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Some Examples of the Liouville Integrability of the Banded Toda Flows

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Some Examples of the Liouville Integrability
of the Banded Toda Flows

by

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Abstract

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The Toda Lattice is a classical Hamiltonian system that describes the exponential interaction of particles on a line with its nearest neighbors. Previous results have shown that the Toda lattice with one band and the full Toda flow are completely integrable. In this paper, we will discuss a more modern approach to studying the Toda lattice by looking at the Lie algebra structure. This approach will then be used to study the Banded Toda flow. The main result will show that the banded Toda lattice up to dimension 10 is completely integrable. We will conclude by stating a few conjectures that, we hope, will be used to prove the Liouville integrability of the banded Toda Flow.

(60 pages)

Public Abstract

Some Examples of the Liouville Integrability

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Zachary A. Youmans

The Toda lattice is a famous integrable system studied by Toda in the 1960s. One can study the Toda lattice using a matrix representation of the system. Previous results have shown that this matrix of dimension n with 1 band and $n-1$ bands is Liouville integrable. In this paper, we lay the foundation for proving the general case of the Toda lattice, where we consider the matrix representation with dimension n and a partially filled lower triangular part. We call this the banded Toda flow. The main theorem is that the banded Toda flow up to dimension 10 is Liouville integrable. To conclude the paper, we will present some conjectures which, we hope, will help us in proving the Liouville integrability of the banded Toda flow of dimension n with k number of bands.

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Zachary A. Youmans

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Chapter 1

Introduction

In the 1960s, Morikazu Toda studied the exponential interaction of particles between its nearest neighbors. The differential equation that arose out of his study has been named the Toda lattice. Since that time, many have worked on and have stated that different cases the Toda flow is Liouville integrable.

The study of the Toda lattice is a small piece in the study of integrable systems. This study dates back to the times of Kepler, whom studied problems such as the two-body problem. The goal of such problems is to study the *integrals of motion* which. Integrals of motion are the conserved quantities of the system in study.

To study these integrals of motion, we will discuss in Chapter 2 the idea of the *Poisson manifold*. Throughout Chapter 2, we will expand on the idea of the Poisson manifold and then look at a specific submanifold, called the *symplectic manifold*. Studying the symplectic manifold is important since we will be able to study the *Casimirs* on the manifold, which the level sets of the Casimirs are referred to as symplectic leaves. On the symplectic leaves are Hamiltonians, which after we pick the Hamiltonian, we can study the integrals of motion by seeing which functions are in involution with the Hamiltonian.

Also in chapter 2, we will give some of the history of the Toda Lattice. We will also look at some of the work that has been done in the study of the Toda lattice. For example, we can extend the ordinary Toda lattice into the full Toda flow, which were studied in [3] and [7].

Finally, we will conclude Chapter 2 by considering the Toda lattice in Konstant form, which is created by using a change of variables called *Flaschka's variables* and a

Poisson map. The Poisson map will guarantee that the work done under the change of variables is still viable.

Chapter 3 will introduce the banded Toda flow. While a lot of work has been done to show the Liouville integrability for the full and ordinary cases of the Toda lattice, there is little literature on the Liouville integrability if we consider the Konstant form of the Toda lattice to be partially filled. One such paper was done in [9]. In chapter 3, we will give a complete construction of how to find possible candidates for the first integrals and Casimirs.

Once we are able to find the first integrals and Casimirs, we will then prove the main theorem, which is as follows

Theorem 1.1. *The banded Toda flow is Liouville integrable up to dimension* $n = 10$*, for all* $2 \leq k \leq n-2$

To prove the main theorem, we will give a complete list of the Casimirs and first integrals that will fulfill the definition of what it means to be Liouville integrable.

In Chapter 4, we will provide some work that has already been done to try to prove the general case of the banded Toda flow. The conjecture is as follows

Conjecture 1.0.1. The banded Toda flow of dimension n with k bands is Liouville integrable.

Current attempts to prove this theorem will be introduced. Some of these observations are made through MAPLE. These observations turn into interesting conjectures which we hope will be able to help us prove conjecture 1.0.1.

Chapter 2
Preliminary Work

This chapter will be to introduce the topics needed in order to understand the banded Toda flow. We will begin with the Poisson manifold and the Poisson bracket. Then, we will view a more specific Poisson manifold, called the symplectic manifold. This will introduce the idea of a symplectic leaf, which are the level sets of the Casimirs. On the symplectic leaf is our first integrals, which are important to us since by definition, if we have *enough* first integrals in involution and functionally independent, then our system is Liouville integrable. We will conclude the chapter with the history of the Toda lattice and the Konstant form of the Toda lattice.

2.1 The Poisson Manifold

2.1.1 Motivation

In the theory of classical Hamiltonian mechanics, the idea of the Poisson manifold and the Poisson bracket are important concepts. One such example of this, as described in [10], is the motion of particles in a potential field.

Let m denote the mass of a particle, and let q denote the position of the particle determined by the vector $q = (q^1, \cdots, q^n)$. Note that this is an n-dimensional space. Then, Newton's equation of the form

$$m\ddot{q}^j = -\frac{\partial V}{\partial q^j} \tag{2.1}$$

describes the motion of the particle. Here, V is the potential. Let p denote the momentum, where $p = (p_1, \cdots, p_n), p_j = m\dot{q}^j$ is the momentum vector. The energy of the particles are defined by the Hamiltonian function

$$H = \frac{1}{2m}p^2 + V(q), \quad p^2 = \sum_{j=1}^{n} p_j p_j \tag{2.2}$$

Newton's equation in (2.1) can be written as

$$\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j} \tag{2.3}$$

Then, to determine if this Hamiltonian system is Liouville integrable, we need to find sufficient conserved quantities which are defined by the Poisson structure. This Poisson structure will form the Poisson bracket in classical Hamiltonian mechanics as

$$\{F(p,q), G(p,q)\} = \sum_{j=1}^{n} (\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q^j} - \frac{\partial F}{\partial q^j}\frac{\partial G}{\partial p_j}) \tag{2.4}$$

where $F(p,q)$ and $G(p,q)$ are smooth functions on $\mathbb{R}^{2n}$. It is important to note that this Poisson bracket will satisfy both skew-symmetry and the Jacobi identity.

While this example is Liouville integrable since the conserved quantities are defined by (2.3), most Hamiltonian system do not have enough conserved quantities. This makes a general Hamiltonian system not Liouville integrable. Therefore, integrable systems are interesting to study.

2.1.2 The Poisson Bracket

The motivating example from above gives us a foundation for what needs to be done, which is to describe precisely the Poisson structure and the integrals of motion. We begin with a definition.

Definition 2.1. Let M be a manifold and $\mathcal{F}(M)$ be the space of smooth functions on M. Then, M is endowed with a Poisson structure if, given two functions $F, G \in \mathcal{F}(M)$, we can define a new function $\{F, G\} \in \mathcal{F}(M)$ that satisfy the following three properties:

1. skew-symmetry, i.e.

$$\{F, G\} = -\{G, F\}$$

2. the Jacobi identity, i.e.

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0$$

3. the Leibniz rule, i.e.

$$\{F,GH\}=\{F,G\}H+\{F,H\}G$$

This new function, $\{F,G\}$, is called the Poisson bracket of F and G. It is important to note here that the three properties follow the same definition as a Lie algebra, which we will take advantage of in section 1.2.

More concretely, let $\omega^{ij}(x)=\{x^i,x^j\}$, where x^i and x^j are local coordinates on a manifold M. Then, the Poisson bracket is of the form

$$\{F(x),G(x)\}=\omega^{jk}(x)\partial_j F\partial_k G,\quad \partial_j=\frac{\partial}{\partial x^j}. \tag{2.5}$$

The Poisson bracket defined in (2.4) is an example of when $\omega^{jk}(x)$ is a constant since the bracket does not depend on x.

2.1.3 Symplectic Structure and Casimirs

Another important idea needed in Hamiltonian mechanics is the symplectic manifold.

Definition 2.2. A symplectic manifold (M,ω) is a smooth manifold M endowed with a closed nondegenerate 2-form ω.

One can view a symplectic manifold as a non-degenerate Poisson manifold, meaning that the matrix $(\omega^{jk})(x)$ is nondegenerate for all $x\in M$. To see that a symplectic manifold is a non-degenerate Poisson manifold, the following proposition is used.

Proposition 2.3. *(ω^{ij}) satisfies the Jacobi identity iff $\omega=\omega_{ij}dx^i\wedge dx^j$ is closed, where $(\omega_{ij})=(\omega^{ij})^{-1}$.*

Proof. Suppose (ω^{ij}) satisfies the Jacobi identity. Let $\omega^{ij}=\{x^i,x^k\}$. Then,

$$\{\{x^i,x^j\},x^k\}+\{\{x^j,x^k\},x^i\}+\{\{x^k,x^i\},x^j\}=0$$

which implies that

$$\{\omega^{ij}, x^k\} + \{\omega^{jk}, x^i\} + \{\omega^{ki}, x^j\} = 0$$

Recall that $\{F, G\} = \omega^{mn}\partial_m F \partial_n G$. Therefore, we have

$$\omega^{mn}\frac{\partial\omega^{ij}}{\partial x^m}\frac{\partial x^k}{\partial x^n} + \omega^{mn}\frac{\partial\omega^{jk}}{\partial x^m}\frac{\partial x^i}{\partial x^n} + \omega^{mn}\frac{\partial\omega^{ki}}{\partial x^m}\frac{\partial x^j}{\partial x^n} = 0$$

which results in

$$\omega^{mk}\frac{\partial\omega^{ij}}{\partial x^m} + \omega^{mi}\frac{\partial\omega^{jk}}{\partial x^m} + \omega^{mj}\frac{\partial\omega^{ki}}{\partial x^m} = 0$$

To continue, the following lemma is used.

Lemma 2.4. $\dfrac{\partial\omega^{jk}}{\partial x^m} = -\omega^{j\alpha}\dfrac{\partial\omega_{\alpha\lambda}}{\partial x^m}\omega^{\lambda k}$

To prove the lemma, we start with $\omega_{\alpha\lambda}\omega^{\alpha\beta} = \delta_\alpha^\lambda$, we have

$$\begin{aligned}
&\frac{\partial(\omega_{\alpha\lambda}\omega^{\lambda\beta})}{\partial x^\gamma} = 0 \\
\implies &\frac{\partial(\omega_{\alpha\lambda})}{\partial x^\gamma}\omega^{\lambda\beta} + \frac{\partial(\omega^{\lambda\beta})}{\partial x^\gamma}\omega_{\alpha\lambda} = 0 \\
\implies &\frac{\partial(\omega^{\lambda\beta})}{\partial x^\gamma}\omega_{\alpha\lambda} = -\frac{\partial(\omega_{\alpha\lambda})}{\partial x^\gamma}\omega^{\lambda\beta} \\
\implies &\omega^{\mu\alpha}\frac{\partial(\omega^{\lambda\beta})}{\partial x^\gamma}\omega_{\alpha\lambda} = -\omega^{\mu\alpha}\frac{\partial(\omega_{\alpha\lambda})}{\partial x^\gamma}\omega^{\lambda\beta} \\
\implies &\delta_\lambda^\mu\frac{\partial(\omega^{\lambda\beta})}{\partial x^\gamma} = -\omega^{\mu\alpha}\frac{\partial(\omega_{\alpha\lambda})}{\partial x^\gamma}\omega^{\lambda\beta} \\
\implies &\frac{\partial(\omega^{\mu\beta})}{\partial x^\gamma} = -\omega^{\mu\alpha}\frac{\partial(\omega_{\alpha\lambda})}{\partial x^\gamma}\omega^{\lambda\beta}
\end{aligned}$$

Therefore, the last line implies the lemma.

Therefore, by Lemma 1, we have

$$\omega^{mk}(-\omega^{i\alpha}\frac{\partial\omega_{\alpha\lambda}}{\partial x^m}\omega^{\lambda j}) + \omega^{mi}(-\omega^{j\alpha}\frac{\partial\omega_{\alpha\lambda}}{\partial x^m}\omega^{\lambda k}) + \omega^{mj}(-\omega^{k\alpha}\frac{\partial\omega_{\alpha\lambda}}{\partial x^m}\omega^{\lambda i}) = 0$$

Through relabeling and using skew-symmetry, we can rewrite this as

$$-\omega^{km}\omega^{i\alpha}\omega^{j\lambda}\frac{\partial\omega_{\alpha\lambda}}{\partial x^m} - \omega^{km}\omega^{i\alpha}\omega^{j\lambda}\frac{\partial\omega_{\lambda m}}{\partial x^\alpha} - \omega^{km}\omega^{i\alpha}\omega^{j\lambda}\frac{\partial\omega_{m\alpha}}{\partial x^\lambda} = 0$$

Therefore, we have

$$-\omega^{km}\omega^{i\alpha}\omega^{j\lambda}(\frac{\partial\omega_{\alpha\lambda}}{\partial x^m}+\frac{\partial\omega_{\lambda m}}{\partial x^\alpha}+\frac{\partial\omega_{m\alpha}}{\partial x^\lambda})=0$$

Though, $\omega^{\mu\nu}$ is non-degenerate, therefore,

$$\frac{\partial\omega_{\alpha\lambda}}{\partial x^m}+\frac{\partial\omega_{\lambda m}}{\partial x^\alpha}+\frac{\partial\omega_{m\alpha}}{\partial x^\lambda}=0$$

which implies that ω is closed, that is, $d\omega = 0$ by the following.

Given

$$d(\omega)=0$$

and if $\omega = \omega_{ij}dx^i \wedge dx^j$, then,

$$d(\omega_{ij}dx^i\wedge dx^j)=0$$

Then, we have

$$\begin{aligned} &d(\omega_{ij}dx^i\wedge dx^j)=0\\ \implies &d\omega_{ij}\wedge dx^i\wedge dx^j+d\omega_{jk}\wedge dx^j\wedge dx^k+d\omega_{ki}\wedge dx^k\wedge dx^i=0\end{aligned}$$

Though, this can be rewritten as

$$\frac{\partial\omega_{ij}}{\partial x^k}dx^k\wedge dx^i\wedge dx^j+\frac{\partial\omega_{jk}}{\partial x^i}dx^i\wedge dx^j\wedge dx^k+\frac{\partial\omega_{ki}}{\partial x^j}dx^j\wedge dx^k\wedge dx^i=0$$

Using skew-symmetry,

$$(dx^i\wedge dx^j\wedge dx^k)(\frac{\partial\omega_{ij}}{\partial x^k}+\frac{\partial\omega_{jk}}{\partial x^i}+\frac{\partial\omega_{ki}}{\partial x^j})=0$$

Though, given that $(dx^i \wedge dx^j \wedge dx^k) \neq 0$, then

$$\frac{\partial\omega_{ij}}{\partial x^k}+\frac{\partial\omega_{jk}}{\partial x^i}+\frac{\partial\omega_{ki}}{\partial x^j}=0$$

The above procedures can be reversed to get the only if direction, therefore the theorem has been proved. □

In order to find the symplectic submanifolds, we use the Casimir functions.

Definition 2.5. A non-constant function C on a manifold M is a Casimir function if

$$\{C, f\} = 0 \text{ for all } f \in C^{\infty}(M)$$

One such example is that for the Lie algebra $so(3)$, which is isomorphic to $\mathbb{R}^3$ with the cross product, the Casimir function will be $x^2 + y^2 + z^2$. The level sets of the Casimirs are symplectic manifolds, and we will call them the symplectic leaves of the Poisson manifold. In the example of $so(3)$, the symplectic leaves are the spheres $x^2 + y^2 + z^2 = r^2$.

2.1.4 First Integrals and Liouville Integrability

On the symplectic leaves of the Poisson manifold are first integrals, the conserved quantities of the system. A Hamiltonian function H on a symplectic manifold defines a mechanical system. Once we pick a Hamiltonian, we can study the first integrals, or sometimes called integrals of motion.

Definition 2.6. Let H be a smooth function on M. Then, the integrals of motion are functions F that satisfy

$$\{H, F\} = 0$$

The existence of enough first integrals guarantees that the system is Liouville integrable.

Definition 2.7. If there are n functionally independent smooth functions in involution on an $2n$-dimensional symplectic manifold M, then the system is called Liouville integrable.

More generally, on a Poisson manifold M, Liouville integrability requires the following formula to hold:

$$\text{Number of independent Casimirs} + 2 \times \text{ Number of independent first integrals } = \dim(M). \tag{2.6}$$

Practically speaking, the appendix will have MAPLE code that will show how one can calculate the amount of first integrals the system in study should have. The dimension of the Poisson manifold will be discussed in section 3.2. Once these two pieces of information is known, (2.6) is used to find the number of Casimirs.

2.2 Lie-Poisson and Symplectic Structure

2.2.1 Lie-Poisson Bracket

Recall the Poisson bracket in (2.5). When $\omega^{jk}(x)$ are linear functions of x, $\omega^{jk}(x)$ can be viewed as the structure constants from a Lie algebra, $\mathfrak{g}$, with

$$\omega^{jk}(x) = C_l^{jk} x^l. \tag{2.7}$$

This turns the Poisson bracket from (2.5) into the Lie-Poisson bracket on $\mathfrak{g}^*$, which is of the form

$$\{F(x), G(x)\} = C_l^{jk} x^l \partial_j F \partial_k G. \tag{2.8}$$

More concretely, if $\mathfrak{g}$ has a non-degenerate invariant form Tr, then the Lie-Poisson bracket on $\mathfrak{g}$ becomes

$$\{F, G\}(X) = \mathrm{Tr}(X.[\mathrm{grad}_X(F), \mathrm{grad}_X(G)]) \tag{2.9}$$

where the gradient is defined by

$$\mathrm{Tr}(Y \cdot \mathrm{grad}_X f) = \lim_{t \to 0} \frac{f(X + tY) - f(X)}{t}, \text{ for all } Y \in \mathfrak{g}. \tag{2.10}$$

The formula for the gradient is

$$\mathrm{grad}_X f = \sum \frac{\partial f}{\partial x_{ij}}\bigg|_X E_{ji}$$

where E_{ji} is a matrix whose only non-trivial entry is 1 at the (j, i) position. Throughout the rest of this paper, we will be using (2.9). For Toda flows, the most relevant Lie algebras are the Borel subalgebras $\mathfrak{b}_+$ and $\mathfrak{b}_+^* = \mathfrak{b}_-$, which are the space of upper triangular matrices and lower triangular matrices, respectively. These Borel subalgebras are discussed more in depth in [4].

As a side note, the symplectic leaves of the Lie-Poisson structure on $\mathfrak{g}^*$ are called coadjoint orbits. The main example used in the theory of the Toda lattice is the coadjoint action of G, where G is the group of real upper triangular matrices with determinant 1. This coadjoint action of G on $\mathfrak{g}^*$ is given by

$$\mathrm{Ad}^*_g : x \mapsto (gxg^{-1})_- \tag{2.11}$$

The minus in the coadjoint action represents a matrix in which the upper triangular elements have been deleted and replaced by zeros. If we consider $G = SO(3)$, the coadjoint orbits are the spheres $x^2 + y^2 + z^2 = r^2$.

2.3 The Toda Lattice

We will now discuss a more specific problem in the theory of integrable systems, that being the Toda Lattice.

2.3.1 History

In the 1960s, Morikazu Toda studied the solutions to a differential equation which explains the exponential interaction of particles between their nearest neighbors. His papers (which one can be seen in [12]) produced soliton solutions to this system, which today is aptly named the Toda lattice. The Hamiltonian for this system has the form

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} \exp[(q_j - q_{j+1})] \tag{2.12}$$

where p_j stands for the momentum of the particle, and q_j is the position. In the 1970s, Henon, Flaschka, and Manakov all made advances in the theory of the Toda lattice. In 1974, both Henon and Flaschka published that the real, finite, and periodic Toda lattice is Liouville integrable. Their results can be seen in [5], [6]. Also in 1974, Flaschka and Manakov announced that they can write the Toda lattice in a Lax pair form through a change of variables. This change of variables is called *Flaschka's variables*.

More concretely, consider an example of the Toda lattice with two particles with positions q_1 and q_1, and momentum p_1 and p_2. Before continuing, consider the following definition.

Definition 2.8. Let M and N be manifolds equipped with the Poisson bracket $\{\ ,\ \}$. Let $f, g \in M$. Then, a Poisson map $\phi : (M, \{\ ,\ \}) \to (N, \{\ ,\ \})$ is defined such that

$$\phi(\{f, g\}) = \{\phi(f), \phi(g)\}$$

A Poisson map guarantees that results found after the change of variables are valid for the original system. Now, consider a 4-dimensional vector field with

$$(p_1, p_2, q_1, q_2) \in \mathbb{R}^4 \tag{2.13}$$

Define a Poisson map such that

$$(p_1, p_2, q_1, q_2) \to \begin{bmatrix} p_1 & 1 \\ e^{q_1 - q_2} & p_2 \end{bmatrix}.$$

Then, let

$$a_i = p_i \text{ and } b_i = e^{q_i - q_{i+1}}. \tag{2.14}$$

These change of variables are the Flaschka's variables. This can be visualized as

$$\begin{bmatrix} p_1 & 1 \\ e^{q_1 - q_2} & p_2 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ b_1 & a_2 \end{bmatrix}.$$

This matrix form will be discussed more in section 2.3.2.

Two other important results worth mentioning are that of Deift, Li, Nanda, and Tomei in 1986, and Gekhtman and Shapiro's in 1999. Deift, et. al. discovered that the full Toda flow is Liouville integrable for A type in [3]. Gekhtman and Shapiro showed that the full Toda flow for all simple Lie algebras are Liouville integrable in [7].

The work of Deift, et. al. was important to this paper since their way of constructing first integrals was used in a modified version to prove the main theorem of this paper. This method will be explained in Chapter 2.

As a side note, much of the history of the Toda lattice can be seen in the work of Kodama and Shipman in [13]. Their work, *Fifty years of the finite nonperiodic Toda lattice: a geometric and topological viewpoint* gives a nice overview of the history and some of the major ideas behind the Toda lattice.

