8-2020

Universal Localizations of Certain Noncommutative Rings

Tyler B. Bowles
Utah State University

UNIVERSAL LOCALIZATIONS OF CERTAIN NONCOMMUTATIVE RINGS

by

Tyler B. Bowles

A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

Approved:

Dariusz M. Wilczyński, Ph.D. Major Professor	Nathan Geer, Ph.D. Committee Member
Zhaohu Nie, Ph.D. Committee Member	Janis L. Boettinger, Ph.D. Acting Vice Provost of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah

2020

ABSTRACT

Universal Localizations of Certain Noncommutative Rings

by

Tyler B. Bowles, Master of Science

Utah State University, 2020

Major Professor: Dariusz M. Wilczyński, Ph.D.
Department: Mathematics and Statistics

Localization is a technique that was originally created to embed a commutative integral domain into its field of fractions. Classical methods of localization can be loosely thought of as a means of adding "denominators" to a ring or module, or more formally, the process of systematically adjoining inverse elements to a ring or module. To overcome difficulties that arise in extending these classical methods to noncommutative rings, P. M. Cohn invented a localization which maps a ring to one over which certain classes of matrices, rather than just elements, become invertible. A modern perspective interprets both methods as instances of universal localization, which is the adjunction to a ring R of universal inverses of a family of morphisms between R-modules. Finding specific models for universal localizations is, in general, difficult. For certain classes of rings, however, explicit constructions are accessible. We introduce such a class of rings, the generalized triangular matrix rings, and provide complete descriptions of their universal localizations with respect to morphisms between their projective ideals. The results demonstrate, in particular, that localizations of matrix rings are themselves matrix rings with an increased degree

of symmetry. This explains the results of Schofield and Sheiham that demonstrated that the localization of a triangular matrix ring of order two is a full matrix ring. Although full symmetry is not usually achieved in localizations of higher order rings, these localizations do have enough idempotents that their structure can be recognized. Nonetheless, for a large enough set of morphisms, i.e., a maximal tree of morphisms, full symmetry is again realized, and the universal localization is a full matrix ring.

(153 pages)

PUBLIC ABSTRACT

Universal Localizations of Certain Noncommutative Rings

Tyler B. Bowles

A common theme throughout algebra is the extension of arithmetic systems to ones over which new equations can be solved. For instance, someone who knows only positive numbers might think that there is no solution to $x + 3 = 0$, yet later learns $x = -3$ to be a feasible solution. Likewise, when faced with the equation $2x = 3$, someone familiar only with integers may declare that there is no solution, but may later learn that $x = \frac{3}{2}$ is a reasonable answer. Many eventually learn that the extension of real numbers to complex numbers unlocks solutions to previously unsolvable equations, such as $x^2 = -1$.

In algebra, a ring is, roughly speaking, any arithmetic system in which addition and multiplication behave "reasonably", while a homomorphism is a function that is compatible with the appropriate arithmetic systems. Some rings are noncommutative, meaning that the order in which one multiplies may change the product (i.e. $ab \neq ba$), in contrast to most grade school arithmetic.

The extension of integers to rational numbers that allows one to solve $2x = 3$ is an example of a more general technique, called localization. For commutative rings, localization is well understood and allows one to reasonably form fraction-like objects with numerators and denominators so that one can solve any equation of the form $ax = b$. However, this process becomes much more difficult for noncommutative rings. A modern perspective on this problem asks more broadly for an extension

of a noncommutative ring which makes any given homomorphism invertible, making it possible to solve certain equations involving the homomorphism. In general, satisfactory descriptions for extensions of this type are elusive. However, there are circumstances in which it is possible to give a concrete answer.

We investigate a class of rings called the generalized triangular matrix rings whose elements are matrix-like. Our study focuses on homomorphisms whose inputs and outputs are each columns from these matrices. The results explicitly describe all of the extensions that result in available inverse homomorphisms. These extensions, called universal localizations of the ring, are also rings whose elements are matrix-like, and these matrices are more symmetric than the ring before localization. To provide some historical context, we also recount the developments in the theory that led to this research. This includes detailed descriptions of classical localization and its counterpart in noncommutative algebra, Ore localization, as well as accounts of modern viewpoints, namely Cohn localization and universal adjunction.

ACKNOWLEDGMENTS

"Our chief want in life is somebody who will make us do what we can."

Ralph Waldo Emerson

I have been fortunate to be surrounded by so many supportive people throughout my life and, in particular, during my graduate studies. I would like to thank a few of those people for pushing me to "do what I can."

First, to my parents, Bryan and Jayme, for their words of encouragement, limitless affection, and belief in me when I doubted myself. I am grateful to my brothers, Parker, Ethan, and Casey, who have provided laughs and good times to make the journey more enjoyable. Parker's endless enthusiasm for life, Ethan's wild sense of humor, and Casey's inquisitive mind have been a rich source of joy and inspiration to me.

My advisor, Dr. Dariusz Wilczyński, deserves as much praise as I could ever give. He has offered me countless pieces of advice, words of wisdom, sentiments of confidence, and encouragement to aim for lofty goals. He has made our work a priority and has never shied away from pursuing and conquering daunting challenges together. His careful and critical eye, mathematical insights, and steady pushes in the right direction are what made this work possible.

My committee members, Dr. Zhaohu Nie and Dr. Nathan Geer, have been wonderful resources throughout my time as a graduate student. Both have provided a wealth of knowledge and experience as my teachers, mentors, and friends. The mathematics and statistics department has given me fantastic opportunities to teach and grow as a mathematician, and my involvement as a graduate teacher has been life-changing. Our graduate program coordinator, Gary Tanner, has been an immense

help and support throughout this process, both in his meticulous work handling program logistics and in his kind, caring, and fun demeanor, which made the graduate program a warm and welcoming place to be.

I owe so much to my friends. Kaitlin Murphy, whose unbounded love and support for me has lifted me up through good times and bad, has been a pillar to lean on during my studies. With Brandon Ashley, I shared many invaluable laughs, heart-to-heart chats, and exchanges of ideas, from which I have learned a lot about myself, about teaching, and about mathematics. I want to thank Patrick Seegmiller and Ian McGahan, whose companionship in our office led to many an interesting discussion, mathematical and otherwise. Their guidance through graduate student life and their ability to brighten the day with hilarity has been indispensable. Appreciation is in order for Thomas Hill, who has been a role model to me as a diligent student and a caring friend. To the rest of my graduate student colleagues, I express gratitude for building such a personal and nurturing community; in particular, a big thanks to Michael Schultz, Zach Youmans, Christian Harris, and Cat Wangen. Finally, I would be remiss not to acknowledge some of my other close friends who have been by my side and encouraged me along the way, including Taylor Bybee, David Sphar, Brad Allen, Sam Schwartz, Jordan Berger, Emma Buckley, Tyler Ely, Camille Wardle, Madeline Smith, Mike Fore, and Breckell Soifua. I couldn't have made it here without them.

Tyler B. Bowles

CONTENTS

LIST OF NOTATION

Algebras, rings, modules, and related matters

$\langle X \mid Y \rangle$	Ring, module, or algebra presentation, page 9
$\sum_i^{\oplus} A_i, \ \ A \oplus B$	Direct sum of modules
$\prod_i A_i, \ \ A \times B$	Product ring
$\coprod_C A_i, \ \ A \sqcup_C B$	Amalgamated free product of rings over C, page 77
$\coprod_{\mathfrak{M}} A_s, \ \ A \sqcup_{(M,x)} B$	Bimodule amalgamated free product of rings A_s over a family $\mathfrak{M}$ of pointed bimodules, page 133
$A \otimes_R B$	Tensor product of bimodules, page 6
$A \otimes B$	Tensor product of bimodules over $\mathbb{Z}$, page 6
$A \otimes_R B$	Tensor product of R-algebras, page 7
$D(\Sigma)$	Ring of (equivalence classes of) displays over Σ, page 55
$\mathrm{End}_R({}_RM)$	Endomorphism ring of a module, page 10
$\mathrm{Hom}_R({}_RM_S, {}_RN_T)$	The (S,T)-bimodule of R-module homomorphisms $M \to T$, page 10
$J(R)$	Jacobson radical of a ring R, page 100
$M^{\otimes n}$	Tensor power of n copies of an (R,R)-bimodule
$\mathcal{M}_n(R)$	Ring of $n \times n$ matrices over R, page 14
${}^nR, \ \ R^n$	The direct sum of n copies of R, viewed as columns or rows (resp.), page 14
$R^{\times}$	The set of nonzero elements in an integral domain or field, page 21
R^{op}	The opposite ring of R, page 10
$R_+\langle M \rangle$	Positive tensor algebra of the (R,R)-bimodule M, page 6
$R\langle M \rangle$	Tensor algebra of the (R,R)-bimodule M, page 7
$R\langle X \rangle R$	Free associative (R,R)-algebra on X, page 7

$R\langle X\rangle$	Free associative R-algebra when R is commutative, page 8
$R\langle X\rangle$	Tensor product ring $R \otimes \mathbb{Z}\langle X\rangle$, page 8
$R[X]$	Polynomial ring in central indeterminates, page 8
$\mathrm{rAnn}_f(S),\ \mathrm{lAnn}_f(T)$	Right and left f-annihilators for a bilinear, balanced map f, page 100
$\mathcal{R}(M, F)$	Matrix ℓ-ring associated with a matrix of bimodules M and family of multiplication maps F, page 88
$\mathcal{R}(\phi_{ij})$	Morphism of matrix ℓ-rings obtained by the matrix of bimodule morphisms (ϕ_{ij}), page 90
$\Sigma^{-1}R$	The universal Σ-inverting localization, where $\Sigma \subset \mathcal{M}(R)$, page 34
$\Sigma^{-1}R$	The universal Σ-inverting localization, where Σ is a collection of R-module morphisms, page 73
$\mathrm{Soc}(M)$	Socle of the left or right R-module M, page 100
$S^{-1}R$	Ring of (left) fractions or universal S-inverting ring, page 21
$T(\Sigma)$	Ring of (equivalence classes of) allowable triples over Σ, page 47
$\mathcal{U}_n(R)$	Ring of $n \times n$ upper triangular matrices over R, page 16

Sets and Functions

$g \circ f$	Composite function with assignment $(g \circ f)(x) = f(g(x))$, page 11
$\lvert X\rvert$	Cardinality of the set X
$\mathbb{1}_X$	Identity function on X
$\bigsqcup_i A_i,\ \ A \sqcup B$	Disjoint union of sets
$\prod_i A_i,\ \ A \times B$	Cartesian product of sets
$\mathbb{C}$	Set of complex numbers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers

$A \setminus B$	Set theoretic difference
$\mathcal{M}_n(f)$	A function $\mathcal{M}_n(R) \to \mathcal{M}_n(R')$ obtained by entry-wise application of $f\colon R \to R'$, page 15
$\mathcal{M}(R)$	Set of matrices of any size over R, page 35
$\mathcal{M}_\square(R)$	Set of square matrices of any size over R, page 35
$\mathcal{M}_n(R)$	Set of square matrices of size $n \times n$ over R, page 35

Categories

$\mathscr{A}(M, F)$	Category of pairs $((X_i), \{g_{ij}\})$ compatible with $\mathcal{R}(M, F)$, page 94
$\mathscr{C}(\mathcal{R}(M, F))$	Category of morphisms from $\mathcal{R}(M, F)$ into full matrix rings, page 104
$\mathscr{C}(\mathcal{R}(M, F), \xi)$	Full subcategory of $\mathscr{C}(\mathcal{R}(M, F))$ of maps $\beta\colon R \to \mathcal{M}_\ell(B)$ such that $\beta_{ij}(\xi) = 1_B$, page 104
$\mathbf{Inv}(R, \Sigma)$	Category of Σ-inverting maps, Σ a collection of R-module morphisms, page 74
${}_R\mathbf{Mod}$, $\mathbf{Mod}_R$, ${}_R\mathbf{Mod}_S$	The categories of unitary left R-modules, unitary right R-modules, and unitary (R, S)-bimodules (resp.), page 17
$\mathbf{Sq}(\alpha, \kappa)$	Category of commutative squares, page 74

Matrices

$A \nabla B$	Determinantal sum of matrices, page 37
$A \oplus B$	Diagonal sum of matrices, page 37
δ_{ij}	Kronecker delta in a ring, equal to 1 if $i = j$ and 0 otherwise
E_j^X	Block column whose j-th block is an identity matrix, page 45
e_i	Column with entry j given by δ_{ij}, page 34
e_{ij}	Matrix units in a matrix ring, page 15
I_n, I	$n \times n$ identity matrix, context-dependent size

Abbreviations

IBN	Invariant Basis Number, page 9
UGN	Unbounded Generating Number, page 69

CHAPTER 1

INTRODUCTION

A useful fact in commutative algebra is that any commutative integral domain can be embedded into a field of fractions. More generally, any commutative ring admits rings of fractions, which are, loosely speaking, rings over which some elements become invertible [20]. This process is called localization, named so due to a particularly useful case of the construction for which the ring of fractions is a local ring.

Attempts to generalize localization to embeddings of arbitrary noncommutative integral domains into skew fields have encountered numerous difficulties. In the early 20th century, A. Malcev showed that there exist integral domains for which such embeddings are not possible [12]. A set of additional hypotheses, proposed by Ø. Ore in 1931, provided a method for embedding some integral domains into a skew field of fractions [17], [25]. However, embeddings, or more generally homomorphisms, of rings into skew fields remained elusive when these conditions failed to hold.

In the 1970s, P. M. Cohn proposed an alternative to these methods. His idea was to construct rings over which certain classes of matrices, rather than just elements, become invertible. Today, these rings are referred to as Cohn localizations of a ring, and they have proved useful in establishing necessary and sufficient conditions for embeddability of an integral domain in a skew field. In addition to this application, Cohn localizations of rings are interesting constructions in their own right, revealing useful noncommutative analogues to objects commonly studied in commutative ring theory, including prime ideals, localizations, ring spectra, principle ideal domains,

and division algorithms [11], [8].

Matrices over a ring R correspond to homomorphisms between free R-modules. As such, a mapping which makes invertible a family of matrices over R is equivalent to a functor between module categories which makes a certain family of mappings between R-modules invertible. This perspective was adopted by G. M. Bergman, who studied a more general question of adjoining to a ring R universal inverses of arbitrary morphisms between R-modules. Such adjunctions are called universal localizations of R. If the modules being considered are finitely generated projective modules, then the universal localization is guaranteed to exist, but explicit constructions are not readily available in this generality [1].

Despite the lack of specific models for universal localizations in general, good descriptions can be given under additional hypotheses. Schofield [27], for example, established that the universal localizations of some interesting triangular matrix rings of the form $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$, with respect to morphisms $\left(\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} M \\ B \end{smallmatrix}\right)$, are full 2×2 matrix rings. For certain choices of bimodules in the triangular matrix ring, the resulting localizations are Morita equivalent to important constructions in ring theory, including the amalgamated free product. In 2006, a result by Sheiham [29] provided a presentation for the entries of such 2×2 matrix rings in order to unify the results of Schofield and others.

Triangular matrix rings have found interesting applications in topology, where their localizations serve to clarify the algebraic K- and L-theory of generalized free products. For example, if (M, N) is a pair of CW-complexes such that N has a neighborhood of the form $N \times [-1, 1] \subseteq M$, then the two morphisms $\pi_1(N \times \{\pm 1\}) \to \pi_1(M \setminus N)$ determine a triangular matrix ring structure whose Cohn localization is Morita equivalent to the group ring $\mathbb{Z}[\pi_1(M)]$. More details on these applications can be found in Ranicki's report [26].

The triangular matrix rings considered by Schofield and others can be considered special cases of a more general class of rings, called generalized matrix rings. Generalized matrix rings of order ℓ, often called matrix ℓ-rings, are built from a family of ℓ^2 bimodules equipped with appropriate multiplication maps. The identity element of a generalized matrix ring R can be decomposed as a sum of ℓ idempotents, denoted e_{ii}. Each idempotent determines a projective left ideal of R, namely Re_{ii}, and R decomposes as the direct sum $\sum_i^{\oplus} Re_{ii}$. A useful fact is that the converse also holds — any ring whose identity decomposes into ℓ idempotents can be given the structure of a matrix ℓ-ring.

The idempotents in a matrix ℓ-ring R provide an interesting way to analyze the ring-theoretic properties of R, especially when R is triangular (in the sense that all bimodules below or above its diagonal are zero). In the case that R is a triangular matrix ℓ-ring, its universal localization with respect to a morphism $Re_{pp} \to Re_{qq}$, mapping its p-th column to its q-th column, can be identified using the images of the idempotents in R. The localization can itself be given the structure of a matrix ℓ-ring whose entry bimodules are describable in terms of certain tensor algebras and tensor product modules of the various bimodules comprising R.

Somewhat surprisingly, the general question of localizing a triangular ℓ-ring with respect to a morphism between its column modules can be reduced to the case of a single morphism $Re_{22} \to Re_{44}$ in a triangular matrix ring of order five, where the localization takes the form

$$\begin{pmatrix} A_1 & U & G & U & W \\ 0 & A & B & A & V \\ 0 & C & D & C & H \\ 0 & A & B & A & V \\ 0 & 0 & 0 & 0 & A_5 \end{pmatrix}.$$

Using this reduction, we can describe the localization of a triangular matrix ring of any size.

The form of this localization also suggests that localizing a triangular matrix ring may yield a more symmetric matrix ring. This explains, in part, why the localization of a triangular 2-ring is a full 2×2 matrix ring. While the localization of a higher order triangular ring with respect to a single morphism does not usually yield a full matrix ring, the increased degree of symmetry begs the question of whether such a ring can be localized to a full matrix ring for a certain choice of morphisms. Indeed, the universal localization of a triangular ring R with respect to a maximal tree of R-module homomorphisms is a full matrix ring.

This thesis is structured as follows. In Chapter 2, we provide a brief overview of the category theoretic and algebraic terminology used for the remainder of the discussion. We introduce and motivate the *left first and right second* convention for composition which is utilized throughout. Classical notions of localization are the focus of Chapter 3, where we describe fields of fractions and Ore localization and discuss some of the problems that arise in noncommutative localization theory. This is followed by an account of Cohn localization in Chapter 4, including a sketch of Cohn's initial construction and a detailed comparison of two modern approaches to the construction, due to P. Malcolmson and Cohn. The adjunctions studied by Bergman are treated in Chapter 5, including a recount of Sheiham's proof of the localization of $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ in the context of localizing a ring which is a direct sum of two left ideals. In Chapter 6, we introduce the class of rings on which our study is focused, namely the generalized matrix rings. Finally, Chapter 7 details work by the author and D. M. Wilczyński on the universal localization of generalized triangular matrix rings. The results in Chapter 7 build on one another in succession, first providing the universal localization of a triangular matrix 3-ring as a stepping stone for the general construction, then describing the localization of a triangular matrix 5-ring. The localization of general matrix rings is given by reducing to the case of 5-rings.

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce some of the conventions that will play an important role in our discourse. We assume that the reader has a working knowledge of some basic category theory, ring theory, and module theory. For more details on the category theory and algebra used, we direct the reader to [10], [17], and [20]. The reader may find utility in the list of notation, which explains all special notation used in this volume, as well as references to the pages where the notation is introduced.

2.1 The Basics

A *rng* is an additive abelian group with an associative multiplication which distributes over addition. A rng with a multiplicative identity element is called a *ring*. Rng homomorphisms are additive and multiplicative maps, while ring homomorphisms must additionally preserve the multiplicative identity elements. An additive subgroup of a rng R which is closed under multiplication in R is called a *subrng* of R. If S is a subrng of a ring R containing the multiplicative identity of R, then S is a *subring* of R. A subrng of R, even if it is a ring in its own right, need not be a subring of R.

Given a rng R, a *left R-module M* is an additive abelian group equipped with a left R-action map $R \times M \to M$. We often write rm for the image of (r, m) under the action map. Such a map is required to be additive in both R and M and satisfies $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$. If R is a ring and $1_R m = m$ for all $m \in M$, we say that M is a *unitary* module. Right modules are defined analogously.

Beyond this chapter, we shall use the term "module" to refer only to unitary modules, though we sometimes use the adjective "unitary" for emphasis.

If an abelian group M is both a left R-module and a right S-module and the action maps commute in the sense that $(rm)s = r(ms)$ for all $r \in R$, $m \in M$, and $s \in S$, then M is called an (R, S)-*bimodule*, sometimes adorned with subscripts as in ${}_RM_S$ when these decorations provide additional clarity. A function $f\colon {}_RM_S \times {}_SN_T \to {}_RP_T$ of appropriate bimodules is called (R, T)-*bilinear* if f is additive in both M and N and for each $r \in R$, $m \in M$, $n \in N$, and $s \in S$, we have $f(rm, n) = rf(m, n)$ and $f(m, ns) = f(m, n)s$. Moreover, the function f is called S-*balanced* if $f(ms, n) = f(m, sn)$ holds for all $s \in S$, $m \in M$, and $n \in N$.

For fixed bimodules ${}_RM_S$ and ${}_SN_T$, there is a universal (R, T)-bilinear and S-balanced map whose target (R, T)-bimodule is called the *tensor product* of M and N, denoted $M \otimes_S N$. The subscript on the tensor product operator is often suppressed in the special case where $S = \mathbb{Z}$.

If A is an (R, R)-bimodule and there is an (R, R)-bilinear and R-balanced multiplication map $A \times A \to A$, then A is called an (R, R)-*algebra*. Morphisms of such algebras are bimodule homomorphisms which preserve multiplication. An (R, R)-algebra is *associative* or *commutative* if the multiplication is associative or commutative, respectively. If the rng R is commutative and the left and right action maps on A are equal, then A is called an R-*algebra*.

The inclusion of an (R, R)-bimodule M into its *positive tensor algebra*, $R_+\langle M\rangle = \sum_{n\geq 1}^{\oplus} M^{\otimes n}$ is universal in the category of (R, R)-bimodule homomorphisms from M into associative (R, R)-algebras, where $M^{\otimes n}$ denotes the n-th *tensor power* of M, namely $\underbrace{M \otimes_R \cdots \otimes_R M}_{n \text{ times}}$.

When R is a ring, an (R, R)-algebra is *unitary* if it is unitary as an (R, R)-bimodule. The *tensor algebra* of a unitary (R, R)-bimodule M is the unitary (R, R)-

algebra $R\langle M\rangle = \sum_{n\geq 0}^{\oplus} M^{\otimes n}$ with the convention $M^{\otimes 0} = R$. The inclusion of M into $R\langle M\rangle$ is universal in the category of (R,R)-bimodule homomorphisms from M into unitary associative (R,R)-algebras. In the special case where $R = \mathbb{Z}$, the tensor algebra $\mathbb{Z}\langle M\rangle$ is referred to as the *tensor ring*.

The tensor product of two R-algebras A and B is denoted by $A\otimes_R B$ and is defined as the bimodule $A \otimes_R B$ with multiplication given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for all $a, a' \in A$ and $b, b' \in B$.

Let R be a ring and M a left unitary R-module. If X is a subset of M such that every element of M can be uniquely written as a finite R-linear combination of elements of X, then X is said to be a *basis* of M. Contrary to vector spaces, not all left R-modules have bases; those that do are said to be *free* left R-modules. A left module which is free on its basis X has the property that every function $X \to N$ into a unitary left R-module uniquely factors through the inclusion $X \hookrightarrow M$ via an R-module homomorphism $M \to N$. Whenever X is a basis of M, there is a decomposition of M into a direct sum of cyclic R-modules, each isomorphic to R itself. More explicitly, $M = \sum_{x\in X}^{\oplus} Rx$.

More generally, an object F in a concrete category is *free* on a set X if there is a set function $i\colon X \to F$ such that for every set function $j\colon X \to A$ into an object A in the category, there is a unique morphism $\phi\colon F \to A$ satisfying $i \circ \phi = j$. In this manner, we may ask about free objects in other familiar categories. For instance, if R and S are rings, then the direct sum $\sum_{x\in X}^{\oplus} R \otimes S$ is free in the category of unitary (R,S)-bimodules, and is referred to as the *free* (R,S)-*bimodule* on X.

Free modules, free bimodules, and the tensor algebra can be used to construct various other free objects. For instance, if R is a ring and M is a free (R,R)-bimodule on the set X, then $R\langle M\rangle$ is a free associative unitary (R,R)-algebra, denoted $R\langle X\rangle R$. In the case where R is a commutative ring and M is a free (one-sided) R-module,

then $R\langle M\rangle$ is a free associative unitary R-algebra, denoted $R\langle X\rangle$. We note that $\mathbb{Z}\langle X\rangle \cong \mathbb{Z}\langle X\rangle\mathbb{Z}$ even though $R\langle X\rangle$ is not, in general, a free associative (R, R)-algebra. Elements of $R\langle X\rangle$ can be thought of as polynomials in noncommuting indeterminates, and this description suggests a construction even when the coefficient ring is noncommutative. That is, when R is noncommutative, then the tensor product ring $R\langle X\rangle = R \otimes \mathbb{Z}\langle X\rangle$ is a ring of polynomials whose indeterminates do not commute with one another but do lie in the centralizer of R. By contrast, elements of $R\langle X\rangle R$ can be thought of as polynomials in noncommuting indeterminates from X which commute with coefficients from the characteristic subring of R (the subring generated by 1_R). Finally, for any ring R, the notation $R[X]$ is used for the polynomial ring in central indeterminates. When R is commutative, $R[X]$ is a free commutative associative unitary R-algebra. For brevity, we shall omit the word "unitary" when discussing free algebras of any type, though it is important to be aware that the free objects in general categories of algebras usually differ from free objects in the various categories of unitary algebras. In other words, a "free algebra" shall always refer to a free object in a category of *unitary* algebras, by convention. Details on polynomial-type algebras such as $R\langle X\rangle R$, $R\langle X\rangle$, and $R[X]$, among others, as well as their associated universal properties can be found in [31], though different notation is used there.

One use of free objects is in the study of objects by generators and relations. For instance, a unitary R-module M is said to be *generated* by X subject to *relations* Y if M is isomorphic to the quotient of a free module on X by the smallest submodule containing Y as a subset. Under these hypotheses, we sometimes say that M has *presentation* $\langle X \mid Y\rangle$. Every module can be realized in this manner. Similarly, a ring R has presentation $\langle X \mid Y\rangle$ if R is isomorphic to the quotient $\mathbb{Z}\langle X\rangle/(Y)$ of the free associative $\mathbb{Z}$-algebra by the two sided ideal generated by Y. As with modules, every

ring can be given a presentation. To see this, for any ring R, we may take $X = R$ and $Y = \{1 - 1_R, x \cdot_R y - x \otimes y, (x +_R y) - (x + y) \mid x, y \in R\}$ to see that R has presentation $\langle X \mid Y \rangle$, where the addition and multiplication in R are denoted by $+_R$ and $\cdot_R$ to distinguish them from the operations in the free algebra. Analogous statements can be made about commutative associative unitary R-algebras, associative unitary (R, R)-algebras, and many other categories of algebraic objects.

The notions of *dimension* or *rank* of a free module M are not, in general, well-defined. A module M may be simultaneously free on X and free on Y, yet $|X| \neq |Y|$. The presence (or lack) of such pathological modules is a property of the ring of scalars. A ring R for which any two bases of a free R-module have the same cardinality is said to have *invariant basis number* (IBN). When R has IBN, we may unambiguously speak of the rank of a free R-module as the cardinality of any of its bases. Given our characterization of free R-modules as direct sums of copies of R, an equivalent way[1] to state the IBN property is to say that $R^n \cong R^m$ implies $n = m$, where R^n denotes the direct sum of n copies of ${}_RR$ or R_R.

Among rings which have the IBN property are commutative rings and skew fields [20], though a proof of this fact is omitted here.

Another important class of modules is the class of *projective modules*. A unitary left R-module P is said to be projective if it is a direct summand of a free module. That is, P is a projective R-module if there exists an R-module Q such that $P \oplus Q$ is free. Every free module is projective, but the converse need not hold in general.

[1] As it turns out, if a free module has a basis of infinite cardinality, any of its bases have the same cardinality. Thus, an ambiguity in rank may only arise with $F(X) \cong F(Y)$ for finite bases X and Y.

2.2 The lf-rs Convention

Given bimodules ${}_RM_S$ and ${}_RN_T$, the set of all R-module homomorphisms $M \to T$ is denoted $\mathrm{Hom}_R({}_RM_S, {}_RN_T)$ and can be enhanced with the structure of an (S,T)-bimodule under pointwise addition and action maps defined by

$$(sf)(x) = f(xs), \quad (ft)(x) = f(x)t$$

for all $s \in S$, $t \in T$, $x \in M$, and $f \in \mathrm{Hom}_R({}_RM_S, {}_RN_T)$. We denote the additive group of morphisms from a module M into itself by $\mathrm{End}_R({}_RM) = \mathrm{Hom}_R({}_RM, {}_RM)$. This group can be given a ring structure, called the *endomorphism ring* of M, but different conventions for the order in which composition is written can give $\mathrm{End}_R({}_RM)$ different ring structures. Using any of these conventions, $\mathrm{End}_R({}_RM_S)$ is an associative unitary (S,S)-algebra using the bimodule structure of $\mathrm{Hom}_R({}_RM_S, {}_RM_S)$ and composition as the multiplication map. The units in $\mathrm{End}_R({}_RM)$ are called *automorphisms* of ${}_RM$.

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Traditionally, the composite map $X \to Z$ with assignment $g(f(x))$ is written $g \circ f$. In the context of noncommutative algebraic structures, this convention of notation can lead to some inconveniences. For instance, students of linear algebra may recall that there is a one-to-one correspondence between linear operators on a vector space and matrices over that space and that composition of operators amounts to multiplication of the corresponding matrices. More concisely, this statement can be written $\mathrm{End}_K({}_KK^n) \cong \mathcal{M}_n(K)$ for a field K if the traditional meaning of $\circ$ is used as the multiplication map in the endomorphism ring. Here, $\mathcal{M}_n(K)$ denotes the ring of $n \times n$ matrices over K. The analogous statement for a noncommutative ring R is no longer true. Rather, it turns out to be the case that $\mathrm{End}_R({}_RR^n) \cong \mathcal{M}_n(R^{\mathrm{op}})$, where R^{op} denotes the *opposite ring* of R. The ring R^{op} has the same additive group as R, but multiplication $\cdot$ defined by $x \cdot y = yx$.

Many authors have found ways to remedy this inconvenience. Some authors

choose to work primarily with right modules and traditional composition, while others choose to write functions to the right of arguments as in $f(x) = xf$ so that the traditional composition, $(g \circ f)(x)$, is written xfg; multiplication in the endomorphism ring is then the map $(f, g) \mapsto fg$. The first of these choices yields $\mathrm{End}_R(R^n_R) \cong \mathcal{M}_n(R)$ while the second yields $\mathrm{End}_R({}_RR^n) \cong \mathcal{M}_n(R)$.

Our choice to bypass this inconvenience is to adopt the *left first and right second* convention (lf-rs, for short) for composition. Given $f\colon X \to Y$ and $g\colon Y \to Z$, we let $f \circ g = gf\colon X \to Z$ be defined by $(f \circ g)(x) = g(f(x))$ for all $x \in X$. The operation $\circ$ using the lf-rs convention is then used as multiplication in the endomorphism ring.

We demonstrate the utility of the lf-rs convention by reproving a result of Cohn [11, Thm. 0.2.1].

Let R be a rng. Suppose $A = \sum^{\oplus}_{i \in I} A_i$, $B = \sum^{\oplus}_{j \in J} B_j$, and $C = \sum^{\oplus}_{k \in K} C_k$ are R-modules (left or right, but all of the same flavor) for some finite index sets I, J, K. With each module A_n ($n \in I$) we can associate the canonical projection $\pi_n\colon A \to A_n$ given by $\pi_n((a_i)) = a_n$ and the canonical monomorphism $\mu_n\colon A_n \to A$ given by $\mu_n(a) = (a_i)$ with $a_i = a$ for $i = n$ and $a_i = 0$ otherwise. These mappings satisfy the equations

$$\sum_i \mu_i \pi_i = \mathbb{1}_A \quad \text{and} \quad \pi_m \mu_n = \delta_{mn}$$

with the Kronecker delta $\delta_{mn} = \mathbb{1}$ for $m = n$ and 0 otherwise. Similar pairs (μ'_n, π'_n) and (μ''_n, π''_n) can be defined for the summands of B and C.

Let $M_{ij} = \mathrm{Hom}_R(A_i, B_j)$. The sets $\mathrm{Hom}_R(A, B)$ can be made into additive groups under pointwise addition, and in fact $\mathrm{Hom}_R(A, B)$ and $\sum^{\oplus}_{i,j} M_{ij}$ are isomorphic, with each morphism f corresponding to the sum $\sum_{i,j} f_{ij}$ where $f_{ij} = \pi'_j f \mu_i$. If the direct sum $\sum^{\oplus}_{i,j} M_{ij}$ is arranged as a matrix of groups (M_{ij}), then $\sum_{i,j} f_{ij}$ corresponds to the matrix (f_{ij}). Similarly, morphisms $g \in \mathrm{Hom}_R(B, C)$ and $h \in \mathrm{Hom}_R(A, C)$ correspond to matrices of morphisms (g_{jk}) in $(N_{jk}) = (\mathrm{Hom}_R(B_j, C_k))$

and (h_{ik}) in $(P_{ik}) = (\mathrm{Hom}_R(A_i, C_k))$, where $g_{jk} = \pi''_k g\mu'_j$, and $h_{ik} = \pi''_k g\mu_i$.

Using these additive isomorphisms, the result of composition of morphisms

$$\mathrm{Hom}_R(A, B) \times \mathrm{Hom}_R(B, C) \to \mathrm{Hom}_R(A, C), \quad (f, g) \mapsto h = f \circ g$$

corresponds to the matrix (h_{ik}) in (P_{ik}) given by

$$h_{ik} = \pi''_k h\mu_i = \pi''_k gf\mu_i = \sum_j \pi''_k g\mu'_j\pi'_j f\mu_i = \sum_j (\pi'_j f\mu_i) \circ (\pi''_k g\mu'_j) = \sum_j f_{ij} \circ g_{jk}.$$

Thus (h_{ik}) is equal to the product matrix $(f_{ij}) \circ (g_{jk})$.

As mentioned previously, the additive group $\mathrm{Hom}_R(A, A)$ is a ring with multiplication given by composition of morphisms, denoted by $\mathrm{End}_R(A)$. The previous discussion with $A = B = C$ allows us to describe this ring in terms of matrices.

Theorem 2.2.1. *For any finite direct sum of R-modules $A = \sum_i^{\oplus} A_i$, the correspondence $f \leftrightarrow (f_{ij})$ defines an isomorphism of the endomorphism ring $\mathrm{End}_R(A)$ with the matrix ring $(\mathrm{Hom}_R(A_i, A_j))$.* □

Although the lf-rs convention was used in the calculation above, a similar isomorphism exists between $\mathrm{End}_R(A)$ and the transposed matrix ring $(\mathrm{Hom}_R(A_i, A_j))^\top$ given by the correspondence $f \leftrightarrow (f_{ji})$, with $f_{ji} = \pi'_j f\mu_i$, when the opposite composition rule is used on both sides of the correspondence

Corollary 2.2.2. *If A is an R-module and $S = \mathrm{End}_R(A)$, then $\mathrm{End}_R(A^n) \cong \mathcal{M}_n(S)$ (using either composition rule).* □

Examples 2.2.3. Let R be a ring.

(i) Let $f, g \in \mathrm{End}_R({}_RR)$ where ${}_RR = Rx$ with $x = 1_R$. Suppose $f(rx) = rf(x) = rax$ and $g(rx) = rg(x) = rbx$. Then $(f \circ g)(rx) = g(f(rx)) = g(rax) = (ra)bx = r(ab)x$. Thus $\mathrm{End}_R({}_RR) \cong R$ under the correspondence $f \leftrightarrow a$. It follows that $\mathrm{End}_R({}_RR^n) \cong \mathcal{M}_n(R)$.

(ii) Let $f, g \in \mathrm{End}_R(R_R)$ where $R_R = xR$ with $x = 1_R$. Suppose $f(xr) = f(x)r = xar$ and $g(xr) = g(x)r = xbr$. Then $(f \circ g)(xr) = g(f(xr)) = g(xar) = xb(ar) = x(ba)r$.

Thus $\mathrm{End}_R(R_R) \cong R^{\mathrm{op}}$ under the correspondence $f \leftrightarrow a$. It follows that $\mathrm{End}_R(R_R^n) \cong \mathcal{M}_n(R^{\mathrm{op}})$.

Again, the lf-rs convention was used in Examples 2.2.3. If the opposite rule is used instead, then the correct statements are $\mathrm{End}_R({}_RR) \cong R^{\mathrm{op}}$ and $\mathrm{End}_R(R_R) \cong R$, and consequently, $\mathrm{End}_R({}_RR^n) \cong \mathcal{M}_n(R^{\mathrm{op}})$ and $\mathrm{End}_R(R_R^n) \cong \mathcal{M}_n(R)$.

