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Classification of Isometry Algebras of Solutions of Einstein's Field Equations

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CLASSIFICATION OF ISOMETRY ALGEBRAS OF SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS

by

Eugene Hwang

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ABSTRACT

Classification of Isometry Algebras of Solutions of Einstein's Field Equations

by

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Solutions of Einstein's equations are classified according to their Lie algebras of isometries and their isotropy subalgebras. Solutions were taken from the USU electronic library of solutions of Einstein's field equations and the classification used Maple code developed at USU. This classification adds to the data contained in the library of solutions and provides additional tools for addressing the equivalence problem for solutions to the Einstein field equations. In this thesis, homogeneous spacetimes, hypersurface-homogeneous spacetimes, Robinson-Trautman solutions, and some famous black hole solutions have been classified.

(75 pages)

PUBLIC ABSTRACT

Classification of Isometry Algebras of Solutions of Einstein's Field Equations

Eugene Hwang

Since Schwarzschild found the first solution of the Einstein's equations, more than 800 solutions were found. Solutions of Einstein's equations are classified according to their Lie algebras of isometries and their isotropy subalgebras. Solutions were taken from the USU electronic library of solutions of Einstein's field equations and the classification used Maple code developed at USU. This classification adds to the data contained in the library of solutions and provides additional tools for addressing the equivalence problem for solutions to the Einstein field equations. In this thesis, homogeneous spacetimes, hypersurface-homogeneous spacetimes, Robinson-Trautman solutions, and some famous black hole solutions have been classified.

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CHAPTER 1

Introduction

Einstein introduced his field equations in 1915; they are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where $g_{\mu\nu}$ is the metric tensor of spacetime, $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar, respectively, Λ is the cosmological constant, G is Newton's constant, the speed of light c is chosen to be 1 for simplicity, and $T_{\mu\nu}$ is the energy-momentum tensor of matter. Physically, the left-hand side of the equations describes the geometry of spacetime and the right-hand side describes the energy-momentum of matter. Mathematically, the equations consist of ten second-order nonlinear partial differential equations for the metric and matter fields. There will often be some additional matter field equations, such as the Maxwell's equations.

Since the Einstein equations were introduced, more than 1000 solutions have been found. Probably the most famous solution is the Schwarzschild metric which describes the gravitational field outside of a spherically symmetric object such as a black hole,

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.2}$$

where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ with $x^\mu = (t, r, \theta, \phi)$ and M is the mass of the object. Here the energy-momentum tensor is zero. Another well-known solution is the Friedmann-Robertson-Walker (FRW) metric, which describes an isotropic and homogeneous universe and is used to study the Big Bang cosmology

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1.3}$$

Here $a(t)$ is the scale factor which tells "how big" the spacelike slice is at the moment t [1].

The scale factor $a(t)$ is usually determined by solving equation (1.1) using a perfect fluid energy-momentum tensor and a given equation of a state.

The majority of the known solutions have been compiled and classified by Stephani et al. largely based upon the dimension of their isometry groups and the nature of the group orbits [2]. These solutions are stored in the USU library of solutions of the Einstein equations [3]. This library contains data which define the solutions and metadata which define properties of the solutions.

Almost all solutions have symmetries which can be described mathematically by a Lie algebra of Killing vector fields. This Lie algebra is called the isometry algebra. Petrov classified all possible gravitational fields i.e. spacetime metrics according to their isometry algebras [4]. More recently, almost every possible isometry algebra in a four-dimensional spacetime has been enumerated by Jesse Hicks in his Ph.D dissertation and he classified some solutions of the Einstein field equations using his classification method [5] [6]. Petrov's and Hicks' classifications do not impose any field equations on the metrics. Hicks' classification verifies, corrects, and extends Petrov's classification. The goal of this thesis is to apply Hicks' classification of isometry algebras to known solutions of the Einstein equations. In other words, the solutions will be classified according to their symmetry.

There are three motivations for this work. First, it is interesting to ask how many isometry groups are allowed when one imposes the Einstein field equations. Second, the precise characterization of the symmetry of the solutions is a very important piece of metadata which needs to be included in the USU library of solutions. This allows researchers to search for solutions with given symmetries which is useful for testing conjectures, finding helpful examples and so forth. Third, such metadata are important for addressing the equivalence problem for solutions of the Einstein equations. The equivalence problem in general relativity is to determine if two solutions are equivalent with respect to a coordinate transformation. Because the isometry group is independent of choice of the coordinates, two solutions are related by a coordinate transformation only if they have the same isometry group. Therefore, knowing the isometry group is helpful in addressing the equivalence problem. As a simple illustration here are three line elements corresponding to three different metrics

$$ds^2 = du^2 + dv^2 \tag{1.4}$$

$$ds^2 = du^2 + u^2 dv^2 \tag{1.5}$$

$$ds^2 = du^2 + (\sin^2 u)dv^2. \tag{1.6}$$

They look similar to each other, but only the first two metrics are line elements on a flat plane $\mathbb{R}^2$ whereas the last line element defines the usual metric on a sphere S^2. Respectively, a basis of Killing vector fields and their Lie brackets are

$$e_1 = -v\partial_u + u\partial_v, \quad e_2 = \partial_v, \quad e_3 = \partial_u \tag{1.7a}$$

$$[e_1, e_2] = e_3, \quad [e_1, e_2] = -e_2, \quad [e_2, e_3] = 0 \tag{1.7b}$$

$$e_1 = \partial_v, \quad e_2 = -\cos(v)\partial_u + \frac{\sin(v)}{u}\partial_v, \quad e_3 = \sin(v)\partial_u + \frac{\cos(v)}{u}\partial_v \tag{1.8a}$$

$$[e_1, e_2] = e_3, \quad [e_1, e_2] = -e_2, \quad [e_2, e_3] = 0 \tag{1.8b}$$

$$e_1 = \partial_v, \quad e_2 = -\cos(v)\partial_u + \frac{\sin(v)\cos(u)}{\sin(u)}\partial_v, \quad e_3 = \sin(v)\partial_u + \frac{\cos(v)\cos(u)}{\sin(u)}\partial_v \tag{1.9a}$$

$$[e_1, e_2] = e_3, \quad [e_1, e_2] = -e_2, \quad [e_2, e_3] = e_1. \tag{1.9b}$$

The first two metrics are equivalent with respect to a coordinate transformation which identifies equation (1.7) and (1.8) and thus have the same isometry algebra. The third metric is not equivalent to the first two and consequently there is no change of coordinates that will identify equation (1.9) to (1.7) or (1.8). In a similar way one can use the isometry algebras to help determine if two solutions of the Einstein equations are equivalent.

This thesis is organized as follows. Following Hicks, chapter two shows how to characterize isometry algebras using Lie algebra-subalgebra pairs. In chapter three, I explain how to classify the solutions of the Einstein equations according to their isometry algebras. The results of the classification are presented in chapter four. Chapter five is a brief conclusion and discussion. An appendix shows a typical MAPLE worksheet that classifies some examples. The electronic documentation consisting of the codes and MAPLE worksheets which show calculations can be found here: (https://digitalcommons.usu.edu/dg/)

CHAPTER 2

Background

When analyzing the symmetry group of a metric it is advantageous to use infinitesimal methods whenever possible. This leads to the analysis of the isometry algebra which is itself characterized by a Lie algebra-subalgebra pair. The possible Lie algebra-subalgebra pairs that can appear for a four-dimensional spacetime have been enumerated by Hicks [6]. The purpose of this chapter is to explain all these ideas.

The first goal is to define the isometry group of a metric. Given a spacetime, a *diffeomorphism* is called an *isometry* if the metric is preserved by *pullback*. The set of isometries of a metric is called its *isometry group* and it satisfies the conditions to be a *Lie group*. I now explain this in more detail.

Definition 2.1. *Let $\Phi : M \to N$ be a smooth map of manifolds, and let f be a differentiable function on N (i.e., $f \in C^\infty(N)$), and let X be a vector field on M. The* <u>*pushforward*</u> *of X by Φ, $\Phi_* X$, is a vector field on N defined by $X \mapsto \Phi_*(X)$ where $\Phi_* X(f) := X(f \circ \Phi)$ for any f.*

Example 2.2. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a map given by

$$\Phi(x, y) = (u = x^2 + y^2, \quad v = x^2 - y^2, \quad w = xy), \tag{2.1}$$

and $f \in C^\infty(\mathbb{R}^3)$. The components of the pushforward of a vector field $X = -y\partial_x + x\partial_y$ is

$$\Phi_*(X)f = (\Phi_* X)^u \frac{\partial f}{\partial u} + (\Phi_* X)^v \frac{\partial f}{\partial v} + (\Phi_* X)^w \frac{\partial f}{\partial w}. \tag{2.2}$$

I calculate these components from the Definition 2.1 as follows

$$X(f \circ \Phi) = X^x \frac{\partial}{\partial x}(f(\Phi)) + X^y \frac{\partial}{\partial y}(f(\Phi)) \tag{2.3}$$

$$= X^x\left(\frac{\partial f}{\partial u}\frac{\partial \Phi^u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial \Phi^v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial \Phi^w}{\partial x}\right) + X^y\left(\frac{\partial f}{\partial u}\frac{\partial \Phi^u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial \Phi^v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial \Phi^w}{\partial y}\right) \tag{2.4}$$

$$= \left(X^x\frac{\partial \Phi^u}{\partial x} + X^y\frac{\partial \Phi^u}{\partial y}\right)\frac{\partial f}{\partial u} + \left(X^x\frac{\partial \Phi^v}{\partial x} + X^y\frac{\partial \Phi^v}{\partial y}\right)\frac{\partial f}{\partial v} + \left(X^x\frac{\partial \Phi^w}{\partial x} + X^y\frac{\partial \Phi^w}{\partial y}\right)\frac{\partial f}{\partial w} \tag{2.5}$$

$$= (-y(2x) + x(2y))\frac{\partial f}{\partial u} + (-y(2x) + x(-2y))\frac{\partial f}{\partial v} + (-y(y) + x(x))\frac{\partial f}{\partial w} \tag{2.6}$$

$$= -4xy\frac{\partial f}{\partial v} + (x^2 - y^2)\frac{\partial f}{\partial w} \tag{2.7}$$

$$= -4w\frac{\partial f}{\partial v} + v\frac{\partial f}{\partial w}. \tag{2.8}$$

Therefore, $\Phi_* X = -4w\partial_v + v\partial_w$.

Definition 2.3. *Let $\Phi : M \to N$ and let f be a function on N. The pullback of f by Φ, $\Phi^* f$, is a smooth function defined on M by*

$$\Phi^* f = f \circ \Phi. \tag{2.9}$$

Example 2.4. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a map given by

$$\Phi(x, y) = (x^2 + y^2, x^2 - y^2, xy), \tag{2.10}$$

and let f be a function on $\mathbb{R}^3$ defined by $f(u, v, w) = u + v - w$. Then the pullback of the function f is

$$\Phi^* f = f(\Phi(x, y)) \tag{2.11}$$

$$= f(x^2 + y^2, x^2 - y^2, xy) \tag{2.12}$$

$$= 2x^2 - xy \tag{2.13}$$

Definition 2.5. *Let $\Phi : M \to N$ and let ω be a covector field on N. The pullback $\Phi^*\omega$ is a covector field on M defined by $\Phi^*\omega(X) := \omega(\Phi_* X)$.*

Example 2.6. Let $x^i = (x, y)$ be coordinates on $\mathbb{R}^2$ and let $u^\alpha = (u, v, w)$ be coordinates on $\mathbb{R}^3$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by

$$\Phi(x, y) = (x + y, x^2 - y^2, xy), \tag{2.14}$$

let $\tau = (vdu - 3dv + dw)$, and K be any vector field in $\mathbb{R}^2$. Then $\Phi^*\tau(K) \equiv (\Phi^*\tau)_\alpha K^\alpha$ is computed as follows. First I use Definition 2.5 to get a formula for the components of a pullback in general.

$$\begin{aligned}
\Phi^*\tau(K) &= \tau(\Phi_* K) && (2.15)\\
&= \tau([\Phi_* K]^i \frac{\partial}{\partial u^i}) && (2.16)\\
&= \tau(K^\alpha \frac{\partial \Phi^i}{\partial x^\alpha} \frac{\partial}{\partial u^i}) && (2.17)\\
&= K^\alpha \frac{\partial \Phi^i}{\partial x^\alpha} \tau(\frac{\partial}{\partial u^i}) && (2.18)\\
&= K^\alpha \frac{\partial \Phi^i}{\partial x^\alpha} \tau_i && (2.19)\\
&= \tau_i \frac{\partial \Phi^i}{\partial x^\alpha} K^\alpha && (2.20)
\end{aligned}$$

Since K is arbitrary,

$$(\Phi^*\tau)_\alpha = \tau_i \frac{\partial \Phi^i}{\partial x^\alpha}. \tag{2.21}$$

Now I use equation (2.21) to calculate the pullback.

$$\Phi^*\tau = (\Phi^*\tau)_x dx + (\Phi^*\tau)_y dy \tag{2.22}$$

$$\begin{aligned}
(\Phi^*\tau)_x &= \tau_u \frac{\partial \Phi^u}{\partial x} + \tau_v \frac{\partial \Phi^v}{\partial x} + \tau_w \frac{\partial \Phi^w}{\partial x} && (2.23)\\
&= (v)(2x) + (-3)(2x) + y && (2.24)\\
&= 2x(x^2 - y^2) - 6x + y && (2.25)
\end{aligned}$$

$$(\Phi^*\tau)_y = \tau_u \frac{\partial \Phi^u}{\partial y} + \tau_v \frac{\partial \Phi^v}{\partial y} + \tau_w \frac{\partial \Phi^w}{\partial y} \tag{2.26}$$

$$= (v)(2y) + (-3)(-2y) + x \tag{2.27}$$

$$= 2y(x^2 - y^2) + 6y + x \tag{2.28}$$

Finally,

$$\Phi^*\tau = (2x(x^2 - y^2) - 6x + y)dx + (2y(x^2 - y^2) + 6y + x)dy \tag{2.29}$$

Definition 2.7. *A metric g on a manifold M is a (0,2)-tensor field satisfying*

(i) symmetry : $g(X,Y) = g(Y,X)$ for all $X, Y \in \Gamma(M)$,

(ii) non-degeneracy: if $g(X,Y) = 0$ for all Y, then X must be 0.