2.3.2 The Konstant Form

Recall that by using Flaschka's variables, a matrix representation can be formed from the Hamiltonian found in (2.12). This matrix, called the Konstant form, can be used to study the integrability of the Toda lattice. Konstant discussed this method in [8]. The Konstant form for the 2-dimensional Toda lattice is

$$\begin{bmatrix} a_1 & 1 \\ b_1 & a_2 \end{bmatrix}$$

In general, for the $n \times n$ Toda lattice, the Konstant form is

$$L = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 \\ 0 & b_2 & a_3 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots\,\ddots & \ddots & \ddots & \ddots \\ & & & & & \\ 0 & \cdots & & 0 & b_{n-1} & a_n \end{bmatrix} \tag{2.15}$$

In (2.15), L is the matrix in Lax pair representation $\dot{L} = [L, M]$ with $M = -\Pi_{\mathfrak{n}_-} L$. Here, $\Pi_{\mathfrak{n}_-} L$ is the projection of L into a strictly lower triangular matrix. We will now verify that the given L does indeed form a Lax pair representation. To do this, recall that the Hamiltonian for the Toda lattice is

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q^i - q^{i+1}}$$

Then,

$$\dot{p}_i = \{H, p_i\} = -e^{q_i - q_{i+1}} + e^{q_{i-1} - q_i} \text{ (for } 1 < i < n)$$
$$\dot{q}_i = \{H, q_i\} = p_i$$

Using Flashka's variables found in (2.14),

$$\dot{a}_i = \dot{p}_i = b_{i-1} - b_i \text{ and } \dot{b}_i = e^{q_i - q_{i+1}}(\dot{q}_i - \dot{q}_{i+1}) = b_i(a_i - a_{i+1}) \tag{2.16}$$

Checking that $\dot{L} = [L, M]$ with L given in (2.15) and

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots\ddots & \ddots & \ddots & \ddots \\ & & & & & \\ 0 & \cdots & & 0 & b_{n-1} & 0 \end{bmatrix} \tag{2.17}$$

then $[L, M] = LM - ML$ is

$$\begin{bmatrix} -b_1 & & & & & \\ -a_2b_1 + b_1a_1 & -b_2 + b_2 & & & & \\ -b_1b_2 + b_1b_2 & -a_3b_2 + a_2b_2 & -b_3 + b_2 & & & \\ & -b_2b_3 + b_2b_3 & -a_4b_3 + a_3b_3 & -b_3 + b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & b_{n-1} \end{bmatrix}$$

Calculating $\dot{L}$ results in

$$\dot{L} = \begin{bmatrix} \dot{a}_1 & & & \\ \dot{b}_1 & \ddots & & \\ & \ddots & \ddots & \\ & & \dot{b}_{n-1} & \dot{a}_n \end{bmatrix} \tag{2.18}$$

Comparing $[L, M]$ with $\dot{L}$ given the relation (2.16) results in the two being equal. Therefore, the Lax pair is verified. Note that when $i = 1$, $\dot{a}_1 = -b_1$ and when $i = n$, $\dot{a}_n = b_{n-1}$

2.3.3 The Full Toda Flow

In Section 2.3.1, it was mentioned that there are complete results on the full Toda flow, done in [3] and [7]. For the ordinary Toda lattice, it was enough to find a sufficient amount of first integrals by considering the characteristic polynomial of the Konstant form. Though, for Deift, et. al., this method did not provide a sufficient amount of first integrals. The Konstant form for the full Toda flow is given by the matrix

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 \\ & b_2 & a_3 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots\ddots & \ddots & \ddots & \ddots \\ & & & & & \\ z_1 & \cdots & & & b_{n-1} & a_n \end{bmatrix}. \tag{2.19}$$

This Konstant form is *completely* filled in the lower triangular part. The Lax Pair representation is similar to (2.15). To find the first integrals, Deift et. al. performed *chopping* on the Konstant form of the Toda flow. The first integrals found by the characteristic polynomial is equivalent to 0-chopping. To do this chopping, consider an $n \times n$ matrix L above and for $0 \leq k \leq \lfloor \frac{1}{2} n \rfloor$, delete the first k rows and the last k columns of the matrix L. For example, consider the 4×4 matrix in Konstant form

$$\begin{bmatrix} a_1 & 1 & 0 & 0 \\ b_1 & a_2 & 1 & 0 \\ c_1 & b_2 & a_3 & 1 \\ d_1 & c_2 & b_3 & a_4 \end{bmatrix}. \tag{2.20}$$

The 1-chop would be

$$\begin{bmatrix} a_1 & 1 & 0 & 0 \\ b_1 & a_2 & 1 & 0 \\ c_1 & b_2 & a_3 & 1 \\ d_1 & c_2 & b_3 & a_4 \end{bmatrix}.$$

This method is similar to finding the determinant. Then, the first integrals and Casimirs can be formed by finding rational expressions. These rational expressions can be found by viewing the polynomials

$$\det(L - \lambda I)_k = \sum_{r=0}^{n-2k} E_{rk}(M) \lambda^{n-2k-r}. \tag{2.21}$$

The first integrals and Casimirs are found using the ratios $E_{rk}(M)/E_{0k}(M)$. The Casimirs are only those of the form E_{1k}/E_{0k}. The first integrals are the rest of the rational functions. It is proved that the first integrals are in involution with each other and functionally independent so that they satisfy the definition of Liouville integrability.

In Chapter 3, we will introduce a slightly different but equivalent method to chopping. Though, with the banded Toda flow, we will run into some difficulties. These issues will present themselves in the sense that there are not enough Casimirs when considering (2.6) and the chopping method.

Chapter 3
The Banded Toda Flow

In this chapter, we turn our attention to the banded Toda flow. We will introduce the main theorem of the paper with a proof based on the definition of Liouville integrability. Before that, we will give the Konstant form of the banded Toda flow. It will also be stated how the first integrals and Casimirs are found. We will provide the MAPLE code used in this chapter in the appendix.

3.1 The Banded Toda flow

Recall from Chapter 2 the ordinary Toda lattice, which has the Konstant form

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 \\ 0 & b_2 & a_3 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots\ddots & \ddots & \ddots & \ddots \\ & & & & & \\ 0 & \cdots & & 0 & b_{n-1} & a_n \end{bmatrix} \tag{3.1}$$

Also, recall the consideration [3] and [7] made for the full Toda flow, which has the Konstant form

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 \\ & b_2 & a_3 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots\ddots & \ddots & \ddots & \ddots \\ & & & & & \\ z_1 & \cdots & & & b_{n-1} & a_n \end{bmatrix}. \tag{3.2}$$

For the *banded* Toda flow, we want to consider the Konstant form of dimension

n with k number of bands. We will denote this the (n, k) Toda flow. By way of example, consider the Konstant form of the $(5, 2)$ banded Toda flow. Instead of using $a_i, b_{i-1}, c_{i-2}, \cdots$ for the change of variables, we will use $x_{i,j}$ where i represents the row and j represents the column. Therefore, the $(5, 2)$ banded Toda flow would have the form

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 \\ 0 & x_{4,2} & x_{4,3} & x_{4,4} & 1 \\ 0 & 0 & x_{5,3} & x_{5,4} & x_{5,5} \end{bmatrix}. \tag{3.3}$$

The work of Henon, Flaschka, and Manakov showed the the ordinary Toda lattice is Liouville integrable. The work of [3] and [7] showed that the full Toda flow is Liouville integrable. The goal is to do the same for the banded Toda flow. Unfortunately, this has not been accomplished for an arbitrary dimension and band, but for this paper, we will show that it is Liouville integrable for each case up to dimension 10.

3.2 The Main Theorem

To establish that the Banded Toda flow is Liouville integrable, the following theorem will be proved.

Theorem 3.1. *The banded Toda flow is Liouville integrable up to dimension* $n = 10$*, for all* $2 \leq k \leq n - 2$*.*

For this theorem, we do not consider the cases when $n = 1, 2$ or 3. These cases are examples of the ordinary Toda lattice, which has been already proven to be Liouville integrable. To prove this theorem, we will rely on the following definition:

If there are d functionally independent smooth functions in involution on an $2d$-dimensional symplectic manifold M, then the system is called completely integrable.

This definition provides the goal of the proof, to find n functionally independent functions so that when we do the Lie-Poisson bracket among them, they are in involution. More concretely, if F_i and F_j are two functionally independent functions, then

$$\{F_i, F_j\}(X) = \mathrm{Tr}(X.[\mathrm{grad}_X(F_i), \mathrm{grad}_X(F_j)]) = 0 \tag{3.4}$$

where X is the banded Toda flow in Konstant form.

Also, recall that the $2d$-dimensional symplectic manifolds in the definition of complete integrability are called symplectic leaves. They are cut out by the Casimir functions. These Casimir functions will be checked by doing the same Lie-Poisson bracket, if C is the Casimir, and F is any other function in the system, then

$$\{C, F\}(X) = \mathrm{Tr}(X.[\mathrm{grad}_X(C), \mathrm{grad}_X(F)]) = 0 \tag{3.5}$$

To find the number of first integrals and Casimirs, recall that from Liouville integrability,

$$\text{Number of Casimirs} + 2 \times \text{ Number of First integrals } = \dim(L).$$

where L is the dimension of the Banded Toda flow in Konstant Form. The dimension of the symplectic leaves (SL) is given by the equation

$$\mathrm{rank}(SL) = \mathrm{rank}(\omega^{ij}). \tag{3.6}$$

As seen in Proposition 2.3, $\omega^{ij} = \{x^i, x^k\}$, where x^i and x^k are the coordinate functions. Then, (3.6) becomes

$$\mathrm{rank}(SL) = \mathrm{rank}(\{x^i, x^j\}) \tag{3.7}$$

To find the dimension of L, we count the number of arbitrary functions in the Konstant form, minus 0 and 1. For example, reconsidering the $(5, 2)$ Banded Toda flow in (3.3), the dimension would be $5 + 4 + 3 = 12$. To count the number of Casimirs, we do

$$\text{Number of Casimirs} = \dim(L) - \dim(SL). \tag{3.8}$$

The number of first integrals has the equation

$$\text{Number of first integrals} = \frac{1}{2}\dim(SL) \tag{3.9}$$

As an example of how to find the gradient of a function, consider the $(5, 2)$ banded Toda flow in (3.3). It turns out that one such first integral of this system is

$x_{1,1}x_{2,2}+x_{1,1}x_{3,3}+x_{1,1}x_{4,4}+x_{1,1}x_{5,5}+x_{2,2}x_{3,3}+x_{2,2}x_{4,4}+x_{2,2}x_{5,5}+x_{3,3}x_{4,4}+x_{3,3}x_{5,5}+x_{4,4}x_{5,5}-x_{2,1}-x_{3,2}-x_{4,3}-x_{5,4}$.

In order to check this this function is actually a first integral, we need to check if it is in involution with other first integrals found by the characteristic polynomial. To do this, we need the gradient of the function which is represented through a matrix. To accomplish this, we first differentiate the function with with respect to each variable in the Konstant form of the matrix. Then, we put the derivative function in the *transpose* position of the resulting gradient matrix. For example, differentiating the above function with respect to $x_{3,2}$ would result in just -1. Then, we would put the -1 in the 2nd row, 3rd column position, which is the transpose position of $x_{3,2}$. Doing this with respect to each variable in the $(5,3)$ Banded Toda flow would result in the gradient matrix

$$\begin{bmatrix} x_{2,2}+x_{3,3}+x_{4,4}+x_{5,5} & -1 & 0 & 0 & 0 \\ 0 & x_{1,1}+x_{3,3}+x_{4,4}+x_{5,5} & -1 & 0 & 0 \\ 0 & 0 & x_{1,1}+x_{2,2}+x_{4,4}+x_{5,5} & -1 & 0 \\ 0 & 0 & 0 & x_{1,1}+x_{2,2}+x_{3,3}+x_{5,5} & -1 \\ 0 & 0 & 0 & 0 & x_{1,1}+x_{2,2}+x_{3,3}+x_{4,4} \end{bmatrix}$$

Knowing how to find the gradient matrix is important since it helps with the proof of the following theorem:

Theorem 3.2. *The function created by taking the trace of the Konstant form of the Toda lattice is a Casimir.*

Proof. The trace of the Konstant form of the Toda lattice is

$$F := x_{1,1}+x_{2,2}+\cdots+x_{n,n} \tag{3.10}$$

where n is the dimension of the matrix. Taking the derivative of this function with respect to each function in the Konstant form results in the matrix

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \tag{3.11}$$

which of course is the $n \times n$ identity matrix. Therefore, the matrix representation of the gradient for the trace of the Konstant form of the Toda lattice is the identity matrix. Now, to verify that the trace is a Casimir, we need to check the Lie-Poisson bracket with this function alongside any other function G. In the following equation, X is the Konstant form of the Toda lattice. Therefore,

$$\{F, G\}(X) = \mathrm{Tr}(X.[\mathrm{grad}_X(F), \mathrm{grad}_X(G)]) \tag{3.12}$$

Though, $\mathrm{grad}_X(F) = I$, therefore for the bracket we have

$$[I, \mathrm{grad}_X(G)] = I.\,\mathrm{grad}_X(G) - \mathrm{grad}_X(G).I = 0. \tag{3.13}$$

Therefore,

$$\{F, G\}(X) = \mathrm{Tr}(X.0) = 0 \tag{3.14}$$

which by definition means that the trace is a Casimir. □

3.3 Constructing First Integrals and Casimirs

As stated in Section 2.3.3, Deift, et. al. in [3] discovered that finding rational functions using the chopping method is one method to find the first integrals and Casimirs of the system. Recall in order to do this, they deleted the first k rows and k columns where k runs from 0 to $\lfloor \frac{1}{2}n \rfloor$ and n is the dimension of the matrix. Then, they found the polynomial expressions, which are of the form found in (2.21). Then, to form the rational expressions, they took the leading coefficient and divided each of the following coefficients of the polynomial by it. The Casimirs of the system are then

$$E_{1k}/E_{0k} \tag{3.15}$$

and the rest are first integrals. For Deift, et. al., these formed enough first integrals so that the definition of Liouville integrability was satisfied.

For this paper, we consider an equivalent form of chopping, which is also discussed in [3]. For this construction, instead of deleting the first k rows and first k columns, we introduce μ_i for $1 \leq i \leq \lfloor \frac{1}{2}n - 1 \rfloor$ in the pivot spots along the secondary diagonal. As an example, we will again view the $(5, 2)$ banded Toda flow. The chopping method results in a matrix that looked like

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 \\ 0 & x_{4,2} & x_{4,3} & x_{4,4} & 1 \\ 0 & 0 & x_{5,3} & x_{5,4} & x_{5,5} \end{bmatrix}$$

The pivot spot refers to the point at which the two lines intersect each other. Using the equivalent form to chopping, we obtain the matrix

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & \mu_1 \\ x_{2,1} & x_{2,2} & 1 & \mu_2 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 \\ 0 & x_{4,2} & x_{4,3} & x_{4,4} & 1 \\ 0 & 0 & x_{5,3} & x_{5,4} & x_{5,5} \end{bmatrix} \tag{3.16}$$

Then, to find the first integrals and Casimirs, we find the characteristic polynomial of the matrix. In this example, the characteristic polynomial is:

$-x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}+x_{1,1}x_{2,2}x_{3,3}x_{5,4}+x_{1,1}x_{2,2}x_{4,3}x_{5,5}+x_{1,1}x_{3,2}x_{4,4}x_{5,5}+x_{2,1}x_{3,3}x_{4,4}x_{5,5}-\mu_1x_{2,1}x_{4,2}x_{5,3}-\mu_1x_{3,1}x_{4,2}x_{5,4}+$
$\mu_2x_{1,1}x_{3,2}x_{5,3}+\mu_2x_{3,1}x_{4,3}x_{5,5}-\mu_2x_{3,1}x_{5,3}+\lambda^5-x_{1,1}x_{2,2}x_{5,3}-x_{1,1}x_{3,2}x_{5,4}-x_{1,1}x_{4,2}x_{5,5}-x_{2,1}x_{3,3}x_{5,4}-x_{2,1}x_{4,3}x_{5,5}-$
$x_{3,1}x_{4,4}x_{5,5}-(x_{4,2}\mu_2-x_{1,1}x_{2,2}-x_{1,1}x_{3,3}-x_{1,1}x_{4,4}-x_{1,1}x_{5,5}-x_{2,2}x_{3,3}-x_{2,2}x_{4,4}-x_{2,2}x_{5,5}-x_{3,3}x_{4,4}-x_{3,3}x_{5,5}-x_{4,4}x_{5,5}+$
$x_{2,1}+x_{3,2}+x_{4,3}+x_{5,4})\lambda^3-(\mu_1x_{3,1}x_{5,3}-\mu_2x_{1,1}x_{4,2}+\mu_2x_{3,2}x_{4,3}-\mu_2x_{3,3}x_{4,2}-\mu_2x_{4,2}x_{5,5}+x_{1,1}x_{2,2}x_{3,3}+x_{1,1}x_{2,2}x_{4,4}+$
$x_{1,1}x_{2,2}x_{5,5}+x_{1,1}x_{3,3}x_{4,4}+x_{1,1}x_{3,3}x_{5,5}+x_{1,1}x_{4,4}x_{5,5}+x_{2,2}x_{3,3}x_{4,4}+x_{2,2}x_{3,3}x_{5,5}+x_{2,2}x_{4,4}x_{5,5}+x_{3,3}x_{4,4}x_{5,5}-x_{1,1}x_{3,2}-$

$x_{1,1}x_{4,3} - x_{5,4}x_{1,1} - x_{2,1}x_{3,3} - x_{2,1}x_{4,4} - x_{2,1}x_{5,5} - x_{2,2}x_{4,3} - x_{5,4}x_{2,2} - x_{3,2}x_{4,4} - x_{3,2}x_{5,5} - x_{5,4}x_{3,3} - x_{4,3}x_{5,5} + x_{3,1} + x_{4,2} + x_{5,3})\lambda^2 - (\mu_1 x_{2,1}x_{3,2}x_{5,3} + \mu_1 x_{2,1}x_{4,2}x_{5,4} - \mu_1 x_{2,2}x_{3,1}x_{5,3} + \mu_1 x_{3,1}x_{4,3}x_{5,4} - \mu_1 x_{3,1}x_{4,4}x_{5,3} - \mu_2 x_{1,1}x_{3,2}x_{4,3} + \mu_2 x_{1,1}x_{3,3}x_{4,2} + \mu_2 x_{1,1}x_{4,2}x_{5,5} - \mu_2 x_{3,2}x_{4,3}x_{5,5} + \mu_2 x_{3,3}x_{4,2}x_{5,5} - x_{1,1}x_{2,2}x_{3,3}x_{4,4} - x_{1,1}x_{2,2}x_{3,3}x_{5,5} - x_{1,1}x_{2,2}x_{4,4}x_{5,5} - x_{1,1}x_{3,3}x_{4,4}x_{5,5} - x_{2,2}x_{3,3}x_{4,4}x_{5,5} + \mu_2 x_{3,1}x_{4,3} + \mu_2 x_{3,2}x_{5,3} + x_{1,1}x_{2,2}x_{4,3} + x_{1,1}x_{2,2}x_{5,4} + x_{1,1}x_{3,2}x_{4,4} + x_{1,1}x_{3,2}x_{5,5} + x_{1,1}x_{3,3}x_{5,4} + x_{1,1}x_{4,3}x_{5,5} + x_{2,1}x_{3,3}x_{4,4} + x_{2,1}x_{3,3}x_{5,5} + x_{2,1}x_{4,4}x_{5,5} + x_{2,2}x_{3,3}x_{5,4} + x_{2,2}x_{4,3}x_{5,5} + x_{3,2}x_{4,4}x_{5,5} - x_{1,1}x_{4,2} - x_{1,1}x_{5,3} - x_{2,1}x_{4,3} - x_{2,1}x_{5,4} - x_{2,2}x_{5,3} - x_{3,1}x_{4,4} - x_{3,1}x_{5,5} - x_{3,2}x_{5,4} - x_{4,2}x_{5,5})\lambda - (x_{5,5} + x_{4,4} + x_{3,3} + x_{2,2} + x_{1,1})\lambda^4 + x_{2,1}x_{5,3} + x_{3,1}x_{5,4} + \mu_1\mu_2 x_{3,1}x_{4,2}x_{5,3} + \mu_1 x_{2,1}x_{3,2}x_{4,4}x_{5,3} + \mu_1 x_{2,1}x_{3,3}x_{4,2}x_{5,4} + \mu_1 x_{2,2}x_{3,1}x_{4,3}x_{5,4} - \mu_1 x_{2,2}x_{3,1}x_{4,4}x_{5,3} - \mu_2 x_{1,1}x_{3,2}x_{4,3}x_{5,5} + \mu_2 x_{1,1}x_{3,3}x_{4,2}x_{5,5} - \mu_1 x_{2,1}x_{3,2}x_{4,3}x_{5,4}$

As one can see, even in the 5 dimensional case, the characteristic polynomial can be pretty horrendous. This turns out to not be too harmful since we have computer code that can parse out the information we need quickly. The code itself can be seen in the appendix, though we will now describe how the code works in order to find the first integrals and Casimirs.

Taking the characteristic polynomial above, sort the polynomial with respect to μ_1 then $\mu_1\mu_2$ and in general with respect to $\mu_1\mu_2\dots\mu_{\lfloor\frac{n}{2}\rfloor}$. When we find the first integrals and Casimirs we do not need consider the factored form of the individual $\mu_2,\dots,\mu_{\lfloor\frac{n}{2}\rfloor}$. In the example above, looking at just the terms with a μ_1 in it, we have
$-c_1c_3\lambda^2 - (-a_2c_1c_3 - a_4c_1c_3 + b_1b_2c_3 + b_1b_4c_2 + b_3b_4c_1)\lambda - b_1c_2c_3 - b_4c_1c_2 - b_1b_2b_3b_4 - a_2a_4c_1c_3 + a_2b_3b_4c_1 + a_3b_1b_4c_2 + a_4b_1b_2c_3$

Then, we form the rational expressions by taking the leading coefficient and dividing it by the remaining coefficients. In the above example, the two polynomials would be

$$\frac{-a_2c_2c_3 - a_3c_1c_3 + b_1b_2c_3 + b_1b_3c_2 + b_3b_4c_1}{-c_1c_3} \tag{3.17}$$

$$\frac{-b_1c_2c_3 - b_4c_1c_2 - b_1b_2b_3b_4 - a_2a_4c_1c_3 + a_2b_3b_4c_1 + a_3b_1b_3c_2 + a_4b_1b_2c_3}{-c_1c_3} \tag{3.18}$$

In the end, only the rational expressions formed by taking the leading coefficient and dividing the next coefficient by it is the only chance one has in order to find a Casimir. The other rational expressions are first integrals. For the full Toda flow, the Casimirs constructed in this manner were enough to determine that the system is Liouville integrable. Though, in the banded Toda flow, this was not the case and will be explained further in chapter 4.

In the example being worked on, one can check that (3.17) is a Casimir and (3.18) is a first integral. For notational purposes, we will denote the rational expressions as F_{kj} where k represents which $\mu_1 \cdots \mu_k$ term we are considering, and j denotes the quotient jth coefficient of λ in descending order over the leading coefficient. In more details,

$$F_k(\lambda) = \frac{\partial^k}{\partial \mu_1 \ldots \partial \mu_k}\bigg|_{\mu_i=0} \det(\lambda I_n - L_\mu), \tag{3.19}$$

$$\frac{F_k(\lambda)}{\text{leading coeff}} = \lambda^{d_k} + \sum_{j=1}^{d_k} F_{kj}\lambda^{d_k - j}. \tag{3.20}$$

3.4 The Proof of Theorem 3.1

Now that we are able to construct the first integrals and Casimirs using the Konstant form of the banded Toda flow, we are now able to prove Theorem 3.1.

