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Teaching Students to Communicate with the Precise Language of Mathematics: A Focus on the Concept of Function in Calculus Courses

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TEACHING STUDENTS TO COMMUNICATE WITH THE PRECISE LANGUAGE OF MATHEMATICS: A FOCUS ON THE CONCEPT OF FUNCTION IN CALCULUS COURSES

by

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ABSTRACT

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Mathematics depends on using precise language that requires equally precise communication. However, this precision of language and communication is often absent in many mathematics classrooms, resulting in formal concept definitions that conflict with students' perceived images of that concept and which could lead to misunderstandings and poor concept constructions. One particular concept that often suffers from a lack of precision is the concept of function. Because this concept is so pervasive throughout mathematics, its importance cannot be understated. In order to explore the development of the concept of function, as it pertains to a calculus course, three articles are presented that (a) introduce the theoretical implications of teaching students to communicate with the precise language of mathematics, (b) relate the action research of a practitioner leading students to discover the concept of function and comprehend its definition with an emphasis on using the precise language of mathematics, and (c) develops and analyzes a suite of assessment tools designed to be relevant to students' higher-cognitive achievement of learning objectives involving the concept of function.

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PUBLIC ABSTRACT

Teaching Students to Communicate with the Precise Language of Mathematics: A Focus on the Concept of Function in Calculus Courses

Derrick S. Harkness

The use of precise language is one of the defining characteristics of mathematics that is often missing in mathematics classrooms. This lack of precision results in poorly constructed concepts that limit comprehension of essential mathematical definitions and notation. One important concept that frequently lacks the precision required by mathematics is the concept of function. Functions are foundational in the study undergraduate mathematics and are essential to other areas of modern mathematics. Because of its pivotal role, the concept of function is given particular attention in the three articles that comprise this study.

A unit on functions that focuses on using precise language was developed and presented to a class of 50 first-semester calculus students during the first two weeks of the semester. This unit includes a learning goal, a set of specific objectives, a collection of learning activities, and an end-of-unit assessment. The results of the implementation of this unit and the administration of the assessment indicated that when students were able to construct the concept of function themselves and formulate a formal definition, they had a deeper and more meaningful understanding of the concept.

In order to demonstrate its validity, the assessment was analyzed as to its relevance, reliability, and its test items' effectiveness in discriminating between different levels of achievement. The results of this analysis indicated that the assessment was relevant to both the mathematical content and learning levels indicated by the unit's objectives and had a high level of reliability. Additionally, the test items contained in the assessment had a reasonable level of effectiveness in discriminating between different levels of student achievement.

For my dearest wife and children whose patience and support were monumental in helping me succeed.

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Derrick S. Harkness

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CHAPTER 1

Introduction

Is the learning of mathematics different from other disciplines, if so, how? What instructional practices help students learn mathematics? What does effective mathematics teaching look like and how can educators provide it? What is mathematical proficiency and how can it be assessed? How can educators know if an assessment is valid? How is a valid and practical assessment developed?

Such questions have been studied since the beginning of the 20th century with the intent of improving instructional practices so students build stronger foundations of meaningful mathematics. Over the years, the performance of U.S. students on a variety of national and international mathematics assessments range from "extremely poor" to "simply mediocre" (Schmidt, 2012, p. 133). Finding answers to these foundational questions will help guide instructional practices to aid students build deeper mathematical foundations and raise their level of mathematical proficiency.

With these questions in mind, we will examine the nature of mathematics and contrast it with the mathematics being taught in schools. Then, using the concept of function as an example, we will (a) lay the theoretical framework to understand the process of student learning (see Chapter 2 on page 13); (b) examine the process of developing and implementing a set of learning activities that lead students to construct the concept of function, develop a definition, and comprehend that definition (see Chapter 3 on page 26); and (c) examine the development and analysis of an assessment designed to measure student achievement of objectives outlining a unit goal (see Chapter 4 on page 59).

Mathematics and Mathematics in Schools

James Milgram (2007) explains that after teaching students to read, the next job of our education system is to teach students mathematics. However, it is here that it fails (Milgram, 2007; National Commission on Excellence in Education, 1983; National Council

of Teachers of Mathematics, 1980; Schmidt, 2012). This failure, William Schmidt (2012) says, has several consequences. First, mathematics and science content knowledge and skills are related to our economic growth and a lack of these skills undermines our technological competitiveness. Second, typical high school students are graduating with little quantitative ability; leaving them unable to to find jobs that require certain technical skills or to enter college and take mathematics courses that offer any credit. One of the reasons for this failure is that mathematics is one of the most misunderstood subjects in school (Fowler, 1994; Milgram, 2007; Stewart, 1992).

The Nature of Mathematics

In order to understand why mathematics is so misunderstood in school, we must have some of idea of what mathematics is. However, when trying to define what mathematics is, the definitions are as varied as the people giving them. These definitions can range from “the study of patterns” to “what mathematicians do” (Milgram, 2007). However, Keith Devlin (2000) points out that the most common definition is that mathematics is the study (or science) of numbers. But, he adds, since this definition ceased to be accurate thousands of years ago, this only increases the misconception about mathematics. With such a misconception it is not surprising that most non-mathematicians don’t realize that mathematics is a “thriving worldwide activity” (p. 1).

Ian Stewart (1992) expounds on the complication of defining mathematics by explaining that “one of the biggest problems of mathematics is to explain to everyone else what it is all about” (p. 9). The reason, he says, is because when people think about mathematics they only focus on its (a) technicalities, (b) symbolism and formality, (c) terminology, and (d) “apparent delight in lengthy calculations” (p. 9). Mathematics, he explains, isn’t about these things; they are only the tools used to explore mathematics. He claims mathematics is about ideas; about how ideas relate to other ideas.

Many people have offered definitions for mathematics, but, as Milgram (2007) instructs, some things, like mathematics, cannot be defined using ordinary language. The best we can do is describe the subject in general terms, focusing on its most important characteristics.

He suggests that the two most important characteristics of mathematics are its (a) precision and (b) stating and solving well-posed problems.

Mathematical Precision

From definitions to theorems and from operations to algorithms, mathematics relies on precision. For in mathematics, what you see is what you get. There are no hidden assumptions or secret operations; everything is laid bare (Milgram, 2007). It is because of the precision inherent in mathematics that allows theorems to be proven and conclusions to be drawn (Cook, 2002). This is part of the inner beauty of mathematics (Devlin, 2000).

While using semantics to study natural language, Roy Cook (2002) illustrates the benefits of the precision found in mathematics. He gives the example of the adjective 'bald'. Certainly a person without any hair on their head is bald, but what if a person had one hair on their head, would they still be considered bald? At what point would a difference of one hair change the categorization of someone being bald? This lack of precision, or vagueness, can cause difficulties and lead to invalid arguments. However, such vagueness doesn't exist in mathematics. The definitions, theorems, and operations used in mathematics are "precisely defined and understood" (Milgram, 2007, p. 33). Even terms without a formal definition (e.g. set or point) are recognized as such and are limited to as few as possible.

Well-Posed Problems

A second characteristic that sets mathematics apart from other disciplines is stating and solving well-posed problems. Milgram (2007) defines a well-posed problem as a problem "where all the terms are precisely defined and refer to a single universe where mathematics can be done" (p. 33). He goes on to claim that almost all of mathematics is solving problems in well-defined environments. Stewart (1992) adds that problems are the driving force of mathematics. He explains that the solution to a good problem opens up new views and opportunities and are not simply an end in themselves. Thus, the aim of mathematics is to strip away the inessential and penetrate to the core of a problem (Stewart, 1992, p. 10).

The Epistemology of Mathematics

Another characteristic of mathematics that differs from other disciplines is its epistemology. Goldin (1990) defines epistemology as the branch of philosophy that deals with how we know what know. In particular, it is the logical bases for ascribing validity or "truth" to what we know (p.32). In other words, epistemology is "concerned with the nature of knowledge and justification of belief" (Muis, 2004, p. 317). Throughout the development of mathematics education, many different perspectives of how mathematical knowledge is gained have battled for dominance (Ernest, 1985; Goldin, 1990; Muis, 2004; Steffe, 2017). However, as Noddings (1990) points out, mathematicians do not need to define knowledge generally, but rather they need only to describe what mathematical knowledge is and establish the tests that a proposition must pass in order to be added to that body of knowledge.

The test that a proposition must pass in order to enter the body of mathematical knowledge is the mathematical proof. Buldt et al. (2008) explain that while people in all disciplines justify their results, mathematicians are some of a few that claim to prove them. For example, those in scientific disciplines justify their results by the use of inductive reasoning. That is, they establish knowledge by repeated experimentation. If they can obtain the same results by performing the same experiment over and over again, then what they have observed must be true. However, in mathematics, inductive reasoning is used to develop propositions and in order to ascertain knowledge, these proportions must be proven. This method of establishing knowledge uses deductive reasoning to chain together a string of arguments that establish the validity of a proposition's claim. In other words, the epistemology of mathematics relies on both inductive and deductive reasoning.

Mathematics in Schools

With an understanding of the characteristics that set mathematics apart from other disciplines, we can turn our attention to the mathematics being taught in school. This mathematics, according to David Fowler (1994), is not really mathematics, but rather another subject altogether; one he terms "schoolmath." The difference, he explains, is that shoolmath has its own terminology, protocol, and set of beliefs. Schoolmath, for example,

calls the rational numbers between 0 and 1 The Fractions and calls their multiplicative inverses The Improper Fractions. Schoolmath also has its own set of procedures to follow for the Story Problem and the Two-Column Proof. Furthermore, schoolmath believes that there is some sort of unstable equilibrium that exists that requires improper fractions to be reduced to Mixed Numbers or that all professions in the working world solve problems using schoolmath. Milgram (2007) also posits that practitioners of schoolmath view mathematics as lists; lists of memorized formulas, rules, and responses to certain triggers.

Perhaps a more substantial difference is that shoolmath doesn't hold the same defining characteristics that belong to mathematics. Schoolmath is riddled with a lack of precision. Vague or missing definitions leave students with the inability to prove or even understand essential theorems (Edwards & Ward, 2008; Vinner, 2002). Schoolmath also lacks well-posed problems. Poorly posed problems can reinforce misconceptions about mathematics such as mathematics being a list of rules that need to be memorized and followed (Milgram, 2007). Lastly, schoolmath often relies on epistemologies that differ from that of mathematics. All too often when a student asks why a theorem or algorithm works, the reply is "because that's the rule." Such an appeal to authority, rather than using logic and reasoning, reinforces the idea that mathematics is nothing but rules that some mythic mathematician made up for the rest of us to follow.

Lack of Precision

One of many examples of the lack of precision in schoolmath is with the use definitions. Mathematical definitions are fundamental to the axiomatic structure that characterizes mathematics, but either are vaguely given, or are missing entirely (Edwards & Ward, 2008; Milgram, 2007; Vinner, 2002). Edwards and Ward (2008) explain that this can be problematic because the "enculturation of college students into the field of mathematics includes their acceptance and understanding of the role of mathematical definitions" (p.223). Many students, they point out, often do not know or do not know how to use the mathematical definitions they need to in order to perform mathematical tasks. "When faced with a task involving a given concept, rigorous mathematics demands that students base their solutions

on the concept definition" (Edwards & Ward, 2008, p. 225). Cangelosi (2003) adds that it is ideal if the definitions of these concepts are formulated by students after using inductive reasoning to construct the concept.

As an example of how a missing or vague definition can affect a student's concept construction, Milgram (2007) shares the prompt in Figure 1.1 on the next page, which aims to measure a student's construction of the concept of rotation. However, he points out that there are several items that lack precision that could cause some misconceptions. First, the phrase "best shows" is imprecise and relies on the notion of "best," which is undefined. Second, the frequent use of the terms "image" and "pre-image" requires students to have constructed those concepts well enough to interpret their usage in this prompt. Thirdly, and perhaps the most problematic, is the definition used for rotation. Milgram (2007) explains that a rotation in space is achieved when an object is rotated around a fixed axis of rotation. From this point of view, Response A is an example of a rotation where the axis of rotation is the line perpendicular to xy-plane and passes through the origin. Response C also is a rotation where the axis of rotation is the y-axis and Response D also demonstrates a rotation where the axis of rotation is the x-axis. So, which diagram best shows a rotation?

Lack of Well-Posed Problems

The prompt in Figure 1.1 is not only an example of the imprecision that exists in schoolmath, it also demonstrates the common issue of poorly posed problems. As long as a problem includes precisely defined terms that belong to a universe where mathematics can be done, the problem is well-posed (Milgram, 2007). However, a lot of problems in schoolmath tend to make hidden assumptions that require students to guess at what is meant or desired. For example, a common prompt found in most algebra textbooks (including those found in college) is found in Figure 1.2 on the following page.

The designers of the prompt want students to recognize that there are no restrictions on the domain since any real number can be squared, then doubled, and then subtracted from 5. Thus, the desired response is $(-\infty, \infty)$. However, there are some hidden assumptions that are being made regarding the definition of a function. For example, it is assumed that

Figure 1.1

A prompt from an eight-grade state assessment measuring a student's construction of the concept of rotation (Milgram, 2007, p. 45)

4. Which diagram below best shows a rotation of the pre-image to the image?

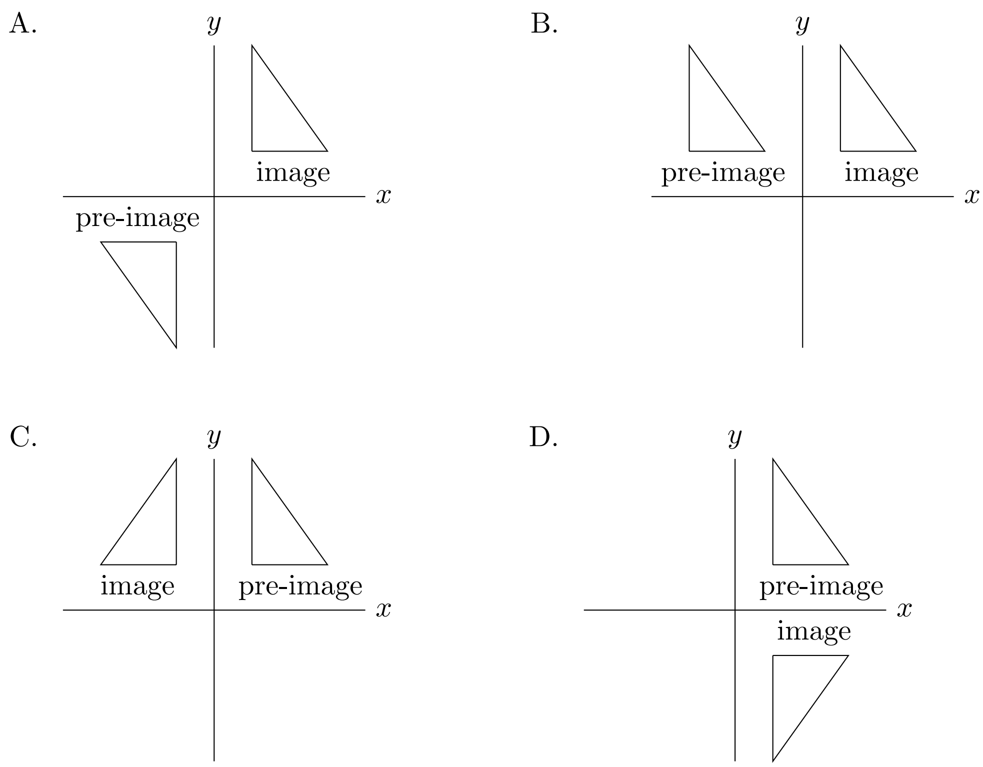


Figure 1.2

A prompt from a college algebra textbook measuring a student's comprehension of the concept of domain of a function (Abramson, 2015).

7. Find the domain of the function

$$f(x) = 5 - 2x^2.$$

the domain of f needs to be the largest subset of the real numbers as possible and that the codomain of f is the set real numbers. These hidden assumptions limit a student's understanding of the concept of function. For, if a student truly comprehends the mathematical definition that a function is a relation between two sets A and B where for every $x \in A$ there exists uniquely a $y \in B$ such that $(x, y) \in f$, they will understand that any subset of the real numbers is a valid domain for the given function. However, if they provided a subset other than $(-\infty, \infty)$, the response would be marked incorrect, even though it meets the definitions of function and domain of a function.

The Epistemology of Religion

This example also illustrates another difference in the defining characteristics between mathematics and schoolmath. Assume, for example, that in response to the prompt in Figure 1.2 a student offers the set $[0, \infty)$ as the domain of f. According the definition of a function, this would be a perfectly valid domain, but according to the key, it would be marked incorrect. In the student's eyes, every element in the set they provided would be a valid input for the given function so, they would certainly wonder why their response was marked incorrect. What are some typical ways an educator might respond to such a concern if voiced by a student? Often, they would refer to the key or explain how we are to "find the domain of a function." In either case, the student usually accepts the educator's response and revises any idea they may have had about what a function is and what its domain might be.

This appeal to authority in order to establish the validity of knowledge or truth works perfectly well in organized religion. After all, isn't faith of any kind a trust or belief in a higher power or authority? However, mathematics isn't based on faith. As discussed previously, mathematics uses an appeal to logic and reasoning in order to establish the validity of a proposition's claim. No mathematician would accept as valid any proof whose only argument was that "so and so said so," so why is it so prevalent in schoolmath? Ask almost any college student why the product of any pair of negative real numbers is positive and the reply is almost certainly, "because that's the rule" or "that's what I was taught."

No logic, no reasoning, only an appeal to some other authority.

In order to transition schoolmath into a subject that better resembles mathematics, a focus on the defining characteristics of mathematics should be attended to. More attention should be given to (a) using precise language, especially in mathematical definitions, (b) using well-posed problems to strengthen student comprehension and communication, and (c) appealing to logic and reasoning as opposed to an appeal to authority. One concept taught in schoolmath that often lacks the characteristic of precision is the concept of function (Carlson, 1995; Oehrtman et al., 2008).

The Concept of Function

The concept of function is not only central to algebra and other undergraduate mathematics courses, but it is foundational to modern mathematics and is also essential to scientific disciplines (Oehrtman et al., 2008). In particular, they explain that if students are wanting to understand calculus, then a strong understanding of the concept of function is needed. However, as Marilyn Carlson (1995) points out, even the more successful students have a narrow view of the concept of function; believing that all functions can be defined by a single algebraic formula. She explains that such a narrow view limits their understanding of the language surrounding functions and prohibits them from representing relationships using functional notation. But why is understanding the concept of function so important to students?

The Concept of Function and Calculus

One reason the concept of function is so vital is because it plays an important role in the study of calculus (Carlson, 1995; Oehrtman et al., 2008; Tall, 1996). David Tall (1996) explains that one of the purposes of the concept of function is to represent how things change (p. 1). This is of particular interest when constructing the concepts of differentiation (rates of change) and integration (cumulative growth). However, he continues to point out that traditionally, calculus courses focus on a mastery of symbolic methods for developing the concepts of differentiation and integration and then apply those methods to solve a range of problems. This focus on symbolic manipulations and procedural techniques, however,

limit students' comprehension of a function as a more general mapping from a set of input values to a set of output values or as a model of a relationships which the function values change continuously as the input values change continuously (Oehrtman et al., 2008).

However, not all approaches to the concept of function are the same in every calculus course (Oehrtman et al., 2008; Tall, 1996). For example, Tall (1996) indicates that calculus courses in some countries focus on an intuitive approach while others focus on a more formal approach. Oehrtman et al. (2008) make the distinction that some calculus texts include a stronger conceptual orientation to learning functions, while others still focus on the more traditional procedural method. No matter the case, if the algebraic and procedural methods were coupled with more of a conceptual method, "students would be better equipped to apply their algebraic techniques appropriately in solving novel problems and tasks" (Oehrtman et al., 2008, p. 28). So, what is it about a procedural approach to the concept of function that makes it difficult for students to construct and comprehend its definition?

Difficulties in Constructing the Concept of Function

Perhaps the difficulties students experience in constructing the concept of function and comprehending its definition stem from the pervasive characteristics carried over from schoolmath. For example, Thompson (1994) explains that the commonly accepted definition of function is the ordered-pair notion that has been around since the 1930's (Figure 1.3 on the next page is an example of such a definition). He remarks that such a definition of function is criticized because some believe that such a definition cannot be meaningful to students who are new to the idea of function. However, without a precise definition that stresses the conceptual aspects of a function, a strong emphasis is placed on procedural fluency. Such an emphasis, according to Oehrtman et al. (2008), is not effective in building strong foundations that allow for meaningful uses and interpretations of functions in novel settings.

Another difficulty that arises due to a lack of using a precise definition is that students struggle with distinguishing between a function that is algebraically defined and an equation (Oehrtman et al., 2008). They state further that this struggle is rooted in the

ambiguity that exists in using terms such as function, equation, and formula. For example, students struggle with distinguishing between the use of the equal sign when it is defining a relationship between two varying quantities and when it is used to establish equality between two expressions. To take this a step further, Thompson (1994), explains that many students think of functions as a set of two written expressions separated by an equal sign. For example, he explains, when a group of undergraduate students were asked to derive and provide a formula for the sum $S_n = 1^2 + 2^2 + \cdots + n^2$, they came up with the poorly formed function $f(x) = \dfrac{n(n+1)(2n+1)}{6}$. However, none of the students thought there was anything wrong with this formulation (p. 6). If students formulate and comprehend precise concept definitions such misunderstandings may not be as prevalent.

The lack of precision is too prevalent in schoolmath and such vagueness leads to ambiguous usage of mathematical terms and operations. As seen with the example of functions, such ambiguity can lead to poor concept constructions that can lead students to struggle with building strong foundations that encourage meaningful engagement with mathematics. To help overcome this obstacle to learning and in order to help remove some of the ambiguity surrounding the concept of function, we will explore the development and implementation of a unit on functions to be given to a class of 50 Calculus I students during the first two weeks of the semester. The aim of this unit is to lead students to construct the concept of function, formulate a precise, set-theoretic definition for the concept where functions are defined as sets of ordered-pairs, and help students develop a deep comprehension of this definition. Additionally, the results and validation study for the assessment designed to

Figure 1.3

A set-theoretic definition of the concept of function.

Definition: Function

Given sets A and B, $f : A \to B \Leftrightarrow f$ is a relation from A to B and $\forall\, x \in A, \exists!\, y \in B$ such that $(x, y) \in f$.

measure student achievement of the set of objectives outlining the unit's learning goal are explored in detail.

CHAPTER 2

Theoretical Implications of Using the Precise Language of Mathematics to Teach the Concept of Function in Calculus

A Constructivist Framework

Kilpatrick (1987) describes constructivism as having two main principles: (a) "knowledge is actively constructed by the cognizing subject, not passively received from the environment" and (b) "coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing work outside the mind of the knower" (p. 7). Lerman (1989) explains that the first principle is generally accepted by mathematics educators and is useful when thinking about how students learn mathematics. However, the second principle is more controversial and is usually where opponents of the theory base their arguments.

Constructivism, however, should be considered a paradigm of how the mind works and not necessarily as a theory of learning (Tsay & Hauk, 2013). Noddings (1990) explains that from a cognitive position constructivism posits that all knowledge is constructed from either innate cognitive structures or structures which are products of developmental construction. He adds, however, that such a position should be offered as a post-epistemological perspective. From this point of view, a constructivist approach should focus on the development of knowledge in the learner and not on the epistemological debates that frequently surround the theory.

Concept Images and Concept Definitions

This particular point of view is apparent in the development of the theoretical framework outlined by Tall and Vinner (2004). They explain that due to the complex nature of the human brain, it is not always logic that leads to insight and knowledge, nor is it always chance that leads us to make mistakes. In order to understand these processes, it is important that a distinction be made between the formally defined mathematical concepts being

studied and the cognitive processes by which they are developed (Tall & Vinner, 2004). This distinction is made by identifying the differences between what they call a *concept image* and a *concept definition*.

Concept Image

Many mathematical concepts that students are exposed to are never formally defined, but rather are recognized by experience in using them in appropriate contexts (Tall & Vinner, 2004). For example, Carlson (1995) points out that many students complete precalculus with a weak understanding of the concept of function. They are able to carry out basic computations involving a function, but have difficulty computing slightly more advanced expressions such as $f(x+a)$ and struggle with components such as inverses and composition of functions. However, even with this limited understanding of functions, these students still have constructed an image of what they perceive a function to be. In this way, a concept image is built up over time and is "the total cognitive structure that is associated with the concept [and] includes all the mental pictures and associated properties and processes" (Tall & Vinner, 2004, p. 99).

Concept Definition

On the other hand, a concept definition is one that defines a concept and is expressed in the form of words or mathematical symbols (Tall & Vinner, 2004). Tall and Vinner further explain that these definitions can be learned through rote memorization or in a more meaning full way and then related to the concept as a whole. Whether the concept definition is given to the student to memorize or whether they formulate it themselves, these perceived definitions can change from time to time and from one individual to another (Nordlander & Nordlander, 2012). Because of this, *personal* concept definitions can vary from the *formal* mathematical definitions established by the mathematical community and the concept definition generates its own concept image in the individual that can sometimes conflict with its previously constructed concept image (Tall & Vinner, 2004). For example, a student may have a concept image of functions that only involve subsets of real numbers. However, when they are introduced to a formal set-theoretic definition of a function, like

the one included in Figure 2.1 on the following page, they may have difficulty recognizing functions which involve arbitrary sets that are not subsets of real numbers. This newly constructed concept image, based on the formal definition of functions involving arbitrary sets, is then at odds with their previous concept image of functions.