Proposition 2.8. *(Sylvester's law of inertia [7]) A metric tensor at each point of a manifold can be diagonalized by a choice of basis where the diagonal values are -1, and 1. The number of -1's and 1's are uniquely determined by the metric.*

Definition 2.9. *If, in a diagonal basis as defined in proposition 2.8, r is the number of 1's in the diagonal and s the number of -1's, I will say the signature of a metric is (r,s). A Riemannian metric has a signature $(r,0)$ which means a positive definite metric tensor. Otherwise a metric is pseudo-Riemannian. In particular, a Lorentzian metric has a signature $(n-1,1)$ or $(1,n-1)$.*

Definition 2.10. *A manifold equipped with a Lorentzian metric is called a spacetime.*

Typically n is 4 in the case of spacetime and in all the solutions in this thesis.

Definition 2.11. *Let γ be a metric on a manifold N, let $\Phi : M \to N$, and let X and Y be vector fields on M. The pullback of γ by Φ denoted by $\Phi^*\gamma$ is the metric on M defined by*

$$\Phi^*\gamma(X,Y) := \gamma(\Phi_* X, \Phi_* Y). \tag{2.30}$$

Now I will derive a formula for the pullback of a metric. Let p^α be coordinates on M and q^i be coordinates on N such that the map $\Phi : M \to N$ takes the form

$$q^i = \Phi^i(p^\alpha). \tag{2.31}$$

Let X and Y be vector fields on M and γ be a metric on N

$$\gamma : N \times N \to C^\infty(N). \tag{2.32}$$

Then $\Phi^*\gamma$ is a metric on M which can be defined as

$$\Phi^*\gamma(X, Y) := \gamma(\Phi_* X, \Phi_* Y) \tag{2.33}$$

$$= \gamma(X^\alpha \frac{\partial \Phi^i}{\partial p^\alpha}\frac{\partial}{\partial q^i}, Y^\beta \frac{\partial \Phi^j}{\partial p^\beta}\frac{\partial}{\partial q^j}) \tag{2.34}$$

$$= X^\alpha \frac{\partial \Phi^i}{\partial p^\alpha} Y^\beta \frac{\partial \Phi^j}{\partial p^\beta} \gamma(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}) \tag{2.35}$$

$$= \frac{\partial \Phi^i}{\partial p^\alpha}\frac{\partial \Phi^j}{\partial p^\beta}\gamma_{ij} X^\alpha Y^\beta. \tag{2.36}$$

Using the notation, $\Phi^*\gamma(X, Y) = (\Phi^*\gamma)_{\alpha\beta} X^\alpha Y^\beta$, it follows that

$$(\Phi^*\gamma)_{\alpha\beta} = \frac{\partial \Phi^i}{\partial p^\alpha}\frac{\partial \Phi^j}{\partial p^\beta}\gamma_{ij}. \tag{2.37}$$

Example 2.12. Let $\Phi : U \to \mathbb{R}^3, U = \{(\theta, \phi) \in \mathbb{R}^2 | 0 < \theta < 2\pi, 0 < \phi < \pi\}$, be the function

$$\Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \tag{2.38}$$

and $\gamma = dx \otimes dx + dy \otimes dy + dz \otimes dz$ in coordinates (x, y, z) in $\mathbb{R}^3$. Then $\Phi^*\gamma$ can be written

$$\Phi^*\gamma = (\Phi^*\gamma)_{\theta\theta} d\theta \otimes d\theta + (\Phi^*\gamma)_{\phi\phi} d\phi \otimes d\phi + 2(\Phi^*\gamma)_{\theta\phi} d\theta \otimes d\phi, \tag{2.39}$$

where

$$(\Phi^*\gamma)_{\theta\theta} = \frac{\partial\Phi^i}{\partial\theta}\frac{\partial\Phi^j}{\partial\theta}\gamma_{ij} \tag{2.40}$$

$$= \frac{\partial\Phi^x}{\partial\theta}\frac{\partial\Phi^x}{\partial\theta}\gamma_{xx} + \frac{\partial\Phi^y}{\partial\theta}\frac{\partial\Phi^y}{\partial\theta}\gamma_{yy} + \frac{\partial\Phi^z}{\partial\theta}\frac{\partial\Phi^z}{\partial\theta}\gamma_{zz} \tag{2.41}$$

$$= (-\sin\theta\sin\phi)(-\sin\theta\sin\phi) + (\cos\theta\sin\phi)(\cos\theta\sin\phi) \tag{2.42}$$

$$= \sin^2\phi \tag{2.43}$$

$$(\Phi^*\gamma)_{\phi\phi} = \frac{\partial\Phi^i}{\partial\phi}\frac{\partial\Phi^j}{\partial\phi}\gamma_{ij} \tag{2.44}$$

$$= \frac{\partial\Phi^x}{\partial\phi}\frac{\partial\Phi^x}{\partial\phi}\gamma_{xx} + \frac{\partial\Phi^y}{\partial\phi}\frac{\partial\Phi^y}{\partial\phi}\gamma_{yy} + \frac{\partial\Phi^z}{\partial\phi}\frac{\partial\Phi^z}{\partial\phi}\gamma_{zz} \tag{2.45}$$

$$= (\cos\theta\cos\phi)(\cos\theta\cos\phi) + (\sin\theta\cos\phi)(\sin\theta\cos\phi) + (-\sin\phi)(-\sin\phi) \tag{2.46}$$

$$= 1 \tag{2.47}$$

$$(\Phi^*\gamma)_{\theta\phi} = \frac{\partial\Phi^i}{\partial\theta}\frac{\partial\Phi^j}{\partial\phi}\gamma_{ij} \tag{2.48}$$

$$= \frac{\partial\Phi^x}{\partial\theta}\frac{\partial\Phi^x}{\partial\phi}\gamma_{xx} + \frac{\partial\Phi^y}{\partial\theta}\frac{\partial\Phi^y}{\partial\phi}\gamma_{yy} + \frac{\partial\Phi^z}{\partial\theta}\frac{\partial\Phi^z}{\partial\phi}\gamma_{zz} \tag{2.49}$$

$$= (-\sin\theta\sin\phi)(\cos\theta\cos\phi) + (\cos\theta\sin\phi)(\sin\theta\cos\phi) \tag{2.50}$$

$$= 0. \tag{2.51}$$

Therefore

$$\Phi^*\gamma = \sin^2\phi d\theta\otimes d\theta + d\phi\otimes d\phi. \tag{2.52}$$

Definition 2.13. *Let $\Phi : M \to N$ be a bijection. Φ is a diffeomorphism if Φ is smooth and Φ^{-1} is smooth.*

Definition 2.14. *Let γ be a metric on M, and let Φ be a diffeomorphism from M to itself. The diffeomorphism Φ is an isometry if*

$$\Phi^*\gamma = \gamma. \tag{2.53}$$

Example 2.15. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism

$$\Phi(x, y) = (\frac{1}{\sqrt{2}}(x - y), \frac{1}{\sqrt{2}}(x + y)), \tag{2.54}$$

and consider a metric tensor in $\mathbb{R}^2$ given by

$$\gamma = \frac{1}{1 + x^2 + y^2}(dx \otimes dx + dy \otimes dy). \tag{2.55}$$

Then the pullback $\Phi^* g$ of the metric g is

$$\Phi^*\gamma = \Phi^*\gamma_{xx} dx \otimes dx + \Phi^*\gamma_{yy} dy \otimes dy \tag{2.56}$$

$$\begin{aligned}
\Phi^*\gamma_{xx} &= \frac{\partial\Phi^i}{\partial x}\frac{\partial\Phi^j}{\partial x}\gamma_{ij} && (2.57)\\
&= \frac{\partial\Phi^x}{\partial x}\frac{\partial\Phi^x}{\partial x}\gamma_{xx} + \frac{\partial\Phi^y}{\partial x}\frac{\partial\Phi^y}{\partial x}\gamma_{yy} && (2.58)\\
&= \frac{1}{1 + x^2 + y^2} && (2.59)
\end{aligned}$$

$$\begin{aligned}
\Phi^*\gamma_{yy} &= \frac{\partial\Phi^i}{\partial y}\frac{\partial\Phi^j}{\partial y}\gamma_{ij} && (2.60)\\
&= \frac{\partial\Phi^x}{\partial y}\frac{\partial\Phi^x}{\partial y}\gamma_{xx} + \frac{\partial\Phi^y}{\partial y}\frac{\partial\Phi^y}{\partial y}\gamma_{yy} && (2.61)\\
&= \frac{1}{1 + x^2 + y^2}. && (2.62)
\end{aligned}$$

From this calculation, one can see

$$\begin{aligned}
\Phi^*\gamma &= \frac{1}{1 + x^2 + y^2} dx \otimes dx + \frac{1}{1 + x^2 + y^2} dy \otimes dy && (2.63)\\
&= \frac{1}{1 + x^2 + y^2}(dx \otimes dx + dy \otimes dy) && (2.64)\\
&= \gamma, && (2.65)
\end{aligned}$$

therefore, $\Phi^*\gamma = \gamma$ and Φ is an isometry.

Definition 2.16. *A nonempty set of elements G is said to form a group if in G there is defined a binary operation, called the product and denoted by $\cdot$, such that*

(i) $a, b \in G$ implies that $a \cdot b \in G$ (closed).

(ii) $a, b, c \in G$ implies that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associative law).

(iii) There exists an element $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$ (the existence of an identity element in G).

(iv) For every $a \in G$ there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (the existence of inverse in G).

Proposition 2.17. *Let (M, γ) be a spacetime. The set of all isometries forms a group under composition.*

Proof. Let Φ, Ψ and Ω be isometries of the metric γ, then $\Phi^*\gamma = \gamma$, $\Psi^*\gamma = \gamma$, and $\Omega^*\gamma = \gamma$. Now let us check the properties of Definition 2.16.

(i) Closure : The composition of isometries is another isometry, because

$$(\Phi \circ \Psi)^*\gamma = (\Psi^* \circ \Phi^*)\gamma \tag{2.66}$$

$$= \Psi^*(\Phi^*\gamma) \tag{2.67}$$

$$= \Psi^*\gamma \tag{2.68}$$

$$= \gamma. \tag{2.69}$$

(ii) Associativity : The composition of isometries is associative because

$$\Phi \circ (\Psi \circ \Omega) = \Phi \circ \Psi(\Omega) \tag{2.70}$$

$$= \Phi(\Psi(\Omega)) \tag{2.71}$$

$$(\Phi \circ \Psi) \circ \Omega = \Phi(\Psi) \circ \Omega \tag{2.72}$$

$$= \Phi(\Psi(\Omega)). \tag{2.73}$$

(iii) Identity element : The identity element, represented by the identity transformation Φ_{id} on M, is an isometry because

$$\Phi_{id}^*\gamma = \gamma. \tag{2.74}$$

(iv) Inverse element : Every isometry Φ has an inverse Φ^{-1} which is also an isometry.

$$\Phi \circ \Phi^{-1} = \Phi_{id} \tag{2.75}$$

$$(\Phi^{-1})^*\gamma = (\Phi^{-1})^*(\Phi^*\gamma) \tag{2.76}$$

$$= (\Phi \circ \Phi^{-1})^*\gamma \tag{2.77}$$

$$= \Phi_{id}^*\gamma \tag{2.78}$$

$$= \gamma \tag{2.79}$$

$\square$

The isometry group is an example of a group action which is defined as follows.

Definition 2.18. *Let G be a group with product $\cdot$, and let S be a set. A group action of G on S is a mapping $\mu : G \times S \to S$ which satisfies*

(i) $\mu(e, x) = x$ for all $x \in S$ and e the identity in G,

(ii) $\mu(a, \mu(b, x)) = \mu(a \cdot b, x)$ for all $a, b \in G, x \in S$.

Example 2.19. Let

$$SE(2) = \left\{ \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right), \theta \in [0, 2\pi), a, b \in \mathbb{R} \right\} \tag{2.80}$$

be the group with group multiplication as follows

$$\left(\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) \cdot \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}, \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \right). \tag{2.81}$$

This group is the Special Euclidean group in 2-dimensions $SE(2)$. The mapping $\mu : SE(2) \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mu(\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right), \begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} \tag{2.82}$$

is an action of the group $SE(2)$ on $\mathbb{R}^2$. Let's check property (ii) of Definition 2.17. Let $G_1, G_2 \in SE(2)$ be

$$G_1 = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \tag{2.83}$$

$$G_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}. \tag{2.84}$$

Then,

$$\mu(G_1, \mu(G_2, x)) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x\cos\theta - y\sin\theta + a \\ x\sin\theta + y\cos\theta + b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \tag{2.85}$$

$$= \begin{bmatrix} x\cos\theta\cos\phi - y\sin\theta\cos\phi + a\cos\phi - x\sin\theta\sin\phi - y\cos\theta\sin\phi - b\sin\phi + c \\ x\cos\theta\sin\phi - y\sin\theta\sin\phi + a\sin\phi + x\cos\phi\sin\theta + y\cos\theta\cos\phi + b\cos\phi + d \end{bmatrix} \tag{2.86}$$

$$= \begin{bmatrix} x\cos(\theta+\phi) - y\sin(\theta+\phi) + a\cos\phi - b\sin\phi + c \\ x\sin(\theta+\phi) + y\cos(\theta+\phi) + a\sin\phi + b\cos\phi + d \end{bmatrix}. \tag{2.87}$$

$$\mu(G_1 \cdot G_2, x) = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a\cos\phi - b\sin\phi + c \\ a\sin\phi + b\cos\phi + d \end{bmatrix} \tag{2.88}$$

$$= \begin{bmatrix} x\cos(\theta+\phi) - y\sin(\theta+\phi) + a\cos\phi - b\sin\phi + c \\ x\sin(\theta+\phi) + y\cos(\theta+\phi) + a\sin\phi + b\cos\phi + d \end{bmatrix}. \tag{2.89}$$

Therefore, $\mu(G_1, \mu(G_2, x)) = \mu(G_1 \cdot G_2, x)$.