For reviewing purposes, in order to prove the theorem, the goal is to give the first integrals and Casimirs so that the system is Liouville integrable. This was guaranteed to us through definition 2.7. Also again, recall that Liouville integrability required the condition that

$$\text{Number of Casimirs} + 2 \times \text{ Number of First integrals } = \dim(L).$$

Therefore, when going through the proof, we will give the Konstant form of the matrix, the dimension of L, the number of Casimirs and first integrals, then explicitly state some of the Casimirs and first integrals. We begin with the $(4, 2)$ case. The Konstant form of the matrix is

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 \\ 0 & x_{4,2} & x_{4,3} & x_{4,4} \end{bmatrix}$$

The dimension of this manifold is 9. There should be 1 Casimir and 4 first integrals. The Casimir is the trace, $x_{1,1} + x_{2,2} + x_{3,3} + x_{4,4}$. The first integrals are

1. $x_{2,2}x_{1,1} + x_{3,3}x_{1,1} + x_{4,4}x_{1,1} + x_{3,3}x_{2,2} + x_{4,4}x_{2,2} + x_{4,4}x_{3,3} - x_{2,1} - x_{3,2} - x_{4,3}$

2. $-x_{1,1}x_{2,2}x_{3,3}-x_{1,1}x_{2,2}x_{4,4}-x_{1,1}x_{3,3}x_{4,4}-x_{2,2}x_{3,3}x_{4,4}+x_{3,2}x_{1,1}+x_{4,3}x_{1,1}+x_{2,1}x_{3,3}+x_{2,1}x_{4,4}+x_{4,3}x_{2,2}+x_{3,2}x_{4,4}-x_{3,1}-x_{4,2}$

3. $x_{1,1}x_{2,2}x_{3,3}x_{4,4}-x_{1,1}x_{2,2}x_{4,3}-x_{1,1}x_{3,2}x_{4,4}-x_{2,1}x_{3,3}x_{4,4}+x_{1,1}x_{4,2}+x_{2,1}x_{4,3}+x_{3,1}x_{4,4}$

4. $\dfrac{-x_{2,1}x_{3,2}x_{4,3}+x_{2,1}x_{3,3}x_{4,2}+x_{2,2}x_{3,1}x_{4,3}-x_{3,1}x_{4,2}}{x_{2,1}x_{4,2}+x_{3,1}x_{4,3}}$

This concludes the dimension 4 banded Toda flow, since the $(4,1)$ and $(4,3)$ cases have already been shown to be Liouville integrable. As a note, recall that all Toda flows of the form $(n,n-1)$ and $(n,1)$ have been shown the be Liouville integrable, so we will not mention those in the rest of the proof.

We move onto the dimension 5 case, beginning with the $(5,2)$ banded Toda flow. This matrix has a Konstant form of the form

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 \\ 0 & x_{4,2} & x_{4,3} & x_{4,4} & 1 \\ 0 & 0 & x_{5,3} & x_{5,4} & x_{5,5} \end{bmatrix}$$

The dimension is 12. The $(5,2)$ Banded Toda Flow has 5 first integrals and 2 Casimirs. The two Casimirs are

1. $x_{1,1}+x_{2,2}+x_{3,3}+x_{4,4}+x_{5,5}$
2. $\dfrac{-x_{2,1}x_{3,2}x_{5,3}-x_{2,1}x_{4,2}x_{5,4}+x_{2,2}x_{3,1}x_{5,3}-x_{3,1}x_{4,3}x_{5,4}+x_{3,1}x_{4,4}x_{5,3}}{x_{3,1}x_{5,3}}$

The first integrals are

1. $x_{2,2}x_{1,1}+x_{3,3}x_{1,1}+x_{4,4}x_{1,1}+x_{5,5}x_{1,1}+x_{3,3}x_{2,2}+x_{4,4}x_{2,2}+x_{5,5}x_{2,2}+x_{4,4}x_{3,3}+x_{5,5}x_{3,3}+x_{5,5}x_{4,4}-x_{2,1}-x_{3,2}-x_{4,3}-x_{5,4}$

2. $-x_{1,1}x_{2,2}x_{3,3}-x_{1,1}x_{2,2}x_{4,4}-x_{1,1}x_{2,2}x_{5,5}-x_{1,1}x_{3,3}x_{4,4}-x_{1,1}x_{3,3}x_{5,5}-x_{1,1}x_{4,4}x_{5,5}-x_{2,2}x_{3,3}x_{4,4}-x_{2,2}x_{3,3}x_{5,5}-x_{2,2}x_{4,4}x_{5,5}-x_{3,3}x_{4,4}x_{5,5}+x_{3,2}x_{1,1}+x_{4,3}x_{1,1}+x_{5,4}x_{1,1}+x_{2,1}x_{3,3}+x_{2,1}x_{4,4}+x_{2,1}x_{5,5}+x_{4,3}x_{2,2}+x_{5,4}x_{2,2}+x_{3,2}x_{4,4}+x_{3,2}x_{5,5}+x_{5,4}x_{3,3}+x_{4,3}x_{5,5}-x_{3,1}-x_{4,2}-x_{5,3}$

3. $x_{1,1}x_{2,2}x_{3,3}x_{4,4}+x_{1,1}x_{2,2}x_{3,3}x_{5,5}+x_{1,1}x_{2,2}x_{4,4}x_{5,5}+x_{1,1}x_{3,3}x_{4,4}x_{5,5}+x_{2,2}x_{3,3}x_{4,4}x_{5,5}-x_{1,1}x_{2,2}x_{4,3}-x_{1,1}x_{2,2}x_{5,4}-x_{1,1}x_{3,2}x_{4,4}-x_{1,1}x_{3,2}x_{5,5}-x_{1,1}x_{3,3}x_{5,4}-x_{1,1}x_{4,3}x_{5,5}-x_{2,1}x_{3,3}x_{4,4}-x_{2,1}x_{3,3}x_{5,5}-x_{2,1}x_{4,4}x_{5,5}-x_{2,2}x_{3,3}x_{5,4}-x_{2,2}x_{4,3}x_{5,5}-x_{3,2}x_{4,4}x_{5,5}+x_{1,1}x_{4,2}+x_{1,1}x_{5,3}+x_{2,1}x_{4,3}+x_{2,1}x_{5,4}+x_{2,2}x_{5,3}+x_{3,1}x_{4,4}+x_{3,1}x_{5,5}+x_{3,2}x_{5,4}+x_{4,2}x_{5,5}$

4. $-x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}+x_{1,1}x_{2,2}x_{3,3}x_{5,4}+x_{1,1}x_{2,2}x_{4,3}x_{5,5}+x_{1,1}x_{3,2}x_{4,4}x_{5,5}+x_{2,1}x_{3,3}x_{4,4}x_{5,5}-x_{1,1}x_{2,2}x_{5,3}-x_{1,1}x_{3,2}x_{5,4}-x_{1,1}x_{4,2}x_{5,5}-x_{2,1}x_{3,3}x_{5,4}-x_{2,1}x_{4,3}x_{5,5}-x_{3,1}x_{4,4}x_{5,5}+x_{2,1}x_{5,3}+x_{3,1}x_{5,4}$

5. $\frac{-x_{2,1}x_{3,2}x_{4,3}x_{5,4}+x_{2,1}x_{3,2}x_{4,4}x_{5,3}+x_{2,1}x_{3,3}x_{4,2}x_{5,4}+x_{2,2}x_{3,1}x_{4,3}x_{5,4}-x_{2,2}x_{3,1}x_{4,4}x_{5,3}-x_{2,1}x_{4,2}x_{5,3}-x_{3,1}x_{4,2}x_{5,4}}{x_{3,1}x_{5,3}}$

The $(5,3)$ Banded Toda flow has the Konstant form

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & 1 \\ 0 & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{bmatrix}$$

The dimension of the matrix is 14. There are 2 Casimirs and 6 First Integrals. The two Casimirs are

1. $x_{1,1}+x_{2,2}+x_{3,3}+x_{4,4}+x_{5,5}$

2. $-\frac{-x_{3,1}x_{4,2}x_{5,3}+x_{3,1}x_{4,3}x_{5,2}+x_{3,2}x_{4,1}x_{5,3}-x_{3,3}x_{4,1}x_{5,2}}{x_{4,1}x_{5,2}}$.

The first integrals are

1. $x_{2,2}x_{1,1}+x_{3,3}x_{1,1}+x_{4,4}x_{1,1}+x_{5,5}x_{1,1}+x_{3,3}x_{2,2}+x_{4,4}x_{2,2}+x_{5,5}x_{2,2}+x_{4,4}x_{3,3}+x_{5,5}x_{3,3}+x_{5,5}x_{4,4}-x_{2,1}-x_{3,2}-x_{4,3}-x_{5,4}$

2. $x_{2,2}x_{1,1}+x_{3,3}x_{1,1}+x_{4,4}x_{1,1}+x_{5,5}x_{1,1}+x_{3,3}x_{2,2}+x_{4,4}x_{2,2}+x_{5,5}x_{2,2}+x_{4,4}x_{3,3}+x_{5,5}x_{3,3}+x_{5,5}x_{4,4}-x_{2,1}-x_{3,2}-x_{4,3}-x_{5,4}$

3. $x_{1,1}x_{2,2}x_{3,3}x_{4,4}+x_{1,1}x_{2,2}x_{3,3}x_{5,5}+x_{1,1}x_{2,2}x_{4,4}x_{5,5}+x_{1,1}x_{3,3}x_{4,4}x_{5,5}+x_{2,2}x_{3,3}x_{4,4}x_{5,5}-x_{1,1}x_{2,2}x_{4,3}-x_{1,1}x_{2,2}x_{5,4}-x_{1,1}x_{3,2}x_{4,4}-x_{1,1}x_{3,2}x_{5,5}-x_{1,1}x_{3,3}x_{5,4}-x_{1,1}x_{4,3}x_{5,5}-$

$x_{2,1}x_{3,3}x_{4,4} - x_{2,1}x_{3,3}x_{5,5} - x_{2,1}x_{4,4}x_{5,5} - x_{2,2}x_{3,3}x_{5,4} - x_{2,2}x_{4,3}x_{5,5} - x_{3,2}x_{4,4}x_{5,5} + x_{1,1}x_{4,2} + x_{1,1}x_{5,3} + x_{2,1}x_{4,3} + x_{2,1}x_{5,4} + x_{2,2}x_{5,3} + x_{3,1}x_{4,4} + x_{3,1}x_{5,5} + x_{3,2}x_{5,4} + x_{4,2}x_{5,5} - x_{4,1} - x_{5,2}$

4. $-x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}+x_{1,1}x_{2,2}x_{3,3}x_{5,4}+x_{1,1}x_{2,2}x_{4,3}x_{5,5}+x_{1,1}x_{3,2}x_{4,4}x_{5,5}+x_{2,1}x_{3,3}x_{4,4}x_{5,5}- x_{1,1}x_{2,2}x_{5,3} - x_{1,1}x_{3,2}x_{5,4} - x_{1,1}x_{4,2}x_{5,5} - x_{2,1}x_{3,3}x_{5,4} - x_{2,1}x_{4,3}x_{5,5} - x_{3,1}x_{4,4}x_{5,5} + x_{1,1}x_{5,2} + x_{2,1}x_{5,3} + x_{3,1}x_{5,4} + x_{4,1}x_{5,5}$

5. $\frac{1}{-x_{2,1}x_{5,2}-x_{3,1}x_{5,3}-x_{4,1}x_{5,4}}(-x_{2,1}x_{3,2}x_{5,3} + x_{2,1}x_{3,3}x_{5,2} - x_{2,1}x_{4,2}x_{5,4} + x_{2,1}x_{4,4}x_{5,2} + x_{2,2}x_{3,1}x_{5,3} + x_{2,2}x_{4,1}x_{5,4} - x_{3,1}x_{4,3}x_{5,4} + x_{3,1}x_{4,4}x_{5,3} + x_{3,3}x_{4,1}x_{5,4} - x_{3,1}x_{5,2} - x_{4,1}x_{5,3})$

6. $\frac{1}{-x_{2,1}x_{5,2}-x_{3,1}x_{5,3}-x_{4,1}x_{5,4}}(-x_{2,1}x_{3,2}x_{4,3}x_{5,4}+x_{2,1}x_{3,2}x_{4,4}x_{5,3}+x_{2,1}x_{3,3}x_{4,2}x_{5,4}-x_{2,1}x_{3,3}x_{4,4}x_{5,2}+ x_{2,2}x_{3,1}x_{4,3}x_{5,4} - x_{2,2}x_{3,1}x_{4,4}x_{5,3} - x_{2,2}x_{3,3}x_{4,1}x_{5,4} - x_{2,1}x_{4,2}x_{5,3} + x_{2,1}x_{4,3}x_{5,2} + x_{2,2}x_{4,1}x_{5,3} - x_{3,1}x_{4,2}x_{5,4} + x_{3,1}x_{4,4}x_{5,2} + x_{3,2}x_{4,1}x_{5,4} - x_{4,1}x_{5,2})$

This concludes the dimension 5 Banded Toda Flow. Now, we move to the dimension 6 Banded Toda flow. Beginning with the $(6,2)$ case, the Konstant form is

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 & 0 \\ 0 & x_{4,2} & x_{4,3} & x_{4,4} & 1 & 0 \\ 0 & 0 & x_{5,3} & x_{5,4} & x_{5,5} & 1 \\ 0 & 0 & 0 & x_{6,4} & x_{6,5} & x_{6,6} \end{bmatrix}.$$

The dimension is 15. There are 1 Casimir (that being the trace) and 7 first integrals. The first integrals are

1. $x_{2,2}x_{1,1} + x_{3,3}x_{1,1} + x_{4,4}x_{1,1} + x_{5,5}x_{1,1} + x_{6,6}x_{1,1} + x_{3,3}x_{2,2} + x_{4,4}x_{2,2} + x_{5,5}x_{2,2} + x_{6,6}x_{2,2} + x_{4,4}x_{3,3} + x_{5,5}x_{3,3} + x_{6,6}x_{3,3} + x_{5,5}x_{4,4} + x_{6,6}x_{4,4} + x_{6,6}x_{5,5} - x_{2,1} - x_{3,2} - x_{4,3} - x_{5,4} - x_{6,5}$

2. $-x_{1,1}x_{2,2}x_{3,3} - x_{1,1}x_{2,2}x_{4,4} - x_{1,1}x_{2,2}x_{5,5} - x_{1,1}x_{2,2}x_{6,6} - x_{1,1}x_{3,3}x_{4,4} - x_{1,1}x_{3,3}x_{5,5} - x_{1,1}x_{3,3}x_{6,6} - x_{1,1}x_{4,4}x_{5,5} - x_{1,1}x_{4,4}x_{6,6} - x_{1,1}x_{5,5}x_{6,6} - x_{2,2}x_{3,3}x_{4,4} - x_{2,2}x_{3,3}x_{5,5} - x_{2,2}x_{3,3}x_{6,6} - x_{2,2}x_{4,4}x_{5,5} - x_{2,2}x_{4,4}x_{6,6} - x_{2,2}x_{5,5}x_{6,6} - x_{3,3}x_{4,4}x_{5,5} - x_{3,3}x_{4,4}x_{6,6} - x_{3,3}x_{5,5}x_{6,6} - x_{4,4}x_{5,5}x_{6,6} + x_{3,2}x_{1,1} + x_{4,3}x_{1,1} + x_{5,4}x_{1,1} + x_{6,5}x_{1,1} + x_{2,1}x_{3,3} + x_{2,1}x_{4,4} +$

$x_{2,1}x_{5,5} + x_{2,1}x_{6,6} + x_{4,3}x_{2,2} + x_{5,4}x_{2,2} + x_{6,5}x_{2,2} + x_{3,2}x_{4,4} + x_{3,2}x_{5,5} + x_{3,2}x_{6,6} + x_{5,4}x_{3,3} + x_{6,5}x_{3,3} + x_{4,3}x_{5,5} + x_{4,3}x_{6,6} + x_{6,5}x_{4,4} + x_{5,4}x_{6,6} - x_{3,1} - x_{4,2} - x_{5,3} - x_{6,4}$

3. $x_{1,1}x_{2,2}x_{3,3}x_{4,4}+x_{1,1}x_{2,2}x_{3,3}x_{5,5}+x_{1,1}x_{2,2}x_{3,3}x_{6,6}+x_{1,1}x_{2,2}x_{4,4}x_{5,5}+x_{1,1}x_{2,2}x_{4,4}x_{6,6}+x_{1,1}x_{2,2}x_{5,5}x_{6,6}+x_{1,1}x_{3,3}x_{4,4}x_{5,5}+x_{1,1}x_{3,3}x_{4,4}x_{6,6}+x_{1,1}x_{3,3}x_{5,5}x_{6,6}+x_{1,1}x_{4,4}x_{5,5}x_{6,6}+x_{2,2}x_{3,3}x_{4,4}x_{5,5}+x_{2,2}x_{3,3}x_{4,4}x_{6,6}+x_{2,2}x_{3,3}x_{5,5}x_{6,6}+x_{2,2}x_{4,4}x_{5,5}x_{6,6}+x_{3,3}x_{4,4}x_{5,5}x_{6,6}-x_{1,1}x_{2,2}x_{4,3}-x_{1,1}x_{2,2}x_{5,4}-x_{1,1}x_{2,2}x_{6,5}-x_{1,1}x_{3,2}x_{4,4}-x_{1,1}x_{3,2}x_{5,5}-x_{1,1}x_{3,2}x_{6,6}-x_{1,1}x_{3,3}x_{5,4}-x_{1,1}x_{3,3}x_{6,5} - x_{1,1}x_{4,3}x_{5,5} - x_{1,1}x_{4,3}x_{6,6} - x_{1,1}x_{4,4}x_{6,5} - x_{1,1}x_{5,4}x_{6,6} - x_{2,1}x_{3,3}x_{4,4} - x_{2,1}x_{3,3}x_{5,5} - x_{2,1}x_{3,3}x_{6,6} - x_{2,1}x_{4,4}x_{5,5} - x_{2,1}x_{4,4}x_{6,6} - x_{2,1}x_{5,5}x_{6,6} - x_{2,2}x_{3,3}x_{5,4} - x_{2,2}x_{3,3}x_{6,5} - x_{2,2}x_{4,3}x_{5,5} - x_{2,2}x_{4,3}x_{6,6} - x_{2,2}x_{4,4}x_{6,5} - x_{2,2}x_{5,4}x_{6,6}-x_{3,2}x_{4,4}x_{5,5}-x_{3,2}x_{4,4}x_{6,6}-x_{3,2}x_{5,5}x_{6,6}-x_{3,3}x_{4,4}x_{6,5}-x_{3,3}x_{5,4}x_{6,6}-x_{4,3}x_{5,5}x_{6,6}+x_{1,1}x_{4,2}+x_{1,1}x_{5,3}+x_{1,1}x_{6,4} + x_{2,1}x_{4,3} + x_{2,1}x_{5,4} + x_{2,1}x_{6,5} + x_{2,2}x_{5,3} + x_{2,2}x_{6,4} + x_{3,1}x_{4,4} + x_{3,1}x_{5,5} + x_{3,1}x_{6,6} + x_{3,2}x_{5,4} + x_{3,2}x_{6,5} + x_{3,3}x_{6,4} + x_{4,2}x_{5,5} + x_{4,2}x_{6,6} + x_{4,3}x_{6,5} + x_{5,3}x_{6,6}$

4. $-x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5} - x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{6,6} - x_{1,1}x_{2,2}x_{3,3}x_{5,5}x_{6,6} - x_{1,1}x_{2,2}x_{4,4}x_{5,5}x_{6,6} - x_{1,1}x_{3,3}x_{4,4}x_{5,5}x_{6,6} - x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6} + x_{1,1}x_{2,2}x_{3,3}x_{5,4} + x_{1,1}x_{2,2}x_{3,3}x_{6,5} + x_{1,1}x_{2,2}x_{4,3}x_{5,5} + x_{1,1}x_{2,2}x_{4,3}x_{6,6} + x_{1,1}x_{2,2}x_{4,4}x_{6,5} + x_{1,1}x_{2,2}x_{5,4}x_{6,6}+x_{1,1}x_{3,2}x_{4,4}x_{5,5}+x_{1,1}x_{3,2}x_{4,4}x_{6,6}+x_{1,1}x_{3,2}x_{5,5}x_{6,6}+x_{1,1}x_{3,3}x_{4,4}x_{6,5}+x_{1,1}x_{3,3}x_{5,4}x_{6,6}+x_{1,1}x_{4,3}x_{5,5}x_{6,6}+x_{2,1}x_{3,3}x_{4,4}x_{5,5}+x_{2,1}x_{3,3}x_{4,4}x_{6,6}+x_{2,1}x_{3,3}x_{5,5}x_{6,6}+x_{2,1}x_{4,4}x_{5,5}x_{6,6}+x_{2,2}x_{3,3}x_{4,4}x_{6,5}+x_{2,2}x_{3,3}x_{5,4}x_{6,6}+x_{2,2}x_{4,3}x_{5,5}x_{6,6}+x_{3,2}x_{4,4}x_{5,5}x_{6,6}-x_{1,1}x_{2,2}x_{5,3}-x_{1,1}x_{2,2}x_{6,4}-x_{1,1}x_{3,2}x_{5,4}-x_{1,1}x_{3,2}x_{6,5}-x_{1,1}x_{3,3}x_{6,4}-x_{1,1}x_{4,2}x_{5,5}-x_{1,1}x_{4,2}x_{6,6}-x_{1,1}x_{4,3}x_{6,5} - x_{1,1}x_{5,3}x_{6,6} - x_{2,1}x_{3,3}x_{5,4} - x_{2,1}x_{3,3}x_{6,5} - x_{2,1}x_{4,3}x_{5,5} - x_{2,1}x_{4,3}x_{6,6} - x_{2,1}x_{4,4}x_{6,5} - x_{2,1}x_{5,4}x_{6,6} - x_{2,2}x_{3,3}x_{6,4} - x_{2,2}x_{4,3}x_{6,5} - x_{2,2}x_{5,3}x_{6,6} - x_{3,1}x_{4,4}x_{5,5} - x_{3,1}x_{4,4}x_{6,6} - x_{3,1}x_{5,5}x_{6,6} - x_{3,2}x_{4,4}x_{6,5} - x_{3,2}x_{5,4}x_{6,6} - x_{4,2}x_{5,5}x_{6,6} + x_{2,1}x_{5,3} + x_{2,1}x_{6,4} + x_{3,1}x_{5,4} + x_{3,1}x_{6,5} + x_{3,2}x_{6,4} + x_{4,2}x_{6,5}$