Tall and Vinner (2004) call the part of the concept image (or concept definition) which conflicts with another part of the concept image (or concept definition) a *potential conflict factor* and, under certain conditions, can become a *cognitive conflict factor*. When a potential conflict factor in a concept image is at variance with the formal concept definition itself, like in the example given above, it can impede a students learning of a formal theory (Tall & Vinner, 2004). This is because, they explain, these potential conflict factors cannot become meaningful cognitive conflict factors unless the student develops the formal concept definition into a concept image which can yield an actual cognitive conflict. The danger that exists with students developing such potential conflict factors without having them become actual cognitive conflict factors is that they may become “secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous” (Tall & Vinner, 2004, p. 101). Thus, Nordlander and Nordlander (2012) explain, during the construction of the concept, the relationship between the concept image and the formal concept definition should be reciprocal and mutual. This can be done, according to Cangelosi (2003), by using inductive reasoning to lead students to construct the concept and then lead them to develop and formulate a formal mathematical definition.

APOS Theory

Based on the work by Sfard (1992), with the hypothesis that mathematical knowledge consists of an individual’s tendency to deal with mathematical situations by constructing mental *actions*, *processes*, and *objects* and then organizing them into *schemas*, Dubinsky and McDonald (2001) introduce what they term *APOS Theory*. The theory is so named in reference to the mental constructs previously mentioned and arises from their attempts to extend the work of Jean Piaget on reflective abstraction in children’s learning to the level of learning mathematics on a collegiate level. Research has shown that this theory has been

Figure 2.1

Set-theoretic definitions of the Cartesian product of two sets, a relation on two sets, and the concept of function.

Definition: Cartesian Product

Given sets A and B, $A \times B = \{(x, y) : x \in A \wedge y \in B\}$.

Definition: Relation

Given sets A and B, r is a relation from A to B if and only if $r \subseteq A \times B$.

Definition: Function

Given sets A and B, $f : A \to B \Leftrightarrow f$ is a relation from A to B and $\forall\, x \in A, \exists!\, y \in B$ such that $(x, y) \in f$.

useful in understanding undergraduate students' learning of a variety of mathematical concepts, including the concept of function (Breidenbach et al., 1992; Dubinsky & McDonald, 2001; Tall, 1996; Thompson, 1994). Table 2.1 on the next page summarizes each of these constructs, but a more detailed explanation follows.

An Action Conception

Dubinsky and McDonald (2001) define an action as a transformation of external content perceived by an individual as needing step-by-step instructions on how to perform an indicated operation. Dubinsky and Harel (1992) clarify that an action is a "repeatable mental or physical manipulation of objects" (p. 85). Thus, an individual operating with an action conception thinks only of the calculations that need to be performed (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Thompson, 1994).

As an example, Thompson (1994) explains that when students view a function as an expression that produces a result from a calculation, they have an action conception of function. He adds that such a conception limits an individuals view of a function to something little more than a recipe that requires a number before it will produce anything. This view of the concept of function is static, explains Dubinsky and Harel (1992), because the individual will tend to think about a function as a calculation that is performed one step at a time. They add that such a view may cause students to struggle when needing to compose two functions if the functions are given by different expressions on different parts of their domains or if they were defined by algorithms rather than expressions.

A Process Conception

When an individual can perform an action in their mind or imagine an action taking place without needing to run through all of the steps, the action has been interiorized to become a process (Breidenbach et al., 1992). When an individual can think about performing a process without actually doing it, they can begin to think about reversing it and composing it with other processes (Dubinsky & McDonald, 2001, p. 3). For example, when students build a process conception of function, they don't feel compelled to evaluate an expression in order to think about the results of its evaluation (Thompson, 1994).

Table 2.1

Dubinsky and McDonald's (2001) APOS Theory references action, process, object, and schema mental constructs in order to extend Jean Piaget's work on reflective abstraction to collegiate mathematics.

Mental Construct	**Summary**
Action	An action is a transformation of external mathematical content requiring step-by-step instructions on how to perform the given operation.
Process	A process is an internal mental construct where an individual can perform an action, but without the need of external stimuli.
Object	An object is a mental construct resulting from an individual becoming aware of a process as a totality and realizing that operations (such as transformations) can act on it.
Schema	A schema is an individual's collection of actions, processes, objects, and other schema that are linked by some characteristics and that forms a framework in the individuals mind about a concept.

Breidenbach et al. (1992) further explain that further classification of functions into injective and/or surjective becomes more accessible when an individual's process conception is strengthened. Table 2.2 on the following page are other examples of action and process conceptions of function proffered by Oehrtman et al. (2008).

Helping students develop a process conception of function has been the focus of several research studies (e.g., Breidenbach et al., 1992; Dubinsky and Harel, 1992; Oehrtman et al., 2008; Tall, 1996; Thompson, 1994). The ability for students to construct processes in their minds is an important requirement for students to understand the concept of function (Breidenbach et al., 1992). However, as Dubinsky and Harel (1992) point out, the process of moving from an action conception to a process conception is not a linear progression and doesn't always move in a single direction. Thus, it is difficult to determine if the concept of function for a given individual is limited to an action conception or if they have constructed a process conception.

An Object Conception

Dubinsky and McDonald (2001) explain that when an individual treats a process as a totality and realizes that they can apply transformations to it, the process has become an object. When a process is transformed into an object, it is said that the process has been encapsulated (Breidenbach et al., 1992). Breidenbach et al. (1992) explain that even though there are several ways an individual can construct a process, the only way an individual can construct an object is by encapsulating a process. This is important, they add, because in many mathematical situations it is essential to be able to transition back and forth from an object to a process.

By way of an example, Thompson (1994) explains that once an individual has solidified a process conception of function in such a way that they are able to use the process to support their reasoning about, they have began to encapsulate the process and are able to begin to reason about functions as if they were objects. Dubinsky and Harel (1992) adds that an individual has an object conception of function if they are able to perform actions on a function, particularly those kind of actions that transform it. One hallmark obtained

Table 2.2

Action conceptions of function differ from process conceptions of function in a variety of ways (Oehrtman et al., 2008, p. 34).

Action Conception	**Process Conception**
A function is tied to a specific rule, formula, or computation and requires the completion of specific computations and/or steps	A function is a generalized input-output process that defines a mapping of a set of input values to a set of output values.
A student must perform each action.	A student can imagine the entire process without having to perform each action.
The response depends on the formula.	The process is independent f the formula.
A student can only imagine a single value at a time as input or output.	A student can imagine all inputs at once or imagine evaluating a continuum of inputs. A function is a transformation of entire sets.
Composition is substituting a formula or expression for the variable x.	Composition is a coordination of two input-output processes; input is processed by one function and then its output is processed by a second function.
Inverse is about algebra (switching the variables x and y and then solving for x) or geometry (reflecting across the line $y = x$).	Inverse is the reversal of a process that defines a mapping from a set of output values to a set of input values.
Functions are conceived as static.	Functions are conceived as dynamic.
A function's graph is a geometric figure.	A function's graph defines a specific mapping of a set of input values to a set of output values.

when an individual has constructed an object conception of function is their ability to reason about operations begin performed on a set of functions (Thompson, 1994).

A Schema Conception

An individual's collection of actions, processes, objects, and other schema linked by certain characteristics that frame a concept is a schema for that concept (Dubinsky & McDonald, 2001). They explain that a schema is similar to the collection of concept images outlined by Tall and Vinner (2004), but that a schema has a requirement of coherence. That is, the schema must provide means of determining which constructs are in the scope of the schema and which are not.

Dubinsky and McDonald (2001) posit that it may be helpful to think about these constructs in an ordered, hierarchical list. That is, an individual must construct an action conception of a concept prior to constructing a process conception and this must be done before an object conception be constructed. In reality, however, when constructing these mental constructs the constructions are not made in such a linear manner. As an example, they explain that with an action conception of function, an individual may relate functions to formulas or expressions that involve at least one variable that can be replaced by a number and then calculated. Since many student's have this concept image of function prior to constructing a process conception, we think of this notion as preceding a process conception, in which a function is treated as machine that takes inputs and returns outputs. However, what is actually happening is that the individual is being restricted to specific formulas, reflecting on calculations, and then starts to think about a process. This is then repeated with more sophisticated formulas (Dubinsky & McDonald, 2001).

Ideally, students would construct a schema conception of function that would include a variety of actions, processes, objects, and other schema. However, such a deep and varied conception of function may not be necessary for students to gain a meaningful understanding of the main conceptual strands of calculus (Breidenbach et al., 1992; Oehrtman et al., 2008). But, Oehrtman et al. (2008) explain that an action conception of function will not be enough to lay a foundation strong enough to understand the essential concepts of calculus; a process

conception is crucial to this endeavor.

Using Precise Language to Teach the Concept of Function in Calculus

Methods from asking specific types of questions (Oehrtman et al., 2008) to the use of technology (Breidenbach et al., 1992; Tall, 1996) have been recommended to foster the development of a process conception among calculus students. But, as was mentioned, the presence and strength of an individual's process conception is difficult to measure. However, Dubinsky and Harel (1992) point out that among a variety of situations, the point of view of a function as a set of ordered pairs is a good indicator for determining the level of an individual's process conception of function. Such a point of view can only be meaningful to students if their constructed concept images of function are in harmony with a formal, precisely stated concept definition that defines a function as a set of ordered pairs. Indeed, if this formal concept definition is at odds with a student's concept image of function, then, as Dubinsky and Harel (1992) mention, the idea of a function as set of ordered pairs may present a number of difficulties for students.

Lead Students to Construct the Concept of Function

In order to reconcile their existing concept image of function to a formal, set-theoretic concept definition (like the one included in Figure 2.1 on page 16) student's need to construct a concept image that resembles the formal definition. For example, if students are to comprehend a definition of function where it is defined as a set of points, students need to first think about a function as a set of points. Cangelosi (2003) explains that without repairing these conceptual gaps, many students are at risk of failing to develop healthy attitudes, algorithmic skills, comprehension and communication skills, and application-level abilities. That is why students should be lead to construct the concept of function with the precise, set-theoretic definition as the guiding aim.

Many calculus students have already constructed a concept image of function, but the image generally contains only action conceptions of the concept (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Oehrtman et al., 2008; Tall, 1996; Thompson, 1994). However, even with this limited view of the concept of function, students can be lead to use inductive

reasoning to distinguish between sets that are examples of functions and sets that are not, allowing them to modify that construction to contain images of a function as a set of points. Choosing appropriate examples and non-examples is vital as they need to be precise and emphasize the defining characteristics of a function. For example, if students are to think about a function as a set of ordered pairs, then some of the examples should be a function represented as a set of ordered pairs. Allowing students to reason inductively in order to identify the characteristics that define a function creates a concept image that provides a basis for subsequent meaningful learning (Cangelosi, 2003).

Lead Students to Develop a Formal Concept Definition of Function

Once students have categorized the examples and non-examples of function and identified the defining characteristics of the concept, students are ready to formulate the definition of function for themselves. Cangelosi (2003) explains that this is done as students formulate a definition of the concept in terms of its defining characteristics, verify their definition by testing it with additional examples and non-examples, refine their definitions as dictated by the outcomes of the tests, and revisit any of the prior stages as necessary. Leading students to develop their own precise definition of function, which with guidance should resemble the formal concept definition, helps them construct a concept image that already includes the formal definition, thereby limiting the dissonance that students sometimes feel when such formal definitions are introduced. The development of a unit for leading calculus students to construct the concept of function and formulate a formal definition is detailed in Chapter 3 on page 26.

Fostering a Process Conception of Function

With a more meaningful concept image of function that includes the formal, precise, mathematical concept definition, students should be better prepared to construct a process conception of function. Consider, for example, two recommendations offered by Oehrtman et al. (2008) that aim to promote students' development of the process conception. They indicate that students should be asked (a) to explain basic function facts in terms of input and output and (b) to make and compare judgements about functions across multiple

representations (pp. 34-35).

Explain Basic Function Facts

The first recommendation proffered by Oehrtman et al. (2008) is to ask students to explain basic function facts in terms of input and output. For example, they assert that students should be asked to determine the domain and range of a function based on the context given in the problem and relate this to responses derived algebraically. When students comprehend the formal definition of function that defines a function as a set of ordered pairs, their understanding of the notation and comprehension of the associated terminology is also enhanced (see Chapter 3). For example, students that recognize the notation $f : A \rightarrow B$ and comprehend the associated definition, are not only able to recognize the set A as the domain and the set B as the codomain of the function f, but are also able choose an appropriate set A and an appropriate set B given a specific condition. This approach of generalizing the set of input and output values is an indication of a process conception (Oehrtman et al., 2008).

Make and Compare Judgements About Functions Across Representations

Another recommendation given by Oehrtman et al. (2008) is to ask students to make and compare judgements about functions across multiple representations. These questions, they explain, should vary in their algebraic representations so that the independence from representations by formula, graph, and table is reinforced. After students make such determinations, they should compare results, justify their conclusions, and explain why the representations are the same. If a student views a function as a set of ordered pairs, the movement from one representation to another is a matter of recognizing an ordered pair as a point on a Cartesian coordinate system or as a paired entry in a table. The ability to move between representations and understand why the representations are the same is another indication that an individual has moved to a process conception of function (Oehrtman et al., 2008).

Conclusion

Leading students to construct the concept of function, formulate their own precise

definition, and then comprehend that definition can help students build concept images that include the formal concept definition that is accepted by the mathematical community at large. Furthermore, when students comprehend a set-theoretic definition of function that define functions as sets of ordered pairs, they are able to enhance the construction of a process conception of function, establishing a meaningful foundation for the introduction of the essential concepts contained in calculus. This is because comprehending such a definition helps students make sense of the additional terminology and definitions associated with the concept of function and helps them recognize and make judgments across various representations of a function.

CHAPTER 3

Leading College Students to Construct the Concept of Function: A Unit Development with a Focus on Precise Language

A central concept to undergraduate mathematics and one essential in the study of calculus is the concept of function. However, as research suggests, students often struggle with building meaningful foundations of the concept of function (Carlson, 1995; Oehrtman et al., 2008; Tall, 1996). One of the reasons students struggle with understanding a concept is because the concept images that they have constructed is at odds with the formal concept definition established by the mathematics community at large (Tall & Vinner, 2004). Chapter 2 suggests this discordance can be overcome by leading students to construct the concept of function and lead them to formulate a precise, mathematical definition that is in agreement with the formal mathematical definition.

With this aim in mind, a unit was developed to lead students to (a) construct the concept of function; (b) formulate a precise mathematical definition using the language of sets; (c) comprehend this formal definition; (d) comprehend the definition of and distinguish between a function's domain, codomain, and range; and (e) comprehend the definition of and distinguish among injective, surjective, and bijective functions. This unit was taught, along with a unit on set theory, as an introduction/review section to a class of 50 calculus students during the first two weeks of the semester. Following is an exposition of the development, implementation and results of the unit along with some reflections about the implementation process.

Developing Goals and Establishing Objectives for a Unit on Functions

Cangelosi (2003) instructs that a teaching unit should consist of four elements: (a) a learning goal, (b) a set of objectives that define the learning goal, (c) a string of planned learning activities that are designed to help students achieve the the objectives, and (d) a means to make a summative evaluation of student achievement of the learning goal (p.

164). While Table 3.1 on the next page summarizes each component of a teaching unit, (a) - (c) are discussed in detail below while (d) is considered extensively in Chapter 4.

Developing a Learning Goal

A unit's learning goal details the overarching purpose of the teaching unit and should indicate what students are expected to gain if the unit is successful (Cangelosi, 2003, p. 164). Thus, in developing a learning goal for a unit on functions, it is necessary to delineate the content students are expected to learn and how they are to interact with it. In preparation for a course on calculus, it was decided that students should construct the concept of function; formulate a set-theoretic definition which defines a function as a set of ordered pairs; comprehend the definitions for a function's domain, codomain, and range; and distinguish between injective, surjective, and bijective functions. The resulting learning goal for this unit on functions is displayed in Figure 3.1.

Establishing Objectives

A learning goal identifies the overall student outcomes, but is not detailed enough to lead students from where they are to where they should be upon the completion of the unit. The achievement of a learning goal requires students to acquire a number of specific skills, abilities, and attitudes that are outlined in a set of specific objectives (Cangelosi, 2003). To be effective, objectives should specify the mathematical content to be covered and the level at which students should mentally interact with that content (referred to as the learning

Figure 3.1

The learning goal established for a unit on functions.

Learning Goal for a Unit on Functions

The goal for this unit on functions is that students will (a) understand and interpret relations between sets, (b) understand functions as relations between sets, (c) distinguish between relations that are functions and those that are not, (d) understand the meaning of and identify a function's domain, codomain, and range, and (e) distinguish among functions that are injective, surjective, and bijective.

Table 3.1

A teaching unit is comprised of a learning goal, a set of objectives that define the learning goal, a string of planned activities designed to help students achieve the objectives, and a means to make a summative evaluation of student achievement of the learning goal (Cangelosi, 2003).

Component	**Description**
Learning Goal	The learning goal is the overall purpose of the teaching unit and should indicate what students are expected to gain.
Objectives	The learning goal is defined by a set of specific objectives that indicate the particular skills, abilities, or attitudes that make up the learning goal. Each objective should specify the mathematical content to be learned and the level at which students are expected to mentally interact with that content.
Learning Activities	Learning activities are the string of lessons designed and conducted for the purpose of helping students achieve the objectives.
Summative Evaluation	Summative evaluations are judgements that are made regarding students' achievement of the learning goal.

level of the objective). Cangelosi (2003) explains that this is because both an objective's mathematical content and learning level will influence how the content will be taught.

Mathematical Content

Objectives should clearly specify the mathematical content so that it is clear what mathematical topics students are expected to learn throughout the unit's planned lesson activities. These lessons are designed differently depending on the content. For example, a lesson designed to teach a concept will be different than one designed to teach an algorithm. Thus, as Cangelosi (2003) points out, before designing a lesson for a particular objective, the content needs to be identified as (a) a concept, (b) a discoverable relationship, (c) a relationship of convention, or (d) an algorithm.

Concepts. According to Cangelosi (2003) a *concept* is a mentally constructed category that organizes unique entities together based on a set of common characteristics. For example, consider the concept of relation. By definition, a relation between two sets A and B is a set of ordered pairs that is a subset of the Cartesian product of A and B. That is r is a relation between A and B if and only if $r \subseteq A \times B$. Thus, if a set is a set of ordered pairs where the first element of each ordered pair belong to the same set and the second element of each ordered pair also belong to the same set, then the set is considered a relation. The concepts covered in the unit on functions are: relation, function, domain, codomain, range, injective function, surjective function, and bijective function.

Discoverable Relationships. A *relationship* is an association between (a) concepts, (b) a concept and a specific, (c) a specific and a concept, or (d) specifics (Cangelosi, 2003). For example, the relationship $\mathbb{Q} \subset \mathbb{R}$ is an association between concepts, $2x \geq 0\ \forall x \in \mathbb{N}$ is an association between a concept and a specific, the relationship $25 \in \{\text{composite numbers}\}$ is an association between a specific and a concept, and $2\pi > 6$ is an example of an association between specifics. If inductive reasoning can be used to discover the association, then the relationship is *discoverable*. For example, students can use examples and reasoning to find that for any triangle, the degree measures of its interior angles sum to 180. The unit on functions doesn't cover any discoverable relationships, but it does include relationships of

convention.

Relationships of Convention. Not all mathematical relationships are discoverable. There are some relationships that have been established through years of tradition or simply by agreement. Cangelosi (2003) defines these relationships as *relationships of convention*. Most notation used in mathematics are relationships of convention. For example, the notation "$f : A \to B$" is read "f is a function from the set A to the set B." There is no amount of examples or reasoning that can derive this interpretation. It has been established by convention. Other relationships of convention discussed in the unit on functions are $f : A \xrightarrow{\text{1-1}} B$, $f : A \xrightarrow{\text{onto}} B$, and $f : A \xrightarrow[\text{onto}]{\text{1-1}} B$.

Algorithms. Ian Stewart (1992), in his book *The Problems of Mathematics*, explains that most people, when thinking about mathematics, think about its "delight in lengthy calculations." These calculations are usually the result of using a mathematical algorithm. An algorithm is defined as a multistep procedure, based on a relationship, for obtaining a result (Cangelosi, 2003, p. 216). Most algorithms used in mathematics can be classified as (a) an arithmetic computation, (b) a reformulation of a symbolic expression, (c) a translation of a statement of relationship, or (d) a measurement. For example, using the distributive property of multiplication to rewrite the polynomial $(x+2)(x-3)$ as $x^2 - x - 6$ is a reformulation of a symbolic expression. Although there are many algorithms associated with the concept of function, there are no algorithms covered in this unit on functions.

Learning Levels

Once the content has been classified, the next step is to identify the learning level of the objective. Cangelosi (2003) defines an objective's learning level as "the manner in which students will mentally interact with the objective's mathematical content once the objective is achieved" (p. 166). There are several published schemes for classifying objectives according to their intended learning levels, but Cangelosi (2003) developed the scheme outlined in Table 3.2 on page 32 for teaching mathematics in accordance with the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000). Kohler and Alibegović (2015) explain that the seven cognitive learning

levels in this scheme describe the thinking that is often required in learning mathematics. Of these seven cognitive levels, only three pertain to the objectives that outline the learning goal for the unit on functions being developed: construct a concept, comprehension and communication, and simple knowledge.

Constructing Concepts. Students achieve an objective at the construct-a-concept learning level when they can use inductive reasoning to distinguish and categorize examples and non-examples of that concept (Cangelosi, 2003; Kohler & Alibegović, 2015). Ideally, students should be lead to construct all concepts, but due to time constraints this isn't always a possibility. As such, there are two concepts in the unit on functions that students should be lead to construct: the concept of relation and the concept of function. The culminating experience of constructing a concept should be the formulation of a definition for the concept. Thus, the construct-a-concept objectives outlined in Figure 3.2 mention the development of such a definition.

Comprehending and Communicating. Once students have formulated or been introduced to a mathematical definition, the next step in their learning process is to comprehend that definition and use it to communicate with and about mathematics. When students can (a) extract and interpret meaning from an expression, (b) use the language of mathematics, and (c) communicate with and about mathematics, they have achieved an objective at the comprehension-and-communication learning level (Cangelosi, 2003). While

Figure 3.2

The two construct-a-concept objectives that make up part of the learning goal for a unit on functions.

Construct-a-Concept Objectives for a Unit on Functions

- Students will distinguish from sets that are relations between two sets (or between a set and itself) and sets that are not and develop a definition.
- Students will distinguish between relations that are functions and those that are not and develop a definition.

Table 3.2

Cangelosi's (2003) scheme for categorizing learning levels specified by objectives.

Learning Level	**Explanation**
Cognitive Domain	
Construct a Concept	Students achieve an objective at the construct-a-concept learning level by using inductive reasoning to distinguish examples of a particular concept from non-examples of that concept.
Discover a Relationship	Students achieve an objective at the discover-a-relationship learning level by using inductive reasoning to discover that a particular relationship exists or why the relationship exists.
Comprehension and Communication	Students achieve an objective at the comprehension-and-communication level by (a) extracting and interpreting meaning from an expression, (b) using the language of mathematics, and (c) communicating with and about mathematics.
Simple Knowledge	Students achieve an objective at the simple-knowledge learning level by remembering a specified response (but not multiple-step process) to a specified stimulus.
Algorithmic Skill	Students achieve an objective at the algorithmic-skill level by remembering and executing a string of steps in a specific procedure.
Application	Students achieve an objective at the application level by using deductive reasoning to decide how to utilize, if at all, a particular mathematical content to solve problems.
Creative Thinking	Students achieve an objective at the creative-thinking learning level by using divergent reasoning to view mathematical content from unusual and novel ways.

Table 3.2 (Cont'd)

Learning Level	**Explanation**
Affective Domain	
Appreciation	Students achieve an objective at the appreciation learning level by believing the mathematical content specified in the objective has value.
Willingness to Try	Students achieve an objective at the willingness-to-try learning level by choosing to attempt a mathematical task specified by the objective.

engaging with the unit on functions, students will construct the definitions for relation and function, but will also be introduced to the concepts of domain, codomain, range, injective function, surjective function, and bijective function. They will then be expected to comprehend the definitions for these concepts and use them to communicate mathematically. Thus, comprehension-and-communication objectives should be developed that cover each of these concepts. Objectives at the comprehension-and-communication learning level for the unit on functions are delineated in Figure 3.3 on the next page.

Acquiring Knowledge. Not all mathematical content needs to be constructed or discovered; there is a practical need for students to remember conventional names for concepts and statements of relationships (Cangelosi, 2003). In other words, there is some mathematical content that students should remember and be able to provide given a certain prompt. Cangelosi (2003) explains that when students remember a specified response (but not a multistep process) to a specified stimulus, they have achieved an objective at the simple-knowledge learning level. One of the things that should be remembered is the conventional notation that is used throughout a given unit. Thus, Figure 3.4 on page 36 outlines the simple-knowledge objectives for the unit on functions.