Endomorphism rings can also be used to add a bimodule structure to a one-sided module. As with the rest of this discussion, the statements in the following proposition rely on the use of the lf-rs convention and must be modified appropriately if the opposite rule is used.

Proposition 2.2.4. *Let A and B be rings.*

(a) If M is a unitary (A, B)-bimodule and N is a unitary (B, A)-bimodule, then $\mathrm{End}_A({}_AM_B)$ is a unitary (B, B)-algebra and $\mathrm{End}_A({}_BN_A)^{\mathrm{op}}$ is a unitary (B, B)-algebra.

(b) If M is a unitary left A-module and there is a ring homomorphism $f\colon B \to \mathrm{End}_A({}_AM)$, then B has a right action on M defined by $mb = f(b)(m)$, thereby making M a unitary (A, B)-bimodule.

(c) If N is a unitary right A-module and there is a ring homomorphism $f\colon B \to \mathrm{End}_A(N_A)^{\mathrm{op}}$, then B has a left action on N defined by $bn = f(b)(n)$, thereby making N a unitary (B, A)-bimodule.

$\square$

2.3 Matrix Rings

For a ring R, the *full matrix ring* of $n \times n$ matrices over R, denoted $\mathcal{M}_n(R)$, is a well-studied object in ring theory. Unitary modules over full matrix rings are completely described by Morita theory, while the Artin–Wedderburn Theorem establishes full matrix rings as the only source of simple Artinian rings [20, Thm. 9.1.14]. As we have already seen, full matrix rings are often used as an easier way to understand the endomorphism ring of a free module.

We proceed to describe a few of the results regarding matrix rings that will be of use to us.

Elements of the Cartesian product R^n can be viewed either as rows ($1 \times n$ matrices) or columns ($n \times 1$ matrices) over R. Which perspective is more useful as the default depends largely on whether one works primarily with left or right modules and which composition rule one uses. For our purposes, we shall take R^n to denote rows over R and use ${}^nR$ to denote columns over R when they are needed. There is a natural right action of $\mathcal{M}_n(R)$ on R^n and left action on ${}^nR$ by matrix multiplication, giving R^n an $(R, \mathcal{M}_n(R))$-bimodule structure and ${}^nR$ an $(\mathcal{M}_n(R), R)$-bimodule structure.

In a matrix ring $\mathcal{M}_n(R)$, there are n^2 elements denoted e_{ij} which are matrices with 1 in the (i, j) position and 0 elsewhere. These elements are called *matrix units*; a full set of matrix units satisfies the equations

$$e_{ij}e_{rs} = \delta_{jr}e_{is}, \qquad \sum_i e_{ii} = 1,$$

where δ is the Kronecker delta in R. The symbols e_{ij} will be reserved for use as matrix units and will mainly be understood as such by context going forward.

The ring $\mathcal{M}_n(R)$ contains R as a subring via the inclusion $a \mapsto aI$, where I denotes the $n \times n$ identity matrix. Matrices of the form aI are called *scalar matrices*, and even in the case where R is noncommutative, the matrix units commute with

all scalar matrices. The next result, from [11, Thm. 0.2.3], shows that the converse holds.

Theorem 2.3.1. *Let R be a ring containing n^2 elements e_{ij} such that*

$$e_{ij}e_{rs} = \delta_{jr}e_{is}, \qquad \sum_i e_{ii} = 1.$$

Then $R \cong \mathcal{M}_n(S)$, where S is the centralizer of $\{e_{ij}\}$.

Proof. Let S be the centralizer of $\{e_{ij}\}$. For each $a \in R$, we define $a_{ij} = \sum_r e_{ri}ae_{jr}$. It is not hard to see that $a_{ij} \in S$, and the correspondence $a \mapsto (a_{ij})$ is a ring isomorphism $R \cong \mathcal{M}_n(S)$. □

Because the images of matrix units under a ring homomorphism also satisfy the equations of a full set of matrix units, we obtain the following useful consequence.

Corollary 2.3.2. *If $f\colon \mathcal{M}_n(S) \to R$ is a ring homomorphism, then $R \cong \mathcal{M}_n(S')$, where S' is the centralizer of $\{f(e_{ij})\}$.* □

Likewise, sets of matrix units also determine ring homomorphisms between matrix rings.

Theorem 2.3.3. *If $f\colon \mathcal{M}_n(R) \to \mathcal{M}_n(R')$ satisfies $f(e_{ij}) = e'_{ij}$ where e_{ij} are the matrix units in $\mathcal{M}_n(R)$ and e'_{ij} are the matrix units in $\mathcal{M}_n(R')$, then f is obtained by applying a ring homomorphism $\theta\colon R \to R'$ entrywise. In other words, we say $f = \mathcal{M}_n(\theta)$.* □

The results discussed so far show that $\mathcal{M}_n(R)$ is a functor of R and, in fact, is an equivalence from the category of rings to the category of $n \times n$ matrix rings, where morphisms are entry-wise application of the same ring homomorphism.

In general, a ring may have more than one set of matrix units. Extending functions between matrix units to ring endomorphisms yields many nontrivial automorphisms of $\mathcal{M}_n(R)$. For instance, we observe that

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset \mathcal{M}_2(\mathbb{Q})$$

is a full set of matrix units which differs from what is perhaps the "obvious" choice. Nonetheless, a set of matrix units $\{e_{ij}\}$ is uniquely determined by the elements e_{ij} with $i \leq j$.

Proposition 2.3.4. *Let R be a ring and $\{e_{ij}\}$, $\{e'_{ij}\}$ be two full sets of matrix units in R such that $e_{ij} = e'_{ij}$ whenever $i \leq j$. Then $e_{ij} = e'_{ij}$ for all i, j.*

Proof. Let $i > j$. Then

$$e_{ij} = e_{ii}e_{ij} = e'_{ii}e_{ij} = e'_{ij}e'_{ji}e_{ij} = e'_{ij}e_{ji}e_{ij} = e'_{ij}e_{jj} = e'_{ij}e'_{jj} = e'_{ij}. \qquad \square$$

This also allows us to prove that a ring homomorphism $\mathcal{M}_n(R) \to T$ from a matrix ring into another ring is uniquely determined by its assignment on the subring, $\mathcal{U}_n(R)$, of upper triangular matrices.

Proposition 2.3.5. *If R is a subring of $S = \mathcal{M}_n(K)$ containing all upper triangular matrices, then the inclusion $\iota \colon R \to S$ is an epimorphism in the category of rings; that is,*

$$\iota \circ f_1 = \iota \circ f_2 \text{ implies } f_1 = f_2$$

for any pair of ring homomorphisms $f_1, f_2 \colon S \to T$. In particular, $\iota \otimes \mathbb{1} \colon R \otimes_R S \to S \otimes_R S$ is an (R, S)-bimodule isomorphism.

Proof. Suppose $f_1, f_2 \colon S \to T$ are ring homomorphisms such that $\iota \circ f_1 = \iota \circ f_2$. Then the images of the standard matrix units e_{ij} in S, namely $\{f_1(e_{ij})\}$ and $\{f_2(e_{ij})\}$, are themselves a set of matrix units in T. However, we have $e_{ij} \in \iota(R)$ for all $i \leq j$,

whence $f_1(e_{ij}) = f_2(e_{ij})$ for all indices $i \leq j$ and by Proposition 2.3.4, for all indices $1 \leq i, j \leq n$. Furthermore, $\iota(R)$ contains all scalar matrices, so

$$f_1\big(\textstyle\sum_{i,j} a_{ij}e_{ij}\big) = \sum_{i,j} f_1(a_{ij})f_1(e_{ij}) = \sum_{i,j} f_2(a_{ij})f_2(e_{ij}) = f_2\big(\sum_{i,j} a_{ij}e_{ij}\big),$$

which shows $f_1 = f_2$ as required. The last statement follows from [11, Prop. 7.2.1]. $\square$

Two rings R and S are said to be *Morita equivalent* if their categories of unitary left modules, denoted ${}_R\mathbf{Mod}$ and ${}_S\mathbf{Mod}$, are equivalent categories. There is a significant amount of literature detailing this concept alone, so we shall discuss only the example which will be of most interest to us, namely that R is Morita equivalent to its $n \times n$ matrix ring.

Lemma 2.3.6. *Let R be any ring. Then there is an $(\mathcal{M}_n(R), \mathcal{M}_n(R))$-bimodule isomorphism ${}^nR \otimes_R R^n \cong \mathcal{M}_n(R)$. Likewise, there is an (R, R)-bimodule isomorphism $R^n \otimes_{\mathcal{M}_n(R)} {}^nR \cong R$.*

Proof. The maps $f\colon {}^nR \otimes_R R^n \cong \mathcal{M}_n(R)$ and $g\colon R^n \otimes_{\mathcal{M}_n(R)} {}^nR \to R$ given by

$$\begin{aligned} f((x_i)^\top \otimes (y_j)) &= (x_i)^\top (y_j) = (x_i y_j), \\ g((x_i) \otimes (y_j)^\top) &= (x_i)(y_j)^\top = \textstyle\sum_i x_i y_i \end{aligned}$$

are isomorphisms of $(\mathcal{M}_n(R), \mathcal{M}_n(R))$-bimodules and (R, R)-bimodules, respectively. $\square$

Theorem 2.3.7. *Let R be any ring. Then R is Morita equivalent to $\mathcal{M}_n(R)$ for $n \geq 1$.*

Proof. Let $F\colon {}_R\mathbf{Mod} \to {}_{\mathcal{M}_n(R)}\mathbf{Mod}$ and $G\colon {}_{\mathcal{M}_n(R)}\mathbf{Mod} \to {}_R\mathbf{Mod}$ be given on objects by $F(X) = {}^nR \otimes_R X$ and $G(Y) = R^n \otimes_{\mathcal{M}_n(R)} Y$. On morphisms, we define $F(f) = \mathbb{1} \otimes f$ and $G(g) = \mathbb{1} \otimes g$. By Lemma 2.3.6, we note that

$$\begin{aligned} (G \circ F)(Y) &= {}^nR \otimes_R R^n \otimes_{\mathcal{M}_n(R)} Y \cong Y, \\ (F \circ G)(X) &= R^n \otimes_{\mathcal{M}_n(R)} {}^nR \otimes_R X \cong X. \end{aligned}$$

and thus F and G are equivalences. $\square$

In particular, every unitary left module over $\mathcal{M}_n(R)$ is a direct sum of n copies of some unitary left R-module X. An analogous statement holds for right modules.

CHAPTER 3
CLASSICAL LOCALIZATIONS

Localization is a process originally invented to embed a commutative integral domain into a field. Over a commutative ring R, one can always construct a ring of fractions whose elements are of the form a/b ($a, b \in R$) with a specified denominator set [20, Sec. III.4]. When the denominator set contains no zero divisors, R embeds into this ring of fractions. In the case where R is a commutative integral domain, one can take the denominator set to be the multiplicative monoid of nonzero elements to obtain an embedding into a field.

For noncommutative rings, the situation grows much more complicated. A noncommutative integral domain need not embed in a skew field, nor even admit a homomorphism into one. In the noncommutative setting, a formal expression of the form a/b is ambiguous; it is unclear whether to interpret the fraction as ab^{-1} or $b^{-1}a$. Imposing equality of these expressions when the base ring is noncommutative seems unnatural and indeed, does not have the universal property we would like from a ring of fractions. We may insist that the ring of fractions consist of only one type of fraction, say $b^{-1}a$, but this immediately presents a difficulty in rewriting sums and products of such elements in the requisite form. A ring whose elements are all of the form $b^{-1}a$ for a specified denominator set is called a (left) ring of fractions, and need not exist in general. A paper by Øystein Ore [25] in 1931 described a necessary and sufficient condition for such a ring of fractions to exist, settling the question of embedding a noncommutative domain into a skew field of fractions.

However, not all skew fields are fields of fractions, and mappings (particularly

embeddings) of noncommutative rings into these more general skew fields or even into rings over which certain elements become invertible are often mysterious. As we shall see in Chapter 4, some of these conundrums were unraveled by Cohn in the 1970s. In the present chapter, we focus on the classical constructions for a ring or field of fractions, including the Ore localization. We also describe some of the difficulties that arise in the localization of rings when Ore's condition does not hold.

3.1 Rings of Fractions and Ore Localization

Let R be a commutative ring and $S \subset R$ any subset. The idea behind a commutative ring of fractions is to consider formal expressions of the type r/s for $r \in R, s \in S$. Working by analogy to the familiar extension from $\mathbb{Z}$ to $\mathbb{Q}$, the sums and products of fractions are defined by

$$r/s + r'/s' = (rs' + sr')/(ss'), \quad (r/s)(r'/s') = (rr')/(ss').$$

Indeed, one can see that if $r/s = s^{-1}r = rs^{-1}$, then the sum and product formulas are a direct consequence of the ring axioms and the commutative property. One particular feature of these formulas is that the product of two denominators is again a denominator; that is if $s, s' \in S$, then ss' must also be an element of S for these operations to be defined. A subset $S \subset R$ such that $1 \in S$ and $s, s' \in S$ implies $ss' \in S$ is called a *multiplicative* subset of R.

A classic result in algebra shows that under a certain equivalence relation on $S \times R$, the quotient set $S \times R/\sim$ can be given the structure of a ring. The equivalence class of a pair (s, r) is intended to represent the fraction r/s. The resulting ring is denoted $S^{-1}R$ and admits a ring homomorphism $R \to S^{-1}R$ which is S-*inverting* in the sense that elements of S map to units in $S^{-1}R$; in fact, this homomorphism is universal among all S-inverting maps into commutative rings. A particularly useful instance of this construction occurs when R is a commutative integral domain and

$S = R^{\times}$, the set of all nonzero elements. In this case, $S^{-1}R$ is a field.

Theorem 3.1.1. *Let S be a multiplicative subset of a commutative ring R. Then*

(a) *the relation defined on $S \times R$ by*

$$(s, r) \sim (s', r') \iff t(rs' - r's) = 0 \text{ for some } t \in S$$

is an equivalence relation;

(b) *the set $S^{-1}R = (S \times R)/\sim$ is a commutative ring with addition and multiplication defined by $r/s + r'/s' = (rs' + sr')/ss'$ and $(r/s)(r'/s') = rr'/ss'$, where r/s denotes the equivalence class of (s, r);*

(c) *the map $\lambda \colon R \to S^{-1}R$ given by $\lambda(r) = r/1$ is the universal S-inverting commutative ring homomorphism; that is, λ itself is S-inverting and if $f \colon R \to T$ is an S-inverting homomorphism into a commutative ring T, then there is a unique ring homomorphism $\phi \colon S^{-1}R \to T$ satisfying $\lambda \circ \phi = f$;*

(d) $\ker \lambda = \{r \in R \mid sr = 0 \text{ for some } s \in S\}$*;*

(e) *if R is an integral domain and $S = R^{\times}$, then $S^{-1}R$ is a field and λ is injective.*

The ring $S^{-1}R$ is called the *ring of fractions* of R with denominators from S. In the last case, where R is an integral domain and S is the complement of the zero ideal, $S^{-1}R$ is called the *total field of fractions* of R.

Example 3.1.2. The total field of fractions for $\mathbb{Z}$ is the field of rational numbers, $\mathbb{Q}$.

Examples 3.1.3. Let k be a field.

(i) For any $S \subset k^{\times}$, there is an isomorphism $S^{-1}k \cong k$.

(ii) The total field of fractions for $k[x]$ is the field of rational functions, $k(x)$.

(iii) Likewise, the total field of fractions for the power series ring, $k[\![x]\!]$, is the ring of formal Laurent series, $k(\!(x)\!)$.

Example 3.1.4. Let R be a commutative ring and $\mathfrak{p}$ be a prime ideal of R. If $S = R \backslash \mathfrak{p}$, then $S^{-1}R$ is a *local ring*. This means that $S^{-1}R$ has a unique maximal ideal. Denoting this maximal ideal by I, the quotient $S^{-1}R/I$ is then a field. Although the quotient map may not be an embedding (e.g. R may not be an integral domain), its existence does raise the question of which rings admit (possibly non-injective) homomorphisms into skew fields. It is this example that gave the process of localization its name.

Proving the theorem is straightforward, albeit tedious. We opt rather to spend time on its generalization by Ore. Ore's motivation was to find an analogue of Theorem 3.1.1 for a noncommutative ring. One way to generalize this theorem is to consider S-inverting homomorphisms. Given a set $S \subset R$, we call a ring homomorphism $R \to T$ an *S-inverting homomorphism* if $\lambda(S)$ is a set of units in T. Such homomorphisms form a category, where a morphism from $\alpha\colon R \to T$ to $\beta\colon R \to T'$ is a ring homomorphism $f\colon T \to T'$ satisfying $\alpha \circ f = \beta$.

Let R be a noncommutative ring with presentation $\langle X \mid Y \rangle$. For a given set of elements $S \subset R$, we define a new set of generators $\bar{S}$ with the same cardinality as S. Let us denote an element of $\bar{S}$ corresponding to $s \in S$ by $\bar{s}$. We may then consider a ring $S^{-1}R$ with presentation $\langle X \sqcup \bar{S} \mid Y \sqcup Y' \rangle$, where $Y' = \{1 - s \otimes \bar{s}, 1 - \bar{s} \otimes s \mid s \in S\}$. There is then an obvious S-inverting ring homomorphism $\lambda\colon R \to S^{-1}R$. In fact, λ is universal in the category of S-inverting homomorphisms. We summarize these results in the following theorem.

Theorem 3.1.5. *Let R be a ring and S any subset of R. There exists a universal S-inverting ring homomorphism $\lambda\colon R \to S^{-1}R$; that is, $\lambda(S)$ consists of units and any ring homomorphism $R \to T$ sending S to a set of units has a unique factorization $R \xrightarrow{\lambda} S^{-1}R \to T$. The map λ is injective if and only if there exists an embedding of R into some ring which maps S to a set of units.* $\square$

Unfortunately, this formulation does not provide much information about the behavior of λ or $S^{-1}R$. For instance, it is unclear under what circumstances the resulting ring $S^{-1}R$ is a skew field, nor whether λ is injective. In general, neither can be expected; as an example, if S contains 0, the localization is a map to the zero ring. However, examples of nontrivial pathologies exist. We will examine one such example in Section 3.2, which will demonstrate that even when S contains neither zero nor any zero divisors, λ may fail to be injective.

In full generality, there is no way to deduce much useful information about $S^{-1}R$. The main difference between this general construction and the one in Theorem 3.1.1 is that Theorem 3.1.5 does not assert a normal form for the elements of $S^{-1}R$.

The idea proposed by Ore was to identify conditions that allow one to write the elements of $S^{-1}R$ in a normal form, say $s^{-1}r$. Such a ring, should it exist, is called a *ring of left fractions* of R with denominators from S. For similar reasons to those in the commutative case, we shall require that S be multiplicative. The global existence of the proposed normal form in $S^{-1}R$ requires at least the ability to rewrite the product $(1^{-1}r)(s^{-1}1) = rs^{-1}$ in the form $s'^{-1}r'$; that is, there must exist $s' \in S$ and $r' \in R$ such that $rs^{-1} = s'^{-1}r'$. Multiplying on the left by s' and on the right by s, we obtain the equation $s'r = r's$. Given that $s'r \in Sr$ and $r's \in Rs$, we obtain a necessary condition for the existence of a ring of left fractions, namely that $Sr \cap Rs \neq \emptyset$ for each $r \in R$ and $s \in S$. In fact, this condition turns out to be nearly sufficient to extend the results of Theorem 3.1.1.

Definition 3.1.6. Let R be any ring and $S \subset R$ a multiplicative set. Suppose that

(i) (left Ore condition) for all $r \in R$ and $s \in S$, $Sr \cap Rs \neq \emptyset$, and

(ii) (left reversibility) for all $r, r' \in R$ and $s \in S$, $rs = r's$ implies that $tr = tr'$ for some $t \in S$.

When these axioms are satisfied, S is called a *left Ore set*.

The left reversibility condition is perhaps best understood in terms of zero divisors. By collecting like terms, the left reversibility condition states that if $r \in R$ and $s \in S$ satisfies $rs = 0$, then there exists $t \in S$ such that $tr = 0$. Morally, we might think of the condition as saying that any left zero divisors of R annihilated by S must also be right zero divisors annihilated by S.

The present form of Ore's theorem and an exposition on its applications to monoids and noncommutative geometry can be found in [30].

Theorem 3.1.7 (Ore, 1931 [25, Thm. 1])**.** *Let R be a ring and S a left Ore set.*[1] *Then*

(a) the relation defined on $S \times R$ by

$$(s, r) \sim (s', r') \iff \widetilde{s}s' = \widetilde{r}s \text{ and } \widetilde{s}r' = \widetilde{r}r \text{ for some } \widetilde{s} \in S, \widetilde{r} \in R$$

is an equivalence relation;

(b) the set $S^{-1}R = (S \times R)/{\sim}$ is a ring with addition and multiplication defined by $s_1\backslash r_1 + s_2\backslash r_2 = s's_1\backslash(s'r_1 + r'r_2)$ and $(s_1\backslash r_1)(s_2\backslash r_2) = s''s_1\backslash r''r_2$, where $s\backslash r$ denotes the equivalence class of (s, r) and $r', r'' \in R$, $s', s'' \in S$ satisfy $r's_2 = s's_1$ and $r''s_2 = s''r_1$;

(c) the map $\lambda\colon R \to S^{-1}R$ given by $\lambda(r) = 1\backslash r$ is the universal S-inverting ring homomorphism; that is, λ itself is S-inverting and if $f\colon R \to T$ is an S-inverting homomorphism into a ring T, then there is a unique ring homomorphism $\phi\colon S^{-1}R \to T$ satisfying $\lambda \circ \phi = f$;

(d) $\ker \lambda = \{r \in R \mid sr = 0$ *for some* $s \in S\}$*;*

(e) if R is a (noncommutative) integral domain and $S = R^\times$ is a left Ore set, then $S^{-1}R$ is a skew field and λ is injective.

[1] A *right Ore set* can be defined analogously. The construction of the Ore localization given here can also be stated and proved for a *ring of right fractions*, a ring whose elements are each of the form rs^{-1}. The notation for such a ring is RS^{-1}.

Proof. Clearly the relation $\sim$ is reflexive, since $1 \in S$. To see symmetry, suppose that $(s, r) \sim (s', r')$. Then there exist $\widetilde{s} \in S$ and $\widetilde{r} \in R$ satisfying $\widetilde{s}s' = \widetilde{r}s$ and $\widetilde{s}r' = \widetilde{r}r$. By the Ore condition, there exist $r_1, r_2 \in R$, and $s_1, s_2 \in S$ such that $r_1 s' = s_1 s$ and $r_2 \widetilde{s}s' = s_2 r_1 s'$. Now we compute

$$r_2 \widetilde{r}s = r_2 \widetilde{s}s' = s_2 r_1 s' = s_2 s_1 s,$$

and so by left reversibility, there exists $t \in S$ such that $tr_2\widetilde{r} = ts_2s_1$. Defining $\widetilde{\widetilde{s}} = ts_2s_1 \in S$ and $\widetilde{\widetilde{r}} = tr_2\widetilde{s} \in R$, we find that

$$\begin{aligned}
\widetilde{\widetilde{s}}s &= ts_2s_1s = tr_2\widetilde{r}s = tr_2\widetilde{s}s' = \widetilde{\widetilde{r}}s', \\
\widetilde{\widetilde{s}}r &= ts_2s_1r = tr_2\widetilde{r}r = tr_2\widetilde{s}r' = \widetilde{\widetilde{r}}r',
\end{aligned}$$

showing that $(s', r') \sim (s, r)$ as required.

For transitivity, let us suppose that $(s, r) \sim (s', r')$ and $(s', r') \sim (s'', r'')$. Then by definition, there exist $\widetilde{s}, \widetilde{\widetilde{s}} \in S$ and $\widetilde{r}, \widetilde{\widetilde{r}} \in R$ satisfying

$$\begin{aligned}
\widetilde{s}s' &= \widetilde{r}s, \quad \widetilde{s}r' = \widetilde{r}r, \\
\widetilde{\widetilde{s}}s'' &= \widetilde{\widetilde{r}}s', \quad \widetilde{\widetilde{s}}r'' = \widetilde{\widetilde{r}}r'.
\end{aligned}$$

By the Ore condition, there exist $s_1 \in S$ and $r_1 \in R$ such that $s_1\widetilde{\widetilde{r}} = r_1\widetilde{s}$. By setting $\widetilde{\widetilde{\widetilde{s}}} = s_1\widetilde{\widetilde{s}} \in S$ and $\widetilde{\widetilde{\widetilde{r}}} = r_1\widetilde{r}$, we find that

$$\begin{aligned}
\widetilde{\widetilde{\widetilde{s}}}s'' &= s_1\widetilde{\widetilde{s}}s'' = s_1\widetilde{\widetilde{r}}s' = r_1\widetilde{s}s' = r_1\widetilde{r}s = \widetilde{\widetilde{\widetilde{r}}}s, \\
\widetilde{\widetilde{\widetilde{s}}}r'' &= s_1\widetilde{\widetilde{s}}r'' = s_1\widetilde{\widetilde{r}}r' = r_1\widetilde{s}r' = r_1\widetilde{r}r = \widetilde{\widetilde{\widetilde{r}}}r,
\end{aligned}$$

thereby proving that $(s, r) \sim (s'', r'')$.

We now show that the operations proposed in (b) are well-defined. The existence of the mediating elements $r', r'' \in R$ and $s', s'' \in S$ for each operation is guaranteed by the Ore condition. We proceed to show that the sum and product of elements is independent of the choice of mediating elements.

Let us consider first addition and show that the proposed formula does not depend on the choice of r' or s'. That is, suppose that $r', r'' \in R$ and $s', s'' \in S$ satisfy

$r's_2 = s's_1$ and $r''s_2 = s''s_1$. By the Ore condition, there exist $\widetilde{s} \in S$ and $\widetilde{r} \in R$ such that $\widetilde{s}s'' = \widetilde{r}s'$. Now we may consider

$$\widetilde{s}r''s_2 = \widetilde{s}s''s_1 = \widetilde{r}s's_1 = \widetilde{r}r's_2,$$

hence by left reversibility, there exists $t \in S$ satisfying $t\widetilde{s}r'' = t\widetilde{r}r'$. From this we may define $\widetilde{\widetilde{s}} = t\widetilde{s}$ and $\widetilde{\widetilde{r}} = t\widetilde{r}$. It is now easy to see that

$$\begin{aligned} \widetilde{\widetilde{s}}s''s_1 &= t\widetilde{s}s''s_1 = t\widetilde{r}s's_1 = \widetilde{\widetilde{r}}s's_1, \\ \widetilde{\widetilde{s}}(s''r_1 + r''r_2) &= \widetilde{\widetilde{s}}s''r_1 + \widetilde{\widetilde{s}}r''r_2 = t\widetilde{s}s''r_1 + t\widetilde{s}r''r_2 = t\widetilde{r}s'r_1 + t\widetilde{r}r'r_2 = \widetilde{\widetilde{r}}(s'r_1 + r'r_2), \end{aligned}$$

proving that $s's_1\backslash(s'r_1 + r'r_2) = s''s_1\backslash(s''r_1 + r''r_2)$. The proof that the formula for addition depends only on equivalence classes rather than representatives is similar.

Likewise, suppose that $r', r'' \in R$ and $s', s'' \in S$ satisfy $r's_2 = s'r_1$ and $r''s_2 = s''r_1$. We shall verify that either choice yields the same product $(s_1\backslash r_1)(s_2\backslash r_2)$. By the Ore condition, there exist $\widetilde{s} \in S$ and $\widetilde{r} \in R$ such that $\widetilde{s}s'' = \widetilde{r}s'$. Now we may observe that

$$\widetilde{s}r''s_2 = \widetilde{s}s''r_1 = \widetilde{r}s'r_1 = \widetilde{r}r's_2,$$

hence by left reversibility, there exists $t \in S$ satisfying $t\widetilde{s}r'' = t\widetilde{r}r'$. Setting $\widetilde{\widetilde{s}} = t\widetilde{s}$ and $\widetilde{\widetilde{r}} = t\widetilde{r}$, it is then clear that

$$\begin{aligned} \widetilde{\widetilde{s}}s''s_1 &= t\widetilde{s}s''s_1 = t\widetilde{r}s's_1 = \widetilde{\widetilde{r}}s's_1, \\ \widetilde{\widetilde{s}}r''r_1 &= t\widetilde{s}r''r_1 = t\widetilde{r}r'r_1 = \widetilde{\widetilde{r}}r'r_1, \end{aligned}$$

proving that $s's_1\backslash r'r_1 = s''s_1\backslash r''r_1$. As with addition, we omit the proof that multiplication depends only on equivalence classes, which is similar to the above verification.

It is worth noting that for any $r \in R$ and $s \in S$, we have $s\backslash r = ts\backslash tr$ whenever $tr \in S$, which can be seen from the equivalence relation simply by taking $\widetilde{s} = 1$ and $\widetilde{r} = r$. In particular, this means that any two left fractions (and by induction, any finite set of fractions) can be brought to a common denominator, for if we consider $s\backslash r$ and $s'\backslash r'$, then the Ore condition provides $t_1 \in R$, $t_2 \in S$ such that $t = t_1s = t_2s' \in S$.

We can then rewrite the fractions with common denominator t, namely

$$s\backslash r = t_1 s\backslash t_1 r = t\backslash t_1 r,$$
$$s'\backslash r' = t_2 s'\backslash t_2 r' = t\backslash t_2 r'.$$

This observation greatly reduces the complexity of verifying the ring axioms.

For instance, to verify associativity, we may assume without loss of generality the three summands have been brought to a common denominator. This gives us the liberty of taking $r' = s' = 1$ in the definition of addition, yielding

$$(s\backslash r_1 + s\backslash r_2) + s\backslash r_3 = s\backslash(r_1 + r_2 + r_3) = s\backslash r_1 + (s\backslash r_2 + s\backslash r_3).$$

Commutativity of addition is verified similarly. We can easily check that $1\backslash 0$ is an additive identity, while $s\backslash(-r)$ is the additive inverse of $s\backslash r$.

To see associativity of multiplication, we note that

$$[(s_1\backslash r_1)(s_2\backslash r_2)](s_3\backslash r_3) = s_5 s_4 s_1\backslash r_5 r_3,$$
$$(s_1\backslash r_1)[(s_2\backslash r_2)(s_3\backslash r_3)] = s_7 s_1\backslash r_7 r_6 r_3,$$

where

$$r_4 s_2 = s_4 r_1, \tag{3.1}$$
$$r_5 s_3 = s_5 r_4 r_2, \tag{3.2}$$
$$r_6 s_3 = s_6 r_2, \tag{3.3}$$
$$r_7 s_6 s_2 = s_7 r_1. \tag{3.4}$$

We invoke the Ore condition to obtain $s \in S$ and $r \in R$ such that $ss_7 = rs_5 s_4 \in S$. Equations (3.1) and (3.4) show that

$$sr_7 s_6 s_2 = ss_7 r_1 = rs_5 s_4 r_1 = rs_5 r_4 s_2.$$

Thus, by left reversibility, there exists $t_1 \in S$ such that $t_1 sr_7 s_6 = t_1 rs_5 r_4$. Likewise,

$$t_1 sr_7 r_6 s_3 = t_1 sr_7 s_6 r_2 = t_1 rs_5 r_4 r_2 = t_1 rr_5 s_3$$

holds by Equations (3.2) and (3.3), and so we may invoke left reversibility to obtain $t_2 \in S$ satisfying $t_2 t_1 sr_7 r_6 = t_2 t_1 rr_5$. Putting this all together, we set $\widetilde{s} = t_2 t_1 s$ and

$\widetilde{r} = t_2 t_1 r$ to see

$$\begin{aligned}\widetilde{s}s_7s_1 &= t_2t_1ss_7s_1 = t_2t_1rs_5s_4s_1 = \widetilde{r}s_5s_4s_1,\\ \widetilde{s}r_7r_6r_3 &= t_2t_1sr_7r_6r_3 = t_2t_1rr_5r_3 = \widetilde{r}r_5r_3.\end{aligned}$$

Therefore, $s_5s_4s_1\backslash r_5r_3 = s_7s_1\backslash r_7r_6r_3$, as claimed. The left and right distributive laws can be verified easily by assuming that the summands have been brought to a common denominator. Explicitly,

$$\begin{aligned}(s_1\backslash r_1)(s_2\backslash r_2 + s_2\backslash r_3) &= s's_1\backslash r'(r_2+r_3)\\ &= s's_1\backslash r'r_2 + s's_1\backslash r'r_3\\ &= (s_1\backslash r_1)(s_2\backslash r_2) + (s_1\backslash r_1)(s_2\backslash r_3)\end{aligned}$$

where $r's_2 = s'r_1$, and so the left distributive property holds. For the right distributive property, let us consider

$$\begin{aligned}(s_1\backslash r_1)(s_2\backslash r_3) + (s_1\backslash r_2)(s_2\backslash r_3) &= s's_1\backslash r'r_3 + s''s_1\backslash r''r_3\\ &= t's's_1\backslash t'r'r_3 + t''s''s_1\backslash t''r''r_3\\ &= t's's_1\backslash(t'r'r_3 + t''r''r_3)\\ &= t's's_1\backslash(t'r' + t''r'')r_3,\end{aligned}$$

where $r's_2 = s'r_1$, $r''s_2 = s''r_2$, and $t's' = t''s''$ for some $r', r'', t' \in R$, $s', s'', t'' \in S$, which can be obtained via the Ore condition. Now we note that

$$(t'r' + t''r'')s_2 = t'r's_2 + t''r''s_2 = t's'r_1 + t''s''r_2 = (t's')r_1 + (t's')r_2 = t's'(r_1+r_2)$$

and so when computing $(s_1\backslash(r_1+r_2))(s_2\backslash r_3)$, we may use $t'r' + t''r''$ and $t's'$ as the mediating factors. Thus,

$$[(s_1\backslash r_1) + (s_1\backslash r_2)](s_2\backslash r_3) = (s_1\backslash(r_1+r_2))(s_2\backslash r_3) = t's's_1\backslash(t'r' + t''r'')r_3,$$

completing the proof of right distributivity. Finally, it is apparent that $1\backslash 1$ is a multiplicative identity. Thus, $S^{-1}R$ is a ring.

Clearly, λ as proposed is an S-inverting ring homomorphism, with $\lambda(s)^{-1} = 1\backslash s$ for each $s \in S$. To see the universal property, suppose that $f\colon R \to T$ is an S-inverting ring homomorphism. We define $\phi\colon S^{-1}R \to T$ by $\phi(s\backslash r) = f(s)^{-1}f(r)$. It is straightforward to verify that ϕ is a well-defined ring homomorphism satisfying

$\lambda \circ \phi = f$, and because every element of $S^{-1}R$ is of the form $s^{-1}r$ for some $s \in S$ and $r \in R$, ϕ is the only map with this property.

To determine the kernel, we note that $r \in \ker \lambda$ if and only if $1\backslash r = 1\backslash 0$. This statement holds precisely when there exist $\widetilde{s} \in S$ and $\widetilde{r} \in R$ such that $\widetilde{s} \cdot 1 = \widetilde{r} \cdot 1$ and $\widetilde{s} \cdot 0 = \widetilde{r}r$, or equivalently, when there exists $\widetilde{r} = \widetilde{s} \in S$ such that $\widetilde{r}r = 0$.

The last conclusion of the theorem statement follows immediately from the previous parts. $\square$

The ring of left fractions, $S^{-1}R$, is sometimes called an *Ore localization* of a ring. If R is commutative, then any multiplicative subset S of R is a left Ore set, since $rs = sr \in Rs \cap Sr$ for all $r \in R$ and $s \in S$. Moreover, the Ore localization is commutative in this case, and admits, in particular, a unique morphism to any commutative S-inverting ring, hence is isomorphic to the ring of fractions constructed in Theorem 3.1.1. Thus, the coincidence of notation and terminology does not lead to any ambiguity.

The circumstance described by part (e) of Theorem 3.1.7 is an important case of Ore localization. An integral domain R whose nonzero elements form a left Ore set is called a (left) *Ore domain*, and as pointed out in the theorem, its Ore localization is an embedding of R into a skew field. This skew field is called the *total skew field of left fractions* of R, although the adjective "total" is sometimes omitted for brevity.

Corollary 3.1.8. *Let R be any ring. Then R embeds into a skew field whose elements are each of the form $s^{-1}r$ for $s \in S$, $r \in R$ if and only if R is a (noncommutative) integral domain and $S = R^{\times}$ is a left Ore set. Under these hypotheses, this skew field of left fractions is unique up to isomorphism.* $\square$

Example 3.1.9. Let R be a commutative ring and let $S \subset \mathcal{M}_n(R)$ be the subset of all regular elements (those elements which are neither left nor right zero divisors). Then $S^{-1}\mathcal{M}_n(R) \cong \mathcal{M}_n(T^{-1}R)$, where T is the set of regular elements in R.