5. $x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6} - x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{6,5} - x_{1,1}x_{2,2}x_{3,3}x_{5,4}x_{6,6} - x_{1,1}x_{2,2}x_{4,3}x_{5,5}x_{6,6} - x_{1,1}x_{3,2}x_{4,4}x_{5,5}x_{6,6} - x_{2,1}x_{3,3}x_{4,4}x_{5,5}x_{6,6} + x_{1,1}x_{2,2}x_{3,3}x_{6,4} + x_{1,1}x_{2,2}x_{4,3}x_{6,5} + x_{1,1}x_{2,2}x_{5,3}x_{6,6} + x_{1,1}x_{3,2}x_{4,4}x_{6,5} + x_{1,1}x_{3,2}x_{5,4}x_{6,6} + x_{1,1}x_{4,2}x_{5,5}x_{6,6}+x_{2,1}x_{3,3}x_{4,4}x_{6,5}+x_{2,1}x_{3,3}x_{5,4}x_{6,6}+x_{2,1}x_{4,3}x_{5,5}x_{6,6}+x_{3,1}x_{4,4}x_{5,5}x_{6,6}-x_{1,1}x_{3,2}x_{6,4}-x_{1,1}x_{4,2}x_{6,5}-x_{2,1}x_{3,3}x_{6,4} - x_{2,1}x_{4,3}x_{6,5} - x_{2,1}x_{5,3}x_{6,6} - x_{3,1}x_{4,4}x_{6,5} - x_{3,1}x_{5,4}x_{6,6} + x_{3,1}x_{6,4}$

6. $\frac{1}{-x_{2,1}x_{4,2}x_{6,4}-x_{3,1}x_{4,3}x_{6,4}-x_{3,1}x_{5,3}x_{6,5}}(-x_{2,1}x_{3,2}x_{4,3}x_{6,4}-x_{2,1}x_{3,2}x_{5,3}x_{6,5}+x_{2,1}x_{3,3}x_{4,2}x_{6,4}-x_{2,1}x_{4,2}x_{5,4}x_{6,5}+x_{2,1}x_{4,2}x_{5,5}x_{6,4}+x_{2,2}x_{3,1}x_{4,3}x_{6,4}+x_{2,2}x_{3,1}x_{5,3}x_{6,5}-x_{3,1}x_{4,3}x_{5,4}x_{6,5}+x_{3,1}x_{4,3}x_{5,5}x_{6,4}+x_{3,1}x_{4,4}x_{5,3}x_{6,5}-x_{3,1}x_{4,2}x_{6,4}-x_{3,1}x_{5,3}x_{6,4})$

7. $\frac{1}{-x_{2,1}x_{4,2}x_{6,4}-x_{3,1}x_{4,3}x_{6,4}-x_{3,1}x_{5,3}x_{6,5}}(-x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{6,5}+x_{2,1}x_{3,2}x_{4,3}x_{5,5}x_{6,4}+x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{6,5}+x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{6,5}-x_{2,1}x_{3,3}x_{4,2}x_{5,5}x_{6,4}+x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{6,5}-x_{2,2}x_{3,1}x_{4,3}x_{5,5}x_{6,4}-x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{6,5}-x_{2,1}x_{3,2}x_{5,3}x_{6,4}-x_{2,1}x_{4,2}x_{5,3}x_{6,5}+x_{2,2}x_{3,1}x_{5,3}x_{6,4} - x_{3,1}x_{4,2}x_{5,4}x_{6,5} + x_{3,1}x_{4,2}x_{5,5}x_{6,4}).$

The $(6,3)$ banded Toda flow is next. The Konstant form of the matrix is

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & 1 & 0 \\ 0 & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} & 1 \\ 0 & 0 & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} \end{bmatrix}$$

The dimension is 18. There are 2 Casimirs, one being the trace. There are 8 first integrals. The other Casimir is

$$\frac{1}{-x_{3,1}x_{5,2}x_{6,3}-x_{4,1}x_{5,2}x_{6,4}}(-x_{3,1}x_{4,2}x_{5,3}x_{6,4}+x_{3,1}x_{4,2}x_{5,4}x_{6,3}+x_{3,1}x_{4,3}x_{5,2}x_{6,4}-x_{3,1}x_{4,4}x_{5,2}x_{6,3}+x_{3,2}x_{4,1}x_{5,3}x_{6,4}-x_{3,2}x_{4,1}x_{5,4}x_{6,3}-x_{3,3}x_{4,1}x_{5,2}x_{6,4}+x_{4,1}x_{5,2}x_{6,3})-\frac{1}{x_{3,1}x_{6,3}+x_{4,1}x_{6,4}}(-x_{2,1}x_{3,2}x_{6,3}-x_{2,1}x_{4,2}x_{6,4}-x_{6,5}x_{5,2}x_{2,1}+x_{2,2}x_{3,1}x_{6,3}+x_{2,2}x_{4,1}x_{6,4}-x_{3,1}x_{4,3}x_{6,4}+x_{3,1}x_{4,4}x_{6,3}-x_{3,1}x_{5,3}x_{6,5}+x_{3,1}x_{5,5}x_{6,3}+x_{3,3}x_{4,1}x_{6,4}-x_{4,1}x_{5,4}x_{6,5}+x_{4,1}x_{5,5}x_{6,4}-x_{4,1}x_{6,3})$$

The first integrals are

1. $x_{2,2}x_{1,1}+x_{3,3}x_{1,1}+x_{4,4}x_{1,1}+x_{5,5}x_{1,1}+x_{6,6}x_{1,1}+x_{3,3}x_{2,2}+x_{4,4}x_{2,2}+x_{5,5}x_{2,2}+x_{6,6}x_{2,2}+x_{4,4}x_{3,3}+x_{5,5}x_{3,3}+x_{6,6}x_{3,3}+x_{5,5}x_{4,4}+x_{6,6}x_{4,4}+x_{6,6}x_{5,5}-x_{2,1}-x_{3,2}-x_{4,3}-x_{5,4}-x_{6,5}$

2. $-x_{1,1}x_{2,2}x_{3,3}-x_{1,1}x_{2,2}x_{4,4}-x_{1,1}x_{2,2}x_{5,5}-x_{1,1}x_{2,2}x_{6,6}-x_{1,1}x_{3,3}x_{4,4}-x_{1,1}x_{3,3}x_{5,5}-x_{1,1}x_{3,3}x_{6,6}-x_{1,1}x_{4,4}x_{5,5}-x_{1,1}x_{4,4}x_{6,6}-x_{1,1}x_{5,5}x_{6,6}-x_{2,2}x_{3,3}x_{4,4}-x_{2,2}x_{3,3}x_{5,5}-x_{2,2}x_{3,3}x_{6,6}-x_{2,2}x_{4,4}x_{5,5}-x_{2,2}x_{4,4}x_{6,6}-x_{2,2}x_{5,5}x_{6,6}-x_{3,3}x_{4,4}x_{5,5}-x_{3,3}x_{4,4}x_{6,6}-x_{3,3}x_{5,5}x_{6,6}-x_{4,4}x_{5,5}x_{6,6}+x_{3,2}x_{1,1}+x_{4,3}x_{1,1}+x_{5,4}x_{1,1}+x_{6,5}x_{1,1}+x_{2,1}x_{3,3}+x_{2,1}x_{4,4}+x_{2,1}x_{5,5}+x_{2,1}x_{6,6}+x_{4,3}x_{2,2}+x_{5,4}x_{2,2}+x_{6,5}x_{2,2}+x_{3,2}x_{4,4}+x_{3,2}x_{5,5}+x_{3,2}x_{6,6}+x_{5,4}x_{3,3}+x_{6,5}x_{3,3}+x_{4,3}x_{5,5}+x_{4,3}x_{6,6}+x_{6,5}x_{4,4}+x_{5,4}x_{6,6}-x_{3,1}-x_{4,2}-x_{5,3}-x_{6,4}$

3. $x_{1,1}x_{2,2}x_{3,3}x_{4,4}+x_{1,1}x_{2,2}x_{3,3}x_{5,5}+x_{1,1}x_{2,2}x_{3,3}x_{6,6}+x_{1,1}x_{2,2}x_{4,4}x_{5,5}+x_{1,1}x_{2,2}x_{4,4}x_{6,6}+x_{1,1}x_{2,2}x_{5,5}x_{6,6}+x_{1,1}x_{3,3}x_{4,4}x_{5,5}+x_{1,1}x_{3,3}x_{4,4}x_{6,6}+x_{1,1}x_{3,3}x_{5,5}x_{6,6}+x_{1,1}x_{4,4}x_{5,5}x_{6,6}+x_{2,2}x_{3,3}x_{4,4}x_{5,5}+x_{2,2}x_{3,3}x_{4,4}x_{6,6}+x_{2,2}x_{3,3}x_{5,5}x_{6,6}+x_{2,2}x_{4,4}x_{5,5}x_{6,6}+x_{3,3}x_{4,4}x_{5,5}x_{6,6}-x_{1,1}x_{2,2}x_{4,3}-x_{1,1}x_{2,2}x_{5,4}-x_{1,1}x_{2,2}x_{6,5}-x_{1,1}x_{3,2}x_{4,4}-x_{1,1}x_{3,2}x_{5,5}-x_{1,1}x_{3,2}x_{6,6}-x_{1,1}x_{3,3}x_{5,4}-x_{1,1}x_{3,3}x_{6,5}-x_{1,1}x_{4,3}x_{5,5}-x_{1,1}x_{4,3}x_{6,6}-x_{1,1}x_{4,4}x_{6,5}-x_{1,1}x_{5,4}x_{6,6}-x_{2,1}x_{3,3}x_{4,4}-x_{2,1}x_{3,3}x_{5,5}-x_{2,1}x_{3,3}x_{6,6}-x_{2,1}x_{4,4}x_{5,5}-x_{2,1}x_{4,4}x_{6,6}-x_{2,1}x_{5,5}x_{6,6}-x_{2,2}x_{3,3}x_{5,4}-x_{2,2}x_{3,3}x_{6,5}-x_{2,2}x_{4,3}x_{5,5}-x_{2,2}x_{4,3}x_{6,6}-x_{2,2}x_{4,4}x_{6,5}-x_{2,2}x_{5,4}x_{6,6}-x_{3,2}x_{4,4}x_{5,5}-x_{3,2}x_{4,4}x_{6,6}-x_{3,2}x_{5,5}x_{6,6}-x_{3,3}x_{4,4}x_{6,5}-x_{3,3}x_{5,4}x_{6,6}-x_{4,3}x_{5,5}x_{6,6}+x_{1,1}x_{4,2}+x_{1,1}x_{5,3}+$

$x_{1,1}x_{6,4} + x_{2,1}x_{4,3} + x_{2,1}x_{5,4} + x_{2,1}x_{6,5} + x_{2,2}x_{5,3} + x_{2,2}x_{6,4} + x_{3,1}x_{4,4} + x_{3,1}x_{5,5} + x_{3,1}x_{6,6} + x_{3,2}x_{5,4} + x_{3,2}x_{6,5} + x_{3,3}x_{6,4} + x_{4,2}x_{5,5} + x_{4,2}x_{6,6} + x_{4,3}x_{6,5} + x_{5,3}x_{6,6} - x_{4,1} - x_{5,2} - x_{6,3}$

4. $-x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5} - x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{6,6} - x_{1,1}x_{2,2}x_{3,3}x_{5,5}x_{6,6} - x_{1,1}x_{2,2}x_{4,4}x_{5,5}x_{6,6} - x_{1,1}x_{3,3}x_{4,4}x_{5,5}x_{6,6} - x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6} + x_{1,1}x_{2,2}x_{3,3}x_{5,4} + x_{1,1}x_{2,2}x_{3,3}x_{6,5} + x_{1,1}x_{2,2}x_{4,3}x_{5,5} + x_{1,1}x_{2,2}x_{4,3}x_{6,6} + x_{1,1}x_{2,2}x_{4,4}x_{6,5} + x_{1,1}x_{2,2}x_{5,4}x_{6,6} + x_{1,1}x_{3,2}x_{4,4}x_{5,5} + x_{1,1}x_{3,2}x_{4,4}x_{6,6} + x_{1,1}x_{3,2}x_{5,5}x_{6,6} + x_{1,1}x_{3,3}x_{4,4}x_{6,5} + x_{1,1}x_{3,3}x_{5,4}x_{6,6} + x_{1,1}x_{4,3}x_{5,5}x_{6,6} + x_{2,1}x_{3,3}x_{4,4}x_{5,5} + x_{2,1}x_{3,3}x_{4,4}x_{6,6} + x_{2,1}x_{3,3}x_{5,5}x_{6,6} + x_{2,1}x_{4,4}x_{5,5}x_{6,6} + x_{2,2}x_{3,3}x_{4,4}x_{6,5} + x_{2,2}x_{3,3}x_{5,4}x_{6,6} + x_{2,2}x_{4,3}x_{5,5}x_{6,6} + x_{3,2}x_{4,4}x_{5,5}x_{6,6} - x_{1,1}x_{2,2}x_{5,3} - x_{1,1}x_{2,2}x_{6,4} - x_{1,1}x_{3,2}x_{5,4} - x_{1,1}x_{3,2}x_{6,5} - x_{1,1}x_{3,3}x_{6,4} - x_{1,1}x_{4,2}x_{5,5} - x_{1,1}x_{4,2}x_{6,6} - x_{1,1}x_{4,3}x_{6,5} - x_{1,1}x_{5,3}x_{6,6} - x_{2,1}x_{3,3}x_{5,4} - x_{2,1}x_{3,3}x_{6,5} - x_{2,1}x_{4,3}x_{5,5} - x_{2,1}x_{4,3}x_{6,6} - x_{2,1}x_{4,4}x_{6,5} - x_{2,1}x_{5,4}x_{6,6} - x_{2,2}x_{3,3}x_{6,4} - x_{2,2}x_{4,3}x_{6,5} - x_{2,2}x_{5,3}x_{6,6} - x_{3,1}x_{4,4}x_{5,5} - x_{3,1}x_{4,4}x_{6,6} - x_{3,1}x_{5,5}x_{6,6} - x_{3,2}x_{4,4}x_{6,5} - x_{3,2}x_{5,4}x_{6,6} - x_{4,2}x_{5,5}x_{6,6} + x_{1,1}x_{5,2} + x_{1,1}x_{6,3} + x_{2,1}x_{5,3} + x_{2,1}x_{6,4} + x_{2,2}x_{6,3} + x_{3,1}x_{5,4} + x_{3,1}x_{6,5} + x_{3,2}x_{6,4} + x_{4,1}x_{5,5} + x_{4,1}x_{6,6} + x_{4,2}x_{6,5} + x_{5,2}x_{6,6}$

5. $x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6} - x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{6,5} - x_{1,1}x_{2,2}x_{3,3}x_{5,4}x_{6,6} - x_{1,1}x_{2,2}x_{4,3}x_{5,5}x_{6,6} - x_{1,1}x_{3,2}x_{4,4}x_{5,5}x_{6,6} - x_{2,1}x_{3,3}x_{4,4}x_{5,5}x_{6,6} + x_{1,1}x_{2,2}x_{3,3}x_{6,4} + x_{1,1}x_{2,2}x_{4,3}x_{6,5} + x_{1,1}x_{2,2}x_{5,3}x_{6,6} + x_{1,1}x_{3,2}x_{4,4}x_{6,5} + x_{1,1}x_{3,2}x_{5,4}x_{6,6} + x_{1,1}x_{4,2}x_{5,5}x_{6,6} + x_{2,1}x_{3,3}x_{4,4}x_{6,5} + x_{2,1}x_{3,3}x_{5,4}x_{6,6} + x_{2,1}x_{4,3}x_{5,5}x_{6,6} + x_{3,1}x_{4,4}x_{5,5}x_{6,6} - x_{1,1}x_{2,2}x_{6,3} - x_{1,1}x_{3,2}x_{6,4} - x_{1,1}x_{4,2}x_{6,5} - x_{1,1}x_{5,2}x_{6,6} - x_{2,1}x_{3,3}x_{6,4} - x_{2,1}x_{4,3}x_{6,5} - x_{2,1}x_{5,3}x_{6,6} - x_{3,1}x_{4,4}x_{6,5} - x_{3,1}x_{5,4}x_{6,6} - x_{4,1}x_{5,5}x_{6,6} + x_{2,1}x_{6,3} + x_{3,1}x_{6,4} + x_{4,1}x_{6,5}$

6. $\frac{1}{-x_{3,1}x_{6,3} - x_{4,1}x_{6,4}}(-x_{2,1}x_{3,2}x_{6,3} - x_{2,1}x_{4,2}x_{6,4} - x_{2,1}x_{5,2}x_{6,5} + x_{2,2}x_{3,1}x_{6,3} + x_{2,2}x_{4,1}x_{6,4} - x_{3,1}x_{4,3}x_{6,4} + x_{3,1}x_{4,4}x_{6,3} - x_{3,1}x_{5,3}x_{6,5} + x_{3,1}x_{5,5}x_{6,3} + x_{3,3}x_{4,1}x_{6,4} - x_{4,1}x_{5,4}x_{6,5} + x_{4,1}x_{5,5}x_{6,4} - x_{4,1}x_{6,3})$

7. $\frac{1}{-x_{3,1}x_{6,3} - x_{4,1}x_{6,4}}(-x_{2,1}x_{3,2}x_{4,3}x_{6,4} + x_{2,1}x_{3,2}x_{4,4}x_{6,3} - x_{2,1}x_{3,2}x_{5,3}x_{6,5} + x_{2,1}x_{3,2}x_{5,5}x_{6,3} + x_{2,1}x_{3,3}x_{4,2}x_{6,4} + x_{2,1}x_{3,3}x_{5,2}x_{6,5} - x_{2,1}x_{4,2}x_{5,4}x_{6,5} + x_{2,1}x_{4,2}x_{5,5}x_{6,4} + x_{2,1}x_{4,4}x_{5,2}x_{6,5} + x_{2,2}x_{3,1}x_{4,3}x_{6,4} - x_{2,2}x_{3,1}x_{4,4}x_{6,3} + x_{2,2}x_{3,1}x_{5,3}x_{6,5} - x_{2,2}x_{3,1}x_{5,5}x_{6,3} - x_{2,2}x_{3,3}x_{4,1}x_{6,4} + x_{2,2}x_{4,1}x_{5,4}x_{6,5} - x_{2,2}x_{4,1}x_{5,5}x_{6,4} - x_{3,1}x_{4,3}x_{5,4}x_{6,5} + x_{3,1}x_{4,3}x_{5,5}x_{6,4} + x_{3,1}x_{4,4}x_{5,3}x_{6,5} - x_{3,1}x_{4,4}x_{5,5}x_{6,3} + x_{3,3}x_{4,1}x_{5,4}x_{6,5} - x_{3,3}x_{4,1}x_{5,5}x_{6,4} - x_{2,1}x_{4,2}x_{6,3} - x_{2,1}x_{5,2}x_{6,4} + x_{2,2}x_{4,1}x_{6,3} - x_{3,1}x_{4,2}x_{6,4} - x_{3,1}x_{5,2}x_{6,5} - x_{3,1}x_{5,3}x_{6,4} + x_{3,1}x_{5,4}x_{6,3} + x_{3,2}x_{4,1}x_{6,4} - x_{4,1}x_{5,3}x_{6,5} + x_{4,1}x_{5,5}x_{6,3})$

8. $\frac{1}{-x_{3,1}x_{6,3} - x_{4,1}x_{6,4}}(-x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{6,5} + x_{2,1}x_{3,2}x_{4,3}x_{5,5}x_{6,4} + x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{6,5} - x_{2,1}x_{3,2}x_{4,4}x_{5,5}x_{6,3} + x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{6,5} - x_{2,1}x_{3,3}x_{4,2}x_{5,5}x_{6,4} - x_{2,1}x_{3,3}x_{4,4}x_{5,2}x_{6,5} + x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{6,5} - x_{2,2}x_{3,1}x_{4,3}x_{5,5}x_{6,4} - x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{6,5} + x_{2,2}x_{3,1}x_{4,4}x_{5,5}x_{6,3} - x_{2,2}x_{3,3}x_{4,1}x_{5,4}x_{6,5} + x_{2,2}x_{3,3}x_{4,1}x_{5,5}x_{6,4} - x_{2,1}x_{3,2}x_{5,3}x_{6,4} + x_{2,1}x_{3,2}x_{5,4}x_{6,3} + x_{2,1}x_{3,3}x_{5,2}x_{6,4} - x_{2,1}x_{4,2}x_{5,3}x_{6,5} + x_{2,1}x_{4,2}x_{5,5}x_{6,3} + x_{2,1}x_{4,3}x_{5,2}x_{6,5} + x_{2,2}x_{3,1}x_{5,3}x_{6,4} - x_{2,2}x_{3,1}x_{5,4}x_{6,3} + x_{2,2}x_{4,1}x_{5,3}x_{6,5} - x_{2,2}x_{4,1}x_{5,5}x_{6,3} - x_{3,1}x_{4,2}x_{5,4}x_{6,5} + x_{3,1}x_{4,2}x_{5,5}x_{6,4} + x_{3,1}x_{4,4}x_{5,2}x_{6,5} + x_{3,2}x_{4,1}x_{5,4}x_{6,5} - x_{3,2}x_{4,1}x_{5,5}x_{6,4} - x_{2,1}x_{5,2}x_{6,3} - x_{3,1}x_{5,2}x_{6,4} - x_{4,1}x_{5,2}x_{6,5})$

The $(6,4)$ Banded Toda flow has Konstant form

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & 1 & 0 \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} & 1 \\ 0 & x_{6,2} & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} \end{bmatrix}.$$

The dimension is 20. There are 2 Casimirs, one being the trace and the other being

$$-\frac{-x_{3,1}x_{5,2}x_{6,3}+x_{3,1}x_{5,3}x_{6,2}+x_{3,2}x_{5,1}x_{6,3}-x_{3,3}x_{5,1}x_{6,2}-x_{4,1}x_{5,2}x_{6,4}+x_{4,1}x_{5,4}x_{6,2}+x_{4,2}x_{5,1}x_{6,4}-x_{4,4}x_{5,1}x_{6,2}}{x_{5,1}x_{6,2}}$$

The first integrals are

1. $x_{1,1}x_{2,2} + x_{1,1}x_{3,3} + x_{1,1}x_{4,4} + x_{1,1}x_{5,5} + x_{1,1}x_{6,6} + x_{2,2}x_{3,3} + x_{2,2}x_{4,4} + x_{2,2}x_{5,5} + x_{2,2}x_{6,6} + x_{3,3}x_{4,4} + x_{3,3}x_{5,5} +$
   $x_{3,3}x_{6,6} + x_{4,4}x_{5,5} + x_{4,4}x_{6,6} + x_{5,5}x_{6,6} - x_{2,1} - x_{3,2} - x_{4,3} - x_{5,4} - x_{6,5}$