Developing Learning Activities

The foremost component of a teaching unit is the string of activities that are designed and conducted in order to help students achieve the defining objectives for the learning goal (Cangelosi, 2003). The order of these activities are not fixed, however, as Cangelosi (2003) and Kohler and Alibegović (2015) point out, it is often beneficial that conventional names for concepts or relationships be introduced after the concept has been constructed or the relationship has been discovered. With this mind, the objectives that define the learning goal for the unit on functions have been placed in the order to be taught and subsequently numbered. Table 3.3 on page 37 shows this ordering.

Leading Students to Construct Concepts

In order to lead students to construct a concept they need to be given the opportunity to use inductive reasoning to distinguish and categorize examples from non-examples of

Figure 3.3

The six comprehension-and-communication objectives that make up part of the learning goal for a unit on functions.

Comprehension-and-Communication Objectives for a Unit on Functions

- Students will explain that a relation is a subset of the cross product of two sets (or the cross product of a set with itself).
- Students will explain that a function is a relation from a set A to a set B where every element in A is mapped to one and only one element in B.
- Given the function $f : A \to B$, students will explain that A is the domain of the function, B is the codomain of the function, and the range of the function is the subset of the codomain which contains all elements in B which are mapped to by the elements of A.
- Students will explain that an injective function is a function where every element in the range of the function is mapped to by one and only one element in the domain.
- Students will explain that a surjective function is a function where every element in the codomain is mapped to by an element in the range, in other words, the range is equal to the codomain.
- Students will explain that a bijective function is both injective and surjective, that is, one-to-one and onto.

Figure 3.4

The four simple-knowledge objectives that make up part of the learning goal for a unit on functions.

Simple-Knowledge Objectives for a Unit on Functions

- Students will associate the notation “$f : A \to B$” with the expression “f is a function from A to B.”
- Students will associate the notation “$f : A \xrightarrow{\text{1-1}} B$” with the expression “$f$ is an injective (one-to-one) function from A to B.”
- Students will associate the notation “$f : A \xrightarrow{\text{onto}} B$” with the expression “$f$ is a surjective (onto) function from A to B.”
- Students will associate the notation “$f : A \xrightarrow[\text{onto}]{\text{1-1}} B$” with the expression “$f$ is a bijective (one-to-one and onto) function from A to B.”

Table 3.3

A set of twelve objectives that define a learning goal for a unit on functions, listed in the order to be covered during instruction.

	Objective	**Learning Level**
2.A.	Students will distinguish from sets that are relations between two sets (or between a set and itself) and sets that are not and develop a definition.	Construct a Concept
2.B.	Students will explain that a relation is a subset of the cross product of two sets (or the cross product of a set with itself).	Comprehension and Communication
2.C.	Students will distinguish between relations that are functions and those that are not and develop a definition.	Construct a Concept
2.D.	Students will explain that a function is a relation from a set A to a set B where every element in A is mapped to one and only one element in B.	Comprehension and Communication
2.E.	Students will associate the notation "$f : A \to B$" with the expression "f is a function from A to B."	Simple Knowledge
2.F.	Given the function $f : A \to B$, students will explain that A is the domain of the function, B is the codomain of the function, and the range of the function is the subset of the codomain which contains all elements in B which are mapped to by the elements of A.	Comprehension and Communication
2.G.	Students will explain that an injective function is a function where every element in the range of the function is mapped to by one and only one element in the domain.	Comprehension and Communication
2.H.	Students will associate the notation "$f : A \xrightarrow{1\text{-}1} B$" with the expression "$f$ is an injective (one-to-one) function from A to B."	Simple Knowledge

Table 3.3 (Cont'd)

	Objective	**Learning Level**
2.I.	Students will explain that a surjective function is a function where every element in the codomain is mapped to by an element in the range, in other words, the range is equal to the codomain.	Comprehension and Communication
2.J.	Students will associate the notation "$f : A \xrightarrow{\text{onto}} B$" with the expression "$f$ is a surjective (onto) function from A to B."	Simple Knowledge
2.K.	Students will explain that a bijective function is both injective and surjective, that is, one-to-one and onto.	Comprehension and Communication
2.L.	Students will associate the notation "$f : A \xrightarrow[\text{onto}]{\text{1-1}} B$" with the expression "$f$ is a bijective (one-to-one and onto) function from A to B."	Simple Knowledge

the concept, determine defining characteristics, generalize their observations, and formulate their own definition (Cangelosi, 2003). This is done by leading students through the four stages of a construct-a-concept lesson. Cangelosi (2003) instructs that the four stages of a construct-a-concept lesson include (a) sorting and categorizing, (b) reflecting and explaining, (c) generalizing and articulating, and (d) verifying and refining. Movement from one stage to another may not always be linear as it may be necessary to revisit prior stages throughout the construction of the concept. This relationship is depicted in Figure 3.5 on the following page. To see how these stages are used to develop a lesson, consider the concepts of relation and function.

The Concept of Relation

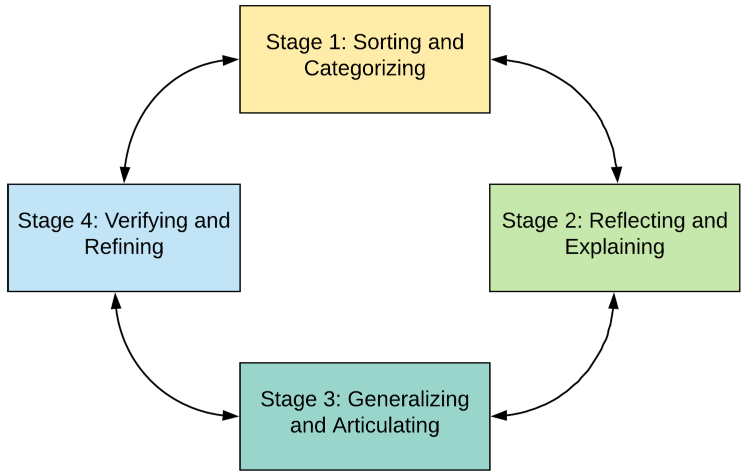
One of the aims of this unit is to lead students to construct the concept of function where a function is defined as set of ordered pairs. In particular, students who achieve the learning goal of this unit will view a function as a relation between sets that meets the uniqueness criteria inherent in a function. Thus, students will first need to construct the concept of relation.

During the first stage of a construct-a-concept lesson, students are given the opportunity to sort and categorize specific examples and non-examples of elements that belong to the concept they are constructing. Selecting appropriate examples and non-examples for students can be challenging as they should provide students with enough information so they can identify the defining characteristics of the concept. For example, the three previously identified characteristics for defining a relation are: (a) it needs to be a set of ordered pairs, (b) the first element of each ordered pair needs to belong to the same set, and (c) the second element of each ordered pair needs to belong to the same set. Thus, examples of a relation should clearly demonstrate these characteristics and the non-examples should be missing at least one of these characteristics. Furthermore, there should be at least one non-example demonstrating a lack of each of the identified characteristics.

For example, students are presented with Table 3.4 on page 41 and are asked to describe how the examples are similar to each other and how they differ from the non-examples.

Figure 3.5

The four stages of a construct-a-concept lesson as outlined by Cangelosi (2003).



Each set in the collection of examples demonstrates the essential attributes of a relation between sets while the non-examples are missing at least one of the characteristics. As an illustration, the first set in the non-examples contains the element (green, Jeep), however, Jeep is not an element of the set B defined in the given conditions and the last set in the non-examples is not a set of ordered pairs. This activity leads students to explore the different characteristics of the given sets and identify those characteristics which define a set as a relation from a set A to a set B.

Once students have identified the defining characteristics of a relation, they are asked to share their thoughts and conjectures with a partner or small group and compare what they discovered with others. This process of reflecting and explaining takes students through the second stage of the lesson. After the pairs or small groups have discussed their conjectures with one another, a class discussion ensues where students are asked to share what they have discovered about relations. During this third stage of the lesson, students are asked to create a list of the attributes and characteristics of a relation, decide which ones are essential, and start to generalize what they have observed in order to start forming a definition. Throughout this process, students should test the definitions they are formulating and refine them as necessary. The goal of this discussion is to lead students to identify the necessary

Table 3.4

Examples and non-examples of sets that are relations from the set A to the set B.

Given Conditions	
$A = \{$green, blue, yellow, red, orange, black$\}$ $B = \{$Honda, Chevrolet, Ford, Porsche, Toyota$\}$	
Examples	**Non-Examples**
{(red, Honda), (red, Ford), (red, Porsche)}	{(green, Ford), (green, Jeep), (green, Honda)}
{(blue, Ford), (red, Ford), (black, Ford)}	{(orange, Toyota), (blue, Toyota), (gray, Toyota)}
{(blue, Porsche), (red, Honda), (black, Chevrolet)}	{(Chevrolet, black), (Porsche, blue), (Honda, red)}
{(yellow, Porsche)}	{yellow, Porsche}

conditions that define a relation and then use them to formulate a precise, mathematically correct definition. For example, students should be lead to formulate a definition for a relation that is similar to the one found in Figure 3.6.

The Concept of Function

Since college algebra and trigonometry (or the equivalent) is required for a first-year calculus course, most students who are enrolled have been exposed to and have experience with the concept of function and have already constructed their own concept image. With

Figure 3.6

Precise, mathematical definition of a relation on two sets.

Definition: Relation

Given sets A and B, r is a relation from A to B if and only if $r \subseteq A \times B$.

the aim of helping students modify their existing concept image, a slightly different method is used to lead students to construct the concept of function. The four stages are still incorporated, but instead of providing examples and non-examples for students to categorize, students are asked to provide the examples and non-examples themselves. Students should be encouraged to include different representations as part of their examples and non-examples and it may be necessary to include additional examples and non-examples where functions are represented as a set of ordered pairs. Students are lead through the various stages of the lesson as they work through the following prompts (included in Appendix A) as a group and as a class:

1. With a partner or small group, list 3 - 5 examples of relations from A to B that are functions (i.e., $f \subseteq A \times B$ such that $f : A \to B$), given A, B are sets.

2. With the same partner or group, list 3 - 5 examples of relations from A to B that are not functions from A to B, given that A, B are sets.

3. As a class, discuss a few examples and non-examples and develop a way to distinguish the two.

4. Formulate a definition for function.

Again, the goal of this process is to lead students to formulate a precise, mathematical definition for the concept of function. This definition should define a function as a set of ordered pairs and should resemble the definition in Figure 3.7.

Figure 3.7

A precise, set-theoretic definition of a function.

Definition: Function

Given sets A and B, $f : A \to B \Leftrightarrow f$ is a relation from A to B and $\forall\, x \in A, \exists!\, y \in B$ such that $(x, y) \in f$.

Leading Students to Comprehend Definitions

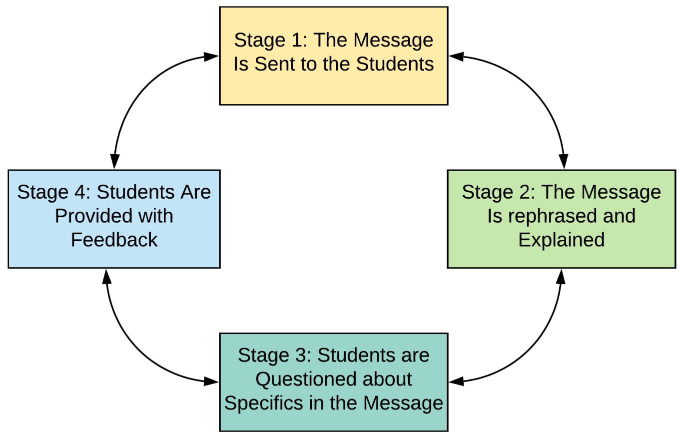
Whether the definitions are formulated by the students or given by the instructor, students are expected to comprehend and use these definitions. In order to lead students to comprehend the definitions and learn to use them, the should be lead through a four-stage direct instruction lesson. Cangelosi (2003) explains that the four stages are: (a) the message is sent to the students, (b) the message is rephrased and explained, (c) students are questioned about specifics in the message, and (d) students are provided with feedback on their responses to questions raised in Stage 3. Like the stages in a construct-a-concept lesson, movement between the stages is sometimes necessary and this relationship is depicted in Figure 3.8 on the following page.

Since students are lead to construct the concepts of relation and function and formulate their definitions, the first stage of a comprehension-and-communication lesson for these definitions is already completed. However, for the definitions that students do not formulate themselves, the definitions need to be given to the students orally and in writing. For example, the definition of the range of a function is given to the students as part of the guided notes they receive for the unit on functions (see Figure 3.9 and Appendix A). Then, during the lesson, the definition is read aloud to the students. Similarly, the definitions for injective, surjective, and bijective function are given.

After the students have been given the definition (or formulated it themselves) the next stage is to have the definition rephrased and explained. Initially, this is done by the instructor, but as students become more familiar with these types of lessons, asking students to rephrase the definition in their own words can be helpful for the students to make sense of the mathematical language and notation being used to define the concept. This stage usually plays a part in constructing a concept and formulating a definition as students begin explaining in their own words and then translate that to mathematical language and notation, however, when students are only given the definition for a concept, they will benefit from rephrasing and explaining the definition. For example, if students are given the set-theoretic definition of the range of a function, like the one in Figure 3.9, they will

Figure 3.8

The four stages of a comprehend-and-communicate lesson as outlined by Cangelosi (2003).



better comprehend its meaning if it is explained as "the set of y's in the codomain that are getting mapped to," or something similar.

The last two stages of a comprehension-and-communication lesson help students strengthen their comprehension of a mathematical message, affords them the opportunity to use the language of mathematics to communicate their comprehension, and gives them timely feedback on their communication. This is done by asking questions about the specifics in the message. For example, in order to enhance students' comprehension of the definitions of function, domain, codomain, and range they are asked to determine the truth value of the given proposition and use the definitions to explain their determination. These prompts, included in the guided notes in Appendix A, emphasize functions as sets of ordered pairs and highlight the link between a function and its domain, codomain, and range.

Figure 3.9

A precise, set-theoretic definition of the range of a function.

Definition: Range of a Function

Given $f : A \to B$, the *range* of the function f is the set $\{y \in B : (x, y) \in f\}$.

1. If $X = \{1, 2, 3, 4\}$, $Y = \{\text{standard lower case letters of the English alphabet}\}$, $r \subseteq X \times Y$ such that $r = \{(2,\text{“c”}), (1,\ \text{“t”}), (4,\ \text{“m”}), (3,\ \text{“c”})\}$, then $r : X \to Y$ such that X is the domain of r, $\{\text{“a,” “b,” “c,”} \ldots, \text{“z”}\}$ is the codomain, and $\{\text{“t,” “c,” “m”}\}$ is the range of r. T F

2. If $f = \left\{\left(q, \frac{q}{q+1}\right) : q \in \mathbb{Q} - \{-1\}\right\}$, then $f : \mathbb{Q} - \{-1\} \to \mathbb{Q}$ such that $\mathbb{Q} - \{-1\}$ is the domain of f and $\mathbb{Q}$ is the range of f. T F

3. If $h = \{(x, h(x)) : x \in \mathbb{R} \text{ and } h(x) = \sqrt{x}\}$, then $h : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R}$ is the domain of h and $\{y : y \in [0, \infty)\}$ is the range of h. T F

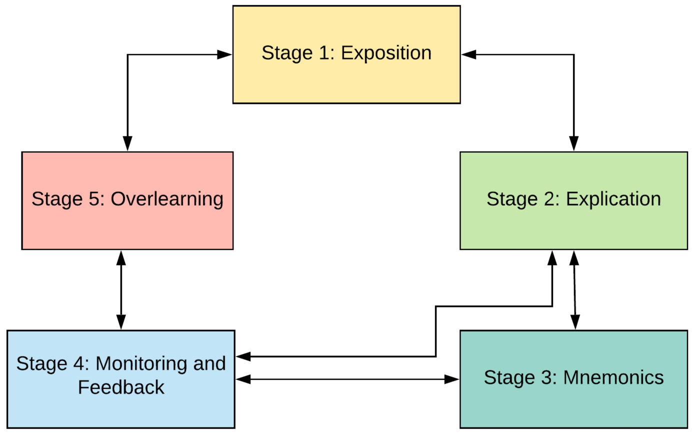
By working through these prompts individually, students reflect on their own comprehension and continue to modify their concept images using the language of mathematics. Then, by comparing their responses to those of a partner or small group, they continue to build their ability to communicate mathematically while also giving feedback to and receiving feedback from their peers. Afterwards, during a group discussion, additional feedback is obtained from peers not in their group and from the instructor. Individual stages are revisited as students refer back to the definition they are using, rephrase it in their own words, and apply the definition to the specifics they are examining.

Helping Students Develop Knowledge

While students are constructing concepts, deepening comprehension, and learning to communicate mathematically, there is mathematical content that they should remember. For example, throughout the unit on functions, students are exposed to a variety of notation and are expected to remember it and its interpretation. Remembering mathematical content is outlined in a unit's simple-knowledge objectives and in order for students to achieve these type of objectives, they need to accurately receive and retain the information (Cangelosi, 2003). Cangelosi (2003) explains that reception and retention are realized through a five-stage process, delivered through direct instruction, that includes: (a) exposition, (b) explication, (c) mnemonics, (d) monitoring and feedback, and (e) overlearning. Movement through the different stages of this process is illustrated in Figure 3.10 on the next page.

Figure 3.10

The five stages of a simple-knowledge lesson as outlined by Cangelosi (2003).



During the first two stages of the process, students are exposed to the information and given a brief explanation of how to respond to the given stimuli. For example, during the lesson where students are constructing the concept of function, they are introduced to the notation "$f : A \to B$" and told that it is read as "f is a function from the set A to the set B." Once students have been exposed to the information and it has been explained to them, in some instances, it can be helpful to provide students with a mnemonic device to help them retain the information. It is important to note, however, that not all simple-knowledge objectives need a mnemonic device. By giving students several opportunities to practice recalling what they have learned, monitoring their progress, and providing feedback throughout the process, they are able to overlearn the information leading to increased resistance to forgetting and long-term retention (Cangelosi, 2000).

Presenting the Lesson

Using the learning activities developed for each of the unit's objectives, the unit on functions was presented to a group of 50 students taking a Calculus I class. Prior to beginning this unit, a unit on the language and theory of sets was presented which included comprehending the set relationships of subset, equality, proper subset, and similar (or equivalent) and the set operations of union, intersection, without (or difference), and Cartesian product. These two units made up the introduction/review section of the course and were

presented during the first two weeks of the semester. After each of the units, students responded to a series of mini-experiments designed to assess their achievement of the objectives outlining each of the unit learning goals. The development and analysis of the set of mini-experiments for the unit on functions is covered in detail in Chapter 4.

Students' Perceptions of the Concept of Function

Before beginning the unit on functions, students were asked to complete a voluntary, anonymous questionnaire designed to discover students' current level of comprehension of the concepts of function, domain, codomain, and range. In order to ensure anonymity, this questionnaire (included in Appendix B) was delivered asynchronously to students through an online learning platform approved by the university. Of the 50 students enrolled in the class, 24 provided responses to all seven of the following prompts:

1. Without disclosing any scores or grades, what prerequisite requirement did you meet?
2. Which mathematics course do you feel helped you the most in learning about functions (this could be any mathematics course you've taken, including the ones you took prior to college)?
3. On a scale of 1 to 5 (1 being not at all and 5 being very) how comfortable are you with functions?
4. What is a function?
5. What is the domain of a function?
6. What is the codomain of a function?
7. What is the range of a function?

The first two prompts were devised to investigate where some of the students' perceptions concerning the concept of function came from. The results from the first two responses, detailed in Figures 3.11 and 3.12 respectively, indicated that most of the students who responded to the questionnaire (54%) took a college algebra and trigonometry class in order

to meet the prerequisites for this course. However, only 46% of respondents felt that their college algebra class was the most helpful in learning about the concept of function while 37% indicated that their high school mathematics classes provided the most help. Four of the respondents indicated that previous calculus courses, taken after high school, were the most helpful for learning about the concept of function.

Students' Perception of a Function

In order to determine how comfortable students are with the concept of function, Prompt 3 asked them to rate their comfort level on a scale of one to five with one being not comfortable at all and five being very comfortable. Table 3.5 on page 51 discloses the frequency of the comfort levels along with a percentage. The average comfort level of the students who responded to the questionnaire was $3.\overline{33}$ with a standard deviation of about 0.8681. These results indicate that the vast majority of respondents (about 88%) feel neutral when dealing with functions. However, their comfort with functions may not be indicative of their understanding of the function concept.

The next prompt in the questionnaire, asks students to describe what they think a function is. The bar chart in Figure 3.13 on page 50 divides the responses into seven categories: (a) input/output, (b) relationship between sets, (c) graph, (d) process of change, (e) other, (f) known, but undescribed, and (g) unknown. Of the 24 responses, eight (33%) described a function as an equation or problem that has a set of inputs and a set of outputs; five (21%) described a function as some sort of relationship between two sets; three (13%) described a function as an equation that can be graphed or as a display of information; two (8%) described a function as a process that changes one variable based on a change in the other; three (13%) described a function in other ways (i.e., a true/false system, a set of numbers, and a process to solve problems); two (8%) claimed to know what a function was, but were unable to describe it; and one (4%) claimed that they didn't understand functions well enough to describe what one was. Furthermore, only three (13%) of the 24 respondents made any reference to the requirement that for every x in the functions domain, there can be one and only one y in the functions codomain.

Figure 3.11

The necessary prerequisites met by a sample of 24 students participating in a Calculus I course.

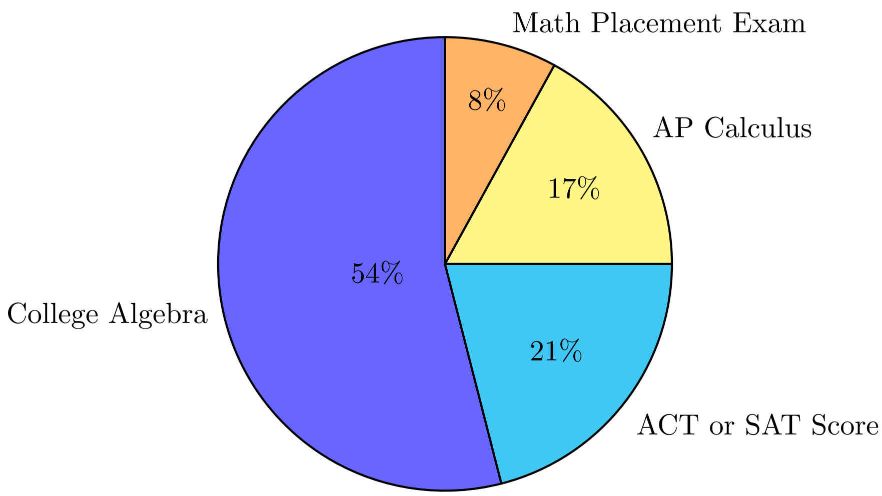


Figure 3.12

The self-identified courses that a sample of 24 students participating in a Calculus I class determined to be the most helpful in learning about the concept of function.

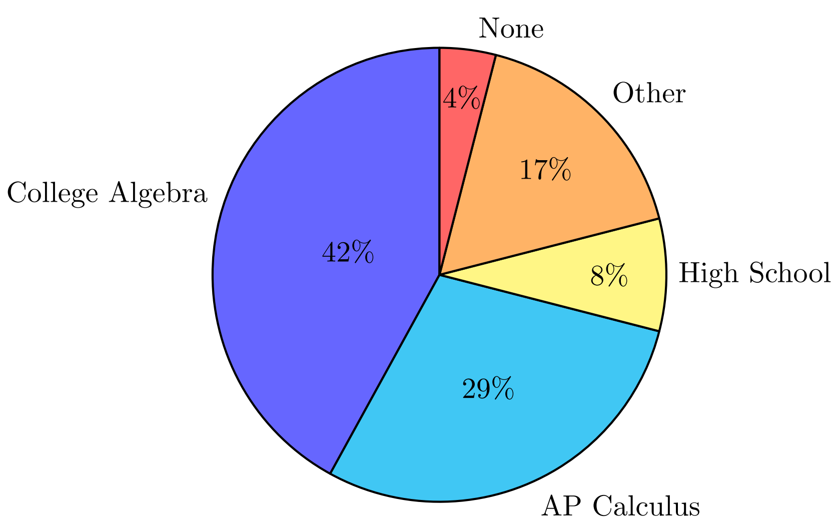


Figure 3.13

The descriptions of a function given by a sample of 24 students in a Calculus I class divided into seven categories: (a) input/output, (b) relationship between sets, (c) graph, (d) process of change, (e) other, (f) known, but undescribed, and (g) unknown.