Definition 2.20. *A* <u>*Lie group*</u> *is a group consisting of a set G and a product $\cdot$ which satisfies*

(i) Four axioms of a group (Closure, Associativity, Identity element, Inverse element).

(ii) G is a smooth manifold.

(iii)The maps $\mu : G \times G \to G, (g_1, g_2) \mapsto g_1 \cdot g_2$ and $\iota : G \to G, g \mapsto g^{-1}$ are smooth.

Example 2.21. Let $G = \mathbb{R}^n$ be a smooth manifold, and let the group operation $+_{\mathbb{R}^n}$ be addition on n-tuples. Then $(\mathbb{R}^n, +_{\mathbb{R}^n})$ is a commutative Lie group, called the n-dimensional translation group. In particular the maps $\mu(g_1, g_2) = g_1 + g_2$ and $\iota(g_1) = -g_1$ are linear and therefore they are smooth.

Definition 2.22. *Let G be a Lie group acting on a manifold M by a smooth action μ. The* <u>*orbit*</u> *of $p \in M$ under the action μ is the set $\mathcal{O}_G(p) = \{\mu(g, p) | g \in G\}$.*

Example 2.23. Let G be the rotation group acting on $\mathbb{R}^2$. If a point p is not the origin, the orbit through p is a circle around the origin. If p is the origin, the orbit is p itself.

Definition 2.24. *Let a group G act on a manifold M by an action μ. The* <u>*isotropy subgroup*</u> *at $x \in M$ is the subgroup*

$$G_x = \{g \in G | \mu(g, x) = x\}. \tag{2.90}$$

If $G_x = \{e\}$ for all x where e is the identity element in G, the group action is called <u>*simply transitive*</u>*. Otherwise, the group action is* <u>*multiply transitive*</u>*.*

Example 2.25. Let $SE(2)$ act on $\mathbb{R}^2$ by μ as defined in example 2.19. Then the isotropy subgroup at $(x, y) \in \mathbb{R}^2$ is the subgroup g_θ where

$$g_\theta = \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} -x\cos\theta + y\sin\theta + x \\ -x\sin\theta - y\cos\theta + y \end{bmatrix} \right). \tag{2.91}$$

To see this is true

$$\mu\left(\left(\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}, \begin{bmatrix}-x\cos\theta+y\sin\theta+x\\ -x\sin\theta-y\cos\theta+y\end{bmatrix}\right), \begin{bmatrix}x\\ y\end{bmatrix}\right)$$

$$= \begin{bmatrix}x\cos\theta-y\sin\theta-x\cos\theta+y\sin\theta+x\\ x\sin\theta+y\cos\theta-x\sin\theta-y\cos\theta+y\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix}. \tag{2.92}$$

Proposition 2.26. *The set of all isometries is a group action called the* <u>*isometry group*</u> *[8].*

I now discuss infinitesimal versions of the previous constructions. The *Killing vector fields* are *infinitesimal generators* of isometries. In other words, a Killing vector field generates a *flow* which is a *one parameter group of isometries.* Killing vector fields are solutions of the Killing equation which is equivalent to saying the *Lie derivative* of the metric tensor vanishes along the Killing vector fields. The Killing vector fields of a metric form a *Lie algebra* where the Lie bracket is the commutator of vector fields. This algebra is called the *isometry algebra.*

The set of Killing vector fields which vanish at a given point p defines a subalgebra of the isometry algebra called the *isotropy subalgebra* at p. The isotropy subalgebra at p defines a representation of $\mathfrak{so}(3,1)$ on the tangent space at p. The Lie algebra and its isotropy subalgebra locally define the isometry group action in Hicks' classification. Let's review these ideas in detail.

Definition 2.27. *A* <u>*flow*</u> *on a manifold M is a smooth map $\Psi : \mathbb{R}\times M \to M$ which satisfies the two properties*

(i) $\Psi(0,p) = p$, *for all* $p \in M$, *and*

(ii) $\Psi(s,\Psi(t,p)) = \Psi(s+t,p)$, *for all* $s,t \in \mathbb{R}, p \in M$.

Another term often used for a flow is a <u>*one parameter group of transformations.*</u>

Example 2.28. A smooth function $\Phi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$,

$$\Phi(t,(x,y)) = (x\cos t - y\sin t, \quad x\sin t + y\cos t) \tag{2.93}$$

is a flow on $\mathbb{R}^2$ with a parameter t because,

$$\Phi(0,(x,y)) = (x\cos(0) - y\sin(0), \quad x\sin(0) + y\cos(0)) \tag{2.94}$$

$$= (x,y) \tag{2.95}$$

and

$$\Phi(\theta_1, \Phi(\theta_2,(x,y))) = \Phi(\theta_1,(x\cos\theta_2 - y\sin\theta_2, \quad x\sin\theta_2 + y\cos\theta_2)) \tag{2.96}$$

$$= ((x\cos\theta_2 - y\sin\theta_2)\cos\theta_1 - (x\sin\theta_2 + y\cos\theta_2)\sin\theta_1,$$

$$(x\cos\theta_2 - y\sin\theta_2)\sin\theta_1 + (x\sin\theta_2 + y\cos\theta_2)\cos\theta_1) \tag{2.97}$$

$$= (x\cos(\theta_1+\theta_2) - y\sin(\theta_1+\theta_2), \quad x\sin(\theta_1+\theta_2) + y\cos(\theta_1+\theta_2)). \tag{2.98}$$

Definition 2.29. *Let $\Psi : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a flow, and let $p \in \mathbb{R}^n$. A vector X_p is called the* <u>*infinitesimal generator*</u> *of the flow Ψ*

$$X_p = \left.\frac{\partial \Psi(t,p)}{\partial t}\right|_{t=0}. \tag{2.99}$$

Definition 2.30. *The* <u>*Lie derivative*</u> *$\mathcal{L}_X T$ of a tensor field T along a vector field X at a point $p \in M$ is defined by*

$$\mathcal{L}_X T = \frac{d}{dt}(\Phi_t^* T)|_{t=0} \tag{2.100}$$

where Φ_t is the flow of X.

Proposition 2.31. *The main properties of the Lie derivative are: [5]*

(i) $\mathcal{L}_X f = X(f)$, for $f \in C^\infty(M)$.

(ii) $\mathcal{L}_X Y = [X,Y]$, for any vector fiels X and Y.

(iii) $\mathcal{L}_X(T+S) = \mathcal{L}_X T + \mathcal{L}_X S$, for any tensor T and S.

(iv) $\mathcal{L}_X(T\otimes S) = (\mathcal{L}_X T)\otimes S + T\otimes \mathcal{L}_X S$.

(v) $\mathcal{L}_{[X,Y]}\gamma = \mathcal{L}_X\mathcal{L}_Y\gamma - \mathcal{L}_Y\mathcal{L}_X\gamma$

Proposition 2.32.

$$\mathcal{L}_{X+Y}\gamma = \mathcal{L}_X\gamma + \mathcal{L}_Y\gamma. \tag{2.101}$$

Lie derivative of the metric tensor γ with respect to a vector field X is

$$\mathcal{L}_X\gamma = \left(g_{il}(\mathbf{x})\frac{\partial X^l(\mathbf{x})}{\partial x^j} + g_{lj}(\mathbf{x})\frac{\partial X^l(\mathbf{x})}{\partial x^i} + X^l(\mathbf{x})\frac{\partial g_{ij}(\mathbf{x})}{\partial x^l} \right) dx^i \otimes dx^j \tag{2.102}$$

where

$$\gamma = g_{ij}(\mathbf{x})dx^i \otimes dx^j. \tag{2.103}$$

Definition 2.33. *A vector field X that satisfies $\mathcal{L}_X\gamma = 0$ is called a Killing vector field.*

Proposition 2.34. *X is a Killing vector field if and only if its flow Φ_t^* is an isometry group.*

Example 2.35. Let $X = (x^2 - y^2)\partial_x + 2xy\partial_y = X^1(\mathbf{x})\partial_x + X^2(\mathbf{x})\partial_y$ and let γ be $ds^2 = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$.

$$\mathcal{L}_X\gamma = \left(g_{i1}\frac{\partial X^1}{\partial x^j} + g_{1j}\frac{\partial X^1}{\partial x^i} + X^1\frac{\partial g_{ij}}{\partial x} + g_{i2}\frac{\partial X^2}{\partial x^j} + g_{2j}\frac{\partial X^2}{\partial x^i} + X^2\frac{g_{ij}}{\partial y} \right) dx^i \otimes dx^j \tag{2.104}$$

$$= \big(\frac{1}{y^2}(2x) + \frac{1}{y^2}(2x) + 2xy(\frac{-2}{y^3})\big)dx \otimes dx + \big(\frac{1}{y^2}(2x) + \frac{1}{y^2}(2x) + 2xy(\frac{-2}{y^3})\big)dy \otimes dy \tag{2.105}$$

$$= 0. \tag{2.106}$$

Definition 2.36. *A Lie algebra $\mathfrak{g}$ is a vector space with a bilinear operator $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies*

(i) the skew-symmetric property: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$

(ii) and the Jacobi identity: $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Proposition 2.37. *The set of all Killing vector fields of a metric equipped with the Lie bracket $[\ ,\ ]$ given by the commutator of vector fields defines a Lie algebra.*

Proof. Let X and Y be the Killing vector fields of a metric γ:

$$\mathcal{L}_X\gamma = 0, \quad \mathcal{L}_Y\gamma = 0. \tag{2.107}$$

Skew-symmetry of the Lie bracket and Jacobi identity are well known properties of the commutators of any vector fields. Therefore, it is sufficient to show that the set of Killing vector fields of a metric is a vector space and that $[X, Y]$ is also a Killing vector field. The sum of two Killing vector fields is also a Killing vector fields as is a constant scalar multiple:

$$\mathcal{L}_{X+Y}\gamma = \mathcal{L}_X\gamma + \mathcal{L}_Y\gamma = 0, \tag{2.108}$$

$$\mathcal{L}_{aX}\gamma = a\mathcal{L}_X\gamma = 0, \quad a \in \mathbb{R}. \tag{2.109}$$

Therefore the set of Killing vector fields form a vector subspace of the set of all vector fields. To see that $[X, Y]$ is also a Killing vector field, I use the following identity satisfied by the Lie derivative.

$$\mathcal{L}_{[X,Y]}\gamma = \mathcal{L}_X\mathcal{L}_Y\gamma - \mathcal{L}_Y\mathcal{L}_X\gamma \tag{2.110}$$

$$= 0. \tag{2.111}$$

□

Example 2.38. Let f be a function in $\mathbb{R}^2$, and let γ be a metric. The following three vector fields are Killing vector fields of a metric $\gamma = dx \otimes dx + dy \otimes dy$

$$X = -y\partial_x + x\partial_y \tag{2.112}$$

$$Y = \partial_x \tag{2.113}$$

$$Z = \partial_y. \tag{2.114}$$

Let us check they form a Lie algebra.

$$[X,Y]f = (XY - YX)f \tag{2.115}$$

$$= (-y\partial_x + x\partial_y)\partial_x f - \partial_x(-y\partial_x + x\partial_y)f \tag{2.116}$$

$$= -\partial_y f \tag{2.117}$$

$$= -Zf \tag{2.118}$$

$$[X,Z]f = (XZ - ZX)f \tag{2.119}$$

$$= (-y\partial_x + x\partial_y)\partial_y f - \partial_y(-y\partial_x + x\partial_y)f \tag{2.120}$$

$$= \partial_x f \tag{2.121}$$

$$= Yf \tag{2.122}$$

$$[Y,Z]f = (YZ - ZY)f \tag{2.123}$$

$$= \partial_x\partial_y f - \partial_y\partial_x f \tag{2.124}$$

$$= 0 \tag{2.125}$$

$$[X,[Y,Z]] - [Z,[X,Y]] - [Y,[Z,X]] = [X,0] - [Z,-Z] - [Y,-Y] = 0 \tag{2.126}$$

Therefore the three Killing vector fields X, Y, and Z satisfy the Jacobi identity.

Definition 2.39. *The Lie algebra of Killing vector fields of a given metric γ is called the* <u>*isometry algebra*</u> *of γ.*

Definition 2.40. *Given a Lie algebra of vector fields Γ, the* <u>*isotropy subalgebra*</u> *of Γ at $p \in M$ is $\Gamma_p = \{X \in \Gamma | X_p = 0\}$.*

Let us check that the isotropy subalgebra is in fact a subalgebra. To do this I should check that it is a vector subspace and that the Lie bracket closes. The vector subspace property follows from the fact that, if X and Y are Killing vectors which vanish at a point p, any linear combination of the vectors $aX + bY$ also vanishes at the point. To show the isotropy subalgebra is closed under Lie bracket, I calculate as follows. Let f be a function. Consider the directional derivative of the function along the Lie bracket.

$$[X, Y](f)|_p = X(Y(f))|_p - Y(X(f))|_p \tag{2.127}$$

$$= 0. \tag{2.128}$$

Example 2.41. Consider $\mathbb{R}^2$ with coordinates (x, y). Let Γ be a Lie algebra with basis $\{-y\partial_x + x\partial_y, \partial_y, \partial_x\}$. The isotropy subalgebra Γ_p of Γ at a point p given by $(x = 1, y = 1)$ is spanned by $-(y-1)\partial_x + (x-1)\partial_y$.

