_\sigma = \partial\phi/\partial y^\sigma - \beta\partial\gamma/\partial y^\sigma$$

and

$$\partial\phi/\partial y^o = \partial\beta/\partial y^o = \partial\gamma/\partial y^o = 0 \tag{3.36}$$

where $y^o \equiv \tau$. Converting back to an arbitrary (cartesian) coordinate system, it is clear that relations (3.36) may be written as

$$\omega_\sigma = \partial_\sigma\phi - \beta\partial_\sigma\gamma \quad \text{and} \quad v^\sigma\partial_\sigma\phi = v^\sigma\partial_\sigma\beta = v^\sigma\partial_\sigma\gamma = 0.$$

Absorbing ϕ into the gauge of α and multiplying the last two equations by ρ_{oe} we conclude finally that there exist functions α, β and γ such that Equations (3.27a,c,d) are satisfied. Hence, the potential set of equations is (locally) equivalent to the Eulerian Maxwell-Lorentz force set.

4. Transformation of I_{STF}

In order to fully transform I_{STF} according to the mathematical transformation (3.18) we must transform both I_F, Equation (3.9), and the definition of A_i^σ, Equation (3.14). Actually, because the equations of motion (i.e., the external Maxwell equations) which follow from the definition of A_i, Equation (3.14), require the specification of N different charge current densities j_i^σ rather than just the one given by Equation (3.18) it is clear that a direct transformation of A_i^σ using (3.18) will not suffice. Instead, it is preferable to find from a more direct approach an equation which expresses A^σ as a function of the Eulerian quantity j^β and which implies both the Lorentz gauge condition, $\partial_\beta A^\beta = 0$, and the external Maxwell equations, Equation (3.8). Since $\partial_\beta F^{\beta\sigma} = \partial_\beta\partial^\beta A^\sigma - \partial^\sigma(\partial_\beta A^\beta)$, the Lorentz gauge condition together with the external Maxwell equations imply that the four-potential A^σ must satisfy the inhomogeneous wave equation

$$\partial_\beta\partial^\beta A^\sigma = \mu_o j^\sigma. \tag{3.8a}$$

Conversely, this inhomogeneous wave equation together with the Lorentz gauge condition imply the external Maxwell equations, Equation (3.8). Hence, we seek a particular solution of the inhomogeneous wave equation (3.8a) which satisfies the Lorentz gauge condition, $\partial_\beta A^\beta = 0$.

A particular solution to Equation (3.8a) may be obtained from the familiar Green function technique,[49] in which one first seeks for a function $G = G(x^\nu, x'^\nu)$ satisfying

$$\partial_\beta \partial^\beta G(x^\nu, x'^\nu) = \delta(x^\nu - x'^\nu). \tag{3.37}$$

In the absence of (space-time) boundary surfaces $G(x^\nu, x'^\nu) = G(x^\nu - x'^\nu)$, and two solutions to the Green function defining relation, Equation (3.37), are given by

$$G_r(x^\nu - x'^\nu) = \frac{\theta(x^o - x'^o}{4\pi|\mathbf{r} - \mathbf{r}'|}\delta(x^o - x'^o - |\mathbf{r} - \mathbf{r}'|)$$

and

$$G_a(x^\nu - x'^\nu) = \frac{\theta(x^o - x'^o)}{4\pi|\mathbf{r} - \mathbf{r}'|}\delta(x^o - x'^o - |\mathbf{r} - \mathbf{r}'|)$$

where

$$\theta(y) \equiv \begin{cases} 1, & \text{if } y > 0 \\ 0, & \text{if } y < 0 \end{cases}$$

denotes the unit step function. G_r and G_a are known as the retarded and advanced Green functions, respectively, because the observed time $t = x^o/c$ which causes the argument of the Dirac delta function to vanish is greater than the source time $t' = x'^o/c$ in the case of G_r and in the case of G_a it is less than the source time t'. A particular solution to the inhomogeneous wave equation may then be formed by taking

$$A_p^\sigma(x^\nu) = \mu_o \int \left[G_p(x^\nu - x'^\nu) j^\sigma(x'^\nu)\right] d^4x' \tag{3.38a}$$

where p stands for either r or a and the integral is taken over all space-time (a general solution is obtained by adding a general solution of the homogeneous wave equation A_g^σ to A_p^σ, where $\partial_\beta \partial^\beta A_g^\sigma = 0$). If charge continuity, $\partial_\beta j^\beta = 0$, is assumed

to hold, then A_p^σ satisfies the Lorentz gauge condition, $\partial_\beta A_p^\beta = 0$, as can be seen from the calculation

$$\begin{aligned}\partial_\beta A_p^\beta(x^\nu) &= \mu_o \int \left[j^\beta(x'^\nu)\partial_\beta G_p(x^\nu - x'^\nu)\right] d^4x' \\ &= -\mu_o \int \left[j^\beta(x'^\nu)\partial G_p(x^\nu - x'^\nu)/\partial x'^\beta\right] d^4x' \\ &= -\mu_o \int \left\{\partial \left[j^\beta(x'^\nu)G_p(x^\nu - x'^\nu)\right]/\partial x'^\beta\right\} d^4x' = 0,\end{aligned}$$

where for the last equality we have assumed that $j^\beta(x'^\nu)G_p(x^\nu - x'^\nu)$ vanishes at space-time infinity. We conclude that A_p^σ as given by Equation (3.38a) is a satisfactory candidate for the transformed Eulerian replacement of A_i^σ, Equation (3.14). Note that integration over x'^o in Equation (3.38a) allows one to write A_p^σ as

$$A_p^\sigma(t, \mathbf{r}) = \frac{\mu_o}{4\pi} \int \left[j^\sigma(t'_p, \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|\right] d^3x' \tag{3.38b}$$

where t'_p is either the advanced, $t'_a = t + \mid \mathbf{r} - \mathbf{r}' \mid /c$, or retarded, $t'_r = t - \mid \mathbf{r} - \mathbf{r}' \mid /c$, time. Because of the symmetry which is usually required between past and future it is often convenient to take $A_s^\sigma(x^\nu) = \frac{1}{2}(A_r^\sigma(x^\nu) + A_a^\sigma(x^\nu))$. We will assume A^σ to be of this symmetrized form in what follows when an explicit determination is required.

Since I_F, Equation (3.9), is precisely equal to the last two terms of I_{C1}, Equation (3.19a), we conclude that under the transformation (3.18) I_F will assume the same form as the last two transformed terms of I_{C1}. Comparing I_{C1} with its transformed version we conclude that I_F(and hence I_{STF}) transforms for a single species fluid to

$$I'_{STF} = -\int \left[A_\sigma j^\sigma + (mc/q)(j_\sigma j^\sigma)^{1/2}\right] d^3x dt, \tag{3.39}$$

where we now take A^σ to be $A_s^\sigma = \frac{1}{2}(A_r^\sigma + A_a^\sigma)$, A_r^σ and A_a^σ given by Equations (3.38) and the appropriate definitions of the Green functions G_r and G_a. The quantities to be varied in I_{STF} consist of the particle trajectories x_i^σ.

5. Variational principle with
A^σ defined in terms of j^σ

An Eulerian variational principle in which A^σ is defined in terms of j^σ is given by the fundamental integral

$$I_{C5} = -\int\left\{\frac{\mu_o}{16\pi}j_\sigma\int\left[(j_r^{\prime\sigma}+j_a^{\prime\sigma})/|\mathbf{r}-\mathbf{r}'|\right]d^3x' \right.$$
$$\left. + (mc/q)(j_\sigma j^\sigma)^{1/2} + j^\sigma(\partial_\sigma\alpha+\beta\partial_\sigma\gamma)\right\}d^3x dt \qquad (3.40a)$$

where $j_p^{\prime\sigma} \equiv j^\sigma(t_p', \mathbf{r}')$, and as before $t_r' \equiv t- \mid \mathbf{r}-\mathbf{r}' \mid /c$ and $t_a' \equiv t+ \mid \mathbf{r}-\mathbf{r} \mid /c$. By defining A_p^σ as in Equations (3.38), and $A_s^\sigma \equiv (A_r^\sigma + A_a^\sigma)/2$ it follows that

$$I_{C5} = -\int\left\{A_s^\sigma j_\sigma/2 + (mc/q)(j_\sigma j^\sigma)^{1/2} + j^\sigma(\partial_\sigma\alpha+\beta\partial_\sigma\gamma)\right\}d^3x dt. \qquad (3.40b)$$

The last term in I_{C5} is the Lin constraint term used in I_{C4} Equation (3.26), while the factor of 1/2 in the field-current interaction term is needed when j^σ is to be varied.

According to our previous rationale, the Lin constraint term must be imposed if one chooses to vary j^σ directly rather than each fluid element trajectory. The field quantities to be varied in I_{C5}, Equations (3.40), are j^σ, α, β and γ. We remark that charge continuity, Equation (3.27b), does not necessarily follow from the definition of A_s^σ and hence even in a "minimally constrained" variational principle the Lin constraint term involving α must be retained in the variational principle in order to obtain from I_{C5} a complete set of equations of motion (recall the discussion concerning a more "minimally constrained" variational principle than I_{C4} following the paragraph in which Equations (3.27) are introduced in Subsection III.D.1). This is apparently due to the fact that definitions (3.38) for A_p^σ automatically restrict the electromagnetic gauge.

Although the conventional Euler-Lagrange equation method may be applied to obtain those equations of motion corresponding to variations of the field quantities α, β and γ (giving Equations (3.27b,c,d) as with I_{C4}), because of the integral over $\mathbf{r}'$ the variation of j^σ must be considered directly. Hence, for $j^\sigma(j_p^{\prime\sigma})$ we substitute

$j^\sigma + \xi J^\sigma$ $(j_p'^\sigma + \xi J_p'^\sigma)$ and then require that to first order in ξ

$$L_{C5}\left(j^\sigma + \xi J^\sigma, j_p'^\sigma + \xi J_p'^\sigma\right) - L_{C5}\left(j^\sigma, j_p'^\sigma\right) \equiv L'_{C5} - L_{C5}$$

vanish for all sufficiently smooth $J^\sigma(J_p'^\sigma)$ which vanish on the boundary of integration. By L_{C5} is meant the argument of the integral I_{C5} (note the correspondence of this method with that used to obtain the Euler-Lagrange equations in Subsection I.C.1). To first order in ξ,

$$\begin{aligned} L'_{C5} - L_{C5} = -\xi\Bigg\{ & \frac{\mu_o}{16\pi} J_\sigma \int \left[(j_r'^\sigma + j_a'^\sigma)/|\mathbf{r} - \mathbf{r}'|\right] d^3x' \\ & + \frac{\mu_o}{16} j_\sigma \int \left[(J_r'^\sigma + J_a'^\sigma)/|\mathbf{r} - \mathbf{r}'|\right] d^3x' \\ & + (mc/q) j_\sigma J^\sigma / (j_\eta j^\eta)^{1/2} + J^\sigma(\partial_\sigma\alpha + \beta\partial_\sigma\gamma)\Bigg\} \end{aligned} \tag{3.41}$$

We now expand $J_p'^\sigma$ in a Taylor series about $t_p' = t$ (assuming such an expansion exists) to get

$$J_p'^\sigma = \sum_{k=0}^{\infty} \frac{1}{k!}(\pm|\mathbf{r} - \mathbf{r}'|/c)^k \frac{\partial^k}{\partial t^k} J^\sigma(t, \mathbf{r}')$$

where the plus sign corresponds to $J_a'^\sigma$ and the minus to $j_r'^\sigma$. These Taylor expansions will allow us to put the second integral term of $L'_{C5} - L_{C5}$, Equation (3.41), in a more suggestive form if we assume that the order of integration in $I'_{C5} - I_{C5} \equiv \int (L'_{C5} - L_{C5}) d^3x dt$ may be readily interchanged. If such is possible, then the time integral may be taken prior to integration over $\mathbf{r}$ or $\mathbf{r}'$. Assuming that not only J^σ, but also partials of J^σ with respect to time of all orders vanish at the (temporal) boundary of integration it is readily demonstrated that

$$\int \left(j_\sigma \frac{\partial^k}{\partial t^k} J^\sigma\right) dt = (-1)^k \int \left(J^\sigma \frac{\partial^k}{\partial t^k} j_\sigma\right) dt$$

after multiple integrations by parts. This fact allows for the expression of

$$\int \left(j_\sigma J_p'^\sigma / |\mathbf{r} - \mathbf{r}'|\right) d^3x' d^3x dt$$

as

$$\int \left(\frac{j_\sigma(t,\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|} \sum_{k=1}^{\infty} \frac{1}{k!}(\pm|\mathbf{r}-\mathbf{r}'|/c)^k \frac{\partial^k}{\partial t^k} J^\sigma(t,\mathbf{r}') \right) d^3x' d^3x dt$$
$$= -\int \left(\frac{J^\sigma(t,\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \sum_{k=0}^{\infty} \frac{1}{k!}(\pm|\mathbf{r}-\mathbf{r}'|/c) \frac{\partial^k}{\partial t^k} j_\sigma(t,\mathbf{r}) \right) d^3x' d^3x dt$$

where the upper (+) sign refers to $J_a'^\sigma$ and the lower (−) to $J_r'^\sigma$. Assuming that a Taylor expansion also exists for j_σ it is apparent from this last equality that

$$\int \left(j_\sigma J_a'^\sigma/|\mathbf{r}-\mathbf{r}'|\right) d^3x' d^3x dt = \sum \left(J^\sigma j_{r\sigma}'/|\mathbf{r}-\mathbf{r}'|\right) d^3x' d^3x dt, \tag{3.42}$$

after performing the change of variables $\mathbf{r} \to \mathbf{r}', \mathbf{r}' \to \mathbf{r}$ and interchanging the order of integration. Similarly, replacement of $J_a'^\sigma$ with $J_r'^\sigma$ requires that $j_{r\sigma}'$ be replaced with $j_{a\sigma}'$ in Equation (3.42).

We may use Equation (3.42) and its $J_r'^\sigma$ companion to obtain the equivalent expression of $L_{C5}' - L_{C5}$, Equation (3.41),

$$L_{C5}' - L_{C5} = -\xi J^\sigma \left\{ A_{s\sigma} + (mc/q) j_\sigma/(j_\eta/j^\eta)^{1/2} + \partial_\sigma \alpha + \beta \partial_\sigma \gamma \right\}, \tag{3.43}$$

where we have utilized the definition of $A_{s\sigma}$. In order for $L_{C5}' - L_{C5}$ to vanish to first order in ξ for all appropriate J^σ it is clear from Equation (3.43) that

$$A_{s\sigma} + (mc/q) j_\sigma/(j_\eta j^\eta)^{1/2} + \partial_\sigma \alpha + \beta \partial_\sigma \gamma = 0. \tag{3.44}$$

Therefore, the equations of motion which result from the fundamental integral I_{C5}, Equations (3.40), include the definition of $A_{s\sigma}$, charge continuity, Equation (3.27b), the potential equations (3.27c,d), and Equation (3.44) which corresponds with the potential representation of $A_\sigma + (mc/q) j_\sigma/(j_\eta j^\eta)^{1/2}$ given in Equation (3.27a). Since $A_{s\sigma}$ satisfies the inhomogeneous wave equation (3.8a) and the Lorentz gauge condition, $\partial_\sigma A^\sigma = 0$, (which follows from charge continuity) by defining the electromagnetic field-strength tensor $F_{\beta\sigma}$ according to $F_{\beta\sigma} \equiv \partial_\beta A_{s\sigma} - \partial_\sigma A_{s\beta}$ the internal and external Maxwell equations (3.5) and (3.8) are satisfied. It follows, then, that the equations of motion which follow from I_{C5}, Equations (3.40), are essentially

(not rigorously because of the inclusion of a definition for A^σ in terms of j^σ and the addition of the Lorentz gauge condition) globally equivalent to the equations of motion which follow from I_{C4}, Equation (3.26), and hence are "essentially" locally equivalent to the Maxwell-Lorentz force set of equations.

6. General relativistic formulation

For completeness we here present an Eulerian variational principle giving rise to the general relativistic version of the Maxwell-Lorentz force set of equations. If we do not concern ourselves with the explicit variation of the metric $g_{\mu\nu}$ a satisfactory general relativistic fundamental integral may be obtained by multiplying the argument of I_{C4} by $\sqrt{-g}$; hence

$$I_{C6} = -\int \Big\{ F_{\mu\nu}F^{\mu\nu}/4\mu_o + A_\mu j^\mu + (mc/q)\left(j_\mu j^\mu\right)^{1/2} + j^\mu\left(\alpha_{,\mu} + \beta\gamma_{,\mu}\right)\Big\}\sqrt{-g}\,d^3x\,dt \tag{3.45}$$

The field quantities to be varied in I_{C6} include $A_\mu, j^\mu, \alpha, \beta$, and γ.

The Euler-Lagrange equations of I_{C6}, Equation (3.45), are (using $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$)

$$\delta A_\mu : \quad \left(F^{\mu\nu}\sqrt{-g}\right)_{,\mu} = \mu_o\sqrt{-g}\,j^\mu, \tag{3.46a}$$

$$\delta j^\mu : \quad A_\mu + (mc/q)j_\mu/\left(j_\nu j^\nu\right)^{1/2} + \alpha_{,\mu} + \beta\gamma_{,\mu} = 0, \tag{3.46b}$$

$$\delta\alpha : \quad \left(\sqrt{-g}\,j^\mu\right)_{,\mu} = 0, \tag{3.46c}$$

$$\delta\beta : \quad j^\mu\gamma_{,\mu} = 0, \tag{3.46d}$$

and

$$\delta\gamma : \quad \left(\beta\sqrt{-g}\,j^\mu\right)_{,\mu} = \sqrt{-g}\,j^\mu\beta_{,\mu} = 0. \tag{3.46e}$$

As usual, we append $j^\mu = \rho_{oe}v^\mu$ and $v_\mu v^\mu = c^2$ to these equations of motion. We will show that this "potential" set of equations is (locally) equivalent to the general

relativistic Maxwell-Lorentz force set of equations consisting of

$$F^{\mu\nu}_{;\mu} = \mu_o j^\nu, \tag{3.47a}$$

$$j^\mu_{;\mu} = 0, \tag{3.47b}$$

and

$$mv^\nu v_{\mu;\nu} = qv^\nu F_{\mu\nu}, \tag{3.47c}$$

as well as the usual $j^\mu = \rho_{oe} v^\mu$, $v_\mu v^\mu = c^2$, and $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, which are also contained in the potential set.

Let $\Gamma^\sigma_{\mu\nu}$ represent the affine connection coefficients corresponding to the (symmetric) metric tensor $g_{\mu\nu}$; i.e., $\Gamma^\sigma_{\mu\nu}$ is the Christoffel symbol of the second kind. Then $\Gamma^\sigma_{\mu\nu}$ is symmetric in μ and ν; whence

$$F^{\mu\nu}_{;\mu} = F^{\mu\nu}_{,\mu} + \Gamma^\mu_{\mu\sigma} F^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} F^{\mu\sigma} = F^{\mu\nu}_{,\mu} + \Gamma^\mu_{\mu\sigma} F^{\sigma\nu} \tag{3.48}$$

for any skew-symmetric tensor $F^{\mu\nu}$. Now,

$$\Gamma^\mu_{\mu\sigma} = \frac{1}{2} g^{\mu\nu} \{g_{\mu\nu,\sigma} + g_{\sigma\nu,\mu} - g_{\sigma\mu,\nu}\} = \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\sigma} = \left(\sqrt{-g}\right)_{,\sigma} / \sqrt{-g} \tag{3.49}$$

where the last equality follows by direct calculation. Substituting this expression for $\Gamma^\mu_{\mu\sigma}$ into Equation (3.48) we obtain the mathematical identity

$$\sqrt{-g} F^{\mu\nu}_{;\mu} = \left(F^{\mu\nu}\sqrt{-g}\right)_{,\mu} \tag{3.50}$$

which is satisfied by any skew-symmetric tensor $F^{\mu\nu}$. Since both sets of equations (set (3.46) and set (3.47)) include the definition $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$ of the electromagnetic field strength tensor $F_{\mu\nu}$ it is clear that $F^{\mu\nu}$ is required to be skew-symmetric by either set, whence $F^{\mu\nu}$ satisfies equality (3.50) for either set, from which follows the equivalence of Equations (3.46a) and (3.47a). Similarly, by using identity (3.49) we obtain

$$j^\mu_{;\mu} = j^\mu_{,\mu} + \Gamma^\mu_{\mu\nu} j^\nu = \left(\sqrt{-g} j^\mu\right)_{,\mu} / \sqrt{-g}$$

from which Equations (3.46c) and (3.47b), the general relativistic versions of charge continuity, are seen to be equivalent. Hence, the proof that the potential set is equivalent to the Maxwell-Lorentz force set is reduced to the demonstration that (i) field quantities which satisfy the potential set of necessity satisfy the general relativistic Lorentz force relation, Equation (3.47c), and (ii) for each set of field quantities which satisfy the Maxwell-Lorentz force set of equations there exist potentials α, β, and γ such that Equations (3.46b,d,e) are satisfied.