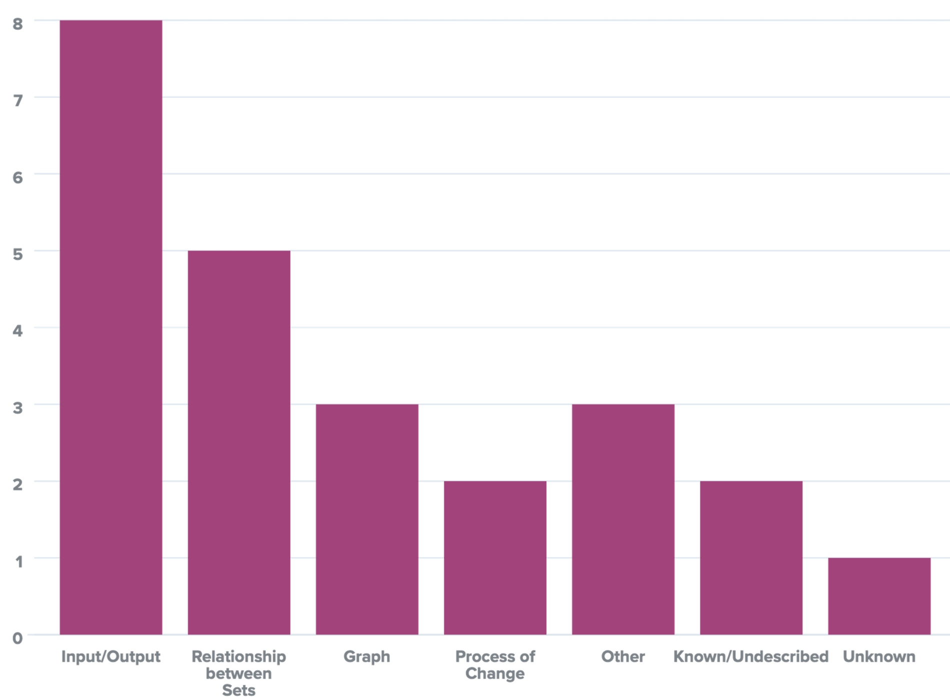


Table 3.5

The self-selected comfort levels of a sample of 24 students participating in a Calculus I class pertaining to the use of the concept of function along with the average comfort level ($\overline{x}$) and standard deviation (s).

Comfort Level	**Frequency**	**Percentage**
1	1	4%
2	2	8%
3	10	42%
4	10	42%
5	1	4%
$\overline{x} = 3.\overline{33}$	$s \approx 0.8681$	

Overall, the responding students' perceptions about the concept of function indicate that the majority of them have an action conception of the concept while a few may have a more advanced process or object conception. Additionally, only very few students described a function as a relation or as a set of ordered pairs that require a pairing from all of the elements in the function's domain with one and only one element in the function's codomain. Student descriptions of a function's domain, codomain, and range also demonstrate this limited perception of the concept of function.

Students' Perception of the Domain of a Function

When asked to describe the domain of a function, all but one described the domain of a function as all possible input or "x" values for the function. The one response that differed was from a student who wasn't sure what the domain of a function was. Furthermore, most of the responses indicated that the domain of a function contains values or numbers that should be graphed or calculated and one indicated that the domain was a restriction placed on the function. Such an image of the domain of a function limits the concept of function

to merely be an equation or formula that operates only on subsets of real numbers.

Students' Perception of the Codomain of a Function

Fourteen (58%) of the 24 respondents expressed that they had no idea what the codomain of a function was, three (13%) described it as somehow relating to the values in the domain of a function, and one (4%) guessed that it was a set of numbers that the function didn't apply to. Only three (13%) of the respondents indicated that the codomain was more than the range of a function; it was a set of all possible output values. In order for students to construct a concept of function that defines a function as a relation between two sets, comprehension of the relationship between a function's domain and codomain is needed.

Students' Perception of the Range of a Function

The majority of the respondents (71%) described the range of a function to be the set of y values (or outputs) that resulted from applying the function on its domain. There were two (8%) respondents who referred to the definition of the range of a data set and described the range of a function as the difference between the maximum and minimum values of the solutions or outputs of the function and two others described the range as the restriction placed on the output of the function. As with the domain of a function, most of the respondents described the range as being a set of values or numbers that were the result of a calculation or part of the graph of a function. In order for students to construct meaningful concept images of injective, surjective, and bijective functions, comprehension of the relationship between a functions codomain and range and the ability to distinguish between the two is indispensable.

Reflections on Leading Students to Construct the Concept of Function

Not all students took the opportunity to respond to the questionnaire prior to beginning the unit on functions. However, during the first few lesson activities outlined in the guided notes (see Appendix A), it became clear that a vast majority of the students had the limited view of a function as an equation, formula, or a graph. For example, when asked as a class to discuss the idea of a function from a set A to a set B, the first few responses were that

it was a formula that required an input and had an output or that it was a graph of an equation. Some even proffered the examples of $y = x^2$ and $f(x) = 5x + 3$. Notwithstanding this limited view, most agreed that a function required a single output for every input.

Just prior to this brief discussion, students constructed the concept of a relation between sets and, consequently, once the idea of a function was discussed, students were informed that a function is a relation between a set A and a set B. With that initial prodding, students were then asked to come up with examples and non-examples of relations that are functions. Almost all the examples presented by students were of the form "$f(x) =$" or were given by a graphical representation. Additionally, several groups of students provided $f(x) = \sqrt{x}$ and $f(x) = \frac{1}{x}$ as non-examples. Of further note was that almost none of examples (or non-examples) included the sets A and B and none of the examples were represented by a set of ordered pairs.

However, during the discussion that followed and with a few additional examples and non-examples represented in a variety of ways (i.e., sets of ordered pairs, tables, etc), most of the students quickly made adjustments to their examples and found situations where a given relation could be modified so that it became a function and vice versa. For example, during the discussion of why students suggested that $f(x) = \sqrt{x}$ is not a function, they discovered that if f was a relation from $\mathbb{R} \to \mathbb{R}$, then $f(x) = \sqrt{x}$ would not be a function as presumed, but, if the sets were changed to either $[0, \infty) \to \mathbb{R}$ or to $\mathbb{R} \to \mathbb{C}$, then f would be a function. Likewise, they found that if f was a relation from $\mathbb{Z} \to \mathbb{Z}$ such that $f = \{(x, y) : y = \frac{1}{2}x\}$ then f would not be a function. Throughout the exploration of the these examples, students identified the essential characteristics that defined a function; namely that a function is a relation from a set A to a set B where every element in A is paired with one and only one element in B. Using these characteristics, students then formulated a set-theoretic definition of a function that was easily modified to resemble the one in Figure 3.7 on page 42.

As students were lead through the remaining learning activities they were exposed to the concepts of domain; codomain; range of a function; and injective, surjective and bijective

functions. After constructing the concept of function and formulating a definition for the concept, these other concepts seemed to develop quickly and most students, within groups, were able to respond to the prompts outlined in the guided notes, providing appropriate examples and non-examples where needed. To further illustrate student achievement and to further reflect on the implementation of this unit, consider the results of the measurement instrument developed for the purpose ascertaining the level of student achievement the objectives outlining the unit's goal.

Results and Conclusions

In order to measure student achievement of the objectives that influenced the design of the lesson activities and that outline the unit's learning goal, an assessment was developed and administered to students upon completion of the unit on functions. This assessment, included in Appendix C, was found to be relevant to the content presented in the unit and to the learning levels students are expected to engage at (see Chapter 4 on page 59). Thus, the results of the assessment provide a useful and valid measurement of student achievement of the outlined learning objectives. Table D.1 in Appendix D displays the item-by-item measurement results for each of the 50 students and Table D.2 on page 131 provides the averages and standard deviations for the scores of each individual item.

Student Achievement of Objectives Regarding the Concept of Function

Objectives 2.C. - 2.F., shown in Table 3.3 on page 37, outline the expectations of student achievement surrounding the concepts of function, domain, codomain, and range. Prompt 2 and the individual parts of Prompt 4 from the assessment address the achievement of these objectives. Table 3.6 on page 56 shows the total possible score for each of the items, the average score obtained by the class, and the average score calculated as a percentage. In particular, Prompt 2 requires students to distinguish between sets that are functions and sets that are not, assessing the level of achievement of the construct-a-concept objective. Prompts 4(a) - 4(c) assess the comprehension-and-communication objective for the concepts of domain, codomain, and range while Prompts 4(d) - 4(g) assess the comprehension-and-communication and simple-knowledge objectives for the concept of function.

Table 3.6

Total possible points, average scores, and the percentage of items associated with determining the level of achievement of objectives related to the concepts of function, domain, codomain, and range.

	Items								**Total**
	2	4(a)	4(b)	4(c)	4(d)	4(e)	4(f)	4(g)	
Possible	5	4	4	4	6	6	6	6	41
Average	3.86	3.44	3.34	2.9	5.04	4.72	4.7	3.72	31.72
Percentage	77%	86%	84%	73%	84%	79%	78%	62%	77%

Overall, students demonstrated a reasonably high level of achievement of these objectives, but, as a class seemed to have more difficulty with prompts 4(c) and 4(g). The first of these, 4(c), was developed to measure a student's comprehension of the definition of the range of a function. The prompt indicates that $h : \mathbb{R} - \{0\} \to \mathbb{R}$ such that $h = \left\{(x, y) : y = \frac{1}{x}\right\}$ and then students are asked if the range of h is equal to the set $\mathbb{R} - \{0\}$. The most common error was not necessarily an indication of lack of comprehension of the range of a function as students noted that the set $\mathbb{R} - \{0\}$ was also the domain of the function and failed to investigate further. Student responses to Prompts 5 - 7, also demonstrate a better comprehension of the range of a function.

The second prompt students tended to struggle with in this section was Prompt 4(g). This prompt was developed to measure a student's comprehension of the definition of a function. Students were given that $j(x) =$ (the ages of all students in a class) and then asked if $j : \{\text{classes at our university}\} \to \mathbb{W}$. Some students claimed that such a relationship was impossible while others claimed that since many students in a particular class may have the same age that it couldn't possible be a function. In either case, these responses demonstrate a struggle with thinking about functions in a slightly more abstract way and indicate that some students still perceive functions as relating only to subsets of real numbers. However,

responses to the other prompts in this section indicate that students' level of comprehension of the definitions of function, domain, and codomain were developing in a positive way.

Student Achievement of Objectives Regarding the Concepts of Injective, Surjective, and Bijective Function

The concepts of injective, surjective, and bijective function are demarcated in Objectives 2.G. - 2.L and the achievement of these objectives is assessed by Prompts 5 - 7 in the assessment. In each of these prompts, students are given a relation without defining the set A or the set B. Then, they are tasked with building sets A and B so that the given relation is an injective, surjective, or bijective function respectively. Additionally, students are asked to provide an explanation of why the sets they have chosen make the relation the indicated type of function. Table 3.7 on the next page displays the total possible score for each of these items, the average score obtained by the class, and the average score calculated as a percentage. These results indicate that students' level of achievement of the objectives outlining these concepts is rather high. Furthermore, such achievement also indicates that students were able to delineate between a function's domain, codomain, and range as these are important concepts when explaining why a function is injective, surjective, or bijective.

Based on the results of the assessment, most students showed a high level of achievement of the all the objectives outlining the unit goal. In particular, students were able to recognize a function as a set of ordered pairs that have the characteristic that for every element in its domain, there is one and only one element in its codomain such that the ordered pair (x, y) is an element of the function. However, even with the promising results, there are some improvements that could be made. For example, many students still struggled with relying on the definition of function to help them make sense of relationships of abstract sets (i.e., sets that did not involve subsets of the real numbers). Perhaps providing more examples of these types of sets would be beneficial in helping students view a function in more abstract terms.

Table 3.7

Total possible points, average scores, and the percentage of items associated with determining the level of achievement of objectives related to the concepts of injective, surjective, and bijective.

	Items			**Total**
	5	6	7	
Possible	8	8	8	24
Average	6.88	7.02	7.24	21.14
Percentage	86%	88%	91%	88%

Implementation for Future Research

The purpose of this action research study was to examine the development and implementation of a unit on functions for a class of Calculus I students. However, very little attention was made to how a deeper understanding of the concept of function and the focus on using precise language assisted them in developing other concepts essential to the study of calculus. The favorable results of this study indicate that exposing students to the precise language of mathematics and allowing them to construct the concepts themselves and formulate their own definitions is one way that students can begin to reconcile self-constructed concept images with the formal concept definitions used throughout calculus.

Additionally, the concept of function is one of the main concepts covered in college algebra classes, but most students are leaving these classes with a limited understanding of what a function is and how its domain, codomain, and range relate to it. Thus, another possible implementation would be to investigate how the introduction of this unit in a college algebra class would effect student achievement of the objectives outlining the goal of this course and other mathematics courses that may follow.

Although the implementation of this unit had a positive effect on students' development

of the concept of function, a single unit that focuses on and uses the precise language of mathematics will be insufficient in building a deep, meaningful foundation for all the concepts within a mathematics course. The methods and attention given to this unit should, in order to be effective, be applied to all the units in a course. Thus, the future development of instruction that can be given to mathematics instructors that focus on developing lessons that use precise language and allow students to construct concepts themselves, formulate definitions, and strengthen their comprehension of those definitions would be beneficial to instructors and students alike.

CHAPTER 4

Assessing Student Achievement of Objectives for Developing Comprehension of the Concept of Function

Assessing Mathematical Achievement

To some, the use of formal assessments (e.g. tests or exams) to ascertain the educational achievement of students may be considered a "necessary evil," while others may view these types of measurements as essential to education effectiveness (Ebel, 1965). No matter the view, most agree that any kind of formal assessment should be relevant to the content being taught, provide reliable results, and be effective in assessing student achievement (Cangelosi, 2000; Ebel, 1965; Gronlund & Linn, 1990; Lewis, 1975; Nitko & Brookhart, 2007; Popham, 1981). Ebel (1965) explains that such assessments (a) help teachers and professors give more reliable and valid grades, (b) motivate and direct student learning, and (c) provide students with extrinsic motivation to learn. However, he points out that assessments that are not effective in promoting student learning have, at best, "questionable utility" (p. 8). Developing assessments that meet these criteria, however, is a complex endeavor.

A Complex Endeavor

One reason for the complexity that exists in developing relevant, reliable, and effective assessments is that the purpose of any assessment is to measure the level of achievement students have attained with respect to a set of learning objectives. However, as Cangelosi (2003) explicates, a student's level of achievement lies on a continuum which, over time, varies due to influences such as learning and forgetting. Thus, identifying the level at which each student is at any given time for any given learning objective is the impossible commission for each assessment given to students. Perhaps the best that can be done is to recognize that there will be some error involved in assessing a student's learning level and try to make that error as close to zero as possible. That is, recognize that each observed score (S_o) is the result of a student's true score (S_t) plus some error (E) (Cangelosi, 2000;

Spearman, 1904). In mathematical terms:

$$S_o = S_t + E \tag{4.1}$$

Developing a Useful Assessment

Another reason assessment development is so complicated is because relevant, reliable, and effective assessments are the end-product of a process that involve many different stages (Lewis, 1975). These stages include (a) developing unit goals, (b) delineating those goals into a string of weighted objectives, (c) choosing a particular kind of assessment, (d) constructing the assessment from a selection of suitable items, (e) administering the assessment at a particular time and place, and (f) scoring the assessment by a particular person (Cangelosi, 2000; Lewis, 1975). Cangelosi (2000) adds that for each step in this process, many different decisions and judgments need to be made in order to develop a useful assessment.

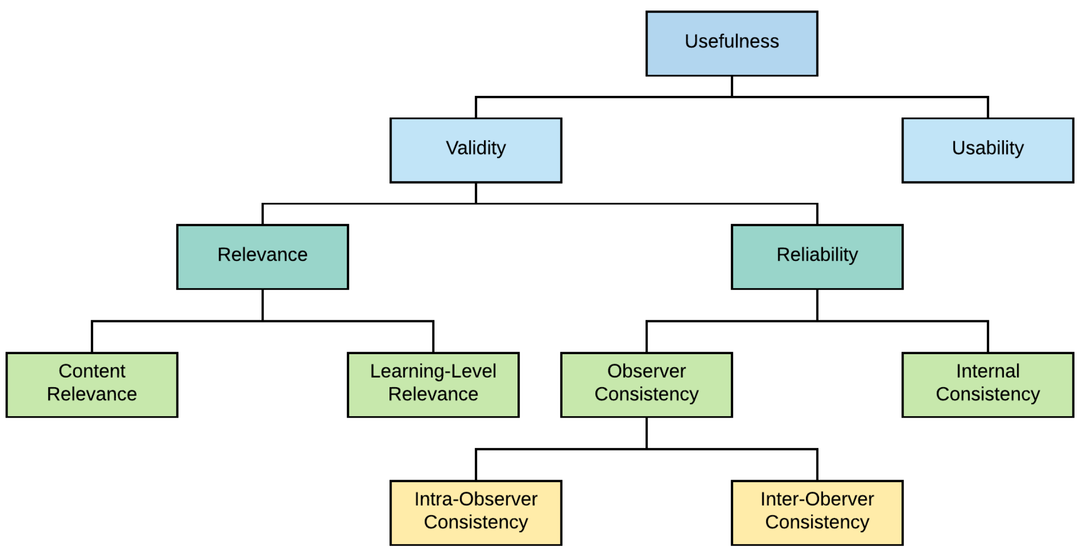
To further complicate the issue, once an assessment is developed, it still may not be clear if its measurements are useful. Cangelosi (2000) explains that the usefulness of a measurement instrument (i.e. an assessment) is determined by its validity and usability. The validity of an instrument requires subject-content and learning-level relevance, internal consistency, and observer consistency. Whereas the usability of an instrument is determined by its cost, ease of administration and scoring, and safety. Figure 4.1 outlines Cangelosi's hierarchy of measurement usefulness (Cangelosi, 2000).

Developing an Assessment for a Unit on Functions

In order to demonstrate the process of developing a useful assessment, we'll investigate an assessment for a unit on functions along with the scores from responses of 50 students taking a first semester course in calculus. Following a unit on using the language of sets, this unit (a) leads students to construct the concepts of relations between sets (or a set and itself) and functions, (b) deepens student comprehension of those concepts, and (c) strengthens communication skills when discussing those concepts. Although an overview of the unit goal and accompanying objectives are outlined, the development and implementation of

Figure 4.1

Measurement usefulness depends on measurement validity and usability; Measurement validity depends on relevance and reliability; Measurement relevance depends on subject-content relevance and learning-level relevance; Measurement reliability depends on internal consistency and observer consistency; Observer consistency depends on intra- and inter-observer consistency.



this unit in a calculus class is discussed in depth in Chapter 3. To determine the usefulness of this assessment, a description of the analysis will also be given.

Establishing a Unit Goal and Objectives

The first step in developing a meaningful assessment is to establish a detailed unit goal that is outlined by weighted objectives that organize expectations of student learning into cognitive types or learning levels (Cangelosi, 2003; Kohler & Alibegović, 2015). The established unit goal for the selected unit on functions is to have students (a) understand and interpret relations between sets, (b) understand functions as relations between sets, (c) distinguish between relations that are functions and those that are not, (d) understand the meaning of and identify a function's domain, codomain, and range, and (e) distinguish between functions that are injective, surjective, and bijective. This goal is outlined by 12 weighted objectives, each with an associated learning level. Table 4.1 states the unit goal and lists the objectives outlining the goal, the objective's associated learning level, and its given weight.

Designing Mini-Experiments

Once a unit goal and its objectives have been established, the next step in developing a useful assessment is to develop a set of prompts and accompanying rubrics to assess the achievement of the outlined objectives. A given prompt and its accompanying rubric is referred to as a *mini-experiment*. This appellation is used instead of the more common "test item" or "test question" for the following reasons: (a) the conventional terms often connote only traditional testing formats (e.g., true/false, or multiple choice). (b) "Mini-experiment" helps remind educators that in order to gather evidence of student achievement they need to create opportunities in which students behave in observable ways that indicate that achievement (or lack thereof). (c) "Mini-experiment" indicates that educators are free to design an information-gathering device that may not look like anything they have seen previously (Cangelosi, 2000, p. G5).

One of the goals of an assessment (i.e., a collection of mini-experiments) is to help both the student and teacher understand what the student knows and to identify areas where

Table 4.1

The goal for a unit on functions along with its weighted objectives, their associated learning levels and weights.

Unit Goal		
The goal for this unit on functions is that students will (a) understand and interpret relations between sets, (b) understand functions as relations between sets, (c) distinguish between relations that are functions and those that are not, (d) understand the meaning of and identify a function's domain, codomain, and range, and (e) distinguish between functions that are injective, surjective, and bijective.		
Objective	**Learning Level**	**Weight**
2.A. Students will distinguish from sets that are relations between two sets (or between a set and itself) and sets that are not and develop a definition.	Construct a Concept	5%
2.B. Students will explain that a relation is a subset of the cross product of two sets (or the cross product of a set with itself).	Comprehension and Communication	15%
2.C. Students will distinguish between relations that are functions and those that are not and develop a definition.	Construct a Concept	5%
2.D. Students will explain that a function is a relation from a set A to a set B where every element in A is mapped to one and only one element in B.	Comprehension and Communication	25%
2.E. Students will associate the notation "$f : A \to B$" with the expression "f is a function from A to B."	Simple Knowledge	5%

Table 4.1 (Cont'd)

Objective		Learning Level	Weight
2.F.	Given the function $f : A \to B$, students will explain that A is the domain of the function, B is the codomain of the function, and the range of the function is the subset of the codomain which contains all elements in B which are mapped to by the elements of A.	Comprehension and Communication	15%
2.G.	Students will explain that an injective function is a function where every element in the range of the function is mapped to by one and only one element in the domain.	Comprehension and Communication	7%
2.H.	Students will associate the notation "$f : A \xrightarrow{\text{1-1}} B$" with the expression "$f$ is an injective (one-to-one) function from A to B."	Simple Knowledge	3%
2.I.	Students will explain that a surjective function is a function where every element in the codomain is mapped to by an element in the range, in other words, the range is equal to the codomain.	Comprehension and Communication	7%
2.J.	Students will associate the notation "$f : A \xrightarrow{\text{onto}} B$" with the expression "$f$ is a surjective (onto) function from A to B."	Simple Knowledge	3%
2.K.	Students will explain that a bijective function is both injective and surjective, that is, one-to-one and onto.	Comprehension and Communication	7%
2.L.	Students will associate the notation "$f : A \xrightarrow[\text{onto}]{\text{1-1}} B$" with the expression "$f$ is a bijective (one-to-one and onto) function from A to B."	Simple Knowledge	3%

the student may need improvement (Schoenfeld, 2007). The process of assessing a student's mathematical achievement eluded to by Schoenfeld has two parts: first is the objective-related prompt that solicits a response from the student and second is the observer's rubric used to evaluate the student's response. The observer's rubric is an essential part of this process as is the set of rules a teacher (or other observer) follows to record an analysis of the student's response to the prompt (Cangelosi, 2003, p. 189). Kohler and Alibegović (2015) note that even though rubrics take longer for teachers to design they offer a more efficient way to communicate detailed feedback to the student.

In this way, a prompt that solicits a student's demonstration of knowledge or understanding about a concept or relationship along with the resulting measurement obtained from the rubric becomes a mini-experiment where, it is hoped, the error in Equation (4.1) on page 60 is as close to zero as possible. However, constructing a mini-experiment that results in an accurate measure of a student's mathematical achievement is a complicated one. As Kohler and Alibegović (2015) point out, asking students to simply state formulas or use mathematical algorithms may not be sufficient as higher levels of cognition are often expected. Thus, developing mini-experiments based on both the content and learning level of a given objective is helpful in creating a relevant and reliable assessment tool.

Cangelosi (2003) defines seven cognitive learning levels: construct a concept, discover a relationship, comprehension and communication, simple knowledge, algorithmic skill, application, and creative thinking (Table 3.2 on page 32 gives details for each of these learning levels). The objectives outlined for the unit on functions are defined using only three of these learning levels: construct a concept, comprehension and communication, and simple knowledge. Thus, we will examine the development of mini-experiments expected to measure achievement of these three learning levels.

Construct a Concept

An objective categorized at the construct-a-concept learning level has students use inductive reasoning to distinguish between examples and non-examples of a particular concept (Cangelosi, 2003). For example, Objective 2.A. is a construct-a-concept objective which re-

quires students to distinguish between examples and non-examples of a relation between sets or between a set and itself. A prompt that is designed to measure the achievement of this objective should lead students to differentiate between examples and non-examples of relations between sets. Prompt 1 in Appendix C is an example of such a mini-experiment.

1. (5 points) Let $A = \{\text{musical instruments}\}$ and $B = \{\text{four legged animals}\}$. Determine which of the following sets are relations from A to B. Indicate your choice(s) by circling the letter in front of the set.

   A. {tuba, dog, trumpet, cat, violin, elephant}
   B. {(cello, zebra), (viola, yak), (trombone, emu), (clarinet, lynx)}
   C. {(piano, horse), (guitar, buffalo), (harp, lion), (oboe, puma)}
   D. {(tiger, French horn), (jackal, keyboard), (wolf, flute)}
   E. {(bassoon, deer), (drums, elk)}

   Observer's Rubric:

   Maximum score is 5 points distributed according to the following criteria:

   +1 if response indicates A is not a relation.
   +1 if response indicates B is not a relation.
   +1 if response indicates C is a relation.
   +1 if response indicates D is not a relation.
   +1 if response indicates E is a relation.