2. $-x_{1,1}x_{2,2}x_{3,3} - x_{1,1}x_{2,2}x_{4,4} - x_{1,1}x_{2,2}x_{5,5} - x_{1,1}x_{2,2}x_{6,6} - x_{1,1}x_{3,3}x_{4,4} - x_{1,1}x_{3,3}x_{5,5} - x_{1,1}x_{3,3}x_{6,6} - x_{1,1}x_{4,4}x_{5,5} -$
   $x_{1,1}x_{4,4}x_{6,6} - x_{1,1}x_{5,5}x_{6,6} - x_{2,2}x_{3,3}x_{4,4} - x_{2,2}x_{3,3}x_{5,5} - x_{2,2}x_{3,3}x_{6,6} - x_{2,2}x_{4,4}x_{5,5} - x_{2,2}x_{4,4}x_{6,6} - x_{2,2}x_{5,5}x_{6,6} -$
   $x_{3,3}x_{4,4}x_{5,5} - x_{3,3}x_{4,4}x_{6,6} - x_{3,3}x_{5,5}x_{6,6} - x_{4,4}x_{5,5}x_{6,6} + x_{1,1}x_{3,2} + x_{1,1}x_{4,3} + x_{1,1}x_{5,4} + x_{1,1}x_{6,5} + x_{2,1}x_{3,3} + x_{2,1}x_{4,4} +$
   $x_{2,1}x_{5,5} + x_{2,1}x_{6,6} + x_{2,2}x_{4,3} + x_{2,2}x_{5,4} + x_{2,2}x_{6,5} + x_{3,2}x_{4,4} + x_{3,2}x_{5,5} + x_{3,2}x_{6,6} + x_{3,3}x_{5,4} + x_{3,3}x_{6,5} + x_{4,3}x_{5,5} +$
   $x_{4,3}x_{6,6} + x_{4,4}x_{6,5} + x_{5,4}x_{6,6} - x_{3,1} - x_{4,2} - x_{5,3} - x_{6,4}$

3. $x_{1,1}x_{2,2}x_{3,3}x_{4,4}+x_{1,1}x_{2,2}x_{3,3}x_{5,5}+x_{1,1}x_{2,2}x_{3,3}x_{6,6}+x_{1,1}x_{2,2}x_{4,4}x_{5,5}+x_{1,1}x_{2,2}x_{4,4}x_{6,6}+x_{1,1}x_{2,2}x_{5,5}x_{6,6}+x_{1,1}x_{3,3}x_{4,4}x_{5,5}+$
   $x_{1,1}x_{3,3}x_{4,4}x_{6,6}+x_{1,1}x_{3,3}x_{5,5}x_{6,6}+x_{1,1}x_{4,4}x_{5,5}x_{6,6}+x_{2,2}x_{3,3}x_{4,4}x_{5,5}+x_{2,2}x_{3,3}x_{4,4}x_{6,6}+x_{2,2}x_{3,3}x_{5,5}x_{6,6}+x_{2,2}x_{4,4}x_{5,5}x_{6,6}+$
   $x_{3,3}x_{4,4}x_{5,5}x_{6,6}-x_{1,1}x_{2,2}x_{4,3}-x_{1,1}x_{2,2}x_{5,4}-x_{1,1}x_{2,2}x_{6,5}-x_{1,1}x_{3,2}x_{4,4}-x_{1,1}x_{3,2}x_{5,5}-x_{1,1}x_{3,2}x_{6,6}-x_{1,1}x_{3,3}x_{5,4}-$
   $x_{1,1}x_{3,3}x_{6,5} - x_{1,1}x_{4,3}x_{5,5} - x_{1,1}x_{4,3}x_{6,6} - x_{1,1}x_{4,4}x_{6,5} - x_{1,1}x_{5,4}x_{6,6} - x_{2,1}x_{3,3}x_{4,4} - x_{2,1}x_{3,3}x_{5,5} - x_{2,1}x_{3,3}x_{6,6} -$
   $x_{2,1}x_{4,4}x_{5,5} - x_{2,1}x_{4,4}x_{6,6} - x_{2,1}x_{5,5}x_{6,6} - x_{2,2}x_{3,3}x_{5,4} - x_{2,2}x_{3,3}x_{6,5} - x_{2,2}x_{4,3}x_{5,5} - x_{2,2}x_{4,3}x_{6,6} - x_{2,2}x_{4,4}x_{6,5} -$
   $x_{2,2}x_{5,4}x_{6,6}-x_{3,2}x_{4,4}x_{5,5}-x_{3,2}x_{4,4}x_{6,6}-x_{3,2}x_{5,5}x_{6,6}-x_{3,3}x_{4,4}x_{6,5}-x_{3,3}x_{5,4}x_{6,6}-x_{4,3}x_{5,5}x_{6,6}+x_{1,1}x_{4,2}+x_{1,1}x_{5,3}+$
   $x_{1,1}x_{6,4} + x_{2,1}x_{4,3} + x_{2,1}x_{5,4} + x_{2,1}x_{6,5} + x_{2,2}x_{5,3} + x_{2,2}x_{6,4} + x_{3,1}x_{4,4} + x_{3,1}x_{5,5} + x_{3,1}x_{6,6} + x_{3,2}x_{5,4} + x_{3,2}x_{6,5} +$
   $x_{3,3}x_{6,4} + x_{4,2}x_{5,5} + x_{4,2}x_{6,6} + x_{4,3}x_{6,5} + x_{5,3}x_{6,6} - x_{4,1} - x_{5,2} - x_{6,3}$

4. $-x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5} - x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{6,6} - x_{1,1}x_{2,2}x_{3,3}x_{5,5}x_{6,6} - x_{1,1}x_{2,2}x_{4,4}x_{5,5}x_{6,6} - x_{1,1}x_{3,3}x_{4,4}x_{5,5}x_{6,6} -$
   $x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6} + x_{1,1}x_{2,2}x_{3,3}x_{5,4} + x_{1,1}x_{2,2}x_{3,3}x_{6,5} + x_{1,1}x_{2,2}x_{4,3}x_{5,5} + x_{1,1}x_{2,2}x_{4,3}x_{6,6} + x_{1,1}x_{2,2}x_{4,4}x_{6,5} +$

$x_{1,1}x_{2,2}x_{5,4}x_{6,6}+x_{1,1}x_{3,2}x_{4,4}x_{5,5}+x_{1,1}x_{3,2}x_{4,4}x_{6,6}+x_{1,1}x_{3,2}x_{5,5}x_{6,6}+x_{1,1}x_{3,3}x_{4,4}x_{6,5}+x_{1,1}x_{3,3}x_{5,4}x_{6,6}+x_{1,1}x_{4,3}x_{5,5}x_{6,6}+$
$x_{2,1}x_{3,3}x_{4,4}x_{5,5}+x_{2,1}x_{3,3}x_{4,4}x_{6,6}+x_{2,1}x_{3,3}x_{5,5}x_{6,6}+x_{2,1}x_{4,4}x_{5,5}x_{6,6}+x_{2,2}x_{3,3}x_{4,4}x_{6,5}+x_{2,2}x_{3,3}x_{5,4}x_{6,6}+x_{2,2}x_{4,3}x_{5,5}x_{6,6}+$
$x_{3,2}x_{4,4}x_{5,5}x_{6,6}-x_{1,1}x_{2,2}x_{5,3}-x_{1,1}x_{2,2}x_{6,4}-x_{1,1}x_{3,2}x_{5,4}-x_{1,1}x_{3,2}x_{6,5}-x_{1,1}x_{3,3}x_{6,4}-x_{1,1}x_{4,2}x_{5,5}-x_{1,1}x_{4,2}x_{6,6}-$
$x_{1,1}x_{4,3}x_{6,5}-x_{1,1}x_{5,3}x_{6,6}-x_{2,1}x_{3,3}x_{5,4}-x_{2,1}x_{3,3}x_{6,5}-x_{2,1}x_{4,3}x_{5,5}-x_{2,1}x_{4,3}x_{6,6}-x_{2,1}x_{4,4}x_{6,5}-x_{2,1}x_{5,4}x_{6,6}-$
$x_{2,2}x_{3,3}x_{6,4}-x_{2,2}x_{4,3}x_{6,5}-x_{2,2}x_{5,3}x_{6,6}-x_{3,1}x_{4,4}x_{5,5}-x_{3,1}x_{4,4}x_{6,6}-x_{3,1}x_{5,5}x_{6,6}-x_{3,2}x_{4,4}x_{6,5}-x_{3,2}x_{5,4}x_{6,6}-$
$x_{4,2}x_{5,5}x_{6,6}+x_{1,1}x_{5,2}+x_{1,1}x_{6,3}+x_{2,1}x_{5,3}+x_{2,1}x_{6,4}+x_{2,2}x_{6,3}+x_{3,1}x_{5,4}+x_{3,1}x_{6,5}+x_{3,2}x_{6,4}+x_{4,1}x_{5,5}+x_{4,1}x_{6,6}+$
$x_{4,2}x_{6,5}+x_{5,2}x_{6,6}-x_{5,1}-x_{6,2}$

5. $-x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}-x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{6,6}-x_{1,1}x_{2,2}x_{3,3}x_{5,5}x_{6,6}-x_{1,1}x_{2,2}x_{4,4}x_{5,5}x_{6,6}-x_{1,1}x_{3,3}x_{4,4}x_{5,5}x_{6,6}-$
$x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6}+x_{1,1}x_{2,2}x_{3,3}x_{5,4}+x_{1,1}x_{2,2}x_{3,3}x_{6,5}+x_{1,1}x_{2,2}x_{4,3}x_{5,5}+x_{1,1}x_{2,2}x_{4,3}x_{6,6}+x_{1,1}x_{2,2}x_{4,4}x_{6,5}+$
$x_{1,1}x_{2,2}x_{5,4}x_{6,6}+x_{1,1}x_{3,2}x_{4,4}x_{5,5}+x_{1,1}x_{3,2}x_{4,4}x_{6,6}+x_{1,1}x_{3,2}x_{5,5}x_{6,6}+x_{1,1}x_{3,3}x_{4,4}x_{6,5}+x_{1,1}x_{3,3}x_{5,4}x_{6,6}+x_{1,1}x_{4,3}x_{5,5}x_{6,6}+$
$x_{2,1}x_{3,3}x_{4,4}x_{5,5}+x_{2,1}x_{3,3}x_{4,4}x_{6,6}+x_{2,1}x_{3,3}x_{5,5}x_{6,6}+x_{2,1}x_{4,4}x_{5,5}x_{6,6}+x_{2,2}x_{3,3}x_{4,4}x_{6,5}+x_{2,2}x_{3,3}x_{5,4}x_{6,6}+x_{2,2}x_{4,3}x_{5,5}x_{6,6}+$
$x_{3,2}x_{4,4}x_{5,5}x_{6,6}-x_{1,1}x_{2,2}x_{5,3}-x_{1,1}x_{2,2}x_{6,4}-x_{1,1}x_{3,2}x_{5,4}-x_{1,1}x_{3,2}x_{6,5}-x_{1,1}x_{3,3}x_{6,4}-x_{1,1}x_{4,2}x_{5,5}-x_{1,1}x_{4,2}x_{6,6}-$
$x_{1,1}x_{4,3}x_{6,5}-x_{1,1}x_{5,3}x_{6,6}-x_{2,1}x_{3,3}x_{5,4}-x_{2,1}x_{3,3}x_{6,5}-x_{2,1}x_{4,3}x_{5,5}-x_{2,1}x_{4,3}x_{6,6}-x_{2,1}x_{4,4}x_{6,5}-x_{2,1}x_{5,4}x_{6,6}-$
$x_{2,2}x_{3,3}x_{6,4}-x_{2,2}x_{4,3}x_{6,5}-x_{2,2}x_{5,3}x_{6,6}-x_{3,1}x_{4,4}x_{5,5}-x_{3,1}x_{4,4}x_{6,6}-x_{3,1}x_{5,5}x_{6,6}-x_{3,2}x_{4,4}x_{6,5}-x_{3,2}x_{5,4}x_{6,6}-$
$x_{4,2}x_{5,5}x_{6,6}+x_{1,1}x_{5,2}+x_{1,1}x_{6,3}+x_{2,1}x_{5,3}+x_{2,1}x_{6,4}+x_{2,2}x_{6,3}+x_{3,1}x_{5,4}+x_{3,1}x_{6,5}+x_{3,2}x_{6,4}+x_{4,1}x_{5,5}+x_{4,1}x_{6,6}+$
$x_{4,2}x_{6,5}+x_{5,2}x_{6,6}-x_{5,1}-x_{6,2}$

6. $\frac{1}{x_{6,2}x_{2,1}+x_{3,1}x_{6,3}+x_{4,1}x_{6,4}+x_{6,5}x_{5,1}}(-x_{3,2}x_{6,3}x_{2,1}+x_{2,1}x_{3,3}x_{6,2}-x_{4,2}x_{6,4}x_{2,1}+x_{2,1}x_{4,4}x_{6,2}-x_{5,2}x_{6,5}x_{2,1}+x_{2,1}x_{5,5}x_{6,2}+$
$x_{2,2}x_{3,1}x_{6,3}+x_{2,2}x_{4,1}x_{6,4}+x_{2,2}x_{5,1}x_{6,5}-x_{3,1}x_{4,3}x_{6,4}+x_{3,1}x_{4,4}x_{6,3}-x_{3,1}x_{5,3}x_{6,5}+x_{3,1}x_{5,5}x_{6,3}+x_{3,3}x_{4,1}x_{6,4}+$
$x_{3,3}x_{5,1}x_{6,5}-x_{4,1}x_{5,4}x_{6,5}+x_{4,1}x_{5,5}x_{6,4}+x_{6,5}x_{5,1}x_{4,4}-x_{3,1}x_{6,2}-x_{4,1}x_{6,3}-x_{5,1}x_{6,4})$

7. $\frac{1}{-x_{2,1}x_{6,2}-x_{3,1}x_{6,3}-x_{4,1}x_{6,4}-x_{5,1}x_{6,5}}(-x_{2,1}x_{3,2}x_{4,3}x_{6,4}+x_{2,1}x_{3,2}x_{4,4}x_{6,3}-x_{2,1}x_{3,2}x_{5,3}x_{6,5}+x_{2,1}x_{3,2}x_{5,5}x_{6,3}+$
$x_{2,1}x_{3,3}x_{4,2}x_{6,4}-x_{2,1}x_{3,3}x_{4,4}x_{6,2}+x_{2,1}x_{3,3}x_{5,2}x_{6,5}-x_{2,1}x_{3,3}x_{5,5}x_{6,2}-x_{2,1}x_{4,2}x_{5,4}x_{6,5}+x_{2,1}x_{4,2}x_{5,5}x_{6,4}+x_{2,1}x_{4,4}x_{5,2}x_{6,5}-$
$x_{2,1}x_{4,4}x_{5,5}x_{6,2}+x_{2,2}x_{3,1}x_{4,3}x_{6,4}-x_{2,2}x_{3,1}x_{4,4}x_{6,3}+x_{2,2}x_{3,1}x_{5,3}x_{6,5}-x_{2,2}x_{3,1}x_{5,5}x_{6,3}-x_{2,2}x_{3,3}x_{4,1}x_{6,4}-x_{2,2}x_{3,3}x_{5,1}x_{6,5}+$
$x_{2,2}x_{4,1}x_{5,4}x_{6,5}-x_{2,2}x_{4,1}x_{5,5}x_{6,4}-x_{2,2}x_{4,4}x_{5,1}x_{6,5}-x_{3,1}x_{4,3}x_{5,4}x_{6,5}+x_{3,1}x_{4,3}x_{5,5}x_{6,4}+x_{3,1}x_{4,4}x_{5,3}x_{6,5}-x_{3,1}x_{4,4}x_{5,5}x_{6,3}+$
$x_{3,3}x_{4,1}x_{5,4}x_{6,5}-x_{3,3}x_{4,1}x_{5,5}x_{6,4}-x_{3,3}x_{4,4}x_{5,1}x_{6,5}-x_{2,1}x_{4,2}x_{6,3}+x_{2,1}x_{4,3}x_{6,2}-x_{2,1}x_{5,2}x_{6,4}+x_{2,1}x_{5,4}x_{6,2}+$
$x_{2,2}x_{4,1}x_{6,3}+x_{2,2}x_{5,1}x_{6,4}-x_{3,1}x_{4,2}x_{6,4}+x_{3,1}x_{4,4}x_{6,2}-x_{3,1}x_{5,2}x_{6,5}-x_{3,1}x_{5,3}x_{6,4}+x_{3,1}x_{5,4}x_{6,3}+x_{3,1}x_{5,5}x_{6,2}+$
$x_{3,2}x_{4,1}x_{6,4}+x_{3,2}x_{5,1}x_{6,5}+x_{3,3}x_{5,1}x_{6,4}-x_{4,1}x_{5,3}x_{6,5}+x_{4,1}x_{5,5}x_{6,3}+x_{4,3}x_{5,1}x_{6,5}-x_{4,1}x_{6,2}-x_{5,1}x_{6,3})$

8. $\frac{1}{-x_{2,1}x_{6,2}-x_{3,1}x_{6,3}-x_{4,1}x_{6,4}-x_{5,1}x_{6,5}}(-x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{6,5}+x_{2,1}x_{3,2}x_{4,3}x_{5,5}x_{6,4}+x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{6,5}-x_{2,1}x_{3,2}x_{4,4}x_{5,5}x_{6,3}+$
$x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{6,5}-x_{2,1}x_{3,3}x_{4,2}x_{5,5}x_{6,4}-x_{2,1}x_{3,3}x_{4,4}x_{5,2}x_{6,5}+x_{2,1}x_{3,3}x_{4,4}x_{5,5}x_{6,2}+x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{6,5}-x_{2,2}x_{3,1}x_{4,3}x_{5,5}x_{6,4}-$
$x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{6,5}+x_{2,2}x_{3,1}x_{4,4}x_{5,5}x_{6,3}-x_{2,2}x_{3,3}x_{4,1}x_{5,4}x_{6,5}+x_{2,2}x_{3,3}x_{4,1}x_{5,5}x_{6,4}+x_{2,2}x_{3,3}x_{4,4}x_{5,1}x_{6,5}-x_{2,1}x_{3,2}x_{5,3}x_{6,4}+$
$x_{2,1}x_{3,2}x_{5,4}x_{6,3}+x_{2,1}x_{3,3}x_{5,2}x_{6,4}-x_{2,1}x_{3,3}x_{5,4}x_{6,2}-x_{2,1}x_{4,2}x_{5,3}x_{6,5}+x_{2,1}x_{4,2}x_{5,5}x_{6,3}+x_{2,1}x_{4,3}x_{5,2}x_{6,5}-x_{2,1}x_{4,3}x_{5,5}x_{6,2}+$
$x_{2,2}x_{3,1}x_{5,3}x_{6,4}-x_{2,2}x_{3,1}x_{5,4}x_{6,3}-x_{2,2}x_{3,3}x_{5,1}x_{6,4}+x_{2,2}x_{4,1}x_{5,3}x_{6,5}-x_{2,2}x_{4,1}x_{5,5}x_{6,3}-x_{2,2}x_{4,3}x_{5,1}x_{6,5}-x_{3,1}x_{4,2}x_{5,4}x_{6,5}+$
$x_{3,1}x_{4,2}x_{5,5}x_{6,4}+x_{3,1}x_{4,4}x_{5,2}x_{6,5}-x_{3,1}x_{4,4}x_{5,5}x_{6,2}+x_{3,2}x_{4,1}x_{5,4}x_{6,5}-x_{3,2}x_{4,1}x_{5,5}x_{6,4}-x_{3,2}x_{4,4}x_{5,1}x_{6,5}-x_{2,1}x_{5,2}x_{6,3}+$

$x_{2,1}x_{5,3}x_{6,2} + x_{2,2}x_{5,1}x_{6,3} - x_{3,1}x_{5,2}x_{6,4} + x_{3,1}x_{5,4}x_{6,2} + x_{3,2}x_{5,1}x_{6,4} - x_{4,1}x_{5,2}x_{6,5} + x_{4,1}x_{5,5}x_{6,2} + x_{4,2}x_{5,1}x_{6,5} -$
$x_{5,1}x_{6,2})$

9. $\frac{1}{x_{5,1}x_{6,2}}(-x_{3,1}x_{4,2}x_{5,3}x_{6,4}+x_{3,1}x_{4,2}x_{5,4}x_{6,3}+x_{3,1}x_{4,3}x_{5,2}x_{6,4}-x_{3,1}x_{4,3}x_{5,4}x_{6,2}-x_{3,1}x_{4,4}x_{5,2}x_{6,3}+x_{3,1}x_{4,4}x_{5,3}x_{6,2}+$
$x_{3,2}x_{4,1}x_{5,3}x_{6,4}-x_{3,2}x_{4,1}x_{5,4}x_{6,3}-x_{3,2}x_{4,3}x_{5,1}x_{6,4}+x_{3,2}x_{4,4}x_{5,1}x_{6,3}-x_{3,3}x_{4,1}x_{5,2}x_{6,4}+x_{3,3}x_{4,1}x_{5,4}x_{6,2}+x_{3,3}x_{4,2}x_{5,1}x_{6,4}-$
$x_{3,3}x_{4,4}x_{5,1}x_{6,2} + x_{4,1}x_{5,2}x_{6,3} - x_{4,1}x_{5,3}x_{6,2} - x_{4,2}x_{5,1}x_{6,3} + x_{4,3}x_{5,1}x_{6,2})$

We now move onto the 7 dimensional case. At this point in the proof, some of the first integrals are becoming quite large. Previously, it was stated that some of the first integrals come from the coefficients of the characteristic polynomial of the Konstant form of the Banded Toda flow. The first integrals will come from all the coefficients except the n and $n-1$ degree coefficient. Therefore, if one wishes to see the first integrals for the system, typing the matrix into MAPLE then taking the characteristic polynomial would be sufficient. Instead of explicitly showing these first integrals, we will give the number of first integrals the characteristic polynomial will give. Beginning with the $(7,2)$, we have the Konstant form

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 & 0 & 0 \\ 0 & x_{4,2} & x_{4,3} & x_{4,4} & 1 & 0 & 0 \\ 0 & 0 & x_{5,3} & x_{5,4} & x_{5,5} & 1 & 0 \\ 0 & 0 & 0 & x_{6,4} & x_{6,5} & x_{6,6} & 1 \\ 0 & 0 & 0 & 0 & x_{7,5} & x_{7,6} & x_{7,7} \end{bmatrix}.$$

The dimension is 18. There are 2 Casimirs and 8 first integrals. The Casimir, beside the trace, is

$\frac{1}{x_{3,1}x_{5,3}x_{7,5}}(x_{2,1}x_{3,2}x_{5,3}x_{7,5}+x_{2,1}x_{4,2}x_{5,4}x_{7,5}+x_{2,1}x_{4,2}x_{6,4}x_{7,6}-x_{2,2}x_{3,1}x_{5,3}x_{7,5}+x_{3,1}x_{4,3}x_{5,4}x_{7,5}+x_{3,1}x_{4,3}x_{6,4}x_{7,6}-$
$x_{3,1}x_{4,4}x_{5,3}x_{7,5} + x_{3,1}x_{5,3}x_{6,5}x_{7,6} - x_{3,1}x_{5,3}x_{6,6}x_{7,5}).$

Six of the first integrals will come from the characteristic polynomial. The other two are