3.2 Difficulties with Embeddings

As we have seen in Corollary 3.1.8, the Ore criteria are necessary and sufficient for embedding an integral domain into a skew field of fractions. Nevertheless, the study of noncommutative localizations does not end with Ore localization, as there may be more general skew fields into which integral domains may be embedded. Whether every integral domain admits such an embedding remained open until 1937, when A. I. Malcev provided the following counterexample, which we state without a complete proof.

Proposition 3.2.1 (Malcev, 1937 [21]). *Let $X = \{a, b, c, d, x, y, u, v\}$ and $R = \mathbb{Z}\langle X\rangle/I$, where $I = (ax - by, cx - dy, cu - dv)$. The ring R is an integral domain, but there is no embedding of R into a skew field.*

The idea of the proof is to establish that every element of Malcev's ring R can be uniquely written as a finite sum of monomials which do not contain any instance of the products by, dy, or dv. One can use this form for elements to prove that the product of nonzero elements is nonzero, thereby making R an integral domain. This form for elements also makes it apparent that $au \neq bv$ and that a, c, y, and v are nonzero. However, the defining relations for R imply that if f is any $\{a, c, y, v\}$-inverting ring homomorphism, then

$$f(a)^{-1}f(b) = f(x)f(y)^{-1}, \quad f(x)f(y)^{-1} = f(c)^{-1}f(d), \quad f(c)^{-1}f(d) = f(u)f(v)^{-1},$$

whence $f(au) = f(bv)$ and so f cannot be injective. In particular, Malcev's ring cannot be embedded in a skew field.

Returning to the more general discussion, suppose that R is an integral domain. Even if R can be embedded in a skew field, there is no guarantee that the universal $R^\times$-inverting ring is a skew field. In essence, the obstruction arises from the fact that upon adjoining inverses to each element of $S = R^\times$, new non-units are introduced in the process. For example, when the ring $S^{-1}R$ is noncommutative, an element $xy^{-1}z + ab^{-1}c$ need not be a unit, even if x, y, z, a, b, c are units lying in the image of the localization $R \to S^{-1}R$.

The Ore condition tames the localization of many rings, though there are a myriad of examples where this condition does not hold. In some sense, failure of the Ore condition means that the ring is quite large, as the following result, first observed by Goldie, demonstrates.

Theorem 3.2.2 (Goldie, 1957 [16, Thm. 1])**.** *If R is an integral domain, then R is either an Ore domain or contains a free algebra of infinite rank.*

Proof. Suppose that $S = R^\times$ is not a left Ore set so that $Ra \cap Sb = \emptyset$ for some $a, b \in R^\times$. Let K denote the (commutative) subring generated by 1. We claim that the K-subalgebra generated by a and b is free. If not, then there exists a nonzero noncommutative polynomial $f \in K\langle x, y\rangle$ such that $f(a, b) = 0$. Assume, without loss of generality, that f is a polynomial of minimum degree with this property. By grouping monomials based on their rightmost indeterminate, we may then write $f = c + f_1x + f_2y$, where c is constant and f_1 and f_2 are either zero (although f has a root, (a, b), so they cannot both be zero) or they are of degree strictly less than f. Assuming, again without loss of generality, that $f_1 \neq 0$, we must have $f_1(a, b) \neq 0$ by minimality of f, yet $f(a, b) = 0$. Recalling that $c \in K$ lies in the center of R, we then have

$$c + f_1(a, b)a + f_2(a, b)b = 0,$$

whence

$$(bf_1(a,b))a = (-c - bf_2(a,b))b.$$

Since R is an integral domain, the left side of this equation must be nonzero, hence the right also. In particular, $-c - bf_2(a,b) \in S$ and the element represented by both sides of this equation lies in $Ra \cap Sb$, which was presumed empty. Thus the K-algebra generated by a and b is free and the subalgebra of R generated by the family $a^n b$ ($n \in \mathbb{N}$) is free of infinite rank. $\square$

A nice corollary to this result is the following, which shows that the class of Ore domains encompasses a broad and important subclass of rings.

Corollary 3.2.3. *Every left Noetherian*[2] *integral domain is a left Ore domain.* $\square$

In addition to Ore localization, other viewpoints from algebra and topology contributed to the overall understanding of noncommutative localizations and embedding problems. For details on the historical development of the subject and a survey of the other algebraic and topological methods, we refer the reader to [5] and [12]. As it turns out, the key to making significant progress towards understanding noncommutative localization beyond the class of Ore domains and topological embedding methods lies in studying maps which invert a set of matrices over a ring R rather than just elements of R. Such was the idea of Cohn. We shall return to this notion in the upcoming chapter.

[2] A ring is called *left Noetherian* if all of its left ideals are finitely generated.

CHAPTER 4
COHN LOCALIZATION

Some of the difficulties that arise in localizing noncommutative rings cannot be easily overcome. There exist many examples of (noncommutative) integral domains which do not embed in a skew field of fractions. It is natural, then, to relax the search for an embedding and ask whether a ring admits any homomorphism into a skew field. More generally, we might consider mappings between rings which take a specified set of elements to units. In Chapter 3, we saw that such mappings always exist, though little can be deduced about them in general.

To address these difficulties, a monograph by P. M. Cohn in 1971 [4] and some associated papers ([3], [5], [6], [7]) exhibited a notion now referred to as Cohn localization. The idea behind Cohn localization is to map a ring R to one over which certain matrices, rather than just elements, become invertible. While this problem appears more broad than the original question, Cohn was successful in showing that such maps can sometimes yield embeddings into skew fields. Contemporaries of Cohn, particularly Malcolmson, demonstrated explicit constructions for these localizations analogous to those for Ore localizations. In the sections that follow, we provide a definition of Cohn localization and detail some of the approaches to the subject.

Throughout this and future chapters, whenever $f\colon R \to S$ is a ring homomorphism and $A = (a_{ii})$ is a matrix with entries from R, we adopt the convention that $f(A) = (f(a_{ii}))$ is a matrix with entries from S obtained by entry-wise application of f. Likewise, the image of a set Σ of matrices under entry-wise application of f to each element of Σ is denoted $f(\Sigma)$. We also adopt the notation $e_i \in {}^nR$ to describe

a column with 1 in the i-th position and 0 elsewhere. The ring R over which e_i is defined will be understood from context.

4.1 Universal Matrix Inversion

In the middle of the 20th century, Schützenberger [28] and Nivat [24] gave conditions for elements of a ring R to be entries of an invertible matrix over R. Cohn recognized that these conditions could be tied to the problem of embedding an integral domain in a skew field [11]. This prompted a consideration of ring homomorphisms $R \to S$ taking classes of matrices over R to invertible matrices over S. The approach was rather fruitful, providing both a new notion of localization of a noncommutative ring, as well as supplying new information about the types of homomorphisms a ring can admit.

Definition 4.1.1. Let R be a ring and Σ a set of matrices over R. A ring homomorphism $f\colon R \to T$ is said to be Σ-*inverting* if $f(\Sigma) \subset \mathcal{M}(T)$ is a set of invertible matrices. A Σ-inverting ring homomorphism $\lambda\colon R \to \Sigma^{-1}R$ is called the *universal* Σ-*inverting localization* of R if every Σ-inverting ring homomorphism $R \to T$ has a unique factorization $R \xrightarrow{\lambda} \Sigma^{-1}R \to T$.

One can form a category of Σ-inverting ring homomorphisms $R \to T$, where a morphism from $\alpha\colon R \to T$ to $\beta\colon R \to T'$ is a ring homomorphism $f\colon T \to T'$ such that $\alpha \circ f = \beta$. The universal Σ-inverting localization of R is then the universal object in this category. It follows that the ring $\Sigma^{-1}R$, often called the *universal* Σ-*inverting ring*, is determined up to isomorphism.

At first glance, it may not be obvious whether a universal Σ-inverting localization always exists. A construction by generators and relations guarantees its existence, though it has similar drawbacks to the description of the ring $S^{-1}R$ given in Theorem 3.1.5.

Theorem 4.1.2. *For any ring R and any set of matrices Σ over R, there exists a universal Σ-inverting localization $\lambda\colon R \to \Sigma^{-1}R$.*

Proof. Let us take a presentation $\langle X \mid Y \rangle$ of R. For each $m \times n$ matrix $A \in \Sigma$, we define $X_A = \{\bar{a}_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and

$$Y_A = \left\{ \delta_{ik} - \sum_{j=1}^{n} a_{ij} \otimes \bar{a}_{jk} \mid A = (a_{ij}) \in \Sigma, \bar{a}_{jk} \in X_A, 1 \leq i, k \leq m \right\},$$

$$Z_A = \left\{ \delta_{ik} - \sum_{j=1}^{n} \bar{a}_{ij} \otimes a_{jk} \mid A = (a_{jk}) \in \Sigma, \bar{a}_{jk} \in X_A, 1 \leq i, k \leq n \right\},$$

where δ_{ik} denotes the Kronecker delta. We may then form the ring $\Sigma^{-1}R$ as the ring with presentation $\langle X \bigsqcup_{A \in \Sigma} X_A \mid Y \bigsqcup_{A \in \Sigma} (Y_A \sqcup Z_A) \rangle$.

There is a natural ring homomorphism $\lambda\colon R \to \Sigma^{-1}R$ taking each generator of R to its corresponding generator in $\Sigma^{-1}R$. Clearly, for each matrix $A \in \Sigma$, the matrix $f(A)$ is invertible with $f(A)^{-1} = (\bar{a}_{ij})$. Finally, for any Σ-inverting ring homomorphism $f\colon R \to T$, we may define a homomorphism $\bar{f}\colon \Sigma^{-1}R \to T$ by assigning $f(x)$ to any $x \in X$ and, for each $A \in \Sigma$, assigning the (i, j) entry of the matrix $f(A)^{-1}$ to any $\bar{a}_{ij} \in X_A$. The relations in $\Sigma^{-1}R$ are relations in R along with necessary relations for invertibility of each A, hence these relations are satisfied in T and the rule of assignment for $\bar{f}$ given on X and each X_A extends to a well-defined ring homomorphism. In fact, the factorization $f = \lambda \circ \bar{f}$ is unique. $\square$

An $m \times n$ matrix A over R corresponds to an R-module homomorphism $R^n \to R^m$. If $A \in \Sigma$, then $\lambda(A)$ is an invertible matrix over $T = \Sigma^{-1}R$, hence corresponds to an isomorphism $T^n \cong T^m$. Thus we see that one consequence of including non-square matrices in Σ is that $\Sigma^{-1}R$ fails to have the IBN property. To avoid this pathology, we often consider only sets of square matrices. For any ring R, we shall denote by $\mathcal{M}(R)$ the set of all matrices over R and use $\mathcal{M}_\square(R) = \bigcup_{n>0} \mathcal{M}_n(R)$ to denote the set of all square matrices over R.

In the case where Σ consists of 1×1 matrices, the definition of universal localization reduces to the inversion of elements. In this sense, Cohn localization generalizes classical localization. Moreover, Cohn localization does not provide new insights into commutative rings. Indeed, when R is a commutative ring and $\Sigma \subset \mathcal{M}_{\square}(R)$, the ring $\Sigma^{-1}R$ can be realized as a classical ring of fractions, as the next proposition shows.

Proposition 4.1.3. *Let R be a commutative ring and $\Sigma \subset \mathcal{M}_{\square}(R)$. Then $\Sigma^{-1}R \cong S^{-1}R$, where S is the multiplicative closure of the set $\{\det A \mid A \in \Sigma\}$.*

Proof. The statement follows immediately from the fact that a square matrix over a commutative ring T is invertible if and only if its determinant is a unit in T. $\square$

4.2 The Construction of Cohn

Although Theorem 4.1.2 guarantees the existence of a Cohn localization for any set Σ of matrices, the nature of the resulting ring homomorphism is unclear. To find an alternative description of this localization, we consider the set of square matrices over R whose images under a homomorphism $f\colon R \to T$ are not invertible, called the *singular kernel* of f. The singular kernel has some structure analogous to an ideal. By considering a set of matrices $\mathcal{P}$ which forms a so-called prime matrix ideal, one can construct a homomorphism $R \to K$ to a skew field which is generated (as a field) by the image of R and whose singular kernel is $\mathcal{P}$. Conversely, every skew field generated by R with singular kernel $\mathcal{P}$ arises in this way. Before giving the definition of a matrix ideal, we first discuss the preliminary notions of a non-full matrix, the diagonal sum, and the determinantal sum.

Suppose that $f\colon R \to T$ is a homomorphism into a skew field. Let A be an $n \times n$ matrix over R which has a factorization $A = PQ$ such that P is $n \times r$ and Q is $r \times n$. The least r for which such a factorization is possible is called the *inner rank* of A. The factorization $f(A) = f(P)f(Q)$ shows that the inner rank of $f(A)$ is not more than

r. We claim that if $r < n$, then $f(A)$ cannot be invertible. To see this, let us note that the matrix $f(Q)$ corresponds to a module homomorphism $g\colon T^n \to T^r$. Since T is a skew field, T has the IBN property and any submodule of T^r is free of rank no greater than r. Consequently, g cannot be injective, else the restriction of g to its image would be an isomorphism between free modules of differing rank. However, if $f(A)$ is invertible, then $f(A)^{-1}f(P)$ is a left inverse for $f(Q)$, contradicting the fact that g is not injective.

We summarize by saying that an $n \times n$ matrix whose inner rank is less than n has no invertible image under a homomorphism into a skew field. This suggests the following definition.

Definition 4.2.1. A square matrix A over a ring R is said to be *non-full* if $A = PQ$ where P is $n \times r$, Q is $r \times n$, and $r < n$.

Let A and B be square matrices of possibly different sizes. We denote by $A \oplus B$ the matrix formed as the block diagonal matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ with appropriately sized blocks of zeros in the off-diagonal positions. The matrix $A \oplus B$ is called the *diagonal sum* of A and B.

Suppose A and B are $n \times n$ matrices which differ only in row i. We define the *determinantal sum* of A and B, denoted $A \nabla B$, as the $n \times n$ matrix whose row i is the sum of row i of A and row i of B and whose remaining rows are the same as the corresponding rows in A and B. We similarly define $A \nabla B$ when A and B differ only in a single column. The determinantal sum is not defined for arbitrary pairs of matrices, so caution should be exercised in its repeated application.

Definition 4.2.2. Let R be a ring and $\mathcal{P} \subset \mathcal{M}_\Box(R)$. The collection $\mathcal{P}$ is called a *matrix ideal* if

M.1) $\mathcal{P}$ includes all non-full matrices;

M.2) whenever $A, B \in \mathcal{P}$ and $A \nabla B$ exists, then $A \nabla B \in \mathcal{P}$;

M.3) whenever $A \in \mathcal{P}$, then $A \oplus B \in \mathcal{P}$ for all square matrices $B \in \mathcal{M}_\Box(R)$;

M.4) $A \oplus 1 \in \mathcal{P}$ implies $A \in \mathcal{P}$.

A matrix ideal is said to be *proper* if it does not contain an identity matrix of any size. If $\mathcal{P}$ is a proper matrix ideal and $A \oplus B \in \mathcal{P}$ implies $A \in \mathcal{P}$ or $B \in \mathcal{P}$, then $\mathcal{P}$ is called a *prime matrix ideal*.

One reason that prime ideals are useful in the theory of commutative localization is that their complements are multiplicatively closed. Similarly, the complement of a prime matrix ideal is multiplicatively closed in a modified sense that we define as follows.

Definition 4.2.3. A set $\Sigma \subset \mathcal{M}_\Box(R)$ is said to be *upper-multiplicative* if $1 \in \Sigma$ and for all $A, B \in \Sigma$ and $C \in \mathcal{M}(R)$ of the appropriate size,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \Sigma.$$

Lower-multiplicative sets are defined similarly, with C in the lower left corner.

Proposition 4.2.4. *If $\mathcal{P}$ is a prime matrix ideal in R, then $\Sigma = \mathcal{M}_\Box(R) \setminus \mathcal{P}$ is both upper-multiplicative and lower-multiplicative.*

Proof. Since $\mathcal{P}$ does not contain an identity matrix of any size, $1 \in \Sigma$. Let $A, B \in \Sigma$ with A an $m \times m$ matrix and B an $n \times n$ matrix. We note that for any $C \in \mathcal{M}(R)$ of the appropriate size, we have

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C' & 0 \\ 0 & B' & B_n \end{pmatrix} \nabla \begin{pmatrix} A & C' & C_n \\ 0 & B' & 0 \end{pmatrix},$$

where C' and B' each denote the submatrix consisting of all but the last column of C and B, respectively, and C_n and B_n each denote the last column of C and B. The second matrix here is non-full, as demonstrated by the equation

$$\begin{pmatrix} A & C' & C_n \\ 0 & B' & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_m \\ B' & 0 \end{pmatrix} \begin{pmatrix} 0 & I_{n-1} & 0 \\ A & C' & C_n \end{pmatrix},$$

where I_k denotes a $k \times k$ identity matrix. This equation is a factorization of the $(m+n) \times (m+n)$ matrix into the product of an $(m+n) \times (m+n-1)$ matrix by an $(m+n-1) \times (m+n)$ matrix. As a matrix ideal, $\mathcal{P}$ contains all non-full matrices, and so $\left(\begin{smallmatrix} 0 & I_{n-1} & 0 \\ A & C' & C_n \end{smallmatrix}\right) \in \mathcal{P}$. By closure of $\mathcal{P}$ under determinantal sums,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{P} \quad \text{if and only if} \quad \begin{pmatrix} A & C' & 0 \\ 0 & B' & B_n \end{pmatrix} \in \mathcal{P}.$$

We may repeat this process as many times as necessary, each time replacing a column of C with a column of zeros. Ultimately, this leads us to conclude that

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{P} \quad \text{if and only if} \quad \begin{pmatrix} A & 0 & 0 \\ 0 & B' & B_n \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{P}.$$

However, since $A, B \in \Sigma = \mathcal{M}_\Box(R) \setminus \mathcal{P}$ and $\mathcal{P}$ is prime, $A \oplus B \in \Sigma$. An analogous argument shows lower-multiplicativity. $\square$

There are many other features of matrix ideals that are interesting to study in their own right. For more on matrix ideals, we refer the reader to [11].

When working with a commutative ring R, we may localize R at a prime ideal $\mathcal{P}$ to obtain the local ring $R_\mathcal{P}$; more precisely, this is the ring of fractions $(R \setminus \mathcal{P})^{-1}R$. The quotient of $R_\mathcal{P}$ by its maximal ideal is then a field and the natural map from R to this field has kernel $\mathcal{P}$.

By considering prime matrix ideals, a parallel analysis exists in the noncommutative case. The existence of a mapping to a skew field with prescribed singular kernel was first proved by Cohn in 1971 [4, Thm. 7.5.3], though the argument given was an indirect one. Cohn's original method involved an axiomatic description of skew fields

as the union of a group of units with a zero element and an involution $x \mapsto 1 - x$ satisfying certain properties.

We give an overview of Cohn's initial approach for historical context, but omit many of the details in favor of discussing more efficient approaches which were later proposed by Cohn and Malcolmson. Readers inclined to see more details of these early arguments may refer to [4, Sec. 7.5] and [8, Sec. 7.5].

Lemma 4.2.5. *Let G be a group (with identity denoted 1) and let $G_{\neq 1} = \{x \in G \mid x \neq 1\}$. Suppose there is a function $\theta \colon G_{\neq 1} \to G_{\neq 1}$ satisfying*

(i) $\theta(yxy^{-1}) = y\theta(x)y^{-1} \quad (x \in G_{\neq 1}, y \in G)$,

(ii) $\theta^2(x) = x \quad (x \in G_{\neq 1})$,

(iii) $\theta(xy^{-1}) = \theta(\theta(x)\theta(y)^{-1})\theta(y^{-1}) \quad (x, y \in G_{\neq 1}, x \neq y)$,

(iv) $e = \theta(x^{-1})x\theta(x)^{-1}$ *is constant for* $x \in G_{\neq 1}$.

Then there exists a unique skew field K whose multiplicative group of nonzero elements is G such that $e = -1$ and $\theta(x) = 1 - x$ for all $x \in G_{\neq 1}$.

Proof Sketch. The statement holds easily for a trivial group G. If G is nontrivial, we define $K = G \cup \{0\}$ and extend the multiplication in G to K by setting $0x = x0 = x$ for all $x \in K$. The mapping θ extends to a map on K by $\theta(1) = 0$ and $\theta(0) = 1$. The additive structure on K is obtained by defining subtraction with the formula

$$x - y = \begin{cases} ey & \text{if } x = 0 \\ \theta(yx^{-1})x & \text{if } x \neq 0 \end{cases}$$

and setting $x + y = x - (0 - y)$. These operations then give K the structure of a ring (and, in fact, a skew field, since its nonzero elements are a multiplicative group) provided subtraction satisfies the following properties for all $x, y, z, w \in K$,

1. $(x - y) - (z - w) = (x - z) - (y - w)$,
2. $x - 0 = x$,
3. $x - x = 0$,

4. $x - (y - z) = z - (y - x)$.

5. $(x - y)z = xz - yz$,

6. $z(x - y) = zx - zy$.

Properties (2) and (3) follow readily from the definition of subtraction. If $x = 0$ or $z = 0$, then both sides of property (5) are eyz or 0, respectively; otherwise, $x, z \neq 0$ and $xz \neq 0$, yielding

$$xz - yz = \theta(yzz^{-1}x^{-1})(xz) = \theta(yx^{-1})(xz) = (x - y)z.$$

This proves property (5) in the remaining cases. After verifying that e is central, property (6) is easily seen when $x = 0$ or $z = 0$. In the remaining cases where $xz \neq 0$, either $yx^{-1} = 0$, $yx^{-1} = 1$, or $yx^{-1} \in G_{\neq 1}$. The property is easily verified in the first two circumstances, while in the last, we find

$$\begin{aligned} z(x - y) &= z\theta(yx^{-1})x = z\theta(yx^{-1})z^{-1}zx \\ &= \theta(zyx^{-1}z^{-1})zx = \theta(zy(zx)^{-1})zx = zx - zy. \end{aligned}$$

Properties (1) and (4) are then straightforward, albeit tedious, verifications. Care must be taken to consider all possible cases at each phase. To see uniqueness of this skew field, we note that any skew field K' whose multiplicative group is G must be of the form $G \cup \{0\}$ with $0x = x0 = 0$ for all $x \in K'$. If $e = -1$ in K' and $\theta(x) = 1 - x$ for all $x \in G_{\neq 1} \subset K'$, then for $x = 0$, we have $x - y = 0 - y = -1y = ey$ and for $x \neq 0$, we have $x - y = (1 - yx^{-1})x = \theta(yx^{-1})x$, where $\theta(0) = 1$ and $\theta(1) = 0$, hence the operations in K' coincide with those in K. $\square$

Given a ring R and a prime matrix ideal $\mathcal{P}$, one can construct a homomorphism $R \to K$ into a skew field whose singular kernel is $\mathcal{P}$. The construction proceeds in several phases, first by showing that the set of solutions to equations $Au = a$ ($a \in {}^nR$ and $A \in \Sigma = \mathcal{M}_\square(R) \setminus \mathcal{P}$) can be given the structure of a semigroup which can be embedded in a group. Lemma 4.2.5 can then be applied to obtain the required skew field.

Theorem 4.2.6. *Let R be any ring and Σ an upper-multiplicative set of square matrices over R. If $\lambda\colon R \to \Sigma^{-1}R$ is the universal Σ-inverting localization, then every element in $\Sigma^{-1}R$ is an entry of the inverse of some matrix in $\lambda(\Sigma)$ and the following are equivalent:*

(a) $x \in \Sigma^{-1}R$,

(b) x *is a component of a solution* u *to some matrix equation of the form*

$$Au = e_j \quad (A \in \lambda(\Sigma)),$$

(c) x *is a component of a solution* u *to some matrix equation of the form*

$$Au = a \quad (A \in \lambda(\Sigma)),$$

(d) $x = bA^{-1}c$ *where* $A \in \lambda(\Sigma)$ *and* b *and* c *are an appropriately sized row and column over* $\Sigma^{-1}R$.

Proof. We note that $\Sigma^{-1}R$ is generated, as a ring, by the image of R along with the entries of inverses of matrices in $\lambda(\Sigma)$. Given an element $x \in R$, we may observe that $A = \left(\begin{smallmatrix} 1 & -x \\ 0 & 1 \end{smallmatrix}\right) \in \Sigma$ by upper-multiplicativity, and $\lambda(x)$ is the (1,2) entry of $\lambda(A)^{-1}$. Furthermore, suppose that a is the (i,k) entry of $A^{-1} = \lambda(A')^{-1}$ and b is the (m,ℓ) entry of $B^{-1} = \lambda(B')^{-1}$ for some $A', B' \in \Sigma$. It is easily verified that

$$\begin{pmatrix} 1 & e_i^\top & 0 & 0 \\ 0 & A & C & e_k \\ 0 & 0 & B & e_\ell \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -e_i^\top A^{-1} & e_i^\top A^{-1}CB^{-1} & a-b \\ 0 & A^{-1} & -A^{-1}CB^{-1} & -A^{-1}e_k + A^{-1}CB^{-1}e_\ell \\ 0 & 0 & B^{-1} & -B^{-1}e_\ell \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where C is the matrix of appropriate size whose m-th column is the i-th column of A and all other entries zero. Thus, the difference $a - b$ appears as an entry of the inverse of some matrix in $\lambda(\Sigma)$. Likewise,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix},$$

where C is the matrix of appropriate size with -1 in the (k,m) position and zero elsewhere. The submatrix $-A^{-1}CB^{-1}$ is then the product of the k-th column of A^{-1}

with the m-th row of B^{-1}. In particular, entry (i,ℓ) of $-A^{-1}CB^{-1}$ is ab, thereby ab appears as an entry of the inverse of some matrix in $\lambda(\Sigma)$. We have now shown that the entries of inverses of matrices in $\lambda(\Sigma)$ form a subring of $\Sigma^{-1}R$ which contains the generators of $\Sigma^{-1}R$, hence every element in $\Sigma^{-1}R$ is of this form.

To see (a) $\Rightarrow$ (b), we use the fact that x appears as the (i,j) entry of A^{-1} for some $A \in \lambda(\Sigma)$; setting u as the j-th column of A^{-1}, we see that $Au = e_j$ and the i-th component of u is x. The implications (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a) are clear. Whenever x is the i-th component of a solution to an equation of the form $Au = a$, we have $x = e_i^\top u = e_i^\top A^{-1}a$, proving that (c) $\Rightarrow$ (d). $\square$

We now state a major result of Cohn which gives a criterion for the existence of an epimorphism into a skew field. As mentioned previously, an outline of Cohn's original proof is provided here, but the constructions in the sections that follow give a framework for a simpler proof that we shall discuss in greater detail in Section 4.5.

Theorem 4.2.7 (Cohn, 1971 [4, Thm. 7.5.3])**.** *Let R be a ring and $\mathcal{P}$ be a prime matrix ideal in R. There exists a homomorphism $f\colon R \to K$ into a skew field such that K is generated (as a field) by the image of R and $\mathcal{P}$ is the singular kernel of f.*

Proof Sketch. Let $\Sigma = \mathcal{M}_\square(R) \setminus \mathcal{P}$. For each $A \in \Sigma$ and $a \in {}^nR$, we consider the augmented matrix $(A|a)$. The matrix obtained by replacing the first column of A with a column a' is denoted $A_{a'}$. When the matrix A is understood, we use a_1 to denote the first column of A. Let $M = \{(A|a) \mid A \in \Sigma \text{ and } A_{a_1} \in \Sigma\}$. Multiplication can be defined on M by the formula

$$(A|a)(B|b) = \left(\begin{array}{cc|c} A & \begin{pmatrix} a & 0 \end{pmatrix} & 0 \\ 0 & B & b \end{array}\right)$$

for a block of zeros of appropriate size. One can verify that the right side of this system is indeed an element of M and that this operation is associative, giving M a

semigroup structure. The subset $M' \subset M$ consisting of all $(A|a)$ for which $A_{a_1+a} \in \mathcal{P}$ is a subsemigroup of M. Furthermore, M' satisfies the following two properties:

(i) $(A|a)(B|b) \in M'$ implies $(B|b)(A|a) \in M'$,

(ii) there exists an involution $\iota\colon M \to M$ such that $(A|a)\iota(A|a) \in M'$.

The first of these properties is easily verified by permuting rows and columns of the products; each of these is an operation that preserves the property of inclusion in M'. The involution $\iota(A|a) = (A_a|a_1)$ satisfies the second property. Conditions (i) and (ii) are sufficient to give $M/{\sim}$ the structure of a group, where $(A|a) \sim (B|b)$ whenever there exists $(C|c) \in M'$ such that $(A|a)(C|c)\iota(B|b) \in M'$. For details on this, we direct the reader to [4, Lem. 7.5.2]. The resulting group $G = M/{\sim}$ will be the multiplicative group of the requisite skew field. On $M_1 = M \setminus M'$, we define a function θ' by

$$\theta'(A|a) = (A_{-a_1}|a_1 + a).$$

This function is well-defined and respects the relation $\sim$, hence descends to an induced map $\theta\colon G_{\neq 1} \to G_{\neq 1}$. After verifying that θ satisfies the conditions of Lemma 4.2.5, we obtain a skew field K whose multiplicative group of units is G. Checking these conditions requires careful manipulation of the augmented matrices involved using row and column operations modulo M'. Finally, a mapping $\lambda\colon R \to K$ is given by

$$\lambda(r) = \begin{cases} 0 & \text{if } r \in \mathcal{P}, \\ (1|-r)_{M'} & \text{otherwise,} \end{cases}$$

where $(A|a)_{M'}$ denotes an equivalence class modulo M'. Upon verifying the details, one finds that the map λ is a ring homomorphism with singular kernel $\mathcal{P}$. $\square$

4.3 The Zigzag Method

The work of Cohn provided a brilliant answer to the important question of classifying homomorphisms into skew fields, but initially had two disadvantages. The first was that the construction was indirect in describing the skew field and the second was that it could not be easily extended to more general Σ-inverting homomorphisms. Two papers by Peter Malcolmson in 1978 [22] and 1982 [23] addressed these concerns by providing a direct construction of the Cohn localization of a ring. Malcolmson referred to his methodology as the *zigzag method*.

For the remainder of this section, we will assume that Σ is upper-multiplicative; occasionally this will be mentioned explicitly for emphasis.

As we have seen in Theorem 4.2.6, if $\lambda\colon R \to \Sigma^{-1}R$ is the universal localization of R, then every element of $\Sigma^{-1}R$ is of the form $\lambda(x)\lambda(A)^{-1}\lambda(u)$ for some $A \in \Sigma$, $x \in R^n$, and $u \in {}^nR$. The idea behind Malcolmson's construction is to represent each element $\lambda(x)\lambda(A)^{-1}\lambda(u)$ by a triple (x, A, u). Given an $n \times n$ matrix $A \in \Sigma$, a row $x \in R^n$, and a column $u \in {}^nR$, the triple (x, A, u) is called an *allowable* triple over Σ. We shall introduce a relation $\sim$ on the set of all allowable triples over Σ by declaring $(x, A, u) \sim (y, B, v)$ if there exist $L, M, P, Q \in \Sigma$, rows ℓ and p, and columns m and q over R of the appropriate sizes such that

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & u \\ 0 & B & 0 & 0 & -v \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & M & m \\ \hline x & y & \ell & 0 & 0 \end{array}\right) = \left(\frac{P}{p}\right)\left(\begin{array}{c|c} Q & q \end{array}\right). \tag{4.1}$$

To assist with computations in the proofs that follow, we introduce some notation. Suppose that (n_i) is a list of m positive integers. For each $1 \le j \le m$, we define $E_j^{(n_i)}$ to be the $\left(\sum_i n_i\right) \times n_j$ matrix consisting of m vertically arranged blocks, with block j the $n_j \times n_j$ identity matrix and block i (for $i \ne j$) a zero matrix of size $n_i \times n_j$. For instance, $E_2^{(1,2,2)^\top} = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}\right)$. If $X = (X_i)$ is a list of matrices in Σ where each

X_i is an $n_i \times n_i$ matrix, then we also write E_j^X to denote $E_j^{(n_i)}$. Throughout this section, I is reserved for an identity matrix whose size can be deduced from context.

Lemma 4.3.1. *If $\Sigma \subset \mathcal{M}_\square(R)$ is upper-multiplicative and the equation*

$$\left(\begin{array}{cc|c} A & 0 & u \\ 0 & B & -v \\ \hline x & y & 0 \end{array}\right) = \left(\begin{array}{c} P \\ \hline p \end{array}\right) \left(\begin{array}{c|c} Q & q \end{array}\right).$$

holds for some $P, Q \in \Sigma$, a row p, and a column q, then $(x, A, u) \sim (y, B, v)$.

Proof. If the equation in the statement holds, then so does

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & u \\ 0 & B & 0 & 0 & -v \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline x & y & 1 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc} P & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline p & 1 & 0 \end{array}\right) \left(\begin{array}{ccc|c} Q & 0 & 0 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

$\square$

Proposition 4.3.2. *If Σ is upper-multiplicative, then the relation $\sim$ on the set of allowable triples over Σ is an equivalence relation.*

Proof. Let $(x, A, u), (y, B, v), (z, C, w)$ be allowable triples over Σ. We note that

$$\left(\begin{array}{cc|c} A & 0 & u \\ 0 & A & -u \\ \hline x & x & 0 \end{array}\right) = \left(\begin{array}{cc} A & -I \\ 0 & I \\ \hline x & 0 \end{array}\right) \left(\begin{array}{cc|c} I & I & 0 \\ 0 & A & -u \end{array}\right),$$

hence $(x, A, u) \sim (x, A, u)$ and so $\sim$ is reflexive. If $(x, A, u) \sim (y, B, v)$ with

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & u \\ 0 & B & 0 & 0 & -v \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & M & m \\ \hline x & y & \ell & 0 & 0 \end{array}\right) = \left(\begin{array}{c} P \\ \hline p \end{array}\right) \left(\begin{array}{c|c} Q & q \end{array}\right),$$

then $(y, B, v) \sim (x, A, u)$, as seen with $E_2 = E_2^{(A,B,L,M)}$ in the block factorization

$$\left(\begin{array}{c|cccc|c|c} B & 0 & 0 & 0 & 0 & 0 & v \\ \hline 0 & A & 0 & 0 & 0 & 0 & -u \\ 0 & 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M & 0 & -m \\ \hline 0 & 0 & 0 & 0 & 0 & B & v \\ \hline y & x & 0 & \ell & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc} B & E_2^\top P & 0 \\ 0 & P & -E_2 \\ 0 & 0 & I \\ \hline y & p & 0 \end{array}\right) \left(\begin{array}{ccc|c} I & -E_2^\top & -I & 0 \\ 0 & Q & QE_2 & -q \\ 0 & 0 & B & v \end{array}\right).$$

This proves that $\sim$ is symmetric. If we additionally assume that $(y, B, v) \sim (z, C, w)$ via the factorization

$$\left(\begin{array}{cccc|c} B & 0 & 0 & 0 & v \\ 0 & C & 0 & 0 & -w \\ 0 & 0 & L' & 0 & 0 \\ 0 & 0 & 0 & M' & m' \\ \hline y & z & \ell' & 0 & 0 \end{array}\right) = \left(\begin{array}{c} P' \\ \hline p' \end{array}\right) \left(\begin{array}{c|c} Q' & q' \end{array}\right),$$

then the following block factorization shows that $(z, C, w) \sim (x, A, u)$, proving transitivity.

$$\left(\begin{array}{c|cccc|c|cccc|c|c} C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \\ \hline 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u \\ 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & L' & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & -v \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C & 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M' & 0 & -m' \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & m \\ \hline z & x & y & \ell & 0 & \ell' & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccccc} C & 0 & 0 & E_2^{Y^\top} P' & 0 \\ 0 & P & 0 & E_2^X E_1^{Y^\top} P' & E_4^X \\ 0 & 0 & L' & E_3^{Y^\top} P' & 0 \\ 0 & 0 & 0 & P' & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline z & p & \ell' & p' & 0 \end{array}\right) \left(\begin{array}{ccccc|c} I & 0 & 0 & -E_2^{Y^\top} & 0 & 0 \\ 0 & Q & 0 & -QE_2^X E_1^{Y^\top} & -QE_4^X & -q \\ 0 & 0 & I & -E_3^{Y^\top} & 0 & 0 \\ 0 & 0 & 0 & Q' & 0 & -q' \\ 0 & 0 & 0 & 0 & M & m \end{array}\right),$$

where $X = (A, B, L, M)$ and $Y = (B, C, L', M')$. $\square$

The equivalence class of an allowable triple (x, A, u) is denoted $[x, A, u]$ and the set of all such equivalence classes is denoted $T(\Sigma)$.

We shall define binary operations on $T(\Sigma)$ in the following proposition.

Proposition 4.3.3. *If Σ is upper-multiplicative, then the following formulas*

$$\begin{aligned} [x,A,u]+[y,B,v] &= \left[\begin{pmatrix} x & y \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right], \\ [x,A,u][y,B,v] &= \left[\begin{pmatrix} x & 0 \end{pmatrix}, \begin{pmatrix} A & -uy \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 \\ v \end{pmatrix}\right], \end{aligned} \tag{4.2}$$

provide well-defined binary operations on $T(\Sigma)$.