According to the associated rubric, a student shows progress toward achievement of this objective if they indicate that C and E are the only sets that are relations from A to B. Prompt 2 is a mini-experiment assessing the achievement of Objective 2.C. and is another example of a construct-a-concept mini-experiment.

2. (5 points) Determine which of the following sets are functions. Indicate your choice(s) by circling the letter in front of the set.

   A. $f = \left\{(x, y) : x, y \in \mathbb{R} \text{ and } y = \dfrac{x-2}{x^2}\right\}$.
   B. g is a relation from {buttons on a vending machine} to {soda flavors in a full vending machine} such that $g = \{(x, y) : y$ is the flavor of soda dispensed when x is pushed$\}$.
   C. h is a relation from $(0, \infty)$ to $\mathbb{R}$ such that $h = \left\{(x, y) : y = \dfrac{1}{\sqrt{x}}\right\}$.

D. j is a relation from {living people} to {people who every lived} such that $j = \{(x, y) : y \text{ is the biological child of } x\}$.

E. k is a relation from {cars} to $\mathbb{W}$ such that $k = \{(x, y) : y \text{ is the number of wheels on car } x\}$.

Observer's Rubric:

Maximum score is 5 points distributed according to the following criteria:

+1 if response indicates A is not a function.

+1 if response indicates B is a function.

+1 if response indicates C is a function.

+1 if response indicates D is not a function.

+1 if response indicates E is a function.

Comprehension and Communication

An objective categorized at the comprehension-and-communication learning level has students (a) extract and interpret meaning from an expression, (b) use the language of mathematics, and (c) communicate with and about mathematics (Cangelosi, 2003). Objectives 2.B., 2.D., 2.F., 2.G., 2.I., and 2.K. are all objectives categorized as comprehension-and-communication objectives. These objectives aim to have students explain why a given expression meets (or doesn't meet) a given definition. For example, in order for a student to achieve Objective 2.D., they must be able to explain (using the definition of a function) why a given expression is or is not a function.

Prompt 4(d) from Appendix C requires students to determine the truthfulness of the following statement:

1. Examine each of the following propositions to determine whether or not it is true; indicate your determination of the truth value by circling either "T" or "F." Then, write a paragraph that explains why your determination is correct.

   (d) (6 points) If $f = \{(x, y) : y = \sqrt{x}\}$, then $f : \mathbb{R} \to \mathbb{R}$.

According to the observer's rubric, a student shows progress toward achievement of Objective 2.D. if they use the definition of function to explain that the set f is not a function from $\mathbb{R}$ to $\mathbb{R}$ because it is not a relation from $\mathbb{R}$ to $\mathbb{R}$. Prompts 4(e), 4(f), and 4(g) are additional mini-experiments assessing the achievement of Objective 2.D..

4. (e) (6 points) If $g = \{(x, y) : y = \sqrt{x}\}$, then $g : [0, \infty) \to \mathbb{R}$.

   Observer's Rubric:
   Maximum score is 6 points distributed according to the following criteria:
   +2 if response indicates why the proposition is true.
   +2 if response uses the the definition of a function.
   +2 if nothing extraneous or erroneous is included in the response.

   (f) (6 points) If $h(x) =$ the result of a fair coin flip, then $h : \{\text{coin flips}\} \to \{\text{H, T}\}$.

   Observer's Rubric:
   Maximum score is 6 points distributed according to the following criteria:
   +2 if response indicates why the proposition is true.
   +2 if response uses the the definition of a function.
   +2 if nothing extraneous or erroneous is included in the response.

   (g) (6 points) If $j(x) =$ the ages of all students in a class, then $j : \{\text{classes at our university}\} \to \mathbb{W}$.

   Observer's Rubric:
   Maximum score is 6 points distributed according to the following criteria:
   +2 if response indicates why the proposition is false.
   +2 if response uses the the definition of a function.
   +2 if nothing extraneous or erroneous is included in the response.

Simple Knowledge

Cangelosi (2003) explains that an objective categorized at the simple-knowledge learning level requires students to remember a specified response (but not a multi-step process) to a specified stimulus. Objectives 2.E., 2.H., 2.J., and 2.L. are examples of simple-knowledge objectives. Since simple-knowledge objectives only require a specified response to a specific stimulus, they can often be assessed as part of a mini-experiment aimed at another objective. For example, Prompt 5 from Appendix C combines Objectives 2.G. and 2.H., a comprehension-and-communication objective with a simple-knowledge objective.

5. (8 points) Let g be a relation from A into B such that $g(x) = x^2 - 2$. Find any set A and any set B such that $g : A \xrightarrow{\text{1-1}} B$. Then write a paragraph explaining why your choices are appropriate.

   Observer's Rubric:

   Maximum score is 8 points distributed according to the following criteria:

+2 if response indicates an appropriate choice for A.

+2 if response indicates an appropriate choice for B.

+2 if response uses the definition of an injective function.

+2 if nothing extraneous or erroneous is included in the response.

In order for student's to accurately respond to to this prompt, they will need to recognize the notation "$g : A \xrightarrow{1\text{-}1} B$" as meaning that g is an injective (or one-to-one) function from the set A to the set B. Whether they recognize this notation is often clear in their response. Based on the given rubric, points for achieving the simple-knowledge objective can be awarded (or not) whether the correct definition was used or whether there was anything erroneous included the student's response.

Validating an Assessment for a Unit on Functions

Once the set of mini-experiments have been designed for an assessment, we want to know if it is useful. The first characteristic of a useful assessment that we'll explore is whether it is valid. The definition of validity often differs from author to author (Ebel, 1965). For example, Popham (1981) explain that most educators think of validity as the degree in which an assessment measures what it is supposed to measure. However, many authors agree that validity should refer to the interpretation of test scores rather than the test itself (Cangelosi, 2000; Gronlund & Linn, 1990; Lewis, 1975; Nitko & Brookhart, 2007; Popham, 1981).

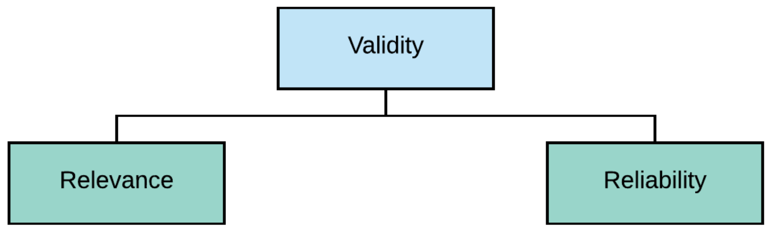
Taking it a step further, Cangelosi (2000) explains that "a measurement is valid to the same degree that it is both relevant and reliable" (p. 181). In this context, a measurement is "the process by which (a) information is collected through empirical observations, and (b) that information is remembered or recorded" (Cangelosi, 2000, p. G4). Thus, the scores students' responses receive are the measurements being considered. Figure 4.2 on the following page shows the hierarchal relationship between a measurement's validity, relevance, and reliability.

Measurement Relevance

A measurement is relevant if it addresses the mathematical content indicated by the unit objectives and requires students to think about and interact with that content at the

Figure 4.2

Measurement validity depends on relevance and reliability.



appropriate learning level (Cangelosi, 2000). Figure 4.3 shows this relationship. In order to determine if a measurement is relevant, each mini-experiment should be examined to determine (a) the mathematical content being assessed, (b) the learning level students are expected to engage with, (c) the weight each content is given in the assessment, and (d) the weight each learning level is given in the assessment.

Table 4.2 on the following page shows the categorization of each of the mini-experiments from the assessment on functions (see Appendix C) into the corresponding content rows and learning-level columns. Table 4.3 shows the weight each objective should be and compares it to the weight that the objective receives in the assessment. Since these mini-experiments were designed with the specific unit objectives in mind, the weights given to the objectives in the assessment are almost identical to the weights that should be assigned to them. This correspondence indicates that the assessment has strong measurement relevance.

Figure 4.3

Measurement relevance depends on content and learning-level relevance.

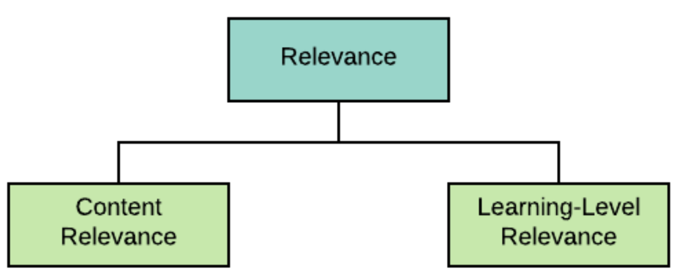


Table 4.2

The categorization of the prompts from the assessment for a unit on functions into content and learning-level categories.

Content	**Learning Level**			**Point Total**	**Weight**
	Simple Knowledge	Comp. and Comm.	Construct a Concept		
set relations		3(a), 3(b)	1	17	21%
functions	4(d) - 4(g)	4(d) - 4(g)	2	29	35%
domain, codomain, and range		4(a) - 4(c)		12	15%
injective functions	5		5	8	10%
surjective functions	6		6	8	10%
bijective functions	7		7	8	10%
Point Total	10	62	10	82	100%
Weight	12%	76%	12%	100%	

Table 4.3

The weights assigned to each objective in a unit on functions compared with the weights each objective is given in the assessment for that unit.

Objective	**Assigned Weight**	**Assessment Weight**	**Objective**	**Assigned Weight**	**Assessment Weight**
2.A.	5%	6%	2.G.	7%	7%
2.B.	15%	15%	2.H.	3%	2%
2.C.	5%	6%	2.I.	7%	7%
2.D.	25%	24%	2.J.	3%	2%
2.E.	5%	5%	2.K.	7%	7%
2.F.	15%	15%	2.L.	3%	2%

Measurement Reliability

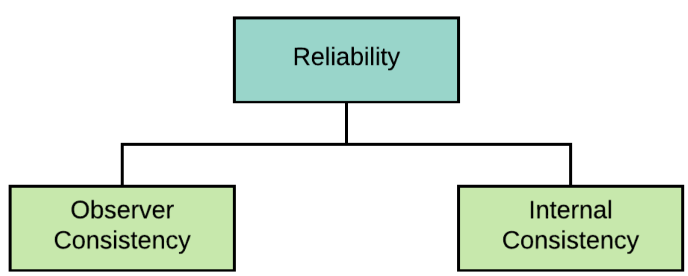
The other condition that is necessary, but insufficient, for a valid measurement is reliability (Popham, 1981). Gronlund and Linn (1990) point out that when speaking about reliability, we are referring to the "*results* obtained with an evaluation instrument and not to the instrument itself" (p. 78). Thus, measurement reliability, generally speaking, refers to the internal consistency of the measurement results or how similar the results are when the instrument is given under different conditions (Gronlund & Linn, 1990; Lewis, 1975; Nitko & Brookhart, 2007; Popham, 1981). In other words, "a measurement is reliable to the same degree that it can be depended upon to provide non-contradictory information" (Cangelosi, 2000, p. 184). Because reliability has to do with the results of an assessment, how those results are obtained should also be a primary concern when considering measurement reliability. Thus, Cangelosi (2000) explains that in order for a measurement to be reliable, it not only needs to have internal-consistency, but it needs to also have observer-consistency. This relationship is detailed in Figure 4.4.

Internal Consistency

When considering the reliability of a measurement, there are several different types of consistency. The most common of these include stability, equivalence, equivalence and stability, and internal consistency (Gronlund & Linn, 1990; Lewis, 1975; Nitko & Brookhart, 2007; Popham, 1981). Stability refers to the consistency between sets of scores that come from the same assessment administered twice. It is also referred to as test-retest consistency.

Figure 4.4

Measurement reliability depends on observer and internal consistency.



Equivalence refers to the consistency between sets of scores that come from two assessments, given at the same time, that are designed to measure the same objectives. If those two assessments are given at different times, then the consistency between the sets of scores is referred to as equivalence and stability (Lewis, 1975; Popham, 1981). Evaluating these types of consistencies requires either multiple assessments designed to measure the same objectives and/or time to administer the assessment multiple times. However, internal consistency focuses on the consistency of an assessment's internal elements (i.e., its mini-experiments) or how consistent these mini-experiments are with each other (Popham, 1981). Since we are only considering one administration of an assessment designed to measure student achievement of objectives outlining a unit on functions, we will focus solely on the measurements' internal consistency.

Cangelosi (2000) explains that "a measurement is internally consistent to the degree results from its mini-experiments agree (i.e., correlate positively)" (p. G3). What is meant by agreement is that the scores for the responses to more difficult prompts were higher for the responses given by students whose test scores are relatively higher than others while the scores for the responses to more difficult prompts were lower for the responses that were given by students whose test scores are relatively lower than others. In other words, easier prompts have a higher proportion of responses receiving higher scores and more difficult prompts have a lower proportion of responses receiving higher scores and those responses receiving higher scores are the responses given by students whose test scores were relatively higher than others.

This idea of "agreement" can be visualized by creating a mini-experiment-by-mini-experiement score matrix where each element of the matrix is the proportion of the maximum possible points obtained by the student's response. Then, reorder the rows and columns of the matrix so the rows are arranged in descending order of test scores and the columns are arranged by ascending order of mini-experiment difficulty. Mini-experiment difficulty will be discussed in more detail when we examine mini-experiment effectiveness, but for now, it suffices to assign mini-experiment difficulty by summing up the scores each

response received for a given mini-experiment. Internally consistent results should have higher scores in the upper-left corner of the matrix and then progress to lower scores as it moves to the lower-right corner.

Figure 4.5 on the following page shows the mini-experiment-by-mini-experiment matrix from the assessment for a unit on functions found in Appendix C with the rows and columns arranged as described. The matrix has been colored in such a way that it easy to see the trend of scores as it moves from upper-left to lower-right. The lighter cells represent the scores of 100% (the responses received the maximum number of points), the slightly darker cells represent the scores between 60% and 100%, while the darkest cells represent the scores of less than 60%. Since the majority of the lighter cells lie in the upper-left, the majority of the darker cells lie in the bottom-right, and the mid-valued cells are focused in the center portion, the results of this particular assessment look internally consistent.

Although a good way to visualize internal consistency, the mini-experiment-by-mini-experiment matrix does little to give us meaningful information about the internal consistency of the results. Fortunately, however, several methods have been developed to help us quantify internal consistency and compute a reliability coefficient. These reliability coefficients "provide the most revealing statistical index of quality that is ordinarily available" (Ebel, 1965, p. 308).

The Split-Halves Method and the Spearman-Brown Prophecy Formula. The split-halves method consists of dividing the assessment into two equal halves and then, after administering the assessment once, comparing the scores from the two different halves and computing the Pearson product-moment correlation coefficient between them (see Appendix E) (Cangelosi, 2000; Gronlund & Linn, 1990; Nitko & Brookhart, 2007; Popham, 1981). The method of halving the assessment can range from combining two assessments into one to splitting the assessment by evens and odds. In any case, the result is a correlation coefficient that measures the internal consistency of the entire assessment.

However, Cangelosi (2000) points out that an assessment's reliability is influenced a great deal by the number of mini-experiments it has. That is, the more mini-experiments

Figure 4.5

Mini-experiment-by-mini-experiment score matrix from an instrument assessing the achievement of objectives from a unit on functions with rows arranged in descending order of exam scores and columns by ascending order of mini-experiment difficulty colored according to the proportion of the maximum possible points obtained by the students' responses.

1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.833	1.000	1.000	0.800	1.000	1.000	1.000
0.800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.600	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	0.750	1.000	1.000	1.000	1.000	1.000	1.000	0.667
1.000	1.000	1.000	1.000	1.000	1.000	0.750	1.000	0.833	0.833	1.000	0.833	1.000	1.000
0.800	1.000	1.000	1.000	0.750	1.000	1.000	1.000	1.000	1.000	0.800	1.000	1.000	0.667
1.000	1.000	0.875	1.000	1.000	0.833	1.000	1.000	0.833	1.000	1.000	1.000	1.000	0.500
0.800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.800	1.000	0.500	0.667
1.000	1.000	1.000	1.000	1.000	0.833	1.000	1.000	0.833	1.000	0.600	0.667	0.750	1.000
1.000	1.000	0.875	0.875	1.000	1.000	0.750	1.000	0.833	1.000	1.000	0.833	0.500	1.000
1.000	1.000	1.000	1.000	0.750	1.000	1.000	0.500	1.000	1.000	0.800	0.500	1.000	1.000
1.000	1.000	1.000	1.000	0.750	0.833	1.000	0.833	1.000	0.833	1.000	0.833	1.000	0.500
1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.833	0.500	0.600	0.667	1.000	1.000
1.000	1.000	0.875	1.000	1.000	0.833	1.000	0.833	1.000	0.833	1.000	0.833	1.000	0.500
1.000	1.000	1.000	1.000	0.250	1.000	0.500	1.000	1.000	1.000	0.600	1.000	1.000	0.833
1.000	1.000	1.000	0.875	1.000	1.000	1.000	1.000	1.000	0.500	0.800	0.833	1.000	0.500
1.000	0.750	1.000	1.000	1.000	0.833	1.000	1.000	0.833	0.667	0.600	1.000	0.750	1.000
0.800	1.000	1.000	0.875	1.000	1.000	1.000	0.833	0.833	1.000	1.000	1.000	0.750	0.333
1.000	1.000	1.000	0.875	1.000	1.000	1.000	0.833	1.000	1.000	0.600	0.833	0.500	0.500
1.000	1.000	0.875	0.875	1.000	0.833	0.750	0.833	0.833	0.833	1.000	1.000	1.000	0.500
1.000	1.000	0.875	1.000	1.000	1.000	1.000	0.667	0.833	0.833	0.600	1.000	0.250	1.000
1.000	1.000	0.875	0.875	1.000	1.000	1.000	0.667	1.000	1.000	1.000	0.333	0.500	1.000
0.800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.667	0.833	0.800	1.000	0.250	0.500
0.800	1.000	1.000	1.000	0.750	0.833	1.000	1.000	1.000	0.667	0.400	1.000	1.000	0.500
0.800	1.000	1.000	0.875	1.000	1.000	1.000	1.000	0.833	0.667	1.000	0.833	0.750	0.333
1.000	1.000	1.000	1.000	1.000	0.667	0.250	1.000	0.667	0.667	0.800	0.833	0.750	1.000
0.800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.833	0.333	0.600	1.000	1.000	0.333
1.000	1.000	0.875	1.000	0.750	1.000	0.500	1.000	1.000	1.000	0.600	0.667	0.500	0.667
1.000	1.000	1.000	0.750	0.500	0.833	1.000	0.833	0.833	0.833	1.000	0.833	0.750	0.500
1.000	1.000	1.000	0.875	1.000	1.000	0.750	0.667	0.833	0.833	0.800	0.500	1.000	0.500
1.000	0.750	0.875	0.875	1.000	0.833	0.750	0.833	0.833	0.667	1.000	1.000	0.750	0.500
1.000	1.000	0.750	0.875	0.750	0.833	0.250	1.000	0.333	0.833	0.800	1.000	0.500	0.500
1.000	0.750	0.750	0.625	1.000	0.833	1.000	0.833	0.833	0.833	0.600	0.833	0.250	0.500
1.000	1.000	1.000	0.875	1.000	0.333	1.000	0.833	0.833	0.500	0.600	0.833	0.250	0.333
0.800	1.000	0.750	1.000	1.000	0.500	1.000	0.500	0.667	0.667	0.800	0.333	1.000	0.500
0.800	0.875	0.375	0.750	1.000	0.833	1.000	0.667	0.833	0.833	0.800	0.667	0.750	0.500
1.000	0.750	0.750	0.625	1.000	0.833	0.750	1.000	0.333	1.000	0.600	1.000	0.250	0.500
1.000	1.000	1.000	1.000	1.000	0.333	1.000	0.500	0.667	0.667	0.800	0.500	0.500	0.333
0.600	1.000	1.000	0.750	0.750	0.833	0.500	0.500	0.833	0.833	0.600	0.500	0.750	0.500
0.800	0.875	1.000	0.750	1.000	1.000	1.000	0.333	0.333	1.000	0.600	0.167	0.250	0.667
1.000	1.000	0.875	0.625	0.750	0.833	0.750	0.500	0.333	0.500	0.600	0.500	1.000	0.500
0.800	0.625	0.750	0.625	0.750	0.500	0.750	0.833	0.333	0.667	1.000	0.833	0.750	0.500
0.800	0.750	0.750	0.750	1.000	0.667	0.750	0.500	0.333	0.333	0.600	0.333	1.000	1.000
0.800	0.750	0.500	0.500	0.750	1.000	0.500	0.333	0.833	1.000	0.600	0.833	0.250	0.500
0.800	0.625	0.500	0.625	1.000	0.333	1.000	0.500	1.000	0.333	0.800	0.333	1.000	0.500
0.800	0.625	0.750	0.625	0.250	0.667	0.500	0.333	0.667	0.667	0.800	0.500	1.000	0.333
0.800	0.750	0.750	0.750	0.500	0.833	0.500	0.333	0.667	0.833	0.600	0.333	0.250	0.167
1.000	0.750	0.750	0.750	0.000	0.500	0.000	1.000	0.500	0.500	0.600	0.333	0.250	0.500
1.000	0.375	0.625	0.625	0.750	1.000	0.750	0.167	0.500	0.500	1.000	0.167	0.500	0.333
1.000	0.250	0.250	0.250	0.250	0.167	1.000	1.000	0.500	0.333	0.600	0.500	0.500	0.333

Proportion of the Maximum Possible Points Obtained by Students' Responses

P = 1 | 1 < P ≤ 0.6 | 0.6 ≤ P ≤ 0

an assessment has, the higher its reliability tends to be. Thus, with only half of the mini-experiments on the assessment, the resulting correlation coefficient may be lower than it really is. Taking this into account, we can adjust the reliability coefficient by using the Spearman-Brown Prophecy formula. This formula produces an estimated correlation coefficient for the entire assessment based on the correlation coefficient achieved by looking at only half of the assessment (Nitko & Brookhart, 2007). In fact, this formula was designed to predict what the correlation coefficient of an assessment would be if it was augmented by a certain multiple of mini-experiments (e.g., the number of mini-experiments were doubled or tripled) (Cangelosi, 2000). The formula is

$$r_t = \frac{tr'}{(t-1)r' + 1} \tag{4.2}$$

where r_t is the estimate of what the reliability coefficient would be if the assessment were t times longer and r' is the reliability coefficient computed for the actual assessment. In terms of using the spit-halves method, we would have

$$r = \frac{2\rho_{a,b}}{\rho_{a,b} + 1} \tag{4.3}$$

where r is the estimate of the reliability coefficient of the whole assessment and $\rho_{a,b}$ is the reliability correlation coefficient computed between the two halves of the assessment.

Using the scores from the assessment of a unit on functions (see Table D.1 in Appendix D), with Split A consisting of prompts 1, 3(a), 4(a), 4(c), 4(e), 4(g), and 6, and Split B consisting of the prompts 2, 3(b), 4(b), 4(d), 4(f), 5, and 7, we have the following data sequences:

$$\begin{aligned} \text{Split } A = \{&28, 35, 33, 22, 34, 27, 26, 32, 37, 33, 28, 38, 21, 36, 28, 35, 28, 33, 20, 31, \\ &36, 29, 39, 24, 34, 25, 29, 37, 38, 26, 38, 27, 32, 34, 37, 30, 27, 30, 21, 33, \\ &34, 33, 31, 34, 23, 35, 35, 38, 34, 32\} \end{aligned} \tag{4.4}$$

$$\text{Split } B = \{33, 39, 39, 32, 40, 34, 35, 36, 38, 39, 33, 41, 26, 34, 32, 40, 24, 37, 29, 38, 37, 33, 43, 23, 42, 33, 26, 36, 42, 30, 36, 30, 40, 42, 42, 41, 36, 32, 17, 37, 38, 36, 42, 40, 27, 39, 42, 40, 37, 39\} \tag{4.5}$$

Using these two sequences, the Pearson product-moment correlation coefficient is calculated to be $\rho_{A,B} = 0.78$. Applying the Spearman-Brown Prediction formula to this coefficient, we get the estimated reliability correlation coefficient to be

$$r = \frac{2\rho_{A,B}}{\rho_{A,B} + 1} \approx \frac{2(0.78)}{0.78 + 1} \approx \frac{1.56}{1.78} \approx 0.88, \tag{4.6}$$

which is a fairly strong correlation and confirms that the internal consistency of the results of the assessment is high.

The Kuder-Richardson Methods. Another method that estimates the reliability of responses from a single administration of an assessment is to use one of the formulas developed by Kuder and Richardson (Cangelosi, 2000; Ebel, 1965; Gronlund & Linn, 1990; Nitko & Brookhart, 2007; Popham, 1981). Nitko and Brookhart (2007) explain that the two most common formulas are the Kuder-Richardson formula 20 (KR20) and the Kuder-Richardson formula 21 (KR21). However, these two formulas can only be used when responses to a mini-experiment are scored dichotomously (i.e., either one point for a correct response and zero points for an incorrect response) (Cangelosi, 2000; Nitko & Brookhart, 2007). Since the mini-experiments in the assessment for a unit on functions are not dichotomous, these two formulas will not be explored.