Before proceeding with the equivalence proof it is convenient to note the identity

$$V_{\mu;\nu} - V_{\nu;\mu} = V_{\mu,\nu} - \Gamma^{\sigma}_{\mu\nu} V_{\sigma} - V_{\nu,\mu} + \Gamma^{\sigma}_{\nu\mu} V_{\sigma} = V_{\mu,\nu} - V_{\nu,\mu} \tag{3.51}$$

for any four-vector V_{μ}, which follows from the symmetry of $\Gamma^{\sigma}_{\mu\nu}$ in μ and ν. From this identity as well as $v^{\nu} v_{\nu;\mu} = \frac{1}{2}\left(v^{\nu} v_{\nu}\right)_{;\mu} = 0$ follows the computation

$$v^{\nu}\left[\left(A_{\mu} + (m/q)v_{\mu}\right)_{;\nu} - \left(A_{\nu} + (m/q)v_{\nu}\right)_{;\mu}\right] = v^{\nu} F_{\nu\mu} + \frac{m}{q} v^{\nu} v_{\mu;\nu}. \tag{3.52}$$

That part (i) of the claim of equivalence is satisfied requires only the direct computation of the left member of equation (3.52) using Equations (3.46b,d,e). By using identity (3.51) it is seen that such a computation may be carried out exactly as in Equation (3.32) so that the left member of (3.52) vanishes. Hence, the general relativistic Lorentz force relation, Equation (3.47c), follows from the potential set.

To show that part (ii) of the claim of equivalence is true we first define a scalar α so that

$$v^{\nu} \alpha_{,\nu} = -v^{\nu}\left(A_{\nu} + (m/q)v_{\nu}\right).$$

Such an α exists locally as long as v^{ν} and A_{ν} are sufficiently well-behaved. Next, compute the Lie derivative with respect to v^{ν} of the velocity-orthogonal quantity

$A_\mu + (m/q)v_\mu + \alpha_{,\mu}$:

$$
\begin{aligned}
L_v\left[A_\mu + (m/q)v_\mu + \alpha_{,\mu}\right] &= \left(A_\mu + (m/q)v_\mu + \alpha_{,\mu}\right)_{,\nu} v^\nu + \left(A_\nu + (m/q)v_\nu + \alpha_{,\nu}\right) v^\nu_{,\mu} \\
&= v^\nu \left[\left(A_\mu + (m/q)v_\mu\right)_{,\nu} - \left(A_\nu + (m/q)v_\nu\right)_{,\mu}\right] \\
&= v^\nu F_{\nu\mu} + \frac{m}{q} v^\nu v_{\mu;\nu}.
\end{aligned}
$$

The second equality is a result of the orthogonality of v^ν and $A_\nu + \frac{m}{q}v_\nu + \alpha_{,\nu}$ and the symmetry of $\alpha_{,\mu\nu}$, while for the last equality account is taken of Equations (3.51) and (3.52). Assuming that the fields satisfy the Lorentz force relation, Equation (3.47c), $L_v\left[A_\mu + (m/q)v_\mu + \alpha_{,\mu}\right]$ vanishes, which together with $v^\nu\left(A_\nu + \frac{m}{q}v_\nu + \alpha_{,\nu}\right) = 0$ implies that locally there exist scalars ϕ, β and γ such that

$$A_\mu + (m/q)\, v_\mu + \alpha_{,\mu} = \phi_{,\mu} - \beta\gamma_{,\mu}$$

where

$$v^\nu \phi_{,\nu} = v^\nu \beta_{,\nu} = v^\nu \gamma_{,\nu} = 0 \tag{3.53}$$

by arguments analogous to those given in previous equivalence proofs. We may then absorb ϕ into the electromagnetic gauge α, then multiply relations (3.53) by ρ_{oe} and thereby obtain Equations (3.46b,d,e). This completes the proof that the potential set, Equations (3.46), is equivalent to the Maxwell-Lorentz force set, Equations (3.47).

7. Alternate general relativistic formulation

Rather than impose the Lin constraint and thereby obtain equations which involve unphysical (Clebsch) potentials and which are only locally equivalent to the general relativistic version of the Maxwell-Lorentz force set of equations, it is possible to start with a fundamental integral in entirely Eulerian form and then perform the Lagrangian variation Δ, Equation (2.33), on it in a manner analogous to the method applied in Subsection II.A.3. The fundamental integral referred to is

$$I_{C7} = -\int \left\{ \sqrt{-g}\left[F_{\mu\nu}F^{\mu\nu}/4\mu_o + A_\mu j^\mu + \frac{mc}{q}\left(j_\mu j^\mu\right)^{1/2}\right]\right\} d^3x dt, \tag{3.54}$$

and involves only the field quantities $g_{\mu\nu}$, A_μ and j^μ, not the Lin potentials α, β, and γ (notice the close similarity between I_{C7} and the originally transformed fundamental integral I_{C3}, Equation (3.25)). Recall that the Lagrangian variation Δ is defined in terms of the Lagrangian displacement vector ξ^α which vanishes at the initial and final times and thus varies a fluid element path while keeping its endpoints fixed in accordance with fluid element path variation requirements.

In a minimally constrained variational principle we need not constrain strict charge conservation, Equation (3.47b); rather, it is sufficient to require the vanishing of the Lagrangian variation of the product of the charge current density j^μ and the square root of the opposite of the determinant of the metric tensor, $\sqrt{-g}$:

$$\Delta\left(\sqrt{-g}j^\mu\right) = 0. \tag{3.55}$$

As will be seen, strict charge conservation follows from the free Lagrangian variation of the electromagnetic four-potential A_μ, so we need not constrain it. Additionally, we vary only the fields A_μ and j^μ and not the metric tensor $g_{\mu\nu}$; that is, we require $\delta g_{\mu\nu} = 0$ where δ represents the Eulerian variation or variation "in place" of the field quantities. Using the relation between Δ and δ given in Equation (2.33) we conclude that Equation (2.36) must be satisfied; that is, $\Delta g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$.

In order to compute ΔI_{C7} we need the following equalities

$$\Delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\Delta g_{\alpha\beta},$$

$$\Delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\Delta g_{\alpha\beta},$$

and

$$\Delta F_{\mu\nu} = \left(\Delta A_\nu\right)_{,\mu} - \left(\Delta A_\mu\right)_{,\nu}.$$

These equalities along with $j^\mu = \rho_{oe}v^\mu$ and $v_\mu v^\mu = c^2$ allow for the expressions

$$\begin{aligned}\Delta\left[\sqrt{-g}\left(F_{\mu\nu}F^{\mu\nu}\right)\right] &= \Delta\left[\sqrt{-g}g^{\mu\sigma}g^{\nu\gamma}F_{\sigma\gamma}F_{\mu\nu}\right]\\ &= \sqrt{-g}\left\{\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta} - 2F^{\alpha\nu}F^{\beta}_{\nu}\right]\Delta g_{\alpha\beta} + 4F^{\mu\nu}(\Delta A_\nu)_{,\mu}\right\},\end{aligned} \tag{3.56a}$$

$$\Delta\left(\sqrt{-g}A_\mu j^\mu\right) = \sqrt{-g}j^\mu \Delta A_\mu + A_\mu \Delta\left(\sqrt{-g}j^\mu\right), \tag{3.56b}$$

and

$$\Delta c\left(-g g_{\mu\nu} j^\mu j^\nu\right)^{1/2} = v_\mu \Delta\left(\sqrt{-g}j^\mu\right) + \frac{1}{2} v^\alpha j^\beta \Delta g_{\alpha\beta}. \tag{3.56c}$$

Equations (3.56), after using the constraint (3.55) and letting L_{C7} represent the argument of the fundamental integral I_{C7}, Equation (3.54), imply

$$\Delta L_{C7} = \frac{1}{2}\sqrt{-g}\left(E^{\alpha\beta} + \frac{m}{q} v^\alpha j^\beta\right)\Delta g_{\alpha\beta} + \left(\sqrt{-g}F^{\mu\nu}\Delta A_\nu/\mu_o\right)_{,\mu}$$
$$+\left[\sqrt{-g}j^\nu - \left(\sqrt{-g}F^{\mu\nu}\right)_{,\mu}/\mu_o\right]\Delta A_\nu$$

where we define the (symmetric) electromagnetic stress-energy tensor $E^{\alpha\beta}$ by

$$4\mu_o E^{\alpha\beta} \equiv F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta} - 4F^{\alpha\nu}F^\beta_\nu. \tag{3.57}$$

Using $\Delta g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha}$ and the symmetry of $E^{\alpha\beta} + \frac{m}{q}v^\alpha j^\beta$ and ignoring the total divergence, since ΔA_μ vanishes at the boundary of integration, we may write

$$I_{C7} = -\int d^3x dt\left\{\sqrt{-g}\left(E^{\alpha\beta} + \frac{m}{q}v^\alpha j^\beta\right)\xi_{\alpha;\beta}\right.$$
$$\left. + \left[\sqrt{-g}j^\mu + \left(\sqrt{-g}F^{\mu\nu}\right)_{,\nu}/\mu_o\right]\Delta A_\mu\right\}.$$

Treating ξ_α and ΔA_μ as arbitrary except for the conditions that they vanish on the boundary of integration and are sufficiently smooth, i.e., varying fluid element paths and field potentials arbitrarily, we then conclude that

$$\left(E^{\alpha\beta} + \frac{m}{q}v^\alpha j^\beta\right)_{;\beta} = 0 \tag{3.58}$$

and

$$\left(\sqrt{-g}F^{\mu\nu}\right)_{,\mu} = \mu_o\sqrt{-g}j^\nu. \tag{3.46a}$$

Hence, the equations of motion which accompany the fundamental integral I_{C7}, Equation (3.54), include the definition of $F^{\mu\nu}$, Equation (3.4), the expressions $j^\nu = \rho_{oe}v^\nu$ and $v_\nu v^\nu = c^2$, and equations (3.46a) and (3.58).

We now show the algebraic and hence global equivalence of the equations of motion associated to I_{C7} with the conventional general relativistic Maxwell-Lorentz force set of equations. Firstly, from Equation (3.50), which is satisfied by any skew-symmetric tensor $F^{\mu\nu}$, we note that (3.46a) implies the general relativistic form of the external Maxwell equations, Equation (3.47a), and vice versa. Secondly, we conclude from the skew-symmetry of $F^{\mu\nu}$ and Equation (3.46a) that charge continuity, Equation (3.46b), must be satisfied and hence we need not consider it in the proof of equivalence since it is not an independent equation of motion. We also note that the internal Maxwell equations, Equation (3.5), which are satisfied identically by the definition $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, imply by algebraic manipulation the equality

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \tag{3.59}$$

which may be considered to be the general relativistic form of the internal Maxwell equations. Now, we calculate the covariant divergence of $E^{\alpha\beta}$ using definition (3.57).

$$\begin{aligned}\mu_o E^{\alpha\beta}_{;\beta} &= \frac{1}{2} F_{\mu\nu;\beta} F^{\mu\nu} g^{\alpha\beta} - F^{\alpha\nu}_{;\beta} F^{\beta}_{\nu} - F^{\alpha\nu} F^{\beta}_{\nu;\beta} \\ &= \frac{1}{2} g^{\alpha\beta} F^{\mu\nu} \left(F_{\mu\nu;\beta} + F_{\beta\mu;\nu} + F_{\nu\beta;\mu} \right) - F^{\alpha}_{\nu} F^{\beta\nu}_{;\beta} = -\mu_o j^{\beta} F^{\alpha}_{\beta}\end{aligned} \tag{3.60}$$

where the last equality follows from the external and internal Maxwell equations (3.47a) and (3.59). Using charge continuity, Equation (3.47b), as well as equality (3.60), Equation (3.58) translates to

$$-j^{\beta} F^{\alpha}_{\beta} + \frac{m}{q} j^{\beta} v^{\alpha}_{;\beta} = 0.$$

By lowering the index α and using $j^{\beta} = \rho_{oe} v^{\beta}$ it is easily seen that this is equivalent to the general relativistic Lorentz force equation, Equation (3.47c) (as long as ρ_{oe} vanishes only at isolated points). We thereby conclude that the standard Maxwell-Lorentz force set of equations is satisfied by solutions to the I_{C7} set, and since all the steps used are algebraic and hence reversible the converse is also true. Therefore, the two sets of equations of motion are (globally) equivalent.

E. Conclusion

In this chapter, a number of variational principles which give rise to the equations of motion for a collection of charged particles or a charged fluid have been presented. The emphasis has been on those variational principles which give rise to a complete set of equations of motion, that is, those principles which generate both Maxwell's equations and the Lorentz force equation. We began by examining variational principles in the particulate (Lagrangian) form then proceeded to variational principles in the mixed particulate (Lagrangian)/Eulerian form. Wishing to motivate the introduction of a completely Eulerian variational principle we then performed a mathematical transformation of variables which introduced the four-current density j^ν while eliminating the particulate notation (see I_{C3}, Equation (3.25)). After performing the transformation of variables we constrained the variations of the field quantities according to the Lin constraint and thereby obtained equations of motion which were subsequently shown to be locally equivalent to the conventional Maxwell-Lorentz force set.

IV. NEGLECT OF THE LIN CONSTRAINT

In the previous two chapters we have concerned ourselves with variational principles leading to the unrestricted equations of motion for a warm perfect fluid and a cold electromagnetic fluid. As a means of obtaining such unrestricted equations of motion from an entirely Eulerian fundamental integral we introduced the "mysterious" Lin constraint. The introduction of the Lin constraint was accomplished initially with very little physical and/or mathematical motivation. In fact, as the Lin constraint was introduced initially in Section II.B the comment was made that the Lin constraint is somewhat mysterious from a mathematical standpoint in that it is not clear from the form of the Lin constraint that it constitutes a constraint at all. That it does constitute a constraint may be verified by comparing the results of applying the "constraint" with those of neglecting the "constraint". If the equations of motion obtained through application of the "constraint" are more general than those obtained through neglect of the "constraint" then the set of variations in the former case is of necessity smaller than the set in the latter case. Hence, if such is found to be the case the "constraint" does indeed constitute a mathematical constraint on the variations of the field quantities. On the other hand, if the set of equations of motion obtained through application and neglect of the "constraint" are equivalent the the "constraint" is vacuous, that is, it does not constitute a mathematical constraint on the variations of the field quantities.

In this chapter we establish that the Lin constraint does indeed constitute a mathematical constraint on the variations of the field quantities. This will be verified by neglecting the Lin constraint as it appears in the Eulerian fundamental integrals of the previous two sections and showing that the equations of motion obtained thereby are restrictive of the expected equations of motion. In Section A we consider the Eulerian fundamental integrals of fluid mechanics introduced in Chapter II, while in Section B we consider those of electromagnetism first introduced in Chapter III.

A. Fluid mechanics

In this section we review the Eulerian fundamental integrals of fluid mechanics introduced in Section II.B., this time neglecting the Lin constraint. Various degrees of restriction on the usual equations of motion for a warm perfect fluid will be obtained by either constraining or neglecting entropy conservation and mass conservation. We begin by considering the non-relativistic case then conclude with an examination of the general relativistic case. Particular attention will be paid to the types of restrictions imposed in the equations of motion through neglect of each usually-imposed constraint, for such information will be valuable as the physical essence of the Lin constraint is examined in the following chapter.

1. Non-relativistic fluid mechanics

Near the outset of Section II.B Hamilton's principle was utilized in order to introduce the fundamental integral I_1, Equation (2.43). That is, the argument of the integral I_1 was taken to be the difference in the kinetic energy density and potential energy density of a perfect fluid:

$$I_1 = \int \left\{ \frac{1}{2}\rho_m v^2 - \rho_m u\left(\rho_m, S\right) \right\} d^3x dt. \tag{2.43}$$

Recall that ρ_m represents the mass density, $\mathbf{v}$ the velocity field, S the entropy and $u(\rho_m, S)$ the specific internal energy of the fluid. As I_1 was initially introduced, the field quantities $\mathbf{v}$ were varied independently giving

$$\delta\mathbf{v}: \quad \mathbf{v} = 0. \tag{4.1a}$$

This equation of motion is obtained through neglect of the Lin constraint and all other constraints. It obviously constitutes a severe restriction on the usual equations of motion for perfect fluids, Equations (2.7), (2.40), (2.41) and (2.42). Variations of the remainder of the field quantities in I_1 give

$$\delta S: \quad -\rho_m \partial u/\partial S = 0 = -\rho_m T, \tag{4.1b}$$

and

$$\delta\rho_m: \quad \frac{1}{2}v^2 - u - \rho_m \partial u/\partial\rho_m = 0 = -u - p/\rho_m. \tag{4.1c}$$

For the last equality of (4.1b) we have utilized the fundamental equation of thermodynamics, Equation (2.7), while for the last equality of (4.1c) we have used both (2.7) and (4.1a). If we require ρ_m to be nonzero Equation (4.1b) requires that the temperature T vanish. This adds an additional restriction to the equations of perfect fluid motion, while (4.1c) is perfectly general given the restrictions imposed by Equations (4.1a,b). Hence, the restrictions imposed on the field quantities of I_1 through the non-imposition of variational constraints are $\mathbf{v} = 0$ and $T = 0$.

These restrictions on physical variables may be relaxed through imposition of variational constraints which restrict the class of field quantities varied over and/or the class of variations themselves. For example, it is clear that not all real-valued functions of space-time suffice as representatives of a perfect fluid's entropy S, for only those functions S for which entropy conservation, Equation (2.42), is satisfied will suffice. This suggests that we constrain the variations of the field quantities of I_1, Equation (2.43), to satisfy entropy conservation. In a similar sense, not all combinations of ρ_m and $\mathbf{v}$ are satisfactory for the modeling of perfect fluid motion, only those which satisfy the equation of continuity, Equation (2.41). hence, we are led to constrain the variations of the field quantities in I_1 in accordance with both entropy conservation and continuity. We do this through the *Lagrange undetermined multiplier method* described in Subsection I.C.3; that is, we multiply each expression which is required to vanish by a Lagrange multiplier (the Lagrange multipliers are in one-to-one correspondence with the independent equations of constraint) and include the product in the Lagrangian. The Lagrange multipliers as well as the other field quantities are then to be varied freely in order to obtain the generalized equations of motion. Although this method works for these two particular constraints it should be mentioned that the two constraints are nonholonomic in nature as they involve derivatives of the field quantities whereas the Lagrange undetermined mul-

tiplier method is only guaranteed to work when used with holonomic constraints. Hence, in this sense entropy conservation and continuity are members of a special class of nonholonomic constraints.

We will first constrain the variations of the field quantities of I_1 with entropy conservation, then with continuity, then with both. In each case the various degrees of physical restrictiveness will be examined. In order to constrain entropy conservation we add to the argument of the integral I_1 the term $\rho_m \mu DS/Dt$, then vary μ as well as ρ_m, $\mathbf{v}$ and S independently (ρ_m is included in this constraint term only for convenience). In this case the modified fundamental integral assumes the form

$$I_1' = \int \left\{ \frac{1}{2}\rho_m v^2 - \rho_m u\left(\rho_m, S\right) + \rho_m \mu \left(\partial S/\partial t + \left(\mathbf{v}\cdot\nabla\right) S\right) \right\} d^3x dt \tag{4.2}$$

where we have made use of the definition of the convective derivate $D/Dt \equiv \partial/\partial t + \mathbf{v}\cdot\nabla$.

The Euler-Lagrange equations which follow from I_1', Equation (4.2), are

$$\delta\mathbf{v}: \quad \mathbf{v} + \mu\nabla S = 0, \tag{4.3a}$$

$$\delta S: \quad -\rho_m \partial u/\partial S - \partial\left(\rho_m \mu\right)/\partial t - \nabla\cdot\left(\rho_m \mu \mathbf{v}\right) = 0, \tag{4.3b}$$

$$\delta\rho_m: \quad \frac{1}{2}v^2 - u - \rho_m \partial u/\partial\rho_m + \mu DS/Dt = 0, \tag{4.3c}$$

and

$$\delta\mu: \quad DS/Dt = 0, \tag{4.3d}$$

the last of which is obviously the equation of constraint. Equation (4.3b) may be rewritten in the form

$$\rho_m T + \rho_m D\mu/Dt + \mu\left[\partial\rho_m/\partial t + \nabla\cdot\left(\rho_m \mathbf{v}\right)\right] = 0, \tag{4.4a}$$

while Equation (4.3c) may be rewritten as

$$\frac{1}{2}v^2 - u - p/\rho_m = 0, \tag{4.4b}$$

after utilizing the equation of thermodynamics (2.7) and the equation of constraint, Equation (4.3d) (entropy conservation). If we impose continuity, Equation (2.41), subsequent to obtaining the Euler-Lagrange equations for I_1', Equation (4.4a) allows us to identify μ with the thermasy of van Dantzig.[33]

Restricting the variations of the field quantities with the imposition of entropy conservation has allowed for the inclusion of an arbitrary temperature field T throughout the fluid, as is clear from Equation (4.4a). In addition, it has slightly generalized the expression for the velocity field over the unconstrained variational principle which allowed for only the trivial solution $\mathbf{v} = 0$. However, it is clear from Equation (4.3a) that even in the entropy conservation-constrained variational principle that allowed velocity fields are severely restrictive, for only those velocity fields which result directly from entropy gradients are included in the solution set. Equation (4.4b) does not appear to be extremely restrictive over and above the restriction on $\mathbf{v}$ imposed by Equation (4.3a).