Next I will show how isotropy subalgebras can be put in correspondence with subalgebras of the Lie algebra of the Lorentz group. The Killing equation is the statement that the Lie derivative of the metric with respect to a vector field X vanishes. From equation (2.102),

$$X^l \partial_l g_{ij} + g_{il}\partial_j X^l + g_{lj}\partial_i X^l = 0. \tag{2.129}$$

If the vector field X is in the isotropy subalgebra, then the equation above can be evaluated at the point p where $X^l(p) = 0$. But note that $J^l{}_i \equiv \partial_i X^l\big|_p \neq 0$. Then,

$$g_{il}\partial_j X^l\big|_p + g_{lj}\partial_i X^l\big|_p = g_{il}J^l{}_j + g_{lj}J^l{}_i = 0. \tag{2.130}$$

Since the metric γ has Lorentzian signature, there is a change of basis which the equation (2.130) can be written as

$$\eta_{il}\hat{J}^l{}_j + \eta_{lj}\hat{J}^l{}_i = 0 \tag{2.131}$$

where η is the Minkowski metric and $\hat{J}$ is J in the new basis.

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Equation (2.131) shows that $\hat{J}$ defines an infinitesimal Lorentz transformation. Therefore, the isotropy subalgebra at a point can be identified with a subalgebra of the Lie algebra of the Lorentz group up to conjugation by the Lorentz group. The subalgebras of $\mathfrak{so}(3,1)$ have been classified up to change of basis by Winternitz and Zassenhaus [9] and labeled F1-F15 according to Table 2.1. In this way I classify the *isotropy type* of a given isometry group.

Table 2.1: Conjugacy classes of Subalgebras of $\mathfrak{so}(3,1)$ considered as 4×4 matrices which preserve the Minkowski metric η as classified in Winternitz and Zassenhaus.

F1:	$\{B_1, B_2, B_3, B_4, B_5, B_6\}$	F9:	$\{B_1, B_2\}$
F2:	$\{B_1, B_2, B_3, B_4\}$	F10:	$\{B_3, B_4\}$
F3:	$\{R_x, R_y, R_z\}$	F11:	$\{B(\theta)\}$
F4:	$\{R_z, K_x, K_y\}$	F12:	$\{R_z\}$
F5:	$\{B(\theta), B_3, B_4\}$	F13:	$\{K_z\}$
F6:	$\{B_1, B_3, B_4\}$	F14:	$\{R_y + K_z\}$
F7:	$\{B_2, B_3, B_4\}$	F15:	$\{0\}$
F8:	$\{B_2, B_3\}$		

$$R_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad R_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$B_1 = 2R_z \qquad B_2 = -2K_z \qquad B_3 = -R_y - K_z$

$B_4 = R_x - K_y \qquad B_5 = R_y - K_x \qquad B_6 = R_x + K_y \qquad B(\theta) = \cos(\theta)R_z - \sin(\theta)K_z$

Example 2.42. Here is an example to show the calculation of isotropy types. Equation (2.132) is from reference [2] equation (13.1)

$$ds^2 = -dt \otimes dt + A^2(t)dx \otimes dx + B^2(t)[dy \otimes dy + \sin^2(y)dz \otimes dz] \tag{2.132}$$

where $A(t)$ and $B(t)$ are arbitrary functions. The space of Killing vectors is four dimensional with the following basis.

$$
\begin{aligned}
&e_1 = \sin(z)\partial_y + \frac{\cos(z)\cos(y)}{\sin(y)}\partial_z, \quad e_2 = -\cos(z)\partial_y + \frac{\sin(z)\cos(y)}{\sin(y)}\partial_z, \\
&e_3 = \partial_z, \quad e_4 = \partial_x.
\end{aligned} \tag{2.133}
$$

The vector field $X = e_1$ is a basis for the isotropy subalgebra of the point p given by $(t = 0, x = 0, y = \frac{\pi}{2}, z = 0)$. From equation (2.130),

$$J^l{}_i \equiv \partial_i X^l\big|_p \tag{2.134}$$

$$J^y{}_z = \partial_z X^y\big|_p = \cos(z)\big|_p = 1 \tag{2.135}$$

$$J^z{}_y = \partial_y X^z\big|_p = \cos(z)(-\csc^2(y))\big|_p = -1 \tag{2.136}$$

$$J^z{}_z = \partial_z X^z\big|_p = -\sin(z)\frac{\cos(y)}{\sin(y)}\bigg|_p = 0. \tag{2.137}$$

Then, the matrix representation of J at the point P is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

which corresponds to infinitesimal rotation. Therefore this is F12 in Winternitz and Zassenhaus classification.

The isometry group determines a Lie algebra-subalgebra pair as shown previously. And it is also determined by the algebra-subalgebra pair [5]. I use Lie algebra-subalgebra pairs enumerated by Hicks up to change of basis to label all isometry groups. Two Lie algebra-subalgebra pairs are considered equivalent if they are related by change of basis.

CHAPTER 3

Classification of solutions

I now classify solutions of the Einstein equations according to their Lie algebra-subalgebra pairs. Following Hicks, the Lie algebra-subalgebra pairs are labeled by three numbers [A, B, C]. The first entry A is the dimension of the Lie algebra of Killing vector fields and it runs from zero to ten (from zero to ten-dimensional Lie algebra of Killing vector fields).[1] The second entry B is the number of linearly independent Killing vector fields at a given point p or equivalently the dimension of the orbit through p. I assume that the dimension of Killing vector fields is constant through out the solution and that the dimension of the orbit is also constant. Only consider when A and B are constant. C is a label based on the structure constants which identifies the Lie algebra-subalgebra pair (up to change of basis) in Hicks' classification.

As an example, consider the Schwarzschild solution in equation (1.2). This metric admits a four dimensional space of Killing vector fields with a basis given by

$$e_1 = \sin\phi\partial_\theta + \frac{\cos\phi\cos\theta}{\sin\theta}\partial_\phi, \quad e_2 = \cos\phi\partial_\theta - \frac{\sin\phi\cos\theta}{\sin\theta}\partial_\phi, \quad e_3 = \partial_\phi, \quad e_4 = \partial_t. \tag{3.1}$$

Therefore, in the label [A, B, C], A is four because the dimension of the Lie algebra of Killing vector fields is four. Next consider a base point p given by $(t = 0, r = 3m, \theta = \frac{\pi}{2}, \phi = 0)$ where

$$e_1|_p = 0|_p, \quad e_2|_p = \partial_\theta|_p, \quad e_3|_p = \partial_\phi|_p, \quad e_4|_p = \partial_t|_p. \tag{3.2}$$

Since e_1 vanishes at p, it is a basis for the isotropy subalgebra at p. The number of linearly independent Killing vectors at the base point is three. Therefore, the second entry B is

[1] Eight and nine-dimension of Lie algebras do not occur in 4D spacetime [4].

three. The Lie brackets of the Killing vector fields are

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_1, e_4] = 0, \tag{3.3}$$

$$[e_2, e_3] = e_1, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = 0. \tag{3.4}$$

By comparing these Lie brackets and isotropy with Hicks [5] as shown in Figure 3.1, it is found that they are identical to [4, 3, 3], therefore the Schwarzschild solution is classified as [4, 3, 3].

[4, 3, 3]

	e_1	e_2	e_3	e_4
e_1	.	e_3	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

Reference : R(4,2) b=0, Bowers

Isotropy: $[e_1]$, F12

Figure 3.1: Hicks' classification of [4, 3, 3]

As another example, consider the following metric [2]

$$ds^2 = Y^2(w)\frac{2d\zeta d\bar{\zeta}}{[1+\frac{1}{2}k\zeta\bar{\zeta}]^2} + \frac{dw^2}{f(w)} - \left[dt + il\frac{\zeta d\bar{\zeta} - \bar{\zeta}d\zeta}{1+\frac{1}{2}k\zeta\bar{\zeta}}\right]^2 \tag{3.5}$$

where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. It defines a class of stationary dust solutions of Ellis' Class I with F12 isotropy type. Here $\Lambda > 0, Y^2 = a\cos\beta w + b\sin\beta w + \frac{k}{2\Lambda}, \Lambda(a^2+b^2) = l^2 + \frac{k^2}{4\Lambda}$. There are three cases: $k < 0, k = 0$, or $k > 0$. Let us consider when $k = 1$ and $l = 1$. The original coordinates $(t, w, \zeta, \bar{\zeta})$ can be changed to new coordinates (t, w, x, y) defined by

$$x = \frac{1}{2}(\zeta + \bar{\zeta}), y = \frac{1}{2i}(\zeta - \bar{\zeta}) \tag{3.6}$$

to remove imaginary numbers. Then the solution is

$$\begin{aligned} ds^2 = &-dt^2 + dw^2 + \frac{8}{x^2+y^2+2}(ydtdx - xdtdy) \\ &+ \frac{8}{(x^2+y^2+2)^2}\Big[\big(Y^2(w) - 2y^2\big)dx^2 + \big(Y^2(w) - 2x^2\big)dy^2 + 4xydxdy\Big]. \end{aligned} \tag{3.7}$$

A basis for the vector space of Killing vector fields of equation (3.7) is

$$\begin{aligned} e_1 &= \frac{y}{16}\partial_x - \frac{x}{16}\partial_y, \quad e_2 = -\frac{y}{16}\partial_t + \frac{x^2-y^2+2}{64}\partial_x + \frac{xy}{32}\partial_y, \\ e_3 &= -\frac{1}{8}\partial_t - \frac{y}{32}\partial_x + \frac{x}{32}\partial_y, \quad e_4 = -\frac{x}{16}\partial_t - \frac{xy}{32}\partial_x + \frac{x^2-y^2-2}{64}\partial_y. \end{aligned} \tag{3.8}$$

Let us pick a base point to be $(t = 0, w = 0, x = 0, y = 0)$ and the vector fields equation (3.8) evaluated at this point become

$$e_1|_p = 0|_p, \quad e_2|_p = \frac{1}{32}\partial_x|_p, \quad e_3|_p = -\frac{1}{8}\partial_t|_p, \quad e_4|_p = -\frac{1}{32}\partial_y|_p. \tag{3.9}$$

So there are three linearly independent vectors at that point. Twenty candidates exist with $[4,\, 3,\, \cdot\,]$, but they can be reduced to seven because Hicks' $[4, 3, 1]$ through $[4, 3, 7]$ are the only ones with F12 isotropy type. The candidates can be narrowed down further by checking whether the Lie algebra can be decomposed. It can be shown that this Lie algebra can be decomposed into $\mathfrak{so}(3) \oplus \mathbb{R}$ Lie algebra with $\mathbb{R}$ being the center by the following change of basis

$$e'_1 = e_1 - 2e_3, \quad e'_2 = e_2, \quad e'_3 = e_4, \quad e'_4 = e_1 + 2e_3. \tag{3.10}$$

The Lie brackets in this basis are

$$[e'_1, e'_2] = -\frac{1}{8}e'_3, \quad [e'_1, e'_3] = \frac{1}{8}e'_2, \quad [e'_1, e'_4] = 0, \tag{3.11}$$

$$[e'_2, e'_3] = -\frac{1}{64}e'_1, \quad [e'_2, e'_4] = 0, \quad [e'_3, e'_4] = 0. \tag{3.12}$$

To compare with Hicks' classification I begin by making the following rescaling

$$e_1'' = 8e_1', \quad e_2'' = 16\sqrt{2}e_2', \quad e_3'' = 16\sqrt{2}e_3', \quad e_4'' = 8e_4'. \tag{3.13}$$

Now the isotropy subalgebra has been changed to $e_1'' + e_4''$ and the Lie brackets are

$$\begin{aligned} &[e_1'', e_2''] = e_3'', \quad [e_1'', e_3''] = -e_2'', \quad [e_1'', e_4''] = 0, \\ &[e_2'', e_3''] = e_1'', \quad [e_2'', e_4''] = 0, \quad [e_3'', e_4''] = 0. \end{aligned} \tag{3.14}$$

The Lie brackets in equation (3.14) and isotropy match those in Hicks [5] labeled [4, 3, 4] as shown in Figure 3.2. Therefore, this solution is classified as [4, 3, 4].

[4, 3, 4]

	e_1	e_2	e_3	e_4
e_1	.	e_3	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

Reference : R(4,2) b=1, Bowers

Isotropy: $[e_1 + e_4]$, F12

Figure 3.2: Hicks' classification of [4, 3, 4]

This classification process has been automated in MAPLE by Hicks and Anderson independently. Hicks' code takes Killing vector fields and a base point as input and Anderson's code takes Killing vector fields, isotropy subalgebra, and isotropy type. About 300 solutions were classified by Hicks' and 40 solutions were classified by Anderson's. Twenty of them were classified by both codes and no discrepancies have been found. There were some cases where the automated classification was not used because no closed form expression for the Killing vector fields was available or other technical difficulty. In those cases, the solutions have been classified manually as shown above.

CHAPTER 4

Results

In this thesis I have classified the isometry algebras of many solutions of the Einstein equations found in Stephani et al [2]. The solutions being classified are taken from chapter 8, and from chapter 12 though chapter 37. Most solutions in the reference [2] chapter 8, 12 (homogeneous space-times), 13 (hypersurface-homogeneous space-times), 14 (spatially-homogeneous perfect fluid cosmologies), and 28 (Robinson-Trautman metrics) have been classified whereas a subset of solutions from other chapters have been classified.

I am using the labeling of solutions in the USU electronic library. This labeling is in turn based upon equation numbers in Stephani et al. [2] where the first number is the chapter and the second number is equation number and the third number is the sub-case number. For example, solution [12, 8, 2] is the second case of equation (12.8) in the reference [2]. Therefore, the first number of the solutions labeling runs from 8 to 37 (from chapter 8 to chapter 37 in reference [2]).