A more general formula developed by Kuder, Richardson, and Cronbach, which is more applicable to other types of instruments than KR20 or KR21, is coefficient α (Cangelosi, 2000; Ebel, 1965; Popham, 1981). Coefficient α is not as easy to compute (at least by hand) as KR20 or KR21, but with the aid of modern technology, it is no longer a concern. Like other reliability coefficients, coefficient α focuses on the degree to which the mini-experiments are in agreement and can be used to estimate the internal consistency of a set of mini-experiments where each could response could receive a range of points (Popham, 1981).

Coefficient α is calculated by using the formula

$$\alpha = \frac{k}{k-1}\left(1 - \frac{\sum_{j=1}^{k} \sigma_j^2}{\sigma^2}\right) \tag{4.7}$$

where α is the reliability coefficient, k is the number of mini-experiments, σ^2 is the variance of the test scores, and σ_j^2 is the variance of the scores from Prompt $\#j$ (Cangelosi, 2000, p. 615).

Using the results from the assessment for a unit on functions (see Appendix D) and a statistics calculator, the variance for each mini-experiment was determined along with the variance of test scores. With a total of 14 mini-experiments, coefficient α for this assessment was determined to be $\alpha \approx 0.85$. Once again, this indicates a strong correlation and we can conclude that the internal consistency of our assessment is fairly high. Table 4.4 shows the results of the various calculations needed to determine coefficient α.

The Standard Error of Measurement. Knowing the reliability coefficient of an assessment is helpful in determining its internal consistency, but recall that the purpose of any assessment is to measure individual student achievement. That is, the goal of any assessment is to accurately measure a student's true score, S_t. However, as demonstrated in Equation (4.1) on page 60 ($S_o = S_t + E$) an individual's true score is not what is observed in a measurement, there is some error involved. If we could somehow estimate E, we could use Spearman's equation to find an estimate for an individual's true score:

Table 4.4

The number of mini-experiments (k), the variance of test scores (σ^2), and the sum of variance for individual mini-experiments ($\sum \sigma_j^2$) used to determine the reliability coefficient α for an assessment for a unit on functions.

k	σ^2	$\sum \sigma_j^2$	α
14	101.59	21.6572	0.85

$$S_t = S_o + E. \tag{4.8}$$

Instead of using reliability coefficients to simply determine whether an assessment provides satisfactory or unsatisfactory results, this coefficient can be used to estimate the error (E) in Spearman's equation. This can be accomplished by calculating the *standard error of measurement* (SEM) (Cangelosi, 2000). Popham (1981) explains the SEM is a reflection of the variability of an individual's scores if they took the assessment over and over again. However, since it is not practical (and perhaps not effective) to continually readminister the same assessment, the SEM can be estimated in the following way:

$$\sigma_E = \sigma_x \sqrt{1 - r_x} \tag{4.9}$$

where σ_E is the estimate for the standard error of measurement for Assessment-x, σ_x is the standard deviation of scores from Assessment-x, and r_x is the reliability coefficient for Assessment-x (Cangelosi, 2000).

The standard deviation for the test scores given in Appendix D is $\sigma_x \approx 10.45$ and if we use the reliability coefficient $\alpha \approx 0.85$, we get the estimated SEM for the assessment for a unit on functions to be

$$\sigma_E = \sigma_x \sqrt{1 - r_x} = 10.45\sqrt{1 - 0.85} = 10.45\sqrt{0.15} \approx 3.92. \tag{4.10}$$

Assuming that the distribution of reliability error is normal, we can use this SEM to make some confidence-band assertions in order to interpret scores from our assessment. For example, using the properties of the normal curve, we can conclude that in a theoretical test/retest experiment we could expect that retest scores should be within about 4 points of the original observed scores 68% of the time (i.e., within ± 1 SEM) and within about 8 points of the original observed scores 95% of the time (i.e., within ± 2 SEM).

Another way that we can use the SEM to interpret scores is to refer to an individual's test score. For example, Student 15's test score was 60. With an estimated SEM of about 4, we can conclude that we are 68% confident that Student 15's true score is between 56

and 64 and we are 95% confident that their true score is between 52 and 68. Thus, with a reliability coefficient and an associated standard error of measurement, we can begin to understand and interpret the results of any given assessment.

Observer Consistency

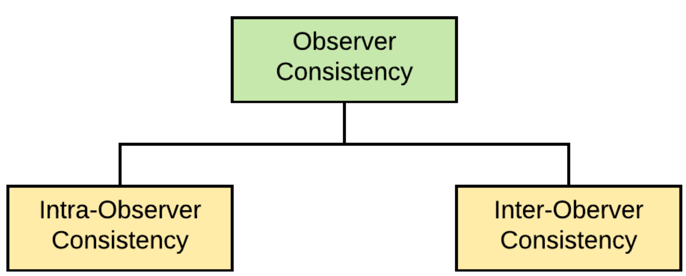
Since reliability refers to the results of a particular assessment, how those results are obtained (i.e., how they are assigned and who is assigning them) should be considered when determining a measurement's reliability. That is why Cangelosi (2000) advises that in order to be reliable, a measurement must also have observer consistency. An observer is the person selected to review the mini-experiments and assign a score to an individual's response. Typically, this is the instructor who is administering the assessment, but can be another person assigned to score the mini-experiments.

Cangelosi (2000) defines observer consistency to be the degree to which an observer faithfully follows the given rubric to a mini-experiment. Since rubrics for dichotomous mini-experiments require very little judgment and are almost always consistent, observer consistency focuses on the mini-experiments that have rubrics that leave some room for judgment by the observer. Figure 4.6 illustrates that observer consistency is dependent on both *intra*-observer consistency and *inter*-observer consistency.

Intra-Observer Consistency. Intra-observer consistency refers to the degree in which a particular observer faithfully follows a mini-experiment's rubric so that the obtained results are not influenced by outside factors (e.g., the observer's mood or the way an ob-

Figure 4.6

Observer consistency depends on intra- and inter-observer consistency.



servation is conducted) (Cangelosi, 2000). In order to examine intra-observer consistency for a particular observer they will need to initially score the responses for a set of mini-experiments and then randomly select a subset of these responses. Then, at a later date, they need to rescore the randomly selected subset and compute a Pearson product-moment correlation coefficient between the scores of the two sets. This correlation coefficient measures how consistent the scoring of the selected responses was between the two different scorings.

After the initial scoring of the responses to the assessment for a unit on functions, a random subset of 30 students was selected and then their responses were rescored several weeks later. The two data sequences O and R are the test scores that resulted from the original scoring and the rescoring respectively.

$$O = \{61, 54, 74, 61, 68, 75, 61, 47, 75, 52, 62, 82, 55, 73, 80, 56, 74, 76, 71, 63, \\ 62, 38, 70, 72, 69, 74, 77, 78, 71, 71\} \tag{4.11}$$

$$R = \{50, 46, 75, 62, 71, 74, 63, 61, 77, 49, 62, 82, 52, 67, 74, 51, 70, 74, 68, 65, \\ 59, 39, 68, 71, 67, 74, 74, 76, 68, 71\} \tag{4.12}$$

When calculating the Pearson product-moment correlation coefficient, we get $r \approx 0.92$, a strong correlation which suggests a high level of intra-observer consistency.

Inter-Observer Consistency. The complement to intra-observer consistency for observer consistency is inter-observer consistency. Cangelosi (2000) explains that inter-observer consistency refers to the degree in which different observers faithfully follow a mini-experiment's rubric so that the obtained results are not influenced by who is making the observations. As with intra-observer consistency, in order to measure inter-observer consistency, we will need to compute a Pearson product-moment coefficient for two sets of scores. The first set should be a randomly selected subset of scores obtained when the original observer scored the responses to the set of mini-experiments. The second set of scores should be the scores obtained when a second, trained observer scores the responses for the subset that was selected. The "trained observer" should be someone who is familiar with the subject-content, learning levels, and, if possible, the students involved in the measurement. At the very

least, they should be someone who has been trained to use the observer's rubric (Cangelosi, 2000).

Using the same randomly selected subset of 30 students, two sets of scores were obtained from the original observer's scores and the scores from a second, trained observer who is familiar with the subject-content, learning levels, and the use of the observer's rubric. The data sequence O is the set of scores from the original observer and the data sequence S is the set of scores from the second observer.

$$O = \{61, 54, 74, 61, 68, 75, 61, 47, 75, 52, 62, 82, 55, 73, 80, 56, 74, 76, 71, 63, 62, 38, 70, 72, 69, 74, 77, 78, 71, 71\} \tag{4.11}$$

$$S = \{36, 38, 48, 44, 51, 66, 50, 34, 74, 60, 63, 78, 52, 75, 71, 33, 68, 66, 50, 65, 49, 21, 66, 71, 60, 66, 69, 72, 48, 64\} \tag{4.13}$$

Although significantly different, the two sets of scores are still relatively consistent (i.e., the responses are receiving corresponding high/low marks). In fact, the Pearson product-moment correlation coefficient is $r \approx 0.81$, which is still a fairly strong correlation and leads us to conclude that the inter-observer consistency is also fairly high.

Examining Mini-Experiment Effectiveness

The purpose of assessing the validity (i.e., the relevance and reliability) of a measurement is to provide information on its validity and overall results. This gives insight into how consistent the set of mini-experiments are with each other, but does little to provide information about individual mini-experiments. Thus, the purpose of this stage of the validation study is to assess whether mini-experiments are (a) contributing to the accuracy of the measurement (b) not contributing to the accuracy of the measurement, or (c) detracting from the accuracy of the measurement (Cangelosi, 2000).

Ebel (1965) teaches that the analysis of student responses to individual mini-experiments is a powerful tool for improving assessments. This is because performing such an analysis helps identify which mini-experiments may be too easy or too difficult and which may fail to discriminate clearly between students who have obtained a high level of achievement from

those students who have not obtained as high a level of achievement (Cangelosi, 2000; Ebel, 1965). Cangelosi (2000) adds that mini-experiments that discriminate in favor of higher levels of achievement of a given objective and against lower levels of achievement of the same objective are considered *effective mini-experiments.*

If a measurement is teeming with ineffective mini-experiments, it can hardly be valid, but a valid measurement may have some faulty or ineffective mini-experiments. Likewise, an invalid measurement may still have some effective mini-experiments. Thus, instead of looking solely at the validity of a measurement, it may be beneficial to examine the effectiveness of its mini-experiments.

In order to assess the effectiveness of a given mini-experiment, a mini-experiment's index of discrimination (D_j), index of difficulty (P_j), and index of efficiency (E_j) will need to discussed and calculated.

Index of Discrimination

Popham (1981) explains that in principle, a mini-experiment's index of discrimination "reflects the relationship between examinees' responses on the total test and their responses on a particular test item" (p. 295). In other words, Nitko and Brookhart (2007) explain, a mini-experiment's index of discrimination is the difference between the fraction of correct responses given by students whose responses generally scored higher and the fraction of correct responses given by students whose responses generally scored lower. They go on to describe the index of discrimination as the extent to which a mini-experiment is able to differentiate between the students who have obtained a high level of achievement and those students who have obtained a lower level of achievement.

If a mini-experiment's index of discrimination is to differentiate between students' levels of achievement, the first step in determining that index is to divide the scores from a measurement into three groups: higher scores (H), in-between scores, and lower scores (L) (Cangelosi, 2000; Ebel, 1965; Gronlund & Linn, 1990; Nitko & Brookhart, 2007; Popham, 1981). Ebel (1965) suggests selecting about 27% of the scores being in the higher group with the same number in the lower group. However, Cangelosi (2000) points out that even

with a highly valid measurement, it will only discriminate between achievement levels if the scores from the measurement are markedly different. Thus, he suggests selecting Groups H and L such that scores from Group H are significantly higher than those from Group L.

Once the groups have been selected, the next step is to determine the proportion of responses in each group that are correct. If the responses are dichotomously scored, simply counting the number of correct responses in each group would suffice. If, however, the responses are given weighted scores then each response will have a degree of correctness and simply counting correct responses will not suffice. In either case, however, the degree of correctness of the responses in each group for a given mini-experiment can be found by the following formulas:

$$PH_j = \frac{\sum_{i=1}^{h} j_i}{hw_j} \tag{4.14}$$

and

$$PL_j = \frac{\sum_{i=1}^{l} j_i}{lw_j} \tag{4.15}$$

where PH_j is the average proportion of points the higher group scored for Mini-experiment j, PL_j is the average proportion of points the lower group scored for Mini-experiment j, h is the number of students in Group H, l is the number of students in Group L, w is the maximum number of points possible for a student's response to Mini-experiment j, and j_i is the score for Mini-experiment j obtained by the i^{th} student's response from the appropriate group (Cangelosi, 2000).

With the formula for the average proportions PH_j and PL_j, calculating the index of discrimination for Mini-experiment j (D_j) is just a matter of finding the difference

$$D_j = PH_j - PL_j. \tag{4.16}$$

For the mini-experiment analysis of the assessment for a unit on functions, Group H was selected such that scores equal to or above 74/82 were included in the group while

scores that were equal to or below 61/82 were added to Group L. The result was that Group H included 15 scores and Group L included 16 scores. Table 4.5 on the following page shows PH_j, PL_j, and D_j for each mini-experiment on the assessment.

Ebel (1965) gives the following guidelines for interpreting a mini-experiment's index of discrimination: (a) a mini-experiment with an index between 0.40 and 1.00 ($0.40 \leq D_j \leq 1.00$) suggests that it is very effective, (b) an index between 0.30 and 0.40 ($0.30 \leq D_j < 0.40$) suggests a moderately effective mini-experiment, but that it could possibly benefit from some minor revisions, (c) an index between 0.20 and 0.30 ($0.20 \leq D_j < 0.30$) suggest the mini-experiment is only marginally effective and refinement should be considered, and (d) an index between -1 and 0.20 ($-1 \leq D_j < .02$) suggests the mini-experiment is inadequately effective and should be modified or rejected (p. 364).

Based on this interpretation of the index of discrimination, there were four mini-experiments from our assessment that should be considered inadequately effective, five that should be considered marginally effective, five that should be considered marginally effective, and none that should be considered very effective. However, as Cangelosi (2000) and Popham (1981) point out, the ability for a mini-experiment to discriminate is highly related to its level of difficulty. This is because the level of difficulty affects the proportion of responses that receive higher marks. For example, if a mini-experiment is very easy then most responses will receive full (or close to full) marks. If this happens, then the index of discrimination for that mini-experiment will be close to 0. That is why it is important to discuss a mini-experiment's index of difficulty.

Index of Difficulty

If a mini-experiment is dichotomously scored, then its index of difficulty (P_j) is simply the proportion of correct responses to the total number of responses and is typically expressed as a percentage (Ebel, 1965; Gronlund & Linn, 1990; Nitko & Brookhart, 2007; Popham, 1981). Determining the measure of a mini-experiment's difficulty in this way, in a sense, makes this an inverse measurement. That is, the higher the numerical value, the lower the difficulty (i.e., the easier it is) (Ebel, 1965).

Table 4.5

The average proportion of points for Mini-experiment j obtained by responses given by the 15 students whose responses were generally higher (PH_j), the average proportion of points for Mini-experiment j obtained by responses given by the 16 students whose responses were generally lower (PL_j), and the index of discrimination ($D_j = PH_j - PL_j$) for Mini-experiment j.

j	PH_j	PL_j	D_j	j	PH_j	PL_j	D_j
1	0.96	0.86	0.10	4(d)	0.96	0.68	0.28
2	0.84	0.71	0.13	4(e)	0.94	0.58	0.36
3(a)	0.93	0.56	0.37	4(f)	0.93	0.67	0.27
3(b)	0.88	0.49	0.39	4(g)	0.82	0.48	0.34
4(a)	0.90	0.73	0.17	5	0.99	0.69	0.30
4(b)	0.96	0.86	0.10	6	0.98	0.71	0.26
4(c)	0.92	0.69	0.29	7	1.00	0.75	0.25

However, from a mini-experiment analysis perspective, the index of difficulty is calculated using only the data from the higher (H) and lower (L) groups, making the assumption that the responses from the students in the in-between group will follow the same pattern (Gronlund & Linn, 1990). In particular, if we use the average proportions of points from each group (i.e., PH_j and PL_j), then we will also take into account those mini-experiments that have a weighted score. Using these statistics, a mini-experiment's item of difficulty is determined by calculating the mean between these two averages. That is,

$$P_j = \frac{PH_j + PL_j}{2} \tag{4.17}$$

where P_j is the index of difficulty for Mini-experiment j, PH_j is the average proportion of points the higher group scored for Mini-experiment j, and PL_j is the average proportion of points the lower group scored for Mini-experiment j (Cangelosi, 2000).

A mini-experiment's index of difficulty necessarily falls between 0 and 1 (i.e., $0 \leq P_j \leq 1$), where an index of 1 indicates every responses received full marks and an index

of 0 indicates that every response failed to score any points. Therefore, the closer to 1 a mini-experiment's index of difficulty is, the easier it is and the closer to 0 the index is, the more difficult it is. If we think about the index of difficulty as a percentage for mini-experiments that are dichotomously scored, then an index of 0.65 indicates that 65% of the responses were correct. Although not completely analogous, this holds similarly with mini-experiments that have a weighted score. With this relationship in mind, mini-experiments with an index of difficulty between 0.80 and 1.00 could be considered easy, an index between 0.50 and 0.80 could be considered moderately difficult, and those with an index less than 0.50 could be considered difficult.

Using the same statistics calculated when determining the index of discrimination, the index of difficulty was determined for each of the mini-experiments in our assessment for a unit on functions. Table 4.6 on the next page shows that eight mini-experiments should be classified as easy, six should be classified as moderately difficult, and none should be classified as difficult.

The Relationship Between D_j and P_j

As discussed earlier, the ability for a mini-experiment to discriminate between higher levels of achievement and lower levels of achievement is highly related to its index of difficulty (Cangelosi, 2000; Popham, 1981). In order to understand this relationship, first consider how the difference between PH_j and PL_j effect D_j. In the case where a mini-experiment has a high level of positive discrimination, $PH_j > PL_j$ and the greater the difference between these statistics, the closer to 1 D_j will become. In the case where $D_j < 0$ (i.e., a negative discrimination), $PH_j < PL_j$ and the greater the difference, the closer to -1 D_j will become. Finally, in the case where the difference between PH_j and PL_j becomes smaller (i.e. $|PH_j - PL_j| \to 0$) the closer to 0 D_j will become.

Now, consider how a mini-experiment's index of difficulty P_j effects the average proportion of points for each group, PH_j and PL_j. If P_j is close to 1, then nearly every response received full marks. That is $PH_j \approx PL_j \approx 1$. If this is the case, then $D_j \approx 0$.Thus, easy mini-experiments will have indices of discrimination close to 0. Likewise, if P_j is close to

Table 4.6

The average proportion of points for Mini-experiment j obtained by responses given by the 15 students whose responses were generally higher (PH_j), the average proportion of points for Mini-experiment j obtained by responses given by the 16 students whose responses were generally lower (PL_j), and the index of difficulty ($P_j = \frac{PH_j + PL_j}{2}$) for Mini-experiment j.

j	PH_j	PL_j	P_j	j	PH_j	PL_j	P_j
1	0.96	0.86	0.91	4(d)	0.96	0.68	0.82
2	0.84	0.71	0.78	4(e)	0.94	0.58	0.76
3(a)	0.93	0.56	0.75	4(f)	0.93	0.67	0.80
3(b)	0.88	0.49	0.68	4(g)	0.82	0.48	0.65
4(a)	0.90	0.73	0.82	5	0.99	0.69	0.84
4(b)	0.96	0.86	0.83	6	0.98	0.71	0.84
4(c)	0.92	0.69	0.77	7	1.00	0.75	0.88

0, then nearly every response failed to score any points (i.e., $PH_j \approx PL_j \approx 0$). If this is the case then $D_j \approx 0$ and it appears that difficult mini-experiments also have indices of discrimination that will be close to 0. In fact, this relationship shows that a mini-experiment's index of discrimination favors those mini-experiments that have an index of difficulty equal to 0.50 ($P_j = 0.50$).

Hofmann (1975), however, developed a method in which the effectiveness of a mini-experiment could still be measured by its index of discrimination even if its index of difficulty varies widely from 0.50. He points out that (a) if $P_j \leq 0.50$, then the maximum possible value for D_j is $2P_j$ and the minimum possible value for D_j is $-2P_j$, and (b) if $P_j > 0.50$, then the maximum possible value for D_j is $2(1 - P_j)$ and the minimum possible value for D_j is $-2(1 - P_j)$. With these relationships, we can introduce a new statistic that will allow us to continue to use a mini-experiment's index of discrimination to inform us about its effectiveness. Let the maximum possible value for a mini-experiment's index of discrimination, described by Hofmann, be designated by "Max$|D_j|$."

Index of Efficiency

Hofmann (1975) noted that a mini-experiment's effectiveness is measured by how closely its index of discrimination approaches its maximum possible value. This new value, a mini-experiment's index of efficiency (E_j), is computed by looking at the ratio of a mini-experiment's index of discrimination with the index's highest possible value. That is

$$E_j = \frac{D_j}{\text{Max}|D_j|} \tag{4.18}$$

where E_j is Mini-experiment j's index of efficiency, D_j is its index of discrimination, and $\text{Max}|D_j|$ is the index of discrimination's highest possible value.

The interpretation of a mini-experiment's index of efficiency is not based on set levels as are the indices of discrimination and difficulty. Rather, the index of efficiency should be interpreted based on how close the value is to one. Similar to the interpretation of a correlation coefficient, an index of efficiency close to one indicates a strong level of efficiency and indicates that the mini-experiment is efficient at discriminating between different levels of achievement. An index near zero would indicate that the mini-experiment was not efficient in discriminating between different levels of achievement and a negative index would indicate that the mini-experiment is detracting from the accuracy of the assessment's measurement. Table 4.7 on the following page shows the index of discrimination (D_j), the index of difficulty (P_j), the maximum possible index of discrimination ($\text{Max}|D_j|$), and the index of efficiency (E_j) for each mini-experiment in the assessment for a unit on functions.

Based on this analysis, the majority of the mini-experiments contributed to the accuracy of the assessment's measurement and had high indices of efficiency. This suggests that most of the mini-experiments were efficient at discriminating against different levels of achievement. However, this analysis also indicates that mini-experiments that had relatively low indices of efficiency, Mini-Experiment 2 for example, should be examined further to determine what, if any, revisions should take place.

Conclusion

In order for a measurement instrument (e.g., an assessment) to be useful, it needs

Table 4.7

The index of discrimination (D_j) for Mini-experiment j, the index of difficulty (P_j) for Mini-experiment j, the maximum possible value for the index of discrimination ($Max|D_j|$), and the index of efficiency ($E_j = \frac{D_j}{Max|D_j|}$) for Mini-experiment j.

j	D_j	P_j	**Max**$\|D_j\|$	E_j	j	D_j	P_j	**Max**$\|D_j\|$	E_j
1	0.10	0.91	0.18	0.55	4(d)	0.28	0.82	0.37	0.76
2	0.13	0.78	0.45	0.28	4(e)	0.36	0.76	0.47	0.76
3(a)	0.37	0.75	0.50	0.74	4(f)	0.27	0.80	0.40	0.67
3(b)	0.39	0.68	0.63	0.61	4(g)	0.34	0.65	0.70	0.49
4(a)	0.17	0.82	0.37	0.45	5	0.30	0.84	0.32	0.95
4(b)	0.18	0.83	0.35	0.52	6	0.26	0.84	0.31	0.84
4(c)	0.29	0.77	0.46	0.64	7	0.25	0.88	0.25	1.00

to be usable and valid. Usability is determined by the ease and safety of administration whereas the validity of the measurement is determined by its relevance to the content it is meant to measure, its relevance to the learning levels students are expected to engage at, the consistency to which the mini-experiments agree with one another, and whether those chosen to observe (i.e., score) the mini-experiments agree with themselves and with one another (Figure 4.1 on page 61 illustrates this relationship).

After developing an assessment that focuses on several objectives for a unit on functions (see Table 4.1 on page 63 for a list of the weighted objectives and Appendix C for the assessment), an assessment analysis was performed to determine the relevance, internal consistency, and observer consistency of the assessment as a whole in order to assess its usefulness. Then, a mini-experiment analysis was performed on the individual mini-experiments to determine their indices of discrimination, difficulty, and efficiency in an effort measure how each mini-experiment contributed to the outcome of the measurement results.

The results of the assessment analysis indicated that the set of mini-experiments had a high level of relevance and reliability. The weights given to the objectives on the assessment matched the weight assigned to the objectives prior to conception of the mini-experiments. However, since the mini-experiments were designed with these weights in mind, it is not surprising to see this correlation. Two reliability coefficients were calculated using the scores from responses given by 50 Calculus I students. The first, using the split-halves method with the Spearman-Brown Prophecy formula gave us an estimated correlation of $r \approx 0.88$. The second, using Cronbach's α, derived from the work of Kuder and Richardson, gave us a correlation of $\alpha \approx 0.85$. In both cases, the high level of correlation between mini-experiments suggests that our assessment have a high level of internal consistency.