Proof. Closure under these operations is apparent from upper-multiplicativity of Σ. Suppose that $(x,A,u) \sim (y,B,v)$ with a factorization

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & u \\ 0 & B & 0 & 0 & -v \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & M & m \\ \hline x & y & \ell & 0 & 0 \end{array}\right) = \left(\begin{array}{c} P \\ \hline p \end{array}\right) \left(\begin{array}{c|c} Q & q \end{array}\right).$$

Now, with $E_j = E_j^{(A,B,L,M)}$, we consider the following two block factorizations. First, we note that

$$\begin{aligned} &\left(\begin{array}{ccccc|cccc|c} A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u \\ 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \\ 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & -v \\ 0 & 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & -w \\ 0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & -v \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & m \\ \hline x & z & y & z & \ell & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ &= \left(\begin{array}{cccccc} A & 0 & 0 & 0 & 0 & E_1^\top P \\ 0 & C & 0 & -I & 0 & 0 \\ 0 & 0 & B & 0 & 0 & E_2^\top P \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & L & E_3^\top P \\ 0 & 0 & 0 & 0 & 0 & P \\ \hline x & z & y & 0 & \ell & p \end{array}\right) \left(\begin{array}{cccccc|c} I & 0 & 0 & 0 & 0 & -E_1^\top & 0 \\ 0 & I & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & -E_2^\top & 0 \\ 0 & 0 & 0 & C & 0 & 0 & -w \\ 0 & 0 & 0 & 0 & I & -E_3^\top & 0 \\ 0 & 0 & 0 & 0 & 0 & Q & q \end{array}\right), \end{aligned} \tag{4.3}$$

and secondly, we may observe that

$$\left(\begin{array}{ccccc|cccc|c|c}
A & -uz & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \\
0 & 0 & B & -vz & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & -w \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & -uz & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & vz & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & -mz & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C & w \\
\hline
x & 0 & y & 0 & \ell & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$$= \left(\begin{array}{ccccccc}
A & -uz & 0 & 0 & 0 & E_1^\top P & 0 \\
0 & C & 0 & 0 & 0 & 0 & I \\
0 & 0 & B & -vz & 0 & E_2^\top P & 0 \\
0 & 0 & 0 & C & 0 & 0 & -I \\
0 & 0 & 0 & 0 & L & E_3^\top P & 0 \\
0 & 0 & 0 & 0 & 0 & P & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I \\
\hline
x & 0 & y & 0 & \ell & p & 0
\end{array}\right)
\left(\begin{array}{ccccccc|c}
I & 0 & 0 & 0 & 0 & -E_1^\top & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & I & 0 & 0 & -E_2^\top & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & -E_3^\top & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q & -qz & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C & w
\end{array}\right). \tag{4.4}$$

The factorization in (4.3) demonstrates that $[x, A, u] + [z, C, w] = [y, B, v] + [z, C, w]$. Likewise, we see $[x, A, u][z, C, w] = [y, B, v][z, C, w]$ by the factorization in (4.4), and so addition and multiplication on the right by a fixed element are well-defined functions of equivalence classes. Similar factorizations show that addition and multiplication on the left are well-defined. □

The operations defined in Equations (4.2) are motivated by our interpretation of the triple $[x, A, u]$ as a formal representation of the element $\lambda(x)\lambda(A)^{-1}\lambda(u)$ in $\Sigma^{-1}R$. Concretely, in any model of the universal localization $\lambda\colon R \to \Sigma^{-1}R$, we have

$$\begin{aligned}
&\lambda(x)\lambda(A)^{-1}\lambda(u) + \lambda(y)\lambda(B)^{-1}\lambda(v) \\
&\qquad = \begin{pmatrix} \lambda(x) & \lambda(y) \end{pmatrix} \begin{pmatrix} \lambda(A)^{-1} & 0 \\ 0 & \lambda(B)^{-1} \end{pmatrix} \begin{pmatrix} \lambda(u) \\ \lambda(v) \end{pmatrix} \\
&\qquad = \lambda \begin{pmatrix} x & y \end{pmatrix} \lambda \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} \lambda \begin{pmatrix} u \\ v \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\lambda(x)\lambda(A)^{-1}\lambda(u)\lambda(y)\lambda(B)^{-1}\lambda(v) \\
&\qquad = \begin{pmatrix}\lambda(x) & 0\end{pmatrix}\begin{pmatrix}\lambda(A)^{-1} & \lambda(A)^{-1}\lambda(uy)\lambda(B)^{-1} \\ 0 & \lambda(B)^{-1}\end{pmatrix}\begin{pmatrix}0 \\ \lambda(v)\end{pmatrix} \\
&\qquad = \lambda\begin{pmatrix}x & 0\end{pmatrix}\lambda\begin{pmatrix}A & -uy \\ 0 & B\end{pmatrix}^{-1}\lambda\begin{pmatrix}0 \\ v\end{pmatrix}.
\end{aligned}$$

Lemma 4.3.4. *The following equations hold in $T(\Sigma)$ for all $A \in \Sigma$, $r \in R$, rows $x, x' \in R^n$, and columns $u, u' \in {}^nR$.*

$$\begin{aligned}
[x, A, u] + [x', A, u] &= [x + x', A, u], \\
[x, A, u] + [x, A, u'] &= [x, A, u + u'], \\
[1, 1, r][x, A, u] &= [rx, A, u], \\
[x, A, u][1, 1, r] &= [x, A, ur].
\end{aligned}$$

Proof. The statements follow from the following factorizations and Lemma 4.3.1

$$\left(\begin{array}{ccc|c} A & 0 & 0 & u \\ 0 & A & 0 & u \\ 0 & 0 & A & -u \\ \hline x & x' & x+x' & 0 \end{array}\right) = \left(\begin{array}{ccc} A & 0 & -I \\ 0 & A & -I \\ 0 & 0 & I \\ \hline x & x' & 0 \end{array}\right)\left(\begin{array}{ccc|c} I & 0 & I & 0 \\ 0 & I & I & 0 \\ 0 & 0 & A & -u \end{array}\right),$$

$$\left(\begin{array}{ccc|c} A & 0 & 0 & u \\ 0 & A & 0 & u' \\ 0 & 0 & A & -(u+u') \\ \hline x & x & x & 0 \end{array}\right) = \left(\begin{array}{ccc} A & -I & -I \\ 0 & I & 0 \\ 0 & 0 & I \\ \hline x & 0 & 0 \end{array}\right)\left(\begin{array}{ccc|c} I & I & I & 0 \\ 0 & A & 0 & u' \\ 0 & 0 & A & -(u+u') \end{array}\right),$$

$$\left(\begin{array}{ccc|c} 1 & -rx & 0 & 0 \\ 0 & A & 0 & u \\ 0 & 0 & A & -u \\ \hline 1 & 0 & rx & 0 \end{array}\right) = \left(\begin{array}{ccc} 1 & -rx & 0 \\ 0 & A & -I \\ 0 & 0 & I \\ \hline 1 & 0 & 0 \end{array}\right)\left(\begin{array}{ccc|c} 1 & 0 & rx & 0 \\ 0 & I & I & 0 \\ 0 & 0 & A & -u \end{array}\right),$$

$$\left(\begin{array}{ccc|c} A & -u & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & A & -ur \\ \hline x & 0 & x & 0 \end{array}\right) = \left(\begin{array}{ccc} A & -u & -I \\ 0 & 1 & 0 \\ 0 & 0 & I \\ \hline x & 0 & 0 \end{array}\right)\left(\begin{array}{ccc|c} I & 0 & I & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & A & -ur \end{array}\right).$$

$\square$

Theorem 4.3.5. *With the operations defined in* (4.2), $T(\Sigma)$ *is a ring.*

Proof. Associativity of both addition and multiplication follows readily from the definition of the operations. To show that addition is commutative, we consider the factorization

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & u \\ 0 & B & 0 & 0 & v \\ 0 & 0 & B & 0 & -v \\ 0 & 0 & 0 & A & -u \\ \hline x & y & y & x & 0 \end{array}\right) = \left(\begin{array}{cccc} A & 0 & 0 & -I \\ 0 & B & -I & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline x & y & 0 & 0 \end{array}\right) \left(\begin{array}{cccc|c} I & 0 & 0 & I & 0 \\ 0 & I & I & 0 & 0 \\ 0 & 0 & B & 0 & -v \\ 0 & 0 & 0 & A & -u \end{array}\right),$$

which shows that $[x, A, u]+[y, B, v] = [y, B, v]+[x, A, u]$. We easily verify that $[1, 1, 0]$ is an additive identity by noting that $[x, A, u] + [1, 1, 0] = [x, A, u]$, as justified by the following factorization

$$\left(\begin{array}{ccc|c} A & 0 & 0 & u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A & -u \\ \hline x & 1 & x & 0 \end{array}\right) = \left(\begin{array}{ccc} A & 0 & -I \\ 0 & 1 & 0 \\ 0 & 0 & I \\ \hline x & 1 & 0 \end{array}\right) \left(\begin{array}{ccc|c} I & 0 & I & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A & -u \end{array}\right).$$

The additive inverse of $[x, A, u]$ is $[x, A, -u]$; the equality $[x, A, u]+[x, A, -u] = [1, 1, 0]$ follows from

$$\left(\begin{array}{ccc|c} A & 0 & 0 & u \\ 0 & A & 0 & -u \\ 0 & 0 & 1 & 0 \\ \hline x & x & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc} A & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \\ \hline x & 0 & 1 \end{array}\right) \left(\begin{array}{ccc|c} I & I & 0 & 0 \\ 0 & A & 0 & -u \\ 0 & 0 & 1 & 0 \end{array}\right).$$

For brevity, we omit the proof of distributivity, though full details can be found in Malcolmson's paper [23]. By Lemma 4.3.4, $[1, 1, 1]$ is a multiplicative identity. $\square$

Theorem 4.3.6 (Malcolmson, 1982 [23])**.** *Let R be any ring and $\Sigma \subset \mathcal{M}_\square(R)$ be upper-multiplicative. The map $\lambda\colon R \to T(\Sigma)$ given by $\lambda(r) = [1, 1, r]$ is the universal Σ-inverting localization. In particular, $\Sigma^{-1}R \cong T(\Sigma)$.*

Proof. Using Lemma 4.3.4, it is clear that λ is a ring homomorphism. To see that λ is Σ-inverting, let $A = (a_{ij}) \in \Sigma$ and consider the matrix $B \in \mathcal{M}(T(\Sigma))$ whose (j, k) entry is $[e_j^\top, A, e_k]$, where $e_\ell \in {}^nR$ denotes a column whose ℓ-th entry is 1 and

all other entries are 0. We claim that B is the inverse of A. It suffices to check that each entry of $\lambda(A)B$ and $B\lambda(A)$ is given by the Kronecker delta in $T(\Sigma)$. To this end, let us compute the (i,k) entry of $\lambda(A)B$. We simplify the computation by repeated application of Lemma 4.3.4 to obtain

$$\begin{aligned}
\sum_j \lambda(a_{ij})[e_j^\top, A, e_k] &= \sum_j [1,1,a_{ij}][e_j^\top, A, e_k] \\
&= \sum_j [a_{ij}e_j^\top, A, e_k] \\
&= \left[\sum_j a_{ij}e_j^\top, A, e_k\right] \\
&= [e_i^\top A, A, e_k].
\end{aligned}$$

The factorization below then reveals that $[e_i^\top A, A, e_k] = [1,1,\delta_{ik}]$, as required.

$$\left(\begin{array}{cc|c} A & 0 & e_k \\ 0 & 1 & -\delta_{ik} \\ \hline e_i^\top A & 1 & 0 \end{array}\right) = \left(\begin{array}{cc} I & 0 \\ 0 & 1 \\ \hline e_i^\top & 1 \end{array}\right) \left(\begin{array}{cc|c} A & 0 & e_k \\ 0 & 1 & -\delta_{ik} \end{array}\right).$$

Universality of λ is also not hard to see. A nearly identical verification reveals that the (i,k) entry of $B\lambda(A)$ is $[1,1,\delta_{ik}]$, as claimed. If $f\colon R \to T$ is a Σ-inverting ring homomorphism, then the map $\bar{f}\colon T(\Sigma) \to T$ given by $\bar{f}([x,A,u]) = f(x)f(A)^{-1}f(u)$ is easily verified to be a well-defined ring homomorphism satisfying $\lambda \circ \bar{f} = f$. Further, we note that $[x,A,u] = [\sum_i x_ie_i^\top, A, \sum_j u_je_j] = \sum_{i,j}[1,1,x_i][e_i^\top, A, e_j][1,1,u_j] = \lambda(x)\lambda(A)^{-1}\lambda(u)$, where x_i and u_j denote the i-th and j-th component of x and u, respectively. In particular, any ring homomorphism $g\colon T(\Sigma) \to T$ satisfying $\lambda \circ g = f$ must satisfy

$$g([x,A,u]) = g(\lambda(x))g(\lambda(A))^{-1}g(\lambda(u)) = f(x)f(A)^{-1}f(u) = \bar{f}([x,A,u]);$$

uniqueness of the map $\bar{f}$ follows. $\square$

4.4 The Display Method

In later writings by Cohn [11], the approach of Malcolmson was adopted and further refined to give an explicit construction of the Cohn localization by defining an equivalence on a set of formal objects called displays. The goal of this section is to give an overview of the display method of constructing $\Sigma^{-1}R$.

Let us recall that a set $\Sigma \subset \mathcal{M}_\square(R)$ of square matrices over R is said to be *upper-multiplicative* if $1 \in \Sigma$ and for all $A, B \in \Sigma$ and $C \in \mathcal{M}(R)$,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \Sigma.$$

We also introduce another mild closure condition on Σ in this section. If, for all square matrices P and Q, the conditions $PQ \in \Sigma$ and $P, Q \in \Sigma$ are equivalent, then the set Σ is called *factor-stable*. When one considers that we seek to make the matrices in Σ invertible, both upper-multiplicativity and factor-stability become quite natural conditions. Neither tends to be very restrictive in practice.

We will assume that Σ is upper-multiplicative and factor-stable for the remainder of this section.

In any Σ-inverting ring, each element is of the form $\lambda(a) - \lambda(x)\lambda(A)^{-1}\lambda(u)$ (see Theorem 4.2.6). Comparable to the zigzag method proposed by Malcolmson, the motivation behind Cohn's construction is to represent each element of the form $\lambda(a) - \lambda(x)\lambda(A)^{-1}\lambda(u)$ in $\Sigma^{-1}R$ using a formal object.

Definition 4.4.1. Let R be a ring and $\Sigma \subset \mathcal{M}_\square(R)$ an upper-multiplicative and factor-stable set. A *display over* Σ is a block matrix

$$\begin{pmatrix} A & u \\ x & a \end{pmatrix} \tag{4.5}$$

where $A \in \Sigma$, $a \in R$, $u \in {}^nR$, and $x \in R^n$ for suitable n. A display of the form

$$\begin{pmatrix} I & 0 \\ 0 & a \end{pmatrix}$$

for an identity matrix I of any size is called a *scalar display*. If a display takes the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 & u_1 \\ 0 & A_2 & \ddots & \vdots & u_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & A_n & u_n \\ x_1 & x_2 & \cdots & x_n & a \end{pmatrix},$$

then it is called a *block-diagonal display*. The matrices $A_1, A_2, \ldots, A_n$ are called the *blocks* of the display.

We define the following *elementary operations* on a display, continuing in the notation of (4.5).

C.1(f) Replace u by $u + Af$ and a by $a + xf$ for any $f \in {}^nR$.

C.2(Q) Replace A by AQ and x by xQ for any $Q \in \Sigma$ of suitable size.

C.3(G, p) Replace the display with

$$\begin{pmatrix} A & 0 & u \\ 0 & G & 0 \\ x & p & a \end{pmatrix} \tag{4.6}$$

where $G \in \Sigma$ and $p \in R^n$.

R.1(g) Replace x by $x + gA$ and a by $a + gu$ for any $g \in {}^nR$.

R.2(P) Replace A by PA and u by Pu for any $P \in \Sigma$ of suitable size.

R.3(F, q) Replace the display with

$$\begin{pmatrix} A & 0 & u \\ 0 & F & q \\ x & 0 & a \end{pmatrix} \tag{4.7}$$

where $F \in \Sigma$ and $q \in {}^nR$.

When the argument to an elementary operation is unspecified, we refer to the operations simply as C.1–C.3 and R.1–R.3. The inverse operation of each of C.1–C.3 and R.1–R.3 is also considered an elementary operation; these are denoted by C.1^{-1}–C.3^{-1} and R.1^{-1}–R.3^{-1}. The operations C.3 and R.3 are each referred to as *inserting*

a trivial block, while the inverse operation of each is referred to as *removing* a trivial block.

The elementary operations define an equivalence relation on the set of all displays over Σ, where two displays are said to be equivalent if one can be obtained from the other by a finite (possibly empty) sequence of elementary operations. The set of equivalence classes of displays over Σ is denoted $D(\Sigma)$ and its elements are denoted with square, rather than round, brackets.

Lemma 4.4.2. *If Σ is upper-multiplicative and factor-stable, then Σ contains all permutation matrices.*

Proof. By upper-multiplicativity, $1 \in \Sigma$, hence the $n \times n$ identity matrix I_n is an element of Σ for all $n > 0$. Furthermore, if P is a permutation matrix, then $P^\top P = I_n \in \Sigma$, whence $P \in \Sigma$ by factor-stability. $\square$

As in the discussion of the zigzag method, I shall denote an identity matrix, with a subscript to resolve size ambiguity where necessary.

Proposition 4.4.3. *The following equalities hold in $D(\Sigma)$,*

$$\begin{bmatrix} G & 0 & 0 \\ 0 & A & u \\ p & x & a \end{bmatrix} = \begin{bmatrix} A & u \\ x & a \end{bmatrix} = \begin{bmatrix} F & 0 & q \\ 0 & A & u \\ 0 & x & a \end{bmatrix}$$

and

$$\begin{bmatrix} A & 0 & 0 & u \\ 0 & G & 0 & 0 \\ 0 & 0 & B & v \\ x & p & y & a \end{bmatrix} = \begin{bmatrix} A & 0 & u \\ 0 & B & v \\ x & y & a \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & u \\ 0 & F & 0 & q \\ 0 & 0 & B & v \\ x & 0 & y & a \end{bmatrix}.$$

That is, insertion or removal of a trivial block before or between the blocks of a block diagonal display (in addition to after the blocks, as in C.3 and R.3) yields an equivalent display.

Proof. We verify the statement only for the first equality; proof of each remaining equality is similar. Let P denote the permutation matrix such that $(G \oplus A)P = A \oplus G$. By Lemma 4.4.2, $P \in \Sigma$, therefore

$$\begin{aligned}
\begin{bmatrix} G & 0 & 0 \\ 0 & A & u \\ p & x & a \end{bmatrix} &= \begin{bmatrix} 0 & G & 0 \\ A & 0 & u \\ x & p & a \end{bmatrix} && \text{(by C.2}(P)) \\
&= \begin{bmatrix} A & 0 & u \\ 0 & G & 0 \\ x & p & a \end{bmatrix} && \text{(by R.2}(P)) \\
&= \begin{bmatrix} A & u \\ x & a \end{bmatrix} && \text{(by C.3}^{-1}(G,p)).
\end{aligned}$$

□

Proposition 4.4.3 is, in some sense, an extension of the elementary operations C.3 and R.3, and so its applications are also referred to as inserting or removing trivial blocks and will be labeled as C.3 or R.3 where appropriate.

Proposition 4.4.4. *The formulas*

$$\begin{aligned}
\begin{bmatrix} A & u \\ x & a \end{bmatrix} + \begin{bmatrix} B & v \\ y & b \end{bmatrix} &= \begin{bmatrix} A & 0 & u \\ 0 & B & v \\ x & y & a+b \end{bmatrix}, \\
\begin{bmatrix} A & u \\ x & a \end{bmatrix} \begin{bmatrix} B & v \\ y & b \end{bmatrix} &= \begin{bmatrix} A & uy & ub \\ 0 & B & v \\ x & ay & ab \end{bmatrix},
\end{aligned} \tag{4.8}$$

provide well-defined binary operations on $D(\Sigma)$.

Proof. It suffices to verify that, for each formula in (4.8), the application of an elementary operation to a representative display in either operand yields an equal output. We provide verification of this for the left operand for each of C.1–C.3; the remaining verifications are similar. Applying C.1(f) to the first summand in the addition

formula, we see that

$$\begin{aligned}
\begin{bmatrix} A & u+Af \\ x & a+xf \end{bmatrix} + \begin{bmatrix} B & v \\ y & b \end{bmatrix} &= \begin{bmatrix} A & 0 & u+Af \\ 0 & B & v \\ x & y & a+b+xf \end{bmatrix} \\
&= \begin{bmatrix} A & 0 & u \\ 0 & B & v \\ x & y & a+b \end{bmatrix} \qquad \text{(by C.1}^{-1}\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right)\text{)} \\
&= \begin{bmatrix} A & u \\ x & a \end{bmatrix} + \begin{bmatrix} B & v \\ y & b \end{bmatrix};
\end{aligned}$$

doing the same in the case of multiplication yields

$$\begin{aligned}
\begin{bmatrix} A & u+Af \\ x & a+xf \end{bmatrix} \begin{bmatrix} B & v \\ y & b \end{bmatrix} &= \begin{bmatrix} A & (u+Af)y & (u+Af)b \\ 0 & B & v \\ x & (a+xf)y & (a+xf)b \end{bmatrix} \\
&= \begin{bmatrix} A & uy+Afy & ub \\ 0 & B & v \\ x & ay+xfy & ab \end{bmatrix} \qquad \text{(by C.1}^{-1}\left(\begin{smallmatrix} fb \\ 0 \end{smallmatrix}\right)\text{)} \\
&= \begin{bmatrix} A & uy & ub \\ 0 & B & v \\ x & ay & ab \end{bmatrix} \qquad \text{(by C.2}\left(\begin{smallmatrix} I & -fy \\ 0 & I \end{smallmatrix}\right)\text{)} \\
&= \begin{bmatrix} A & u \\ x & a \end{bmatrix} \begin{bmatrix} B & v \\ y & b \end{bmatrix}.
\end{aligned}$$

We note that in the last application of C.2, the matrix $\left(\begin{smallmatrix} I & -fy \\ 0 & I \end{smallmatrix}\right)$ lies in Σ by upper-multiplicativity. Next, we consider the application of C.2(Q) to the first operand in each operation and compute

$$\begin{aligned}
\begin{bmatrix} AQ & u \\ xQ & a \end{bmatrix} + \begin{bmatrix} B & v \\ y & b \end{bmatrix} &= \begin{bmatrix} AQ & 0 & u \\ 0 & B & v \\ xQ & y & a+b \end{bmatrix} \\
&= \begin{bmatrix} A & 0 & u \\ 0 & B & v \\ x & y & a+b \end{bmatrix} \qquad \text{(by C.2}^{-1}\left(\begin{smallmatrix} Q & 0 \\ 0 & I \end{smallmatrix}\right)\text{)} \\
&= \begin{bmatrix} A & u \\ x & a \end{bmatrix} + \begin{bmatrix} B & v \\ y & b \end{bmatrix};
\end{aligned}$$

in the case of multiplication, we find

$$\begin{bmatrix} AQ & u \\ xQ & a \end{bmatrix} \begin{bmatrix} B & v \\ y & b \end{bmatrix} = \begin{bmatrix} AQ & uy & ub \\ 0 & B & v \\ xQ & ay & ab \end{bmatrix}$$

$$= \begin{bmatrix} A & uy & ub \\ 0 & B & v \\ x & ay & ab \end{bmatrix} \qquad \text{(by C.2}^{-1}\left(\begin{smallmatrix} Q & 0 \\ 0 & I \end{smallmatrix}\right))$$

$$= \begin{bmatrix} A & u \\ x & a \end{bmatrix} \begin{bmatrix} B & v \\ y & b \end{bmatrix}.$$

Again, the direct sum matrix $Q \oplus I$ lies in Σ by upper-multiplicativity. Finally, when a summand or factor is altered by C.3(G, p), then we may verify

$$\begin{bmatrix} A & 0 & u \\ 0 & G & 0 \\ x & p & a \end{bmatrix} + \begin{bmatrix} B & v \\ y & b \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & u \\ 0 & G & 0 & 0 \\ 0 & 0 & B & v \\ x & p & y & a+b \end{bmatrix} = \begin{bmatrix} A & 0 & u \\ 0 & B & v \\ x & y & a+b \end{bmatrix} \qquad \text{(by C.3}^{-1}(G,p))$$

$$= \begin{bmatrix} A & u \\ x & a \end{bmatrix} + \begin{bmatrix} B & v \\ y & b \end{bmatrix},$$

$$\begin{bmatrix} A & 0 & u \\ 0 & G & 0 \\ x & p & a \end{bmatrix} \begin{bmatrix} B & v \\ y & b \end{bmatrix} = \begin{bmatrix} A & 0 & uy & ub \\ 0 & G & 0 & 0 \\ 0 & 0 & B & v \\ x & p & ay & ab \end{bmatrix} = \begin{bmatrix} A & uy & ub \\ 0 & B & v \\ x & ay & ab \end{bmatrix} \qquad \text{(by C.3}^{-1}(G,p))$$

$$= \begin{bmatrix} A & u \\ x & a \end{bmatrix} \begin{bmatrix} B & v \\ y & b \end{bmatrix},$$

as claimed. ☐

Theorem 4.4.5. *With the operations defined in Equations* (4.8), $D(\Sigma)$ *is a ring.*

Proof. Closure under addition is clear, while closure under multiplication follows from upper-multiplicativity of Σ. Associativity of addition is obvious. To see that addition is commutative, we note that there is an appropriate permutation matrix $P \in \Sigma$ by Lemma 4.4.2 such that applying C.2(P) and R.2(P) yields

$$\begin{bmatrix} A & 0 & u \\ 0 & B & v \\ x & y & a+b \end{bmatrix} = \begin{bmatrix} 0 & A & u \\ B & 0 & v \\ y & x & a+b \end{bmatrix} = \begin{bmatrix} B & 0 & v \\ 0 & A & u \\ y & x & b+a \end{bmatrix}$$

The scalar display $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ is an additive identity in $D(\Sigma)$, as the sum of any display with this scalar display can be reduced by removal of a trivial block. We also observe that

$$\begin{aligned}
\begin{bmatrix}A & u\\ x & a\end{bmatrix} + \begin{bmatrix}A & -u\\ x & -a\end{bmatrix} &= \begin{bmatrix}A & 0 & u\\ 0 & A & -u\\ x & x & 0\end{bmatrix}\\
&= \begin{bmatrix}A & A & 0\\ 0 & A & -u\\ x & x & 0\end{bmatrix} && \left(\text{by R.2}\left(\begin{smallmatrix}I & I\\ 0 & I\end{smallmatrix}\right)\right)\\
&= \begin{bmatrix}A & 0 & 0\\ 0 & A & -u\\ x & 0 & 0\end{bmatrix} && \left(\text{by C.2}\left(\begin{smallmatrix}I & -I\\ 0 & I\end{smallmatrix}\right)\right)\\
&= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & A & 0 & 0\\ 0 & 0 & A & -u\\ 0 & x & 0 & 0\end{bmatrix} && (\text{by C.3}(1,0) \text{ or R.3}(1,0))\\
&= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} && (\text{by C.3}^{-1}(A,x) \text{ and R.3}^{-1}(A,-u)),
\end{aligned}$$

and so every display has an additive inverse. Multiplication is easily seen to be associative, as

$$\begin{aligned}
\begin{bmatrix}A & u\\ x & a\end{bmatrix}\left(\begin{bmatrix}B & v\\ y & b\end{bmatrix}\begin{bmatrix}C & w\\ z & c\end{bmatrix}\right) &= \begin{bmatrix}A & u\\ x & a\end{bmatrix}\begin{bmatrix}B & vz & vc\\ 0 & C & w\\ y & bz & bc\end{bmatrix}\\
&= \begin{bmatrix}A & uy & ubz & ubc\\ 0 & B & vz & vc\\ 0 & 0 & C & w\\ x & ay & abz & abc\end{bmatrix}\\
&= \begin{bmatrix}A & uy & ub\\ 0 & B & v\\ x & ay & ab\end{bmatrix}\begin{bmatrix}C & w\\ z & c\end{bmatrix}\\
&= \left(\begin{bmatrix}A & u\\ x & a\end{bmatrix}\begin{bmatrix}B & v\\ y & b\end{bmatrix}\right)\begin{bmatrix}C & w\\ z & c\end{bmatrix}.
\end{aligned}$$

It is also easy to see that the scalar display $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ is the multiplicative identity in $D(\Sigma)$, as its product with any display can be reduced to that display by the removal of a trivial block. Finally, to see that multiplication distributes over addition, we consider

$$\begin{aligned}
&\begin{bmatrix} A & u \\ x & a \end{bmatrix}\begin{bmatrix} B & v \\ y & b \end{bmatrix} + \begin{bmatrix} A & u \\ x & a \end{bmatrix}\begin{bmatrix} C & w \\ z & c \end{bmatrix} \\
&\qquad = \begin{bmatrix} A & uy & ub \\ 0 & B & v \\ x & ay & ab \end{bmatrix} + \begin{bmatrix} A & uz & uc \\ 0 & C & w \\ x & az & ac \end{bmatrix} \\
&\qquad = \begin{bmatrix} A & uy & 0 & 0 & ub \\ 0 & B & 0 & 0 & v \\ 0 & 0 & A & uz & uc \\ 0 & 0 & 0 & C & w \\ x & ay & x & az & ab+ac \end{bmatrix} \\
&\qquad = \begin{bmatrix} A & uy & A & uz & u(b+c) \\ 0 & B & 0 & 0 & v \\ 0 & 0 & A & uz & uc \\ 0 & 0 & 0 & C & w \\ x & ay & x & az & ab+ac \end{bmatrix} && \text{(by R.2}\left(\begin{smallmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{smallmatrix}\right)\text{)} \\
&\qquad = \begin{bmatrix} A & uy & 0 & uz & u(b+c) \\ 0 & B & 0 & 0 & v \\ 0 & 0 & A & uz & uc \\ 0 & 0 & 0 & C & w \\ x & ay & 0 & az & a(b+c) \end{bmatrix} && \text{(by C.2}\left(\begin{smallmatrix} I & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{smallmatrix}\right)\text{)} \\
&\qquad = \begin{bmatrix} A & u \\ x & a \end{bmatrix}\left(\begin{bmatrix} B & v \\ y & b \end{bmatrix} + \begin{bmatrix} A & u \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C & w \\ z & c \end{bmatrix}\right) \\
&\qquad = \begin{bmatrix} A & u \\ x & a \end{bmatrix}\left(\begin{bmatrix} B & v \\ y & b \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C & w \\ z & c \end{bmatrix}\right) && \text{(by R.3}(1,0)\text{ and R.3}^{-1}(A,u)\text{)} \\
&\qquad = \begin{bmatrix} A & u \\ x & a \end{bmatrix}\left(\begin{bmatrix} B & v \\ y & b \end{bmatrix} + \begin{bmatrix} C & w \\ z & c \end{bmatrix}\right),
\end{aligned}$$

as required. Proof of the right-distributive property is similar. □

As we mentioned previously, the display in (4.5) is meant to represent the expression $\lambda(a) - \lambda(x)\lambda(A)^{-1}\lambda(u)$. Through this interpretation, the sum and product of displays appropriately represents the sum and product of the formal expressions $\lambda(a) - \lambda(x)\lambda(A)^{-1}\lambda(u)$ and $\lambda(b) - \lambda(y)\lambda(B)^{-1}\lambda(v)$. More explicitly, one can verify

that each of the following equations holds in any model of $\Sigma^{-1}R$:

$$\begin{aligned}
&\lambda(a) - \lambda(x)\lambda(A)^{-1}\lambda(u) + \lambda(b) - \lambda(y)\lambda(B)^{-1}\lambda(v) \\
&\qquad = \lambda(a+b) - \begin{pmatrix}\lambda(x) & \lambda(y)\end{pmatrix}\begin{pmatrix}\lambda(A)^{-1} & 0 \\ 0 & \lambda(B)^{-1}\end{pmatrix}\begin{pmatrix}\lambda(u) \\ \lambda(v)\end{pmatrix} \\
&\qquad = \lambda(a+b) - \lambda\begin{pmatrix}x & y\end{pmatrix}\lambda\begin{pmatrix}A & 0 \\ 0 & B\end{pmatrix}^{-1}\lambda\begin{pmatrix}u \\ v\end{pmatrix}, \\
&(\lambda(a) - \lambda(x)\lambda(A)^{-1}\lambda(u))(\lambda(b) - \lambda(y)\lambda(B)^{-1}\lambda(v)) \\
&\qquad = \lambda(ab) - \begin{pmatrix}\lambda(x) & \lambda(ay)\end{pmatrix}\begin{pmatrix}\lambda(A)^{-1} & -\lambda(A)^{-1}\lambda(uy)\lambda(B)^{-1} \\ 0 & \lambda(B)\end{pmatrix}\begin{pmatrix}\lambda(ub) \\ \lambda(v)\end{pmatrix} \\
&\qquad = \lambda(ab) - \lambda\begin{pmatrix}x & ay\end{pmatrix}\lambda\begin{pmatrix}A & uy \\ 0 & B\end{pmatrix}^{-1}\lambda\begin{pmatrix}ub \\ v\end{pmatrix}.
\end{aligned}$$

Similarly, the elementary operations on a display can each be seen as a realization of some identity that must hold in $\Sigma^{-1}R$. For instance, C.1(f) corresponds to the following identity: $\lambda(a + xf) - \lambda(x)\lambda(A)^{-1}\lambda(u + Af) = \lambda(a) - \lambda(x)\lambda(A)^{-1}\lambda(u)$.

The following lemma sometimes simplifies the computation of the sum in $D(\Sigma)$.

Lemma 4.4.6. *The following equalities hold in* $D(\Sigma)$,

$$\begin{bmatrix}A & u \\ x & a\end{bmatrix} + \begin{bmatrix}A & u \\ y & b\end{bmatrix} = \begin{bmatrix}A & u \\ x+y & a+b\end{bmatrix}, \tag{4.9}$$

$$\begin{bmatrix}A & u \\ x & a\end{bmatrix} + \begin{bmatrix}A & v \\ x & b\end{bmatrix} = \begin{bmatrix}A & u+v \\ x & a+b\end{bmatrix}. \tag{4.10}$$

Proof. Given the displays in the statement, we have

$$\begin{aligned}
\begin{bmatrix}A & u \\ x & a\end{bmatrix} + \begin{bmatrix}A & u \\ y & b\end{bmatrix} &= \begin{bmatrix}A & 0 & u \\ 0 & A & u \\ x & y & a+b\end{bmatrix} \\
&= \begin{bmatrix}A & A & u \\ 0 & A & u \\ x & x+y & a+b\end{bmatrix} \qquad \text{(by C.2}\left(\begin{smallmatrix}I & I \\ 0 & I\end{smallmatrix}\right)\text{)} \\
&= \begin{bmatrix}A & 0 & 0 \\ 0 & A & u \\ x & x+y & a+b\end{bmatrix} \qquad \text{(by R.2}\left(\begin{smallmatrix}I & -I \\ 0 & I\end{smallmatrix}\right)\text{)} \\
&= \begin{bmatrix}A & u \\ x+y & a+b\end{bmatrix} \qquad \text{(by C.3}(A, x)\text{)}.
\end{aligned}$$

Likewise,

$$\begin{bmatrix} A & u \\ x & a \end{bmatrix} + \begin{bmatrix} A & v \\ x & b \end{bmatrix} = \begin{bmatrix} A & 0 & u \\ 0 & A & v \\ x & x & a+b \end{bmatrix}$$

$$= \begin{bmatrix} A & A & u+v \\ 0 & A & v \\ x & x & a+b \end{bmatrix} \qquad \text{(by R.2}\left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)\text{)}$$

$$= \begin{bmatrix} A & 0 & u+v \\ 0 & A & v \\ x & 0 & a+b \end{bmatrix} \qquad \text{(by C.2}\left(\begin{smallmatrix} I & -I \\ 0 & I \end{smallmatrix}\right)\text{)}$$

$$= \begin{bmatrix} A & u+v \\ x & a+b \end{bmatrix} \qquad \text{(by R.3}(A, v)\text{).}$$

□

We are now in a position to prove that $D(\Sigma)$ is the universal Σ-inverting ring.

Theorem 4.4.7. *Let R be any ring and $\Sigma \subset \mathcal{M}_\square(R)$ an upper-multiplicative and factor-stable set of matrices. The map $\lambda\colon R \to D(\Sigma)$ given by*

$$\lambda(r) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$$

is the universal Σ-inverting localization. In particular, $\Sigma^{-1}R \cong D(\Sigma)$.