We next modify I_1 by constraining the variations of its field quantities in accordance with continuity, Equation (2.41). Hence, we add the term

$$\lambda \left[\partial \rho_m / \partial t + \nabla \cdot (\rho_m \mathbf{v})\right]$$

to the argument of the integral I_1, then vary λ as well as the other field quantities ρ_m, S, and $\mathbf{v}$. The modified fundamental integral is

$$I_1'' = \int \left\{ \frac{1}{2}\rho_m v^2 - \rho_m u\left(\rho_m, S\right) + \lambda\left[\partial \rho_m/\partial t + \nabla\cdot\left(\rho_m \mathbf{v}\right)\right]\right\} d^3x dt \tag{4.5}$$

which gives the Euler-Lagrange equations

$$\delta \mathbf{v}: \quad \mathbf{v} = \nabla \lambda, \tag{4.6a}$$

$$\delta S: \quad \partial u/\partial S = 0 = T, \tag{4.6b}$$

$$\delta \rho_m: \quad \frac{1}{2}v^2 - u - p/\rho_m - D\lambda/Dt = 0 \tag{4.6c}$$

and

$$\delta\lambda: \quad \partial\rho_m/\partial t + \nabla \cdot (\rho_m \mathbf{v}) = 0. \tag{4.6d}$$

The Euler-Lagrange equations (4.6) which follow from I_1'', Equation (4.5), are clearly restrictive of the usual equations of perfect fluid mechanics, for they require the fluid to have zero temperature and the velocity field to have vanishing curl everywhere. They are, however, more general than those obtained from the unconstrained fundamental integral I_1, Equations (4.1). It is also notable that equation (4.6c) appears to be slightly more general than the similar expression (4.4b) obtained from I_1' (note the presence of $D\lambda/Dt$).

The fundamental integrals I_1', Equation (4.2), and I_1'' Equation (4.5), have been presented not for their physical validity, but for their instructive content. The unconstrained variational principle associated with I_1, Equation (2.43), gives four restrictions on physical variables, namely $\mathbf{v} = 0$ and $T = 0$. By constraining the variations of the field quantities using only entropy conservation as in I_1' the restriction on the temperature T is reduced significantly while $\mathbf{v}$ remains severely restricted, being expressed in terms of the gradient of the entropy, a physical scalar (as opposed to an "unphysical" potential such as the Lagrange multiplier scalars μ and λ). On the other hand, by constraining the variations of the field quantities using only continuity as in I_1'' the restriction on the velocity field is slightly relaxed while the restriction $T = 0$ remains. The expression of $\mathbf{v}$ as the gradient of an arbitrary scalar gives $\mathbf{v}$ one degree of freedom. In essence, then, the constraining of the variations of the field quantities using a single equation of constraint adds one degree of freedom to the physical quantities $\mathbf{v}$ and T. As one would expect, most of the freedom produced through the entropy conservation-constrained variational principle is manifest in the temperature, the thermodynamic variable conjugate to the entropy. Similarly, most of the freedom produced through the continuity-constrained variational principle is manifest in the velocity, as one would expect since continuity constrains $\mathbf{v}$ and not S.

According to the above argument, by constraining the variations of the field quantities in I_1, Equation (2.43), using both entropy conservation and continuity one should provide but two degrees of freedom between the four quantities $\mathbf{v}$ and T. Hence, the Euler-Lagrange equations so obtained should be physically restrictive by two degrees of freedom. This is indeed the case and the essence of the Lin constraint is to provide two additional constraints on the field quantity variations so as to allow for a completely unrestricted description of the physical variables $\mathbf{v}$ and T. Throughout the remainder of this section we will consider those Eulerian variational principles in fluid mechanics for which only the Lin constraint is neglected. That is, where appropriate, both entropy conservation and continuity will be constrained, while neglecting the other terms associated to the Lin constraint. This will allow us to "home in" on the effect of neglecting only the Lin constraint which information will be valuable as the physical interpretation of the Lin constraint is explored in the following chapter.

In Section II.B the fundamental integral

$$I_{N1} \equiv I_2 = I_1 + \int \Big\{ \lambda \Big(\partial \rho_m / \partial t + \nabla \cdot (\rho_m \mathbf{v}) \Big) + \mu \Big(\partial \left(\rho_m S \right) / \partial t + \nabla \cdot (\rho_m S \mathbf{v}) \Big) \Big\} d^3x dt \tag{4.7}$$

which constrains the variations of the field quantities in I_1 using both entropy conservation and continuity was introduced. At that time only those Euler-Lagrange equations resulting from variations of $\mathbf{v}$ were presented in order to motivate the need for the Lin constraint. Here, we present the entire set of Euler-Lagrange equations associated with I_2. They are

$$\delta \mathbf{v} : \quad \mathbf{v} = \nabla \lambda + S \nabla \mu, \tag{4.8a}$$

$$\delta S : \quad D\mu / Dt = -\partial u / \partial S = -T, \tag{4.8b}$$

$$\delta \rho_m : \quad \frac{1}{2} v^2 - u - p/\rho_m - D\lambda / Dt - S D\mu / Dt = 0, \tag{4.8c}$$

$$\delta\lambda: \quad \partial\rho_m/\partial t + \nabla\cdot(\rho_m \mathbf{v}) = 0, \tag{4.8d}$$

and

$$\delta\mu: \quad \partial(\rho_m S)/\partial t + \nabla\cdot(\rho_m S\mathbf{v}) = 0. \tag{4.8e}$$

Although Equation (4.8e) is not the usual form of entropy conservation it is clear that Equations (4.8d,e) taken together are equivalent to entropy conservation and continuity. Comparing Equations (4.8) with the potential set of equations (2.52), which were shown in Section II.B to be locally equivalent to the Eulerian equations of motion for a perfect fluid, Equations (2.40), (2.41), and (2.42), it is clear that the only equation which might be physically restrictive is the expression for the velocity field, Equation (4.8a), for all other equations in the (4.8) set are included in the (2.52) set.

That Equation (4.8a) is indeed physically restrictive may be seen by computing the fluid's vorticity $\mathbf{w}$ using expression (4.8a) for the velocity field

$$\mathbf{w} \equiv \nabla\times\mathbf{v} = \nabla S \times \nabla\mu. \tag{4.9}$$

Equation (4.9) implies that isentropic fluids (fluids for which S does not vary spatially) may not possess any vorticity. It is known, however, that isentropic fluids may possess non-vanishing vorticity. In such isentropic fluids the vorticity may be introduced initially (perhaps through entropy variations prior to t_0) and subsequently convect with the fluid. In order to overcome this physical restriction it is necessary to supply two additional degrees of freedom for $\mathbf{v}$. In Section II.B these two additional degrees of freedom were provided through imposition of the Lin constraint.

2. Relativistic fluid mechanics

To neglect the Lin constraint in general relativistic perfect fluid mechanics we must modify the fundamental integral I''_{rel}, Equation (2.68). As with the non-relativistic case just considered the neglect of the Lin constraint here consists in

dropping the scalars β and γ from the integrand of I''_{rel}. After so doing one obtains the fundamental integral

$$I_{N2} = \int \Big\{ R - 2K\rho_o \Big[c^2 + U_o + \frac{1}{2}\lambda \left(g_{\sigma\nu} u^\sigma v^\nu - 1 \right) + u^\nu \left(S_o \mu_{,\nu} + \alpha_{,\gamma} \right) \Big] \Big\} (-g)^{1/2} \, d^3x dt. \tag{4.10}$$

The Euler-Lagrange equations associated to I_{N2}, Equation (4.10), are

$$\delta g_{\sigma\nu} : \quad -(-g)^{1/2} \Big\{ G^{\sigma\nu} + K\rho_o \Big[\lambda u^\sigma u^\nu + g^{\sigma\nu} \Big[c^2 + U_o + \frac{1}{2}\lambda \left(g_{\eta\delta} u^\eta u^\delta - 1 \right) + u^\sigma \left(S_o \mu_{,\sigma} + \alpha_{,\sigma} \right) \Big] \Big] \Big\} = 0, \tag{4.11a}$$

$$\delta u^\nu : \quad \lambda u_\nu + S_o \mu_{,\nu} + \alpha_{,\nu} = 0, \tag{4.11b}$$

$$\delta \rho_o : \quad c^2 + U_o + \rho_o \left(\partial U_o / \partial \rho_o \right) S_o + \frac{1}{2}\lambda \left(g_{\sigma\nu} u^\sigma u^\nu - 1 \right) + u^\nu \left(S_o \mu_{,\nu} + \alpha_{,\nu} \right) = 0, \tag{4.11c}$$

$$\delta S_o : \quad \left(\partial U_o / \partial S_o \right)_{\rho_o} + u^\nu \mu_{,\nu} = 0. \tag{4.11d}$$

$$\delta \lambda : \quad g_{\sigma\nu} u^\sigma u^\nu = 1, \tag{4.11e}$$

$$\delta \mu : \quad \left(\rho_o S_o u^\nu \right)_{;\nu} = 0, \tag{4.11f}$$

and

$$\delta \alpha : \quad \left(\rho_o u^\nu \right)_{;\nu} = 0, \tag{4.11g}$$

These equations may be readily reduced to the equivalent set

$$G^{\sigma\nu} + KT^{\sigma\nu} = 0, \tag{4.12a}$$

$$\lambda u_\nu + S_o \mu_{,\nu} + \alpha_{,\nu} = 0, \tag{4.12b}$$

$$u^\nu \alpha_{,\nu} = -\left(c^2 + U_o + p/\rho_o - S_o T_o \right) \tag{4.12c}$$

$$u^{\nu}\mu_{,\nu} = -T_o, \tag{4.12d}$$

$$g_{\sigma\nu}u^{\sigma}u^{\nu} = 1, \tag{4.12e}$$

$$u^{\nu}S_{o,\nu} = 0, \tag{4.12f}$$

and

$$(\rho_o u^{\nu})_{;\nu} = 0, \tag{4.12g}$$

by using the Euler-Lagrange equations (4.11), the equations and definitions of thermodynamics (see Equation (2.7)) and the definition of the momentum-energy tensor

$$T^{\sigma\nu} = \rho_o \lambda u^{\sigma}u^{\nu} - pg^{\sigma\nu}.$$

Recalling that Equations (2.70) are locally equivalent to the standard set of equations governing general relativistic perfect fluid motion (which in essence consist of Equations (4.12a,e,f,g)), comparison of Equations (4.12) with Equations (2.70) suggests that only Equation (4.12b) may be physically restrictive. This potential representation of the four-velocity, Equation (4.12b), is indeed physically restrictive. This may be seen by computing the vorticity tensor

$$\omega_{\sigma\nu} \equiv \partial_{\sigma}(\lambda u_{\nu}) - \partial_{\nu}(\lambda u_{\sigma}) = S_{o,\nu}\mu_{,\sigma} - S_{o,\sigma}\mu_{,\nu}. \tag{4.13}$$

In the isentropic fluid case Equation (4.13) suggests that $\omega_{\sigma\nu} \equiv 0$. This constitutes a restriction on the physical variables λ and u^{ν} which may be removed through imposition of the Lin constraint.

B. Electromagnetics

In this section we establish that the Lin constraint as imposed upon the variations of the field quantities of the Eulerian electromagnetic fundamental integrals in Section III.D is a valid mathematical constraint by illustrating that the equations of

motion obtained through neglect of the constraint are restrictive of those obtained when it is imposed. Moreover, it will be seen that there are similarities between the restrictions imposed upon the equations of fluid mechanics and those imposed on the equations of electromagnetics through neglect of the constraint. These similarities suggest a relation with the equations of superconductivity. We first consider the two special relativistic fundamental integrals of Section III.D, then conclude with the general relativistic fundamental integral of that section. In each case we restrict the variations of the field quantities so as to satisfy charge continuity and the homogeneous (or internal) Maxwell equations (see Eq. (3.4)).

1. Special relativistic formulation

Neglect of the Lin constraint in I_{C4} Eq. (3.26), consists in dropping the term involving the Clebsch potentials β and γ. The term which includes α is optional here just as it is when the Lin constraint is imposed, for inclusion of the term is equivalent to constraining the variations of j^σ so as to satisfy charge continuity, $\partial_\sigma j^\sigma = 0$, a vacuous constraint since variations of A_σ give rise to the external Maxwell equations (3.8) which in turn imply charge continuity due to the skew-symmetry of $F_{\sigma\beta}$. In the minimally constrained case we may therefore also drop the term involving the Clebsch potential α. In this minimally constrained case the fundamental integral assumes the form

$$I_{N3} \equiv I_{C3} = -\int \left[F_{\beta\sigma}F^{\beta\sigma}/4\mu_o + A_\sigma j^\sigma + (mc/q)\left(j_\sigma j^\sigma\right)^{1/2}\right] d^3x dt, \tag{4.14}$$

where A_σ and j^σ are the field quantities to be varied.

The Euler-Lagrange equations associated to I_{N3}, Eq. (4.14), are

$$\delta A_\sigma : \quad \partial_\sigma F^{\beta\sigma} = \mu_o j^\sigma \tag{3.8}$$

and

$$\delta j^\sigma : \quad A_\sigma + (mc/q)\, j_\sigma / \left(j_\beta j^\beta\right)^{1/2} = 0. \tag{4.15}$$

Use of the relations $j^\sigma = \rho_{oe} v^\sigma$ and $v_\beta v^\beta = c^2$ allows for the equivalent representation of Eq. (4.15) as

$$A_\sigma + (m/q)v_\sigma = 0. \tag{4.16}$$

As is the case in I_{C4}, Eq. (3.26) — the Lin constraint-imposed version of I_{N3}, dropping the term involving the Clebsch potential α from the fundamental integral results in an electromagnetic gauge restriction. The gauges in both the present and the Lin constraint-imposed cases coincide and may be characterized by the condition $qv^\sigma A_\sigma = -mc^2$ (see e.g., Eq. (3.29)). This gauge restriction arises because without the term involving the potential α the arguments of the fundamental integrals are not gauge-invariant. Inclusion of the α term in each fundamental integral, although it allows for the expression of A_σ in an arbitrary gauge, does not alter the physical content of the Euler-Lagrange equations. The gauge restriction may be removed from the Euler-Lagrane equations of I_{N3} Eqs. (3.8) and (4.16), by performing an arbitrary gauge transformation on A_σ according to $A_\sigma \rightarrow A_\sigma + \partial_\sigma \alpha$ where α is an arbitrary scalar. By so doing it is found that (3.8) is unaltered (i.e., (3.8) is *gauge invariant*) while (4.16) assumes the more general form

$$A_\sigma + \partial_\sigma \alpha + (m/q)\, v_\sigma = 0. \tag{4.17}$$

Eq. (3.8) constitutes the external (inhomogeneous) set of Maxwell's Equations and as such is not physically restrictive. This is to be expected, for the variations of the electromagnetic field A_σ are properly constrained in the I_{N3} variational principle. Variations of j^σ in I_{N3}, on the other hand, are not subjected to the Lin constraint. Hence, if the Lin constraint is a valid mathematical and physical constraint in this case the equation of motion resulting from the variations of j^σ, Eq.(4.17) (or, equivalently, Eq. (4.15) and (4.16)), should be mathematically and physically restrictive.

That Eq. (4.17) is mathematically restrictive may be seen immediately upon comparing it with the analogous Lin constraint-imposed Euler-Lagrange equation

(3.31). The potentials β and γ in (3.31) allow the sum $A_\sigma + (m/q)v_\sigma$ two additional degrees of freedom over the single degree of freedom allowed in (4.17). The physical restriction imposed by (4.17) may be seen by using (4.17) to compute the fluid's vorticity $\omega_{\sigma\nu}$:

$$\omega_{\sigma\nu} \equiv \partial_\sigma v_\nu - \partial_\nu v_\sigma = (q/m)\left(\partial_\nu A_\sigma - \partial_\sigma A_\nu\right) = (q/m)F_{\nu\sigma}.$$

Hence, the only vorticity allowed by (4.17) results directly in the production of fields, or is a direct consequence of the presence of fields. In particular, in the uncharged fluid limit the vorticity must vanish. This is evidently restrictive of the general Lorentz force relation which requires only that $v^\sigma \omega_{\sigma\nu} = (q/m)v^\sigma F_{\nu\sigma}$, which, in the neutral fluid limit implies only that $v^\sigma \omega_{\sigma\nu} = 0$, i.e., vorticity is convected with the fluid. We thereby conclude that Eq. (4.17) is mathematically and physically restrictive, and hence that the Lin constraint is a valid mathematical and physical constraint on the variations of j^σ in I_{C4}, Eq. (3.26).

The second special relativistic fundamental integral of interest I_{C5}, Eq. (3.40), may be altered so as to exclude the Lin constraint by dropping the term involving β and γ, just as with the I_{C4} case considered above. The term involving α must be retained in this case, however, for charge continuity is not otherwise guaranteed by I_{C5}. In I_{C4} it is guaranteed through the external Maxwell Equations. Hence, the Lin constraint-neglected version of I_{C5} assumes the form

$$I_{N4} = -\int \left(\frac{1}{2}A_s^\sigma j_\sigma + (mc/q)(j^\sigma j_\sigma)^{1/2} + j^\sigma \partial_\sigma \alpha\right) d^3x dt \qquad (4.18)$$

where

$$A_s^\sigma \equiv \frac{\mu_o}{8\pi}\int \left(\left(j_r^\sigma + j_a^\sigma\right)/|\mathbf{r} - \mathbf{r}'|\right) d^3x.$$

By performing variations of j^σ according to the method prescribed in Section III.D it is found that the resultant Euler-Lagrange equation is identical with Eq.(4.17) after using the usual relations $j^\sigma = \rho_{oe}v^\sigma$ and $v_\beta v^\beta = c^2$. Hence, this resultant Euler-Lagrange equation differs from its Lin constraint-imposed counterpart

Eq.(3.44) in that it does not contain the term involving β and γ. By the same arguments given above for the I_{C4}/I_{N3} case, the Lin constraint is therefore a valid mathematical and physical constraint in I_{C5}.

2. General relativistic formulation

The neglect of the Lin constraint in the general relativistic electromagnetic fundamental integral I_{C6}, Eq.(3.45), may be effected analogously to the special relativistic I_{C4} case considered above. That is, the term involving β and γ should be excluded from the fundamental integral I_{C6} in the event that the Lin constraint is to be neglected. Also, the term involving α is optional in that it constrains charge continuity which is already guaranteed by the external Maxwell equations which are included in the Euler-Lagrange equations derived from I_{C6}. Nevertheless, the α term must be retained in order to guarantee gauge invariance of the fundamental integral, which in turn allows for the expression of A_σ in terms of an arbitrary electromagnetic gauge. Neglect or retention of the α term, however, leads to physically equivalent Euler-Lagrange equations. We choose here to negelect the α term so as to obtain a minimally constrained variational principle and one which is most easily compared to with I_{N3}, Eq.(4.14). The Lin constraint-neglected analog of I_{C6} then assumes the form

$$I_{N5} = -\int \left[F_{\mu\nu}F^{\mu\nu}/4\mu_o + A_\mu j^\mu + (mc/q)\left(j_\mu j^\mu\right)^{1/2}\right]\sqrt{-g}d^3x dt. \tag{4.19}$$

The Euler-Lagrange equations which follow from the variations of A_μ and j^μ in I_{N5} are

$$\delta A_\nu: \quad \left(F^{\mu\nu}\sqrt{-g}\right)_{,\mu} = \mu_o\sqrt{-g}j^\nu \tag{3.46a}$$

and

$$\delta j^\sigma: \quad A_\sigma + \frac{mc}{q}j_\sigma/\left(j_\beta j^\beta\right)^{1/2} = 0, \tag{4.15}$$

which are equivalent to

$$F^{\mu\nu}_{;\mu} = \mu_o j^\nu \tag{3.47a}$$

and

$$A_\sigma + \frac{m}{q} v_\sigma = 0. \tag{4.16}$$

Eq.(3.47a) is the general relativistic version of the external Maxwell equations and as such is non-restrictive, whereas Eq.(4.16) is identical to the restrictive Euler-Lagrange equation obtained from I_{N3} and as such carries the same mathematical and physical restrictiveness. That is, mathematically speaking, Eq.(4.16) has two less degrees of freedom than does its analogous Lin constraint-imposed version, Eq.(3.46b), because β and γ do not appear in it. Physically, fluid vorticity and the presence of fields are inextricably intertwined according to Eq.(4.16) as was discussed earlier in the special relativistic case (recall from that discussion that $\omega_{\sigma\nu} = (q/m)F_{\sigma\nu}$, where $\omega_{\sigma\nu}$ is fluid vorticity, results from (4.16)). We conclude that, in the case of I_{C6}, the Lin constraint constitutes a valid constraint on the variations of j^σ, both physically and mathematically.

This concludes the demonstration that for every variational principle considered to this point in which the Lin constraint is imposed the constraint is valid, both physically and mathematically. Furthermore, neglect of the constraint gives rise to Euler-Lagrange equations which exhibit some commonalities. For example, in every case fluid vorticity is restricted through neglect of the constraint. The commonalities may be pursued further by restricting attention to zero entropy (i.e., zero temperature) fluids. For this case the restrictive equations of fluid mechanics obtained through neglect of the Lin constraint require zero vorticity, $\omega_{\sigma\nu} = 0$, while those of electromagnetics require $\omega_{\sigma\nu} = -(q/m)F_{\sigma\nu}$. The electromagnetics version is obviously more general and it reduces to the neutral fluid mechanics version in the absence of electromagnetic fields and/or charge.