Since the purpose of any measurement instrument is to assess a student's true level of achievement (i.e., their true score) and can only be estimated by

$$S_t = S_o + E \tag{4.8}$$

where S_t is the student's true score, S_o is the student's observed score (i.e., the result of a measurement), and E is the error present in the measurement, it would be beneficial to be able to estimate E. Using the reliability coefficient this can be done by calculating the standard error of measurement (SEM). Using the reliability coefficient α, the SEM for our assessment came out to be $\sigma_E \approx 3.92$. This allowed us to estimate, with a 68% confidence interval, each student's true score to be $S_t \approx S_o \pm 4$ and, with a 95% confidence interval, to be $S_t \approx S_o \pm 8$.

The two types of observer consistency (intra-observer and inter-observer) were also assessed by computing the Pearson's product-moment correlation coefficient between the scores of a randomly selected subset of 30 students' responses and (a) the scores of those same responses after rescoring by the same observer, (b) the scores of those same responses after scoring by another trained observer. The first correlation coefficient, measuring intra-observer consistency, came out to be $r \approx 0.92$. The second, measuring inter-observer consistency came out to be $r \approx 0.81$. In both cases, the correlation is pretty high, indi-

cating a high level of observer consistency. Thus, with high levels of internal and observer consistency, it follows that the assessment being evaluated has a high level of reliability. Coupled with a high level of relevance, we can conclude that the assessment has a high level of validity.

Since the assessment was a take-home, paper and pencil assessment with a moderate scoring time the usability of the assessment should also be considered high. Since the assessment was shown to be both usable and valid, it can be concluded that the designed set of mini-experiments provides useful measurements pertaining to the achievement of the outlined objectives for a unit on functions. However, just because a measurement is useful as a whole, doesn't mean all of the mini-experiments provided accurate information regarding the level of achievement. Thus, an analysis of the individual mini-experiments is needed in order to gather all the information provided by the assessment.

A mini-experiment's index of discrimination (D_j) measures the ability of Mini-experiment j to discriminate between high and low levels of achievement. In other words, can a mini-experiment determine if a student has achieved the objective, at the learning level indicated, it was designed to measure. Indices can range between -1 and 1, where an index of $0.40 \leq D_j \leq 1$ indicates an effective mini-experiment. If indices fall below 0.2, then it is suggestive of a mini-experiment that is ineffective and should be removed or revised. The results of the mini-experiment analysis indicated that Mini-experiments 1, 2, 4(a), and 4(b) are ineffective and that revision or removal should be considered. However, a mini-experiment's index of discrimination is tied directly to its index of difficulty.

A mini-experiment's index of difficulty (P_j) measures how difficult a mini-experiment is. This index is determined by the number of correct responses (or by the proportion of points earned if it is a weighted mini-experiment) and can very from easy ($P_j \geq 0.8$) to difficult ($P_j < 0.5$). Based on the results of the mini-experiment analysis, Mini-experiments 1, 4(a), 4(b), 4(d), 4(f), 5, 6, and 7 are considered easy, Mini-experiments 2, 3(a), 3(b), 4(c), 4(e), and 4(g) are considered moderately easy ($0.50 \leq P_j < 0.8$). Since there were no difficult mini-experiments, this suggests that some of these mini-experiments should

be modified in order to assess higher levels of achievement. However, since the index of discrimination favors mini-experiments with an index of difficulty closer to 0.50, another measurement is needed for those mini-experiments whose indices of difficulty vary widely from this ideal.

A mini-experiment's index of efficiency (E_j) takes into account its index of difficulty in order to assess its efficiency. Because the maximum possible index of discrimination is limited to a certain value based on a mini-experiment's index of difficulty, the index of efficiency is a measure of the how much of the maximum possible discrimination index is achieved by the mini-experiment. The maximum possible index of discrimination (Max$|D_j$—) was determined for each mini-experiment (using P_j) and then E_j was calculated for each. The calculated indices of efficiency for the mini-experiments indicated that almost all were efficient in discriminating between different levels of achievement; only one mini-experiment had an index of efficiency that was relatively lower than the others and that was Mini-experiment 2, with $E_2 = 0.28$. This suggests that this mini-experiment may not be as effective as it could be and should be considered for revision.

Overall, the mini-experiments seemed to be quite efficient at discriminating between higher and lower levels of achievement. However, with the high number of mini-experiments classified as easy and with no mini-experiments classified as difficult, future versions of this assessment will require some adjustments. One such possible adjustment would be to remove or revise Mini-experiment 2. With $D_2 = 0.13$, $P_2 = 0.78$, and $E_2 = 0.28$, it would suggest that is a moderately easy mini-experiment that isn't very effective or efficient. This mini-experiment asks students to distinguish between examples and non-examples of sets that are functions. Perhaps one possible revision would be to change the examples and non-examples given. Depending on the choices, we could increase or decrease the index of difficulty which will then change its indices of discrimination and efficiency.

The development of a useful assessment is a challenging endeavor because attention needs to be given to establishing goals and objectives, developing mini-experiments that are relevant to those goals and objectives, develop rubrics that allow for scoring consistency,

assess the validity and usability of the developed measurement instrument, and then revise any ineffective or inefficient mini-experiments. However, with a useful set of efficient mini-experiments, learning the true level of achievement for each student becomes slightly more attainable.

CHAPTER 5

Summary

Mathematics is best described by exploring its defining characteristics rather than by a formal definition (Milgram, 2007). The characteristics that set mathematics apart from other disciplines, according to Milgram (2007), are (a) its dedication to precision and (b) stating and then solving well-posed problems. The precision found in mathematics is the foundation for formulating definitions, establishing conjectures, proving theorems, and is part of what mathematics beautiful (Cook, 2002; Devlin, 2000). Additionally, problems that precisely defined and whose solutions open up new views and opportunities are the driving force behind building the foundations of mathematics and exploring new mathematical frontiers (Milgram, 2007; Stewart, 1992). Thus, any endeavor to teach mathematics should be couched in these defining attributes and establish mathematical knowledge using the foundations of logic and reasoning.

However, the imitation mathematics being taught in schools, termed "schoolmath" by David Fowler (1994), lacks the precision and well-posed problems demanded by the mathematical community at large (Edwards & Ward, 2008; Fowler, 1994; Milgram, 2007; Vinner, 2002). For example, many mathematical definitions, essential to the axiomatic structure of mathematics, are either lacking in precision, or are not even given to students (Edwards & Ward, 2008; Milgram, 2007; Vinner, 2002). Furthermore, Milgram (2007) points out that many problems in schoolmath are not well-posed as they frequently make hidden assumptions not apparent to students and often fail to adhere to appropriate mathematical definitions. Then, to further distance schoolmath from mathematics, too many educators rely on the epistemology of religion, rather than that of mathematics, by appealing to "rules" and textbooks as explanations to the why's of mathematics rather than using the logic and reasoning that established those "rules" to begin with.

The concept of function, essential to the study of calculus, is one of the concepts that

frequently lacks the precision required by mathematics (Carlson, 1995; Oehrtman et al., 2008; Tall, 1996; Thompson, 1994). Carlson (1995) explains that many students, even those considered successful, view functions as single algebraic formulas and have difficulty understanding the language and notation surrounding this important concept. Oehrtman et al. (2008) add that such a view is not effective in building strong mathematical foundations and interpretations of the concept. What is needed then, is an approach to teaching about functions that focuses on the conceptual aspects of a function and that allow students to construct the concept themselves and formulate their own definition.

Advantages of Leading Students to Construct the Concept of Function

The theory that an individual's mathematical knowledge is actively constructed and not passively received is generally accepted by the mathematics education community as a basis for how students best learn mathematics (Kilpatrick, 1987; Lerman, 1989). In fact, Tall and Vinner (2004) use this theory as the basis for explaining how students construct their own images of a given concept based on experiences they have with that concept. However, these concept images can, at times, conflict with with the formal concept definition established by the mathematics community. When this happens, they explain, students can struggle with learning about a given concept or formal theory. That is why Nordlander and Nordlander (2012) explains that when students construct a given concept, the relationship between the concept image and the formal concept definition should be emphasized. In other words, leading student to formulate a precise, mathematical definition should be a part of their concept construction.

Focusing on how students deal with these concept images mentally, Dubinsky, Harel, and McDonald (see Dubinsky and Harel, 1992 and Dubinsky and McDonald, 2001) extend the work of Jean Piaget to establish what they term APOS theory. Their theory categorizes an individual's mental constructions hierarchically as either a(n) (a) action, (b) process, (c) object, or (d) schema. An individual with an action conception of a concept thinks only of the calculations or physical manipulations that pertain to that concept (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Thompson, 1994). Breidenbach et al. (1992) explain that

with a process conception, students begin to think of a concept as an action, but without the need to run through a string of individual steps. An object conception is achieved when an individual can treat a process as a totality and apply any number of transformations to it (Dubinsky & McDonald, 2001). Finally, Dubinsky and McDonald (2001) explain that a schema is the collection of actions, process, objects, and other schema that are joined together by a set of characteristics that frame a given concept.

In order to foster deep, meaningful foundations of the concept of function, students should move from an action conception of function to at least a process conception (Breidenbach et al., 1992; Oehrtman et al., 2008; Tall, 1996; Thompson, 1994). However, moving from an action to a process conception is rarely made in a linear fashion so measuring the extent to which a student's conception has changed is difficult (Dubinsky & Harel, 1992). One way to overcome this difficulty is to consider the student's ability to think of a function as a set of ordered pairs. Dubinsky and Harel (1992) explain that this point of view is a good indicator for measuring an individual's process conception of function. Thompson (1994) indicates that there is a lot of criticism with the ordered-pair notation of a function because some believe that this definition is not meaningful to students. However, such a point of view can be meaningful if students' concept image of a function is in harmony with this precise, formal mathematical definition.

Students begin to reconcile their preexisting concept image with the formal, ordered-pair definition of function when they are lead to construct the concept of function using examples and non-examples that follow this definition. By allowing students to use inductive reasoning to distinguish among examples and non-examples of functions, they are lead to recognize the characteristics that define the concept of function. With these defining characteristics delineated, students then use them to formulate, for themselves, the formal, ordered-pair definition of function. In this way, students are mentally interacting with the concept of function at a higher level and moving from an action conception to a process or even an object conception. In order to help students make this mental progress, they need lessons that lead them to construct concepts and formulate and comprehend definitions.

Development and Implementation of a Unit on Functions

Prior to developing lessons that lead students to construct concepts and comprehend definitions, a learning goal and a set of objectives that outline that goal need to be established. In fact, Cangelosi (2003) instructs that every teaching unit should have (a) a learning goal, (b) a set of objectives defining the learning goal, (c) a string of learning activities planned to help students achieve the objectives, and (d) means to assess student achievement of the learning goal. The learning goal declares the overall purpose of the teaching unit and should indicate what students are expected to learn. Figure 3.1 on page 27 states the learning goal for the unit on functions that was developed for this study. Additionally, a set of objectives, detailing the mathematical content and learning levels students are expected to interact with, should outline the learning goal. Table 3.3 on page 37 present the objectives that outline the learning goal for the unit of functions.

Once the learning goal and accompanying objectives have been established, the next step is to develop the string of learning activities that lead students to achieve those objectives. Although a particular order is not required, Cangelosi (2003) and Kohler and Alibegović (2015) explain that it is often beneficial to construct concepts before introducing formal conceptual names or providing formal definitions. Thus, learning activities that lead students to construct concepts will typically be presented before learning activities that lead students to deepen their comprehension of formal definitions or lead them to develop knowledge about specific notation. Following this recommendation, the lessons in the unit on functions first lead students to construct the concepts of relation and function and then help them comprehend the definitions resulting from this construction. Students are then lead to deepen their comprehension of the definitions of domain, codomain, and range of a function followed by instruction on the concepts of injective, surjective, and bijective function.

Cangelosi (2003) asserts that lessons for both construct-a-concept and comprehension-and-communication objectives follow a four-stage process, although the stages differ significantly. Construct-a-concept lessons consist of (a) sorting and categorizing, (b) reflecting and

explaining, (c) generalizing and articulating, and (d) verifying and refining while the stages of a comprehension-and-communication lesson are (a) sending the message to the students, (b) rephrasing and explaining the message, (c) questioning students about specifics in the message, and (d) providing feedback to students about their responses to questions raised in the previous stage. The learning activities designed to lead students to construct the concepts of relation and function and help students comprehend these and other definitions are developed using these stages and are included as a set of guided notes in Appendix A.

The unit on functions, together with a unit on the language and theory of sets, was implemented as a review unit during the first two weeks of the semester to a class of 50 Calculus I students. During the presentation, it became clear the the majority of students had a concept image of function that only entailed an action conception. By including examples of functions, using a variety of representations, that portrayed a function as a relation from a set A to a set B (i.e., a set of ordered pairs that is a subset of $A \times B$), students began to identify the necessary characteristics that make a set a function, namely: (a) the set needed to be a relation from a set A to a set B and (b) every element x in the set A needed to be paired with one and one element y in the set B such that the ordered pair (x, y) was an element of the function. With these attributes identified, students formulated a set-theoretic definition of function that resembled the ordered-pair definition found in Figure 3.7 on page 42.

Through group and whole-class discussions, students also deepened their comprehension of the definitions for domain, codomain, and range of a function and for the definitions of injective, surjective, and bijective function. Their progress was evidenced by the addition of sets, identified as possible domains of a given function, that could modify a set from being a function to not being a function and vice versa. Additionally, choices of sets that were proffered as examples and non-examples of injective, surjective, and bijective functions indicated a deepening comprehension of not only the different types of functions, but of the definitions for domain, codomain and range of a function.

The final element of a teaching unit is a means to assess student achievement of the

stated learning goal. With this in mind, after students completed the string of learning activities for this unit on functions, they were given an assessment that measured their achievement of the objectives outlining the unit's learning goal. The results of this assessment indicated that, not only did most of the students have a high level of achievement for all of the established objectives, but that they also progressed in developing a higher level of conception for the concept of function. For example, student responses demonstrated an understanding of the concept of function and a comprehension of the formal definition by scoring relatively high on prompts that required students to discriminate among examples and non-examples of functions and that required them to use the definition of function to determine and explain if a given set of ordered pairs was or was not a function. However, these results also indicated that some students still viewed functions as relations between subsets of real numbers and had difficulty viewing functions in a more abstract way.

Developing a Valid Measurement Instrument

In order for an assessment to provide valid and usable results, it must be relevant and reliable and the mini-experiments it contains should be effective (Cangelosi, 2000). Cangelosi (2000) explains that relevant measurements address the mathematical content indicated by the unit's objectives and require students to interact with that content at the indicated learning level. Table 4.2 on page 71 categorizes each of the mini-experiments in the assessment for the unit on functions by the content it covers and the learning level its designed to measure. Additionally, the weight for each category (content and learning level) is calculated and compared to the assigned weight established when the objectives were designed. Thus, the assessment administered to the class of 50 calculus students is relevant with respect to content and learning level achievement.

Reliable instruments provide consistent or similar measurement results when the instruments are given under different conditions (Cangelosi, 2000; Gronlund & Linn, 1990; Lewis, 1975; Nitko & Brookhart, 2007; Popham, 1981). In particular, since reliability is determined by measurement results, Cangelosi (2000) explains that in order for an instrument to be reliable, it needs to have internal and observer consistency. By internal consistency,

he means the extent that the measurement results correlate positively. In other words, an internally consistent instrument would have a higher proportion of responses receiving higher scores for easier prompts and a lower proportion of responses receiving higher scores for more difficult prompts. Since the mini-experiments in the assessment for the unit on functions are weighted (i.e. not dichotomously scored) the more applicable coefficient α, designed by Kuder, Richardson, and Cronbach, was used to measure the instruments reliability coefficient (Cangelosi, 2000; Ebel, 1965; Popham, 1981). With a reliability coefficient of $\alpha \approx 0.85$ and a standard error of measurement about equal to 3.92, the results indicate that the assessment for the unit on functions has a very high level of internal consistency.

Cangelosi (2000) explains that observer consistency is the degree to which an observer faithfully follows a mini-experiment's rubric and is divided into intra-observer consistency (the level of faithfulness to a rubric by an observer regardless of any outside factors) and inter-observer consistency (the level of faithfulness to a rubric regardless of who the observer is). Intra-observer consistency is calculated by determining the correlation coefficient between the scores given by an observer of a sample of responses and the scores given by the same observer of the same sample responses scored at a different time. Inter-observer consistency is calculated by determining the correlation coefficient between the scores given by an observer of a sample of responses and the scores given by a different observer of the same sample responses. The results of the observer consistency analysis on the assessment for a unit of functions found that it had a high intra-observer consistency of $r \approx 0.92$ and a high inter-observer consistency of $r \approx 0.81$.

The results of the analysis indicate that the the assessment developed to measure the level of student achievement of the objectives outlining the learning goal for the unit on functions is relevant and reliable and is therefore a valid measurement instrument. However, individual mini-experiments that make up the assessment can be examined to determine if they are effective. The measure of a mini-experiment's effectiveness is derived from its indices of discrimination (D_j), difficulty (P_j), and efficiency (E_j) (Cangelosi, 2000; Ebel, 1965; Gronlund & Linn, 1990; Nitko & Brookhart, 2007; Popham, 1981).

Nitko and Brookhart (2007) explain that a mini-experiment's index of discrimination is the extent to which a mini-experiment can discriminate between students who have developed a high level of achievement of a particular objective and students who have developed a lower level of achievement of the same objective. The index is calculated by dividing the measurement results from an instrument into three groups: a high group (H), an in-between group, and a low group (L) based on the total score the responses received. Then, the difference of the proportion of points the higher group scored for Mini-experiment j (PH_j) and the proportion of points the lower group scored for Mini-experiment j (PL_j) is calculated. The result is Mini-experiment j's index of discrimination. Table 4.5 on page 86 shows the index of discrimination for each of the mini-experiments in the assessment for the unit on functions.

Ebel (1965) describes a mini-experiment's index of difficulty is a measure of how difficult a particular mini-experiment is with a higher index indicating a lower level of difficulty. That is, the closer to one an mini-experiment's index of difficulty is, the easer it is. Cangelosi (2000) instructs that the index of difficulty is simply the average between the proportion of points the higher group scored for Mini-experiment j (PH_j) and the proportion of points the lower group scored for Mini-experiment j (PL_j). Table 4.6 on page 88 shows the index of difficulty for each of the mini-experiments that make up the assessment for the unit on functions.

Since a mini-experiment's index of discrimination is directly related to its index of difficulty, additional means of examining the effectiveness of a mini-experiment are needed (Cangelosi, 2000; Popham, 1981). Hofmann (1975) determined that the effectiveness of a mini-experiment (called its index of efficiency) is measured by how close the mini-experiment's index of discrimination is to its highest possible value. This maximum value ($\text{Max}|D_j|$), he explains, is based on the mini-experiment's index of difficulty. In particular, if $P_j \leq 0.50$, then the maximum possible value for D_j is $2P_j$ and if $P_j > 0.50$, the maximum possible value for D_j is $2(1 - P_j)$. Thus, Hofmann (1975) finds the index of efficiency (E_j) for Mini-experiment j by calculating the ratio between its index of discrimination (D_j) and its

maximum possible vale for D_j (Max$|D_j|$). Table 4.7 on page 90 shows the index of discrimination (D_j), the index of difficulty (P_j), the maximum possible index of discrimination (Max$|D_j|$), and the index of efficiency (E_j) for each mini-experiment in the assessment for a unit on functions.

The results of the mini-experiment analysis indicate that most of the mini-experiments are efficient at discriminating between higher and lower levels of achievement. However, several of the mini-experiments can be classified as easy and very few can be classified as difficult. Thus, future versions of this assessment would benefit from revising some of the mini-experiments to make them more difficult or by including additional prompts with a higher difficulty level. Even though there is a need for some minor revisions, the assessment that was developed to assess the level of student achievement of objectives defining a unit on functions was found to be valid and useful and the mini-experiments it contains were effective and discriminating between various levels of achievement.

Conclusions

The need for an emphasis on precise language in the mathematics classroom cannot be understated. Such a focus portrays mathematics as it really is and allows students to build deep, meaningful foundations to their mathematical learning. Without this precision, ambiguous usage of mathematical terms, definitions, and notation lead to poor concept image constructions that are frequently at odds with the formal definitions used by the mathematics community. This discordance limits an individual's ability to form important connections and develop adequate understanding of essential concepts and have sufficient comprehension of the definitions surrounding those concepts.

One way to focus attention on the precision required by mathematics and help students merge their pre-constructed concept images with the formal mathematical concept definitions is to lead students to construct concepts themselves, formulate definitions that resemble the formal ones, help them deepen their comprehension of those definitions, and aid them in learning to communicate with and about mathematics. In order to be effective, this process should carry through all units within a given course and, wherever possible,

through a given string of courses. This, of course, would necessitate instruction and professional development for those with the responsibility of teaching mathematics.

Finally, in order to accurately measure student achievement, valid and useable measurement instruments need to be developed that are relevant to the mathematical content and learning levels outlined by a unit's objectives and provide results that are both internally consistent and have observer consistency. Furthermore, the mini-experiments that make up the measurement instrument should have a high level of efficiency in discriminating between high and low levels of achievement. These useful instruments can help lower the error inherent in measuring student achievement and get closer to knowing what student really know about a given topic.

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APPENDICES

APPENDIX A

A Set of Guided Notes for a Unit Designed to Lead Students to Construct the Concept of Function and Comprehend Its Definition

Unit 2: A Set-Theoretic Development of Functions

Relations Between Sets

Let $A = \{\text{green, blue, yellow, red, orange, black}\}$ and $B = \{\text{Honda}, \text{Chevrolet}, \text{Ford}, \text{Porsche}, \text{Toyota}\}$. The table below shows some examples and non-examples of a relation from A to B.

Examples	**Non-Examples**
{(red, Honda), (red, Ford), (red, Porsche)}	{(green, Ford), (green, Jeep), (green, Honda)}
{(blue, Ford), (red, Ford), (black, Ford)}	{(orange, Toyota), (blue, Toyota), (gray, Toyota)}
{(blue, Porsche), (red, Honda), (black, Chevrolet)}	{(Chevrolet, black), (Porsche, blue), (Honda, red)}
{(yellow, Porsche)}	{yellow, Porsche}

1. On your own, briefly describe how the examples are similar to each other and how they differ from the non-examples.

2. With a partner or small group, compare your conjectures and briefly explain their rationale.

3. As a class, let's compare some conjectures and develop an agreed upon, commonly worded generalization of relations on the set of sets and find a way to test for relational attributes.

4. Now, let's agree upon a formal definition for a relation between two sets (or between a set and itself).

Functions

Given A, B are sets, discuss (as a class) the idea of a function from A to B.

Notation

Given A, B are sets, note that "$f : A \to B$" is read "f is a function from A to B."

With a partner or small group, list 3 - 5 examples of relations from A to B that are functions (i.e., $f \subseteq A \times B$ such that $f : A \to B$), given A, B are sets.

With the same partner or group, list 3 - 5 examples of relations from A to B that are not functions from A to B, given that A, B are sets.

As a class, discuss a few examples and non-examples and develop a way to distinguish the two.

Formulate a definition for *function*:

Definition: Function

Given $A, B \in \{\text{sets}\}$, $f : A \to B$ if and only if

Function Terminology

Given $f : A \to B$, note the related terminology:

1. The set A is the *domain* of f.
2. The set B is the *codomain* of f.
3. Note the definition of the *range* of f:

Definition: Range of a Function

Given $f : A \to B$, the *range* of the function f is the set $\{y \in B : (x, y) \in f\}$.

Examples

Examine each of the following propositions, determine whether or not it is true, indicate our choice by circling either "T" to "F," and explain why our choice is correct.

1. If $X = \{1, 2, 3, 4\}$, $Y = \{$standard lower case letters of the English alphabet$\}$, $r \subseteq X \times Y$ such that $r = \{(2,\text{“c”}), (1, \text{“t”}), (4, \text{“m”}), (3, \text{“c”})\}$, then $r : X \to Y$ such that X is the domain of r, $\{$"a," "b," "c," ..., "z"$\}$ is the codomain, and $\{$"t," "c," "m"$\}$ is the range of r.

   T F

2. If $f = \left\{\left(q, \frac{q}{q+1}\right) : q \in \mathbb{Q} - \{-1\}\right\}$, then $f : \mathbb{Q} - \{-1\} \to \mathbb{Q}$ such that $\mathbb{Q} - \{-1\}$ is the domain of f and $\mathbb{Q}$ is the range of f.

   T F

3. If $h = \{(x, h(x)) : x \in \mathbb{R}$ and $h(x) = \sqrt{x}\}$, then $h : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R}$ is the domain of h and $\{y : y \in [0, \infty)\}$ is the range of h.

   T F

Injective (One-to-One) Functions

Examine the notation and definition of an injective (or one-to-one) function:

Notation

Given $A, B \in \{\text{sets}\}$, note that "$f : A \xrightarrow{\text{1-1}} B$" is read "$f$ is an injective (or one-to-one) function from A to B."

Definition: Injective Function

Given $A, B \in \{\text{sets}\}$, $f : A \xrightarrow{\text{1-1}} B \Leftrightarrow f : A \to B$ such that if $(x_1, y_1), (x_2, y_1) \in f$, then $x_1 = x_2$.

Write the definition out in words (i.e., don't use any mathematical notation).

Explain the definition in your own words.

Compare your explanation with a partner or small group and note any similarities and differences.