I also indicate in the results the type of solutions, vacuum, Einstein, Einstein-Maxwell, pure radiation, perfect fluid, and generic type. I will briefly describe the meaning of these types. A vacuum solution satisfies

$$R_{\mu\nu} = 0, \tag{4.1}$$

which is equivalent to equation (1.1) with $T_{\mu\nu} = 0 = \Lambda$. An Einstein solution means $T_{\mu\nu} = 0$ but $\Lambda \neq 0$. Einstein-Maxwell solutions have the following energy-momentum tensor

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{4.2}$$

where F is anti-symmetric and satisfies the source free Maxwell equation.

$$F_{\mu\nu} = -F_{\nu\mu} \tag{4.3}$$

$$F_{\mu\ ;\nu}^{\ \nu} = 0 = F_{[\mu\nu;\sigma]}. \tag{4.4}$$

Pure radiation solutions have an energy-momentum tensor of the form

$$T_{\mu\nu} = \Phi^2 k_\mu k_\nu, \quad k_\mu k^\mu = 0. \tag{4.5}$$

Perfect fluid solutions have an energy-momentum tensor of the form

$$T_{\mu\nu} = (\alpha + p)u_\mu u_\nu + p g_{\mu\nu}, \quad u_\mu u^\mu = -1. \tag{4.6}$$

where α and p are functions and u satisfies the normalization condition. Occasionally, I have classified metrics of type generic which means no field equations have been imposed.

The solutions in reference [2] chapter 8 have ten-dimensional isometry algebras. This is the maximum dimension the isometry algebra can have in four-dimensional spacetime. A ten-dimensional isometry algebra means the spacetime has constant curvature. If the curvature is positive, the spacetime is de Sitter space and the isometry algebra is the de Sitter algebra. If the curvature is negative, it is anti de Sitter space and the isometry algebra is the anti de Sitter algebra. If the curvature is zero, the spacetime is Minkowski spacetime and the isometry algebra is the Poincaré Lie algebra.

Some solutions have two dimensional isometry algebras. There are only two possibilities up to change of basis for such algebras, abelian and non-abelian. The abelian algebras have been labeled as [2, 2, 1] and non-abelian as [2, 2, 2]. There can be no isotropy subalgebra for two dimensional Lie algebra [4]. One dimensional isometry algebras are labeled as [1-dim.] and zero dimensional isometry algebras or solutions with no Killing vector fields are labeled as [0-dim.].

Hicks classified simple G spacetimes, but $[12, 7, 1], [12, 37, 1], [12, 37, 2]$ and $[15, 18, 1]$ are not simple G spacetimes so they cannot be classified using Hicks' method (algebra-subalgebra pairs). To see that these are not simple G spaces, I use the fact [5] that at each point of a simple G space the group orbit through that point will admit a complement with a G-invariant basis to its tangent space. In solutions [12, 7, 1], [12, 37, 1], [12, 37, 2], and

[15, 18, 1] it is easily verified that no such complement exists. Therefore these solutions are not simple G spaces and are not covered by Hicks' classification.

As a simple G space example, let us consider the Schwarzschild spacetime in equation (1.2). It has four dimensional tangent space and three dimensional tangent space to the orbit. The three dimensional tangent space to the orbit through any point is spanned by $\partial_\theta, \partial_\phi, \partial_t$. It follows that the complement to the tangent space to the orbit is one dimensional and the general form of a basis of the complement is of the form

$$X = \partial_r + a\partial_t + b\partial_\theta + c\partial_\phi \tag{4.7}$$

for any choice of a, b, and c. The Lie derivative of the isotropy subalgebra with respect to the complement is

$$-c\cos\phi\partial_\theta + \frac{c\sin\phi\cos\theta\sin\theta + b\cos\phi}{\sin^2\theta}\partial_\phi \tag{4.8}$$

and it vanishes when b and c are zero.

As an example of the solutions which is not simple G, consider the solution [12, 37, 1], as shown in equation (4.9)

$$ds^2 = C^2(u)(dx^2 + dy^2) - 2dudv, \tag{4.9}$$

where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ and $C(u)$ is a second order differentiable function. It has orbits when u is constant and the tangent space at each point to the orbit is spanned by ∂_x, ∂_y, ∂_v. The complement to the tangent space to the orbit is one dimensional and the general form of a basis of the complement is

$$X = \partial_u + a\partial_v + b\partial_x + c\partial_y \tag{4.10}$$

for any choice of a, b, and c. For the vector (4.10) to be invariant, the Lie derivative of the isotropy subalgebra with respect to the complement must vanish. The basis of the isotropy subalgebra is

$$e_1 = y\partial_v + \left(\int_0^u \frac{1}{C(s)^2}ds\right)\partial_y, \quad e_2 = -y\partial_x + x\partial_y, \quad e_3 = x\partial_v + \left(\int_0^u \frac{1}{C(s)^2}ds\right)\partial_x \tag{4.11}$$

and Lie brackets are

$$[X, e_1] = -c\partial_v - \frac{1}{C^2(u)}\partial_y, \quad [X, e_2] = c\partial_x - b\partial_y, \quad [X, e_3] = -b\partial_v - \frac{1}{C^2(u)}\partial_x. \tag{4.12}$$

Because $\frac{1}{C(u)}$ does not vanish, the complement cannot be chosen to be G-invariant.

I have found that certain solutions in the library need to be split into sub-cases because they have distinct isometry groups.

$$\begin{aligned} ds^2 &= 2x^{-2}[\left(\frac{dx}{A}\right)^2 + A^2dy^2 + \left(\frac{dz}{B}\right)^2 - B^2dt^2], \\ A^2 &= ax^2 + cx^3 - 2e^2x^4, \quad B^2 = b - az^2, \quad a, b, c, e = const. \end{aligned} \tag{4.13}$$

$$\begin{aligned} ds^2 &= 2x^{-2}[\left(A^2 - \frac{2}{3}\Lambda\right)^{-1} dx^2 + \left(A^2 - \frac{2}{3}\Lambda\right) dy^2 + \left(\frac{dz}{B}\right)^2 - B^2dt^2], \\ A^2 &= ax^2 + cx^3 - 2e^2x^4, \quad B^2 = b - az^2, \quad a, b, c, e = const. \end{aligned} \tag{4.14}$$

where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Equation (4.13) and (4.14) are solutions [31, 60, 1] and [31, 60, 2] in the reference [2] respectively. These solutions need to be split into sub-cases depending on the value of a. Equation (4.13) and equation (4.14) are classified as [4, 3, 1] with $a > 0$, and [4, 3, 8] with $a < 0$.

$$ds^2 = 2d\zeta d\bar{\zeta}[1 + \lambda\zeta\bar{\zeta}]^{-2} - 2dudv[1 + \lambda uv]^{-2} \tag{4.15}$$

where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Equation (4.15) is solution [35, 35, 1] in reference [2] and it is classified as [6, 4, 1] with $\lambda > 0$, and [6, 4, 2] with $\lambda < 0$.

Table 4.1 through Table 4.13 shows the classification results of the solutions from Stephani et al. [2]

Table 4.1: Stephani et al. Chapter 8. Maximally Symmetric Solutions

Solution	Type	Classification
[8, 33, 1]	Vacuum	[Poincare]
[8, 33, 2]	Einstein	[de Sitter]
[8, 33, 3]	Einstein	[anti de Sitter]

Table 4.2: Stephani et al. Chapter 12. Homogeneous Spacetimes

Solution	Type	Classification
[12, 6, 1]	Pure Radiation	[6, 4, -1]
[12, 7, 1]	Einstein-Maxwell	[N/A]
[12, 8, 1]	Einstein	[6, 4, 1]
[12, 8, 2]	Generic	[6, 4, 4]
[12, 8, 3]	Generic	[6, 4, 1]
[12, 8, 4]	Generic	[6, 4, 3]
[12, 8, 5]	Generic	[6, 4, 3]
[12, 8, 6]	Generic	[6, 4, 2]
[12, 8, 7]	Generic	[6, 4, 5]
[12, 8, 8]	Einstein	[6, 4, 2]
[12, 9, 1]	Generic	[6, 3, 1]
[12, 9, 2]	Generic	[6, 3, 2]
[12, 9, 3]	Generic	[6, 3, 3]
[12, 12, 1]	Einstein-Maxwell	[6, 4, 6]
[12, 12, 2]	Einstein-Maxwell	[7, 4, 5]
[12, 12, 3]	Einstein-Maxwell	[6, 4, 6]
[12, 12, 4]	Einstein-Maxwell	[7, 4, 5]
[12, 13, 1]	Vacuum	[6, 4, 6]
[12, 14, 1]	Vacuum	[4, 4, 15]
[12, 16, 1]	Einstein-Maxwell	[6, 4, 1]
[12, 18, 1]	Einstein-Maxwell	[6, 4, 1]
[12, 19, 1]	Einstein-Maxwell	[6, 4, 1]
[12, 21, 1]	Einstein-Maxwell	[4, 4, 18]
[12, 23, 1]	Perfect Fluid	[7, 4, 1]
[12, 23, 2]	Perfect Fluid	[7, 4, 2]
[12, 24.1, 1]	Perfect Fluid	[7, 4, 1]

Solution	Type	Classification
[12, 24.2, 1]	Perfect Fluid	[7, 4, 1]
[12, 24.3, 1]	Perfect Fluid	[7, 4, 1]
[12, 26, 1]	Perfect Fluid	[5, 4, 1]
[12, 27, 1]	Perfect Fluid	[4, 4, 1]
[12, 28, 1]	Perfect Fluid	[4, 4, 2]
[12, 29, 1]	Perfect Fluid	[4, 4, 2]
[12, 30, 1]	Perfect Fluid	[4, 4, 10]
[12, 31, 1]	Perfect Fluid	[4, 4, 12]
[12, 32, 1]	Perfect Fluid	[4, 4, 15]
[12, 34, 1]	Einstein	[5, 4, -2]
[12, 35, 1]	Einstein	[4, 4, 10]
[12, 36, 1]	Einstein-Maxwell	[5, 4, -2]
[12, 37, 1]	Pure Radiation	[N/A]
[12, 37, 2]	Einstein-Maxwell	[N/A]
[12, 37, 3]	Pure Radiation	[7, 4, 5]
[12, 37, 4]	Einstein-Maxwell	[7, 4, 5]
[12, 37, 5]	Pure Radiation	[7, 4, 5]
[12, 37, 6]	Einstein-Maxwell	[7, 4, 5]
[12, 37, 7]	Einstein-Maxwell	[7, 4, 5]
[12, 37, 8]	Einstein-Maxwell	[7, 4, 5]
[12, 37, 9]	Einstein-Maxwell	[7, 4, 5]
[12, 38, 1]	Pure Radiation	[5, 4, -2]
[12, 38, 2]	Pure Radiation	[5, 4, -4]
[12, 38, 3]	Einstein-Maxwell	[5, 4, -2]
[12, 38, 4]	Pure Radiation	[5, 4, -2]
[12, 38, 5]	Einstein	[5, 4, -2]

Table 4.3: Stephani et al. Chapter 13. Hypersurface-homogeneous Spacetimes

Solution	Type	Classification
[13, 1, 1]	Generic	[4, 3, 3]
[13, 1, 2]	Generic	[4, 3, 6]
[13, 1, 3]	Generic	[4, 3, 1]
[13, 2, 1]	Generic	[4, 3, 4]
[13, 2, 2]	Generic	[4, 3, 5]
[13, 2, 3]	Generic	[4, 3, 2]
[13, 3, 1]	Generic	[4, 3, 7]
[13, 9, 1]	Generic	[4, 3, 9]
[13, 9, 2]	Generic	[4, 3, 10]
[13, 9, 3]	Generic	[4, 3, 9]
[13, 14, 1]	Generic	[4, 3, 8]
[13, 14, 2]	Generic	[4, 3, 11]
[13, 14, 3]	Generic	[4, 3, 8]
[13, 15, 1]	Generic	[4, 3, 12]
[13, 15, 2]	Generic	[4, 3, 10]
[13, 15, 3]	Generic	[3, 3, 8]
[13, 15, 4]	Generic	[3, 3, 8]
[13, 17, 1]	Generic	[4, 3, 9]
[13, 17, 2]	Generic	[4, 3, 10]
[13, 17, 3]	Generic	[4, 3, 9]
[13, 19, 1]	Generic	[4, 3, 20]
[13, 20, 1]	Generic	[3, 3, 2]
[13, 20, 2]	Generic	[3, 3, 3]
[13, 20, 3]	Generic	[3, 3, 1]
[13, 20, 4]	Generic	[3, 3, 6]
[13, 20, 5]	Generic	[3, 3, 5]

Solution	Type	Classification
[13, 20, 6]	Generic	[3, 3, 4]
[13, 20, 7]	Generic	[3, 3, 7]
[13, 20, 8]	Generic	[3, 3, 8]
[13, 20, 9]	Generic	[3, 3, 9]
[13, 31, 1]	Generic	[3, 3, 2]
[13, 32, 1]	Generic	[3, 3, 5]
[13, 46, 1]	Einstein-Maxwell	[4, 3, 7]
[13, 48, 1]	Einstein-Maxwell	[4, 3, 4]
[13, 48, 2]	Einstein-Maxwell	[4, 3, 5]
[13, 48, 3]	Einstein-Maxwell	[4, 3, 2]
[13, 49, 1]	Vacuum	[4, 3, 4]
[13, 49, 2]	Vacuum	[4, 3, 4]
[13, 51, 1]	Vacuum	[3, 3, 2]
[13, 53, 1]	Vacuum	[3, 3, 2]
[13, 55, 1]	Vacuum	[3, 3, 3]
[13, 55, 2]	Vacuum	[3, 3, 3]
[13, 56, 1]	Vacuum	[3, 3, 3]
[13, 56, 2]	Vacuum	[3, 3, 3]
[13, 57, 1]	Vacuum	[3, 3, 4]
[13, 57, 2]	Vacuum	[3, 3, 4]
[13, 58, 1]	Vacuum	[3, 3, 4]
[13, 58, 2]	Vacuum	[3, 3, 4]
[13, 59, 1]	Vacuum	[3, 3, 5]
[13, 59, 2]	Vacuum	[3, 3, 5]
[13, 60, 1]	Vacuum	[3, 3, 4]
[13, 60, 2]	Vacuum	[3, 3, 7]
[13, 61, 1]	Vacuum	[3, 3, 7]
[13, 61, 2]	Vacuum	[3, 3, 4]