Utilization of the definitions of the electric and magnetic fields in terms of the scalar potentials (see Eqs.(3.2)), allows for $\omega_{\sigma\nu} = -(q/m)F_{\sigma\nu}$ to be written in the

non-relativistic limit as

$$\mathbf{B} = -(m/q)\nabla \times \mathbf{v} \tag{4.20a}$$

and

$$\mathbf{E} = (m/q)\left(\frac{\partial v}{\partial t} + \frac{1}{2}\nabla v^2\right). \tag{4.20b}$$

These constitute the celebrated London Equations of superconductivity first proposed by F. and H. London[50] as *ad hoc* restrictions on the Maxwell-Lorentz force set of equations necessary to account for the Meissner effect.[51] The Meissner effect is that hallmark of superconductivity in which magnetic lines of flux are expelled from a superconducting medium. The London equations are derivable from quantum mechanics.[50] The derivation suggests the interpretation of the Meissner effect as a macroscopic manifestation of quantum mechanics with no classical analog. Here, however, we have obtained the London equations (4.20) from a classical variational principle through neglect of the Lin constraint. Our derivation suggests that in every classical and/or quatum mechanical system for which the Lin constraint may be neglected one should expect to observe the Meissner effect. The question then arises as to what quantum mechanical connection, if any, the neglect of the Lin constraint might have. If there is no connection between the Lin constraint and quantum mechanics, then the above analysis indicates that one need not resort to quantum mechanics to explain the Meissner effect, one need only neglect the Lin constraint.

V. PHYSICAL INTERPRETATIONS OF THE LIN CONSTRAINT

Throughout the preceding sections we have examined the Lagrangian method of obtaining, from a variational principle, the equations of motion for a neutral, perfect fluid and for a charged, zero-entropy fluid. We have also examined the effect of imposing and neglecting the Lin constraint in those variational principles. In those sections it was demonstrated that the Lagrangian variational principles yield Euler-Lagrange equations equivalent to the generally accepted equations of motion. The Eulerian variational principles, on the other hand, give rise to restrictive equations of motion unless a constraint term such as the Lin constraint is imposed upon the variations of the field quantities. The Lin constraint (or any constraint having the same effect as the Lin constraint) evidently carries information pertaining to the distinction between the Lagrangian and Eulerian descriptions of a fluid since it is essential to Eulerian variational principles whereas it is unnecessary in Lagrangian variational principles. A clear understanding of the Lin constraint should promote a greater understanding of the differing physical implications of the two fluid descriptions.

Some authors have probed the physical subtleties involved in the Lin constraint, most notably Schutz and Sorkin,[5] Edwards[12,13] and Putterman.[15] To our knowledge, however, few others have sought to physically motivate and understand alternate additional variational constraints which have the same effect as the Lin constraint. The development and study of such constraints may add valuable insight concerning the distinctions between Lagrangian and Eulerian fluid descriptions and concerning the Lin constraint itself.

This chapter begins with a revisitation of the traditional interpretation of the Lin constraint, cited upon the introduction of the constraint in Section II.B. An alternate interpretation of the constraint is then developed by an appeal to the Eulerian transformation of variables of Section III.D. The necessity of including the Lin constraint in certain situations as dictated by Subsection I.C.4's Theorem of

Schutz and Sorkin as well as other implications of this theorem are then discussed.

A. Traditional interpretation

Recall that the Lagrangian description of matter models a fluid as being composed of a continuum of infinitesimal elements called "fluid elements" or "fluid particles". These fluid particles behave much like true classical particles in that each is governed by its own evolutionary equation of motion. The Eulerian description, on the other hand, is completely determined through the specification of the values of all vector and scalar observables at every point of some space-time coordinate system. The Eulerian description makes no mention of fluid particles. Since a fluid is in reality a collection of many particles, such as electrons, atoms or molecules, it is generally conceeded that the Lagrangian description more accurately models physical reality than does the Eulerian. However, the same feature that makes the Eulerian description unattractive from a physical standpoint makes it essentially mandatory from a practical standpoint, for proper implementation of the Lagrangian description requires information concerning every fluid element, while implementation of the Eulerian description only requires knowledge of macroscopically averaged observables.

The traditional physical motivation behind the imposition of the Lin constraint is that the Lagrangian description of a fluid is physically more fundamental than is the Eulerian description. According to this motivation, the imposition of the Lin constraint introduces Lagrangian coordinates into an otherwise Eulerian variational integral. The mere presence of these Lagrangian coordinates is then said to be responsible for the recovery of the fully general equations of fluid motion as the Euler-Lagrange equations following from the variational principle. This motivation is perhaps satisfactory for most applications; however, it is too vague to illuminate any specific distinction between the Lagrangian and Eulerian descriptions of matter. The motivation has evolved into a more specific one based mainly on the distinction between Lagrangian and Eulerian fluid descriptions. That is, since the Lin

constraint introduces Lagrangian coordinates into an otherwise Eulerian variational principle and since the Lagrangian description of matter differs from the Eulerian mainly in that it describes completely the motional evolution of every fluid element, the Lin constraint may thereby be considered as constraining the variations of the Eulerian field quantities so as to guarantee that each fluid element may be followed continuously. This physical interpretation has gained nearly universal acceptance, yet it does not preclude the existence of alternate variational constraints motivated through somewhat different physical considerations. The development of such constraints may shed additional light upon the Lagrangian-Eulerian relationship and the Lin constraint itself.

1. Form of the constraint

In order to motivate a form for the Lin constraint some authors[52] use the heuristic argument that the Eulerian field quantities should depend on the Lagrangian coordinates $\mathbf{a}$. They then introduce the equation of constraint $D\mathbf{a}/Dt = 0$ where D/Dt represents the convective (or "total") derivative; i.e., they require that $\mathbf{a}$ convect with the fluid. The rationale behind such an equation of constraint is that the Lagrangian coordinates may be taken as the initial conditions of the fluid and that the initial conditions must convect with the fluid. The constraint is imposed on the variations of the field quantities according to the usual Lagrange multiplier technique (see Subsection I.C.3).

A closer examination of the equation of constraint reveals one of its apparently paradoxical attributes. The field quantities $\mathbf{a}$ are introduced in the constraint term and are then varied; they do not appear elsewhere in any of the variational principles upon which the constraint is imposed. Normally an equation of constraint includes only field quantities that are already in existence within a fundamental integral and demonstrates the precise manner (in equational form) in which those field quantities are restricted. In fact, when one introduces new field quantities into a variational principle then varies them freely one rarely alters the physical and mathematical

content of the resultant Euler-Lagrange equations as they pertain to the original variational field quantities. That is, inclusion of the new field quantities in the fundamental integral generally leads to what might be called a "vacuous" constraint: one which does not alter the resultant Euler-Lagrange equations as they pertain to the previously existent variables. Nevertheless, as was seen in the previous chapter, the Lin constraint is not vacuous and indeed has the desired result of allowing for completely general equations of motion.

Some of this apparent paradox may be removed by noting that the term $D\mathbf{a}/Dt$ does not involve $\mathbf{a}$ exclusively, but also involves the fluid velocity field $\mathbf{v}$ (recall the operator definition $D/Dt = \partial/\partial t + v_x\partial/\partial x + v_y\partial/\partial y + v_z\partial/\partial z$). In fact, in order to convert it into the form of a density for inclusion in the fundamental integral the equation of constraint must be multiplied by the mass or charge density. The result is an overall dependence of the constraint term on the the mass or charge current density (which is the product of density and velocity). Since the term does involve the current density it is less paradoxical that it has the effect of constraining the variations of the current density.

Additionally, if the quantities $\mathbf{a}$ truly represent initial/ boundary conditions for the fluid, their variations must vanish on the boundary of integration. This may be insured by varying them freely: hence the paradox of varying the newly introduced variables freely is somewhat relieved.

B. Implications of a transformation of variables

The adequacy of the Lin constraint for most applications has perhaps formed a barrier to investigations of alternate Eulerian variational constraints which are as or more appropriate than the Lin constraint itself. This lack of investigation into alternate constraints is apparently responsible for the almost universal acceptance of the motivation for the Lin constraint exclusive of any other physical motivation which might lead to other variational constraints of equal or greater validity than the Lin constraint. Fortunately, there has been some relatively recent activity in

the literature[12,13,14] which has revived the Eulerian variational constraint issue. The basis of the activity is an Eulerian transformation of variables in a mixed Lagrangian/Eulerian variational integral. We next present an alternate Eulerian variational constraint which is motivated by that activity as well as a discussion of conserved quantities associated with the variational principle.

1. A Lagrangian to Eulerian variable transformation

We begin our discussion by revisiting the complete electromagnetic variational principle in mixed Lagrangian/Eulerian notation introduced in Section III.C. This familiar variational principle models the interaction of a system of charged particles with masses m_i and charges q_i where i is the particle index. The σ^{th} component of the four-position of particle i is designated as x_i^σ and the position is parameterized by τ_i. For convenience τ_i is taken as the proper time of particle i so that the σ^{th} component of the particle's four-velocity v_i^σ is given by $v_i^\sigma = dx_i^\sigma/d\tau_i$. As usual, the electromagnetic field-strength tensor $F_{\beta\sigma}$ is given a component-wise definition in terms of the electromagnetic four-potential A_σ:

$$F_{\beta\sigma} = \partial_\beta A_\sigma - \partial_\sigma A_\beta. \tag{3.4}$$

Recall that the fundamental integral has the following form,

$$I_{C2} = -\int \left(F_{\beta\sigma}F^{\beta\sigma}/4\mu_o + A_\sigma j^\sigma \right) d^3x dt - \sum_{i=1}^{N} \int m_i c \left(v_{i\sigma} v_i^\sigma \right)^{1/2} d\tau_i, \tag{3.19b}$$

where

$$j^\sigma(x^\nu) \equiv c \sum_{i=1}^{N} q_i \int_{-\infty}^{+\infty} \delta\left(x^\nu - x_i^\nu\right) v_i^\sigma d\tau_i. \tag{3.18}$$

The Euler-Lagrange equations which result from I_{C2} through variations of the field quantities A_σ and x_i^σ are the external (or inhomogeneous) Maxwell's equations,

$$\delta A_\sigma : \quad \partial_\beta F^{\beta\sigma} = \mu_o j^\sigma, \tag{3.8}$$

and the Lorentz force relation for each of the N particles,

$$\delta x_i^\sigma : \quad m_i \frac{dv_{i\sigma}}{d\tau_i} = q_i v_i^\beta F_{i\sigma\beta}, \quad i = 1, ..., N, \tag{3.11}$$

where $F_{i\sigma\beta}$ is the electromagnetic field tensor evaluated at the position of particle i, that is, $F_{i\sigma\beta} = F_{\sigma\beta}\left(x_i^\alpha\right)$.

As in the presentation of Subsection III.D.1, we perform a transformation of variables in I_{C2} so as to obtain a completely Eulerian variational principle. The transformation of variables is based on the definition of the charge current density j^σ. We first make two simplifying assumptions and two fundamental definitions. We assume that (i) the mass-to-charge ratio is the same for all system particles (that is, $m_i/q_i = m/q$ for all i), and (ii) $q_i > 0$ for all i. We next define a fluid four-velocity field $v^\sigma(x^\nu)$ such that $v^\sigma v_\sigma = c^2$ and $v^\sigma(x^\nu)|_{x^\nu = x_i^\nu} = v_i^\sigma$. Finally, we define the fluid rest charge density ρ_{oe} in accordance with

$$\rho_{oe}(x^\nu) = c \sum_{i=1}^{N} q_i \int \delta\left(x^\nu - x_i^\nu\right) d\tau_i. \tag{3.23}$$

Then the charge current density j of Eq.(3.18) may be written as

$$j^\sigma(x^\nu) = \rho_{oe}(x^\nu) v^\sigma(x^\nu)$$

and the equality

$$\frac{mc}{q} \int \left(j^\sigma j_\sigma\right)^{1/2} d^3x dt = mc \sum_{i=1}^{N} \int \left(v_i^\sigma v_{i\sigma}\right)^{1/2} d\tau_i \tag{3.24}$$

is satisfied. Hence, the transformation of variables based on the definition of the charge current density j leads us to the fundamental integral

$$I_{C3} = -\int \left(F_{\beta\sigma} F^{\beta\sigma}/4\mu_o + A_\sigma j^\sigma + \frac{mc}{q} \left(j_\sigma j^\sigma\right)^{1/2} \right) d^3x dt. \tag{3.25}$$

As Henyey[14] points out in response to a paper by Edwards,[12] the transformation of I_{C2} according to conventional techniques is completed by adding a constraint term

to I_{C3}. Henyey cites Courant and Hilbert[53] in indicating that the necessary equation of constraint is none other than the mathematical equation of transformation: the definition of the charge current density, Eq.(3.18). The variational constraint is necessary to establish equivalence of the variational principles. In the absence of the constraint the two fundamental integrals are only guaranteed to be equal along solutions to their respective Euler-Lagrange equations, the Euler-Lagrange equations themselves may not be equivalent. Without imposing the constraint, one loses direct control over the variations of the individual particle trajectories, for they then no longer appear in the fundamental integral.

The variational constraint may be imposed through the Lagrange undetermined multiplier technique. If $\lambda_\sigma = \lambda_\sigma(x^\nu)$ represents the Lagrange undetermined multiplier, the conventionally transformed fundamental integral is

$$I_T = I_{C3} + \int \lambda_\sigma \left(j^\sigma - cq_i \sum_{i=1}^{N} \int \delta\left(x^\nu - x_i^\nu\right) v_i^\sigma d\tau_i \right) d^3x dt. \tag{5.1}$$

I_T and I_{C2} are expected to be equivalent in the sense that their Euler-Lagrange equations should be equivalent.

2. *Euler-Lagrange equations of I_T*

The Euler-Lagrange equations obtained through the variation of A in I_T, Eq.(5.1), are the inhomogeneous Maxwell equations, Eq.(3.8). The equation of transformation, that is, the defining relation for the current density j, Eq.(3.18), follows from the variation of the Lagrange undetermined multiplier λ. Variations of j and x_i yield the Euler-Lagrange equations

$$\delta j^\sigma : \quad A_\sigma + \frac{mc}{q} j_\sigma / (j^\beta j_\beta)^{1/2} = \lambda_\sigma, \tag{5.2a}$$

and

$$\delta x_i^\sigma : \quad d\lambda_\sigma(x_i^\nu)/d\tau_i - v_i^\beta \partial\lambda_\beta(x_i^\nu)/\partial x_i^\sigma = 0, \quad i = 1, ..., N. \tag{5.2b}$$

Eq.(5.2a) follows from a straightforward application of the standard form of the Euler-Lagrange equations. Eqs.(5.2b) may be obtained by first integrating the only x_i^ν-dependent term of I_T over space-time in the following manner:

$$-\int \lambda_\sigma(x^\nu)\Big[c\sum_{i=1}^{N} q_i \int v_i^\sigma \delta^4(x^\nu - x_i^\nu)d\tau_i\Big]d^3x dt = -c\sum_{i=1}^{N} q_i \int v_i^\sigma \lambda_\sigma(x_i^\nu)d\tau_i.$$

Application of the standard Euler-Lagrange equations to the revised term assuming for each term in the summation a single "independent variable" τ_i, and keeping in mind that $x_i^\nu = x_i^\nu(\tau_i)$ and $v_i^\sigma = dx_i^\sigma(\tau_i)/d\tau_i$, then results in the Euler-Lagrange equations (5.2b).

Recall that j may be expressed in the form $j^\sigma(x^\nu) = \rho_{oe}(x^\nu)v^\sigma(x^\nu)$, from whence Eq.(5.2a) may be rewritten as

$$A_\sigma + \frac{m}{q}v_\sigma = \lambda_\sigma. \tag{5.3a}$$

Application of the differential chain rule and the definition of v_i^σ allows for the re-expression of Eqs.(5.2b) in the equivalent form

$$v_i^\beta(\partial\lambda_\sigma(x_i^\nu)/\partial x_i^\beta - \partial\lambda_\beta(x_i^\nu)/\partial x_i^\sigma) = 0. \tag{5.3b}$$

The Euler-Lagrange equations associated with I_T are therefore Eqs.(3.8), (3.18), and (5.2a,b), or equivalently, Eqs.(3.8), (3.18) and (5.3a,b).

3. Equivalence of I_{C2} and I_T

We here demonstrate the equivalence of I_{C2} and I_T directly; that is, by showing that their Euler-Lagrange equations are (globally) equivalent. The demonstration consists of two parts. The first part shows that if the I_{C2} set of Euler-Lagrange equations (Eqs.(3.8) and (3.11)) together with the equation of transformation of j (Eq.(3.18)) are satisfied by a specific set of field quantities $\{A, x_i, j = \rho_{oe}v\}$, then a vector-valued function of space-time λ may be defined such that the I_T set of Euler-Lagrange equations (Eqs.(3.8), (3.18) and (5.3a,b)) is satisfied by that A, x_i, j and

λ. The second part shows the converse: if field quantities $A, x_i, j = \rho_{oe}v$ and λ satisfy the I_T set of Euler-Lagrange equations, then A, x_i, and j satisfy the I_{C2} set of Euler-Lagrange equations. Taken together, the two parts demonstrate a one-to-one correspondence between the solutions of the I_{C2} and I_T sets of Euler-Lagrange equations and are therefore sufficient to demonstrate the equivalence of the Euler-Lagrange equations and hence of the fundamental integrals themselves.

To establish the first part we let A, x_i, and $j = \rho_{oe}v$ be a set of field quantities satisfying the I_{C2} set of Euler-Lagrange equations together with the equation of transformation (Eq.(3.18)). We now define a new variable λ in accordance with (5.3a). Next, we take the generalized curl of both sides of (5.3a), then project the curl onto v so as to find

$$v^\beta(\partial_\beta A_\sigma + \frac{m}{q}\partial_\beta v_\sigma) - v^\beta(\partial_\sigma A_\beta + \frac{m}{q}\partial_\sigma v_\beta) = v^\beta(\partial_\beta\lambda_\sigma - \partial_\sigma\lambda_\beta). \tag{5.4}$$

Next, we note that v is defined such that $v^\beta v_\beta = c^2$ so that

$$v^\beta\partial_\sigma v_\beta = \frac{1}{2}\partial_\sigma(v^\beta v_\beta) = 0.$$

Accordingly, we rearrange (5.4) to obtain

$$\frac{m}{q}v^\beta\partial_\beta v_\sigma - v^\beta F_{\sigma\beta} = v^\beta(\partial_\sigma\lambda_\beta - \partial_\beta\lambda_\sigma) \tag{5.5}$$

through use of the electromagnetic field-strength tensor defining relation, Eq.(3.4). Finally, we evaluate Eq.(5.5) at particle position x_i after noting $v_i^\beta\partial v_{i\sigma}/\partial x_i^\beta = dv_{i\sigma}/d\tau_i$ and find[54]

$$\frac{m}{q}dv_{i\sigma}/d\tau_i - v_i^\beta F_{i\sigma\beta} = v_i^\beta\Big(\partial\lambda_\sigma(x_i^\nu)/\partial x_i^\beta - \partial\lambda_\beta(x_i^\nu)/\partial x_i^\sigma\Big). \tag{5.6}$$

Eq.(5.3b) then follows immediately after recalling $m/q = m_i/q_i$ and after applying the Lorentz force relation for particle i, Eq.(3.11), which is one of the Euler-Lagrange equations of I_{C2}. Hence, λ *defined* by Eq.(5.3a) necessarily *satisfies* Eq.(5.3b) whenever A, x_i, and $j = \rho_{oe}v$ are solutions to the I_{C2} Euler-Lagrange equations. Futhermore, the remaining I_T Euler-Lagrange equations are contained in the I_{C2} set and

hence are satisfied whenever the I_{C2} set is. Therefore, if A, x_i, and j satisfy the I_{C2} set of Euler-Lagrange equations we may define λ according to Eq.(5.3a) and thereby obtain a solution set of the I_T Euler-Lagrange equations. This establishes the first part of the demonstration of equivalence.

For the second part, assume that $A, x_i, j = \rho_{oe} v$ and λ comprise a solution set of the I_T Euler-Lagrange equations. Recall that the derivation of Eq.(5.6) relies only on the universal definitions of the four-velocity v and the electromagnetic field-strength tensor F, and on Eq.(5.3a); hence, since Eq.(5.3a) is a member of the set of I_T Euler-Lagrange equations Eq.(5.6) must be satisfied under our hypothesis. Next, we apply Eq.(5.3b) to Eq.(5.6) and conclude that the Lorentz force relation is satisfied for each particle i. Since all other I_{C2} Euler-Lagrange equations are contained in the I_T set we conclude that A, x_i and j satisfy the I_{C2} set of Euler-Lagrange equations. This concludes the demonstration of the equivalence of the I_{C2} and I_T Euler-Lagrange equations, and hence the equivalence of the fundamental integrals I_{C2} and I_T.