With your partner or small group, list 3 - 5 examples of functions that are injective (one-to-one) and 3 - 5 examples of functions that are not.

As a class, let's discuss some of these examples and non-examples.

Surjective (Onto) Functions

Examine the notation and definition of a surjective (or onto) function:

Notation

Given a function f, "$f : A \xrightarrow{\text{onto}} B$" is read "$f$ is an *onto* function from A to B." It is also read, "f is a *surjection* from A to B."

Definition: Surjective Function

Given $A, B \in \{\text{sets}\}$, $f : A \xrightarrow{\text{onto}} B \Leftrightarrow f : A \to B$ such that the range of f is equal to B.

Write the definition out in words (i.e., don't use any mathematical notation).

Explain the definition in your own words.

Compare your explanation with a partner or small group and note any similarities and differences.

With your partner or small group, list 3 - 5 examples of functions that are surjective (onto) and 3 - 5 examples of functions that are not.

As a class, let's discuss some of these examples and non-examples.

Bijective (One-to-One and Onto) Functions

Examine the notation and definition of a bijective (or one-to-one and onto) function:

Notation

Given a function f, "$f : A \xrightarrow[\text{onto}]{\text{1-1}} B$" is read "$f$ is a *one-to-one and onto* function from A to B." It is also read, "f is a *bijection* from A to B."

Definition: Bijective Function

Given $A, B \in \{\text{sets}\}$, $f : A \xrightarrow[\text{onto}]{\text{1-1}} B \Leftrightarrow f : A \xrightarrow{\text{1-1}} B$ and $f : A \xrightarrow{\text{onto}} B$.

Write the definition out in words (i.e., don't use any mathematical notation).

Explain the definition in your own words.

Compare your explanation with a partner or small group and note any similarities and differences.

With your partner or small group, list 3 - 5 examples of functions that are bijective (one-to-one and onto) and 3 - 5 examples of functions that are not.

As a class, let's discuss some of these examples and non-examples.

APPENDIX B

A Pre-Unit Questionnaire Designed to Discover Students' Current Perceptions of the Concepts of Function, Domain, Codomain, and Range

Function Questionnaire

On your own, please respond to the following prompts using the knowledge you currently have. Don't worry about being "right" or "wrong," just respond as best you can.

1. The prerequisite for this course was a(n)
   - ACT Math score of at least 27 or equivalent SAT Math score,
   - AP Calculus AB score of at least 3 ,
   - grade of C- or better in BOTH MATH 1050 AND MATH 1060 within the last year or three consecutive semesters (including summer), or
   - satisfactory score on the Math Placement Exam.

   Without disclosing any scores or grades, which prerequisite requirement did you meet?

2. Which mathematics course do you feel helped you the most in learning about functions (this could be any mathematics course you've taken, including the ones you took prior to college)?

3. On a scale of 1 to 5 (1 being not at all and 5 being very) how comfortable are you with functions?

   1 2 3 4 5

4. What is a function?

5. What is the domain of a function?

6. What is the codomain of a function?

7. What is the range of a function?

APPENDIX C

A Set of Mini-Experiments for a Unit on Functions

Function Assessment

1. (5 points) Let $A = \{$musical instruments$\}$ and $B = \{$four legged animals$\}$. Determine which of the following sets are relations from A to B. Indicate your choice(s) by circling the letter in front of the set.

   A. {tuba, dog, trumpet, cat, violin, elephant}
   B. {(cello, zebra), (viola, yak), (trombone, emu), (clarinet, lynx)}
   C. {(piano, horse), (guitar, buffalo), (harp, lion), (oboe, puma)}
   D. {(tiger, French horn), (jackal, keyboard), (wolf, flute)}
   E. {(bassoon, deer), (drums, elk)}

Observers Rubric:
Maximum score is 5 points distributed according to the following criteria:

+1 if response indicates A is not a relation.
+1 if response indicates B is not a relation.
+1 if response indicates C is a relation.
+1 if response indicates D is not a relation.
+1 if response indicates E is a relation.

2. (5 points) Determine which of the following sets are functions. Indicate your choice(s) by circling the letter in front of the set.

   A. $f = \left\{(x, y) : x, y \in \mathbb{R} \text{ and } y = \dfrac{x-2}{x^2}\right\}$.
   B. g is a relation from {buttons on a vending machine} to {soda flavors in a full vending machine} such that $g = \{(x, y) : y$ is the flavor of soda dispensed when x is pushed$\}$.
   C. h is a relation from $(0, \infty)$ to $\mathbb{R}$ such that $h = \left\{(x, y) : y = \dfrac{1}{\sqrt{x}}\right\}$.
   D. j is a relation from {living people} to {people who every lived} such that $j = \{(x, y) :$ y is the biological child of $x\}$.
   E. k is a relation from {cars} to $\mathbb{W}$ such that $k = \{(x, y) : y$ is the number of wheels on car $x\}$.

Observers Rubric:
Maximum score is 5 points distributed according to the following criteria:

+1 if response indicates A is not a function.
+1 if response indicates B is a function.
+1 if response indicates C is a function.
+1 if response indicates D is not a function.
+1 if response indicates E is a function.

3. Let $A = \mathbb{R}$ and $B = \mathbb{N}$. Examine each of the following propositions to determine whether or not it is true; indicate your determination of the truth value by circling either "T" or "F." Then, write a paragraph that explains why your determination is correct.

   (a) (6 points) If $D = \{a + b : a \in A, b \in B\}$ then D is a relation from A to B. T F

   Observers Rubric:
   Maximum score is 6 points distributed according to the following criteria:
   +2 if response indicates why the proposition is false.
   +2 if response uses the definition of relation.
   +2 if nothing extraneous or erroneous is included in the response.

   (b) (6 points) If $E = \{(x, y) : x > y, x, y \in B\}$ then E is a relation from B to B. T F

   Observers Rubric:
   Maximum score is 6 points distributed according to the following criteria:
   +2 if response indicates why the proposition is true.
   +2 if response uses the definition of relation.
   +2 if nothing extraneous or erroneous is included in the response.

4. Examine each of the following propositions to determine whether or not it is true; indicate your determination of the truth value by circling either "T" or "F." Then, write a paragraph that explains why your determination is correct.

   (a) (4 points) Let f be a relation from $\mathbb{N}$ into $\mathbb{Q}$ such that $f = \left\{(x, y) : y = \frac{1}{x}\right\}$. If $f : \mathbb{N} \to \mathbb{Q}$ then $\mathbb{N}$ is the domain of f.

   T F

   Observers Rubric:
   Maximum score is 4 points distributed according to the following criteria:
   +2 if response indicates why the proposition is true.
   +2 if nothing extraneous or erroneous is included in the response.

(b) (4 points) Let g be a relation from $\mathbb{R} - \{0\}$ into $\mathbb{R}$ such that $g = \left\{(x, y) : y = \frac{1}{x}\right\}$. If $g : \mathbb{R} - \{0\} \to \mathbb{R}$ then $\mathbb{Z}$ is the codomain of g.

T F

Observers Rubric:
Maximum score is 4 points distributed according to the following criteria:

+2 if response indicates why the proposition is false.
+2 if nothing extraneous or erroneous is included in the response.

(c) (4 points) Let h be a relation from $\mathbb{R} - \{0\}$ into $\mathbb{R}$ such that $h = \left\{(x, y) : y = \frac{1}{x}\right\}$. If $h : \mathbb{R} - \{0\} \to \mathbb{R}$ then $\mathbb{R} - \{0\}$ is the range of h.

T F

Observers Rubric:
Maximum score is 4 points distributed according to the following criteria:

+2 if response indicates why the proposition is true.
+2 if nothing extraneous or erroneous is included in the response.

(d) (6 points) If $f = \{(x, y) : y = \sqrt{x}\}$, then $f : \mathbb{R} \to \mathbb{R}$. T F

Observers Rubric:
Maximum score is 6 points distributed according to the following criteria:

+2 if response indicates why the proposition is false.
+2 if response uses the the definition of a function.
+2 if nothing extraneous or erroneous is included in the response.

(e) (6 points) If $g = \{(x, y) : y = \sqrt{x}\}$, then $g : [0, \infty) \to \mathbb{R}$. T F

Observers Rubric:
Maximum score is 6 points distributed according to the following criteria:

+2 if response indicates why the proposition is true.
+2 if response uses the the definition of a function.
+2 if nothing extraneous or erroneous is included in the response.

(f) (6 points) If $h(x) =$ the result of a fair coin flip, then $h : \{\text{coin flips}\} \to \{\text{H, T}\}$.
T F

Observers Rubric:
Maximum score is 6 points distributed according to the following criteria:

+2 if response indicates why the proposition is true.
+2 if response uses the the definition of a function.
+2 if nothing extraneous or erroneous is included in the response.

(g) (6 points) If $j(x) =$ the ages of all students in a class, then $j : \{\text{classes at our university}\} \to \mathbb{W}$.

T F

Observers Rubric:
Maximum score is 6 points distributed according to the following criteria:

+2 if response indicates why the proposition is false.
+2 if response uses the the definition of a function.
+2 if nothing extraneous or erroneous is included in the response.

5. (8 points) Let g be a relation from A into B such that $g(x) = x^2 - 2$. Find any set A and any set B such that $g : A \xrightarrow{\text{1-1}} B$. Then write a paragraph explaining why your choices are appropriate.

Observers Rubric:
Maximum score is 8 points distributed according to the following criteria:

+2 if response indicates an appropriate choice for A.
+2 if response indicates an appropriate choice for B.
+2 if response uses the definition of an injective function.
+2 if nothing extraneous or erroneous is included in the response.

6. (8 points) Let h be a relation from A into B such that $h(x) = 3 - x^2$. Find any set A and any set B such that $h : A \xrightarrow{\text{onto}} B$. Then write a paragraph explaining why your choices are appropriate.

Observers Rubric:
Maximum score is 8 points distributed according to the following criteria:

+2 if response indicates an appropriate choice for A.
+2 if response indicates an appropriate choice for B.
+2 if response uses the definition of a surjective function.
+2 if nothing extraneous or erroneous is included in the response.

7. (8 points) Let j be a relation from A into B such that $j(x) = x^2 - 15$. Find any set A and any set B such that $j : A \xrightarrow[\text{onto}]{\text{1-1}} B$. Then write a paragraph explaining why your choices are appropriate.

Observers Rubric:
Maximum score is 8 points distributed according to the following criteria:

+2 if response indicates an appropriate choice for A.
+2 if response indicates an appropriate choice for B.
+2 if response uses the definition of a bijective function.
+2 if nothing extraneous or erroneous is included in the response.

APPENDIX D

Scores from the Set of Mini-Experiments for a Unit on Functions

Table D.1

Item-by-item scores from an assessment for a unit on functions given to 50 calculus students.

Student i	**Items**														**Test Score**
	1	2	3(a)	3(b)	4(a)	4(b)	4(c)	4(d)	4(e)	4(f)	4(g)	5	6	7	
1	5	4	3	3	4	4	2	2	4	4	2	8	8	8	61
2	5	4	3	3	3	4	4	6	6	6	6	8	8	8	74
3	5	3	5	5	4	4	2	6	6	6	3	7	8	8	72
4	4	3	2	5	3	2	1	6	5	6	3	4	4	6	54
5	5	5	5	5	3	4	4	5	6	5	3	8	8	8	74
6	5	3	6	6	4	3	1	5	2	6	3	5	6	6	61
7	4	4	4	4	4	4	3	5	5	5	3	6	3	7	61
8	5	5	5	6	4	3	3	5	5	4	3	7	7	6	68
9	5	3	6	4	4	4	3	5	5	6	6	8	8	8	75
10	5	5	5	6	4	3	4	5	5	5	3	7	7	8	72
11	4	4	3	2	4	4	4	3	4	4	3	8	6	8	61
12	4	3	6	6	4	4	4	6	6	6	6	8	8	8	79
13	5	5	1	1	3	3	2	6	3	3	2	5	5	3	47
14	5	4	6	5	4	1	3	4	4	4	6	8	8	8	70

Table D.1 (Cont'd)

Student i	Items														Test Score
	1	2	3(a)	3(b)	4(a)	4(b)	4(c)	4(d)	4(e)	4(f)	4(g)	5	6	7	
15	3	3	3	3	3	2	3	5	5	5	3	6	8	8	60
16	5	5	6	5	4	3	2	6	5	6	6	7	7	8	75
17	4	4	3	2	4	4	4	2	6	2	3	5	4	5	52
18	4	3	6	6	4	4	4	6	5	2	2	8	8	8	70
19	4	3	2	2	2	2	1	5	4	5	1	6	6	6	49
20	5	5	5	5	2	4	3	5	5	5	3	6	8	8	69
21	5	4	6	5	4	4	4	6	6	3	3	7	8	8	73
22	5	3	5	5	4	4	1	5	5	5	3	5	6	6	62
23	5	5	6	6	4	4	4	6	6	6	6	8	8	8	82
24	5	3	6	2	0	0	1	3	3	3	3	6	6	6	47
25	5	5	6	6	4	4	4	5	5	6	3	8	7	8	76
26	4	3	2	1	4	4	1	6	2	6	4	6	8	7	58
27	4	3	3	2	4	3	4	4	2	2	6	6	6	6	55
28	5	3	6	6	4	4	3	5	5	4	6	8	8	6	73
29	5	4	5	6	4	4	4	6	6	6	6	8	8	8	80

Table D.1 (Cont'd)

Student i	Items														Test Score
	1	2	3(a)	3(b)	4(a)	4(b)	4(c)	4(d)	4(e)	4(f)	4(g)	5	6	7	
30	4	5	5	5	3	3	3	3	2	4	3	5	6	5	56
31	5	3	6	4	4	4	4	6	5	3	6	8	8	8	74
32	5	3	3	3	3	3	4	5	2	3	3	5	7	8	57
33	5	3	4	6	4	4	1	6	5	5	6	8	7	8	72
34	4	4	6	6	4	4	2	6	6	6	4	8	8	8	76
35	5	5	6	6	4	3	4	6	6	6	4	8	8	8	79
36	4	4	6	6	4	4	1	6	4	5	3	8	8	8	71
37	5	4	6	6	3	1	2	5	2	5	3	7	6	8	63
38	5	3	5	5	4	4	1	2	5	3	2	7	8	8	62
39	5	3	6	3	1	4	2	1	3	2	2	2	2	2	38
40	5	3	6	4	3	2	2	6	6	6	4	8	7	8	70
41	5	5	4	2	4	4	2	6	6	6	6	7	7	8	72
42	5	4	4	3	4	3	4	6	5	5	3	7	8	8	69
43	4	5	5	6	4	4	3	6	5	6	2	7	8	8	73
44	5	5	5	5	4	4	4	5	6	5	3	8	7	8	74

Table D.1 (Cont'd)

Student i	Items 1	2	3(a)	3(b)	4(a)	4(b)	4(c)	4(d)	4(e)	4(f)	4(g)	5	6	7	Test Score
45	4	4	2	3	1	2	4	4	4	4	2	5	6	5	50
46	5	3	6	6	1	2	4	6	6	6	5	8	8	8	74
47	4	4	6	6	3	4	4	6	6	6	4	8	8	8	77
48	5	5	6	5	4	3	4	6	5	5	6	8	8	8	78
49	4	2	6	6	3	4	4	5	6	4	3	8	8	8	71
50	4	5	6	5	4	4	3	6	5	4	2	7	8	8	71

Table D.2

Total possible points, average scores and standard deviations for each item in an assessment for a unit on functions given to 50 calculus students.

	Items 1	2	3(a)	3(b)	4(a)	4(b)	4(c)	4(d)	4(e)	4(f)	4(g)	5	6	7	**Test Score**
Possible	5	5	6	6	4	4	4	6	6	6	6	8	8	8	82
Average	4.62	3.86	4.78	4.48	3.44	3.34	2.90	5.04	4.72	4.70	3.72	6.88	7.02	7.24	66.74
SD	0.53	0.88	1.46	1.58	0.97	0.98	1.16	1.29	1.31	1.30	1.51	1.36	1.42	1.38	10.18

APPENDIX E

Correlation Coefficients

Correlation coefficients are statistics "that indicate the degree of relationship between any two sets of scores obtained from the same group of individuals" (Gronlund & Linn, 1990, p. 79). In particular, these coefficients are calculated to assess the relationship between the results of two assessments administered tot he same group of students. Perhaps the most common correlation coefficient and the one on which a lot of other formulas are based is the Pearson product-moment correlation coefficient (Cangelosi, 2000).

The Pearson product-moment correlation coefficient was devised by Karl Pearson just prior to 1900 and was a principal outcome of the work Pearson and his colleagues had done while developing a way to quantify the consistency between two score sequences (Cangelosi, 2000). As a parameter, the Greek letter "ρ" ("rho") is used to symbolize this coefficient and when calculated as a statistic, the lowercase "r" is typically used. So the conventional symbol used to denote the correlation coefficient between two data sequences $\{a_i\}$ and $\{b_i\}$ is $\rho_{a,b}$ or $r_{a.b}$.

The Pearson product-moment correlation coefficient is typically computed using statistical calculators but can be computed by hand using the following formula:

$$\rho_{a,b} = \frac{1}{N} \sum_{i=1}^{N} z_{a_i} z_{b_i} \tag{E.1}$$

where $\rho_{a,b}$ is the correlation coefficient for $\{a_i\}$ and $\{b_i\}$; z_{a_i} is the z-score derived from a_i, μ_a, and σ_a; z_{b_i} is the z-score derived from b_i, μ_b, and σ_b; and N is the number of students (Cangelosi, 2000).

Derrick S. Harkness

Curriculum Vitae

EDUCATION

May 2020 **PH.D.**, *Utah State University*, Logan.
Mathematical Sciences (Emphasis in Education).

May 2016 **M.MATH**, *Utah State University*, Logan.
Master of Mathematics.

May 2014 **B.S.**, *Utah State University*, Logan.
Mathematics and Statistics Teaching Composite (Secondary Teaching License with Mathematics Endorsement).

DISSERTATION

Title *Teaching Students to Communicate with the Precise Language of Mathematics: A Focus on the Concept of Function in Calculus Courses*

Committee Dr. James Cangelosi (Advisor), Dr. David Brown, Dr. Nathan Geer, Dr. Zhaohu Nie, Dr. Beth MacDonald

Abstract Mathematics depends on using precise language that requires equally precise communication. However, this precision of language and communication is often absent in many mathematics classrooms, resulting in formal concept definitions that conflict with students' perceived images of that concept and which could lead to misunderstandings and poor concept constructions. One particular concept that often suffers from a lack of precision is the concept of function. Because this concept is so pervasive throughout mathematics, its importance cannot be understated. In order to explore the development of the concept of function, as it pertains to a calculus course, three articles are presented that (a) introduce the theoretical implications of teaching students to communicate with the precise language of mathematics, (b) relate the action research of a practitioner leading students to discover the concept of function and comprehend its definition with an emphasis on using the precise language of mathematics, and (c) develops and analyzes a suite of assessment tools designed to be relevant to students' higher-cognitive achievement of learning objectives involving the concept of function.

EMPLOYMENT HISTORY

Utah State University

2019-2020 **Faculty**, *Temporary Lecturer*, MATH 1210, MATH 2010, MATH 2020
Developed content for both MATH 1210 and MATH 2020. Also developed a professional development program for graduate teaching assistants to help improve their teaching of undergraduate courses. Assisted in the development of course content for MATH 2010.

2018-2019 **Graduate Teaching Assistant**, *Instructor*, MATH 1210, MATH 2020, MATH 2250.

2017-2018 **Graduate Teaching Assistant**, *Instructor*, MATH 1051, MATH 2020, MATH 2250.

2016-2017 **Graduate Teaching Assistant**, *Instructor*, MATH 2020, MATH 2250.

2015-2016 **Graduate Research Assistant**.
Exploring nontraditional undergraduates' resistance to active learning in an online support forum in calculus (see publications).

2013-2014 **Mathematics and Statistics Tutor**.
Tutored and aided students in both mathematics and statistics classes. Classes ranged from algebra through linear algebra and differential equations.

2013-2014 **Recitation Leader**.
Lead recitations for MATH 1050 and STAT 1040.

Public School Teaching Experience

2014-2015 **Faculty**, *Bear River High School*, Garland, UT.
Sophomore Mathematics, Sophomore Honors Mathematics, Junior Mathematics, and Junior Honors Mathematics.

2013 **Student Teacher**, *Cedar Ridge Middle School*, Hyde Park, UT.
Seventh-grade mathematics.

2013 **Permanent Substitute**, *Logan River Academy*, Logan, UT.
Temporary instructor for a high school geometry course.

COURSES TAUGHT

STAT 1040 Introduction to Statistics (Lead Recitations)
Descriptive and inferential statistical methods. Emphasis on conceptual understanding and statistical thinking. Examples presented from many different areas.

MATH 1050 College Algebra (Lead Recitations)
Functions: graphs, transformations, combinations, and inverses. Polynomial, rational, exponential, logarithmic functions, and applications. Systems of equations and matrices. Partial fractions.

MATH 1051 Classical Algebra for Elementary School Teachers
For pre-service elementary school teachers. To lead preservice elementary school teachers to develop a deep conceptual understanding of classical algebra necessary to succeed in MATH 2010, 2020, and 1210. Topics include naive set theory, the field axioms of classical algebra, functions (including binary operations; sequences and strings; bijections; and polynomial, rational, exponential, and logarithmic functions), set cardinality, and analytical geometry.

MATH 2010 Algebraic Thinking and Number Sense for Elementary School Teachers
Pre-service elementary school teachers develop a deep conceptual understanding of foundations of algebra and numeration necessary for them to teach elementary school students mathematics in a manner consistent with NCTM's Principles and Standards of School Mathematics.

MATH 2020 Euclidean Geometry and Statistics for Elementary School Teachers
For pre-service elementary school teachers. To help students develop a deep conceptual understanding of Euclidean geometry and statistics necessary for them to teach elementary school students mathematics in a manner consistent with NCTM's Principles and Standards of School Mathematics.

MATH 2250 Linear Algebra and Differential Equations
Linear systems, abstract vector spaces, matrices through eigenvalues and eigenvectors, solution of ode's, Laplace transforms, first order systems.

AWARDS AND RECOGNITION

College of Science Scholarship for Academic Excellence

Ray L. And Eloise Hoopes Lillywhite University Scholars Endowment

RESEARCH INTERESTS

Mathematics Teacher Preparation

Assessment of Mathematical Achievement

PUBLICATIONS

Conference Proceedings (Refereed)

Minichiello, A., Hood, J. R., & Harkness, D. S. (2017). Work in progress: Methodological considerations for constructing nontraditional student personas with scenarios from online forum usage data in calculus, In *2017 ASEE Annual Conference and Exposition*, Columbus, OH.

Harkness, D. S., Minichiello, A., & Marquit, J. (2016). Exploring nontraditional undergraduates' resistance to active learning in an online support forum in calculus, In *2016 ASEE Annual Conference and Exposition*, New Orleans, LA.

Journal Articles (Refereed)

Harkness, D. S. (2020a). *Assessing student achievement of objectives for constructing and comprehending the concept of functions: The development and analysis of a measurement instrument*. Manuscript in preparation.

Harkness, D. S. (2020c). *Theoretical implications of using the precise language of mathematics to teach the concept of function in calculus*. Manuscript in preparation.

Harkness, D. S. (2020b). *Leading college students to construct the concept of function: A unit development with a focus on precision language*. Manuscript in preparation.

Harkness, D. S. (2020d). *Uncovering ambiguity: Creating a unit for defining fractions in a preservice elementary education course*. Manuscript in preparation.

Minichiello, A., Hood, J. R., & Harkness, D. S. (2018). Bringing user experience design to bear on stem education: A narrative literature review. *Journal for STEM Education Research*.

PRESENTATIONS

2019 **Together We Teach Conference**, Logan, UT.
Teaching Students to Communicate Mathematics; Developing and Following a Lesson Plan; and Scoring, Grading, Rubrics, Oh My!

2018 **Together We Teach Conference**, Logan, UT.
Unit Goals and Objectives: A Learning Level Approach

2018 **Utah Council of Teachers of Mathematics**, Draper, UT.
Emphasizing Precision in Mathematical Language with a Focus on the Concept of Fractions.

2018 **Utah Association of Mathematics Teacher Educators**, Salt Lake City, UT.
MATH 1051 - Classical Algebra for Elementary School Teachers: Student Perceptions

2016 **American Society for Engineering Education**, New Orleans, LA.

Harkness, D. S., Minichiello, A., & Marquit, J. (2016). Exploring nontraditional undergraduates' resistance to active learning in an online support forum in calculus, In *2016 ASEE Annual Conference and Exposition*, New Orleans, LA.

LEADERSHIP AND SERVICE

2019 **Organizer**, *Utah State University* Logan, UT.
Together We Teach Conference

2018 **Co-Organizer**, *Utah State University* Logan, UT.
Together We Teach Conference

2019 **Committee Member**, *Utah System of Higher Education* Salt Lake City, UT.
Mathematics for Elementary Education Majors Committee

AFFILIATIONS

AMS, American Mathematical Society

AMTE, Association of Mathematics Teacher Educators

MAA, Mathematical Association of America

NCTM, National Council of Teachers of Mathematics