Solution	Type	Classification
[13, 62, 1]	Vacuum	[3, 3, 7]
[13, 63, 1]	Einstein	[3, 3, 2]
[13, 64, 1]	Einstein	[3, 3, 4]
[13, 64, 2]	Vacuum	[3, 3, 4]
[13, 64, 3]	Einstein	[3, 3, 4]
[13, 64, 4]	Einstein	[3, 3, 4]
[13, 65, 1]	Einstein	[3, 3, 4]
[13, 67, 1]	Einstein	[3, 3, 4]
[13, 67, 2]	Einstein	[3, 3, 4]
[13, 69, 1]	Einstein-Maxwell	[3, 3, 2]
[13, 71, 1]	Einstein-Maxwell	[3, 3, 3]
[13, 71, 2]	Einstein-Maxwell	[3, 3, 3]
[13, 71, 3]	Einstein-Maxwell	[3, 3, 3]
[13, 72, 1]	Einstein-Maxwell	[3, 3, 4]
[13, 73, 1]	Einstein-Maxwell	[3, 3, 7]
[13, 74, 1]	Einstein-Maxwell	[3, 3, 3]
[13, 74, 2]	Einstein-Maxwell	[3, 3, 3]
[13, 74, 3]	Einstein-Maxwell	[3, 3, 3]
[13, 74, 4]	Einstein-Maxwell	[3, 3, 3]
[13, 74, 5]	Einstein-Maxwell	[3, 3, 3]
[13, 74, 6]	Einstein-Maxwell	[3, 3, 3]
[13, 76, 1]	Einstein-Maxwell	[3, 3, 3]
[13, 77, 1]	Einstein-Maxwell	[3, 3, 3]
[13, 77, 2]	Einstein-Maxwell	[3, 3, 3]
[13, 79, 1]	Perfect Fluid	[4, 3, 4]
[13, 79, 2]	Perfect Fluid	[4, 3, 5]
[13, 79, 3]	Perfect Fluid	[4, 3, 2]
[13, 80, 1]	Perfect Fluid	[4, 3, 4]

Solution	Type	Classification
[13, 80, 2]	Perfect Fluid	[4, 3, 5]
[13, 80, 3]	Perfect Fluid	[4, 3, 2]
[13, 81, 1]	Perfect Fluid	[4, 3, 4]
[13, 81, 2]	Perfect Fluid	[4, 3, 5]
[13, 81, 3]	Perfect Fluid	[4, 3, 2]
[13, 82, 1]	Perfect Fluid	[4, 3, 4]
[13, 82, 2]	Perfect Fluid	[4, 3, 5]
[13, 82, 3]	Perfect Fluid	[4, 3, 2]
[13, 83, 1]	Perfect Fluid	[4, 3, 4]
[13, 84, 1]	Perfect Fluid	[3, 3, 3]
[13, 84, 2]	Perfect Fluid	[3, 3, 3]
[13, 84, 3]	Perfect Fluid	[3, 3, 3]
[13, 85, 1]	Perfect Fluid	[3, 3, 3]
[13, 86, 1]	Perfect Fluid	[3, 3, 4]
[13, 87, 1]	Perfect Fluid	[3, 3, 7]

Table 4.4: Stephani et al. Chapter 14. Spatially-homogeneous Perfect Fluid Cosmologies

Solution	Type	Classification
[14, 6.1, 1]	Dust	[6, 3, 1]
[14, 6.2, 1]	Dust	[6, 3, 2]
[14, 6.3, 1]	Dust	[6, 3, 3]
[14, 7, 1]	Perfect Fluid	[6, 3, 2]
[14, 8.1, 1]	Perfect Fluid	[6, 3, 2]
[14, 8.2, 1]	Perfect Fluid	[6, 3, 2]
[14, 8.3, 1]	Perfect Fluid	[6, 3, 2]
[14, 9.1, 1]	Perfect Fluid	[6, 3, 1]
[14, 9.2, 1]	Perfect Fluid	[6, 3, 3]
[14, 10, 1]	Perfect Fluid	[6, 3, 1]
[14, 10, 2]	Perfect Fluid	[6, 3, 1]
[14, 12, 1]	Perfect Fluid	[6, 3, 1]
[14, 12, 2]	Perfect Fluid	[6, 3, 2]
[14, 12, 3]	Perfect Fluid	[6, 3, 3]
[14, 14, 1]	Perfect Fluid	[4, 3, 6]
[14, 14, 2]	Perfect Fluid	[4, 3, 6]
[14, 15, 1]	Perfect Fluid	[4, 3, 3]
[14, 15, 2]	Perfect Fluid	[4, 3, 1]
[14, 15, 3]	Perfect Fluid	[4, 3, 1]
[14, 15, 4]	Perfect Fluid	[4, 3, 1]
[14, 16, 1]	Perfect Fluid	[4, 3, 3]
[14, 16, 2]	Perfect Fluid	[4, 3, 1]
[14, 17, 1]	Perfect Fluid	[4, 3, 3]
[14, 18, 1]	Perfect Fluid	[4, 3, 3]
[14, 18, 2]	Perfect Fluid	[4, 3, 1]
[14, 19, 1]	Perfect Fluid	[4, 3, 1]

Solution	Type	Classification
[14, 20, 1]	Perfect Fluid	[4, 3, 5]
[14, 21, 1]	Perfect Fluid	[4, 3, 4]
[14, 21, 2]	Perfect Fluid	[4, 3, 5]
[14, 21, 3]	Perfect Fluid	[4, 3, 2]
[14, 22, 1]	Perfect Fluid	[4, 3, 7]
[14, 23, 1]	Perfect Fluid	[4, 3, 2]
[14, 24, 1]	Perfect Fluid	[4, 3, 7]
[14, 25, 1]	Perfect Fluid	[4, 3, 4]
[14, 26, 1]	Perfect Fluid	[3, 3, 2]
[14, 26, 2]	Perfect Fluid	[3, 3, 2]
[14, 26, 3]	Perfect Fluid	[3, 3, 2]
[14, 27, 1]	Perfect Fluid	[3, 3, 2]
[14, 28, 1]	Perfect Fluid	[3, 3, 2]
[14, 28, 2]	Perfect Fluid	[3, 3, 2]
[14, 28, 3]	Perfect Fluid	[3, 3, 2]
[14, 29, 1]	Perfect Fluid	[3, 3, 3]
[14, 30, 1]	Perfect Fluid	[3, 3, 3]
[14, 31, 1]	Perfect Fluid	[3, 3, 3]
[14, 32, 1]	Perfect Fluid	[3, 3, 5]
[14, 33, 1]	Perfect Fluid	[3, 3, 4]
[14, 34, 1]	Generic	[3, 3, 4]
[14, 35, 1]	Perfect Fluid	[3, 3, 4]
[14, 37, 1]	Perfect Fluid	[3, 3, 4]
[14, 38, 1]	Perfect Fluid	[3, 3, 4]
[14, 38, 2]	Perfect Fluid	[3, 3, 4]
[14, 38, 3]	Perfect Fluid	[3, 3, 4]
[14, 39, 1]	Perfect Fluid	[3, 3, 4]
[14, 39, 2]	Perfect Fluid	[3, 3, 4]

Solution	Type	Classification
[14, 39, 3]	Perfect Fluid	[3, 3, 4]
[14, 39, 4]	Perfect Fluid	[3, 3, 4]
[14, 39, 5]	Perfect Fluid	[3, 3, 4]
[14, 39, 6]	Perfect Fluid	[3, 3, 4]
[14, 40, 1]	Perfect Fluid	[3, 3, 4]
[14, 41, 1]	Perfect Fluid	[3, 3, 4]
[14, 42, 1]	Perfect Fluid	[3, 3, 4]
[14, 44, 1]	Perfect Fluid	[3, 3, 4]
[14, 45, 1]	Perfect Fluid	[3, 3, 7]
[14, 46, 1]	Perfect Fluid	[3, 3, 4]

Table 4.5: Stephani et al. Chapter 15. Spherical and Plane Symmetry

Solution	Type	Classification
[15, 12, 1]	Einstein-Maxwell	[4, 3, 8]
[15, 12, 2]	Einstein-Maxwell	[4, 3, 11]
[15, 12, 3]	Einstein-Maxwell	[4, 3, 8]
[15, 17, 1]	Einstein-Maxwell	[3, 2, 3]
[15, 17, 2]	Einstein-Maxwell	[3, 2, 1]
[15, 17, 3]	Einstein-Maxwell	[3, 2, 2]
[15, 18, 1]	Einstein-Maxwell	[N/A]
[15, 19, 1]	Vacuum	[4, 3, 3]
[15, 19, 2]	Vacuum	[4, 3, 3]
[15, 20, 1]	Pure Radiation	[3, 2, 3]
[15, 21, 1]	Einstein-Maxwell	[4, 3, 3]
[15, 21, 2]	Einstein-Maxwell	[4, 3, 3]
[15, 22, 1]	Vacuum	[4, 3, 3]
[15, 23, 1]	Vacuum	[4, 3, 3]
[15, 23, 2]	Vacuum	[4, 3, 3]
[15, 24, 1]	Vacuum	[4, 3, 3]
[15, 24, 2]	Vacuum	[4, 3, 3]
[15, 26, 1]	Vacuum	[4, 3, 3]
[15, 26, 2]	Vacuum	[4, 3, 3]
[15, 27, 1]	Generic	[3, 2, 1]
[15, 27, 2]	Einstein-Maxwell	[4, 3, 6]
[15, 27, 3]	Einstein-Maxwell	[4, 3, 6]
[15, 27, 4]	Einstein-Maxwell	[3, 2, 1]
[15, 27, 5]	Einstein-Maxwell	[4, 3, 7]
[15, 27, 6]	Einstein-Maxwell	[4, 3, 7]
[15, 27, 7]	Vacuum	[4, 3, 6]

Solution	Type	Classification
[15, 27, 8]	Vacuum	[4, 3, 6]
[15, 28, 1]	Einstein-Maxwell	[4, 3, 6]
[15, 29, 1]	Vacuum	[4, 3, 6]
[15, 30, 1]	Vacuum	[4, 3, 6]
[15, 31, 1]	Einstein	[4, 3, 6]
[15, 32, 1]	Einstein-Maxwell	[3, 2, 1]
[15, 50, 1]	Perfect Fluid	[4, 3, 6]
[15, 50, 2]	Perfect Fluid	[4, 3, 6]
[15, 50, 3]	Perfect Fluid	[4, 3, 3]
[15, 50, 4]	Perfect Fluid	[4, 3, 3]
[15, 50, 5]	Perfect Fluid	[4, 3, 1]
[15, 50, 6]	Perfect Fluid	[4, 3, 1]
[15, 59, 3]	Perfect Fluid	[3, 2, 2]
[15, 65, 1]	Perfect Fluid	[3, 2, 3]
[15, 65, 2]	Perfect Fluid	[3, 2, 3]
[15, 75, 1]	Perfect Fluid	[3, 2, 3]
[15, 75, 2]	Perfect Fluid	[3, 2, 1]
[15, 75, 3]	Perfect Fluid	[3, 2, 2]
[15, 78, 1]	Perfect Fluid	[4, 3, 6]
[15, 79, 1]	Perfect Fluid	[4, 3, 6]
[15, 81, 1]	Perfect Fluid	[3, 2, 1]
[15, 81, 2]	Perfect Fluid	[3, 2, 1]
[15, 81, 3]	Perfect Fluid	[3, 2, 1]
[15, 88, 1]	Perfect Fluid	[3, 2, 1]

Table 4.6: Stephani et al. Chapter 16. Spherically-symmetric Perfect Fluid Solutions

Solution	Type	Classification
[16, 1, 1]	Perfect Fluid	[4, 3, 3]
[16, 1, 2]	Perfect Fluid	[4, 3, 3]
[16, 1, 3]	Perfect Fluid	[4, 3, 3]
[16, 1, 4]	Perfect Fluid	[4, 3, 3]
[16, 1, 5]	Perfect Fluid	[4, 3, 3]
[16, 1, 6]	Perfect Fluid	[4, 3, 3]
[16, 1, 7]	Perfect Fluid	[4, 3, 3]
[16, 1, 8]	Perfect Fluid	[4, 3, 3]
[16, 1, 9]	Perfect Fluid	[4, 3, 3]
[16, 1, 10]	Perfect Fluid	[4, 3, 3]
[16, 1, 11]	Perfect Fluid	[4, 3, 3]
[16, 1, 12]	Perfect Fluid	[4, 3, 3]
[16, 1, 13]	Perfect Fluid	[4, 3, 3]
[16, 1, 14]	Perfect Fluid	[4, 3, 3]
[16, 1, 15]	Perfect Fluid	[4, 3, 3]
[16, 1, 16]	Perfect Fluid	[4, 3, 3]
[16, 1, 17]	Perfect Fluid	[4, 3, 3]
[16, 1, 18]	Perfect Fluid	[4, 3, 3]
[16, 1, 19]	Perfect Fluid	[4, 3, 3]
[16, 1, 20]	Perfect Fluid	[4, 3, 3]
[16, 1, 21]	Perfect Fluid	[4, 3, 3]
[16, 1, 22]	Perfect Fluid	[4, 3, 3]
[16, 1, 23]	Perfect Fluid	[4, 3, 3]
[16, 1, 24]	Perfect Fluid	[4, 3, 3]
[16, 1, 25]	Perfect Fluid	[4, 3, 3]
[16, 1, 26]	Perfect Fluid	[4, 3, 3]