4. Canonical momentum as a constant of motion

Having obtained the Euler-Lagrange equations associated to I_{C2} and I_T and having demonstrated their equivalence we are now in a position to discuss some constants of the motion. For the Lagrangian coordinates used in I_{C2} and I_T the canonical momentum $P_{i\sigma}$ conjugate to the position (i.e., trajectory) x_i^σ of particle i is defined by

$$P_{i\sigma} = \partial L / \partial v_i^\sigma \tag{5.7}$$

where L is the Lagrangian and v_i^σ is the velocity of particle i, $v_i^\sigma \equiv dx_i^\sigma / d\tau_i$. The canonical momentum $P_{i\sigma}^{(C2)}$ obtained from I_{C2} using definition (5.7) and $v_i^\sigma v_{i\sigma} = c^2$ is

$$P_{i\sigma}^{(C2)} = -q_i A_\sigma(x_i^\nu) - m_i v_{i\sigma}. \tag{5.8a}$$

The canonical momentum $P_{i\sigma}^{(T)}$ obtained from I_T is

$$P_{i\sigma}^{(T)} = -cq_i \int \lambda_\sigma \delta^4(x^\nu - x_i^\nu) d^3x dt = -q_i \lambda_\sigma(x_i^\nu). \tag{5.8b}$$

Using (5.3a) and the property $v^\sigma|_{x^\nu = x_i^\nu} = v_i^\sigma$ of the velocity field it is clear that $P_{i\sigma}^{(C2)} = P_{i\sigma}^{(T)}$, which is expected since I_{C2} and I_T are equivalent.

The Euler-Lagrange equation resulting from the variation of the trajectory x_i^σ of particle i may be cast into the general form

$$\partial L/\partial x_i^\sigma - dP_{i\sigma}/d\tau_i = 0,$$

where $P_{i\sigma}$ is defined by (5.7) and $L = L(x_i^\sigma, v_i^\sigma)$ is an arbitrary Lagrangian. Hence, for any such Lagrangian, $P_{i\sigma}$ is a *constant of the motion* of particle i, that is, $dP_i^\sigma/d\tau_i = 0$ if and only if $\partial L/x_i^\sigma$ vanishes (this is a special case of Noether's Theorem - see Section I.C.2). In the particular cases of I_{C2} and I_T, this necessary and sufficient condition implies that $P_{i\sigma}^{(C2)}$ is a constant of the motion of particle i if and only if

$$q_i v_i^\beta \partial A_\beta(x_i^\nu)/\partial x_i^\sigma = 0, \tag{5.9a}$$

while $P_{i\sigma}^{(T)}$ is a constant of the motion of particle i if and only if

$$q_i v_i^\beta \partial \lambda_\beta(x_i^\nu)/\partial x_i^\sigma = 0. \tag{5.9b}$$

By using the "definition" of λ, Eq.(5.3a), and $v_i^\sigma v_{i\sigma} = c^2$ it is readily verified that Eqs.(5.9) are equivalent. This again is expected since I_{C2} is equivalent to I_T, so that any condition regarding the field quantities of one should apply to the field quantities of the other.

From Eq.(5.8a) or from Eq.(5.9a) it is evident that the notion of a *conserved* canonical momentum, i.e., a canonical momentum that is a *constant of the motion*, is electromagnetic gauge dependent. That is, the addition of the four-gradient of some scalar-valued function α to the electromagnetic four-potential A may destroy

the equivalence indicated in Eq.(5.9a). We say that the canonical momentum $P_{i\sigma}$ is *conserved to physical significance* if an electromagnetic gauge can be found for A such that $P_{i\sigma}$ is a constant of the motion.

5. Henyey's equations of motion

In his response to Edwards[12] Henyey[14] correctly notes that Edwards' variable transformation in the fundamental integral I_{C2} is not performed according to traditionally accepted techniques. He then transforms variables in I_{C2} according to the technique prescribed in Courant and Hilbert[53] and thereby obtains our I_T, Eq.(5.1). However, Henyey's derivation of the Euler-Lagrange equations which follow from I_T is faulty. He correctly obtains Maxwell's equations, Eq.(3.8), and the expression relating λ to A and v, Eq.(5.2a). As Allen, Clifton and Edwards[17] point out, though, his final Euler-Lagrange equation

$$d\lambda_\sigma(x_i^\nu)/d\tau_i = 0, \tag{5.10}$$

is a misrepresentation of Eq.(5.2b). This incorrect Euler-Lagrange equation prompted Henyey to claim the "constancy of λ along particle trajectories." As can be seen from Eq.(5.9b), this claim is equivalent to the claim that the canonical momentum of particle i is "constant along particle trajectories," i.e., is *conserved*, for $\lambda_{i\sigma}$ and $P_{i\sigma}$ are directly proportional to one another, the proportionality constant being the charge on particle i, q_i. Following Allen, Clifton and Edwards[17] we demonstrate in the following subsection that there exist (locally) physically realizable solutions to the Maxwell/Lorentz force set of equations for which the canonical momentum is not a constant of the motion *to physical significance.*

6. Non-conserved canonical momentum

To expedite the following discussion we shall consider the equations of motion for a single particle and shall drop the subscript i throughout. The relevant equations of motion are the I_T set of Euler-Lagrange equations. The condition for conservation

of the canonical momentum P_σ conjugate to the trajectory of the particle is given by Eq.(5.9b), which in our current notation is

$$v^\beta \partial_\sigma \lambda_\beta = 0. \tag{5.9c}$$

This is also directly evident from the expression $P_\sigma = -q\lambda_\sigma$ (Eq.(5.8b)), from the Euler-Lagrange equation $v^\beta(\partial_\beta - \partial_\sigma \lambda_\beta) = 0$ (Eq.(5.3b)) and from the identity $dP_\sigma/d\tau = -qd\lambda_\sigma/d\tau = -qv^\beta \partial_\beta \lambda_\sigma$.

We will show, by specific example, that there exists a v and a λ which satisfy Eq.(5.3b) but not Eq.(5.9c). From these field quantities we will then construct a solution set $\{A, v\}$ to the Maxwell-Lorentz force set of equations for which the canonical momentum is not conserved *to physical significance* by using Eq.(5.3a), $A_\sigma = -\frac{m}{q}v_\sigma + \lambda_\sigma$.

A general gauge transformation on A is obtained by adding the gradient of a scalar-valued function of space-time, α, to A : $\tilde{A}_\sigma = A_\sigma + \partial_\sigma \alpha$. From Eq.(5.3a) it is evident that A and λ have the same gauge freedom. Recall that the canonical momentum is a constant of the motion to physical significance if a gauge transformation may be performed on A (equivalently λ) thereby allowing for the satisfaction of $dP/d\tau = 0$.

As a prelude to our counterexample, we multiply Eq.(5.9c) by an arbitrary scalar-valued function of space-time f, then take the (generalized) curl of the result so as to obtain[55]

$$D[\lambda_\eta] \equiv \partial_\nu(fv^\beta)\partial_\sigma \lambda_\beta - \partial_\sigma(fv^\beta)\partial_\nu \lambda_\beta. \tag{5.11a}$$

In view of Eq.(5.11a) we may then conclude that if P is conserved to physical significance, a scalar-valued function α must exist such that

$$D[\partial\alpha/\partial x^\eta] + D[\lambda_\eta] = 0, \tag{5.11b}$$

where the operator D is as defined in Eq.(5.11a). Equivalently, if there exists a solution set $\{A, v, \lambda\}$ to the I_T Euler-Lagrange equations for which no α may be

found resulting in the satisfaction of Eq.(5.11b), then the corresponding canonical momentum P is not conserved to physical significance.

Our counterexample follows:

Let $fv^0 = C_1$, $fv^1 = x + C_2$, $fv^2 = y$, and $fv^3 = z$ where C_1 and C_2 are positive constants and $|x| < C_2$ (the usual convention $x^1 = x$, $x^2 = y$, $x^3 = z$ is used). The only restriction that must be imposed on fv is that it be time-like since v is time-like and f is an arbitrary scalar-valued function of space-time. This requirement is satisfied in a neighborhood of the origin where $(x + C_2)^2 + y^2 + z^2 < C_1^2$ (take $C_2 < C_1$).[56]

Define λ by $\lambda_0 = \lambda_2 = 0$, $\lambda_1 = -\frac{mc}{q}[C_1 yz/(x + C_2)^3]$ and $\lambda_3 = \frac{mc}{q}[C_1 y/(x + C_2)^2]$. A straightforward computation verifies that $fv^\beta \left(\partial_\beta \lambda_\sigma - \partial_\sigma \lambda_\beta\right) = 0$, and hence that this choice for v and λ satisfies (locally) the I_T Euler-Lagrange equation, Eq.(5.3b).[57] By defining $A_\sigma = -(m/q)v_\sigma + \lambda_\sigma$, $\{v, A\}$ then becomes a solution set to the Maxwell-Lorentz force equations (the I_{C2} Euler-Lagrange equations). However, calculation of the $\nu = 2$, $\sigma = 3$ component of Eq.(5.11b) for this fv and λ yields

$$\partial^2 \alpha/\partial z \partial y - \partial^2 \alpha/\partial y \partial z - \frac{mc}{q}[C_1/(x + C_2^2)] = 0. \tag{5.12}$$

The order of differentiation is commutative for any twice continuously differentiable function. Hence, the first two terms of Eq.(5.12) should combine to give zero. This, however, contradicts the fact that the last term is nowhere zero on the domain of definition.

We conclude that there does not exist an α which satisfies (5.12) and hence (5.11b) cannot be satisfied for this choice of fv and λ. Therefore, $P = q\lambda$ is not a constant of the motion to physical significance for this particular solution of the Maxwell-Lorentz force equations.

In Appendix A we show that the Eulerian generalization of the canonical momentum P may not be a constant of a charged fluid's motion even when the volume integral of the familiar canonical momentum density is. Additionally, the volume

integral of the canonical momentum density lends itself more naturally to the imposition of physical boundary conditions. As such, the volume integral of the canonical momentum density is a more natural place to look for a constant of a fluid's motion than is the canonical momentum P.

7. I_T constraint term

I_T is obviously not an entirely Eulerian variational principle. The Lagrangian to Eulerian variable transformation of the field quantities of I_{C2} effectively transforms the the Lagrangian terms of I_{C2} to Eulerian form. However, according to the conventional rules of functional variable transformations it is required to constrain the equation of transformation in order to obtain an equivalent functional. Since the equation of transformation, Eq.(3.18), contains Lagrangian notation the equation of constraint required in this case introduces Lagrangian notation into I_T. Hence, the Lagrangian to Eulerian variable transformation does not completely accomplish the goal of casting I_{C2} in Eulerian form.

Edwards[12] performs the Lagrangian to Eulerian variable transformation on I_{C2} then neglects the mixed Lagrangian/Eulerian constraint term in an effort to obtain an entirely Eulerian analog of I_{C2}. Edwards' method results in our I_{C3}, Eq.(3.25). Neglect of the equation of constraint is a reasonable first attempt at finding an entirely Eulerian analog of I_{C2}, for it is possible that an equation of constraint may be vacuous. As Edwards demonstrates and as is shown in the previous chapter, the Euler-Lagrange equations obtained from I_{C3} are restrictive of the Maxwell-Lorentz force set of equations which result from variation of the field quantities of I_{C2}. This is true even though I_{C2}, Eq.(3.25), is equal to I_T on solutions since the constraint term vanishes on solutions (recall that the Euler-Lagrange equations of I_{C2} and I_T are equivalent). This establishes that the I_T constraint is not vacuous and that one must constrain the variations of the field quantities of I_{C3} in order to obtain unrestrictive dynamical equations. Since it is the charge current density j which is involved in the equation of constraint, Eq.(3.18), we conclude that the class of

variations of j is larger in the absence of the constraint than when it is imposed. It is by varying over the larger class of variations that the restrictiveness is introduced into the Euler-Lagrange equations.

The differences in Lagrangian and Eulerian variational principles may be elucidated by examining the differences in the classes of charge current densities varied over in the I_{C3} and I_T variational principles. The class of charge current densities varied over in the case of I_T is given explicitly by the equation of constraint, Eq.(3.18), and hence consists of those current densities produced by the (classical) motion of a fixed number of charged point particles. In the case of I_{C3} the class apparently consists of all (time-like, Eulerian) four-vectors j, since no constraint is imposed on the variations of j in that case. The question, "Do there exist classes of Eulerian charge current densities j which may not be cast in the familiar Lagrangian form given by Eq.(3.18)," may be answered in the affirmative. One class may be immediately recognized as those Eulerian current densities which do not satisfy the equation of charge conservation, $\partial_\sigma j^\sigma = 0$, for such current densities may not be considered as created from the motion of a *fixed* number of particles.

Imposing the constraint of charge conservation on the variations of the charge current density in I_{C3} does not result in physically less restrictive Euler-Lagrange equations. The reason is that the equation of charge conservation may be derived from Maxwell's equations, the Euler-Lagrange equation which follows from variation of the electromagnetic four-vector A. This may be shown by taking the divergence of Eq.(3.8) so as to obtain $\partial_\sigma \partial_\beta F^{\beta\sigma} = \mu_o \partial_\sigma j^\sigma$. Since the order of differentiation is unimportant and since $F^{\beta\sigma}$ is skew-symmetric, the left hand member of the expression vanishes thereby requiring the satisfaction of the equation of charge conservation. Since the Euler-Lagrange equations require the satisfaction of the equation of charge conservation, the equation may be considered as constrained. It is therefore evident that charge conservation (equivalently, conservation of particle number) is not the key physical distinction between Lagrangian and Eulerian variational principles.

Further examination of the equation of constraint, Eq.(3.18), with consideration for the definition of the charge density, Eq.(3.23), and the velocity field v^σ which allow for the expression of the charge current density as $j^\sigma = \rho_{oe} v^\sigma$, suggests no additional class of charge current density j involved in the variations of I_{C3} but not I_T. For, by appropriately varying particle trajectories one is able to vary over virtually all velocity fields and charge densities (after utilizing conventional smoothing techniques). However, examination of the variational processes themselves leads to the discovery of a larger class of *variations* in the entirely Eulerian variational principle I_{C3}.

By appropriately varying particle trajectories one is able to vary over virtually all velocity fields and charge densities. Hence, examination of the equation of constraint, Eq.(3.18), suggests that the classes of charge current densities j involved in the variations of I_{C3} and I_T are essentially the same. However, examination of the variational processes themselves leads to the discovery of a larger class of variations in the entirely Eulerian variational principle I_{C3}.

A general variation of any field quantity consists in varying the field quantity arbitrarily (but smoothly) within the volume of integration while keeping the field quantities fixed on the boundary of the volume. Note that A and j are held fixed on the boundary of integration in both the I_{C3} and I_T variational principles since both field quantities are varied freely. On comparing the variational integrals I_{C3} and I_T via the equation of constraint, Eq.(3.18), it is evident that additional field quantities, the particle trajectories, are introduced into I_T through the constraint term. Hence for I_T the particle trajectories must also remain fixed on the boundary of the integration volume.

Fixing a collection of particle trajectories or position labels on a specified boundary is a stronger condition than holding the charge current density fixed on that boundary. The same current density results from an arbitrary *permutation* of particle position labels when the velocity at each position is held fixed. Hence, holding the charge current density fixed on the boundary of a volume does not necessitate

a particular labeling of fluid elements on that boundary, as fixing particle position labels does.

The I_T equation of constraint, Eq.(3.18), as well as requiring charge conservation, requires what we shall refer to as "fluid element identity". The charge conservation constraint reduces the size of the set of charge current densities varied over. On the other hand, the fluid element identity constraint restricts the class of variations themselves. The two constraints taken together (and packaged in the single Eq.(3.18)) lead to completely general Euler-Lagrange equations. The charge conservation constraint alone does not physically generalize the Euler-Lagrange equations that result from I_{C3}. It is therefore apparent that the fluid element identity constraint is the key element in the physical distinction between the two variational principles. It should also contain a key element in the distinction between the Lagrangian and Eulerian descriptions of matter. "Fluid element identity" is a fundamental concept in the conventionally held view of the distiction between the two descriptions of matter. Here, however, it is motivated through the investigation of a variational equation of constraint that generalizes the Euler-Lagrange equations obtained from an entirely Eulerian variational principle to the conventional equations of charged fluid motion.

8. The Lagrange multiplier λ

Conventionally, a Lagrange multiplier is viewed as a force or potential which imposes the equation of constraint on particle (or system) motion. According to this view, the Lagrange multiplier λ appearing in the variational integral I_T is a potential which requires the charge current density j to be expressible as the result of the motion of a collection of particles. The quantity λ may therefore be considered as a potential giving rise to the Poincare stresses[58] which hold fluid particles together, i.e., a potential requiring "fluid element integrity". This interpretation for λ is strongly correlated to the "fluid element identity" interpretation of the equation of constraint, as one would expect.

An alternative interpretation for λ is that of "the canonical momentum of the fluid". By "canonical momentum of the fluid" is meant the quantity $\mathbf{P}_\sigma(x^\nu)$ which restricts to the canonical momentum of particle i along its trajectory; that is,

$$\mathbf{P}_\sigma(x^\nu) = qA_\sigma + mv_\sigma.$$

The identity

$$\lambda_\sigma = -q\mathbf{P}_\sigma$$

is evident from Eq.(5.3a). Hence λ, to within a constant $(-q)$, is the charged fluid's "canonical momentum".

A third physical interpretation for the Lagrange multiplier λ is as follows. Define a field $\tilde{F}$ in terms of the potential λ by

$$\tilde{F}_{\sigma\beta} = \partial_\sigma \lambda_\beta - \partial_\beta \lambda_\sigma. \tag{5.13}$$

This definition for $\tilde{F}$ is exactly analogous to the definition of the usual electromagnetic field tensor F in terms of the potential A (see Eq.(3.5)). Assuming that the Eulerian form of the Lorentz force relation,

$$mv^\beta \partial_\beta v_\sigma = qv^\beta F_{\sigma\beta}, \tag{5.14}$$

is satisfied, Eq.(5.5) requires that

$$qv^\beta \tilde{F}_{\sigma\beta} = 0. \tag{5.15}$$

Interpreting $mv^\beta \partial_\beta v_\sigma$ as the force exerted on a fluid element of mass m, and noting its absence from Eq.(5.15) suggests that $\tilde{F}$ is a forceless field; i.e., the presence of $\tilde{F}$ does not result in a net acceleration of fluid. Hence, we interpret λ as a "ghost potential"; i.e., a potential giving rise to a forceless field.

An important "ghost potential" existent in nature is that which gives rise to a surface force. Surface forces are characterized by their ability to maintain fluid

element integrity and by their lack of influence on bulk fluid motion. A surface force may be viewed as the force exerted by a single fluid element on neighboring fluid elements which prevents fluid element collapse, thereby insuring fluid element integrity. Because the forces exerted by neighboring elements are equal and opposite they do not contribute to bulk fluid motion. Interpreting λ as a surface force potential is entirely consistent with the first physical interpretation given above for λ, that λ is a potential which guarantees fluid element identity.

Integration of the three physical essences for λ listed above suggests that the "canonical momentum" of this charged fluid is a potential which gives rise to the surface forces that maintain fluid element integrity. Note that in this summary we have loosely disregarded some constants and dimensionality.

9. Entirely Eulerian variational principles

It is frequently desirable to have an entirely Eulerian variational principle which leads to general equations of fluid motion. In variational fluid mechanics there exist three basic means of obtaining "entirely Eulerian" variational principles. In all three methods one must first obtain the appropriate Lagrangian density through Hamilton's principle, that the Lagrangian density is the kinetic minus potential energy densities.

In one method, the "Lagrangian variation" method, Lagrangian variations are performed on the entirely Eulerian fundamental integral (the space-time integral of the Hamilton's principle Lagrangian density).[59] Since the Lagrangian aspect of such variational principles is not evident in the form of the fundamental integral these principles are "entirely Eulerian".

In a second method, the "Lin constraint" method, the variations of the Hamilton's principle Lagrangian density are constrained through imposition of the Lin constraint.[60] That is, three functions of space-time $f_i, i = 1, 2, 3$ are introduced as "fluid particle labels". To prevent variation of fluid element world lines on the integration volume boundary,the variation of the labels on the boundary must vanish,

$\delta f_i|_S = 0$ (S denotes the surface of the volume of integration). This is accomplished by including the f_i in the Lagrangian density and varying them freely. The condition that the particle labels be convected with the fluid,

$$v^\sigma \partial_\sigma f_i = 0, \tag{5.16}$$

is used as the equation of constraint. Eq.(5.16) is incorporated into the Lagrangian density through the usual Lagrange multiplier technique. As Schutz and Sorkin[5] point out in their appendix, the six new variables introduced through the imposition of this constraint (three particle labels and three Lagrange multipliers) may be reduced to three through application of Pfaff's Theorem (see Chapter I.D). The new variables together with the other field quantities of the Lagrangian density are then varied freely. Euler-Lagrange equations then result which are locally equivalent to the appropriate general equations of fluid dynamics. In this method the particle labels are actually functions of space-time, hence the resultant variational principle is entirely Eulerian.

In a third method, the "velocity potential" method, a general expression for the fluid velocity field in terms of "Clebsch variables" (or "Schutz potentials" in relativistic fluid dynamics) is first obtained.[61] The velocity potential representation of the fluid velocity is substituted for all occurrences of the velocity in the Hamilton's principle density. The potentials rather than the velocity are then varied and Euler-Lagrange equations are obtained which are locally equivalent to the appropriate fluid dynamical equations. In this method, the appearance of an entirely Eulerian variational principle is achieved through introduction of unphysical potentials.