1. $(-x_{2,1}x_{3,2}x_{5,3}x_{7,5} - x_{2,1}x_{4,2}x_{5,4}x_{7,5} - x_{2,1}x_{4,2}x_{6,4}x_{7,6} + x_{2,2}x_{3,1}x_{5,3}x_{7,5} - x_{3,1}x_{4,3}x_{5,4}x_{7,5} - x_{3,1}x_{4,3}x_{6,4}x_{7,6} +$
$x_{3,1}x_{4,4}x_{5,3}x_{7,5} - x_{3,1}x_{5,3}x_{6,5}x_{7,6} + x_{3,1}x_{5,3}x_{6,6}x_{7,5})^{-1}(-x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{7,5} - x_{2,1}x_{3,2}x_{4,3}x_{6,4}x_{7,6} + x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{7,5} -$
$x_{2,1}x_{3,2}x_{5,3}x_{6,5}x_{7,6} + x_{2,1}x_{3,2}x_{5,3}x_{6,6}x_{7,5} + x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{7,5} + x_{2,1}x_{3,3}x_{4,2}x_{6,4}x_{7,6} - x_{2,1}x_{4,2}x_{5,4}x_{6,5}x_{7,6} + x_{2,1}x_{4,2}x_{5,4}x_{6,6}x_{7,5} +$
$x_{2,1}x_{4,2}x_{5,5}x_{6,4}x_{7,6} + x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{7,5} + x_{2,2}x_{3,1}x_{4,3}x_{6,4}x_{7,6} - x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{7,5} + x_{2,2}x_{3,1}x_{5,3}x_{6,5}x_{7,6} - x_{2,2}x_{3,1}x_{5,3}x_{6,6}x_{7,5} -$
$x_{3,1}x_{4,3}x_{5,4}x_{6,5}x_{7,6} + x_{3,1}x_{4,3}x_{5,4}x_{6,6}x_{7,5} + x_{3,1}x_{4,3}x_{5,5}x_{6,4}x_{7,6} + x_{3,1}x_{4,4}x_{5,3}x_{6,5}x_{7,6} - x_{3,1}x_{4,4}x_{5,3}x_{6,6}x_{7,5} - x_{2,1}x_{4,2}x_{5,3}x_{7,5} -$
$x_{2,1}x_{4,2}x_{6,4}x_{7,5} - x_{3,1}x_{4,2}x_{5,4}x_{7,5} - x_{3,1}x_{4,2}x_{6,4}x_{7,6} - x_{3,1}x_{4,3}x_{6,4}x_{7,5} - x_{3,1}x_{5,3}x_{6,4}x_{7,6})$

2. $(-x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{6,5}x_{7,6} + x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{6,6}x_{7,5} + x_{2,1}x_{3,2}x_{4,3}x_{5,5}x_{6,4}x_{7,6} + x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{6,5}x_{7,6} - x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{6,6}x_{7,5} +$
$x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{6,5}x_{7,6} - x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{6,6}x_{7,5} - x_{2,1}x_{3,3}x_{4,2}x_{5,5}x_{6,4}x_{7,6} + x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{6,5}x_{7,6} - x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{6,6}x_{7,5} -$
$x_{2,2}x_{3,1}x_{4,3}x_{5,5}x_{6,4}x_{7,6} - x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{6,5}x_{7,6} + x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{6,6}x_{7,5} - x_{2,1}x_{3,2}x_{4,3}x_{6,4}x_{7,5} - x_{2,1}x_{3,2}x_{5,3}x_{6,4}x_{7,6} +$
$x_{2,1}x_{3,3}x_{4,2}x_{6,4}x_{7,5} - x_{2,1}x_{4,2}x_{5,3}x_{6,5}x_{7,6} + x_{2,1}x_{4,2}x_{5,3}x_{6,6}x_{7,5} + x_{2,2}x_{3,1}x_{4,3}x_{6,4}x_{7,5} + x_{2,2}x_{3,1}x_{5,3}x_{6,4}x_{7,6} - x_{3,1}x_{4,2}x_{5,4}x_{6,5}x_{7,6} +$
$x_{3,1}x_{4,2}x_{5,4}x_{6,6}x_{7,5} + x_{3,1}x_{4,2}x_{5,5}x_{6,4}x_{7,6} - x_{3,1}x_{4,2}x_{6,4}x_{7,5})(-x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{7,5} - x_{2,1}x_{3,2}x_{4,3}x_{6,4}x_{7,6} + x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{7,5} -$
$x_{2,1}x_{3,2}x_{5,3}x_{6,5}x_{7,6} + x_{2,1}x_{3,2}x_{5,3}x_{6,6}x_{7,5} + x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{7,5} + x_{2,1}x_{3,3}x_{4,2}x_{6,4}x_{7,6} - x_{2,1}x_{4,2}x_{5,4}x_{6,5}x_{7,6} + x_{2,1}x_{4,2}x_{5,4}x_{6,6}x_{7,5} +$
$x_{2,1}x_{4,2}x_{5,5}x_{6,4}x_{7,6} + x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{7,5} + x_{2,2}x_{3,1}x_{4,3}x_{6,4}x_{7,6} - x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{7,5} + x_{2,2}x_{3,1}x_{5,3}x_{6,5}x_{7,6} - x_{2,2}x_{3,1}x_{5,3}x_{6,6}x_{7,5} -$
$x_{3,1}x_{4,3}x_{5,4}x_{6,5}x_{7,6} + x_{3,1}x_{4,3}x_{5,4}x_{6,6}x_{7,5} + x_{3,1}x_{4,3}x_{5,5}x_{6,4}x_{7,6} + x_{3,1}x_{4,4}x_{5,3}x_{6,5}x_{7,6} - x_{3,1}x_{4,4}x_{5,3}x_{6,6}x_{7,5} - x_{2,1}x_{4,2}x_{5,3}x_{7,5} -$
$x_{2,1}x_{4,2}x_{6,4}x_{7,5} - x_{3,1}x_{4,2}x_{5,4}x_{7,5} - x_{3,1}x_{4,2}x_{6,4}x_{7,6} - x_{3,1}x_{4,3}x_{6,4}x_{7,5} - x_{3,1}x_{5,3}x_{6,4}x_{7,6})^{-1}$

The $(7,3)$ Banded Toda flow has Konstant form

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 & 0 & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & 1 & 0 & 0 \\ 0 & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} & 1 & 0 \\ 0 & 0 & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} & 1 \\ 0 & 0 & 0 & x_{7,4} & x_{7,5} & x_{7,6} & x_{7,7} \end{bmatrix}$$

The dimension is 22. There are 2 Casimirs, one being the trace, and 10 first integrals, 6 of which come from the characteristic polynomial. The other Casimir is

$\frac{1}{x_{4,1}x_{7,4}}(-x_{2,1}x_{4,2}x_{7,4} - x_{7,5}x_{5,2}x_{2,1} + x_{2,2}x_{4,1}x_{7,4} - x_{3,1}x_{4,3}x_{7,4} - x_{3,1}x_{5,3}x_{7,5} - x_{3,1}x_{6,3}x_{7,6} + x_{3,3}x_{4,1}x_{7,4} -$
$x_{4,1}x_{5,4}x_{7,5} + x_{4,1}x_{5,5}x_{7,4} - x_{4,1}x_{6,4}x_{7,6} + x_{4,1}x_{6,6}x_{7,4})$

The other 4 first integrals are

1. $\frac{1}{x_{4,1}x_{7,4}}(-x_{2,1}x_{3,2}x_{4,3}x_{7,4}-x_{2,1}x_{3,2}x_{5,3}x_{7,5}-x_{2,1}x_{3,2}x_{6,3}x_{7,6}+x_{2,1}x_{3,3}x_{4,2}x_{7,4}+x_{2,1}x_{3,3}x_{5,2}x_{7,5}-x_{2,1}x_{4,2}x_{5,4}x_{7,5}+$
$x_{2,1}x_{4,2}x_{5,5}x_{7,4}-x_{2,1}x_{4,2}x_{6,4}x_{7,6}+x_{2,1}x_{4,2}x_{6,6}x_{7,4}+x_{2,1}x_{4,4}x_{5,2}x_{7,5}-x_{2,1}x_{5,2}x_{6,5}x_{7,6}+x_{2,1}x_{5,2}x_{6,6}x_{7,5}+x_{2,2}x_{3,1}x_{4,3}x_{7,4}+$
$x_{2,2}x_{3,1}x_{5,3}x_{7,5}+x_{2,2}x_{3,1}x_{6,3}x_{7,6}-x_{2,2}x_{3,3}x_{4,1}x_{7,4}+x_{2,2}x_{4,1}x_{5,4}x_{7,5}-x_{2,2}x_{4,1}x_{5,5}x_{7,4}+x_{2,2}x_{4,1}x_{6,4}x_{7,6}-x_{2,2}x_{4,1}x_{6,6}x_{7,4}-$
$x_{3,1}x_{4,3}x_{5,4}x_{7,5}+x_{3,1}x_{4,3}x_{5,5}x_{7,4}-x_{3,1}x_{4,3}x_{6,4}x_{7,6}+x_{3,1}x_{4,3}x_{6,6}x_{7,4}+x_{3,1}x_{4,4}x_{5,3}x_{7,5}+x_{3,1}x_{4,4}x_{6,3}x_{7,6}-x_{3,1}x_{5,3}x_{6,5}x_{7,6}+$
$x_{3,1}x_{5,3}x_{6,6}x_{7,5}+x_{3,1}x_{5,5}x_{6,3}x_{7,6}+x_{3,3}x_{4,1}x_{5,4}x_{7,5}-x_{3,3}x_{4,1}x_{5,5}x_{7,4}+x_{3,3}x_{4,1}x_{6,4}x_{7,6}-x_{3,3}x_{4,1}x_{6,6}x_{7,4}-x_{4,1}x_{5,4}x_{6,5}x_{7,6}+$
$x_{4,1}x_{5,4}x_{6,6}x_{7,5} + x_{4,1}x_{5,5}x_{6,4}x_{7,6} - x_{4,1}x_{5,5}x_{6,6}x_{7,4} - x_{2,1}x_{5,2}x_{7,4} - x_{3,1}x_{4,2}x_{7,4} - x_{3,1}x_{5,2}x_{7,5} - x_{3,1}x_{5,3}x_{7,4} -$
$x_{3,1}x_{6,3}x_{7,5} + x_{3,2}x_{4,1}x_{7,4} - x_{4,1}x_{5,3}x_{7,5} - x_{4,1}x_{6,3}x_{7,6} - x_{4,1}x_{6,4}x_{7,5} + x_{4,1}x_{6,5}x_{7,4})$

2. $\frac{1}{x_{4,1}x_{7,4}}(-x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{7,5}+x_{2,1}x_{3,2}x_{4,3}x_{5,5}x_{7,4}-x_{7,6}x_{2,1}x_{3,2}x_{4,3}x_{6,4}+x_{2,1}x_{3,2}x_{4,3}x_{6,6}x_{7,4}+x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{7,5}+$
$x_{2,1}x_{3,2}x_{4,4}x_{6,3}x_{7,6}-x_{7,6}x_{2,1}x_{3,2}x_{5,3}x_{6,5}+x_{2,1}x_{3,2}x_{5,3}x_{6,6}x_{7,5}+x_{2,1}x_{3,2}x_{5,5}x_{6,3}x_{7,6}+x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{7,5}-x_{2,1}x_{3,3}x_{4,2}x_{5,5}x_{7,4}+$
$x_{2,1}x_{3,3}x_{4,2}x_{6,4}x_{7,6}-x_{2,1}x_{3,3}x_{4,2}x_{6,6}x_{7,4}-x_{2,1}x_{3,3}x_{4,4}x_{5,2}x_{7,5}+x_{2,1}x_{3,3}x_{5,2}x_{6,5}x_{7,6}-x_{2,1}x_{3,3}x_{5,2}x_{6,6}x_{7,5}-x_{7,6}x_{2,1}x_{4,2}x_{5,4}x_{6,5}+$
$x_{2,1}x_{4,2}x_{5,4}x_{6,6}x_{7,5}+x_{2,1}x_{4,2}x_{5,5}x_{6,4}x_{7,6}-x_{2,1}x_{4,2}x_{5,5}x_{6,6}x_{7,4}+x_{2,1}x_{4,4}x_{5,2}x_{6,5}x_{7,6}-x_{2,1}x_{4,4}x_{5,2}x_{6,6}x_{7,5}+x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{7,5}-$
$x_{2,2}x_{3,1}x_{4,3}x_{5,5}x_{7,4}+x_{2,2}x_{3,1}x_{4,3}x_{6,4}x_{7,6}-x_{2,2}x_{3,1}x_{4,3}x_{6,6}x_{7,4}-x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{7,5}-x_{2,2}x_{3,1}x_{4,4}x_{6,3}x_{7,6}+x_{2,2}x_{3,1}x_{5,3}x_{6,5}x_{7,6}-$
$x_{2,2}x_{3,1}x_{5,3}x_{6,6}x_{7,5}-x_{2,2}x_{3,1}x_{5,5}x_{6,3}x_{7,6}-x_{2,2}x_{3,3}x_{4,1}x_{5,4}x_{7,5}+x_{2,2}x_{3,3}x_{4,1}x_{5,5}x_{7,4}-x_{2,2}x_{3,3}x_{4,1}x_{6,4}x_{7,6}+x_{2,2}x_{3,3}x_{4,1}x_{6,6}x_{7,4}+$
$x_{2,2}x_{4,1}x_{5,4}x_{6,5}x_{7,6}-x_{2,2}x_{4,1}x_{5,4}x_{6,6}x_{7,5}-x_{2,2}x_{4,1}x_{5,5}x_{6,4}x_{7,6}+x_{2,2}x_{4,1}x_{5,5}x_{6,6}x_{7,4}-x_{7,6}x_{3,1}x_{4,3}x_{5,4}x_{6,5}+x_{3,1}x_{4,3}x_{5,4}x_{6,6}x_{7,5}+$
$x_{3,1}x_{4,3}x_{5,5}x_{6,4}x_{7,6}-x_{3,1}x_{4,3}x_{5,5}x_{6,6}x_{7,4}+x_{3,1}x_{4,4}x_{5,3}x_{6,5}x_{7,6}-x_{3,1}x_{4,4}x_{5,3}x_{6,6}x_{7,5}-x_{3,1}x_{4,4}x_{5,5}x_{6,3}x_{7,6}+x_{3,3}x_{4,1}x_{5,4}x_{6,5}x_{7,6}-$
$x_{3,3}x_{4,1}x_{5,4}x_{6,6}x_{7,5}-x_{3,3}x_{4,1}x_{5,5}x_{6,4}x_{7,6}+x_{3,3}x_{4,1}x_{5,5}x_{6,6}x_{7,4}-x_{2,1}x_{3,2}x_{5,3}x_{7,4}-x_{2,1}x_{3,2}x_{6,3}x_{7,5}+x_{2,1}x_{3,3}x_{5,2}x_{7,4}-$
$x_{2,1}x_{4,2}x_{5,3}x_{7,5}-x_{7,6}x_{2,1}x_{4,2}x_{6,3}-x_{2,1}x_{4,2}x_{6,4}x_{7,5}+x_{2,1}x_{4,2}x_{6,5}x_{7,4}+x_{2,1}x_{4,3}x_{5,2}x_{7,5}-x_{7,6}x_{2,1}x_{5,2}x_{6,4}+x_{2,1}x_{5,2}x_{6,6}x_{7,4}+$
$x_{2,2}x_{3,1}x_{5,3}x_{7,4}+x_{2,2}x_{3,1}x_{6,3}x_{7,5}+x_{2,2}x_{4,1}x_{5,3}x_{7,5}+x_{2,2}x_{4,1}x_{6,3}x_{7,6}+x_{2,2}x_{4,1}x_{6,4}x_{7,5}-x_{2,2}x_{4,1}x_{6,5}x_{7,4}-x_{3,1}x_{4,2}x_{5,4}x_{7,5}+$
$x_{3,1}x_{4,2}x_{5,5}x_{7,4}-x_{7,6}x_{3,1}x_{4,2}x_{6,4}+x_{3,1}x_{4,2}x_{6,6}x_{7,4}-x_{3,1}x_{4,3}x_{6,4}x_{7,5}+x_{3,1}x_{4,3}x_{6,5}x_{7,4}+x_{3,1}x_{4,4}x_{5,2}x_{7,5}+x_{3,1}x_{4,4}x_{6,3}x_{7,5}-$
$x_{7,6}x_{3,1}x_{5,2}x_{6,5}+x_{3,1}x_{5,2}x_{6,6}x_{7,5}-x_{7,6}x_{3,1}x_{5,3}x_{6,4}+x_{3,1}x_{5,3}x_{6,6}x_{7,4}+x_{3,1}x_{5,4}x_{6,3}x_{7,6}+x_{3,2}x_{4,1}x_{5,4}x_{7,5}-x_{3,2}x_{4,1}x_{5,5}x_{7,4}+$
$x_{3,2}x_{4,1}x_{6,4}x_{7,6}-x_{3,2}x_{4,1}x_{6,6}x_{7,4}+x_{3,3}x_{4,1}x_{6,4}x_{7,5}-x_{3,3}x_{4,1}x_{6,5}x_{7,4}-x_{7,6}x_{4,1}x_{5,3}x_{6,5}+x_{4,1}x_{5,3}x_{6,6}x_{7,5}+x_{4,1}x_{5,5}x_{6,3}x_{7,6}-$
$x_{3,1}x_{5,2}x_{7,4} - x_{3,1}x_{6,3}x_{7,4} - x_{4,1}x_{5,2}x_{7,5} - x_{4,1}x_{6,3}x_{7,5})$

3. $\frac{1}{x_{4,1}x_{7,4}}(-x_{7,6}x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{6,5} + x_{2,1}x_{3,2}x_{4,3}x_{5,4}x_{6,6}x_{7,5} + x_{2,1}x_{3,2}x_{4,3}x_{5,5}x_{6,4}x_{7,6} - x_{2,1}x_{3,2}x_{4,3}x_{5,5}x_{6,6}x_{7,4} +$
$x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{6,5}x_{7,6}-x_{2,1}x_{3,2}x_{4,4}x_{5,3}x_{6,6}x_{7,5}-x_{2,1}x_{3,2}x_{4,4}x_{5,5}x_{6,3}x_{7,6}+x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{6,5}x_{7,6}-x_{2,1}x_{3,3}x_{4,2}x_{5,4}x_{6,6}x_{7,5}-$
$x_{2,1}x_{3,3}x_{4,2}x_{5,5}x_{6,4}x_{7,6}+x_{2,1}x_{3,3}x_{4,2}x_{5,5}x_{6,6}x_{7,4}-x_{2,1}x_{3,3}x_{4,4}x_{5,2}x_{6,5}x_{7,6}+x_{2,1}x_{3,3}x_{4,4}x_{5,2}x_{6,6}x_{7,5}+x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{6,5}x_{7,6}-$
$x_{2,2}x_{3,1}x_{4,3}x_{5,4}x_{6,6}x_{7,5}-x_{2,2}x_{3,1}x_{4,3}x_{5,5}x_{6,4}x_{7,6}+x_{2,2}x_{3,1}x_{4,3}x_{5,5}x_{6,6}x_{7,4}-x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{6,5}x_{7,6}+x_{2,2}x_{3,1}x_{4,4}x_{5,3}x_{6,6}x_{7,5}+$
$x_{2,2}x_{3,1}x_{4,4}x_{5,5}x_{6,3}x_{7,6}-x_{2,2}x_{3,3}x_{4,1}x_{5,4}x_{6,5}x_{7,6}+x_{2,2}x_{3,3}x_{4,1}x_{5,4}x_{6,6}x_{7,5}+x_{2,2}x_{3,3}x_{4,1}x_{5,5}x_{6,4}x_{7,6}-x_{2,2}x_{3,3}x_{4,1}x_{5,5}x_{6,6}x_{7,4}-$
$x_{7,5}x_{2,1}x_{3,2}x_{4,3}x_{6,4}+x_{2,1}x_{3,2}x_{4,3}x_{6,5}x_{7,4}+x_{2,1}x_{3,2}x_{4,4}x_{6,3}x_{7,5}-x_{7,6}x_{2,1}x_{3,2}x_{5,3}x_{6,4}+x_{2,1}x_{3,2}x_{5,3}x_{6,6}x_{7,4}+x_{2,1}x_{3,2}x_{5,4}x_{6,3}x_{7,6}+$
$x_{2,1}x_{3,3}x_{4,2}x_{6,4}x_{7,5}-x_{2,1}x_{3,3}x_{4,2}x_{6,5}x_{7,4}+x_{2,1}x_{3,3}x_{5,2}x_{6,4}x_{7,6}-x_{2,1}x_{3,3}x_{5,2}x_{6,6}x_{7,4}-x_{7,6}x_{2,1}x_{4,2}x_{5,3}x_{6,5}+x_{2,1}x_{4,2}x_{5,3}x_{6,6}x_{7,5}+$
$x_{2,1}x_{4,2}x_{5,5}x_{6,3}x_{7,6}+x_{2,1}x_{4,3}x_{5,2}x_{6,5}x_{7,6}-x_{2,1}x_{4,3}x_{5,2}x_{6,6}x_{7,5}+x_{2,2}x_{3,1}x_{4,3}x_{6,4}x_{7,5}-x_{2,2}x_{3,1}x_{4,3}x_{6,5}x_{7,4}-x_{2,2}x_{3,1}x_{4,4}x_{6,3}x_{7,5}+$
$x_{2,2}x_{3,1}x_{5,3}x_{6,4}x_{7,6}-x_{2,2}x_{3,1}x_{5,3}x_{6,6}x_{7,4}-x_{2,2}x_{3,1}x_{5,4}x_{6,3}x_{7,6}-x_{2,2}x_{3,3}x_{4,1}x_{6,4}x_{7,5}+x_{2,2}x_{3,3}x_{4,1}x_{6,5}x_{7,4}+x_{2,2}x_{4,1}x_{5,3}x_{6,5}x_{7,6}-$
$x_{2,2}x_{4,1}x_{5,3}x_{6,6}x_{7,5}-x_{2,2}x_{4,1}x_{5,5}x_{6,3}x_{7,6}-x_{7,6}x_{3,1}x_{4,2}x_{5,4}x_{6,5}+x_{3,1}x_{4,2}x_{5,4}x_{6,6}x_{7,5}+x_{3,1}x_{4,2}x_{5,5}x_{6,4}x_{7,6}-x_{3,1}x_{4,2}x_{5,5}x_{6,6}x_{7,4}+$
$x_{3,1}x_{4,4}x_{5,2}x_{6,5}x_{7,6}-x_{3,1}x_{4,4}x_{5,2}x_{6,6}x_{7,5}+x_{3,2}x_{4,1}x_{5,4}x_{6,5}x_{7,6}-x_{3,2}x_{4,1}x_{5,4}x_{6,6}x_{7,5}-x_{3,2}x_{4,1}x_{5,5}x_{6,4}x_{7,6}+x_{3,2}x_{4,1}x_{5,5}x_{6,6}x_{7,4}-$

$x_{7,4}x_{2,1}x_{3,2}x_{6,3} - x_{7,5}x_{2,1}x_{4,2}x_{6,3} - x_{7,6}x_{2,1}x_{5,2}x_{6,3} + x_{2,2}x_{3,1}x_{6,3}x_{7,4} + x_{2,2}x_{4,1}x_{6,3}x_{7,5} - x_{7,5}x_{3,1}x_{4,2}x_{6,4} + x_{3,1}x_{4,2}x_{6,5}x_{7,4} -$

$x_{7,6}x_{3,1}x_{5,2}x_{6,4} + x_{3,1}x_{5,2}x_{6,6}x_{7,4} + x_{3,2}x_{4,1}x_{6,4}x_{7,5} - x_{3,2}x_{4,1}x_{6,5}x_{7,4} - x_{7,6}x_{4,1}x_{5,2}x_{6,5} + x_{4,1}x_{5,2}x_{6,6}x_{7,5})$

4. $\frac{1}{-x_{3,1}x_{4,2}x_{6,3}x_{7,4} - x_{3,1}x_{5,2}x_{6,3}x_{7,5} + x_{3,2}x_{4,1}x_{6,3}x_{7,4} - x_{4,1}x_{5,2}x_{6,4}x_{7,5} + x_{4,1}x_{5,2}x_{6,5}x_{7,4}}(x_{3,1}x_{4,2}x_{5,3}x_{6,4}x_{7,5} - x_{7,4}x_{3,1}x_{4,2}x_{5,3}x_{6,5} -$

$x_{7,5}x_{3,1}x_{4,2}x_{5,4}x_{6,3} + x_{3,1}x_{4,2}x_{5,5}x_{6,3}x_{7,4} - x_{7,5}x_{3,1}x_{4,3}x_{5,2}x_{6,4} + x_{3,1}x_{4,3}x_{5,2}x_{6,5}x_{7,4} + x_{3,1}x_{4,4}x_{5,2}x_{6,3}x_{7,5} - x_{7,5}x_{3,2}x_{4,1}x_{5,3}x_{6,4} +$

$x_{3,2}x_{4,1}x_{5,3}x_{6,5}x_{7,4} + x_{3,2}x_{4,1}x_{5,4}x_{6,3}x_{7,5} - x_{3,2}x_{4,1}x_{5,5}x_{6,3}x_{7,4} + x_{3,3}x_{4,1}x_{5,2}x_{6,4}x_{7,5} - x_{3,3}x_{4,1}x_{5,2}x_{6,5}x_{7,4} - x_{7,4}x_{3,1}x_{5,2}x_{6,3} -$

$x_{7,5}x_{4,1}x_{5,2}x_{6,3})$

Some of the other rational expressions for the first integrals are getting quite large as well. There is a pattern for a few of them here as well. Using the chopping method as described in section 3.3 and sorting the characteristic polynomial with respect to μ_1, the rational expressions formed will always be first integrals (sometimes Casimirs). Therefore, one can check whether these rational expressions using only the μ_1 term are first integrals. For the remainder of the proof, we will not explicitly show these first integrals.