Proof. Clearly λ is identity-preserving. Additivity follows from Lemma 4.4.6, while multiplicativity can be seen by removing trivial blocks as follows

$$\lambda(r)\lambda(r') = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & r' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & rr' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & rr' \end{bmatrix} = \lambda(rr').$$

To see that λ is Σ-inverting, let $A = (a_{ij}) \in \Sigma$. Entrywise application of λ to A yields the matrix $A' = (a'_{ij}) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & a_{ij} \end{smallmatrix}\right]\right) \in \mathcal{M}(D(\Sigma))$. We define $B = (b_{jk}) \in \mathcal{M}(D(\Sigma))$ where $b_{jk} = \left[\begin{smallmatrix} A & -e_k \\ e_j^\top & 0 \end{smallmatrix}\right]$. Let us consider $A'B = (c_{ik})$ where $c_{ik} = \sum_j a'_{ij} b_{jk}$. We may then evaluate

$$c_{ik} = \sum_j \begin{bmatrix} 1 & 0 \\ 0 & a_{ij} \end{bmatrix} \begin{bmatrix} A & -e_k \\ e_j^\top & 0 \end{bmatrix} = \sum_j \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & -e_k \\ 0 & a_{ij} e_j^\top & 0 \end{bmatrix} = \sum_j \begin{bmatrix} A & -e_k \\ a_{ij} e_j^\top & 0 \end{bmatrix}$$

by removing trivial blocks in the last step. Repeated application of Lemma 4.4.6 then yields

$$\begin{aligned}
\sum_j \begin{bmatrix} A & -e_k \\ a_{ij}e_j^\top & 0 \end{bmatrix} &= \begin{bmatrix} A & -e_k \\ \sum_{j=1}^n a_{ij}e_j^\top & 0 \end{bmatrix} \\
&= \begin{bmatrix} A & -e_k \\ e_i^\top A & 0 \end{bmatrix} \\
&= \begin{bmatrix} I & -e_k \\ e_i^\top & 0 \end{bmatrix} && \text{(by C.2}^{-1}(A)) \\
&= \begin{bmatrix} I & 0 \\ e_i^\top & e_i^\top e_k \end{bmatrix} && \text{(by C.1}(e_k)) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & e_i^\top e_k \end{bmatrix} && \text{(by C.3}(1,0) \text{ and C.3}^{-1}(I, e_i^\top)) \\
&= \delta_{ik},
\end{aligned}$$

where δ_{ik} denotes the Kronecker delta in $D(\Sigma)$. It follows that $A'B = I$. A similar verification shows that $BA' = I$. Thus, B is the inverse of $A' = \lambda(A)$ and so $\lambda(A)$ is invertible in $D(\Sigma)$.

All that remains to be seen is the universal property of $D(\Sigma)$. Suppose that $f\colon R \to T$ is a Σ-inverting ring homomorphism. We define $\bar{f}\colon D(\Sigma) \to T$ by

$$\bar{f}\left(\begin{bmatrix} A & u \\ x & a \end{bmatrix}\right) = f(a) - f(x)f(A)^{-1}f(u). \tag{4.11}$$

By assumption, f is Σ-inverting, so the formula in (4.11) is at least sensible. It is easy to verify that the application of elementary operations to a representative display does not change the $\bar{f}$ value, thus $\bar{f}$ is a well-defined function on equivalence classes.

We may further observe that

$$\begin{aligned}
\bar{f}\left(\begin{bmatrix} A & 0 & u \\ 0 & B & v \\ x & y & a+b \end{bmatrix}\right) &= f(a+b) - f\begin{pmatrix} x & y \end{pmatrix} f\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} f\begin{pmatrix} u \\ v \end{pmatrix} \\
&= f(a) + f(b) - \begin{pmatrix} f(x) & f(y) \end{pmatrix} \begin{pmatrix} f(A)^{-1} & 0 \\ 0 & f(B)^{-1} \end{pmatrix} \begin{pmatrix} f(u) \\ f(v) \end{pmatrix} \\
&= f(a) - f(x)f(A)^{-1}f(u) + f(b) - f(y)f(B)^{-1}f(v) \\
&= \bar{f}\left(\begin{bmatrix} A & u \\ x & a \end{bmatrix}\right) + \bar{f}\left(\begin{bmatrix} B & v \\ y & b \end{bmatrix}\right)
\end{aligned}$$

and

$$\begin{aligned}
\bar{f}\left(\begin{bmatrix} A & uy & ub \\ 0 & B & v \\ x & ay & ab \end{bmatrix}\right) &= f(ab) - f\begin{pmatrix} x & ay \end{pmatrix} f\begin{pmatrix} A & uy \\ 0 & B \end{pmatrix}^{-1} f\begin{pmatrix} ub \\ v \end{pmatrix} \\
&= f(a)f(b) - \begin{pmatrix} f(x) & f(ay) \end{pmatrix} \begin{pmatrix} f(A)^{-1} & -f(A)^{-1}f(uy)f(B)^{-1} \\ 0 & f(B) \end{pmatrix} \begin{pmatrix} f(ub) \\ f(v) \end{pmatrix} \\
&= f(a)f(b) - f(x)f(A)^{-1}f(u)f(b) \\
&\qquad + f(x)f(A)^{-1}f(u)f(y)f(B)^{-1}f(v) - f(a)f(y)f(B)^{-1}f(v) \\
&= \big(f(a) - f(x)f(A)^{-1}f(u)\big)\big(f(b) - f(y)f(B)^{-1}f(v)\big) \\
&= \bar{f}\left(\begin{bmatrix} A & u \\ x & a \end{bmatrix}\right)\bar{f}\left(\begin{bmatrix} B & v \\ y & b \end{bmatrix}\right),
\end{aligned}$$

thereby showing that $\bar{f}$ is additive and multiplicative.

Finally, for all $r \in R$, we have

$$\bar{f}(\lambda(r)) = \bar{f}\left(\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}\right) = f(r) - f(0)f(1)^{-1}f(0) = f(r).$$

To show uniqueness of $\bar{f}$, suppose now that $g\colon D(\Sigma) \to T$ is an alternative ring homomorphism satisfying $\lambda \circ g = f$. One can quickly verify by Lemma 4.4.6 that

$$\begin{bmatrix} A & u \\ x & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} - \sum_{i,j} \begin{bmatrix} 1 & 0 \\ 0 & x_i \end{bmatrix} \begin{bmatrix} A & -e_j \\ e_i^\top & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u_j \end{bmatrix}$$

where x_i and u_j are the i-th component of x and j-th component of u, respectively. Let us recall that the (i,j) entry of $\lambda(A)^{-1}$ is $\left[\begin{smallmatrix} A & -e_j \\ e_i^\top & 0 \end{smallmatrix}\right]$. Using the fact that g is a ring homomorphism, we then see that

$$\begin{aligned}
g\left(\begin{bmatrix} A & u \\ x & a \end{bmatrix}\right) &= g\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} - \sum_{i,j} \begin{bmatrix} 1 & 0 \\ 0 & x_i \end{bmatrix} \begin{bmatrix} A & -e_j \\ e_i^\top & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u_j \end{bmatrix}\right) \\
&= g\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right) - g\left(\sum_{i,j} \begin{bmatrix} 1 & 0 \\ 0 & x_i \end{bmatrix} \begin{bmatrix} A & -e_j \\ e_i^\top & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u_j \end{bmatrix}\right) \\
&= g(\lambda(a)) - g(\lambda(x)\lambda(A)^{-1}\lambda(u)) \\
&= g(\lambda(a)) - g(\lambda(x))g(\lambda(A))^{-1}g(\lambda(u)) \\
&= f(a) - f(x)f(A)^{-1}f(u) \\
&= \bar{f}\left(\begin{bmatrix} A & u \\ x & a \end{bmatrix}\right).
\end{aligned}$$

Thus, λ is universally Σ-inverting and it follows that $\Sigma^{-1}R \cong D(\Sigma)$. $\square$

4.5 Generalities

The main goal of the constructions in Sections 4.3 and 4.4 is to provide a method for detecting the kernel of the map $\lambda\colon R \to \Sigma^{-1}R$.

Theorem 4.5.1. *Let R be a ring and $\Sigma \subset \mathcal{M}_{\square}(R)$ an upper-multiplicative and factor-stable set of matrices. The following are equivalent:*

(a) the display $\left[\begin{smallmatrix} A & u \\ x & a \end{smallmatrix}\right]$ is the zero element of $D(\Sigma)$, and

(b) there exist $F, G, P, Q \in \Sigma$, rows f and p, and columns g and q such that

$$\begin{pmatrix} A & 0 & 0 & u \\ 0 & F & 0 & 0 \\ 0 & 0 & G & g \\ x & f & 0 & a \end{pmatrix} = \begin{pmatrix} P \\ p \end{pmatrix} \begin{pmatrix} Q & q \end{pmatrix}. \tag{4.12}$$

Furthermore, if we take $a = 0$ in both (a) and (b), then each of these conditions is equivalent to

(c) the triple $[x, A, u]$ is the zero element of $T(\Sigma)$.

Proof. We first prove (a) $\Leftrightarrow$ (b). If the display in (a) represents the zero element of $D(\Sigma)$, then there is some finite sequence of elementary operations that take the display $\left(\begin{smallmatrix} A & u \\ x & a \end{smallmatrix}\right)$ to the scalar display $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The latter display is clearly non-full. It is straightforward to verify that each of the elementary operations except R.3^{-1} and C.3^{-1} preserves the property of being non-full. Thus, by reversing the sequence of elementary operations and omitting any steps that remove trivial blocks, we obtain a non-full display which can be obtained from $\left(\begin{smallmatrix} A & u \\ x & a \end{smallmatrix}\right)$ only by insertion of trivial blocks. Since this display is non-full, it either provides a factorization of the form (4.12), possibly after permuting the rows and columns, or if too few trivial blocks were added to be a factorization of this form, then we may invoke Lemma 4.3.1 to complete the factorization. Conversely, given a factorization (4.12), the display on the left is equivalent to $\left(\begin{smallmatrix} A & u \\ x & a \end{smallmatrix}\right)$ by removal of two trivial blocks, so it suffices to show that the

larger display is equivalent to zero. The given factorization means that this larger display is simply $\left[\begin{smallmatrix} PQ & Pq \\ pQ & pq \end{smallmatrix}\right]$. We note that in $D(\Sigma)$,

$$\begin{aligned}
\begin{bmatrix} PQ & Pq \\ pQ & pq \end{bmatrix} &= \begin{bmatrix} P & Pq \\ p & pq \end{bmatrix} && \text{(by C.}2^{-1}(Q)) \\
&= \begin{bmatrix} I & q \\ p & pq \end{bmatrix} && \text{(by R.}2^{-1}(P)) \\
&= \begin{bmatrix} I & 0 \\ p & 0 \end{bmatrix} && \text{(by C.1}(-q)) \\
&= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} && \text{(by R.1}(-p)) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} && \text{(by removal of trivial blocks if necessary).}
\end{aligned}$$

To complete the proof, we show that (b) and (c) are equivalent when $a = 0$. If $[x, A, u] = [1, 1, 0]$ in $T(\Sigma)$ and $a = 0$, then there exist $F, G \in \Sigma$, rows f and p, and columns g and q such that

$$\begin{pmatrix} A & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & F & 0 & 0 \\ 0 & 0 & 0 & G & g \\ x & 1 & f & 0 & 0 \end{pmatrix} = \begin{pmatrix} P \\ p \end{pmatrix} \begin{pmatrix} Q & q \end{pmatrix}.$$

Viewing this left matrix as a block matrix

$$\left(\begin{array}{c|cc|c|c} A & 0 & 0 & 0 & u \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & F & 0 & 0 \\ \hline 0 & 0 & 0 & G & g \\ \hline x & 1 & f & 0 & 0 \end{array}\right)$$

shows that this factorization is of the form (4.12). Conversely, if (4.12) holds with $a = 0$, then so does

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & u \\ 0 & 0 & F & 0 & 0 \\ 0 & 0 & 0 & G & g \\ 1 & x & f & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & P \\ 1 & p \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q & q \end{pmatrix},$$

proving that $[1, 1, 0] = [x, A, u]$. $\square$

Corollary 4.5.2. *Let Σ be an upper-multiplicative and factor-stable set of matrices over R and $\lambda\colon R \to \Sigma^{-1}R$ the universal localization. An element $r \in R$ belongs to* $\ker\lambda$ *if and only if there exist $F, G \in \Sigma$, rows f and p, and columns g and q such that*

$$\begin{pmatrix} F & 0 & 0 \\ 0 & G & g \\ f & 0 & r \end{pmatrix} = \begin{pmatrix} P \\ p \end{pmatrix} \begin{pmatrix} Q & q \end{pmatrix}.$$

Proof. We may pass the problem to either of the models we have constructed for $\Sigma^{-1}R$. Let $\lambda'\colon R \to D(\Sigma)$ be the universal Σ-inverting map given by

$$\lambda'(r) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}.$$

Thus $r \in \ker\lambda' = \ker\lambda$ if and only if an equation of the form

$$\left(\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ \hline 0 & 0 & G & g \\ \hline 0 & f & 0 & r \end{array}\right) = \begin{pmatrix} P \\ p \end{pmatrix} \begin{pmatrix} Q & q \end{pmatrix}$$

holds. Viewing the left side as a block matrix, we obtain the result. $\square$

Displays heuristically represent elements of the form $\lambda(a) - \lambda(x)\lambda(A)^{-1}\lambda(u)$, while triples represent elements of the form $\lambda(x)\lambda(A)^{-1}\lambda(u)$. Theorem 4.2.6 implies that any element of $\Sigma^{-1}R$ can be represented in either form, although it is worth seeing a correspondence between the objects in $D(\Sigma)$ and those in $T(\Sigma)$.

A display whose scalar term is 0 is called a *homogeneous* display, and can be thought of as representing an element of the form $\lambda(x)\lambda(A)^{-1}\lambda(u)$, just as in Malcolmson's triples. There is then an obvious bijection between the set of homogeneous displays and the set of triples given by $\left(\begin{smallmatrix} A & u \\ x & 0 \end{smallmatrix}\right) \leftrightarrow (x, A, u)$. This bijection descends to one on equivalence classes, as the next proposition shows.

Proposition 4.5.3. *In $D(\Sigma)$, the equation*

$$\begin{bmatrix} A & u \\ x & 0 \end{bmatrix} = \begin{bmatrix} B & v \\ y & 0 \end{bmatrix} \tag{4.13}$$

holds if and only if $[x, A, u] = [y, B, v]$ is satisfied in $T(\Sigma)$.

Proof. Equation (4.13) holds precisely when the difference between these displays is zero. By Theorem 4.5.1, the difference is zero if and only if there exist $F, G, P, Q \in \Sigma$, rows f, p, and columns g, q such that

$$\begin{pmatrix} A & 0 & 0 & 0 & u \\ 0 & B & 0 & 0 & -v \\ 0 & 0 & F & 0 & 0 \\ 0 & 0 & 0 & G & g \\ x & y & f & 0 & 0 \end{pmatrix} = \begin{pmatrix} P \\ p \end{pmatrix} \begin{pmatrix} Q & q \end{pmatrix},$$

which is precisely the definition of equivalence of the triples (x, A, u) and (y, B, v). $\square$

To fully understand the correspondence, we address how non-homogeneous displays are related to homogeneous displays. In doing so, we will have described an isomorphism $D(\Sigma) \cong T(\Sigma)$. That these rings are abstractly isomorphic is no surprise, since each has the universal property of the universal Σ-inverting ring. Nonetheless, seeing a direct correspondence provides additional clarity to each approach.

Proposition 4.5.4. *Any display is equivalent to a homogeneous display.*

Proof. We note that

$$\begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & A & 0 & u \\ 0 & 0 & A & -u \\ 1 & x & x & -a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & -I \\ 0 & 0 & I \\ 1 & x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & I & I & 0 \\ 0 & 0 & A & -u \end{pmatrix},$$

whence by Theorem 4.5.1,

$$\begin{bmatrix} 1 & 0 & -a \\ 0 & A & u \\ 1 & x & 0 \end{bmatrix} - \begin{bmatrix} A & u \\ x & a \end{bmatrix} = 0,$$

proving that the representative display in the second summand is equivalent to the homogeneous representative display in the first summand. $\square$

The results of this section, in particular Corollary 4.5.2, provide a useful framework for proving Theorem 4.2.7 more constructively, and allow one to give more explicit conditions under which λ is an embedding. In addition, the models $T(\Sigma)$ and $D(\Sigma)$ lead to criteria for a ring to admit a homomorphism into a skew field. The full details of these expositions would take us too far astray, though we close this chapter by stating (without proof) some of these major results to illustrate the success of the schemes introduced here and to reemphasize the importance of Cohn localization in the study of noncommutative rings. These results can be found in [11]. The inquisitive reader may find interest in some of the other powerful results regarding $\Sigma^{-1}R$, especially in the case where R is a semifir, fir, or Sylvester domain. Many results of that nature can be found in [7], [9], [13], [14], and [15].

Theorem 4.5.5. *If R is a ring such that all identity matrices over R are full*[1] *and Σ is an upper-multiplicative factor-stable set of matrices over R, then $\Sigma^{-1}R$ is nonzero.*

Theorem 4.5.6. *If R is a ring such that all identity matrices over R are full and Σ is the set of all full matrices over R, then $\Sigma^{-1}R$ is a skew field.*

Theorem 4.5.7. *If $\mathcal{P}$ is a prime matrix ideal of R and $\Sigma = \mathcal{M}_{\square}(R) \setminus \mathcal{P}$, then $\Sigma^{-1}R$ is a local ring. If I is the unique maximal ideal of $\Sigma^{-1}R$, then the composite mapping $R \xrightarrow{\lambda} \Sigma^{-1}R \to (\Sigma^{-1}R)/I$ is a homomorphism into a skew field with singular kernel $\mathcal{P}$.*

Theorem 4.5.8. *Let R be any ring. Then there exists a homomorphism of R into a skew field if and only if none of the identity matrices over R can be written as a determinantal sum of non-full matrices.*

Theorem 4.5.9. *A ring R can be embedded in a skew field if and only if it is an integral domain and no non-zero scalar matrix can be written as a determinantal sum of non-full matrices.*

[1] A ring with this property is said to have *unbounded generating number* (UGN)

CHAPTER 5

PUSHOUTS AND AMALGAMATED FREE PRODUCTS

We recall that for any ring R, there is a correspondence between matrices over R and morphisms of free R-modules. From this viewpoint, we may interpret Cohn localization as the process of adjoining inverses of a collection of morphisms between free modules to a category of R-modules. Such was the thought process of G. M. Bergman, who studied the universal adjunction of additional morphisms to this category subject to relations [1]. When the morphisms of interest are mappings between finitely generated projective modules, these universal adjunctions always exist.

Using Morita equivalence, Bergman reduced many of the questions about these adjunctions to the case where the modules involved are left ideals of R. Under this reduction, extensions of the category of R-modules can be examined via certain amalgamated free products of rings.

On this front, Bergman made significant progress describing the module theory of these constructions. In general, however, it remains difficult to provide explicit models for many adjunctions. In particular, little has been established regarding the general adjunction of a two-sided inverse of a morphism between projective modules for an arbitrary ring, also called a universal localization of the ring.

When we restrict our view to more specific classes of rings and morphisms, concrete constructions of the universal localizations are sometimes available. For example, work of A. H. Schofield indicated that for a ring of the form $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$, where M is an (A, B)-bimodule, the universal localization of R with respect to a morphism of the type $\sigma\colon \left(\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} M \\ B \end{smallmatrix}\right)$ is $\mathcal{M}_2(T)$, a full matrix ring [27]. In 2006, D. Sheiham

described a presentation for the entry ring T, unifying some of the results of Schofield and others [29]. Sheiham's approach, in the manner of Bergman, was to recognize the universal localization of R as part of a certain pushout diagram, from which the universal property of the entry ring, T, could be extracted.

In this chapter, our main goal is to give an overview of Sheiham's theorem and provide a detailed version of his proof. To do so, we introduce Bergman's viewpoint on universal adjunctions. We then clarify the relationship between these adjunctions and pushouts in the category of rings. Finally, we conclude the chapter by putting these mechanisms to work in proving Sheiham's theorem.

5.1 Adjunctions to Module Categories and Universal Localizations

Given a ring R, the idea of a universal adjunction is to append new morphisms to the category of R-modules subject to some prescribed relations. Loosely speaking, we would also like for the resulting category to itself be a category of modules over a ring R' and for R to admit a ring homomorphism to R' so that the categories are related by the extension-of-scalars functor $R' \otimes_R -$. Furthermore, the resulting category should be "universal" in a sense that we make precise in the theorems that follow. The desired category may not exist in general, but is guaranteed whenever the codomain of the appended morphism is a finitely generated projective module. The results of this section are due to Bergman. For their proofs, we direct the reader to [1, Thms. 3.1, 3.2].

Theorem 5.1.1. *Let R be any ring, $\{M_i\}$ any family of unitary R-modules, and $\{P_i\}$ any family of unitary finitely generated projective R-modules. There exists a ring homomorphism $R \to R'$ and a family of R'-module homomorphisms $\{f_i \colon R' \otimes_R M_i \to R' \otimes_R P_i\}$ which is universal; that is, for any ring homomorphism $R \to T$ and any*

family of T-module homomorphisms $\{g_i\colon T\otimes_R M_i \to T\otimes_R P_i\}$, there exists a unique ring homomorphism $R' \to T$ which makes the following diagrams commute for all i.

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R' \arrow[d] \\ & T \end{tikzcd} \qquad \begin{tikzcd} T\otimes_R M_i \arrow[r, "g_i"] \arrow[d, "\cong"'] & T\otimes_R P_i \arrow[d, "\cong"] \\ T\otimes_{R'} R'\otimes_R M_i \arrow[r, "\mathbb{1}_T\otimes f_i"'] & T\otimes_{R'} R'\otimes_R P_i \end{tikzcd}$$

Theorem 5.1.1 describes the process of adjoining a family of morphisms to a module category. In addition to adjoining morphisms universally, we may wish to impose relations on morphisms in a such a category.

Theorem 5.1.2. *Let R be any ring, $\{M_i\}$ any family of unitary R-modules, $\{P_i\}$ any family of unitary finitely generated projective R-modules, and $\{f_i\colon M_i \to P_i\}$ a family of R-module homomorphisms. There exists a ring homomorphism $R \to R'$ universal with respect to the property that each R'-module homomorphism $\mathbb{1}_{R'}\otimes f_i\colon R'\otimes_R M_i \to R'\otimes_R P_i$ satisfies $\mathbb{1}_{R'}\otimes f_i = 0$; that is, for any ring homomorphism $R\to T$ such that $\mathbb{1}_T\otimes f_i = 0\colon T\otimes_R M_i \to T\otimes_R R'_i$, there exists a unique ring homomorphism $R'\to T$ which makes the following diagram commute.*

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R' \arrow[d] \\ & T \end{tikzcd}$$

We shall not need the full generality of these theorems, though they ensure the existence of the construction we proceed to discuss, namely the universal localization of a ring with respect to morphisms of projective modules.

Definition 5.1.3. Let R be a ring and $\Sigma = \{\sigma_i\}$ a set of R-module homomorphisms, $\sigma_i\colon P_i \to Q_i$. A ring homomorphism $\mu\colon R\to T$ is said to be Σ-*inverting* if, for each $\sigma_i \in \Sigma$, the induced T-module homomorphism $\mathbb{1}_T\otimes\sigma_i\colon T\otimes_R P_i \to T\otimes_R Q_i$ is an isomorphism. A Σ-inverting ring homomorphism $\lambda\colon R\to\Sigma^{-1}R$ is called the *universal Σ-inverting localization* of R if every Σ-inverting ring homomorphism $\mu\colon R\to T$ has

a unique factorization $R \xrightarrow{\lambda} \Sigma^{-1}R \to T$. In the case where Σ consists of a single morphism, σ, the ring $\Sigma^{-1}R$ is also denoted by $\sigma^{-1}R$. The ring $\Sigma^{-1}R$ is said to be the *universal* Σ*-inverting ring.*

It is worth remarking that if Σ' is a set of matrices over R, each $m \times n$ matrix in Σ' corresponds to a module homomorphism $\sigma\colon R^n \to R^m$. Denoting by Σ the set of these homomorphisms, the universal Σ'-inverting localization (that is, the Cohn localization) is isomorphic to the universal Σ-inverting localization. The existence of a Cohn localization, which we proved in Theorem 4.1.2, is subsumed by the following proposition, since R^n is a unitary finitely generated projective R-module for each $n > 0$.

Proposition 5.1.4. *If R is any ring and $\Sigma = \{\sigma_i\colon P_i \to Q_i\}$ is a set of morphisms between finitely generated projective modules, then the universal Σ-inverting localization of R exists.*

Proof. We apply first Theorem 5.1.1 to obtain a ring homomorphism $R \to R'$ with a family of maps $\{f_i\colon R' \otimes_R Q_i \to R' \otimes_R P_i\}$, then use Theorem 5.1.2 to obtain a ring homomorphism $R' \to \Sigma^{-1}R$ such that

$$\mathbb{1}_{\Sigma^{-1}R} \otimes (f_i \circ (\mathbb{1}_{R'} \otimes \sigma_i) - \mathbb{1}_{R'} \otimes \mathbb{1}_{Q_i}) = 0\colon \Sigma^{-1}R \otimes_{R'} R' \otimes_R Q_i \to \Sigma^{-1}R \otimes_{R'} R' \otimes_R Q_i$$

and

$$\mathbb{1}_{\Sigma^{-1}R} \otimes ((\mathbb{1}_{R'} \otimes \sigma_i) \circ f_i - \mathbb{1}_{R'} \otimes \mathbb{1}_{P_i}) = 0\colon \Sigma^{-1}R \otimes_{R'} R' \otimes_R P_i \to \Sigma^{-1}R \otimes_{R'} R' \otimes_R P_i$$

hold for each i. Imposing these relations means that each $\mathbb{1}_{\Sigma^{-1}R} \otimes f_i$ is the inverse of $\mathbb{1}_{\Sigma^{-1}R} \otimes \mathbb{1}_{R'} \otimes \sigma_i$, thereby the composite mapping $R \to R' \to \Sigma^{-1}R$ is Σ-inverting. Universality follows from Theorems 5.1.1 and 5.1.2. $\square$

5.2 Induced Morphisms and Pushouts

As we have done in the discussion of element-inverting localizations and Cohn localizations, we may consider a category of Σ-inverting homomorphisms from a ring R for a fixed collection Σ of R-module homomorphisms. If $\alpha\colon R \to T$ and $\beta\colon R \to T'$ are a pair of Σ-inverting ring homomorphisms, then a morphism from α to β is a ring homomorphism $f\colon T \to T'$ such that $\alpha \circ f = \beta$. As usual, this implies that the universal Σ-inverting ring is uniquely determined up to isomorphism. The category of Σ-inverting homomorphisms is denoted $\mathbf{Inv}(R, \Sigma)$ or, in the case where σ is the sole morphism in Σ, $\mathbf{Inv}(R, \sigma)$.

Definition 5.2.1. Suppose S is a ring admitting a morphism to a ring R and let $\tau\colon P_0 \to Q_0$ be an S-module homomorphism. Given a morphism $\sigma\colon P \to Q$ of R-modules, if there exist R-module isomorphisms $f_1\colon R \otimes_S P_0 \to P$ and $f_2\colon R \otimes_S Q_0 \to Q$ such that $f_1 \circ \sigma = (\mathbb{1}_R \otimes \tau) \circ f_2$, as in the diagram below, then σ is said to be *induced* by τ.

$$\begin{tikzcd} R \otimes_S P_0 \arrow[r, "\mathbb{1}_R \otimes \tau"] \arrow[d, "f_1"', "\cong"] & R \otimes_S Q_0 \arrow[d, "\cong"', "f_2"] \\ P \arrow[r, "\sigma"'] & Q \end{tikzcd}$$

We introduce another category, this time consisting of commutative squares. Let $\alpha\colon S \to R$ be a ring homomorphism and suppose $\sigma\colon P \to Q$ is an R-module homomorphism which is induced by an S-module homomorphism $\tau\colon P_0 \to Q_0$. If there is a universal τ-inverting localization of S, say $\kappa\colon S \to \tau^{-1}S$, then there is a category $\mathbf{Sq}(\alpha, \kappa)$ whose objects are triples (μ, R', α') making the following square of ring homomorphisms commute.

$$\begin{tikzcd} S \arrow[r, "\kappa"] \arrow[d, "\alpha"'] & \tau^{-1}S \arrow[d, "\alpha'"] \\ R \arrow[r, "\mu"'] & R' \end{tikzcd} \tag{5.1}$$

A morphism $(\mu, R', \alpha') \to (\mu', T, \beta')$ in this category is a ring homomorphism $\phi\colon R' \to T$ making the following diagram commute.

$$\begin{tikzcd} S \arrow[r, "\kappa"] \arrow[d, "\alpha"'] & \tau^{-1}S \arrow[d, "\alpha'"] \arrow[ddr, bend left, "\beta'"] & \\ R \arrow[r, "\mu"'] \arrow[drr, bend right, "\mu'"'] & R' \arrow[dr, "\phi"] & \\ & & T \end{tikzcd} \tag{5.2}$$

Theorem 5.2.2. *The categories* $\mathbf{Inv}(R, \sigma)$ *and* $\mathbf{Sq}(\alpha, \kappa)$ *are equivalent.*

Proof. First, we establish that the forgetful functor $\Phi\colon \mathbf{Sq}(\alpha, \kappa) \to \mathbf{Inv}(R, \sigma)$, which sends each object (μ, R', α') to μ and each morphism ϕ to the underlying ring homomorphism ϕ, is well-defined.

It suffices to show that for any object (μ, R', α') in $\mathbf{Sq}(\alpha, \kappa)$, the ring homomorphism μ is σ-inverting. By assumption, κ is τ-inverting, and so the induced mapping $\mathbb{1}_{\tau^{-1}S} \otimes \tau\colon \tau^{-1}S \otimes_S P_0 \to \tau^{-1}S \otimes_S Q_0$ is an isomorphism. Utilizing commutativity of the square (5.1) associated with (μ, R', α'), the following maps are isomorphisms, with each of the last three following from the previous one by a canonical isomorphism:

$$\begin{aligned} \mathbb{1}_{\tau^{-1}S} \otimes \tau\colon \tau^{-1}S \otimes_S P_0 &\cong \tau^{-1}S \otimes_S Q_0, \\ \mathbb{1}_{R'} \otimes \mathbb{1}_{\tau^{-1}S} \otimes \tau\colon R' \otimes_{\tau^{-1}S} \tau^{-1}S \otimes_S P_0 &\cong R' \otimes_{\tau^{-1}S} \tau^{-1}S \otimes_S Q_0, \\ \mathbb{1}_{R'} \otimes \tau\colon R' \otimes_S P_0 &\cong R' \otimes_S Q_0, \\ \mathbb{1}_{R'} \otimes \mathbb{1}_R \otimes \tau\colon R' \otimes_R R \otimes_S P_0 &\cong R' \otimes_R R \otimes_S Q_0. \end{aligned} \tag{5.3}$$

By assumption, σ is induced by τ; that is, $f_1 \circ \sigma \circ f_2^{-1} = \mathbb{1}_R \otimes \tau$ for some pair of isomorphisms $f_1\colon R \otimes_S P_0 \cong P$ and $f_2\colon R \otimes_S Q_0 \cong Q$. The last isomorphism of (5.3) can then be written $\mathbb{1}_{R'} \otimes (f_1 \circ \sigma \circ f_2^{-1})$. Composing with the isomorphisms $\mathbb{1}_{R'} \otimes f_1^{-1}$ and $\mathbb{1}_{R'} \otimes f_2$ on the appropriate sides shows that $\mathbb{1}_{R'} \otimes \sigma\colon R' \otimes_R P \cong R' \otimes_R Q$ is an isomorphism, making μ a σ-inverting map as claimed.

Since any morphism $\phi\colon (\mu, R', \alpha') \to (\mu', T, \beta')$ in $\mathbf{Sq}(\alpha, \kappa)$ makes the diagram (5.2) commute, such a map satisfies, in particular, $\mu \circ \phi = \mu'$, thereby making it a morphism in $\mathbf{Inv}(R, \sigma)$.

The inverse functor $\Psi\colon \mathbf{Inv}(R,\sigma) \to \mathbf{Sq}(\alpha,\kappa)$ shall be constructed via the universal property of $\tau^{-1}S$. To this end, suppose that $\mu\colon R \to R'$ is a σ-inverting ring homomorphism. In order to invoke the aforementioned universal property, we will prove that $\alpha \circ \mu$ is τ-inverting. By assumption, there are isomorphisms f_1 and f_2 such that $\mathbb{1}_{R'} \otimes \sigma = \mathbb{1}_{R'} \otimes (f_1^{-1} \circ (\mathbb{1}_R \otimes \tau) \circ f_2)\colon R' \otimes P \to R' \otimes Q$ is an isomorphism. Composition with the isomorphisms $\mathbb{1}_{R'} \otimes f_1$ and $\mathbb{1}_{R'} \otimes f_2^{-1}$ on the appropriate sides shows that $\mathbb{1}'_R \otimes \mathbb{1}_R \otimes \tau\colon R' \otimes_R R \otimes_S P_0 \to R' \otimes_R R \otimes_S Q_0$ is an isomorphism, and this map is naturally isomorphic to the map $\mathbb{1}_{R'} \otimes \tau\colon R' \otimes_S P_0 \to R' \otimes_S Q_0$. Thus, $\alpha \circ \mu$ is τ-inverting. By the universal property of κ, there is then a unique ring homomorphism $\alpha'\colon \tau^{-1}S \to R'$ satisfying $\kappa \circ \alpha' = \alpha \circ \mu$, making the triple (μ, R', α') an object of $\mathbf{Sq}(\alpha,\kappa)$. Consequently, to the object α in $\mathbf{Inv}(R,\sigma)$, Ψ shall assign (μ, R', α').

Finally, we show that a morphism $\phi\colon \mu \to \mu'$ in $\mathbf{Inv}(R,\sigma)$, which is a ring homomorphism $\phi\colon R \to T'$, is also a morphism $(\mu, R', \alpha') \to (\mu', T, \beta')$ in $\mathbf{Sq}(\alpha,\kappa)$, where α' and β' are ring homomorphisms uniquely satisfying $\kappa \circ \alpha' = \alpha \circ \mu$ and $\kappa \circ \beta' = \alpha \circ \mu'$, respectively. Let us observe that

$$\kappa \circ \alpha' \circ \phi = \alpha \circ \mu \circ \phi = \alpha \circ \mu'.$$

However, β' is the unique ring homomorphism with this property, hence $\beta' = \alpha' \circ \phi$, and so ϕ is indeed a morphism in $\mathbf{Sq}(\alpha,\kappa)$, completing the definition of Ψ.

The functors Φ and Ψ are clearly inverses, proving the claimed equivalence. □

The initial object $(\iota_1, R \sqcup_S \tau^{-1}S, \iota_2)$ in $\mathbf{Sq}(\alpha,\kappa)$ is called the *pushout* of α and κ in the category of rings. The ring $R \sqcup_S \tau^{-1}S$ is said to be the *amalgamated free product* of the rings R and $\tau^{-1}S$. Now is an opportune time to point out that the category $\mathbf{Sq}(\alpha,\kappa)$ is an instance of the following more general notion, though the generalization will not be needed in this chapter.

Definition 5.2.3. Given a family of rings $\mathcal{A} = \{A_i\}$ and a family of ring homomorphisms $\{\phi_i \colon C \to A_i\}$, there is a category whose objects are rings B equipped with a family of ring homomorphisms $\{\beta_i \colon A_i \to B\}$ that are *amalgamated* over C. This condition means that $\phi_i \circ \beta_i = \phi_j \circ \beta_j$ for all i, j. The initial object in this category is denoted $\coprod_C A_i$ and is called the *amalgamated free product* of the family $\mathcal{A}$ over C.

The equivalence proved in Theorem 5.2.2 gives us an alternative description for $\sigma^{-1}R$, as the following corollary reveals.

Corollary 5.2.4. *If $\sigma \colon P \to Q$ is an R-module homomorphism induced by an S-module homomorphism $\tau \colon P_0 \to Q_0$, then $\sigma^{-1}R \cong R \sqcup_S \tau^{-1}S$.*

Proof. The initial object in $\mathbf{Inv}(R, \sigma)$ is the universal localization $\lambda \colon R \to \sigma^{-1}R$, while the initial object in $\mathbf{Sq}(\alpha, \kappa)$ is $(\iota_1, R \sqcup_S \tau^{-1}S, \iota_2)$. Since initial objects in equivalent categories correspond, the statement follows. $\square$

5.3 Morphisms Between Complementary Summands

Projective modules are direct summands of free modules. In the special case where the ring R is a direct sum of two left ideals, say $R = P \oplus Q$, the universal localization of R with respect to a single morphism $\sigma \colon P \to Q$ can be recognized as a full matrix ring. The strategy of proof is to identify a homomorphism between modules over a familiar ring that induces σ and utilize the mechanism of Corollary 5.2.4. The familiar ring we will use is $\mathcal{U}_2(K)$, presuming R is a (K, K)-algebra and σ is a morphism of (R, K)-bimodules. This does not restrict the class of rings and morphisms to which the results apply, as we may always choose $K = \mathbb{Z}$.