Solution	Type	Classification
[16, 1, 27]	Perfect Fluid	[4, 3, 3]
[16, 14, 1]	Perfect Fluid	[4, 3, 3]
[16, 14, 2]	Perfect Fluid	[4, 3, 3]
[16, 14, 3]	Perfect Fluid	[4, 3, 3]
[16, 14, 4]	Perfect Fluid	[4, 3, 3]
[16, 14, 5]	Perfect Fluid	[4, 3, 3]
[16, 14, 6]	Perfect Fluid	[4, 3, 3]
[16, 14, 7]	Perfect Fluid	[4, 3, 3]
[16, 14, 8]	Perfect Fluid	[4, 3, 3]
[16, 14, 9]	Perfect Fluid	[4, 3, 3]
[16, 14, 10]	Perfect Fluid	[4, 3, 3]
[16, 14, 11]	Perfect Fluid	[4, 3, 3]
[16, 14, 12]	Perfect Fluid	[4, 3, 3]
[16, 14, 13]	Perfect Fluid	[4, 3, 3]
[16, 14, 14]	Perfect Fluid	[4, 3, 3]
[16, 14, 15]	Perfect Fluid	[4, 3, 3]
[16, 14, 16]	Perfect Fluid	[4, 3, 3]
[16, 14, 17]	Perfect Fluid	[4, 3, 3]
[16, 14, 18]	Perfect Fluid	[4, 3, 3]
[16, 14, 19]	Perfect Fluid	[4, 3, 3]
[16, 14, 20]	Perfect Fluid	[4, 3, 3]
[16, 18, 1]	Perfect Fluid	[4, 3, 3]
[16, 19, 1]	Perfect Fluid	[4, 3, 3]
[16, 24, 1]	Generic	[3, 2, 3]
[16, 43, 1]	Perfect Fluid	[3, 2, 3]
[16, 67, 1]	Perfect Fluid	[3, 2, 3]
[16, 76, 1]	Perfect Fluid	[3, 2, 3]

Table 4.7: Stephani et al. Chapter 21. Non-empty Stationary Axisymmetric Solutions

Solution	Type	Classification
[21, 5, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 6, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 7, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 10, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 11, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 16, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 17, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 20, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 22, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 24, 1]	Einstein-Maxwell	[2, 2, 1]
[21, 31, 1]	Einstein-Maxwell	[4, 3, 3]
[21, 35, 1]	Einstein-Maxwell	[2, 2, 1]

Table 4.8: Stephani et al. Chapter 22. Cylindrical Symmetry

Solution	Type	Classification
[22, 11, 1]	Einstein-Maxwell	[3, 3, 2]
[22, 12, 1]	Einstein-Maxwell	[3, 3, 2]
[22, 13, 1]	Einstein-Maxwell	[4, 3, 11]
[22, 14, 1]	Einstein-Maxwell	[3, 3, 2]
[22, 15, 1]	Einstein-Maxwell	[3, 3, 2]
[22, 16, 1]	Einstein-Maxwell	[3, 3, 2]
[22, 17, 1]	Einstein-Maxwell	[3, 3, 2]
[22, 64, 1]	Einstein-Maxwell	[2, 2, 1]
[22, 67, 1]	Einstein-Maxwell	[2, 2, 1]
[22, 67, 2]	Einstein-Maxwell	[2, 2, 1]

Table 4.9: Stephani et al. Chapter 24. Plane Waves

Solution	Type	Classification
[24, 21, 1]	Einstein-Maxwell	[2, 2, 1]
[24, 22, 1]	Einstein-Maxwell	[4, 3, 10]
[24, 35, 1]	Einstein-Maxwell	[1-dim.]
[24, 37, 1]	Einstein-Maxwell	[1-dim.]
[24, 37, 2]	Einstein-Maxwell	[2, 2, 1]
[24, 37, 3]	Einstein-Maxwell	[2, 2, 1]
[24, 37, 4]	Einstein-Maxwell	[2, 2, 1]
[24, 37, 5]	Einstein-Maxwell	[2, 2, 2]
[24, 37, 6]	Einstein-Maxwell	[3, 3, 4]
[24, 37, 7]	Einstein-Maxwell	[3, 3, 2]
[24, 37, 8]	Einstein-Maxwell	[3, 3, 2]
[24, 37, 9]	Vacuum	[4, 3, 11]
[24, 38, 1]	Vacuum	[Poincare]
[24, 38, 2]	Einstein-Maxwell	[2, 2, 1]
[24, 46, 1]	Einstein-Maxwell	[1-dim.]
[24, 47, 1]	Einstein-Maxwell	[5, 4, -2]
[24, 51, 1]	Einstein-Maxwell	[7, 4, 5]

Table 4.10: Stephani et al. Chapter 26. Various Classes of Algebraically Special Solutions

Solution	Type	Classification
[26, 6, 1]	Einstein-Maxwell	[1-dim.]

Table 4.11: Stephani et al. Chapter 28. Robinson-Trautman Solutions

Solution	Type	Classification
[28, 16, 1]	Vacuum	[3, 3, 4]
[28, 17, 1]	Vacuum	[2, 2, 1]
[28, 21, 1]	Vacuum	[4, 3, 3]
[28, 21, 2]	Vacuum	[4, 3, 3]
[28, 21, 3]	Vacuum	[4, 3, 3]
[28, 21, 4]	Vacuum	[4, 3, 6]
[28, 21, 5]	Vacuum	[4, 3, 6]
[28, 21, 6]	Vacuum	[4, 3, 1]
[28, 21, 7]	Vacuum	[4, 3, 1]
[28, 24, 1]	Vacuum	[2, 2, 1]
[28, 25, 1]	Vacuum	[2, 2, 1]
[28, 26, 1]	Vacuum	[2, 2, 1]
[28, 26, 2]	Vacuum	[2, 2, 1]
[28, 26, 3]	Vacuum	[2, 2, 1]
[28, 41, 1]	Einstein-Maxwell	[1-dim.]
[28, 43, 1]	Einstein-Maxwell	[3, 2, 1]
[28, 44, 1]	Einstein-Maxwell	[4, 3, 3]
[28, 44, 2]	Einstein-Maxwell	[4, 3, 3]
[28, 44, 3]	Einstein-Maxwell	[4, 3, 1]
[28, 44, 4]	Einstein-Maxwell	[4, 3, 6]
[28, 44, 5]	Einstein-Maxwell	[4, 3, 1]
[28, 44, 6]	Einstein-Maxwell	[4, 3, 1]
[28, 45, 1]	Einstein-Maxwell	[2, 2, 1]
[28, 45, 2]	Einstein-Maxwell	[2, 2, 1]
[28, 46, 1]	Einstein-Maxwell	[2, 2, 1]
[28, 46, 2]	Einstein-Maxwell	[2, 2, 1]

Solution	Type	Classification
[28, 53, 1]	Einstein-Maxwell	[1-dim.]
[28, 53, 2]	Einstein-Maxwell	[1-dim.]
[28, 55, 1]	Einstein-Maxwell	[2, 2, 1]
[28, 55, 2]	Einstein-Maxwell	[2, 2, 1]
[28, 56.1, 1]	Einstein-Maxwell	[2, 2, 1]
[28, 56.2, 2]	Einstein-Maxwell	[2, 2, 1]
[28, 56.2, 3]	Einstein-Maxwell	[1-dim.]
[28, 56.3, 1]	Einstein-Maxwell	[1-dim.]
[28, 56.4, 1]	Einstein-Maxwell	[1-dim.]
[28, 56.5, 1]	Einstein-Maxwell	[1-dim.]
[28, 56.6, 1]	Einstein-Maxwell	[1-dim.]
[28, 58.2, 1]	Einstein-Maxwell	[0-dim.]
[28, 58.3, 1]	Einstein-Maxwell	[2, 2, 1]
[28, 58.3, 2]	Einstein-Maxwell	[2, 2, 1]
[28, 58.4, 1]	Einstein-Maxwell	[0-dim.]
[28, 60, 1]	Einstein-Maxwell	[1-dim.]
[28, 61, 1]	Einstein-Maxwell	[1-dim.]
[28, 64, 1]	Einstein-Maxwell	[1-dim.]
[28, 66, 1]	Einstein-Maxwell	[1-dim.]
[28, 67, 1]	Einstein-Maxwell	[1-dim.]
[28, 68, 1]	Einstein-Maxwell	[1-dim.]
[28, 72, 1]	Pure Radiation	[3, 3, 4]
[28, 73, 1]	Pure Radiation	[0-dim.]
[28, 74, 1]	Pure Radiation	[1-dim.]

Table 4.12: Stephani et al. Chapter 31. Non-diverging Solutions

Solution	Type	Classification
[31, 34, 1]	Vacuum	[1-dim.]
[31, 34, 2]	Vacuum	[1-dim.]
[31, 40, 1]	Vacuum	[2, 2, 2]
[31, 50, 1]	Einstein-Maxwell	[6, 4, 2]
[31, 57, 1]	Einstein-Maxwell	[0-dim.]
[31, 60, 1[1]]	Einstein-Maxwell	[4, 3, 1]
[31, 60, 1[2]]	Einstein-Maxwell	[4, 3, 8]
[31, 60, 2[1]]	Einstein-Maxwell	[4, 3, 1]
[31, 60, 2[2]]	Einstein-Maxwell	[4, 3, 8]
[31, 61, 1]	Einstein-Maxwell	[4, 3, 9]
[31, 61, 2]	Einstein-Maxwell	[4, 3, 9]

[1] When $a > 0$. [2] When $a < 0$. See chapter 4 for detail.

Table 4.13: Stephani et al. Chapter 35. Special Vector and Tensor Fields and Chapter 37. Local isometric embedding of four-dimensional Riemannian manifolds

Solution	Type	Classification
[35, 33, 1]	Einstein-Maxwell	[2, 2, 2]
[35, 35, 1[1]]	Einstein-Maxwell	[6, 4, 1]
[35, 35, 1[2]]	Einstein-Maxwell	[6, 4, 2]
[37, 98, 1]	Einstein-Maxwell	[6, 4, 1]

[1] When $\lambda > 0$. [2] When $\lambda < 0$. See chapter 4 for detail.

CHAPTER 5

Conclusion

Table 5.1 shows solutions as organized by Stephani et al. [2] and how many of them have been classified in this thesis. As a result, 421 out of 802 solutions have been classified and 54 out of 97 Hicks' Lie algebra-sub algebra pairs have appeared. Two possible avenues for future research would be to complete the classifications of missing cases and try to understand why certain symmetry types do not appear. It could be that some matter field types do not allow certain types of symmetries. Or it could be that some symmetry groups grow in dimension when field equations are imposed.

Three solutions, [31, 60, 1], [31, 60, 2], and [35, 35, 1], in the electronic library need to be split into sub cases depending on values of certain parameters. These results are detailed in chapter 4.

It has been shown in chapter 4 of this thesis that four solutions, [12, 7, 1], [12, 37, 1], [12, 37, 2], and [15, 18, 1], cannot be classified using Hicks' method because they are not simple G spaces. The classification of non simple G cases will be left for future work.

Table 5.1: Number of Classified Solutions in Stephani et al.

Chapter	**Description**	**Number of Solutions**	**Number of Classified Solutions**
8	Spaces of constant curvature	3	3
12	Homogeneous space-times	52	52
13	Hypersurface-homogeneous space-times	97	97
14	Spatially-homogeneous perfect fluid cosmologies	64	64

15	Group G_3 on non-null orbits V_2. Spherical and plane symmetry	71	50
16	Spherically-symmetric perfect fluid solutions	81	53
17	Group G_2 and G_1 on non-null orbits	16	0
18	Stationary gravitational fields	19	0
19	Stationary axisymmetric fields	1	0
20	Stationary axisymmetric vacuum solutions	31	0
21	Non-empty stationary axisymmetric solutions	26	12
22	Group G_2I on spacelike orbits	40	10
23	Inhomogeneous perfect fluid solutions with symmetry	38	0
24	Groups on null orbits	29	17
25	Collision of plane waves	17	0
26	Some algebraically general solutions	16	1
28	Robinson-Trautman solutions	50	50
29	Twisting vacuum solutions	9	0
30	Twisting Einstein-Maxwell and pure radiation fields	18	0
31	Non-diverging solutions	18	9
32	Kerr-Schild metrics	23	0
33	Algebraically special perfect fluid solutions	26	0
35	Special vector and tensor fields	19	2

36	Solutions with special subspaces	15	0
37	Local isometric embedding of four-dimensional Riemannian manifolds	23	1

Table 5.2: For each symmetry class in Hicks, indicated whether the symmetry appears in solutions of the Einstein equation.

Classification	Appears
[0-dim.]	Yes
[1-dim.]	Yes
[Poincare]	Yes
[de Sitter]	Yes
[anti de Sitter]	Yes
[2, 2, 1]	Yes
[2, 2, 2]	Yes
[3, 2, 1]	Yes
[3, 2, 2]	Yes
[3, 2, 3]	Yes
[3, 2, 4]	No
[3, 2, 5]	No
[3, 3, 1]	Yes
[3, 3, 2]	Yes
[3, 3, 3]	Yes
[3, 3, 4]	Yes
[3, 3, 5]	Yes
[3, 3, 6]	Yes
[3, 3, 7]	Yes
[3, 3, 8]	Yes
[3, 3, 9]	Yes
[4, 3, 1]	Yes
[4, 3, 2]	Yes
[4, 3, 3]	Yes
[4, 3, 4]	Yes
[4, 3, 5]	Yes
[4, 3, 6]	Yes
[4, 3, 7]	Yes
[4, 3, 8]	Yes
[4, 3, 9]	Yes
[4, 3, 10]	Yes

Continued . . .

Classification	Appears
[4, 3, 11]	Yes
[4, 3, 12]	Yes
[4, 3, 13]	No
[4, 3, 14]	No
[4, 3, 15]	No
[4, 3, 16]	No
[4, 3, 17]	No
[4, 3, 18]	No
[4, 3, 19]	No
[4, 3, 20]	Yes
[4, 4, 1]	Yes
[4, 4, 2]	Yes
[4, 4, 3]	No
[4, 4, 4]	No
[4, 4, 5]	No
[4, 4, 6]	No
[4, 4, 7]	No
[4, 4, 8]	No
[4, 4, 9]	No
[4, 4, 10]	Yes
[4, 4, 11]	No
[4, 4, 12]	Yes
[4, 4, 13]	No
[4, 4, 14]	No
[4, 4, 15]	Yes
[4, 4, 16]	No
[4, 4, 17]	No
[4, 4, 18]	Yes
[4, 4, 19]	No
[4, 4, 20]	No
[4, 4, 21]	No
[4, 4, 22]	No
[4, 4, 23]	No
[5, 4, -6]	No
[5, 4, -5]	No
[5, 4, -4]	Yes

Continued . . .