Any one of the three methods is sufficient to properly generalize the Euler-Lagrange equations resulting from I_{C3}, Eq.(3.25), just as the I_T equation of constraint is. The three methods yield entirely Eulerian variational principles whereas I_T contains Lagrangian coordinates explicitly in the constraint term. However, our analysis of the I_T Lagrange multiplier in the previous subsections enables us to introduce a fourth method of modifying I_{C3} (and extendable for use in other

variational principles) so as to obtain a variational principle leading to the general Maxwell-Lorentz force set of equations.[62] The method as we introduce it is most akin to the Lin constraint method as it involves the imposition of an equation of constraint, but it could also lead to a modification of the velocity potential method.

10. Surface force potential constraint method

Take I_{C3}, Eq.(3.25), as the starting point for an entirely Eulerian variational principle for a perfect, zero-temperature electromagnetic fluid. Rather than use one of the conventional means (as outlined in the previous subsection and illustrated throughout the previous chapters) to obtain a variational principle leading to Euler-Lagrange equations equivalent to the general equations of motion for this fluid, let us assume the existence of a surface force potential Λ. The surface force derivable from Λ must not generate any bulk fluid acceleration. Hence the zero-acceleration expression

$$qv^{\beta}\tilde{F}^{surf}_{\sigma\beta} = 0$$

must be satisfied, where $\tilde{F}^{surf}$ is defined in terms of the potential Λ as

$$\tilde{F}^{surf}_{\sigma\beta} = \partial_{\sigma}\Lambda_{\beta} - \partial_{\beta}\Lambda_{\sigma}$$

(compare Eqs.(5.13) and (5.15)). To ensure the existence of such a potential Λ we must constrain the variations of the other field quantities, in particular the charge current density j, so as to provide for its existence.

The variational constraint can be cast in terms of an equation of constraint by substituting the definition of $\tilde{F}^{surf}$ in terms of Λ into the zero-acceleration expression. Post multiplication by ρ_{oe} yields the following equation of constraint.

$$j^{\beta}(\partial_{\sigma}\Lambda_{\beta} - \partial_{\beta}\Lambda_{\sigma}) = 0. \tag{5.17}$$

Besides satisfying Eq.(5.17) the surface force potential Λ must be allowed to interact directly with the charge current density j just as the electromagnetic potential A

does. Hence the revised fundamental integral should contain a field-current interaction term $\Lambda_\beta j^\beta$.

Let γ be the Lagrange multiplier which constrains the variations of the I_{C3} field quantities to satisfy Eq.(5.17). The revised fundamental integral then becomes

$$I_{SF} = I_{C3} - \int \left[\Lambda_\beta j^\beta + \gamma^\sigma j^\beta (\partial_\sigma \Lambda_\beta - \partial_\beta \Lambda_\sigma)\right] d^3x dt, \tag{5.18}$$

where all the field quantities A, j, Λ and γ are to be varied independently.

The Euler-Lagrange equations which follow from I_{SF}, Eq.(5.18), are

$$\delta A_\sigma : \quad \partial_\beta F^{\beta\sigma} = \mu_o j^\sigma, \tag{3.8}$$

$$\delta j^\sigma : \quad A_\sigma + \frac{m}{q} v_\sigma + \Lambda_\sigma + \gamma^\beta (\partial_\beta \Lambda_\sigma - \partial_\sigma \Lambda_\beta) = 0, \tag{5.19a}$$

$$\delta \Lambda_\sigma : \quad j^\sigma + \partial_\beta (\gamma^\sigma j^\beta - \gamma^\beta j^\sigma) = 0, \tag{5.19b}$$

and

$$\delta \gamma^\sigma : \quad j^\beta (\partial_\sigma \Lambda_\beta - \partial_\beta \Lambda_\sigma). \tag{5.19c}$$

The equations of motion for a perfect, zero temperature charged fluid consist of Maxwell's equations, Eqs.(3.5) and (3.8), and the Lorentz force relation, Eq.(3.11). The Euler-Lagrange equations associated with I_{SF} include Maxwell's equations. Hence, to demonstrate that the I_{SF} Euler-Lagrange equations are equivalent to the equations of motion for a perfect charged fluid one need only consider two questions: (i) Does the Lorentz force relation follow from the I_{SF} Euler-Lagrange equations, and (ii) Given a solution set $\{A, j = \rho_{oe} v\}$ to the Maxwell-Lorentz force set of equations does there exist a Λ and a γ such that Eqs.(5.19) are satisfied?

In Appendix B we establish that the I_{SF} Euler-Lagrange equations are equivalent to the perfect charged fluid equations of motion by demonstrating that both the above questions may be answered in the affirmative. Moreover, the equivalence is

stronger than that claimed for the traditional Lin constraint method. Not only may the equivalence be established on a local basis, as is possible for the traditional Lin constraint method, but it may also be established globally. Hence, the surface force potential constraint method is a viable, and perhaps preferable, alternative to the conventional means of obtaining Eulerian entirely variational principles leading to the general equations of fluid motion.

11. Conclusion

A Lagrangian to Eulerian variable transformation in a standard classical electromagnetic particle action which transforms occurrences of particle position and velocity into charge current density motivates the following conclusions.

1) A simple term-by-term variable transformation of the Lagrangian is not always sufficient to obtain an equivalent variational principle, that is, a fundamental integral the variation of whose variables leads to equivalent Euler-Lagrange equations. With some variable transformations, in particular the Lagrangian to Eulerian transformation presented in this section, it is necessary to constrain the variations of the transformed field quantities in order to obtain an equivalent variational principle. A s standard method of fundamental integral variable transformation[53] suggests that the variation of the field quantities be constrained to satisfy the equation of variable transformation. By so doing the transformed electromagnetic variational principle presented in this section yields Euler-Lagrange equations globally equivalent to the initial variational principle's Euler-Lagrange equations (the Maxwell-Lorentz force set of equations).

2) The Lagrange multiplier introduced to constrain the equation of variable transformation becomes (within a constant factor) the canonical momentum of particle i along the trajectory of particle i. Examination of the equation of motion of the Lagrange multiplier (one of the transformed variational principle's Euler-Lagrange equations) reveals that there exist (at least locally) physically realizable solutions to the Maxwell-Lorentz force set of equations which do not allow for conservation of

the canonical momentum, regardless of the gauge representation of the electromagnetic four-potential A. It is also true that for any solution of the Maxwell-Lorentz force set there exists a gauge representation for A in which the canonical momentum is not a constant of the motion because the canonical momentum is not a gauge invariant quantity. Moreover, the volume integral of the canonical momentum density may be a constant of the fluid's motion when the Lagrange multiplier canonical momentum is not.

3) Examination of the transformed variational principle equation of constraint (the equation of variable transformation) and the associated Lagrange multiplier (the canonical momentum of the fluid mentioned in item 2 above) leads to the following physical insights. Firstly, the constraint equation requires fluid element identity. That is to say, a fluid element trajectory must be identicle to a particle trajectory in order for the charge current density to maintain its required form. The constraint is therefore very similar in nature to the familiar Lin constraint of fluid mechanics which is designed to accomplish much the same thing, but which has a very different form. Secondly, view the Lagrange multiplier in its conventional image as the potential which gives rise to forces of constraint. The Lagrange multiplier satisfies an equation of motion identicle to the Lorentz force relation except for the absence of a fluid acceleration term. Hence, the Lagrange multiplier may be viewed as a potential which generates forces that although they do not accelerate the fluid do maintain fluid particle identity. These forces may be thought of as the Poincaré stresses that hold particles together, or as surface forces.

4) The physical insights outlined in item 3 above may be exploited in the development of an entirely Eulerian constraint term for the transformed but unconstrained variational principle. The interpretation of the Lagrange multiplier as a potential giving rise to surface forces is the most appropriate point of departure for this endeavor. Constrain the variations of the field quantities such that a surface force potential exists which is allowed to interact with the charge current density, but which does not alter bulk fluid motion. By so doing an entirely Eulerian variational

principle is obtained which yields Euler-Lagrange equations equivalent (either locally or globally) to the Eulerian Maxwell-Lorentz force set of equations. This approach has similarities to the familiar Lin constraint method of fluid mechanics in that both approaches are said to require fluid element identity or integrity. However, the new approach has the advantage of being physically more specific in that it refers directly to a physical force whereas the Lin constraint method relies entirely on the somewhat nebulous concept of a fluid element label. In addition, the new approach allows for a stronger equivalence of its Euler-Lagrange equations to the Maxwell-Lorentz force set of equations.

C. Necessity of the Lin constraint

In Chapter IV it was demonstrated that the Lin constraint is essential to the specific variational principles considered therein in the sense that neglect of the constraint leads to Euler-Lagrange equations restrictive of the usual equations of fluid motion. Although we showed the necessity of the Lin constraint in those particular cases, we did not show the general necessity of the Lin constraint. That is, we have not yet demonstrated that there does not exist a variational principle expressed entirely in terms of the Eulerian variables $\{\rho, p, \mathbf{v}\}$ or $\{F^{\sigma\beta}, j_\sigma\}$ which gives rise to the appropriate equations of fluid motion. We do so now by invoking the Theorem of Schutz and Sorkin which we stated and proved in Subsection I.C.4. The Theorem of Schutz and Sorkin demonstrates the necessity of the Lin constraint and other constraints such as continuity.

We first derive the $0-0$ components of the Energy-Momentum (or Stress-Energy) Tensors from several of the variational principles considered earlier: non-relativistic neutral fluid mechanics principles and special relativistic electromagnetic principles. We do this for both the Lin constraint-imposed versions of Chapters II and III and the unconstrained versions of Chapter IV. As required, the energy component of these tensors corresponds to the expected expression of total energy for each type of fluid.

We then apply the Theorem of Schutz and Sorkin to deduce the necessity of imposing constraints. At the same time, we conclude that solutions to the restrictive Euler-Lagrange equations that result from an unconstrained variational principle extremize the total energy while solutions to the more general Euler-Lagrange equations which result from constrained variational principles do not always extremize the energy.

1. Energy-momentum tensors

In this subsection, we compute the energy density of several of the variational principles considered earlier. We do this by evaluating the $0-0$ component of the energy-momentum tensor derivable from the variational principle. The ***canonical stress-energy tensor*** is defined to be

$$T^i_{ck} = \sum_{j=1}^{m} \frac{\partial L}{\partial Q_{j,i}} Q_{j,k} - L\delta^i_k. \tag{1.34}$$

Our derivations of the energy densities are based on this definition.

Consider the variational principle I_3, Eq.(2.50). By inspection, this variational principle has the Lagrangian density

$$L_3 = \rho_m \left[\frac{v^2}{2} - u(\rho_m, S) + \frac{\mu}{\rho_m} \left(\frac{\partial(\rho_m S)}{\partial t} + \nabla \cdot (\rho_m S \mathbf{v}) \right) + \frac{D\alpha}{Dt} + \beta \frac{D\gamma}{Dt} \right]. \tag{5.20}$$

Recall that the Euler-Lagrange equations which follow from the I_3 variational principle are

$$\mathbf{v} = S\nabla\mu - \nabla\alpha - \beta\nabla\gamma, \tag{2.52a}$$

$$\frac{D\alpha}{Dt} = g(\rho_m, S) - \frac{v^2}{2}, \tag{2.52b}$$

$$\frac{D\mu}{Dt} = -T(\rho_m, S), \tag{2.52c}$$

$$\frac{D\gamma}{Dt} = \frac{D\beta}{Dt} = \frac{DS}{Dt} = 0, \tag{2.52d}$$

and

$$\frac{D\rho_m}{Dt} = -\rho_m \nabla \cdot \mathbf{v}. \tag{2.52e}$$

From the expression for L_3, Eq.(5.20), we compute

$$\frac{\partial L_3}{\partial v_{j,i}} = \rho_m \mu \delta^i_j,$$

where $j = 1, 2, 3$ and $i = 0, 1, 2, 3$,

$$\frac{\partial L_3}{\partial \rho_{m,i}} = \begin{cases} \mu S c, & \text{if } i = 0, \\ \mu S v_i, & \text{otherwise,} \end{cases}$$

$$\frac{\partial L_3}{\partial \mu_{,i}} = 0,$$

$$\frac{\partial L_3}{\partial S_{,i}} = \begin{cases} \mu \rho_m c, & \text{if } i = 0, \\ \mu \rho_m v_i, & \text{otherwise,} \end{cases}$$

$$\frac{\partial L_3}{\partial \alpha_{,i}} = \begin{cases} \rho_m c, & \text{if } i = 0, \\ \rho_m v_i, & \text{otherwise,} \end{cases}$$

$$\frac{\partial L_3}{\partial \beta_{,i}} = 0,$$

$$\frac{\partial L_3}{\partial \gamma_{,i}} = \begin{cases} \beta \rho_m c, & \text{if } i = 0, \\ \beta \rho_m v_i, & \text{otherwise.} \end{cases}$$

We now define

$$E_k = \sum_{j=1}^{m} \frac{\partial L}{\partial Q_{j,0}} Q_{j,k} \tag{5.21}$$

for $k = 0, 1, 2, 3$. Then

$$\begin{aligned} E_0 &= \mu S \frac{\partial \rho_m}{\partial t} + \mu \rho_m \frac{\partial S}{\partial t} + \rho_m \frac{\partial \alpha}{\partial t} + \rho_m \beta \frac{\partial \gamma}{\partial t} \\ &= \frac{\partial (\mu \rho_m S)}{\partial t} + \rho_m \left(\frac{\partial \alpha}{\partial t} + \beta \frac{\partial \gamma}{\partial t} - S \frac{\partial \mu}{\partial t} \right). \end{aligned}$$

Utilizing the appropriate Euler-Lagrange equations we find

$$\begin{aligned} E_0 &= \frac{\partial(\mu\rho_m S)}{\partial t} + \rho_m\left(g - \frac{v^2}{2} - \mathbf{v}\cdot\nabla\alpha - \beta\mathbf{v}\cdot\nabla\gamma + ST + S\mathbf{v}\cdot\nabla\mu\right) \\ &= \frac{\partial(\mu\rho_m S)}{\partial t} + \rho_m\left(g + ST + \frac{v^2}{2}\right), \end{aligned}$$

$$\begin{aligned} \frac{\partial(\mu\rho_m S)}{\partial t} &= (\mathbf{v}\cdot\nabla\mu - T)\,\rho_m S - \mu S\,(\nabla\cdot(\rho_m\mathbf{v})) - \mu\rho_m\mathbf{v}\cdot\nabla S, \\ &= -\rho_m ST - \nabla\cdot(\rho_m S\mu\mathbf{v}), \end{aligned}$$

and

$$\begin{aligned} L_3 &= \rho_m\left(\frac{v^2}{2} - u + g - \frac{v^2}{2}\right) \\ &= \rho_m\,(g-u). \end{aligned}$$

The gauge freedom in E_0 allows us to drop total divergences; hence, we may write

$$E_0 = \rho_m\left(g + \frac{v^2}{2}\right).$$

Therefore,

$$T^0_{0\,(3)} = E_0 - L_3 = \rho_m\left(\frac{v^2}{2} + u\right), \tag{5.22}$$

which is the expected energy density of a perfect fluid.

Next, consider the unconstrained analog of I_3; that is, I_{N1}, Eq.(4.7). By inspection, we write the Lagrangian density of I_{N1} as

$$\begin{aligned} L_{N1} = \rho_m &\left(\frac{v^2}{2} - u(\rho_m, S)\right) \\ &+ \lambda\left[\partial\rho_m/\partial t + \nabla\cdot(\rho_m\mathbf{v})\right] \\ &+ \mu\left[\partial\left(\rho_m S\right)/\partial t + \nabla\cdot(\rho_m S\mathbf{v})\right]. \end{aligned} \tag{5.23}$$

The Euler-Lagrange equations that follow from the I_{N1} variational principle are

$$\mathbf{v} = \nabla\lambda + S\nabla\mu, \tag{4.8a}$$

$$D\mu/Dt = -T, \tag{4.8b}$$

$$\frac{1}{2}v^2 - u - p/\rho_m - D\lambda/Dt - SD\mu/Dt = 0, \tag{4.8c}$$

$$\partial\rho_m/\partial t + \nabla\cdot(\rho_m \mathbf{v}) = 0, \tag{4.8d}$$

and

$$\partial(\rho_m S)/\partial t + \nabla\cdot(\rho_m S\mathbf{v}) = 0. \tag{4.8e}$$

From the expression for L_{N1}, Eq.(5.23), we compute

$$\frac{\partial L_{N1}}{\partial v_{j,i}} = \rho_m(\lambda + \mu S)\,\delta^i_j,$$

where $j = 1,2,3$ and $i = 0,1,2,3$,

$$\frac{\partial L_{N1}}{\partial \rho_{m,i}} = \begin{cases} (\lambda + \mu S)\,c, & \text{if } i = 0, \\ (\lambda + \mu S)\,v_i, & \text{otherwise,} \end{cases}$$

$$\frac{\partial L_{N1}}{\partial \mu_{,i}} = 0,$$

$$\frac{\partial L_{N1}}{\partial S_{,i}} = \begin{cases} \mu\rho_m c, & \text{if } i = 0, \\ \mu\rho_m v_i, & \text{otherwise.} \end{cases}$$

Using definition (5.21) for E_k and the appropriate Euler-Lagrange equations we then compute

$$\begin{aligned} E_0 &= (\lambda + \mu S)\,\partial\rho_m/\partial t + \mu\rho_m\partial S/\partial t \\ &= (\lambda + \mu S)\,(-\nabla\cdot(\rho_m\mathbf{v})) + \mu\rho_m\,(-\mathbf{v}\cdot\nabla S) \\ &= -\nabla\cdot[(\lambda + \mu S)\,\rho_m\mathbf{v}] + \rho_m\mathbf{v}\cdot\nabla(\lambda + \mu S) - \rho_m\mu\mathbf{v}\cdot\nabla S \\ &= \rho_m\mathbf{v}\cdot\nabla\lambda + \rho_m S\mathbf{v}\cdot\nabla\mu \\ &= \rho_m v^2, \end{aligned}$$

where, as before, we absorb total divergences into the gauge of E_0. Also,

$$L_{N1} = \rho_m\left(\frac{1}{2}v^2 - u\right),$$

since continuity and entropy conservation are satisfied. Therefore,

$$T^0_{0\,(N1)} = E_0 - L_{N1} = \rho_m \left(\frac{1}{2}v^2 + u\right), \tag{5.24}$$

as before. It is interesting to note that whereas the total energy ascribed to I_3 and I_{N1} is the same, the Lagrangian densities L_3 and L_{N1} are not equal.