Lemma 5.3.1. *Let R be a unitary associative (K,K)-algebra and suppose additionally that ${}_RR_K = {}_RP_K \oplus {}_RQ_K$. Given an (R,K)-bimodule homomorphism $\sigma\colon P \to Q$, the map*

$$\alpha_\sigma\colon \mathcal{U}_2(K) \to \mathrm{End}_R({}_RR) = \begin{pmatrix} \mathrm{Hom}_R(P,P) & \mathrm{Hom}_R(P,Q) \\ \mathrm{Hom}_R(Q,P) & \mathrm{Hom}_R(Q,Q) \end{pmatrix}$$

given by

$$\alpha_\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a\mathbb{1}_P & b\mathbb{1}_\sigma \\ 0 & c\mathbb{1}_Q \end{pmatrix}$$

is a ring homomorphism and σ is induced by the $\mathcal{U}_2(K)$-module homomorphism $\tau\colon \mathcal{U}_2(K)e_{11} \to \mathcal{U}_2(K)e_{22}$ whose assignment is $\tau(ue_{11}) = ue_{12}$.

Proof. Throughout this proof, we denote $M = \mathcal{M}_2(K)$ and $U = \mathcal{U}_2(K)$. Theorem 2.2.1 provides the decomposition of $\mathrm{End}_R({}_RR)$ which allows us to define α_σ. It is not hard to see that α_σ is a ring homomorphism; it is clearly additive and identity-preserving, and we have

$$\begin{aligned}\alpha_\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}\right) = \alpha_\sigma\begin{pmatrix} aa' & ab'+bc' \\ 0 & cc' \end{pmatrix} &= \begin{pmatrix} aa'\mathbb{1}_P & (ab'+bc')\sigma \\ 0 & cc'\mathbb{1}_Q \end{pmatrix} \\ &= \begin{pmatrix} a\mathbb{1}_P & b\sigma \\ 0 & c\mathbb{1}_Q \end{pmatrix} \circ \begin{pmatrix} a'\mathbb{1}_P & b'\sigma \\ 0 & c'\mathbb{1}_Q \end{pmatrix}.\end{aligned}$$

The last equality requires $a\mathbb{1}_P \circ b'\sigma + b\sigma \circ c'\mathbb{1}_Q = (ab'+bc')\sigma$, a straightforward verification if one keeps in mind the assumption that σ is a right K-module homomorphism.

To see that σ is induced by τ, we first note that the mapping $f\colon R \otimes_U U \cong R$ given by $f(r \otimes u) = r\gamma_R^{-1}(\alpha_\sigma(u))$ can be restricted to a pair of R-module isomorphisms, $R \otimes_U Ue_{11} \cong P$ and $R \otimes_U Ue_{22} \cong Q$, where $\gamma\colon R \cong \mathrm{End}_R({}_RR)$ denotes the canonical isomorphism. On an arbitrary tensor, $(p+q) \otimes ae_{11} \in R \otimes_U Ue_{11}$, we compute

$$\begin{aligned}((\mathbb{1}_R \otimes \tau) \circ f)((p+q) \otimes ae_{11}) &= f((p+q) \otimes ae_{12}) \\ &= (p+q)\gamma^{-1}(\alpha_\sigma(ae_{12})) \\ &= (\alpha_\sigma(ae_{12}))(p+q) \\ &= \sigma(pa) \\ &= \sigma(p)a.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}(f\circ\sigma)((p+q)\otimes ae_{11}) &= \sigma((p+q)\gamma^{-1}(\alpha_\sigma(ae_{11})))\\ &= \sigma((\alpha_\sigma(ae_{11}))(p+q))\\ &= \sigma(pa)\\ &= \sigma(p)a.\end{aligned}$$

Thus we see that σ and $\mathbb{1}_R\otimes\tau$ have the same rule of assignment when composed in the appropriate order with the isomorphism f. The diagram below then commutes, thereby proving that σ is induced by τ.

$$\begin{tikzcd} R\otimes_U Ue_{11} \arrow[r, "\mathbb{1}_R\otimes\tau"] \arrow[d, "f|_{R\otimes_U Ue_{11}}"', "\cong"] & R\otimes_U Ue_{22} \arrow[d, "\cong"', "f|_{R\otimes_U Ue_{22}}"] \\ P \arrow[r, "\sigma"'] & Q \end{tikzcd}$$

□

In particular, under the hypotheses of the lemma, $\sigma^{-1}R\cong R\sqcup_{\mathcal{U}_2(K)}\tau^{-1}\mathcal{U}_2(K)$. We thereby shift the problem of describing $\sigma^{-1}R$ to the problem of describing $\tau^{-1}\mathcal{U}_2(K)$.

Proposition 5.3.2. *Let $\tau\colon\mathcal{U}_2(K)e_{11}\to\mathcal{U}_2(K)e_{22}$ be given by $\tau(ue_{11})=ue_{12}$. The inclusion map $\mathcal{U}_2(K)\to\mathcal{M}_2(K)$ is universally τ-inverting.*

Proof. Throughout this proof, we denote $M=\mathcal{M}_2(K)$ and $U=\mathcal{U}_2(K)$. We define $g\colon M\otimes_U Ue_{22}\to M\otimes_U Ue_{11}$ by

$$g(x\otimes ue_{22}))=xue_{21}\otimes e_{11}.$$

It is straightforward to verify that this map is a well-defined M-module homomorphism. We claim that g and $\mathbb{1}_M\otimes\tau$ are inverse morphisms. On tensors, we may verify that

$$\begin{aligned}g((\mathbb{1}_M\otimes\tau)(x\otimes ue_{11}))&=g(x\otimes ue_{12})=xue_{12}e_{21}\otimes e_{11}=xue_{11}\otimes e_{11}=x\otimes ue_{11},\\ (\mathbb{1}_M\otimes\tau)(g(x\otimes ue_{22}))&=xue_{21}\otimes e_{12}=xue_{21}e_{12}\otimes e_{22}=xue_{22}\otimes e_{22}=x\otimes ue_{22}.\end{aligned}$$

Thus, $\mathbb{1}_M\otimes\tau$ is an isomorphism and so the inclusion $i\colon U\to M$ is τ-inverting.

To see that i is universal among τ-inverting maps, suppose that $f\colon U \to T$ is a τ-inverting ring homomorphism and let η be the inverse of $\mathbb{1}_T \otimes \tau$. Rather than define a map $M \to T$ directly, we will instead define a ring homomorphism

$$\phi\colon M \to \mathrm{End}_T({}_T T) = (\mathrm{Hom}_T(T \otimes_U Ue_{ii}, T \otimes_U Ue_{jj}))_{1\le i,j\le 2}.$$

Denoting by $\gamma\colon T \cong \mathrm{End}_T({}_T T)$ the canonical isomorphism, $\phi \circ \gamma^{-1}\colon M \to T$ will be the desired morphism. The question of uniqueness will also be addressed this way; that is, if ϕ uniquely satisfies $i \circ \phi = f \circ \gamma$, then $\phi \circ \gamma^{-1}$ uniquely satisfies $i \circ \phi \circ \gamma^{-1} = f$.

In order to define such a map, we note that $T \otimes_U Ue_{ii}$ is a $(T, e_{ii}Ue_{ii})$-bimodule for $i \in \{1, 2\}$, and since $e_{ii}Ue_{ii} \cong K$, we can view $T \otimes_U Ue_{ii}$ as a (T, K)-bimodule and $\mathrm{Hom}_T(T \otimes_U Ue_{ii}, T \otimes_U Ue_{jj})$ as a (K, K)-bimodule. This allows us to define the following map[1]

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\mathbb{1}_{Te_{11}} & b(\mathbb{1}_T \otimes \tau) \\ c\eta & d\mathbb{1}_{Te_{22}} \end{pmatrix}.$$

One can then verify that

$$c\eta \circ b'(1_T \otimes \tau) = (cb')(\eta \circ (1 \otimes \tau)) \quad \text{and} \quad b(1_T \otimes \tau) \circ c'\eta = (bc')((1_T \otimes \tau) \circ \eta)$$

hold for all $b, c, b', c' \in K$; from here, checking that ϕ is a ring homomorphism is routine.

For a given $t \in T$, the endomorphism $\gamma(t)$ decomposes as a matrix of morphisms $\gamma(t) = (\gamma_{ij}(t))$, where $\gamma_{ij}(t) \in \mathrm{Hom}_T(T \otimes Ue_{ii}, T \otimes Ue_{jj})$ has the action

$$\gamma_{ij}(t)(x \otimes ue_{ii}) = xue_{ii}te_{jj} \otimes e_{jj} = xf(ue_{ii})tf(e_{jj}) \otimes e_{jj}$$

on tensors. In particular, we find that $\gamma_{ij}(f\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right))(x \otimes ue_{ii}) = xf\big(ue_{ii}\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)e_{jj}\big) \otimes e_{jj}$. To see that $f \circ \gamma = i \circ \phi$, it suffices to check that these morphisms agree on tensors.

[1] Though we choose in this proof to use the left K-module structure of the Hom groups to define ϕ, we could just as well choose to use the right module structure. Both choices yield the same rule of assignment for ϕ, as is easy to verify. The left-right symmetry is essentially due to the fact that K centralizes matrix units in $\mathcal{U}_2(K)$.

To this end, let us observe that

$$\begin{aligned}\gamma\left(f\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\right)(x\otimes e_{11}+x\otimes e_{22}) &= \sum_{i,j\in\{1,2\}}\gamma_{ij}\left(f\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\right)(x\otimes e_{ii})\\ &= \sum_{i,j\in\{1,2\}} xf\left(e_{ii}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}e_{jj}\right)\otimes e_{jj}\\ &= xf(ae_{11})\otimes e_{11}+xf(be_{12})\otimes e_{22}\\ &\quad + xf(0)\otimes e_{11}+xf(de_{22})\otimes e_{22}\\ &= x\otimes ae_{11}+x\otimes be_{12}+x\otimes de_{22}\\ &= x\otimes\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\phi\left(i\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\right)(x\otimes e_{11}+x\otimes e_{22}) &= (a\mathbb{1}_{Ue_{11}})(x\otimes e_{11})+(b(\mathbb{1}_T\otimes\tau))(x\otimes e_{11})\\ &\quad + (0\eta)(x\otimes e_{22})+(d\mathbb{1}_{Ue_{22}})(x\otimes e_{22})\\ &= ax\otimes e_{11}+bx\otimes e_{12}+dx\otimes e_{22}\\ &= x\otimes\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}.\end{aligned}$$

In order to conclude uniqueness, it is enough to recall that by Proposition 2.3.5, $i\colon U\to M$ is a ring epimorphism, hence any ring homomorphism $g\colon M\to \mathrm{End}_T({}_TT)$ satisfying $i\circ g = f\circ\gamma = i\circ\phi$ satisfies $g=\phi$. $\square$

Finally, we are in a position to identify the localization $\sigma^{-1}R$ in the case where ${}_RR_K = {}_RP_K\oplus {}_RQ_K$.

Theorem 5.3.3. *Let R be a unitary associative (K,K)-algebra and suppose additionally that R is a direct sum of (R,K)-bimodules P and Q; that is, ${}_RR_K = {}_RP_K\oplus {}_RQ_K$. Given an (R,K)-bimodule homomorphism $\sigma\colon P\to Q$, the universal σ-inverting ring is the pushout of the inclusion $\mathcal{U}_2(K)\hookrightarrow\mathcal{M}_2(K)$ and the map*

$\alpha_\sigma \colon \mathcal{U}_2(K) \to \mathrm{End}_R({}_RR)$, as indicated in the following diagram,

$$\begin{tikzcd} \mathcal{U}_2(K) \arrow[r, "\subset"] \arrow[d, "\alpha_\sigma"'] & \mathcal{M}_2(K) \arrow[d] \\ \mathrm{End}_R({}_RR) \arrow[r, "\widetilde{\lambda}"'] & \sigma^{-1}R \end{tikzcd} \tag{5.4}$$

where

$$\alpha_\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a\mathbb{1}_P & b\mathbb{1}_\sigma \\ 0 & c\mathbb{1}_Q \end{pmatrix}.$$

The universal localization $\lambda \colon R \to \sigma^{-1}R$ is given by $\gamma \circ \widetilde{\lambda}$, where $\gamma \colon R \cong \mathrm{End}_R({}_RR)$ is the canonical isomorphism.

Proof. Lemma 5.3.1 and Proposition 5.3.2 and provide the setup for invoking Corollary 5.2.4. The result follows immediately. □

One may notice that the mapping $\alpha_\sigma \colon \mathcal{U}_2(K) \to \mathrm{End}_R({}_RR)$ depends not only on σ, but also on K. If a ring satisfies the hypotheses of Theorem 5.3.3 for different choices of the ring K, then the diagram (5.4) varies with K. Universal σ-inverting rings are necessarily isomorphic, so one consequence of Theorem 5.3.3 is that the same ring $\sigma^{-1}R$ serves as the amalgamated free product of $\mathcal{M}_2(K)$ and $\mathrm{End}_R({}_RR)$ for various choices of K. As we noticed previously, when we take $K = \mathbb{Z}$, the hypotheses of Theorem 5.3.3 are easily satisfied, which provides the following useful corollary.

Corollary 5.3.4. *If R is the direct sum of two left ideals, $R = P \oplus Q$, and $\sigma \colon P \to Q$ is an R-module homomorphism, then $\sigma^{-1}R = \mathcal{M}_2(A)$ for some ring A.*

Proof. By Theorem 5.3.3, $\sigma^{-1}R \cong \mathcal{M}_2(\mathbb{Z}) \sqcup_{\mathcal{U}_2(\mathbb{Z})} R$, hence $\sigma^{-1}R$ admits a ring homomorphism from $\mathcal{M}_2(\mathbb{Z})$. By Corollary 2.3.2, $\sigma^{-1}R \cong \mathcal{M}_2(A)$ where A is the centralizer of the image of the matrix units $\{e_{ij}\} \subset \mathcal{M}_2(\mathbb{Z})$ in $\sigma^{-1}R$. □

5.4 Universal Localization of Triangular Matrix 2-Rings

A *triangular 2-ring* is a ring of the form $R = \left(\begin{smallmatrix} A_1 & M_{12} \\ 0 & A_2 \end{smallmatrix}\right)$ for any rings A_1 and A_2 and an (A_1, A_2)-bimodule M_{12}. We may recognize that ${}_R R = \left(\begin{smallmatrix} A_1 \\ 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} M_{12} \\ A_2 \end{smallmatrix}\right)$, and thus by Theorem 5.3.3, the universal localization with respect to a map $\sigma\colon Re_{11} \to Re_{22}$ can be identified as a full matrix ring, $\mathcal{M}_2(A)$, by Corollary 5.3.4. By exploiting the structure of the triangular 2-ring, Sheiham [29] exhibited the defining property of the entry ring, A. His work describes the concept of an (A_1, M_{12}, A_2)-ring, namely a ring T together with a pair of ring homomorphisms $A_1 \to T$ and $A_2 \to T$ along with a bimodule homomorphism $M_{12} \to T$. Such objects form a category; a morphism in this category is a ring homomorphism compatible with the aforementioned maps into T. For a fixed element $\xi \in M$, there is a full subcategory of (A_1, M_{12}, A_2)-rings whose map $M_{12} \to T$ sends ξ to 1_T.

The initial object in this subcategory can be given a ring presentation or equivalently, described as the quotient of a tensor algebra. Specifically, the universal object is given by $A = \mathbb{Z}\langle A_1 \oplus M_{12} \oplus A_2 \rangle / J$, together with the obvious maps from each summand to A, where J is the two-sided ideal generated by $1 - 1_{A_1}$, $1 - 1_{A_2}$, $1 - \xi$, and all elements of the forms $xy - x \otimes y$ $(x, y \in A_i, i \in \{1, 2\})$, $am - a \otimes m$, or $mb - m \otimes b$ $(a \in A_1, b \in A_2, m \in M_{12})$.

Theorem 5.4.1 (Sheiham, 2006 [29, Thm 2.4])**.** *Let $R = \left(\begin{smallmatrix} A_1 & M_{12} \\ 0 & A_2 \end{smallmatrix}\right)$ be a triangular 2-ring and let $\sigma\colon Re_{11} \to Re_{22}$ be an R-module homomorphism. The universal σ-inverting localization of R is then*

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{22} \end{pmatrix} : R \to \mathcal{M}_2(A),$$

where A is the initial object in the full subcategory of (A_1, M_{12}, A_2)-rings whose map from M_{12} sends $\sigma(e_{11}) = \xi$ to 1.

Proof. Let us consider the two diagrams of (5.5), where $\lambda = \left(\begin{smallmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{22} \end{smallmatrix}\right)$, $\tilde{\lambda} = \gamma^{-1} \circ \lambda$, α' is the obvious map, and $\alpha = \alpha_\sigma \circ \gamma^{-1}$. The rule of assignment for α is then $\alpha \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b\xi \\ 0 & d \end{smallmatrix}\right)$. The diagram on the left differs from the diagram on the right only by the canonical isomorphism $\gamma\colon R \cong \mathrm{End}_R({}_RR)$. From this, we deduce that if either diagram is a pushout diagram, then so is the other. In light of Theorem 5.3.3, it is then enough to show that the right diagram in (5.5) is a pushout diagram.

$$\begin{tikzcd}
\mathcal{U}_2(\mathbb{Z}) \arrow[r, "i"] \arrow[d, "\alpha_\sigma"'] & \mathcal{M}_2(\mathbb{Z}) \arrow[d, "\alpha'"] \\
\mathrm{End}_R({}_RR) \arrow[r, "\tilde{\lambda}"'] & \mathcal{M}_2(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{U}_2(\mathbb{Z}) \arrow[r, "i"] \arrow[d, "\alpha"'] & \mathcal{M}_2(\mathbb{Z}) \arrow[d, "\alpha'"] \\
R \arrow[r, "\lambda"'] & \mathcal{M}_2(A)
\end{tikzcd}
\tag{5.5}$$

To verify commutativity of the square, let us note that

$$\lambda\left(\alpha\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \lambda\begin{pmatrix} a & b\xi \\ 0 & d \end{pmatrix} = \begin{pmatrix} \lambda_{11}(a) & \lambda_{12}(b\xi) \\ 0 & \lambda_{22}(d) \end{pmatrix} = \begin{pmatrix} a1_A & b1_A \\ 0 & d1_A \end{pmatrix} = \alpha'\left(i\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right).$$

Given any pair of ring homomorphisms $\mu\colon R \to T$ and $\beta'\colon \mathcal{M}_2(\mathbb{Z}) \to T$, we aim to show that there is a unique ring homomorphism ϕ completing the diagram below.

$$\begin{tikzcd}
\mathcal{U}_2(\mathbb{Z}) \arrow[r, "i"] \arrow[d, "\alpha"'] & \mathcal{M}_2(\mathbb{Z}) \arrow[d, "\alpha'"] \arrow[ddr, bend left, "\beta'"] & \\
R \arrow[r, "\lambda"'] \arrow[drr, bend right, "\mu"'] & \mathcal{M}_2(A) \arrow[dr, dashed, "\phi"] & \\
 & & T
\end{tikzcd}$$

The map β' induces a decomposition of T as a matrix ring. Thus, $T = \mathcal{M}_2(T')$, where T' is the centralizer of the elements $\beta'(e_{ij})$. Furthermore, by commutativity of the diagram, μ can be written as a matrix, $\mu = \left(\begin{smallmatrix} \mu_{11} & \mu_{12} \\ 0 & \mu_{22} \end{smallmatrix}\right)$, for some entry functions μ_{11}, μ_{12}, and μ_{22}. Utilizing the fact that μ is a ring homomorphism, one easily sees that μ_{11} and μ_{22} are ring homomorphisms and μ_{12} is an (A_1, A_2)-bimodule homomorphism.[2] Furthermore,

$$\begin{pmatrix} 0 & \mu_M(\xi) \\ 0 & 0 \end{pmatrix} = \mu(\alpha(e_{12})) = \beta'(i(e_{12})) = \begin{pmatrix} 0 & 1_{T'} \\ 0 & 0 \end{pmatrix}.$$

[2] We later elaborate on this idea and prove it in greater generality. See Theorem 6.1.10.

In particular, $\mu_M(\xi) = 1_{T'}$, and so T' is an (A_1, M_{12}, A_2)-ring with $\mu_M(\xi) = 1$. By the universal property of A, there exists a unique (A_1, M_{12}, A_2)-ring morphism $\phi' \colon A \to T'$. This means that $\phi = \mathcal{M}_2(\phi')$ is a ring homomorphism which makes the diagram commute, and any other ring homomorphism making the diagram commute must decompose as a matrix of morphisms $A \to T'$ which are compatible with the (A_1, M_{12}, A_2)-ring structure of T'. Uniqueness of ϕ' then implies that this morphism must be $\mathcal{M}_2(\phi') = \phi$. $\square$

CHAPTER 6

GENERALIZED MATRIX RINGS AND THEIR PROPERTIES

The triangular rings considered by Schofield, Sheiham, and others, that is, rings of the form $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ for rings A, B and a unitary (A, B)-bimodule M, have enough structure that their universal localizations can be identified. Ranicki identified some applications of these rings and their 3×3 counterparts to geometry and topology in [26]. These interesting rings suggest a more general type of ring, one consisting of a matrix of rings and bimodules with a multiplication map mimicking the structure of multiplication in full matrix rings.

In this chapter, we provide such a generalization, namely the generalized matrix ℓ-rings, which are introduced by the author and D. M. Wilczyński in [2]. Our focus will be on the notable subclass of triangular matrix ℓ-rings. These rings serve as interesting and accessible examples of noncommutative rings, and their idempotents provide a means of determining their localizations. We shall give examples of these rings as well as demonstrate their ubiquity by relating them to the Peirce decomposition of a ring. Some of the basic ring-theoretic properties of generalized matrix rings are discussed.

6.1 Generalized Matrix Rings

To motivate the definition of a generalized matrix ring, we first consider a full matrix ring over a ring K. There is a natural decomposition of $\mathcal{M}_n(K)$ into a direct sum of (K, K)-bimodules using some of the matrix units, namely the idempotents[1] e_{ii}. That is, $\mathcal{M}_n(K) = \sum_{i,j}^{\oplus} e_{ii}\mathcal{M}_n(K)e_{jj}$.

[1] An element x of a ring R is said to be *idempotent* if $x^2 = x$.

The product of two matrices is given by $(a_{ij})(b_{jk}) = (c_{ik})$ where $c_{ik} = \sum_j a_{ij}b_{jk}$. In particular, we see that the product of matrices involves an interaction between the direct summands. Restricting multiplication in $\mathcal{M}_n(K)$ to each of the summands shows that there is a natural (K, K)-bilinear and K-balanced multiplication map

$$e_{ii}\mathcal{M}_n(K)e_{jj} \times e_{jj}\mathcal{M}_n(K)e_{kk} \to e_{ii}\mathcal{M}_n(K)e_{kk}.$$

This observation suggests that we could form a ring of formal matrices by replacing the summands $e_{ii}\mathcal{M}_n(K)e_{jj}$ with more general bimodules. Formal matrix multiplication can be defined if there is a family of appropriate bilinear and balanced multiplication maps between the entry bimodules.

Let $A_1, A_2, \ldots, A_\ell$ be rings. For $1 \leq i, j \leq \ell$, let M_{ij} be a unitary (A_i, A_j)-module, where $M_{ii} = A_i$ with its natural bimodule structure defined by multiplication in A_i. Suppose that for each index triple (i, j, k), there is an (A_i, A_k)-bilinear and A_j-balanced map $f_{ijk}\colon M_{ij} \times M_{jk} \to M_{ik}$. If $i = j$ or $j = k$, we assume that f_{ijk} is the action map of A_j on the left A_j-module M_{jk} or the right A_j-module M_{ij}. Furthermore, we require that the following diagram be commutative

$$\begin{tikzcd}
M_{is} \times M_{st} \times M_{tk} \arrow[r, "f_{ist} \times \mathbb{1}"] \arrow[d, "\mathbb{1} \times f_{stk}"'] & M_{it} \times M_{tk} \arrow[d, "f_{itk}"] \\
M_{is} \times M_{sk} \arrow[r, "f_{isk}"'] & M_{ik}
\end{tikzcd}$$

for all $1 \leq i, s, t, k \leq \ell$. That is,

$$(\mathbb{1} \times f_{stk}) \circ f_{isk} = (f_{ist} \times \mathbb{1}) \circ f_{itk}\colon M_{is} \times M_{st} \times M_{tk} \to M_{ik} \tag{6.1}$$

holds for all indices, where $\mathbb{1}$ stands for the appropriate identity map. Let M denote the $\ell \times \ell$ matrix of bimodules (M_{ij}), $F = \{f_{ijk} \mid 1 \leq i, j, k \leq \ell\}$, and $\mathcal{R}(M, F)$ denote the abelian group $\sum_{i,j}^{\oplus} M_{ij}$ whose elements are arranged as $\ell \times \ell$ matrices (x_{ij}) with $x_{ij} \in M_{ij}$. Addition and multiplication of such matrices are defined by

$$(x_{ij}) + (y_{ij}) = (x_{ij} + y_{ij}) \quad \text{and} \quad (x_{ij})(y_{jk}) = (z_{ik}),$$

where $z_{ik} = \sum_j f_{ijk}(x_{ij}, y_{jk})$.

Theorem 6.1.1. *With the operations defined as above, $\mathcal{R}(M, F)$ is a ring.*

Proof. The distributive law follows readily from the bilinear properties of the multiplication maps f_{ijk} while the associative condition for multiplication follows from their balanced properties together with conditions (6.1). $\square$

Definition 6.1.2. The ring $\mathcal{R}(M, F)$ is called the *matrix ℓ-ring* associated with the pair (M, F). Furthermore, $\mathcal{R}(M, F)$ is an *upper* (resp. *lower*) *triangular* matrix ℓ-ring whenever $M_{ij} = 0$ for all $i > j$ (resp. $i < j$).

Remark 6.1.3. When there is no danger of confusion, a matrix ℓ-ring may be referred to as simply a matrix ring. Additionally, we will frequently shorten *upper triangular matrix ℓ-ring* to *triangular matrix ℓ-ring.* All results which hold for upper triangular matrix rings also hold, *mutatis mutandis*, for lower triangular matrix rings.

Example 6.1.4. If $M_{ij} = 0$ for all $i \neq j$, then the matrix ℓ-ring $\mathcal{R}(M, F)$ is isomorphic to the product ring $A_1 \times A_2 \times \cdots \times A_\ell$.

Example 6.1.5. Suppose K is a ring and $M_{ij} = K$, viewed as a (K, K)-bimodule, for all $1 \leq i, j \leq \ell$. If each f_{ijk} is the (ring) multiplication map $K \times K \to K$, then $\mathcal{R}(M, F) = \mathcal{M}_\ell(K)$, the ring of all $\ell \times \ell$ matrices over K. If K is replaced by the zero module for all indices $i > j$, then $\mathcal{R}(M, F) = \mathcal{U}_\ell(K)$, the ring of upper triangular $\ell \times \ell$ matrices over K.

Example 6.1.6. Given a unitary (A_i, A_{i+1})-bimodule $M_{i,i+1}$ for each $1 \leq i \leq \ell - 1$, an upper triangular matrix ℓ-ring whose superdiagonal entries are $M_{i,i+1}$ can be constructed by setting $M_{ii} = A_i$, $M_{ij} = 0$ for all $i > j$, and

$$M_{ij} = M_{i,i+1} \otimes_{A_{i+1}} M_{i+1,i+2} \otimes_{A_{i+2}} \cdots \otimes_{A_{j-1}} M_{j-1,j}$$

for $j-i>1$. There is a canonical (A_i, A_k)-bimodule isomorphism $M_{ij} \otimes_{A_j} M_{jk} \cong M_{ik}$ for all $i \leq j \leq k$. For such indices, the multiplication map f_{ijk} is defined by the composite map

$$M_{ij} \times M_{jk} \to M_{ij} \otimes_{A_j} M_{jk} \xrightarrow{\cong} M_{ik},$$

while $f_{ijk} = 0$ for all other indices. The family $F = \{f_{ijk}\}$ and matrix of bimodules $M = (M_{ij})$ provide the requisite upper triangular matrix ℓ-ring, $\mathcal{R}(M, F)$.

One important viewpoint when working with matrix ℓ-rings is that of collapsing or expanding a matrix ring to a matrix ring of a different size. In the following two examples, we elaborate on this principle for triangular 2- and 3-rings.

Example 6.1.7. Suppose $R_1 = \mathcal{R}(M', F')$ and $R_2 = \mathcal{R}(M'', F'')$ are matrix ℓ_1- and ℓ_2-rings, respectively, and let (N_{st}) be an $\ell_1 \times \ell_2$ matrix of bimodules, where N_{st} is a unitary (A'_s, A''_t)-bimodule for any $1 \leq s \leq \ell_1$ and $1 \leq t \leq \ell_2$. Suppose further that there is an (R_1, R_2)-bimodule structure on $N = \sum_{s,t}^{\oplus} N_{st}$ that extends the given bimodule structures of the summands so that $M'_{is} N_{st} \subseteq N_{it}$ and $N_{st} M''_{tj} \subseteq N_{sj}$ for all i, j, s, t. Then, by expanding the blocks of the matrix 2-ring $R = \left(\begin{smallmatrix} R_1 & N \\ 0 & R_2 \end{smallmatrix}\right)$, the latter can be viewed as a matrix ℓ-ring with $\ell = \ell_1 + \ell_2$. In particular, all equations (6.1) hold in this case under no additional assumptions. If R_1 and R_2 are upper triangular matrix rings, then so is the expanded ring R. Conversely, every upper triangular matrix ℓ-ring with $\ell \geq 3$ can be viewed as the expansion of a matrix 2-ring $\left(\begin{smallmatrix} R_1 & N \\ 0 & R_2 \end{smallmatrix}\right)$ for some upper triangular matrix rings R_1 and R_2.

Example 6.1.8. By the remarks in the last example, if R_i is a matrix ℓ_i-ring for $i \in \{1, 2, 3\}$, then any triangular matrix of bimodules $\left(\begin{smallmatrix} R_1 & M & P \\ 0 & R_2 & N \\ 0 & 0 & R_3 \end{smallmatrix}\right)$, with suitable bimodule matrices M, N, P of appropriate size and a bilinear, balanced matrix multiplication map $M \times N \to P$, has the structure of a matrix ℓ-ring with $\ell = \ell_1 + \ell_2 + \ell_3$. Higher order generalizations (*i.e.*, those with $\ell = \sum_{i=1}^{n} \ell_i$ for $n \geq 4$) or nontriangular examples of any order require additional hypotheses in the form of equations (6.1) for

an $n \times n$ matrix of bimodules to be a matrix n-ring, even when matrix multiplication is formally defined. When these conditions hold, however, the n-ring is defined and has the structure of a matrix ℓ-ring.

We may also consider *morphisms* of matrix ℓ-rings, a notion slightly stronger than that of ring homomorphisms, as follows.

Theorem 6.1.9. *Suppose $\psi_{ij}\colon M_{ij} \to M'_{ij}$ are functions such that*

(a) $\psi_{ii}\colon A_i \to A'_i$ is a ring homomorphism for all i,

(b) $\psi_{ij}\colon M_{ij} \to M'_{ij}$ is an (A_i, A_j)-bimodule homomorphism[2] *for all $i \neq j$.*

(c) $(\psi_{ij} \times \psi_{jk}) \circ f'_{ijk} = f_{ijk} \circ \psi_{ik}$ for all i, j, k.

Then the function $\psi = \mathcal{R}(\psi_{ij})\colon \mathcal{R}(M, F) \to \mathcal{R}(M', F')$, mapping (x_{ij}) to $(\psi_{ij}(x_{ij}))$, is a ring homomorphism. $\square$

The conditions of Theorem 6.1.9 are sometimes implied by one another and the utility of each depends upon the circumstance. For instance, if the functions ψ_{ij} are additive homomorphisms that preserve the multiplicative identity whenever $i = j$, then conditions (a) and (b) are redundant as they are implied by the relations of type (c), namely those with $i = j = k$ for condition (a) and those with $i = j$ or $j = k$ for condition (b). Furthermore, when $\ell = 2$, condition (c) is equivalent to the conjunction of (a) and (b) whenever the matrix M is triangular. For $\ell = 3$, condition (c) is implied by relations (a) and (b) for all but a single triple of indices whenever M is triangular, namely $(i, j, k) = (1, 2, 3)$.

It is evident that the assignment of $\mathcal{R}(M, F)$ to the pair (M, F) is functorial.

If we denote by e_{ii} the idempotent matrix in $\mathcal{R}(M, F)$ whose only nonzero entry is 1_{A_i} at position (i, i), then the ℓ-ring morphisms from $\mathcal{R}(M, F)$ are easily characterizable.

[2] Note that M'_{ij} is an (A_i, A_j)-bimodule via the ring homomorphism pair (ψ_{ii}, ψ_{jj}), with actions defined by $amb = \psi_{ii}(a)m\psi_{jj}(b)$.

Theorem 6.1.10. *If $\mathcal{R}(M, F)$ and $\mathcal{R}(M', F')$ are matrix ℓ-rings, then every ring homomorphism between them that maps e_{ii} to e'_{ii} for all $1 \le i \le \ell$ is a morphism of the form $\mathcal{R}(\psi_{ij})$ for some matrix of bimodule homomorphisms (ψ_{ij}).*

Proof. If $\psi \colon \mathcal{R}(M, F) \to \mathcal{R}(M', F')$ is a ring homomorphism such that $\psi(e_{ii}) = e'_{ii}$ for all $1 \le i \le \ell$, then

$$\psi(x) = \psi(e_{ii} x e_{jj}) = \psi(e_{ii})\psi(x)\psi(e_{jj}) = e'_{ii}\psi(x)e'_{jj} \in M'_{ij}$$

for all $x \in M_{ij}$. Therefore the functions $\psi_{ij} \colon M_{ij} \to M'_{ij}$, obtained by restricting ψ, are additive homomorphisms, preserve the multiplicative identity elements whenever $i = j$, and satisfy condition (c) of Theorem 6.1.9. It follows that $\psi = \mathcal{R}(\psi_{ij})$. $\square$

6.2 The Role of Idempotents

As mentioned in the previous section, for a matrix ℓ-ring $R = \mathcal{R}(M, F)$ and $1 \le i \le \ell$, we let $e_{ii} \in R$ be the matrix whose only nonzero entry is 1_{A_i} at position (i, i). While a matrix ℓ-ring does not, in general, possess a full set of matrix units, the matrices e_{ii} are mutually orthogonal and have the property that $\sum_i e_{ii} = 1$, which gives them similar power in determining the structure of R and its modules.

Each idempotent e_{ii} determines a projective left ideal, Re_{ii}, which can be understood as a single column of $\mathcal{R}(M, F)$; that these modules are projective is clear from the fact that $R = \sum_i^{\oplus} Re_{ii}$. This is sometimes called the *Peirce decomposition* of R. Homomorphisms between the modules Re_{ii} are each determined by a single choice of element, as follows.

Theorem 6.2.1. *Let $R = \mathcal{R}(M, F)$. For all $1 \le i, j \le \ell$, there is an (A_i, A_j)-bimodule isomorphism*

$$\mathrm{Hom}_R(Re_{ii}, Re_{jj}) \cong e_{ii} R e_{jj} = M_{ij} \tag{6.2}$$

which is also a ring isomorphism whenever $i = j$.

Proof. We first note that Re_{ii} is a unitary (R, A_i)-bimodule. The map given by $f(\varphi) = \varphi(e_{ii})$ is clearly additive, while

$$f(a_i\varphi) = (a_i\varphi)(e_{ii}) = \varphi(e_{ii}a_i) = \varphi(e_{ii}a_ie_{ii}) = a_i\varphi(e_{ii}) = a_if(\varphi),$$
$$f(\varphi a_j) = (\varphi a_j)(e_{ii}) = \varphi(e_{ii})a_j = f(\varphi)a_j,$$

for all $a_i \in A_i$ and $a_j \in A_j$. Since Re_{ii} is a cyclic R-module generated by e_{ii}, any map $\varphi \in \mathrm{Hom}_R(Re_{ii}Re_{jj})$ is uniquely determined by its assignment on e_{ii}, whence f is a bimodule isomorphism. In the case $i = j$, we have additionally that

$$(\varphi_1 \circ \varphi_2)(e_{ii}) = \varphi_2(\varphi_1(e_{ii})) = \varphi_2(\varphi_1(e_{ii})e_{ii}) = \varphi_1(e_{ii})\varphi_2(e_{ii}),$$

proving that f is a ring isomorphism in this case. $\square$

The presence of n^2 matrix units can be used to recognize a full matrix ring. Comparably, the presence of ℓ mutually orthogonal idempotents whose sum is the identity can be used to recognize a matrix ℓ-ring structure in a ring.