The $(7,4)$ Banded Toda Flow has Konstant form

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 & 0 & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & 1 & 0 & 0 \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} & 1 & 0 \\ 0 & x_{6,2} & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} & 1 \\ 0 & 0 & x_{7,3} & x_{7,4} & x_{7,5} & x_{7,6} & x_{7,7} \end{bmatrix}.$$

The dimension of is 25. There are 3 Casimirs, one being the trace, and 11 first integrals, again 6 coming from the characteristic polynomial. The other two Casimirs are

1. $\frac{1}{x_{5,1}x_{6,2}x_{7,3}}(-x_{4,1}x_{5,2}x_{6,3}x_{7,4} + x_{4,1}x_{5,2}x_{6,4}x_{7,3} + x_{4,1}x_{5,3}x_{6,2}x_{7,4} - x_{4,1}x_{5,4}x_{6,2}x_{7,3} + x_{4,2}x_{5,1}x_{6,3}x_{7,4} - x_{4,2}x_{5,1}x_{6,4}x_{7,3} -$

$x_{4,3}x_{5,1}x_{6,2}x_{7,4} + x_{4,4}x_{5,1}x_{6,2}x_{7,3})$

2. $\frac{1}{-x_{3,1}x_{6,2}x_{7,3} - x_{4,1}x_{6,2}x_{7,4} - x_{5,1}x_{6,2}x_{7,5}}(-x_{3,1}x_{4,2}x_{6,3}x_{7,4} + x_{3,1}x_{4,2}x_{6,4}x_{7,3} + x_{3,1}x_{4,3}x_{6,2}x_{7,4} - x_{3,1}x_{4,4}x_{6,2}x_{7,3} -$

$x_{3,1}x_{5,2}x_{6,3}x_{7,5} + x_{3,1}x_{5,2}x_{6,5}x_{7,3} + x_{3,1}x_{5,3}x_{6,2}x_{7,5} - x_{3,1}x_{5,5}x_{6,2}x_{7,3} + x_{3,2}x_{4,1}x_{6,3}x_{7,4} - x_{3,2}x_{4,1}x_{6,4}x_{7,3} + x_{3,2}x_{5,1}x_{6,3}x_{7,5} -$

$$x_{3,2}x_{5,1}x_{6,5}x_{7,3}-x_{3,3}x_{4,1}x_{6,2}x_{7,4}-x_{3,3}x_{5,1}x_{6,2}x_{7,5}-x_{4,1}x_{5,2}x_{6,4}x_{7,5}+x_{4,1}x_{5,2}x_{6,5}x_{7,4}+x_{4,1}x_{5,4}x_{6,2}x_{7,5}-x_{4,1}x_{5,5}x_{6,2}x_{7,4}+$$
$$x_{4,2}x_{5,1}x_{6,4}x_{7,5}-x_{4,2}x_{5,1}x_{6,5}x_{7,4}-x_{4,4}x_{5,1}x_{6,2}x_{7,5}+x_{4,1}x_{6,2}x_{7,3}+x_{5,1}x_{6,2}x_{7,4})-\frac{1}{x_{3,1}x_{7,3}+x_{4,1}x_{7,4}+x_{5,1}x_{7,5}}(-x_{2,1}x_{3,2}x_{7,3}-$$
$$x_{2,1}x_{4,2}x_{7,4}-x_{2,1}x_{5,2}x_{7,5}-x_{7,6}x_{6,2}x_{2,1}+x_{2,2}x_{3,1}x_{7,3}+x_{2,2}x_{4,1}x_{7,4}+x_{2,2}x_{5,1}x_{7,5}-x_{3,1}x_{4,3}x_{7,4}+x_{3,1}x_{4,4}x_{7,3}-$$
$$x_{3,1}x_{5,3}x_{7,5}+x_{3,1}x_{5,5}x_{7,3}-x_{3,1}x_{6,3}x_{7,6}+x_{3,1}x_{6,6}x_{7,3}+x_{3,3}x_{4,1}x_{7,4}+x_{3,3}x_{5,1}x_{7,5}-x_{4,1}x_{5,4}x_{7,5}+x_{4,1}x_{5,5}x_{7,4}-$$
$$x_{4,1}x_{6,4}x_{7,6}+x_{4,1}x_{6,6}x_{7,4}+x_{4,4}x_{5,1}x_{7,5}-x_{5,1}x_{6,5}x_{7,6}+x_{5,1}x_{6,6}x_{7,5}-x_{4,1}x_{7,3}-x_{5,1}x_{7,4})$$

There will be 4 first integrals coming from the rational expressions made by coefficients from μ_1. The other first integral is

$$\frac{1}{(x_{3,1}x_{6,2}x_{7,3}+x_{4,1}x_{6,2}x_{7,4}+x_{5,1}x_{6,2}x_{7,5})^2}(-x_{3,1}x_{4,2}x_{6,3}x_{7,4}+x_{3,1}x_{4,2}x_{6,4}x_{7,3}+x_{3,1}x_{4,3}x_{6,2}x_{7,4}-x_{3,1}x_{4,4}x_{6,2}x_{7,3}-$$
$$x_{3,1}x_{5,2}x_{6,3}x_{7,5}+x_{3,1}x_{5,2}x_{6,5}x_{7,3}+x_{3,1}x_{5,3}x_{6,2}x_{7,5}-x_{3,1}x_{5,5}x_{6,2}x_{7,3}+x_{3,2}x_{4,1}x_{6,3}x_{7,4}-x_{3,2}x_{4,1}x_{6,4}x_{7,3}+x_{3,2}x_{5,1}x_{6,3}x_{7,5}-$$
$$x_{3,2}x_{5,1}x_{6,5}x_{7,3}-x_{3,3}x_{4,1}x_{6,2}x_{7,4}-x_{3,3}x_{5,1}x_{6,2}x_{7,5}-x_{4,1}x_{5,2}x_{6,4}x_{7,5}+x_{4,1}x_{5,2}x_{6,5}x_{7,4}+x_{4,1}x_{5,4}x_{6,2}x_{7,5}-x_{4,1}x_{5,5}x_{6,2}x_{7,4}+$$
$$x_{4,2}x_{5,1}x_{6,4}x_{7,5}-x_{4,2}x_{5,1}x_{6,5}x_{7,4}-x_{4,4}x_{5,1}x_{6,2}x_{7,5}+x_{4,1}x_{6,2}x_{7,3}+x_{5,1}x_{6,2}x_{7,4})$$

For the $(7,5)$ case, the Konstant form is

$$\begin{bmatrix} x_{1,1} & 1 & 0 & 0 & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & 1 & 0 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 & 0 & 0 & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & 1 & 0 & 0 \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} & 1 & 0 \\ x_{6,1} & x_{6,2} & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} & 1 \\ 0 & x_{7,2} & x_{7,3} & x_{7,4} & x_{7,5} & x_{7,6} & x_{7,7} \end{bmatrix}.$$

The dimension is 27. The number of Casimirs is 3, and the number of first integrals is 12. The two other Casimirs are

1. $\frac{1}{x_{6,1}x_{7,2}}(-x_{3,1}x_{6,2}x_{7,3}+x_{3,1}x_{6,3}x_{7,2}+x_{3,2}x_{6,1}x_{7,3}-x_{3,3}x_{6,1}x_{7,2}-x_{4,1}x_{6,2}x_{7,4}+x_{4,1}x_{6,4}x_{7,2}+x_{4,2}x_{6,1}x_{7,4}-$
$x_{4,4}x_{6,1}x_{7,2}-x_{5,1}x_{6,2}x_{7,5}+x_{5,1}x_{6,5}x_{7,2}+x_{5,2}x_{6,1}x_{7,5}-x_{5,5}x_{6,1}x_{7,2})$

2. $\frac{1}{x_{5,1}x_{6,2}x_{7,3}-x_{5,1}x_{6,3}x_{7,2}-x_{5,2}x_{6,1}x_{7,3}+x_{5,3}x_{6,1}x_{7,2}}(-x_{4,1}x_{5,2}x_{6,3}x_{7,4}+x_{4,1}x_{5,2}x_{6,4}x_{7,3}+x_{4,1}x_{5,3}x_{6,2}x_{7,4}-x_{4,1}x_{5,3}x_{6,4}x_{7,2}-$
$x_{4,1}x_{5,4}x_{6,2}x_{7,3}+x_{4,1}x_{5,4}x_{6,3}x_{7,2}+x_{4,2}x_{5,1}x_{6,3}x_{7,4}-x_{4,2}x_{5,1}x_{6,4}x_{7,3}-x_{4,2}x_{5,3}x_{6,1}x_{7,4}+x_{4,2}x_{5,4}x_{6,1}x_{7,3}-x_{4,3}x_{5,1}x_{6,2}x_{7,4}+$
$x_{4,3}x_{5,1}x_{6,4}x_{7,2}+x_{4,3}x_{5,2}x_{6,1}x_{7,4}-x_{4,3}x_{5,4}x_{6,1}x_{7,2}+x_{4,4}x_{5,1}x_{6,2}x_{7,3}-x_{4,4}x_{5,1}x_{6,3}x_{7,2}-x_{4,4}x_{5,2}x_{6,1}x_{7,3}+x_{4,4}x_{5,3}x_{6,1}x_{7,2}).$

In this case, 6 first integrals come from the characteristic polynomial and 4 come from the μ_1 coefficients, therefore the other 2 are

1. $\frac{1}{x_{7,2}x_{6,1}}(x_{3,1}x_{6,2}x_{7,3}-x_{3,1}x_{6,3}x_{7,2}-x_{3,2}x_{6,1}x_{7,3}+x_{3,3}x_{6,1}x_{7,2}+x_{4,1}x_{6,2}x_{7,4}-x_{4,1}x_{6,4}x_{7,2}-x_{4,2}x_{6,1}x_{7,4}+x_{4,4}x_{6,1}x_{7,2}+$
$x_{5,1}x_{6,2}x_{7,5} - x_{5,1}x_{6,5}x_{7,2} - x_{5,2}x_{6,1}x_{7,5} + x_{5,5}x_{6,1}x_{7,2})$

2. $\frac{1}{x_{7,2}x_{6,1}}(-x_{3,1}x_{4,2}x_{6,3}x_{7,4}+x_{3,1}x_{4,2}x_{6,4}x_{7,3}+x_{3,1}x_{4,3}x_{6,2}x_{7,4}-x_{3,1}x_{4,3}x_{6,4}x_{7,2}-x_{3,1}x_{4,4}x_{6,2}x_{7,3}+x_{3,1}x_{4,4}x_{6,3}x_{7,2}-$
$x_{3,1}x_{5,2}x_{6,3}x_{7,5}+x_{3,1}x_{5,2}x_{6,5}x_{7,3}+x_{3,1}x_{5,3}x_{6,2}x_{7,5}-x_{3,1}x_{5,3}x_{6,5}x_{7,2}-x_{3,1}x_{5,5}x_{6,2}x_{7,3}+x_{3,1}x_{5,5}x_{6,3}x_{7,2}+x_{3,2}x_{4,1}x_{6,3}x_{7,4}-$
$x_{3,2}x_{4,1}x_{6,4}x_{7,3}-x_{3,2}x_{4,3}x_{6,1}x_{7,4}+x_{3,2}x_{4,4}x_{6,1}x_{7,3}+x_{3,2}x_{5,1}x_{6,3}x_{7,5}-x_{3,2}x_{5,1}x_{6,5}x_{7,3}-x_{3,2}x_{5,3}x_{6,1}x_{7,5}+x_{3,2}x_{5,5}x_{6,1}x_{7,3}-$
$x_{3,3}x_{4,1}x_{6,2}x_{7,4}+x_{3,3}x_{4,1}x_{6,4}x_{7,2}+x_{3,3}x_{4,2}x_{6,1}x_{7,4}-x_{3,3}x_{4,4}x_{6,1}x_{7,2}-x_{3,3}x_{5,1}x_{6,2}x_{7,5}+x_{3,3}x_{5,1}x_{6,5}x_{7,2}+x_{3,3}x_{5,2}x_{6,1}x_{7,5}-$
$x_{3,3}x_{5,5}x_{6,1}x_{7,2}-x_{4,1}x_{5,2}x_{6,4}x_{7,5}+x_{4,1}x_{5,2}x_{6,5}x_{7,4}+x_{4,1}x_{5,4}x_{6,2}x_{7,5}-x_{4,1}x_{5,4}x_{6,5}x_{7,2}-x_{4,1}x_{5,5}x_{6,2}x_{7,4}+x_{4,1}x_{5,5}x_{6,4}x_{7,2}+$
$x_{4,2}x_{5,1}x_{6,4}x_{7,5}-x_{4,2}x_{5,1}x_{6,5}x_{7,4}-x_{4,2}x_{5,4}x_{6,1}x_{7,5}+x_{4,2}x_{5,5}x_{6,1}x_{7,4}-x_{4,4}x_{5,1}x_{6,2}x_{7,5}+x_{4,4}x_{5,1}x_{6,5}x_{7,2}+x_{4,4}x_{5,2}x_{6,1}x_{7,5}-$
$x_{4,4}x_{5,5}x_{6,1}x_{7,2}+x_{4,1}x_{6,2}x_{7,3}-x_{4,1}x_{6,3}x_{7,2}-x_{4,2}x_{6,1}x_{7,3}+x_{4,3}x_{6,1}x_{7,2}+x_{5,1}x_{6,2}x_{7,4}-x_{5,1}x_{6,4}x_{7,2}-x_{5,2}x_{6,1}x_{7,4}+$
$x_{5,4}x_{6,1}x_{7,2})$

This concludes the dimension 7 case for the banded Toda flow. For the remaining cases, the first integrals start to as well get large. As with the other cases, there is a pattern. As mentioned in section 3.3, when we find the characteristic polynomial of the Konstant matrix with μ's down the off-diagonal, we sort it with respect to $\mu_1, \mu_1\mu_2, \cdots$. Therefore, finding all the first integrals amounts to forming the rational functions of the polynomial coefficients to $\mu_1, \mu_1\mu_2, \cdots$. In section 3.5, we will describe how to do this in MAPLE so the reader can test for him or herself whether these rational expressions are indeed first integrals. Though, for the remaining cases, the first integrals will not be explicitly given. The Casimirs are less obvious to find, therefore they will be given explicitly if there are more than 1 and are reasonable in length. The following table will recap the findings for dimension 8 and higher. Also, we will not be giving the Konstant matrix for each case.

Case	Dimension	Number of Casimirs	Number of First Integrals
(8,2)	21	1	10
(8,3)	26	2	12
(8,4)	30	2	14
(8,5)	33	3	15
(8,6)	35	3	16
(9,2)	24	2	11
(9,3)	30	2	14
(9,4)	35	3	16
(9,5)	39	3	18
(9,6)	42	4	19
(9,7)	44	4	20
(10,2)	27	1	13
(10,3)	34	2	16
(10,4)	40	2	19
(10,5)	45	3	21
(10,6)	49	3	23
(10,7)	52	4	24
(10,8)	54	4	25

FIGURE 3.1: Casimirs and First Integrals for Dimension 8 and higher

The Casimirs for some of the cases are as follows:

$$(8,3): \quad \frac{1}{x_{4,1}x_{5,2}x_{7,4}x_{8,5}}(-x_{3,1}x_{4,2}x_{5,3}x_{7,4}x_{8,5}-x_{3,1}x_{4,2}x_{6,3}x_{7,4}x_{8,6}+x_{3,1}x_{4,3}x_{5,2}x_{7,4}x_{8,5}-x_{3,1}x_{5,2}x_{6,3}x_{7,5}x_{8,6}+x_{3,1}x_{5,2}x_{6,3}x_{7,6}x_{8,5}+$$
$$x_{3,2}x_{4,1}x_{5,3}x_{7,4}x_{8,5}+x_{3,2}x_{4,1}x_{6,3}x_{7,4}x_{8,6}-x_{3,3}x_{4,1}x_{5,2}x_{7,4}x_{8,5}-x_{4,1}x_{5,2}x_{6,4}x_{7,5}x_{8,6}+x_{4,1}x_{5,2}x_{6,4}x_{7,6}x_{8,5}+x_{4,1}x_{5,2}x_{6,5}x_{7,4}x_{8,6}-$$
$$x_{4,1}x_{5,2}x_{6,6}x_{7,4}x_{8,5})$$

$$(8,4): \quad (x_{2,1}x_{3,2}x_{5,3}x_{8,5}+x_{2,1}x_{3,2}x_{6,3}x_{8,6}-x_{2,1}x_{3,3}x_{5,2}x_{8,5}+x_{2,1}x_{4,2}x_{5,4}x_{8,5}+x_{2,1}x_{4,2}x_{6,4}x_{8,6}+x_{2,1}x_{4,2}x_{7,4}x_{8,7}-x_{2,1}x_{4,4}x_{5,2}x_{8,5}+$$
$$x_{2,1}x_{5,2}x_{6,5}x_{8,6}-x_{2,1}x_{5,2}x_{6,6}x_{8,5}+x_{2,1}x_{5,2}x_{7,5}x_{8,7}-x_{2,1}x_{5,2}x_{7,7}x_{8,5}-x_{2,2}x_{3,1}x_{5,3}x_{8,5}-x_{2,2}x_{3,1}x_{6,3}x_{8,6}-x_{2,2}x_{4,1}x_{5,4}x_{8,5}-$$
$$x_{2,2}x_{4,1}x_{6,4}x_{8,6}-x_{2,2}x_{4,1}x_{7,4}x_{8,7}+x_{3,1}x_{4,3}x_{5,4}x_{8,5}+x_{3,1}x_{4,3}x_{6,4}x_{8,6}+x_{3,1}x_{4,3}x_{7,4}x_{8,7}-x_{3,1}x_{4,4}x_{5,3}x_{8,5}-x_{3,1}x_{4,4}x_{6,3}x_{8,6}+$$
$$x_{3,1}x_{5,3}x_{6,5}x_{8,6}-x_{3,1}x_{5,3}x_{6,6}x_{8,5}+x_{3,1}x_{5,3}x_{7,5}x_{8,7}-x_{3,1}x_{5,3}x_{7,7}x_{8,5}-x_{3,1}x_{5,5}x_{6,3}x_{8,6}+x_{3,1}x_{6,3}x_{7,6}x_{8,7}-x_{3,1}x_{6,3}x_{7,7}x_{8,6}-$$
$$x_{3,3}x_{4,1}x_{5,4}x_{8,5}-x_{3,3}x_{4,1}x_{6,4}x_{8,6}-x_{3,3}x_{4,1}x_{7,4}x_{8,7}+x_{4,1}x_{5,4}x_{6,5}x_{8,6}-x_{4,1}x_{5,4}x_{6,6}x_{8,5}+x_{4,1}x_{5,4}x_{7,5}x_{8,7}-x_{4,1}x_{5,4}x_{7,7}x_{8,5}-$$
$$x_{4,1}x_{5,5}x_{6,4}x_{8,6}-x_{4,1}x_{5,5}x_{7,4}x_{8,7}+x_{4,1}x_{6,4}x_{7,6}x_{8,7}-x_{4,1}x_{6,4}x_{7,7}x_{8,6}-x_{4,1}x_{6,6}x_{7,4}x_{8,7}+x_{3,1}x_{5,2}x_{8,5}+x_{3,1}x_{6,3}x_{8,5}+$$

$x_{4,1}x_{5,3}x_{8,5} + x_{4,1}x_{6,3}x_{8,6} + x_{4,1}x_{6,4}x_{8,5} + x_{4,1}x_{7,4}x_{8,6})(-x_{2,1}x_{5,2}x_{8,5} - x_{3,1}x_{5,3}x_{8,5} - x_{3,1}x_{6,3}x_{8,6} - x_{4,1}x_{5,4}x_{8,5} -$

$x_{4,1}x_{6,4}x_{8,6} - x_{4,1}x_{7,4}x_{8,7})^{-1} - \frac{1}{x_{5,1}x_{6,2}x_{7,3} - x_{5,1}x_{6,3}x_{7,2} - x_{5,2}x_{6,1}x_{7,3} + x_{5,3}x_{6,1}x_{7,2}}(-x_{4,1}x_{5,2}x_{6,3}x_{7,4} + x_{4,1}x_{5,2}x_{6,4}x_{7,3} +$