Now consider the electromagnetic fundamental integral I_{SF}, Eq. (5.18), which yields Euler-Lagrange equations globally equivalent to the Maxwell-Lorentz force set of equations. I_{SF} has the Lagrangian density

$$L_{SF} = -F_{\beta\sigma}F^{\beta\sigma}/4\mu_o - (A_\sigma + \Lambda_\sigma)\, j^\sigma - \frac{mc}{q}\left(j_\sigma j^\sigma\right)^{1/2} - \gamma^\sigma j^\beta \left(\partial_\sigma \Lambda_\beta - \partial_\beta \Lambda_\sigma\right). \tag{5.25}$$

The Euler-Lagrange equations that follow from the I_{SF} variational principle are

$$\partial_\beta F^{\beta\sigma} = \mu_o j^\sigma, \tag{3.8}$$

$$A_\sigma + \frac{m}{q}v_\sigma + \Lambda_\sigma + \gamma^\beta \left(\partial_\beta \Lambda_\sigma - \partial_\sigma \Lambda_\beta\right) = 0, \tag{5.19a}$$

$$j^\sigma + \partial_\beta \left(\gamma^\sigma j^\beta - \gamma^\beta j^\sigma\right) = 0, \tag{5.19b}$$

and

$$j^\beta \left(\partial_\sigma \Lambda_\beta - \partial_\beta \Lambda_\sigma\right). \tag{5.19c}$$

From Eq. (5.25), the expression of the Lagrangian density L_{SF}, we compute

$$\begin{aligned}
\frac{\partial L_{N1}}{\partial\left(\partial_\sigma A_\beta\right)} &= \frac{\partial}{\partial\left(\partial_\sigma A_\beta\right)}\left[-F_{\alpha\gamma}F^{\alpha\gamma}/4\mu_o\right] \\
&= -\frac{1}{2\mu_o}F^{\alpha\gamma}\frac{\partial}{\partial\left(\partial_\sigma A_\beta\right)}\left[\partial_\alpha A_\gamma - \partial_\gamma A_\alpha\right] \\
&= -\frac{1}{\mu_o}F^{\alpha\gamma}\frac{\partial\left(\partial_\alpha A_\gamma\right)}{\partial\left(\partial_\sigma A_\beta\right)} \\
&= -\frac{1}{\mu_o}F^{\sigma\beta},
\end{aligned}$$

$$\frac{\partial L_{N1}}{\partial\left(\partial_\sigma j_\beta\right)}=0,$$

$$\frac{\partial L_{N1}}{\partial\left(\partial_\sigma \gamma_\beta\right)}=0,$$

and

$$\frac{\partial L_{N1}}{\partial\left(\partial_\sigma \Lambda_\beta\right)}=-\left(\gamma^\sigma j^\beta-\gamma^\beta j^\sigma\right).$$

From the definition of the canonical stress-energy tensor, Eq. (1.34), we evaluate

$$T^{\beta}_{\sigma\,(SF)}=-\frac{1}{\mu_o}F^{\beta\nu}\partial_\sigma A_\nu-\left(\gamma^\beta j^\nu-\gamma^\nu j^\beta\right)\partial_\sigma\Lambda_\nu-L\delta^\beta_\sigma.$$

The Euler-Lagrange equations imply the following identities.

$$\partial_\nu\left(F^{\nu\beta}A_\sigma\right)=\mu_o j^\beta A_\sigma+F^{\nu\beta}F_{\nu\sigma}+F^{\nu\beta}\partial_\sigma A_\nu,$$

from whence

$$F^{\nu\beta}\partial_\sigma A_\nu=\mu_o j^\beta A_\sigma+F^{\beta\nu}F_{\sigma\nu}+\partial_\nu\left(F^{\beta\nu}A_\sigma\right).$$

$$\left(\gamma^\beta j^\nu-\gamma^\nu j^\beta\right)\partial_\sigma\Lambda_\nu=\left(\gamma^\beta j^\nu-j^\beta\gamma^\nu\right)\partial_\nu\Lambda_\sigma-\left(A_\sigma+\frac{m}{q}v_\sigma+\Lambda_\sigma\right)j^\beta,$$

and

$$\begin{aligned}\left(\gamma^\beta j^\nu-j^\beta\gamma^\nu\right)\partial_\nu\Lambda_\sigma&=\partial_\nu\left[(\gamma^\beta j^\nu-j^\beta\gamma^\nu)\Lambda_\sigma\right]-\Lambda_\sigma\partial_\nu(\gamma^\beta j^\nu-j^\beta\gamma^\nu)\\&=\Lambda_\sigma j^\beta,\end{aligned}$$

from whence

$$\left(\gamma^\beta j^\nu-\gamma^\nu j^\beta\right)\partial_\sigma\Lambda_\nu=-A_\sigma j^\beta-\frac{m}{q}v_\sigma j^\beta+\partial_\nu\left[(\gamma^\beta j^\nu-j^\beta\gamma^\nu)\Lambda_\sigma\right].$$

$$\begin{aligned}L_{SF}&=-F^{\alpha\nu}F_{\alpha\nu}/4\mu_o-\frac{mc}{q}\left(j^\alpha j_\alpha\right)^{1/2}-\left(A_\alpha+\Lambda_\alpha\right)j^\alpha-j^\nu\gamma^\alpha\left(\partial_\alpha\Lambda_\nu-\partial_\nu\Lambda_\alpha\right)\\&=-F^{\alpha\nu}F_{\alpha\nu}/4\mu_o-\frac{mc}{q}\left(j^\alpha j_\alpha\right)^{1/2}+\frac{m}{q}j^\nu v_\nu\\&=F^{\alpha\nu}F_{\alpha\nu}/4\mu_o.\end{aligned}$$

Recall that $T^{\beta}_{c\sigma}$ has the gauge freedom $\partial_\nu \Psi^{\beta\nu}_{\sigma}$ with $\Psi^{\beta\nu}_{\sigma} = -\Psi^{\nu\beta}_{\sigma}$ so that such terms may be dropped from the collection of equations given above. Using the above equations we conclude that

$$T^{\beta}_{\sigma\ (SF)} = F^{\beta\nu}F_{\nu\sigma}/\mu_o + \frac{m}{q}v_\sigma j^\beta + \frac{1}{4\mu_o}F^{\alpha\nu}F_{\alpha\nu}\delta^{\beta}_{\sigma},$$

so that

$$\begin{aligned} T^{0}_{0\ (SF)} &= \varepsilon_o E^2 + \frac{B^2}{2\mu_o} - \varepsilon_o\frac{E^2}{2} + \frac{mc^2}{q}\gamma\rho_e \\ &= \varepsilon_o\frac{E^2}{2} + \frac{B^2}{2\mu_o} + \frac{mc^2}{q}\gamma\rho_e, \end{aligned} \tag{5.26}$$

where use has been made of the definition of $F_{\sigma\beta}$ in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, Eq. (3.6).

Finally, consider the electromagnetic variational principle I_{N3}, Eq.(4.14). The Lagrangian density associated with I_{N3} is

$$L_{N3} = -F_{\beta\sigma}F^{\beta\sigma}/4\mu_o - (A_\sigma + \Lambda_\sigma)\, j^\sigma - \frac{mc}{q}\left(j_\sigma j^\sigma\right)^{1/2}. \tag{5.27}$$

The Euler-Lagrange equations that follow from the I_{N3} variational principle are

$$\partial_\beta F^{\beta\sigma} = \mu_o j^\sigma, \tag{3.8}$$

and

$$A_\sigma + \frac{m}{q}v_\sigma = 0. \tag{4.16}$$

From Eq. (5.27), the expression of the Lagrangian density L_{N3}, we compute

$$\frac{\partial L_{N1}}{\partial\left(\partial_\sigma A_\beta\right)} = -\frac{1}{\mu_o}F^{\sigma\beta},$$

and

$$\frac{\partial L_{N1}}{\partial\left(\partial_\sigma j_\beta\right)} = 0.$$

From the Euler-Lagrange equations we deduce the following identities.

$$L_{N3} = F^{\alpha\nu} F_{\alpha\nu} / 4\mu_o,$$

and

$$F^{\nu\beta} \partial_\sigma A_\nu = \mu_o j^\beta A_\sigma + F^{\beta\nu} F_{\sigma\nu} + \partial_\nu \left(F^{\beta\nu} A_\sigma \right).$$

Hence,

$$T^{\beta}_{\sigma\ (N3)} = F^{\beta\nu} F_{\nu\sigma} / \mu_o + \frac{m}{q} v_\sigma j^\beta + \frac{1}{4\mu_o} F^{\alpha\nu} F_{\alpha\nu} \delta^\beta_\sigma,$$

so that

$$T^{0}_{0\ (N3)} = \frac{B^2}{2\mu_o} + \varepsilon_o \frac{E^2}{2} + \frac{mc^2}{q} \gamma \rho_e. \tag{5.28}$$

Note that the stress-energy tensors and the evaluated Lagrangian densities are identical for I_{N3} and I_{SF}.

2. Application of the theorem of Schutz and Sorkin

One of the possibly many applications of the Theorem of Schutz and Sorkin is to determine the necessity of variational constraints. As the theorem was stated and proved in Subsection I.C.4 we indicated that the theorem implies, among other things, that if the field quantities of a variational principle are time-independent and if they satisfy the Euler-Lagrange equations then the total system energy (the volume integral of the $0-0$ component of the stress-energy tensor associated to the variational principle) is an extremum against all variations of the field quantities of compact support. If, on the other hand, the total energy is not an extremum against all variations of the field quantities then there does not exist a variational principle in those field quantities which will result in unrestrictive Euler-Lagrange equations; i.e., variations of the field quantities must be restricted by either constraining the variations themselves or by introducing new field quantities (perhaps through the imposition of equations of constraint by the Lagrange multiplier method) whose free variation does extremize the total energy.

Consider now the specific case of a perfect fluid. The total energy for a perfect fluid is given by Eqs. (5.22) and (5.24). Examination of the total energy expression indicates that there are several ways to change the total energy to first order. Firstly, the potential energy may be changed to first order if either (i) heat is added to the system (this causes a first order change in entropy), or (ii) a particle is added to the system (this causes a first order change in the mass density). Also, the kinetic energy may be changed to first order for a uniformly moving fluid by (iii) a change in velocity along the direction of motion. Hence, according to the Theorem of Schutz and Sorkin, no unconstrained variational principle exists in the field quantities $\{\rho_m, S, \mathbf{v}\}$ which gives rise to a general set of equations of motion.

Condition (i) implies that one variational constraint that should be imposed is entropy conservation. Condition (ii) implies that mass conservation, or continuity, should be imposed as a variational constraint. What condition (iii) implies is somewhat less obvious from a physical standpoint, but imposition of the Lin constraint or some other constraint having the same effect as the Lin constraint satisfies the demands of the condition.

It is interesting to note in light of the Theorem of Schutz and Sorkin that if none of the constraints are imposed the Euler-Lagrange equations reduce to

$$\mathbf{v} = 0, \quad T = 0, \quad p = 0,$$

a solution to which clearly extremizes the total energy of the fluid. Similarly, if any of the constraits is neglected, the total energy is extremized *subject to* the remaining constraints.

For the specific case of an electromagnetic fluid the total energy is given by Eqs. (5.26) and (5.28) (of course, this expression for the total energy excludes thermodynamic and internal magnetization contributions). There are four apparent means whereby the expression for the total energy may be changed to first order. The energy can be changed to first order if (i) the charge density is changed to

first order (a particle can be added to the fluid). The electric field energy in a uniform electric field may be changed to first order by (ii) varying the electric field along the direction of the field. The magnetic field energy in a uniform magnetic field may be changed to first order by (iii) varying the magnetic field along the direction of the field. Finally, the kinetic energy may be changed to first order for a uniformly moving fluid by (iv) a change in velocity along the direction of motion. Hence, according to the Theorem of Schutz and Sorkin, no unconstrained variational principle exists in the field quantities $\{\rho_m, \mathbf{E}, \mathbf{B}, \mathbf{v}\}$ which gives rise to a general set of equations of motion.

Condition (i) indicates the need for the imposition of charge continuity as a variational constraint. Conditions (ii) and (iii) may be satisfied through the introduction of the electromagnetic vector and scalar potentials; that is, by defining $\mathbf{E}$ and $\mathbf{B}$ in terms of A_σ. Condition (iv) again requires the imposition of the Lin constraint, or some other constraint having the same effect.

If no variational constraints are imposed, the Euler-Lagrange equations become

$$\mathbf{E} = 0, \quad \mathbf{B} = 0, \quad j^\sigma = 0,$$

solutions to which clearly extremize the energy (note that $j^\sigma = 0$ implies $\rho_{oe} = 0$). Similarly, if any of the constraits is neglected, the total energy is extremized *subject to* the remaining constraints.

D. Conclusion

Application of the Theorem of Schutz and Sorkin clearly demonstrates the necessity of imposing a constraint on the variations of the velocity field in variational principles of both fluid mechanics and electromagnetism in order to obtain from the principles Euler-Lagrange equations which are completely general. There are three means whereby this constraint may be imposed. In the first, the "Lagrangian Variation" method, a special variational technique is set up whereby one avoids

the free variation of the velocity (or current density). In the second, the "Velocity Potential" method, the velocity is expressed in terms of "Clebsch Variables" or "Schutz Potentials" by appealing to Pfaff's Theorem, then the potentials are varied freely rather than the velocity. In the third, the "Lin Constraint" method, the Lin constraint is imposed upon the variation of the velocity field (or current density) through the Lagrange multiplier technique. The main emphasis of this chapter has been the Lin Constraint method.

The Lin constraint is usually imposed after the heuristic argument that the Lagrangian description of matter is preferable to the Eulerian description, at least insofar as classical mechanics is concerned, for the Lagrangian description requires that an observer may follow the trajectory of each fluid element whereas the Eulerian description has no such requirement built into it. One therefore introduces fluid element labels into the variational principle and requires that they "convect with the fluid". This is accomplished through the usual Lagrange multiplier technique.

In this chapter we have demonstrated that the Lin constraint is not unique. Constraining the existence of a "Surface Force Potential" within an electromagnetic variational principle results in Euler-Lagrange equations globally equivalent to the Maxwell-Lorentz force set. This provides an even stronger equivalence than allowed by the conventional Lin constraint. It is quite possible that there exist still more Eulerian constraints which may be imposed upon the variations of the velocity field or current density which allow for generalization of the Euler-Lagrange equations. If additional constraints are found, they may help to shed additional light upon the variational constraint which has previously been considered "mysterious".

REFERENCES AND FOOTNOTES

1. The notion of a variational principle along with a number of other essential concepts from the Calculus of Variations are defined thoroughly in Section I.C.
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3. A. H. Taub, Phys. Rev. **94**, 1468 (1954).
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10. We define a perfect fluid as one with isentropic pressure p and no viscosity or electromagnetic interactions.
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18. H. Ito, Progress of Theor. Phys. **48**, 1142 (1972).

19. For more about the Calculus of Variations see, e.g., I. M. Gelfand and S. V. Fomin, Calculus of Variations, (Prentice-Hall, New Jersey, 1963). For a function theoretical development see, e.g., J. T. Oden, and J. N. Reddy, Variational Methods in Theoretical Mechanics (Springer-Verlag New York, 1976).

20. See, e.g., R. G. Bartle, The Elements of Real Analysis (Wiley, New York, 1964), or any other good Advanced Calculus text.

21. See, e.g., I. M. Gelfand and S. V. Fomin, Calculus of Variations (Prentice-Hall, New Jersey, 1963).

22. The canonical stress-energy tensor is often defined as the symmetrized tensor obtained from (1.34) by the addition of a quantity of the form $\sum_{j=1}^{n} \Phi^{ij}_{k,j}$ where $\Phi^{ij}_{k} = -\Phi^{ji}_{k}$. Note that if $\sum_{i=1}^{n} dT^{i}_{ck}/dx^{i} = 0$ then $\sum_{i=1}^{n} d/dx^{i} \left[T^{i}_{ck} + \sum_{j=1}^{n} \Phi^{ij}_{k,j}\right] = 0$, that is, Φ may be thought of as an unphysical addition to the canonical stress-energy tensor, i.e., a *gauge transformation.* The gauge transformation shall always be chosen so as to give a symmetric stress-energy tensor.

23. For a fairly comprehensive account of physical and mathematical constraints see, e.g., C. W. Kilmister and J. E. Reeve, Rational Mechanics (Elsevier, New York, 1966).

24. We define an Asymptotic Regular Hypersurface as one that is spacelike and asymptotically flat, following Schutz and Sorkin, ref. 5.

25. The support of a function is the closure of the set upon which it is non-vanishing. (see, e.g., J. T. Oden and J. N. Reddy, Variational Methods of Theoretical Mechanics (Springer-Verlag, New York, 1976)).

26. See, e.g., R. Abraham and J. E. Marsden, Foundations of Mechanics, 2nd ed. (Addison-Wesley, New York, 1978), H. Flanders, Differential Forms (Academic Press, New York, 1963), or R. L. Bishop and S. I. Goldberg, Tensor Analysis on Manifolds (Dover, New York, 1980).

27. J. F. Pfaff, Abh. Akad. Wiss., Berlin (1814).

28. The many uses of the names of Euler and Lagrange which have been dictated to us by history often invoke a good deal of confusion. Here the phrase "Euler's equation" should not be confused with the "Euler-Lagrange equations" introduced in Section I.C. Euler's equation is an equation which governs perfect fluid motion and is given in Eqs.(2.1) (Lagrangian description) and (2.40) (Eulerian description).

29. Here again we wish to avoid the possible confusion that may arise due to the plethora of uses for the names of Euler and Lagrange. Here we use the name Lagrangian to refer to the argument of the fundamental integral I (see Section I.C) and do not refer to the Lagrangian description of a fluid.

30. W. van Saarloos, Physica **108A**, 557 (1981).

31. For a definition of generalized coordinates see, e.g., Section I.C.

32. In order to make this variational principle correspond more closely with Hamilton's principle it would be better to replace H_o with U_o, assume S_o to be an independent variable rather than T_o and then constrain entropy conservation. As we shall see (see Eq.(2.32)), Taub gets entropy conservation from this variational principle, but he makes an *ad hoc* assumption to do so (see footnote 31).

33. This scalar-valued function was originally dubbed the *thermasy* by D. van Dantzig, Physica **6**, 693 (1939). For a physical discussion of this "temperature potential" see L. A. Schmidt, in A Critical Review of Thermodynamics, edited by E. B. Stuart, B. Gal-Or, and A. J. Brainard (Mono, Baltimore, 1970).

34. For a complete discussion of the Lie derivative or derivation see, e.g., L. Auslander and R. E. MacKenzie, Introduction to Differentiable Manifolds (McGraw-Hill, New York, 1963), or R. Hermann, Differential Geometry and the Calculus of Variations (Academic Press, New York, 1968).

35. The quantity μ is the *thermasy* by D. van Dantzig (see footnote 32). It

was introduced by Taub[3] to get entropy conservation from his fundamental integral, Eq.(2.12).

36. See, e.g., R. Adler, M. Bazin and M. Schiffer, Introduction to General Relativity (McGraw-Hill, New York, 1975), or C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (Freeman, San Francisco, 1973).

37. Note that α and μ always exist at least locally so long as T_o, S_o and γ satisfy sufficient "smoothness" requirements. For these requirements see any good book on differential equations such as P. Garabedian, Partial Differential Equations (Wiley, New York, 1964).

38. This condition is obviously satisfied when the charge distribution is assumed to be a continuous fluid or solid. If, on the other hand, we think of the charge distribution as being composed of a finite number of point particles, the fact that two particles must not occupy the same spacial position at the same instant allows us to smooth the velocities of the particles into a continuously differentiable velocity field.

39. See, e.g., J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1975), page 596.

40. The equivalence of the two variational principles is seen more clearly by defining $\mathbf{E}$ and $\mathbf{B}$ in terms of the potentials $\mathbf{A}$ and ϕ according to Eqs.(3.2) *a priori*, and substituting these defining relations into the fundamental integral to get $I_{EM} = \int \frac{\varepsilon_o}{2} \left\{ c^2 (\nabla \times \mathbf{A})^2 - [-\nabla\phi - \partial\mathbf{A}/\partial t]^2 \right\} d^3x dt$ (note that we need not impose the two equations of constraint). By varying $\mathbf{A}$ and ϕ freely one then recovers the two equations which were before imposed as constraints on the variations of $\mathbf{E}$ and $\mathbf{B}$. Alternatively, by several integrations by parts after neglecting any quantity evaluated on the boundary of integration and using Eqs.(3.2), one may show that $I_{EM1} = -\int \frac{\varepsilon_o}{2} \left\{ c^2 B^2 - E^2 \right\} d^3x dt$. The presence of the minus sign will not affect the equations of motion and hence the two fundamental integrals I_{EM1} and $I'_{EM2} = -\int F_{\beta\sigma} F^{\beta\sigma} / 4\mu_o d^4x$ are equivalent.

41. See, e.g., ref. 39, page 577, or A. O. Barut, Electrodynamics and Classical Theory of Fields and Particles (Macmillan, New York, 1964), page 62.

42. Variation of a particle trajectory is completely analogous to the point of view maintained in a Lagrangian description of fluids, that each fluid element trajectory be varied.

43. J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).

44. K. Schwarzschild, Goittinger Nachrichten, 128, 132 (1903).

45. H. Tetrode, Zeits. f. Physik **10**, 317 (1922).

46. A. D. Fokker, Zeits. f. Physik **58**, 386 (1929).

47. P. A. M. Dirac, Proc. Roy. Soc. London **A167**, 148 (1938).

48. $j^\beta(j_m^\beta)$ may always be written as the product of the charge (mass) density and the four-velocity field v^β in a fluid composed of a finite number of particles. See the discussion preceeding Eqs.(3.22) and (3.23).

49. See, e.g., ref. 39, pages 608-612, for greater detail.

50. The London equations of superconductivity were first proposed by F. and H. London (see, e.g., F. and H. London, Proc. Roy. Soc. London A **149**, 71 (1935), or F. London, in Superfluids (Wiley, New York, 1950)Vol. 1.) as *ad hoc* restrictions on Maxwell's equations in order to describe superfluid motion. The London equations were later derived from quatum mechanics by Bardeen, Cooper and Schrieffer (Phys. Rev. 108, 1175 (1957)) and have ever since been classified as illustrative of macroscopic quantum mechanical phenomena.

51. W. Meissner and R. Oshsenfeld, Naturwissenschaften **21**, 787 (1933).

52. See, e.g., P. Penfield Jr., Phys. of Fluids **9**, 1184 (1966).

53. R. Courant and D. Hilbert, Methods of Mathematical Physics (Interscience, New York, 1965), Vol. 1, Chapt. 6, Sec. 9.

54. The reader may find the derivation of Eq.(5.6) more compeling by first evaluating Eq.(5.3a) at particle position x_i, taking the curl with respect to x_i, then following the remainder of the steps as outlined. Also, note that the

assumption that $v_{i\sigma}$ does not depend explicitly on τ_i, used to conclude the form of the total derivative of $v_{i\sigma}$ with respect to τ_i, $dv_{i\sigma}/d\tau_i = v_i^\beta \partial v_{i\sigma}/\partial x_i^\beta$, is valid since none of the equations involved depends explicitly on τ_i.

55. In general, the trajectory x^ν of the particle in question satisfies a (autonomous) dynamical equation $dx^\nu/d\tau = v^\nu(x^\alpha)$. That is, one may *not* assume that v is independent of x.

56. By imposing the initial condition that the particle is at the origin at time zero, the solution given for v and λ results in the particle trajectory $x(t) = C_2[\exp(ct/C_1) - 1], y(t) = z(t) = 0$. The local nature of the problem requries that this trajectory be maintained for $x < C_1 - C_2$. Hence, the solution can only be valid for $t < (C_1/c)\ln\,(C_1/C_2)$.