That is, if R is a ring with orthogonal idempotents $e_1, \ldots, e_\ell$ such that $\sum_i e_i = 1_R$, then Corollary 2.2.2 and Theorem 2.2.1 can be applied to the Peirce decomposition $R = \sum_i^{\oplus} Re_i$. This provides an isomorphism $R \cong \mathrm{End}_R({}_RR) \cong \mathcal{R}(M, F)$, where $M_{ij} = \mathrm{Hom}_R(Re_i, Re_j)$ and the multiplication maps f_{ijk} are defined by composition of morphisms.

From this, any ring homomorphism $\mathcal{R}(M, F) \to S$ gives rise to a useful matrix ℓ-ring decomposition of S. We first recall that for any ring S, there exists a ring isomorphism $\gamma\colon S \to \mathrm{End}_S({}_SS)$ such that $\gamma(s)$ is right multiplication by s for each $s \in S$.

Theorem 6.2.2. *If S is a ring and $\mu\colon \mathcal{R}(M,F) \to S$ is a ring homomorphism, then there exist a ring isomorphism $\phi\colon \mathcal{R}(M',F') \to \operatorname{End}_S({}_SS)$ and a morphism $\psi = \mathcal{R}(\psi_{ij})\colon \mathcal{R}(M,F) \to \mathcal{R}(M',F')$ such that $\mu \circ \gamma = \psi \circ \phi$.*

$$\begin{tikzcd} \mathcal{R}(M,F) \arrow[r, "\mu"] \arrow[d, "\psi"'] & S \arrow[d, "\gamma", "\cong"'] \\ \mathcal{R}(M',F') \arrow[r, "\cong", "\phi"'] & \operatorname{End}_S({}_SS) \end{tikzcd}$$

Proof. The elements $e'_i = \mu(e_{ii})$ are mutually orthogonal idempotents in S such that $\sum_i e'_i = 1_S$. Thus, $S = \sum_i^{\oplus} Se'_i$, and there is a ring isomorphism $\phi\colon \mathcal{R}(M',F') \to \operatorname{End}_S({}_SS)$, where $M'_{ij} = \operatorname{Hom}_S(Se'_i, Se'_j)$ is a unitary $(e'_iSe'_i, e'_jSe'_j)$-bimodule and the multiplication maps f'_{ijk} are defined by composition of morphisms. Theorem 6.1.10 applies then to the ring homomorphism $\psi = \mu \circ \gamma \circ \phi^{-1}$. $\square$

In the case where $\mu\colon \mathcal{R}(M,F) \to R$ is the identity homomorphism, the morphism $\psi = \mathcal{R}(\psi_{ij})$ guaranteed by Theorem 6.2.2 is an isomorphism, where each $\psi_{ij}\colon M_{ij} \to M'_{ij} = \operatorname{Hom}_R(Re_{ii}, Re_{jj})$ is the inverse of the isomorphism (6.2).

6.3 Modules Over Generalized Matrix Rings

For a full matrix ring, $\mathcal{M}_\ell(K)$, a classic result of Morita theory states that any unitary left $\mathcal{M}_\ell(K)$-module is isomorphic to a direct sum of copies of some unitary left K-module X (see Theorem 2.3.7). While a statement this strong cannot be expected for a more general matrix ℓ-ring $\mathcal{R}(M,F)$, the elements $e_{ii} \in \mathcal{R}(M,F)$ do provide a way to describe any unitary $\mathcal{R}(M,F)$-module as a sum of modules over the entry rings A_i with some additional conditions. E. Green gave a structure result of this nature for matrix 2- and 3-rings [18] and we provide a natural generalization.

Suppose that $\mathcal{R}(M,F)$ is a matrix ℓ-ring. We introduce the category $\mathscr{A}(M,F)$ whose objects are pairs (X,G) where $X=(X_i)_{1\le i\le \ell}$, each X_i is a left unitary A_i-module, and $G=\{g_{ij}\mid 1\le i,j\le \ell\}$ consists of biadditive maps $g_{ij}\colon M_{ij}\times X_j\to X_i$ which make the following diagram commute

$$\begin{tikzcd}
M_{ij}\times M_{jk}\times X_k \arrow[r, "f_{ijk}\times\mathbb{1}"] \arrow[d, "\mathbb{1}\times g_{jk}"'] & M_{ik}\times X_k \arrow[d, "g_{ik}"] \\
M_{ij}\times X_j \arrow[r, "g_{ij}"'] & X_i
\end{tikzcd}$$

for all $1\le i,j,k\le \ell$. That is,

$$(f_{ijk}\times\mathbb{1})\circ g_{ik}=(\mathbb{1}\times g_{jk})\circ g_{ij} \tag{6.3}$$

holds for all indices, where $\mathbb{1}$ stands for the appropriate identity map. A morphism $\varphi\colon (X,G)\to(X',G')$ in $\mathscr{A}(M,F)$ is an ℓ-tuple (φ_i) where each $\varphi_i\colon X_i\to X'_i$ is an A_i-module homomorphism making the following diagram commute

$$\begin{tikzcd}
M_{ij}\times X_j \arrow[r, "g_{ij}"] \arrow[d, "\mathbb{1}\times\varphi_j"'] & X_j \arrow[d, "\varphi_j"] \\
M_{ij}\times X'_j \arrow[r, "g'_{ij}"'] & X'_j
\end{tikzcd}$$

for all $1\le i,j\le \ell$. That is,

$$g_{ij}\circ\varphi_i=(\mathbb{1}\times\varphi_j)\circ g'_{ij} \tag{6.4}$$

holds for all indices, where $\mathbb{1}$ stands for the appropriate identity map.

It is worth noting that in the case $i=j$ or $j=k$, conditions (6.3) are equivalent to the statement that that the maps g_{ij} are left A_i-linear and A_j-balanced.

Theorem 6.3.1. *Let $\mathcal{R}(M,F)$ be a matrix ℓ-ring. The category ${}_R\mathbf{Mod}$ is equivalent to the category $\mathscr{A}(M,F)$.*

Proof. Letting e_{ii} denote the usual idempotents in $R=\mathcal{R}(M,F)$, we proceed to define a functor $\Phi\colon {}_R\mathbf{Mod}\to\mathscr{A}(M,F)$. On objects, let us consider the assignment $\Phi(X)=(X,G)$, where $X=(e_{ii}X)$, $G=\{g_{ij}\}$, and $g_{ij}(m_{ij},e_{jj}x)=m_{ij}x$ by viewing

M_{ij} as a subrng of R. Conditions (6.3) hold since X is an R-module. For any R-module homomorphism $\varphi\colon X \to X'$, we note that

$$\varphi(e_{ii}X) = e_{ii}\varphi(e_{ii}X) \subset e_{ii}X',$$

hence the restrictions $\varphi_i\colon e_{ii}X \to e_{ii}X'$ are well-defined A_i-module homomorphisms. It is easily checked that conditions (6.4) hold for the ℓ-tuple (φ_i), making $\Phi(\varphi) = (\varphi_i)$ a valid assignment on morphisms. Functoriality of this assignment is clear.

To define the inverse functor $\Psi\colon \mathscr{A}(M,F) \to {}_R\mathbf{Mod}$, consider a pair $(X,G) \in \mathscr{A}(M,F)$. For any $(x_j) \in X$ and $(m_{ij}) \in R$, we define $(m_{ij})(x_j) = (y_i)$, where $y_i = \sum_j g_{ij}(m_{ij}, x_j)$. To see that this makes X an R-module, we can easily verify that

$$\begin{aligned}
\left((m_{ij})(m'_{jk})\right)(x_k) &= \left(\sum_j f_{ijk}(m_{ij}, m'_{jk})\right)(x_k) \\
&= \left(\sum_{j,k} g_{ik}(f_{ijk}(m_{ij}, m'_{jk}), x_k)\right) \\
&= \left(\sum_{j,k} g_{ij}(m_{ij}, g_{jk}(m'_{jk}, x_k))\right) \\
&= (m_{ij})\left((m'_{jk})(x_k)\right),
\end{aligned}$$

since G satisfies conditions (6.3). Consequently, we let $\Psi((X,G)) = X$ with the module structure defined above. On a morphism $(\varphi_i)\colon (X,G) \to (X',G')$, we define $\Psi((\varphi_i))(x_i) = (\varphi_i)(x_i) = (\varphi_i(x_i))$. Using conditions (6.4), it is easy to verify that this is a well-defined R-module homomorphism, as

$$\begin{aligned}
(\varphi_i)((m_{ij})(x_j)) &= (\varphi_i)(\textstyle\sum_j g_{ij}(m_{ij}, x_j)) \\
&= (\varphi_i(\textstyle\sum_j g_{ij}(m_{ij}, x_j))) \\
&= (\textstyle\sum_j \varphi_i(g_{ij}(m_{ij}, x_j))) \\
&= (\textstyle\sum_j g'_{ij}(m_{ij}, \varphi_j(x_j))) \\
&= (m_{ij})(\varphi_j(x_j)).
\end{aligned}$$

One can easily verify that Φ and Ψ are inverse functors. $\square$

6.4 Ring-Theoretic Properties of Triangular Rings

Theorem 6.3.1 sheds some light on the structure of modules over matrix rings. We may additionally ask about the structure of the matrix ring itself. Some of ring-theoretic properties of $\mathcal{R}(M, F)$ are within reach when $\mathcal{R}(M, F)$ is triangular. Specifically, we will identify the Jacobson radical and socles of the ring $\mathcal{R}(M, F)$. The following are generalizations of results by Haghany and Varadarajan on triangular matrix 2-rings [19].

Lemma 6.4.1. *Let $R = \mathcal{R}(M, F)$ be an upper triangular matrix 2-ring. Each left (resp. right) ideal of R is of the form $M' = \left(\begin{smallmatrix} A'_1 & M'_{12} \\ 0 & A'_2 \end{smallmatrix}\right)$ where A'_1 is a left (resp. right) ideal of A_1, A'_2 is a left (resp. right) ideal of A_2, M'_{12} is a left A_1-submodule (resp. right A_2-submodule) of M_{12}, and $M_{12}A'_2 \subset M'_{12}$ (resp. $A'_1 M_{12} \subset M'_{12}$).*

Proof. Let I be a left ideal of R. We define $M'_{ij} = e_{ii}Ie_{jj} \subset e_{ii}Re_{jj} = M_{ij}$ for $i, j \in \{1, 2\}$. We also denote $M'_{11} = A'_1$ and $M'_{22} = A'_2$. Utilizing the fact that I is a left ideal in R, we may observe

$$\begin{aligned} A_1A'_1 &= e_{11}Re_{11}^2Ie_{11} \subset e_{11}Ie_{11} = A'_1, \\ A_2A'_2 &= e_{22}Re_{22}^2Ie_{22} \subset e_{22}Ie_{22} = A'_2, \\ A_1M'_{12} &= e_{11}Re_{11}^2Ie_{22} \subset e_{11}Ie_{22} = M'_{12}, \\ M_{12}A'_2 &= e_{11}Re_{22}^2Ie_{22} \subset e_{11}Ie_{22} = M'_{12}. \end{aligned}$$

Thus, A'_i is a left ideal in A_i for $i \in \{1, 2\}$, M'_{12} is a left A_1-submodule of M_{12}, $M'_{21} = 0$, and $M_{12}A'_2 \subset M'_{12}$, as claimed. Similar observations can be made about right ideals of R. □

Proposition 6.4.2. *Let $R = \mathcal{R}(M, F)$ be an upper triangular matrix ℓ-ring. The maximal left (resp. right) ideals of R are precisely the left (resp. right) ideals of the form $M' = (M'_{ij})$, where $A'_k = M'_{kk}$ is a maximal left (resp. right) ideal of A_k for some $1 \leq k \leq \ell$ and $M'_{ij} = M_{ij}$ for all other indices.*

Proof. It is straightforward to see that any subrng of R of the form proposed by the theorem statement is a maximal left ideal. To prove the converse, we proceed by induction on ℓ. Clearly, in the case $\ell = 1$, any maximal left ideal is of the requisite form. In the case $\ell = 2$, suppose that I is a maximal left ideal of R. By Lemma 6.4.1, $I = M' = \left(\begin{smallmatrix} A_1' & M_{12}' \\ 0 & A_2' \end{smallmatrix}\right)$ where A_1' is a left ideal of A_1, A_2' is a left ideal of A_2, M_{12}' is a left A_1-submodule of M_{12}, and $M_{12}A_2' \subset M_{12}'$.

If $A_2' \neq A_2$, then A_2' is contained in some maximal left ideal A'' of A_2. We may now consider the proper left ideal $J = \left(\begin{smallmatrix} A_1 & M_{12} \\ 0 & A'' \end{smallmatrix}\right)$ of R. Clearly $I \subset J$, and so by maximality of I, we have $I = J$, proving that the statement holds in this case.

On the other hand, if $A_2' = A_2$, then we have

$$M_{12} = M_{12}A_2 = M_{12}A_2' \subset M_{12}',$$

and therefore $M_{12}' = M_{12}$. By maximality of I, A_1' must then be a maximal left ideal of A_1, and therefore $I = M'$ is of the requisite form.

Finally, let $\ell > 2$ and assume inductively that for any triangular matrix $(\ell - 1)$-ring, the maximal left ideals are of the required form. We may view R as the expansion of a matrix 2-ring, say

$$R = \begin{pmatrix} \mathcal{R}(N, G) & \sum_{1 \leq i < \ell}^{\oplus} M_{i\ell} \\ 0 & A_\ell \end{pmatrix}$$

where N is the upper left $(\ell - 1) \times (\ell - 1)$ block of M and $G \subset F$ is the obvious family of multiplication maps. By the induction hypothesis and the statement for triangular 2-rings, any maximal left ideal I of R is then of one of the following forms,

$$\begin{pmatrix} N' & \sum_{1 \leq i < \ell}^{\oplus} M_{i\ell} \\ 0 & A_\ell \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{R}(N, G) & \sum_{1 \leq i < \ell}^{\oplus} M_{i\ell} \\ 0 & A_\ell' \end{pmatrix},$$

where in the first form, $N' = (N_{ij}')$ with N_{kk}' a maximal left ideal of A_k for some $1 \leq k \leq \ell$ and $N_{ij}' = N_{ij}$ for all other indices, and in the second form, A_ℓ' is a maximal left ideal of A_ℓ. In either case, I is then of the claimed form.

The proof of the analogous statement for maximal right ideals of R is similar. $\square$

Proposition 6.4.3. *Let $R = \mathcal{R}(M, F)$ be an upper triangular matrix ℓ-ring. The minimal left (resp. right) ideals of R are precisely the left (resp. right) ideals of the form $M' = (M'_{ij})$ where, for some $1 \leq s \leq t \leq \ell$, M'_{st} is minimal among nonzero left A_s-submodules (resp. right A_t-submodules) of M_{st} satisfying $M_{ks}M'_{st} = 0$ for all $1 \leq k < s$ (resp. $M'_{st}M_{tk} = 0$ for all $t < k \leq \ell$) and $M'_{ij} = 0$ for all other indices.*

Proof. It is straightforward to see that any subrng of R of the proposed form is a minimal left ideal of R. To prove the converse, we proceed by induction on ℓ. The statement apparently holds in the case $\ell = 1$. For $\ell = 2$, let I be a minimal left ideal of R. By Lemma 6.4.1, $I = M' = \left(\begin{smallmatrix} A'_1 & M'_{12} \\ 0 & A'_2 \end{smallmatrix}\right)$ where A'_1 is a left ideal of A_1, A'_2 is a left ideal of A_2, M'_{12} is a left A_1-submodule of M_{12}, and $M_{12}A'_2 \subset M'_{12}$.

Suppose first that $M'_{12} \neq 0$. Under this assumption, we may consider the nonzero left ideal $J = \left(\begin{smallmatrix} 0 & M'_{12} \\ 0 & 0 \end{smallmatrix}\right)$ of R. Clearly, $J \subset I$, and so by minimality of I, we have $I = J$, proving that $A'_1 = 0$ and $A'_2 = 0$. Any nonzero left A_1-submodule M''_{12} of M'_{12} similarly gives rise to a nonzero left ideal of R, namely $\left(\begin{smallmatrix} 0 & M''_{12} \\ 0 & 0 \end{smallmatrix}\right)$, which is contained in I. Thus, minimality of I implies that M'_{12} is a minimal submodule of M_{12}, and I of the claimed form.

In the complementary case where $M'_{12} = 0$, we may consider the left ideals $J_1 = \left(\begin{smallmatrix} A'_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $J_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A'_2 \end{smallmatrix}\right)$, each of which is contained in I. By minimality of I, either $A'_2 = 0$ and A'_1 is a minimal left ideal of A_1 or $A'_1 = 0$ and A'_2 is a minimal left ideal of A_2 such that $M_{12}A'_2 = 0$. In either circumstance, I is of the requisite form.

Assuming inductively that minimal ideals of any triangular $(\ell - 1)$-ring are of the proposed form, we may view the triangular ℓ-ring R as the expansion of a matrix 2-ring, say

$$R = \begin{pmatrix} \mathcal{R}(N, G) & \sum^{\oplus}_{1 \leq i < \ell} M_{i\ell} \\ 0 & A_\ell \end{pmatrix}$$

where N is the upper left $(\ell-1)\times(\ell-1)$ block of M and $G \subset F$ is the obvious family of multiplication maps. By the induction hypothesis and the statement for triangular 2-rings, any minimal left ideal I of R is then of one of the following forms,

$$\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & A'_\ell \end{pmatrix},$$

where the entry bimodules are as follows. In the first form, $N' = (N'_{ij})$ with some N'_{st} minimal among nonzero left A_s-submodules of N_{st} satisfying $N_{ks}N'_{st} = 0$ for each $1 \leq k \leq s$, and $N'_{ij} = N_{ij}$ for all other indices, proving that I is of the required form. In the second form, W is a minimal left $\mathcal{R}(N,G)$-submodule of $\sum^{\oplus}_{1\leq i<\ell} M_{i\ell}$; we claim that $W = \sum^{\oplus}_{1\leq i<\ell} M'_{i\ell}$ where each $M'_{i\ell}$ is left A_i-submodule of $M_{i\ell}$ and $M'_{s\ell} \neq 0$ for precisely one index $1 \leq s < \ell$. To obtain the decomposition, we note that $W = \sum^{\oplus}_{1\leq i<\ell} e_{ii}W$ and set $M'_{i\ell} = e_{ii}W$. Certainly $M'_{s\ell} \neq 0$ for some s, since I is nonzero by assumption. Let s be the smallest index such that $M'_{s\ell} \neq 0$. In particular, since W is a submodule of $\sum^{\oplus}_{1\leq i<\ell} M_{i\ell}$, we have

$$M_{ks}M'_{s\ell} \subset M_{ks}M_{s\ell} \subset M_{k\ell} = 0$$

for all $k < s$; the left ideal J formed by replacing the summands $M'_{i\ell}$ with 0 whenever $i \neq s$ then satisfies $J \subset I$, and so minimality of I implies $I = J$. In particular, $M'_{s\ell}$ is the only nonzero summand of W, and must be minimal among left A_s-submodule of $M_{s\ell}$ satisfying $M_{ks}M'_{s\ell}$ for all $1 \leq k < s$, proving that I is of the requisite form in this case. Finally, the third form has A'_ℓ a minimal left ideal of A_ℓ which satisfies $\left(\sum^{\oplus}_{1\leq i<\ell} M_{i\ell}\right)A'_\ell = \sum_{1\leq i<\ell} M_{i\ell}A'_\ell = 0$ by Lemma 6.4.1. This shows that in the third case, R is of the proposed form, completing the proof of the induction step.

The proof of the analogous statement for minimal right ideals of R is similar. $\square$

With these tools, we can illuminate some of the properties of triangular matrix rings. Let us recall the Jacobson radical and socles of a ring.

Definition 6.4.4. Let R be a ring. The *Jacobson radical* of R is the intersection of all maximal left (or right[3]) ideals and is denoted $J(R)$.

The *socle* of a left or right R-module M is the submodule generated by all minimal submodules of M and is denoted $\mathrm{Soc}(M)$. Since R can be interpreted as either a left or right module over itself, it has a *left socle* and a *right socle*, denoted $\mathrm{Soc}({}_RR)$ and $\mathrm{Soc}(R_R)$, respectively.

Before providing descriptions of the Jacobson radical and the left and right socles of a triangular matrix ring, we first provide the notion of the left and right f-annihilators of the entry bimodules. These annihilators capture some of the key conditions which determine the minimal ideals of a triangular ring, for instance, those of the form $M_{ks}M'_{st} = 0$. These conditions strongly resemble the definition of an annihilator in the classic sense, that is, the set of scalars from a ring R which act on an R-module M by zero. Since the f-annihilators are closely related to the minimal ideals of a triangular ring, they can be used to describe the left and right socles of such a ring.

Definition 6.4.5. Let A, B, C be rings and $f\colon {}_AM_B \times {}_BN_C \to {}_AP_C$ be an (A, C)-bilinear and B-balanced map. For $S \subset M$ and $T \subset N$, the *right f-annihilator of S* and the *left f-annihilator of T* are defined as

$$\begin{aligned} \mathrm{rAnn}_f(S) &= \{n \in N \mid f(S, n) = 0\}, \\ \mathrm{lAnn}_f(T) &= \{m \in M \mid f(m, T) = 0\}, \end{aligned}$$

respectively.

Continuing the notation used in the definition, one can verify that $\mathrm{rAnn}_f(S)$ is a submodule of N_C for any subset $S \subset M$. If, in addition, S is a submodule of M_B,

[3]The intersection must be taken across maximal ideals of the same flavor, but the notion turns out to be left-right symmetric, so it does not matter whether one intersects the maximal left ideals or the maximal right ideals. Proof of the left-right symmetry is available in [20].

then $\mathrm{rAnn}_f(S)$ is a subbimodule of ${}_BN_C$. Similarly, $\mathrm{lAnn}_f(T)$ is a submodule of ${}_AM$ for any subset $T \subset N$ and is additionally a subbimodule of ${}_AM_B$ if T is a submodule of ${}_BN$.

Corollary 6.4.6. *Let $R = \mathcal{R}(M, F)$ be an upper triangular matrix ℓ-ring. The following hold:*

(a) $J(R) = (M'_{ij})$, *where* $M'_{ii} = J(M'_{ii})$ *and* $M'_{ij} = M_{ij}$ *for* $1 \leq i < j \leq \ell$,

(b) $\mathrm{Soc}({}_RR) = (M'_{ij})$ *where* $M'_{ij} = \mathrm{Soc}\left(\bigcap_{k<i} \mathrm{rAnn}_{f_{kij}}(M_{ki})\right)$,

(c) $\mathrm{Soc}(R_R) = (M'_{ij})$ *where* $M'_{ij} = \mathrm{Soc}\left(\bigcap_{k>j} \mathrm{lAnn}_{f_{ijk}}(M_{jk})\right)$, *and*

(d) $R/J(R) \cong \prod_i A_i/J(A_i)$.

Proof. These properties follow immediately from Propositions 6.4.2 and 6.4.3. $\square$

CHAPTER 7

UNIVERSAL LOCALIZATION OF TRIANGULAR RINGS

The results of Sheiham that were discussed in Section 5.4 provided a presentation for the entry ring of the localization of a triangular matrix 2-ring, adding to the results of Schofield, who established that such rings localize to full matrix rings.

In Chapter 6, we provided a class of rings enveloping those investigated by Schofield and Sheiham. The aim of this chapter is to generalize their results by giving a construction for the universal localization of a generalized triangular matrix ring with respect to a morphism between its column modules. That is, for a generalized triangular matrix ring R, we consider morphisms of the type $\sigma\colon Re_{pp} \to Re_{qq}$ and show that the universal σ-inverting localization of R is a generalized matrix ring of the same size whose entry bimodules can be described in terms of certain tensor algebras and tensor product modules of the various bimodules extracted from R.

The general question of a morphism $\sigma\colon Re_{pp} \to Re_{qq}$ can be reduced by collapsing the structure of the matrix ℓ-ring to a matrix 5-ring structure. This is conducive to discussing the morphism as a map $\sigma\colon Re'_{22} \to Re'_{44}$, where the localization takes the form

$$\sigma^{-1}R = \begin{pmatrix} A_1 & U & G & U & W \\ 0 & A & B & A & V \\ 0 & C & D & C & H \\ 0 & A & B & A & V \\ 0 & 0 & 0 & 0 & A_5 \end{pmatrix}.$$

This reduction allows us to describe the localization for a triangular matrix ring of any size.

In contrast to the case of triangular 2-rings, the localization does not usually yield

a full matrix ring. We shall address the question of whether the localization of higher-order triangular rings is ever a full matrix ring and describe a class of morphisms for which there is a positive answer.

The results in this chapter are the outcome of collaborative work between the author and D. M. Wilczyński [2].

7.1 Universal Localization of Triangular Matrix 3-Rings

In an initial attempt to generalize Theorem 5.4.1, a seemingly simple case to consider first is that of a triangular matrix 3-ring, $R = \mathcal{R}(M, F)$, and a morphism $\sigma\colon Re_{11} \to Re_{33}$. One approach to determining the localization of a triangular 3-ring might be to generalize the mechanism used in proving Sheiham's result; that is, we may try to realize σ as an induced morphism by a mapping of modules over a more familiar ring. However, this effort immediately runs into a difficulty — it is not clear what morphism might induce σ. An attempt to use $\mathcal{U}_3(K)$ does not work for two reasons, namely that there is no clear morphism $\mathcal{U}_3(K) \to \mathcal{R}(M, F)$, and it is not apparent what the universal localization of $\mathcal{U}_3(K)$ with respect to a morphism $\tau\colon \mathcal{U}_3(K)e_{11} \to \mathcal{U}_3(K)e_{33}$ is.

Given that $R \neq Re_{11} \oplus Re_{33}$, the results of Section 5.3 do not immediately apply. Rather, since $R = Re_{11} \oplus Re_{22} \oplus Re_{33}$, it turns out to be possible to induce σ from a mapping, τ, between $\mathcal{U}_2(K) \times K$ modules, and it is not too hard to show that the universal τ-inverting localization of this product is $\mathcal{M}_2(K) \times K$. This reveals that $\sigma^{-1}R \cong R \sqcup_{\mathcal{U}_2(K)\times K} (\mathcal{M}_2(K) \times K)$, but this formulation does not shed much light on the structure of $\sigma^{-1}R$. The pushout diagram that would arise from this setup does not have as much utility in identifying $\sigma^{-1}R$ as the setup in Section 5.3 because $\mathcal{M}_2(K) \times K$ does not contain a full set of matrix units. Unfortunately, this means that a ring homomorphism $\mathcal{M}_2(K) \times K \to \sigma^{-1}R$ does not necessarily endow $\sigma^{-1}R$

with a full matrix ring structure. In the end, we will see that this difficulty cannot be avoided, as the localization $\sigma^{-1}R$ is not, in general, a full matrix ring. Its true structure turns out to be more complicated than we might initially expect.

Our approach will instead be to use the idempotents $e_{ii} \in R$ to recognize a generalized matrix ring structure in any codomain of a homomorphism originating from R. By identifying the properties that the entry bimodules must have to be σ inverting, we can construct the universal localization.

We begin with the definition of a certain category. For any matrix ℓ-ring $R = \mathcal{R}(M,F)$, there is a category of morphisms from R to full matrix ℓ-rings, denoted $\mathscr{C}(\mathcal{R}(M,F))$. Specifically, the objects of $\mathscr{C}(\mathcal{R}(M,F))$ are morphisms of matrix ℓ-rings of the form $\beta = \mathcal{R}(\beta_{ij})\colon R \to \mathcal{M}_\ell(B)$. If $\gamma\colon R \to \mathcal{M}_\ell(C)$ is another object, then a morphism $\beta \to \gamma$ is a ring homomorphism $\theta\colon B \to C$ such that $\beta \circ \mathcal{M}_\ell(\theta) = \gamma$.

The category $\mathscr{C}(\mathcal{R}(M,F))$ has an initial object $\alpha\colon R \to \mathcal{M}_\ell(A_*)$, where A_* is a quotient of the tensor ring, $\mathbb{Z}\langle\sum_{i,j}^{\oplus} M_{ij}\rangle$ of the additive group of R modulo the relations $1 - 1_{A_i}$ and $xy - x \otimes y$ for all $x \in M_{ij}$, $y \in M_{jk}$ $(1 \leq i,j,k \leq \ell)$. Given an element $\xi \in M_{ij}$, the quotient map

$$\pi\colon A_* \to A_\xi = A_*/(1 - \alpha_{ij}(\xi))$$

induces the composite morphism $\alpha^\xi = \alpha \circ \mathcal{M}_\ell(\pi)\colon R \to \mathcal{M}_\ell(A_\xi)$ which can, in turn, be interpreted as the initial object in the full subcategory, $\mathscr{C}(\mathcal{R}(M,F),\xi)$, of those morphisms $\beta\colon R \to \mathcal{M}_\ell(B)$ in $\mathscr{C}(\mathcal{R}(M,F))$ satisfying $\beta_{ij}(\xi) = 1_B$.

Theorem 7.1.1. *Let $R = \mathcal{R}(M,F)$ be a triangular matrix 3-ring and suppose $\sigma\colon Re_{11} \to Re_{33}$ is an R-module homomorphism. The following morphism of matrix 3-rings*

$$\lambda = \mathcal{R}(\lambda_{ij})\colon R = \begin{pmatrix} A_1 & M_{12} & M_{13} \\ 0 & A_2 & M_{23} \\ 0 & 0 & A_3 \end{pmatrix} \longrightarrow \sigma^{-1}R = \begin{pmatrix} A & B & A \\ C & D & C \\ A & B & A \end{pmatrix} \tag{7.1}$$

is the universal σ-inverting localization of R, where $\sigma(e_{11}) = \xi \in M_{13}$ and

$$\alpha^{\xi} = \mathcal{R}\begin{pmatrix} \alpha^{\xi}_{11} & \alpha^{\xi}_{13} \\ 0 & \alpha^{\xi}_{33} \end{pmatrix} : S_{13} = \begin{pmatrix} A_1 & M_{13} \\ 0 & A_3 \end{pmatrix} \longrightarrow \mathcal{M}_2(A_{\xi})$$

is the initial object in $\mathscr{C}(S_{13}, \xi)$. The entry bimodules of the $\sigma^{-1}R$ matrix and the corresponding λ-values are listed below.

$$\begin{array}{ll} A = A_{\xi} & \lambda_{ij}(x) = \alpha^{\xi}_{ij}(x) \quad (i \le j,\ i, j \in \{1,3\}); \\ B = A \otimes_{A_1} M_{12} & \lambda_{12}(x) = 1 \otimes x; \\ C = M_{23} \otimes_{A_3} A & \lambda_{23}(x) = x \otimes 1; \\ D = A_2\langle C \otimes_A B\rangle / E & \lambda_{22}(x) = x + E; \\ E = \big(c(bc') \otimes b' - (c \otimes b) \otimes (c' \otimes b')\big) & \qquad\qquad (b, b' \in B,\ c, c' \in C). \end{array} \tag{7.2}$$

The convention $bc = a\alpha^{\xi}_{13}(xy)a' \in A$ for $(b, c) = (a \otimes x, y \otimes a') \in B \times C$ is assumed throughout. The quotient ring D, of the tensor algebra of the (A_2, A_2)-bimodule $C \otimes_A B$, has a right action on B and a left action on C defined by the equations

$$b(c \otimes b' + E) = (bc)b' \text{ and } (c \otimes b + E)c' = c(bc'), \tag{7.3}$$

thereby making B a unitary (A, D)-bimodule and C a unitary (D, A)-bimodule. The multiplication maps $f_{ijk} \colon R_{ij} \times R_{jk} \to R_{ik}$ of the ring $\sigma^{-1}R = (R_{ij})$, other than the module action maps, are defined as follows

$$f_{121}(b, c) = f_{123}(b, c) = f_{321}(b, c) = f_{323}(b, c) = bc, \tag{7.4}$$

$$f_{212}(c, b) = f_{232}(c, b) = c \otimes b + E. \tag{7.5}$$

We remark that the result of Sheiham (Theorem 5.4.1) follows from Theorem 7.1.1 in the special case where A_2 is the zero ring (in which case D is the zero ring as well).

Before proving the result, we will provide a few examples.

Examples 7.1.2. Let $\mathbb{E}$ denote the abelian group of even integers. By taking as R each of the following subrings of $\mathcal{U}_3(\mathbb{Z})$ and $\sigma(e_{11}) = e_{13}$ in $M_{13} = \mathbb{Z}$, we obtain the

following universal σ-inverting rings $\sigma^{-1}R$.

$$\begin{array}{rccccccccc}
R: & \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{E} \\ 0 & 0 & \mathbb{Z} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & \mathbb{E} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix} \\
\sigma^{-1}R: & \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ \mathbb{E} & \mathbb{Z} & \mathbb{E} \\ \mathbb{Z} & 0 & \mathbb{Z} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & \mathbb{E} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}[2t]/(4t-4t^2) & \mathbb{Z} \\ \mathbb{Z} & \mathbb{E} & \mathbb{Z} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}[t]/(t-t^2) & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}
\end{array}$$

The module structures are given by $pq = p(1)q \in \mathbb{Z}$ for all $p \in \mathbb{Z}[t]/(t-t^2)$ and $q \in \mathbb{Z}$. In the last two examples, we have $f_{212}(c,b) = f_{232}(c,b) = cbt$. All other multiplication maps are obvious.

Examples 7.1.3. Suppose $\sigma(e_{11}) = e_{13}$ in $M_{13} = \mathbb{Q}$. Let $\mathbb{Q}_{\mathbb{Z}}[t]$ denote the ring of those polynomials in $\mathbb{Q}[t]$ with integer constant terms.

$$\begin{array}{rccccccccc}
R: & \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Q} \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Q} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Q} \\ 0 & \mathbb{Z} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \end{pmatrix} \\
\sigma^{-1}R: & \begin{pmatrix} \mathbb{Q} & 0 & \mathbb{Q} \\ 0 & \mathbb{Z} & 0 \\ \mathbb{Q} & 0 & \mathbb{Q} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Q} & 0 & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Z} & \mathbb{Q} \\ \mathbb{Q} & 0 & \mathbb{Q} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q}_{\mathbb{Z}}[t]/(t-t^2) & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q}_{\mathbb{Z}}[t]/(t-t^2) & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix} & \subset & \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q}[t]/(t-t^2) & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}
\end{array}$$

The module structures are given by $pq = p(1)q \in \mathbb{Q}$ for all $p \in \mathbb{Q}[t]/(t-t^2)$ and $q \in \mathbb{Q}$. In the last three examples, we have $f_{212}(c,b) = f_{232}(c,b) = cbt$.

Proof of Theorem 7.1.1. We note first that the map $m\colon B \times C \to A$, $m(b,c) = bc$, defined initially on tensor pairs (b,c), is an (A,A)-bilinear and A_2-balanced function. Let us consider the formulas

$$b(c \otimes b) = (bc)b' \ \text{ and } (c \otimes b)c' = c(bc'). \tag{7.6}$$

The first is a well-defined, (A, A_2)-bilinear, and A_2-balanced function of $(b, c \otimes b')$. Similarly, the second formula of (7.6) is a well-defined, (A_2, A)-bilinear, and A_2-balanced function of $(c \otimes b, c')$. Consequently, these functions induce morphisms

$$B \otimes_{A_2} (C \otimes_A B) \to B \ \text{ and } (C \otimes_A B) \otimes_{A_2} C \to C$$

of (A, A_2)-bimodules and (A_2, A)-bimodules, respectively, whose adjoint morphisms

$$C \otimes_A B \to \mathrm{End}_A({}_AB) \ \text{ and } C \otimes_A B \to \mathrm{End}_A(C_A)$$

are (A_2, A_2)-bimodule homomorphisms. By the universal property of the tensor algebra $A_2\langle C \otimes_A B\rangle$, these bimodule homomorphisms can be extended uniquely to (A_2, A_2)-algebra homomorphisms

$$A_2\langle C \otimes_A B\rangle \to \mathrm{End}_A({}_AB) \quad \text{and } A_2\langle C \otimes_A B\rangle \to \mathrm{End}_A(C_A)^{\mathrm{op}} \tag{7.7}$$

which define a right action of $A_2\langle C\otimes_A B\rangle$ on B and a left action on C, as in Proposition 2.2.4. These actions satisfy an additional condition. For example, the action on B satisfies

$$\begin{aligned} b_1((c \otimes b) \otimes (c' \otimes b')) &= (b_1(c \otimes b))\,(c' \otimes b') \\ &= ((b_1c)b)\,(c' \otimes b') \\ &= (b_1c)\,(b(c' \otimes b')) \\ &= (b_1c)\,((bc')b') \\ &= b_1(c \otimes (bc')b') \\ &= b_1(c(bc') \otimes b') \end{aligned}$$

for all $b_1 \in B$. Similarly, for the action on C the following holds

$$\begin{aligned} ((c \otimes b) \otimes (c' \otimes b'))c_1 &= (c \otimes b)\,((c' \otimes b')c_1) \\ &= (c \otimes b)\,(c'(b'c_1)) \\ &= ((c \otimes b)c')\,(b'c_1) \\ &= (c(bc'))\,(b'c_1) \\ &= (c(bc') \otimes b')c_1 \end{aligned}$$

for all $c_1 \in C$. Therefore the elements

$$c(bc') \otimes b' - (c \otimes b) \otimes (c' \otimes b') \in A_2\langle C \otimes_A B\rangle$$

are in the kernel of either morphism (7.7) for all $b, b' \in B$ and $c, c' \in C$. This shows that the actions of $A_2\langle C \otimes_A B\rangle$ on each of B and C factor through the quotient ring D. Consequently, formulas (7.3) are well-defined D-actions, making B a unitary (A, D)-bimodule and C a unitary (D, A)-bimodule, as claimed.