Classification	Appears
[5, 4, -3]	No
[5, 4, -2]	Yes
[5, 4, -1]	No
[5, 4, 1]	Yes
[5, 4, 2]	No
[5, 4, 3]	No
[5, 4, 4]	No
[5, 4, 5]	No
[5, 4, 6]	No
[5, 4, 7]	No
[5, 4, 8]	No
[5, 4, 9]	No
[5, 4, 10]	No
[5, 4, 11]	No
[6, 3, 1]	Yes
[6, 3, 2]	Yes
[6, 3, 3]	Yes
[6, 3, 4]	No
[6, 3, 5]	No
[6, 3, 6]	No
[6, 4, -1]	Yes
[6, 4, 1]	Yes
[6, 4, 2]	Yes
[6, 4, 3]	Yes
[6, 4, 4]	Yes
[6, 4, 5]	Yes
[6, 4, 6]	Yes
[7, 4, 1]	Yes
[7, 4, 2]	Yes
[7, 4, 3]	No
[7, 4, 4]	No
[7, 4, 5]	Yes

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APPENDICES

APPENDIX A

MAPLE example worksheet

This section demonstrates how to retrieve a solution from the electronic library and classify its isometry algebra. Then it also shows how to identify one of Hicks classifications, [6, 4, 1], as shown in Figure A.1 by multiple changes of the basis.

[6, 4, 1]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	.	.	.
e_2		.	e_1	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

Reference : so(3)+so(2,1), Snobl

Isotropy: $[-e_1, -e_4 - e_5]$, F9

Figure A.1: Hicks' classification of [6, 4, 1]

This worksheet demonstrates how to classify the Lie algebra of a solution, then change the basis to identify one of Hicks classifications.

Load Differential Geometry packages.

```
> with(DifferentialGeometry):with(Library):with(Tools):with(Tensor)
  :with(LieAlgebras):with(GroupActions):
```

Load Killing vector fields, metric, basepoint, and isotropy type into KV, g, Bp, IT respectively.

```
> KV, g, Bp, IT := op(Retrieve("Stephani", 1, [12, 16, 1],
  manifoldname = M, output= ["KillingVectors", "Metric",
  "BasePoints", "IsotropyType"]));
```

$$KV, g, Bp, IT := \Big[-\cos(\phi)\, \partial_\theta + \frac{\sin(\phi)\cos(\theta)}{\sin(\theta)}\, \partial_\phi,\ \sin(\phi)\, \partial_\theta + \frac{\cos(\phi)\cos(\theta)}{\sin(\theta)}\, \partial_\phi,\ \partial_\phi, -\frac{e^t \cosh(x)}{\sinh(x)}\, \partial_t + e^t\, \partial_x,\ \frac{e^{-t}\cosh(x)}{\sinh(x)}\, \partial_t + e^{-t}\, \partial_x,\ \partial_t \Big],\ -\_k^2 \sinh(x)^2\, dt \otimes dt + \_k^2\, dx \otimes dx + \_k^2\, d\theta \otimes d\theta + \_k^2 \sin(\theta)^2\, d\phi \otimes d\phi,\ \Big[\Big[t = 0, x = \ln(2), \theta = \frac{1}{2}\pi, \phi = 0\Big]\Big], \text{"F9"} \quad \textbf{(1)}$$

Calculate the structure constants of the Lie algebra and store into LD1.

```
M > LD1 := LieAlgebraData(KV, L1)
```

$$LD1 := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = 2\, e6,\ [e4, e6] = -e4,\ [e5, e6] = e5 \quad \textbf{(2)}$$

Set up the Lie algebra LD1.

```
M > DGsetup(LD1)
```

$$\textit{Lie algebra: L1} \quad \textbf{(3)}$$

Find the isotrpy subalgebra at the basepoint Bp.

```
L1 > GroupActions:-IsotropySubalgebra(KV, Bp[1], output = [L1])
```

$$\Big[e2,\ e4 - e5 + \left(\frac{10}{3}\right) e6\Big] \quad \textbf{(4)}$$

Display the multiplication table of the Lie algebra LD1.

```
L1 > MultiplicationTable()
```

(5)

L1	e1	e2	e3	e4	e5	e6
e1	0	e3	− e2	0	0	0
e2	− e3	0	e1	0	0	0
e3	e2	− e1	0	0	0	0
e4	0	0	0	0	2 e6	− e4
e5	0	0	0	− 2 e6	0	e5
e6	0	0	0	e4	− e5	0

(5)

Change the basis to match the multiplication table with Hicks' classification.

```
L1 > KV2:=[KV[2],-KV[1],KV[3],KV[6],-KV[4]/2,KV[5]]
```

$$KV2 := \left[\sin(\phi)\,\partial_\theta + \frac{\cos(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi, -\cos(\phi)\,\partial_\theta + \frac{\sin(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi, \partial_\phi, \partial_t, -\frac{1}{2}\,\partial_t, -\frac{e^t\cosh(x)}{\sinh(x)}\,\partial_t + e^t\,\partial_x, \frac{e^{-t}\cosh(x)}{\sinh(x)}\,\partial_t + e^{-t}\,\partial_x\right] \quad \mathbf{(6)}$$

Calculate the structure constants of the Lie algebra with a new basis and store into LD2.

```
M > LD2 := LieAlgebraData(KV2, L2)
```

$LD2 := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = e5,\ [e4, e6] = -e6,\ [e5, e6] = -e4$ **(7)**

Set up the Lie algebra LD2.

```
M > DGsetup(LD2)
```

Lie algebra: L2 **(8)**

Display the multiplication table of the Lie algebra LD2.

```
M > MultiplicationTable()
```

L2	e1	e2	e3	e4	e5	e6
e1	0	e3	− e2	0	0	0
e2	− e3	0	e1	0	0	0
e3	e2	− e1	0	0	0	0
e4	0	0	0	0	e5	− e6
e5	0	0	0	− e5	0	− e4
e6	0	0	0	e6	e4	0

(9)

Find the isotropy subalgebra at the basepoint with a new basis.

```
M > GroupActions:-IsotropySubalgebra(KV2, Bp[1], output = [L2])
```

$$\left[e1, e4-\left(\frac{3}{5}\right)e5-\left(\frac{3}{10}\right)e6\right] \tag{10}$$

Find a constant for a new basis which preserves the multiplication table.

```
L1 > A4:=LinearAlgebra[MatrixExponential](c*dv[4])
```

$$A4 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c & \frac{1}{2}c^2 & 1 \end{bmatrix} \tag{11}$$

Find a new basis which preserves the multiplication table.

```
L2 > B := convert(A4^+.Vector([e1, e2, e3, e4, e5, e6]), list)
```

$$B := \left[e1, e2, e3, e4+c\,e6, c\,e4+e5+\frac{1}{2}c^2\,e6, e6\right] \tag{12}$$

More change of basis to match the multiplication table with Hicks' classification.

```
L1 > KV3:=[KV2[1], KV2[2], KV2[3], KV2[4]+c*KV2[6], c*KV2[4]+KV2
     [5]+(c^2/2)*KV2[6], KV2[6]]
```

$$KV3 := \left[\sin(\phi)\,\partial_\theta+\frac{\cos(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi, --\cos(\phi)\,\partial_\theta+\frac{\sin(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi, \partial_\phi,\right. \tag{13}$$
$$c\,\frac{e^{-t}\cosh(x)}{\sinh(x)}\,\partial_t+e^{-t}\,\partial_x+\partial_t, c\,\partial_t-\frac{1}{2}-\frac{e^{t}\cosh(x)}{\sinh(x)}\,\partial_t+e^{t}\,\partial_x$$
$$\left.+\frac{1}{2}c^2\,\frac{e^{-t}\cosh(x)}{\sinh(x)}\,\partial_t+e^{-t}\,\partial_x, \frac{e^{-t}\cosh(x)}{\sinh(x)}\,\partial_t+e^{-t}\,\partial_x\right]$$

Calculate the structure constants of the Lie algebra with a new basis and store into LD3.

```
L2 > LD3 := LieAlgebraData(KV3, L3)
```

$$LD3 := [e1, e2]=e3, [e1, e3]=-e2, [e1, e4]=0, [e1, e5]=0, [e1, e6]=0, [e2, e3]=e1, [e2, e4]=0, [e2, e5]=0, [e2, e6]=0, [e3, e4]=0, [e3, e5]=0, [e3, e6]=0, [e4, e5]=e5, [e4, e6]=-e6, [e5, e6]=-e4 \tag{14}$$

Set up the Lie algebra LD3.

```
L2 > DGsetup(LD3)
```

$$\text{Lie algebra: L3} \tag{15}$$

Find the isotropy subalgebra at the basepoint with a new basis.

```
L2 > GroupActions:-IsotropySubalgebra(KV3, Bp[1], output = [L3])
```

$$\left[e1, e4 - \frac{3}{3\,c+5}\,e5 - \frac{3\,c^2+10\,c+3}{2\,(3\,c+5)}\,e6\right] \tag{16}$$

Display the multiplication table of the Lie algebra LD3.

```
L2 > MultiplicationTable()
```

L3	e1	e2	e3	e4	e5	e6
e1	0	e3	− e2	0	0	0
e2	− e3	0	e1	0	0	0
e3	e2	− e1	0	0	0	0
e4	0	0	0	0	e5	− e6
e5	0	0	0	− e5	0	− e4
e6	0	0	0	e6	e4	0

(17)

Additional change of basis to match the isotropy subalgebra with Hicks' classification

```
L3 > KV4:=eval(KV3, c=-3)
```

$$KV4 := \left[\sin(\phi)\,\partial_\theta + \frac{\cos(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi, - - \cos(\phi)\,\partial_\theta + \frac{\sin(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi, \partial_\phi, -3\,\frac{e^{-t}\cosh(x)}{\sinh(x)}\,\partial_t + e^{-t}\,\partial_x + \partial_t, -3\,\partial_t - \frac{1}{2} - \frac{e^{t}\cosh(x)}{\sinh(x)}\,\partial_t + e^{t}\,\partial_x + \frac{9}{2}\,\frac{e^{-t}\cosh(x)}{\sinh(x)}\,\partial_t + e^{-t}\,\partial_x, \frac{e^{-t}\cosh(x)}{\sinh(x)}\,\partial_t + e^{-t}\,\partial_x\right] \tag{18}$$

Calculate the structure constants of the Lie algebra with a new basis and store into LD3.

```
L3 > LD4 := LieAlgebraData(KV4, L4)
```

$$LD4 := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e5, [e4, e6] = -e6, [e5, e6] = -e4 \tag{19}$$

Set up the Lie algebra LD4.

```
L3 > DGsetup(LD4)
```

$$\text{Lie algebra: L4} \tag{20}$$

Find the isotropy subalgebra at the basepoint with a new basis.

```
L3 > Iso4 := GroupActions:-IsotropySubalgebra(KV4, Bp[1], output =
     [L4])
```

$$Iso4 := \left[e1, e4 + \left(\frac{3}{4}\right) e5\right] \tag{21}$$

Display the multiplication table of the Lie algebra LD4.

```
L3 > MultiplicationTable()
```

(22)

L4	e1	e2	e3	e4	e5	e6
e1	0	e3	− e2	0	0	0
e2	− e3	0	e1	0	0	0
e3	e2	− e1	0	0	0	0
e4	0	0	0	0	e5	− e6
e5	0	0	0	− e5	0	− e4
e6	0	0	0	e6	e4	0

Additional change of basis to match the isotropy subalgebra with Hicks' classification

```
L3 > KV5:=[KV4[1], KV4[2], KV4[3], KV4[4], KV4[5]*3/4, KV4[6]*4/3]
```

$$KV5 := \left[\sin(\phi)\, \partial_\theta + \frac{\cos(\phi)\cos(\theta)}{\sin(\theta)}\, \partial_\phi, -- \cos(\phi)\, \partial_\theta + \frac{\sin(\phi)\cos(\theta)}{\sin(\theta)}\, \partial_\phi, \partial_\phi, -3\, \frac{e^{-t}\cosh(x)}{\sinh(x)}\, \partial_t + e^{-t}\, \partial_x + \partial_t, -\frac{9}{4}\, \partial_t - \frac{3}{8}\, - \frac{e^{t}\cosh(x)}{\sinh(x)}\, \partial_t + e^{t}\, \partial_x + \frac{27}{8}\, \frac{e^{-t}\cosh(x)}{\sinh(x)}\, \partial_t + e^{-t}\, \partial_x, \frac{4}{3}\, \frac{e^{-t}\cosh(x)}{\sinh(x)}\, \partial_t + e^{-t}\, \partial_x\right] \tag{23}$$

Calculate the structure constants of the Lie algebra with a new basis and store into LD3.

```
L3 > LD5 := LieAlgebraData(KV5, L5)
```

$$LD5 := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e5, [e4, e6] = -e6, [e5, e6] = -e4 \tag{24}$$

Set up the Lie algebra LD4.

```
L3 > DGsetup(LD5)
```

$$\textit{Lie algebra: L5} \tag{25}$$

Find the isotropy subalgebra at the basepoint with a new basis.

```
L3 > Iso4 := GroupActions:-IsotropySubalgebra(KV5, Bp[1], output =
```

```
[L5])
```

$$Iso4 := [e1, e4 + e5] \tag{26}$$

Display the multiplication table of the Lie algebra LD4.

```
L3 > MultiplicationTable()
```

L5	e1	e2	e3	e4	e5	e6
e1	0	e3	− e2	0	0	0
e2	− e3	0	e1	0	0	0
e3	e2	− e1	0	0	0	0
e4	0	0	0	0	e5	− e6
e5	0	0	0	− e5	0	− e4
e6	0	0	0	e6	e4	0

(27)