57. Note that f is uniquely determined (up to sign) by the condition $(fv^\beta)(fv_\beta) = f^2c^2$. Hence, $f = c^{-1}[C_1^2 - (x+C_2)^2 - y^2 - z^2]^{1/2}$.

58. See, e.g., J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1975), pp. 792–3.

59. The Lagrangian variation method is described in B. F. Schutz and R. Sorkin, Ann. of Phys. **107**, 1 (1972) and in Subsections II.A.3 and III.D.7 of the dissertation.

60. The Lin constraint method is described and utilized in refs.29,15,52, and the appendix of ref.5 as well as Subsections II.B.1, II.B.3, III.D.2, III.D.4, and III.D.6 of this dissertation.

61. The velocity potential method is exercised in refs.9 and 11.

62. Note that our analysis of the I_T constraint term might also be used as a motivation for the introduction of the Lin constraint itself.

APPENDICES

Appendix A. Canonical momentum versus canonical momentum density

We choose to refer to the quantity

$$\mathbf{P}_\sigma = qA_\sigma + mv_\sigma, \tag{A1}$$

which restricts to the canonical momentum of particle i along the trajectory of particle i, as the canonical momentum of a perfect, zero-temperature electromagnetic fluid. By methods analogous to those used in Subsection V.B.6, it may be shown that there exist local solutions to the Eulerian Maxwell-Lorentz force set of equations for which the canonical momentum $\mathbf{P}$, Eq.(A1), is not conserved for any electromagnetic gauge representation of A.

The canonical momentum *density* P is defined in terms of the symmetric canonical stress-energy tensor T by

$$P_\mu = \frac{1}{c} T^o_\mu$$

where

$$T^\mu_\nu = \sum_{j=1}^{N} \left[\partial L / \partial \left(\frac{\partial \eta_j}{\partial x^\mu} \right) \right] \partial \eta_j / \partial x^\nu - L \delta^\mu_\nu + \partial \Psi^{\alpha\mu}_\nu / \partial x^\alpha. \tag{A2}$$

Here, L is the Lagrangian density with field quantities η_j, the sum runs over all field quantities, and $\Psi^{\alpha\mu}_\nu = -\Psi^{\mu\alpha}_\nu$ represents the gauge freedom in the canonical stress-energy tensor T. We choose Ψ in such a way that $T^{\mu\nu}$ is symmetric.

The highest order partial derivative appearing in the Lagrangian densities of interest is first order. Hence,

$$\frac{dL}{dx^\alpha} = \frac{\partial L}{\partial x^\alpha} + \sum_{j=1}^{N} \left[\frac{\partial L}{\partial \eta_j} \right] \frac{\partial \eta_j}{\partial x^\alpha} + \sum_{j=1}^{N} \left[\partial L / \partial \left(\frac{\partial \eta_j}{\partial x^\beta} \right) \right] \frac{\partial^2 \eta j}{\partial x^\alpha \partial x^\beta}. \tag{A3}$$

When combined with the Euler-Lagrange equations

$$\frac{\partial L}{\partial \eta_j} = \frac{d}{dx^\beta} \left[\partial L / \partial \left(\frac{\partial \eta_j}{\partial x^\beta} \right) \right]$$

and definition (A2), Eq.(A3) becomes

$$\frac{\partial L}{\partial x^\alpha} + \frac{d}{dx^\beta} T_\alpha^\beta = 0. \tag{A4}$$

Expression (A4) suggests that the divergence of the four-vector T_α (T_α^β for fixed α) vanishes if and only if the Lagrangian density L does not depend explicitly on x^α. In the event that $\frac{\partial L}{\partial x^\alpha} = 0$, one therefore finds that $\frac{d}{dx^\beta} T_\alpha^\beta = 0$, which in turn implies

$$\frac{1}{c}\frac{d}{dt}\int T_\alpha^o d^3x = -\int \frac{d}{dx^i} T_\alpha^i d^3x = -\oint T_\alpha^i dS_i, \tag{A5}$$

where dS_i is a surface area element. Taking the boundary at infinity and assuming that T_α^i asymptotically approaches zero faster than $\frac{1}{r^2}$ assures that the integral over the surface vanishes. Hence, if the Lagrangian density L does not depend explicitly on x^α the volume integral of the α-component of the canonical momentum density P_α is a constant of the fluid's motion under a mild generality condition on the asymptotic behavior of T.

For both of the entirely Eulerian variational principles of Section V.B, that is, for I_{C3} and I_{SF}, Eqs.(3.25) and (5.18), the canonical stress-energy tensor T is found from definition (A2) and the appropriate Euler-Lagrange equations to be

$$T_\nu^\mu = F^{\mu\alpha} F_{\alpha\nu}/\mu_o + \frac{m}{q}\rho_{oe} v^\mu v_\nu + \left(F^{\alpha\beta} F_{\alpha\beta}/4\mu_o\right)\delta_\nu^\mu.$$

Neither one of the variational principles contains explicit space-time dependence, hence the volume integral of the canonical momentum density $P_\nu = \frac{1}{c} T_\nu^o$ is a constant of the motion in the case that the generality condition is satisfied. The generality condition is satisfied for non-radiative fluids of compact spacial support. Note that $P_o = \frac{1}{c} T_o^o$ is the usual energy density, the sum of the mechanical momentum density and the Poynting flux.

Consider now the specific solution given in Subsection V.B.5 that demonstrates that the canonical momentum $\mathbf{P}$ is not necessarily a constant of the motion. For that particular solution $\mathbf{P}$ has no explicit time dependence. As such, the solution

is non-radiative. Assuming the fluid is of compact spacial support (which does not violate the local nature of the existence of the solution) we may conclude that the volume integral of the canonical momentum density P is a constant of the fluid's motion for that particular solution even though the canonical momentum $\mathbf{P}$ is not.

This discussion demonstrates that the volume integral of the canonical momentum density P is a constant of a fluid's motion in a case for which the canonical momentum $\mathbf{P}$ is not. In general, P is preferable to $\mathbf{P}$ for two reasons. First, P is independent of the electromagnetic gauge whereas $\mathbf{P}$ is not. Second, global boundary conditions may be considered in the process of determining conservation of the energy or momentum in the event that the canonical momentum *density* is chosen as the point of departure. These global boundary conditions are often more amenable to physical interpretation than are the local conditions required for conservation of $\mathbf{P}$.

Appendix B. Generality of the surface force potential equations

We wish to demonstrate that the Surface Force Potential equations of motion are equivalent to the Maxwell-Lorentz force set of equations. To do so we first form three sets of equations: (1) the set of equations unique to the Surface Force Potential equations; (2) the set of equations unique to the Maxwell-Lorentz force equations; and (3) the set of equations common to both sets. We cast the equations into their general relativistic forms in order to insure complete generality.

Set (1) consists of the following equations.

$$A_\sigma + \frac{m}{q} v_\sigma + \Lambda_\sigma + \gamma^\beta (\Lambda_{\sigma;\beta} - \Lambda_{\beta;\sigma}) = 0, \tag{B1a}$$

$$j^\sigma + (\gamma^\sigma j^\beta - \gamma^\beta j^\sigma)_{;\beta} = 0, \tag{B1b}$$

and

$$j^\beta (\Lambda_{\beta;\sigma} - \Lambda_{\sigma;\beta}) = 0. \tag{B1c}$$

Set (2) consists of

$$\frac{m}{q} v^\beta v_{\sigma;\beta} = v^\beta F_{\sigma\beta}. \tag{B2a}$$

Set (3) consists of

$$j^\sigma = \rho_{oe} v^\sigma, \tag{B3a}$$

$$\rho_{om} = \frac{m}{q} \rho_{oe}, \tag{B3b}$$

$$F_{\sigma\beta} = A_{\beta;\sigma} - A_{\sigma;\beta}, \tag{B3c}$$

$$F^{\beta\sigma}_{;\beta} = \mu_o j^\sigma, \tag{B3d}$$

and

$$j^\beta_{;\beta} = 0. \tag{B3e}$$

The proof of equivalence consists of two parts: (1) the proof that the Maxwell-Lorentz force set of equations follows from the Surface Force Potential set, and (2) the proof that when the Maxwell- Lorentz force set of equations is satisfied, functions Λ and γ may be found such that the Surface Force Potential set of equations are satisfied. We now proceed with the demonstration of equivalence.

Theorem 1: The set of functions which satisfy the Surface Force Potential equations of motion necessarily satisfy the Maxwell- Lorentz force set of equations.

Proof: We first prove the following Lemma.

Lemma:

$$j^{\beta}\left[\Lambda_{\sigma}+\gamma^{\nu}(\Lambda_{\sigma;\nu}-\Lambda_{\nu;\sigma})\right]_{;\beta}=j^{\beta}\left[\Lambda_{\beta}+\gamma^{\nu}(\Lambda_{\beta;\nu}-\Lambda_{\nu;\beta})\right]_{;\sigma}.$$

This is shown by direct computation of the left hand member with the aid of the (1) and (3) sets of equations as follows.

$$\begin{aligned}
j^{\beta}\left[\Lambda_{\sigma}+\gamma^{\nu}(\Lambda_{\sigma;\nu}-\Lambda_{\nu;\sigma})\right]_{;\beta} &= j^{\beta}\Lambda_{\sigma;\beta}+j^{\beta}\gamma^{\nu}_{;\beta}(\Lambda_{\sigma;\nu}-\Lambda_{\nu;\sigma}) \\
&\quad + j^{\beta}\gamma^{\nu}(\Lambda_{\sigma;\nu\beta}-\Lambda_{\nu;\sigma\beta}) \\
&= j^{\beta}\Lambda_{\sigma;\beta}+(j^{\beta}\gamma^{\nu})_{;\beta}(\Lambda_{\sigma;\nu}-\Lambda_{\nu;\sigma}) \\
&\quad + j^{\beta}\gamma^{\nu}\Lambda_{\sigma;\nu\beta}-j^{\beta}\gamma^{\nu}\Lambda_{\nu;\sigma\beta} \\
&= j^{\beta}\Lambda_{\beta;\sigma}+\left[(j^{\nu}\gamma^{\beta})_{;\beta}-j^{\nu}\right](\Lambda_{\sigma;\nu}-\Lambda_{\nu;\sigma}) \\
&\quad + j^{\beta}\gamma^{\nu}\Lambda_{\sigma;\nu\beta}-j^{\beta}\gamma^{\nu}\Lambda_{\nu;\sigma\beta} \\
&= j^{\beta}\Lambda_{\beta;\sigma}+\left[(j^{\nu}\gamma^{\beta})_{;\beta}-j^{\nu}\right](\Lambda_{\sigma;\nu}-\Lambda_{\nu;\sigma}) \\
&\quad + j^{\beta}\gamma^{\nu}\Lambda_{\sigma;\nu\beta}-j^{\beta}\gamma^{\nu}\Lambda_{\nu;\sigma\beta} \\
&= j^{\beta}\Lambda_{\beta;\sigma}+\left[j^{\nu}\gamma^{\beta}(\Lambda_{\sigma;\nu}-\Lambda_{\nu;\sigma})\right]_{;\beta}-j^{\nu}\gamma^{\beta}\Lambda_{\sigma;\nu\beta} \\
&\quad + j^{\nu}\gamma^{\beta}\Lambda_{\nu;\sigma\beta}+j^{\beta}\gamma^{\nu}\Lambda_{\sigma;\nu\beta}-j^{\beta}\gamma^{\nu}\Lambda_{\nu;\sigma\beta} \\
&= j^{\beta}\left[\Lambda_{\beta;\sigma}+\gamma^{\nu}(\Lambda_{\sigma;\nu\beta}-\Lambda_{\sigma;\beta\nu})+\gamma^{\nu}(\Lambda_{\beta;\sigma\nu}-\Lambda_{\nu;\sigma\beta})\right] \\
&= j^{\beta}\Big[\Lambda_{\beta;\sigma}+\gamma^{\nu}(\Lambda_{\alpha}R^{\alpha}_{\sigma\nu\beta}-[\Lambda_{\beta;\nu\sigma}-(\Lambda_{\beta;\nu\sigma}-\Lambda_{\beta;\sigma\nu}) \\
&\quad -\Lambda_{\nu;\beta\sigma}-(\Lambda_{\nu;\beta\sigma}-\Lambda_{\nu;\sigma\beta})])\Big]
\end{aligned}$$

$$
\begin{aligned}
&= j^\beta \Big[\Lambda_{\beta;\sigma} + \gamma^\nu (\Lambda_\alpha R^\alpha_{\sigma\nu\beta} + [\Lambda_{\beta;\nu\sigma} - \Lambda_{\nu;\beta\sigma}] \\
&\quad - \Lambda_\alpha R^\alpha_{\beta\nu\sigma} + \Lambda_\alpha R^\alpha_{\nu\beta\sigma}) \Big] \\
&= j^\beta \Big[\Lambda_{\beta;\sigma} + \gamma^\nu ([\Lambda_{\beta;\nu\sigma} - \Lambda_{\nu;\beta\sigma}] \\
&\quad + \Lambda_\alpha [R^\alpha_{\sigma\nu\beta} + R^\alpha_{\beta\sigma\nu} + R^\alpha_{\nu\beta\sigma}]) \Big] \\
&= j^\beta \left[\Lambda_\beta + \gamma^\nu (\Lambda_{\beta;\nu} - \Lambda_{\nu;\beta}) \right]_{;\sigma} - \gamma^\nu_{;\sigma} j^\beta (\Lambda_{\beta;\nu} - \Lambda_{\nu;\beta}) \\
&= j^\beta \left[\Lambda_\beta + \gamma^\nu (\Lambda_{\beta;\nu} - \Lambda_{\nu;\beta}) \right]_{;\sigma}
\end{aligned}
$$

The first equality results from the properties of the covariant derivative. The second follows from Eqs.(B1c) and (B3e). The third equality results from application of Eq.(B1b). The fourth, from Eq.(B1c) and the properties of the covariant derivative. The fifth equality is a consequence of Eq.(B1c) and a renaming of dummy sum indices. The sixth follows from the definition of the curvature tensor $R^\alpha_{\sigma\nu\beta}$ and from the introduction of a new term. The seventh equality results from the definition of the curvature tensor, the eighth from a regrouping of terms. The ninth follows from the First Bianci identity and from the properties of the covariant derivative. The final equality is a result of Eq.(B1c).

The remainder of the proof of the theorem is straightforward. First, we use Eq.(B1a) and the Lemma to deduce

$$j^\beta (A_\sigma + \frac{m}{q} v_\sigma)_{;\beta} = j^\beta (A_\beta + \frac{m}{q} v_\beta)_{;\sigma}.$$

We then rearrange terms and use Eq.(B3c) to obtain

$$j^\beta (F_{\beta\sigma} + \frac{m}{q} v_{\sigma;\beta}) = j^\beta \frac{m}{q} v_{\beta;\sigma}.$$

The term on the right of this last equation vanishes since $v^\beta v_\beta = c^2$ which implies

$$
\begin{aligned}
(v^\beta v_\beta)_{;\sigma} &= 2 v^\beta v_{\beta;\sigma} \\
&= 0.
\end{aligned}
$$

With the aid of Eqs.(B3a) and (B3b) one then obtains

$$\rho_{oe} v^{\beta} F_{\beta\sigma} = \rho_{om} v^{\beta} v_{\sigma;\beta},$$

which, assuming non-vanishing fluid density, leads directly to Eq.(B2a). This completes the proof of Theorem 1.

Theorem 2: If a set of functions satisfy the Maxwell- Lorentz force set of equations, then functions Λ and γ may be found which, together with the solution set of the Maxwell- Lorentz force equations, form a solution set to the Surface Force Potential set of equations. Moreover, Λ and γ may be defined globally.

Proof: We assume a positively charged fluid such that $\rho_{oe} > 0$, and demonstrate the result in a comoving coordinate system denoted by (y^0, y^1, y^2, y^3). In this system $v^0 = c$ and $v^i = 0$ for $i = 1, 2, 3$. We wish to find functions Λ and γ such that the equations in Set (1) are satisfied given that Sets (2) and (3) are satisfied. We choose a γ such that $\gamma^{\sigma} = 0$ for $\sigma = 0, 2, 3$. That is, we choose a γ in our comoving coordinate system with only one (possibly) non-zero element, γ^1.

We note that in our comoving coordinate system and with our choice for γ the equations from the three sets (1), (2) and (3) take the following forms. Set (1):

$$A_0 + \frac{mc}{q} + \Lambda_0 + \gamma^1(\Lambda_{0,1} - \Lambda_{1,0}) = 0, \qquad (B1a')$$

$$A_i + \Lambda_i + \gamma^1(\Lambda_{i,1} - \Lambda_{1,i}) = 0, \qquad (B1a'')$$

$$\sqrt{-g}\, j^{0} - (\sqrt{-g}\,\gamma^1 j^{0})_{,1} = 0, \qquad (B1b')$$

$$(\sqrt{-g}\,\gamma^1 j^{0})_{,0} = 0, \qquad (B1b'')$$

and

$$\Lambda_{0,\sigma} - \Lambda_{\sigma,0} = 0. \qquad (B1c')$$

Set (2):

$$F_{\sigma 0} = A_{\sigma,0} - A_{0,\sigma} = 0. \qquad (B2a')$$

Set (3):

$$j^{\cdot 0} = \rho_{oe} c, \qquad (B3a')$$

$$F_{\sigma\beta} = A_{\beta,\sigma} - A_{\sigma,\beta}, \qquad (B3c')$$

and

$$j^{\cdot\beta}_{;\beta} = \frac{1}{\sqrt{-g}} (\sqrt{-g} j^{\cdot 0})_{,0} = 0. \qquad (B3e')$$

Eqs.(B3b), and (B3d) remain as previously displayed.

We define u by $u = \sqrt{-g} j^{\cdot 0}$. Then $u = u(y^1, y^2, y^3)$ (from Eq.(B3e$'$)) and $u > 0$ everywhere (since $\rho_{oe} > 0$). We next define U by

$$U(y^1, y^2, y^3) = \int_0^{y^1} u(s, y^2, y^3) ds.$$

We set $\gamma^1 = \frac{U}{u}$. This choice for γ satisfies both $\gamma^1_{,0} = 0$, as required by Eqs.(B1b$''$) and (B3e$'$), and $u - (\gamma^1 u)_{,1} = 0$, as required by Eq.(B1b$'$).

Next, we set

$$\Lambda_0 = -A_0 - \frac{mc}{q},$$

$$\Lambda_1 = -A_1,$$

and

$$\Lambda_k + \gamma^1 \Lambda_{k,1} = -A_k - \gamma^1 A_{1,k},$$

where $k = 2, 3$. Λ_k may be solved for explicitly by utilizing the expression of γ^1 in terms of u and U. We rewrite the Λ_k equation as

$$u\Lambda_k + U\Lambda_{k,1} = (U\Lambda_k)_{,1} = -uA_k - UA_{1,k}.$$

This last equation admits the solution

$$\Lambda_k = -\frac{1}{U}\int_0^{y^1} (uA_k + UA_{1,k})(y^0, s, y^2, y^3)ds,$$

which we take as the final form for Λ_k.

With our choice for Λ and γ it is clear that Eqs.(B1a$'$, B1a$''$) and (B1b$'$, B1b$''$) are satisfied. Hence, it only remains for us to show that when A_σ satisfies Eq.(B2a$'$) Λ satisfies Eq.(B1c$'$). That $\Lambda_{0,1} - \Lambda_{1,0} = 0$ is satisfied is immediate from the definition of Λ_0 and Λ_1 after using Eq.(B2a$'$). We now note the following.

$$\begin{aligned}
\Lambda_{k,0} &= -\frac{1}{U}\int_0^{y^1} (uA_{k,0} + UA_{1,k0})ds \\
&= -\frac{1}{U}\int_0^{y^1} (uA_{0,k} + UA_{0,k1})ds \\
&= -\frac{1}{U}\int_0^{y^1} (UA_{0,k})_{,1}ds \\
&= -\frac{1}{U}(UA_{0,k}) \\
&= -A_{0,k}.
\end{aligned}$$

The first equality follows from the fact that u and U do not depend on y^0. The second follows from Eq.(B2a$'$) and the fact that partial differentiation is commutative. The third equality results from the definitions of u and U and the properties of differentiation. The fourth is a result of the Fundamental Theorem of Calculus and the last is immediate. From this equation and the definition of Λ_0 we deduce the following identity.

$$\Lambda_{0,k} - \Lambda_{k,0} = -A_{0,k} - (-A_{0,k}) = 0.$$

This concludes the demonstration that our definition of Λ satisfies Eq.(B1c$'$), which therefore concludes the proof of Theorem 2. Noting that all definitions are defined globally, we conclude the global equivalence of the Surface Force Potential equations and the Maxwell-Lorentz force set of equations.

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Publications:

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2. Implications of a Lagrangian to Eulerian variable transformation in an electromagnetic action: Canonical momentum and variational constraints, in preparation (with W. F. Edwards and Y. H. Clifton).
3. An Extended Kalman Filter for radar tracking of targets in Earth orbit, *Hughes Aircraft Company Interdepartmental Correspondence.*