$x_{4,1}x_{5,3}x_{6,2}x_{7,4} - x_{4,1}x_{5,3}x_{6,4}x_{7,2} - x_{4,1}x_{5,4}x_{6,2}x_{7,3} + x_{4,1}x_{5,4}x_{6,3}x_{7,2} + x_{4,2}x_{5,1}x_{6,3}x_{7,4} - x_{4,2}x_{5,1}x_{6,4}x_{7,3} - x_{4,2}x_{5,3}x_{6,1}x_{7,4} +$

$x_{4,2}x_{5,4}x_{6,1}x_{7,3} - x_{4,3}x_{5,1}x_{6,2}x_{7,4} + x_{4,3}x_{5,1}x_{6,4}x_{7,2} + x_{4,3}x_{5,2}x_{6,1}x_{7,4} - x_{4,3}x_{5,4}x_{6,1}x_{7,2} + x_{4,4}x_{5,1}x_{6,2}x_{7,3} - x_{4,4}x_{5,1}x_{6,3}x_{7,2} -$

$x_{4,4}x_{5,2}x_{6,1}x_{7,3} + x_{4,4}x_{5,3}x_{6,1}x_{7,2})$

$(9,2):\ \frac{1}{x_{7,1}x_{8,2}}(-x_{3,1}x_{7,2}x_{8,3} + x_{3,1}x_{7,3}x_{8,2} + x_{3,2}x_{7,1}x_{8,3} - x_{3,3}x_{7,1}x_{8,2} - x_{4,1}x_{7,2}x_{8,4} + x_{4,1}x_{7,4}x_{8,2} + x_{4,2}x_{7,1}x_{8,4} -$

$x_{4,4}x_{7,1}x_{8,2} - x_{5,1}x_{7,2}x_{8,5} + x_{5,1}x_{7,5}x_{8,2} + x_{5,2}x_{7,1}x_{8,5} - x_{5,5}x_{7,1}x_{8,2} - x_{6,1}x_{7,2}x_{8,6} + x_{6,1}x_{7,6}x_{8,2} + x_{6,2}x_{7,1}x_{8,6} -$

$x_{6,6}x_{7,1}x_{8,2})$

$(10,3)\ \frac{1}{x_{4,1}x_{7,4}x_{10,7}}(x_{2,1}x_{4,2}x_{7,4}x_{10,7} + x_{2,1}x_{5,2}x_{7,5}x_{10,7} + x_{2,1}x_{5,2}x_{8,5}x_{10,8} - x_{2,2}x_{4,1}x_{7,4}x_{10,7} + x_{3,1}x_{4,3}x_{7,4}x_{10,7} + x_{3,1}x_{5,3}x_{7,5}x_{10,7} +$

$x_{3,1}x_{5,3}x_{8,5}x_{10,8} + x_{3,1}x_{6,3}x_{7,6}x_{10,7} + x_{3,1}x_{6,3}x_{8,6}x_{10,8} + x_{3,1}x_{6,3}x_{9,6}x_{10,9} - x_{3,3}x_{4,1}x_{7,4}x_{10,7} + x_{4,1}x_{5,4}x_{7,5}x_{10,7} +$

$x_{4,1}x_{5,4}x_{8,5}x_{10,8} - x_{4,1}x_{5,5}x_{7,4}x_{10,7} + x_{4,1}x_{6,4}x_{7,6}x_{10,7} + x_{4,1}x_{6,4}x_{8,6}x_{10,8} + x_{4,1}x_{6,4}x_{9,6}x_{10,9} - x_{4,1}x_{6,6}x_{7,4}x_{10,7} +$

$x_{4,1}x_{7,4}x_{8,7}x_{10,8} - x_{4,1}x_{7,4}x_{8,8}x_{10,7} + x_{4,1}x_{7,4}x_{9,7}x_{10,9} - x_{4,1}x_{7,4}x_{9,9}x_{10,7})$

$(10,4)\ \frac{1}{x_{5,1}x_{6,2}x_{9,5}x_{10,6}}(-x_{3,1}x_{5,2}x_{6,3}x_{9,5}x_{10,6} - x_{3,1}x_{5,2}x_{7,3}x_{9,5}x_{10,7} + x_{3,1}x_{5,3}x_{6,2}x_{9,5}x_{10,6} - x_{3,1}x_{6,2}x_{7,3}x_{9,6}x_{10,7} +$

$x_{3,1}x_{6,2}x_{7,3}x_{9,7}x_{10,6} + x_{3,2}x_{5,1}x_{6,3}x_{9,5}x_{10,6} + x_{3,2}x_{5,1}x_{7,3}x_{9,5}x_{10,7} - x_{3,3}x_{5,1}x_{6,2}x_{9,5}x_{10,6} - x_{4,1}x_{5,2}x_{6,4}x_{9,5}x_{10,6} -$

$x_{4,1}x_{5,2}x_{7,4}x_{9,5}x_{10,7} - x_{4,1}x_{5,2}x_{8,4}x_{9,5}x_{10,8} + x_{4,1}x_{5,4}x_{6,2}x_{9,5}x_{10,6} - x_{4,1}x_{6,2}x_{7,4}x_{9,6}x_{10,7} + x_{4,1}x_{6,2}x_{7,4}x_{9,7}x_{10,6} -$

$x_{4,1}x_{6,2}x_{8,4}x_{9,6}x_{10,8} + x_{4,1}x_{6,2}x_{8,4}x_{9,8}x_{10,6} + x_{4,2}x_{5,1}x_{6,4}x_{9,5}x_{10,6} + x_{4,2}x_{5,1}x_{7,4}x_{9,5}x_{10,7} + x_{4,2}x_{5,1}x_{8,4}x_{9,5}x_{10,8} -$

$x_{4,4}x_{5,1}x_{6,2}x_{9,5}x_{10,6} - x_{5,1}x_{6,2}x_{7,5}x_{9,6}x_{10,7} + x_{5,1}x_{6,2}x_{7,5}x_{9,7}x_{10,6} + x_{5,1}x_{6,2}x_{7,6}x_{9,5}x_{10,7} - x_{5,1}x_{6,2}x_{7,7}x_{9,5}x_{10,6} -$

$x_{5,1}x_{6,2}x_{8,5}x_{9,6}x_{10,8} + x_{5,1}x_{6,2}x_{8,5}x_{9,8}x_{10,6} + x_{5,1}x_{6,2}x_{8,6}x_{9,5}x_{10,8} - x_{5,1}x_{6,2}x_{8,8}x_{9,5}x_{10,6})$

We will now consider the Casimirs that are only made through the next degree coefficient is divided by the leading coefficient, that is, they are the F_{k1}'s using the notation in section 3.3. The F_{0k}'s are left out since that is the trace function. Some of these cases will highlight that some Casimirs are formed by taking the difference of two first integrals. This observation will be discussed more in chapter 4.

Dimension	Casimirs
$(5,2)$	F_{11}
$(5,3)$	F_{21}
$(6,3)$	$F_{21}-F_{11}$
$(6,4)$	F_{21}
$(7,2)$	F_{11}
$(7,3)$	F_{11}
$(7,4)$	$F_{31}, F_{21}-F11$
$(7,5)$	F_{21}, F_{31}
$(8,3)$	F_{21}
$(8,4)$	$F_{31}-F_{11}$
$(8,5)$	$F_{31}, F_{21}-F_{11}$
$(8,6)$	F_{21}, F_{31}
$(9,3)$	$F_{21}-F_{11}$
$(9,4)$	$F_{11}, F_{31}-F_{21}$
$(9,5)$	$F_{41}, F_{31}-F_{11}$
$(9,6)$	$F_{21}-F_{11}, F_{31}, F_{41}$
$(9,7)$	F_{21}, F_{31}, F_{41}
$(10,5)$	$F_{41}-F_{11}, F_{31}-F_{21}$
$(10,6)$	$F_{41}, F_{31}-F_{11}$
$(10,7)$	$F_{21}-F_{11}, F_{31}, F_{41}$
$(10,8)$	$F_{21},\ F_{31},\ F_{41}$

FIGURE 3.2: Casimirs for Each Case

This concludes the proof of the main theorem.

3.5 Using MAPLE

In order to use MAPLE, one will need the programs that were being run. These programs can be found in the appendix. For this section, we will walk through an example of how MAPLE was used to calculate the first integrals and Casimirs so that the reader will feel comfortable going to MAPLE and check the other cases that were omitted.

The example that we will do is a low-dimension one, the $(5,2)$ case. Recall from section 3.4 that the $(5,3)$ case had 6 first integrals and 2 Casimirs. To find these, first create the Konstant form of the matrix using (1) from the appendix. Then, one can automatically find some of the first integrals by taking the characteristic polynomial of the matrix. To extract the coefficients of the polynomial, one can use the **coeff** command. Also, to calculate the characteristic polynomial, one needs to load in the linear algebra package from MAPLE.

To check that these functions are in involution with each other, one can use the code $(2), (3)$, and (4) in the appendix. From the $(5,3)$ case, in order to find the other first integrals, one needs to form a new matrix, either (5) or (6) in the appendix depending on whether the dimension of the matrix is even or odd. This code will also create the characteristic polynomial. Next, to sort with respect to the variable one wants, one can use the **sort** command or differentiate the characteristic polynomial with respect to what μ_i term that is being considered. One can find the rational expressions by using the **coeff** command. For the example, the only other first integral is coming from the μ_1 term, therefore the sort command was used as so:

F1 := sort(coeff(Toda1,mu[1]),lambda);

Then, we formed the rational expressions by doing:

F2 := subs(mu[2]=0, F1);

coeff(F2,lambda)/lcoeff(F2,lambda);

subs(lambda=0,F2)/lcoeff(F2,lambda);

Then, to check if they are in involution with the other first integrals, use the **LiePoissonBracket** command again. The hope is that if the function is a first integral, the Lie-Poisson Bracket will output a 0.

To find the Casimirs, one can use the code found in (7). Though, the code in order to find the rational expressions for the Casimirs can be found in (11). Then, after running (11), the following command will check to see if the functions are Casimirs.

for i from 0 to nm do ckcsm(F||i||1) od;

This command will output a 1 if the function is not a Casimir, and a 0 if the function is a Casimir. If running this on the $(5, 3)$ example, the following would output if the above expression was run.

$$0$$

$$1$$

$$0$$

This would tell the user that F_{01} and F_{21} are Casimirs, and since there were 2 Casimirs, one would be done with this case. Special cases of this output is discussed in Chapter 4.

One can also use MAPLE to determine how many first integrals and Casimirs are needed. To do this, first define the matrix with dimension n and band k using the **TM** command. Then, run the following command:

a:= convert(convert(TM(n, k), set) minus 0, 1, list).

Then, the command **nops(a)** tells you the dimension. This command creates a list of all the coordinates in the matrix, minus 0 and 1. This is the $\dim(L)$ in (2.6). To determine the number of first integrals, use the following command:

PBM := Matrix(dm, (i, j)->PBF(a[i], a[j], n, k)).

Then, running the following command

tPBM := subs(seq(a[i]=$i^2 + 7 * i$, i=1..dm), PBM)

and running **Rank(tPBM)** will result in two times the number of first integrals. Therefore, the number of first integrals is

$$\frac{\text{Rank(tPMB)}}{2}. \tag{3.21}$$

Knowing this will result in the number of Casimirs since

$$\text{Number of Casimirs} = \dim(L) - 2 \times \text{Number of First Integrals.}$$

Combining both finding the number of Casimirs and first integrals and then explicitly seeing what the Casimirs and first integrals are can be done using the steps above. Still, there are cases in figure 3.2 where the difference between two first integrals is a Casimir, which the reasoning is further discussed in Chapter 4.

Chapter 4
Future Work

In Chapter 4, we will discuss some of the results in Chapter 3 that need further explanation. As seen in some of the Casimirs, the difference between two functions ended up a Casimir. We will discuss what led to this discovery using MAPLE. Another conjecture is presented to see what the end goal for the project will be. To end the chapter, we will discuss how the Differential Geometry package on MAPLE can be used in this work.

4.1 The General Proof

Although our main theorem in this paper only proved up the dimension 10, it is the feeling that this work can be expanded to prove the general case of the (n, k) banded Toda flow. The conjecture is as follows:

Conjecture 4.1.1. The (n, k) banded Toda flow is Liouville integrable.

As already presented, some cases have already been proven to be Liouville integrable, that being when $k = 1$ and when $k = n - 1$. Both cases have also been shown to be Liouville integrable for all simple Lie algebras. For Theorem 3.1 and the work that has been done in this paper, the underlying Lie algebra has been $gl(n, \mathbb{R})$. So, it is the hope that Conjecture 4.1.1 will be true for all simple Lie algebras, but the main focus is to first prove the Liouville integrability of the banded Toda flow for $gl(n, \mathbb{R})$.

As nice as it would be, the general proof cannot just state the first integrals and Casimirs for each n and for each k. There would definitely be a lot of cases to work through (in fact, infinity many)! Instead, the general proof should account for all cases at once. As of this writing, work is being done to try and do this. Though, one of the beliefs to unlocking the proof of the conjecture is to understanding why in some cases in the proof of Theorem 3.1 there were a lack of Casimirs when running the code. This is discussed in the next section.

4.2 The Lack of Casimirs

Looking back on the proof of Theorem 3.1, when giving the Casimirs formed by cases such as $(9,4)$ and $(10,5)$, it can be seen that some of the Casimirs were found by taking the difference of two functions. Though, the problem of a lack of Casimirs came about as early as $(6,3)$. As an example of the lack of Casimirs, we will focus on the $(6,3)$.

When running the code that was discussed in section 3.5, it was found that for the $(6,3)$ case, there were supposed to be 2 Casimirs and 8 first integrals. As a side note, finding the first integrals was never an issue due to the chopping method that has been discussed. The chopping method and the alternative chopping method used in this paper has been proven to account for all the first integrals, which was shown in [3].

When running the **ckcsm** code, this was the output:

$$\begin{gathered} 0 \\ 1 \\ 1 \end{gathered}$$

This tells us that the only Casimir when running the code is F_{01}, which is the trace of the matrix. Though, we knew there had to be two Casimirs according to the definition of Liouville integrability. It was discovered that by taking the difference between the functions that produced the 1's is also a Casimir, which was checked through MAPLE. So for this example the other Casimir is $F_{21} - F_{11}$.

Though, another interesting situation arose, such as $(10,5)$ case. Reading the table in Section 3.4, the number of Casimirs was supposed to be 3. Running the code to check for Casimirs, the output was

$$\begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}$$

which tells us again that the only Casimir found was F_{01}, which was the trace. In this case though, it is not obvious what functions to take the difference between. Taking the difference between the outer-most string of 1's results in a Casimir, therefore the two other Casimirs are $F_{41} - F_{11}$ and $F_{31} - F_{21}$. It is important to note that taking the difference in other manners will result in a Casimir as well, but they will not be functionally independent as required.

This idea of taking the difference between the outer-most string 1's and seeing if it is a Casimir was done on all such examples in which there were a lack a Casimirs. This led to the following conjecture.

Conjecture 4.2.1. Functions which are produced by taking the difference between the outer-most string of 1's is a Casimir.

While Conjecture 4.2.1 definitely seems hard to believe, work is currently being done in order to see why such a statement must be true.

4.3 Using the Differential Geometry Package on MAPLE

During this project, Dr. Ian Anderson suggested we try finding the first integrals of the Toda flow using packages that were already coded for MAPLE. Dr. Anderson, alongside Dr. Charles Torre, are two prominent figures in MAPLE since they have written many packages for users who want to use MAPLE for work in differential geometry. The package itself can be found in [1].

A fellow Utah State University student named Tyler Hansen developed some codes for the Differential Geometry package, more specifically for Hamiltonian mechanics. Different tools that Tyler developed include the **LiePoisson**, **PoissonBracket**, and **HamiltonianVectorFields**, all of which could be useful for the Toda Flow. As an example of how this could be useful, recall that a symplectic manifold is defined using a 2-form. Define this 2-form such that

$$\omega = dx_1 \wedge dp_1 + dx_2 \wedge dp_2 \tag{4.1}$$

where x_1, x_2, p_1, and p_2 are local coordinates on a manifold M. Then, recall that to find first integrals, one must first pick a Hamiltonian, H. In this example, pick $H = p_1^2 + p_2^2$.

One can study that Hamiltonian Vector Field of H, which in this case would be

$$2p_1\partial_{x_1} + 2p_2\partial_{x_2}. \tag{4.2}$$

Finding the vector field helps us study the flow, which will lead us to understanding the conserved quantities of the system, namely the first integrals.

Also, recall the first integrals are defined so that they are in involution with the Hamiltonian H. Therefore, one can use the command **PoissonBracket** to study the first integrals. In the example we are working on, if we take H and use the Poisson bracket command on H with p_1, the result is 0 which tells us that p_1 is a first integral.

Though, say one did not know the first integrals to begin with. Then, doing the Poisson bracket between H and an arbitrary function will result in a partial differential equation that one can solve in order to find the first integrals. In the example, if we do the Poisson bracket between H and an arbitrary function $f(x_1, x_2, p_1, p_2)$, then the result is

$$-2(\frac{\partial}{\partial x_1} f(x_1, x_2, p_1, p_2))p_1 - 2(\frac{\partial}{\partial x_2} f(x_1, x_2, p_1, p_2))p_2 \tag{4.3}$$

which then using the **pdsolve** command on (4.3) results in three first integrals, which are

$$p_1, \quad p_2, \quad p_1x_2 - p_2x_1. \tag{4.4}$$

These first integrals though do not quite line up to the definition in Chapter 2 of a system being Liouville integrable. While they are in involution with the Hamiltonian, they are not in involution with each other. For example,

$$\{p_1, p_1x_2 - p_2x_1\} \neq 0.$$

Though, in this example, p_1 and p_2 are in involution, which would fit the definition. One might also tell if a function is a first integral by again studying the Hamiltonian vector Field and finding the Lie derivative of the function. The Lie derivative calculates the flow of the vector field, which if the result is zero, then the function being studied a conserved quantity, that being the first integral. In this case, if we take the Hamiltonian Vector Field of the third equation in (4.4), we get

$$x_2\partial_{x_1} - x_1\partial_{x_2} + p_2\partial_{p_1} - p_1\partial_{p_2} \tag{4.5}$$

which then calculating the Lie derivative of this with respect to ω results in 0.

Of course, the worry with many of the programs is the practicality of the software and whether it will work for more specific examples. One good example that one might try is the Toda lattice, since one of the main concerns is to find the first integrals which make the system Liouville integrable. Therefore, future work is to check whether the packages on MAPLE will be able to handle such examples as the Toda flow. If they do not or there are some problems, we want to be able to fix the code so that people who are wanting to use MAPLE in their projects will feel secure in knowing that the code will produce the correct outputs.

4.4 Conclusion

The goal of this paper was to introduce the ideas needed in order to study the Toda lattice, and then provide a basis in order to prove that the banded Toda flow of dimension n and k number of bands is Liouville integrable. While the general case has not yet been proven, this paper discussed at least the Liouville integrability up to dimension 10. This was successful in that the number of first integrals and Casimirs were obtained and then found using MAPLE.

It is possible to find the Casimirs and first integrals for dimension higher than 10. Though, for this paper, we only proved up to dimension 10 since the computation time using MAPLE for higher dimensions was extremely long. It is the belief though that even though only up through dimension 10 was proven that the general case of n dimensions with k number of bands is Liouville integrable.

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APPENDIX

In this appendix, we will be giving and explaining the MAPLE code that were used in order to find the Casimirs and First Integrals for each case of the Banded Toda Flow. We begin with a simple command that produces the (n, k) Toda flow that one desires.

```
1. TM := proc(n, k)
     Matrix(n, seq((i,i+1)=1, i=1..n-1), seq(seq((i+j, i)=x[i+j, i], i=1..n-j), j=0..k))
end:
```

The inputs into this function are just n which is the dimension and k which is the number of bands. Next, in order to see if functions are in involution, one needs to calculate the Lie-Poisson bracket between the functions. To do this, one can use the following code

```
2. LiePoissonBracket := proc(f,g,n,k)
     simplify(Trace(TM(n,k) . LieBracket(GradientCalculation(f,n),GradientCalculation(g,n))));
end;
```

The inputs into this are functions f, g and the dimension n and number of bands k. Though, in order to use this program, one also needs the LieBracket command and GradientCalculation command, which can be seen below.

```
3. LieBracket := proc(M,N)
     simplify( M.N-N.M);
end;
```

```
4. GradientCalculation := proc(f,n)
     local i,j;
     Matrix(n, seq(seq((i, j)=diff(f, x[j,i]), i=1..n), j=1..n));
end;
```

The inputs for (3) are two matrices M, N which will be automatically formed using the GradientCalculation command. The inputs for (4) are just the function f, and the dimension n. Now, in order to form a new matrix with the μ's down the off-diagonal, one can use the following two codes depending on if the matrix dimension is even or odd. This command will also form the characteristic polynomial as long as the Linear Algebra package is loaded into the MAPLE file.

```
5.TMmu := proc(n,k)
    local a;
    a := TM(n,k) + Matrix(n, seq((i, n+1-i)=mu[i], i=1..floor(n/2)));
    CharacteristicPolynomial(a,lambda);
end;
```

```
6. TMmuEven := proc(n,k)
    local a;
    a := TM(n,k) + Matrix(n, seq((i, n+1-i)=mu[i], i=1..floor((n-1)/2)));
    CharacteristicPolynomial(a,lambda);
end;
```

The inputs for both commands are the dimension of the matrix n and the number of bands k. Next, the following code will check to see if functions are Casimirs or not.

```
7. ckcsm := proc(f)
    local cs, j:
    cs := 0:
    for j to dm do
    if PBF(f, a[j], n, k)<>0
    then cs := 1: break: fi:
    od:
cs:
end:
```

The input for this function is just a function f. In order to use this code though, a slightly different version of the Lie Bracket and the gradient matrix is used. To following three codes are used in the **ckcsm** code. The first one will be the gradient matrix calculation, the second will be the Lie bracket, and the third will calculate the Lie-Poisson bracket between the functions.

```
8. GM := (f, n) -> Matrix(n, seq(seq((i, j)=diff(f, x[j, i]), i=1..n), j=1..n))
```

```
9. lb := (a, b) -> simplify( a.b - b.a ):
```

10. PBF := (f, g, n, k) -> simplify(Trace(TM(n, k) . lb(GM(f, n), GM(g, n))))

In order to form the rational expressions, the following code is used

11. DD := Determinant(Total):

```
for i from 0 to nm do
F||i := sort(subs(seq(mu[j]=0, j=1..nm), DD), lambda);
d||i := degree(F||i, lambda):
a||i := coeff(F||i, lambda, d||i):
for j to d||i do
F||i||j := coeff(F||i, lambda, d||i-j)/a||i:
od:
DD := diff(DD, mu[i+1]):
od:
seq(d||i, i=0..nm), add(d||i, i=0..nm);
```