We now turn to the multiplication maps (7.4) and (7.5). Since $f_{121} = m$, we see that f_{121} is (A, A)-bilinear. We also have

$$\begin{aligned} f_{121}(b(c \otimes b' + E), c') &= f_{121}((bc)b', c') = (bc)f_{121}(b', c') = (bc)(b'c') \\ &= f_{121}(b, c)(b'c') = f_{121}(b, c(b'c')) = f_{121}(b, (c \otimes b' + E)c'), \end{aligned}$$

proving that f_{121} is D-balanced. Furthermore, f_{212} is A-balanced and

$$\begin{aligned}
(c_1 \otimes b_1 + E)f_{212}(c,b)(c_2 \otimes b_2 + E) &= (c_1 \otimes b_1 + E)(c \otimes b + E)(c_2 \otimes b_2 + E)\\
&= (c_1 \otimes b_1) \otimes (c \otimes b) \otimes (c_2 \otimes b_2) + E\\
&= (c_1 \otimes b_1) \otimes (c(bc_2) \otimes b_2) + E\\
&= (c_1 \otimes b_1) \otimes (c \otimes (bc_2)b_2) + E\\
&= c_1(b_1c) \otimes (bc_2)b_2 + E\\
&= f_{212}(c_1(b_1c), (bc_2)b_2)\\
&= f_{212}((c_1 \otimes b_1 + E)c, b(c_2 \otimes b_2 + E)),
\end{aligned}$$

proving that f_{212} is (D, D)-bilinear. It follows that all multiplication maps f_{ijk} of (7.4) and (7.5) are bilinear and balanced, as required in their positions.

We need to verify, however, that the proposed multiplication in $\sigma^{-1}R$ is associative; that is, that equations (6.1) hold for all choices of indices $1 \leq i, s, t, k \leq 3$. Whenever two neighboring indices in the quadruple (i, s, t, k) differ by 0 or 2, equation (6.1) involves a module action map and is thereby satisfied. In the remaining cases, equation (6.1) takes one of the following two forms

$$b(cb') = (bc)b' \text{ or } (cb)c' = c(bc')$$

with $b, b' \in B$ and $c, c' \in C$. These equations hold, however, by (7.6).

Each λ_{ii} is clearly a ring homomorphism and the remaining λ_{ij} are bimodule homomorphisms. Thus, to prove the multiplicative homomorphism property of λ we need only observe that the identity

$$\lambda(x)\lambda(y) = f_{123}(1 \otimes x, y \otimes 1) = \alpha^{\xi}_{13}(xy) = \lambda(xy)$$

holds for all $x \in M_{12}$ and $y \in M_{23}$. Therefore λ is a morphism of matrix 3-rings, as required.

For the σ-inverting property of λ, we note that in addition to the idempotents e'_{ii} $(1 \leq i \leq 3)$, the ring $R' = \sigma^{-1}R$ also contains the matrix units e'_{13} and e'_{31} which are featured in the calculation below. It is worth noting as well that $\lambda(\sigma(e_{11})) = e'_{13}$ since $\alpha^{\xi}_{13}(\xi) = 1_A$. Consider the R'-module homomorphism $\rho\colon R' \otimes_R Re_{33} \to R' \otimes_R Re_{11}$

given by $\rho(r'\otimes re_{33}) = r'\lambda(r)e'_{31}\otimes e_{11}$. Then we have

$$\begin{aligned}
[\rho\circ(\mathbb{1}\otimes\sigma)]\,(r'\otimes re_{33}) &= (\mathbb{1}\otimes\sigma)(r'\lambda(r)e'_{31}\otimes e_{11})\\
&= r'\lambda(r)e'_{31}\otimes\sigma(e_{11})\\
&= r'\lambda(r)e'_{31}\lambda(\sigma(e_{11}))\otimes e_{33}\\
&= r'\lambda(r)e'_{31}e'_{13}\otimes e_{33}\\
&= r'\lambda(r)e'_{33}\otimes e_{33}\\
&= r'\lambda(r)\lambda(e_{33})\otimes e_{33}\\
&= r'\otimes re_{33}.
\end{aligned}\tag{7.8}$$

Similarly, we have

$$\begin{aligned}
[(\mathbb{1}\otimes\sigma)\circ\rho]\,(r'\otimes re_{11}) &= \rho(r'\otimes r\sigma(e_{11}))\\
&= \rho(r'\lambda(r)\lambda(\sigma(e_{11}))\otimes e_{33})\\
&= r'\lambda(r)\lambda(\sigma(e_{11}))e'_{31}\otimes e_{11}\\
&= r'\lambda(r)e'_{13}e'_{31}\otimes e_{11}\\
&= r'\lambda(r)e'_{11}\otimes e_{11}\\
&= r'\lambda(r)\lambda(e_{11})\otimes e_{11}\\
&= r'\otimes re_{11}.
\end{aligned}\tag{7.9}$$

Thus, $\rho = (\mathbb{1}\otimes\sigma)^{-1}$, and we see that λ is σ-inverting.

Suppose now that $\mu\colon R\to T$ is a σ-inverting ring homomorphism. Using the canonical ring isomorphism $\gamma\colon T\to \mathrm{End}_T({}_TT)$ together with Theorem 6.2.2, we obtain a natural ring isomorphism $\phi\colon \mathcal{R}(M',F')\to \mathrm{End}_T({}_TT)$ and a morphism $\nu = \mathcal{R}(\nu_{ij})\colon \mathcal{R}(M,F)\to\mathcal{R}(M',F')$ making the following diagram commute, where $M'_{ij} = \mathrm{Hom}_T(T\otimes_R Re_{ii}, T\otimes_R Re_{jj})$ and the multiplication maps in F' are given by composition.

$$\begin{tikzcd}
\mathcal{R}(M,F) \arrow[r, "\mu"] \arrow[d, "\nu"'] & T \arrow[d, "\gamma", "\cong"'] \\
\mathcal{R}(M',F') \arrow[r, "\cong", "\phi"'] & \mathrm{End}_T({}_TT)
\end{tikzcd}$$

By assumption, since μ is σ-inverting, $\tau = \mathbb{1}_T\otimes\sigma$ is an isomorphism of T-modules. Denoting $N_{ij} = K = M'_{11}$ $(i,j\in\{1,3\})$, $N_{22} = L = M'_{22}$, $N_{12} = N_{32} = U = M'_{12}$ and $N_{21} = N_{23} = V = M'_{23}$, we obtain an isomorphism of matrix 3-rings

$$\eta = \mathcal{R}(\eta_{ij})\colon \mathcal{R}(M',F')\xrightarrow{\cong}\mathcal{R}(N,G) = \begin{pmatrix} K & U & K\\ V & L & V\\ K & U & K\end{pmatrix}\tag{7.10}$$

where $\eta_{13}(f) = f\circ\tau^{-1}$, $\eta_{21}(f) = f\circ\tau$, $\eta_{31}(f) = \tau\circ f$, $\eta_{32}(f) = \tau\circ f$, $\eta_{33}(f) = \tau\circ f\circ\tau^{-1}$ and the remaining η_{ij} are the respective identity morphisms. The multiplication maps in $G = \{g_{ijk}\}$ are given by the obvious modifications of the maps in F' to compensate for the change of labels. In particular, for those maps g_{ijk} that are not module action maps we have the following relations

$$g_{121} = g_{123} = g_{321} = g_{323} \colon U \times V \to K \quad \text{and } g_{212} = g_{232} \colon V \times U \to L. \tag{7.11}$$

It follows that $\omega = \gamma\circ\phi^{-1}\circ\eta \colon T \to \mathcal{R}(N, G)$ is a ring isomorphism and $\psi = \nu\circ\eta \colon R \to \mathcal{R}(N, G)$ is a morphism of matrix 3-rings such that $\mu = \psi \circ \omega^{-1}$. Thus $\psi = \mathcal{R}(\psi_{ij})$ with $\psi_{ij} = \nu_{ij} \circ \eta_{ij}$. In particular,

$$\widetilde{\psi} = \mathcal{R}\begin{pmatrix} \psi_{11} & \psi_{13} \\ 0 & \psi_{33} \end{pmatrix} : S_{13} = \begin{pmatrix} A_1 & M_{13} \\ 0 & A_3 \end{pmatrix} \to \mathcal{M}_2(K)$$

is an object in $\mathscr{C}(S_{13})$. We can identify the element $\psi_{13}(\xi)$ by noting that $\gamma(\mu(\xi))$ is an endomorphism of ${}_TT$ whose action is right multiplication by $\mu(\xi)$. Since

$$(\mu \circ \gamma \circ \phi^{-1})(\xi) = \nu(\xi) = \nu_{13}(\xi) \in M'_{13} = \mathrm{Hom}_T(T \otimes_R Re_{11}, T \otimes_R Re_{33}),$$

we have

$$\nu_{13}(\xi)(t \otimes re_{11}) = \nu_{13}(\xi)(t\mu(r) \otimes e_{11}) = t\mu(r)\mu(\xi) \otimes e_{33} = t \otimes r\xi e_{33} = t \otimes \sigma(re_{11}),$$

so that $\nu_{13}(\xi) = \mathbb{1}_T \otimes \sigma = \tau$ and $\psi_{13}(\xi) = \eta_{13}(\tau) = \tau \circ \tau^{-1} = 1_K$, proving that $\widetilde{\psi}$ is an object in $\mathscr{C}(S_{13}, \xi)$. By the universal property of $\alpha^\xi \colon S_{13} \to \mathcal{M}_2(A)$, there exists a unique ring homomorphism $\theta\colon A \to K$ such that $\alpha^\xi \circ \mathcal{M}_2(\theta) = \widetilde{\psi}$. That is,

$$\lambda_{ij} \circ \theta = \alpha^\xi_{ij} \circ \theta = \psi_{ij} \tag{7.12}$$

for $i, j \in \{1, 3\}$ with $i \le j$.

We proceed to define a morphism

$$\psi' = \mathcal{R}(\psi'_{ij}) \colon R' = \begin{pmatrix} A & B & A \\ C & D & C \\ A & B & A \end{pmatrix} \longrightarrow \mathcal{R}(N, G) = \begin{pmatrix} K & U & K \\ V & L & V \\ K & U & K \end{pmatrix}$$

as follows. Let

$$\begin{aligned}
\psi'_{ij}(a) &= \theta(a) \qquad\qquad\qquad (i,j \in \{1,3\}),\\
\psi'_{12}(a\otimes x) &= \psi'_{32}(a\otimes x) = \theta(a)\psi_{12}(x),\\
\psi'_{21}(x\otimes a) &= \psi'_{23}(x\otimes a) = \psi_{23}(x)\theta(a),\\
\psi'_{22}(c\otimes b + E) &= g_{212}(\psi'_{21}(c), \psi'_{12}(b)).
\end{aligned} \tag{7.13}$$

Clearly, ψ'_{11} and ψ'_{33} are ring homomorphisms which, along with ψ_{22}, give U a unitary (A, A_2)-bimodule structure and V a unitary (A_2, A)-bimodule structure. One can verify that $\psi'_{12} = \psi'_{32}$ and $\psi'_{21} = \psi'_{23}$ are well-defined (A, A_2)-bimodule and (A_2, A)-bimodule homomorphisms, respectively. Since ψ is a ring homomorphism, we then have

$$\begin{aligned}
\psi'_{12}(a\otimes x)\psi'_{23}(y\otimes a') &= g_{123}(\psi'_{12}(a\otimes x), \psi'_{23}(y\otimes a'))\\
&= g_{123}(\theta(a)\psi_{12}(x), \psi_{23}(y)\theta(a'))\\
&= \theta(a)g_{123}(\psi_{12}(x), \psi_{23}(y))\theta(a')\\
&= \theta(a)\psi_{13}(xy)\theta(a')\\
&= \theta(a)\theta(\alpha^{\xi}_{13}(xy))\theta(a')\\
&= \theta(a\alpha^{\xi}_{13}(xy)a')\\
&= \psi'_{13}(f_{123}(a\otimes x, y\otimes a'))
\end{aligned}$$

for all generator pairs $(a\otimes x, y\otimes a') \in B\times C$. Therefore

$$\psi'_{12}(b)\psi'_{23}(c) = \psi'_{13}(bc) = \theta(bc) \tag{7.14}$$

holds for all pairs $(b, c) \in B\times C$.

The last formula of (7.13) initially defines only an (A_2, A_2)-bimodule homomorphism $C\otimes_A B \to L$ which, by the universal property of the tensor algebra, extends uniquely to an (A_2, A_2)-algebra homomorphism $\psi''_{22}\colon A_2\langle C\otimes_A B\rangle \to L$. To prove that $E \subseteq \ker\psi''_{22}$, we use relation (7.14) to see that

$$\begin{aligned}
\psi''_{22}(c(bc')\otimes b' - (c\otimes b)\otimes(c'\otimes b')) &= \psi''_{22}(c(bc')\otimes b') - \psi''_{22}(c\otimes b)\psi''_{22}(c'\otimes b')\\
&= \psi'_{21}(c(bc'))\psi'_{12}(b') - \psi'_{21}(c)\psi'_{12}(b)\psi'_{21}(c')\psi'_{12}(b')\\
&= \psi'_{21}(c)\theta(bc')\psi'_{12}(b') - \psi'_{21}(c)\psi'_{12}(b)\psi'_{23}(c')\psi'_{12}(b')\\
&= \psi'_{21}(c)\left(\theta(bc') - \psi'_{12}(b)\psi'_{23}(c')\right)\psi'_{12}(b')\\
&= 0.
\end{aligned}$$

Thus, the algebra morphism ψ'_{22} factors uniquely through the quotient ring to give a ring homomorphism $\psi'_{22}\colon D \to L$.

Now that we have shown each ψ'_{ii} to be a ring homomorphism, we proceed to show that the remaining ψ'_{ij} are bimodule homomorphisms. In light of the previous discussion, all that remains to be seen is that ψ'_{12} is right D-linear and ψ'_{23} is left D-linear. The computations are similar and one is provided below.

$$\begin{aligned}
\psi'_{12}(b(c \otimes b' + E)) &= \psi'_{12}((bc)b') \\
&= \theta(bc)\psi'_{12}(b') \\
&= \psi'_{12}(b)\psi'_{23}(c)\psi'_{12}(b') \\
&= \psi'_{12}(b)\psi'_{21}(c)\psi'_{12}(b') \\
&= \psi'_{12}(b)\psi'_{22}(c \otimes b + E).
\end{aligned}$$

We also have to verify that the matrix, (ψ'_{ij}), of bimodule homomorphisms is a ring homomorphism. That is, that the equations

$$(\psi'_{ij} \times \psi'_{jk}) \circ g_{ijk} = f_{ijk} \circ \psi'_{ik} \tag{7.15}$$

hold for all i, j, k. Taking into account relations (7.4), (7.5) and (7.11), it suffices to verify these equations in the case of only two index triples (i, j, k), say $(2, 1, 2)$ and $(1, 2, 3)$. In the first case,

$$g_{212}(\psi'_{21}(c), \psi'_{12}(b)) = \psi'_{22}(c \otimes b + E) = \psi'_{22}(f_{212}(c, b))$$

holds by the definition of ψ'_{22}. In the second case,

$$g_{123}(\psi'_{12}(b), \psi'_{23}(c)) = \psi'_{13}(bc) = \psi'_{13}(f_{123}(b, c))$$

holds by (7.14). Therefore all equations (7.15) are satisfied and ψ' is a ring homomorphism, as claimed. It follows from equations (7.2), (7.12), and (7.13) that $\lambda \circ \psi' = \psi$.

$$\begin{tikzcd}
R \arrow[r, "\lambda"] \arrow[d, "\mu"'] \arrow[dr, "\psi"] & R' \arrow[d, "\psi'"] \\
T \arrow[r, "\cong", "\omega"'] & \mathcal{R}(N, G)
\end{tikzcd}$$

Suppose now that $\chi = \mathcal{R}(\chi_{ij}) \colon R' \to \mathcal{R}(N, G)$ is an alternative morphism such that $\lambda \circ \chi = \psi$. We note the following submatrices of the $\mathcal{R}(N, G)$ matrix (7.10)

$$Q_1 = \begin{pmatrix} K & 0 & K \\ 0 & 0 & 0 \\ K & 0 & K \end{pmatrix}, \ Q_2 = \begin{pmatrix} U \\ 0 \\ U \end{pmatrix}, \ Q_3 = \begin{pmatrix} V & 0 & V \end{pmatrix}.$$

The first matrix is a subrng of $\mathcal{R}(N,G)$ which is isomorphic to $\mathcal{M}_2(K)$ as a ring. The other two are additive subgroups and left (resp. right) unitary Q_1-modules. Similar statements hold for the corresponding submatrices of the R' matrix (7.1)

$$Q_1' = \begin{pmatrix} A & 0 & A \\ 0 & 0 & 0 \\ A & 0 & A \end{pmatrix}, \; Q_2' = \begin{pmatrix} B \\ 0 \\ B \end{pmatrix}, \; Q_3' = \begin{pmatrix} C & 0 & C \end{pmatrix}.$$

Furthermore, any morphism $R' \to \mathcal{R}(N,G)$, such as χ, maps each Q_i' to Q_i as a ring or module homomorphism. In particular, we have $\chi_{ii}(1_A) = 1_K$ for $i \in \{1,3\}$ and $\chi_{13}(1_A) = \chi_{13}(\lambda_{13}(\xi)) = \psi_{13}(\xi) = 1_K$. Thus if E_{ij} $(i,j \in \{1,3\})$ denote the matrix units of Q_1, then $\chi(e_{ij}') = E_{ij}$ for $i \le j$. By Proposition 2.3.4 $\chi(e_{ij}') = E_{ij}$ for all i,j. It then follows by Morita equivalence that $\chi_{ij} = \chi_{11}$ for $i,j \in \{1,3\}$, $\chi_{12} = \chi_{32}$, and $\chi_{21} = \chi_{23}$. In particular, the equations

$$\alpha_{ij}^{\xi} \circ \chi_{11} = \alpha_{ij}^{\xi} \circ \psi_{11}' \qquad (i \le j,\ i,j \in \{1,3\})$$

imply, by the universal property of α^{ξ}, that $\chi_{11} = \psi_{11}'$. The bimodule homomorphism properties of χ_{ij} and ψ_{ij}' together with

$$\lambda_{12} \circ \chi_{12} = \lambda_{12} \circ \psi_{12}' \;\text{ and }\; \lambda_{23} \circ \chi_{23} = \lambda_{23} \circ \psi_{23}'$$

imply that $\chi_{12} = \psi_{12}'$ and $\chi_{23} = \psi_{23}'$. Finally, since the algebra D is generated by the image of f_{212}, the equation $\lambda_{22} \circ \chi_{22} = \lambda_{22} \circ \psi_{22}'$ implies $\chi_{22} = \psi_{22}'$. It follows that $\chi = \psi'$.

If $\delta\colon R' \to T$ is a ring homomorphism satisfying $\lambda \circ \delta = \mu$, then δ maps the idempotents $e_{ii}' = \lambda(e_{ii})$ of R' to the idempotents $\mu(e_{ii}) \in T$. It follows by Theorem 6.1.10 that $\chi = \delta \circ \omega\colon R' \to \mathcal{R}(N,G)$ is an instance of the alternative morphism χ such that $\lambda \circ \chi = \psi$. Therefore $\delta = \psi' \circ \omega^{-1}$. In particular, there is exactly one such homomorphism δ. $\square$

We conclude this section with several examples demonstrating some of the phenomena that can occur in the various entry bimodules of the localization. We may

observe that the middle ring, D, grows quite large even when the bimodules in R are relatively small. In the following examples, we recall that $\mathbb{Z}\langle X\rangle$ and $\mathbb{Z}[X]$ denote respectively the free associative and free commutative associative $\mathbb{Z}$-algebras generated by X and, for any ring K, we denote the tensor product ring $K\langle X\rangle = K \otimes \mathbb{Z}\langle X\rangle$ and the polynomial ring $K[X] = K \otimes \mathbb{Z}[X]$.

Example 7.1.4. Let K be a ring with a fixed central element $b \in K$ and

$$R = \begin{pmatrix} K & K[x] & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$$

where $f_{123}(p, k) = p(b)k$. If $\sigma(e_{11}) = e_{13}$, then

$$\sigma^{-1}R = \begin{pmatrix} K & K[x] & K \\ K & K\langle X\rangle/W & K \\ K & K[x] & K \end{pmatrix}$$

where $X = \{x^i \mid i \geq 0\}$ and W is the two-sided ideal in $K\langle X\rangle = K\langle K \otimes_K K[x]\rangle$ generated by the elements $b^j u - vu$ for $u = x^i$, $v = x^j \in X$. The quotient ring $K\langle X\rangle/W$ acts on $K[x]$ and K by the equations $p(q \otimes p' + W) = (p(b)q)p'$ and $(q \otimes p + W)q' = q(p(b)q')$ for all $p, p' \in K[x], q, q' \in K$.

Example 7.1.5. Let K be a ring with fixed central elements $b, c \in K$ and

$$R = \begin{pmatrix} K & K[x] & K \\ 0 & K & K[y] \\ 0 & 0 & K \end{pmatrix}$$

where $f_{123}(p, q) = p(b)q(c)$. If $\sigma(e_{11}) = e_{13}$, then

$$\sigma^{-1}R = \begin{pmatrix} K & K[x] & K \\ K[y] & K\langle X \times Y\rangle/W & K[y] \\ K & K[x] & K \end{pmatrix}$$

where $X = \{x^i \mid i \geq 0\}$, $Y = \{y^j \mid j \geq 0\}$, and W is the ideal in $K\langle X \times Y\rangle = K\langle K[y] \otimes_K K[x]\rangle$ generated by the elements $b^s c^j w - vu$ for $u = (x^i, y^j)$, $v = (x^s, y^t)$, $w = (x^i, y^t) \in X \times Y$. The quotient ring $K\langle X \times Y\rangle/W$ acts on $K[x]$ and $K[y]$ by the equations $p(q \otimes p' + W) = (p(b)q(c))p'$ and $(q \otimes p + W)q' = q(p(b)q'(c))$ for all $p, p' \in K[x]$ and $q, q' \in K[y]$.

Example 7.1.6. Let K be a ring with fixed central elements $b, c \in K$ and

$$R = \begin{pmatrix} K & K[x] & K^2 \\ 0 & K & K[y] \\ 0 & 0 & K \end{pmatrix}$$

where $f_{123}(p,q) = (p(b)q(c), 0) \in K^2$. If $\sigma(e_{13}) = (1,0)$, then

$$\sigma^{-1}R = \begin{pmatrix} K[z] & K[x,z] & K[z] \\ K[y,z] & K\langle X \times Y \times Z\rangle/W & K[y,z] \\ K[z] & K[x,z] & K[z] \end{pmatrix}$$

where $X = \{x^i \mid i \geq 0\}, Y = \{y^j \mid j \geq 0\}, Z = \{z^s \mid s \geq 0\}$, and W is the ideal in

$$K\langle X \times Y \times Z\rangle = K\langle K[y,z] \otimes_{K[z]} K[x,z]\rangle$$

generated by the elements $b^r c^j w - vu$ for $u = (x^i, y^j, z^k)$, $v = (x^r, y^s, z^t)$, $w = (x^i, y^s, z^{k+t}) \in X \times Y \times Z$. The quotient ring $K\langle X \times Y \times Z\rangle/W$ acts on $K[x,z]$ and $K[y,z]$ by the equations $p(q \otimes p' + W) = (p|_{x=b})(q|_{y=c})p'$ and $(q \otimes p + W)q' = q(p|_{x=b})(q'|_{y=c})$ for all $p, p' \in K[x,z], q, q' \in K[y,z]$. The only nonobvious multiplication maps are given by $f'_{i23}(p,q) = (p|_{x=b})(q|_{y=c}) \in K[z]$ for all $p \in K[x,z]$, $q \in K[y,z]$, and $i \in \{1,3\}$. The morphism $\lambda = \mathcal{R}(\lambda_{ij})\colon R \to \sigma^{-1}R$ is given by $\lambda_{13}(g,h) = g + hz$ while the remaining λ_{ij} are the appropriate inclusions.

Example 7.1.7. Let K be a ring with fixed central elements $b, c, d \in K[z]$ and

$$R = \begin{pmatrix} K & K[x] & K[z] \\ 0 & K & K[y] \\ 0 & 0 & K \end{pmatrix}$$

where $f_{123}(p,q) = p(b)q(c)d$. If $\sigma(e_{11}) = e_{13}$, then

$$\sigma^{-1}R = \begin{pmatrix} K\langle Z\rangle & K[x]\langle Z\rangle & K\langle Z\rangle \\ K[y]\langle Z\rangle & K[x,y]\langle Z\rangle/W & K[y]\langle Z\rangle \\ K\langle Z\rangle & K[x]\langle Z\rangle & K\langle Z\rangle \end{pmatrix}$$

where $Z = \{z^s \mid s > 0\}$ and W is the ideal in

$$K[x,y]\langle Z\rangle = K\langle K[y]\langle Z\rangle \otimes_{K\langle Z\rangle} K[x]\langle Z\rangle\rangle$$

generated by the elements $b^s c^j dw - vu$ for $u = x^i y^j z_U, v = x^s y^t z_V, w = x^i y^t z_V z_U \in K[x,y]\langle Z\rangle$, with $z_U = (z^{s_1})^{u_1} \cdots (z^{s_m})^{u_m}, z_V = (z^{t_1})^{v_1} \cdots (z^{t_n})^{v_n}$.

7.2 Universal Localization of Triangular Matrix 5-Rings

Determining the universal localization of a triangular matrix 3-ring is a stepping stone in our main construction. In this section, we examine the localization of a triangular matrix 5-ring. Although this may at first seem like an arbitrary choice, the structure of matrix 5-rings captures all of the detail necessary to handle the general case and so, for our purposes, is the most useful case to consider.

Our strategy for localizing a matrix 5-ring,

$$R = \begin{pmatrix} A_1 & M_{12} & M_{13} & M_{14} & M_{15} \\ 0 & A_2 & M_{23} & M_{24} & M_{25} \\ 0 & 0 & A_3 & M_{34} & M_{35} \\ 0 & 0 & 0 & A_4 & M_{45} \\ 0 & 0 & 0 & 0 & A_5 \end{pmatrix},$$

with respect to a morphism $\sigma\colon Re_{22} \to Re_{44}$, is quite similar to the mechanism used to identify the localization in the case of a 3-ring. The key to this strategy is the realization that the middle 3×3 submatrix,

$$Z = \begin{pmatrix} A_2 & M_{23} & M_{24} \\ 0 & A_3 & M_{34} \\ 0 & 0 & A_4 \end{pmatrix}$$

admits a related morphism ζ between its first and third columns. We then deduce that any σ-inverting homomorphism will necessarily be ζ inverting, whence the middle 3×3 block of the localization is determined by Theorem 7.1.1. Identifying the remaining bimodules is then done by examining a matrix 5-ring structure that can be found in any σ-inverting ring. Thus, the proof of the main result of this section builds upon Theorem 7.1.1 in a substantial way.

Theorem 7.2.1. *Let $R = \mathcal{R}(M, F)$ be a triangular matrix 5-ring and suppose $\sigma\colon Re_{22} \to Re_{44}$ is an R-module homomorphism. The following morphism of matrix 5-rings*

$$\lambda = \mathcal{R}(\lambda_{ij})\colon R = \begin{pmatrix} A_1 & M_{12} & M_{13} & M_{14} & M_{15} \\ 0 & A_2 & M_{23} & M_{24} & M_{25} \\ 0 & 0 & A_3 & M_{34} & M_{35} \\ 0 & 0 & 0 & A_4 & M_{45} \\ 0 & 0 & 0 & 0 & A_5 \end{pmatrix} \longrightarrow \sigma^{-1}R = \begin{pmatrix} A_1 & U & G & U & W \\ 0 & A & B & A & V \\ 0 & C & D & C & H \\ 0 & A & B & A & V \\ 0 & 0 & 0 & 0 & A_5 \end{pmatrix} \tag{7.16}$$

is the universal σ-inverting localization of R, where $\sigma(e_{22}) = \xi \in M_{24}$ and

$$\alpha^\xi = \mathcal{R}\begin{pmatrix} \alpha^\xi_{22} & \alpha^\xi_{24} \\ 0 & \alpha^\xi_{44} \end{pmatrix} : S_{24} = \begin{pmatrix} A_2 & M_{24} \\ 0 & A_4 \end{pmatrix} \longrightarrow \mathcal{M}_2(A_\xi)$$

is the initial object in $\mathscr{C}(S_{24}, \xi)$. The entry bimodules of the $\sigma^{-1}R$ matrix and the corresponding λ-values are listed below.

$$\begin{array}{lll}
A = A_\xi & \lambda_{ij}(x) = \alpha^\xi_{ij}(x) & (i \le j,\ i,j \in \{2,4\}); \\
B = A \otimes_{A_2} M_{23} & \lambda_{23}(x) = 1 \otimes x; & \\
C = M_{34} \otimes_{A_4} A & \lambda_{34}(x) = x \otimes 1; & \\
D = A_3\langle C \otimes_A B\rangle / E & \lambda_{33}(x) = x + E & \\
E = \big(c(bc') \otimes b' - (c \otimes b) \otimes (c' \otimes b')\big) & & (b, b' \in B, c, c' \in C); \\
A_i & \lambda_{ii}(x) = x & (i \in \{1,5\}); \\
U = (M_{12} \otimes_{A_2} A \oplus M_{14} \otimes_{A_4} A)/I & \lambda_{1j}(x) = x \otimes 1 + I & (j \in \{2,4\}) \\
I = A_1(x \otimes \alpha^\xi_{24}(y) - xy \otimes 1)A & & (x \in M_{12},\ y \in M_{24}); \\
V = (A \otimes_{A_2} M_{25} \oplus A \otimes_{A_4} M_{45})/J & \lambda_{i5}(x) = 1 \otimes x + J & (i \in \{2,4\}) \\
J = A(1 \otimes xy - \alpha^\xi_{24}(x) \otimes y)A_5 & & (x \in M_{24},\ y \in M_{45}); \\
G = (M_{13} \oplus U \otimes_A B)/K & \lambda_{13}(x) = x + K & \\
K = A_1(xy - x \otimes 1 \otimes 1 \otimes y)A_3 & & (x \in M_{12},\ y \in M_{23}); \\
H = (M_{35} \oplus C \otimes_A V)/L & \lambda_{35}(x) = x + L & \\
L = A_3(xy - x \otimes 1 \otimes 1 \otimes y)A_5 & & (x \in M_{34},\ y \in M_{45}); \\
W = (M_{15} \oplus U \otimes_A V)/N & \lambda_{15}(x) = x + N & \\
N = A_1(xy - x \otimes 1 \otimes 1 \otimes y)A_5 & & (x \in M_{1s}, y \in M_{s5}, \\
 & & \ s \in \{2,4\}).
\end{array} \tag{7.17}$$

The quotient ring D, of the tensor algebra of the (A_3, A_3)-bimodule $C \otimes_A B$, has right actions on B and G and left actions on C and H defined by the equations

$$\begin{aligned} b(c \otimes b' + E) &= (bc)b', \\ (x + u \otimes b + K)(c \otimes b' + E) &= (xc + I) \otimes b' + u \otimes (bc)b' + K, \\ (c \otimes b + E)c' &= c(bc'), \\ (c \otimes b + E)(x + c' \otimes v + L) &= c \otimes (bx + J) + c(bc') \otimes v + L, \end{aligned} \tag{7.18}$$

where $bc = a\alpha^{\xi}_{24}(xy)a' \in A$ for $(b, c) = (a \otimes x, y \otimes a')$, thereby making B an (A, D)-bimodule, G an (A_1, D)-bimodule, C a (D, A)-bimodule and H a (D, A_5)-bimodule (all modules unitary). The multiplication maps $f_{ijk} \colon R_{ij} \times R_{jk} \to R_{ik}$ of the ring $\sigma^{-1}R = (R_{ij})$, other than the module action maps, are defined as follows

$$\begin{aligned} f_{232}(b, c) = f_{234}(b, c) &= f_{432}(b, c) = f_{434}(b, c) = bc, \\ f_{323}(c, b) &= f_{343}(c, b) = c \otimes b + E, \\ f_{123}(u, b) &= f_{143}(u, b) = u \otimes b + K, \\ f_{125}(u, v) &= f_{145}(u, v) = u \otimes v + N, \\ f_{132}(x + u \otimes b + K, c) &= f_{134}(x + u \otimes b + K, c) \\ &= (xc + I) + u(bc), \\ f_{135}(x + u \otimes b + K, y + c \otimes v + L) &= xy + (xc + I) \otimes v + u \otimes (by + J) \\ &\quad + u(bc) \otimes v + N, \\ f_{235}(b, y + c \otimes v + L) &= f_{435}(b, y + c \otimes v + L) \\ &= (by + J) + (bc)v, \\ f_{325}(c, v) &= f_{345}(c, v) = c \otimes v + L. \end{aligned} \tag{7.19}$$

Proof. We note first that both matrices of (7.1) match the middle 3×3 submatrices of (7.16) under the position correspondence $(i, j) \leftrightarrow (i+1, j+1)$. This match extends to the corresponding morphisms λ_{ij}, α^{ξ}, D-actions (7.3) as well as the multiplication maps (7.4) and (7.5). We also find, by a routine verification, that the D-actions (7.18) are well-defined and that the multiplication maps (7.19) are well-defined, bilinear and balanced, as appropriate in their positions. It follows, in particular, that the middle 3×3 submatrix of (7.16)

$$Z' = \begin{pmatrix} A & B & A \\ C & D & C \\ A & B & A \end{pmatrix}$$

is an associative ring. Using this fact, we can view the $\sigma^{-1}R$ matrix (7.16) as a 3×3 matrix

$$\begin{pmatrix} A_1 & U'_{12} & W \\ 0 & Z' & V'_{23} \\ 0 & 0 & A_5 \end{pmatrix} \tag{7.20}$$

where $U'_{12} = (U\ G\ U)$ and $V'_{23} = (V\ H\ V)^\top$. From this perspective, one can verify that U'_{12} is a unitary (A_1, Z')-bimodule, V'_{23} is a unitary (Z', A_5)-bimodule and the multiplication $U'_{12} \times V'_{23} \to W$ induced by the maps f_{125}, f_{135}, and f_{145} is (A_1, A_5)-bilinear and Z'-balanced. Since the 3×3 matrix (7.20) is upper triangular, all equations of type (6.1) must then hold for it, proving that the multiplication in $R' = \sigma^{-1}R$ is associative. The ring R can similarly be viewed as a matrix 3-ring

$$\begin{pmatrix} A_1 & U_{12} & M_{15} \\ 0 & Z & V_{23} \\ 0 & 0 & A_5 \end{pmatrix}$$

where Z is the middle 3×3 submatrix of the R matrix (7.16), $U_{12} = (M_{12}\ M_{13}\ M_{14})$ and $V_{23} = (M_{25}\ M_{35}\ M_{45})^\top$.

Each function λ_{ii} is clearly a ring homomorphism and the remaining λ_{ij} are bimodule homomorphisms. Using the 3-ring descriptions of R and R', however, we can further observe that λ respects these block decompositions and maps Z into Z' as a ring homomorphism, U_{12} into U'_{12} as an (A_1, Z)-bimodule homomorphism, and V_{23} into V'_{23} as a (Z, A_3)-bimodule homomorphism. Consequently, verification of the multiplicative homomorphism property of λ can be reduced to calculating the products $\lambda_{1j}(x)\lambda_{j5}(y)$ for $x \in M_{1j}$ and $y \in M_{j5}$ $(2 \le j \le 4)$ which, by (7.19) , are equal to $\lambda_{15}(xy)$. Therefore, $\lambda\colon R \to R'$ is a morphism of matrix 5-rings, as claimed. Furthermore, λ is σ-inverting by calculations similar to (7.8) and (7.9) but utilizing instead the matrix units e'_{24} and e'_{42} in R'.

Suppose now that $\mu\colon R \to T$ is a σ-inverting ring homomorphism. Using the canonical ring isomorphism $\gamma\colon T \to \mathrm{End}_T({}_TT)$ together with Theorem 6.2.2,

we obtain a natural ring isomorphism $\phi\colon \mathcal{R}(M',F') \to \mathrm{End}_T({}_TT)$ and a morphism $\nu = \mathcal{R}(\nu_{ij})\colon \mathcal{R}(M,F) \to \mathcal{R}(M',F')$ making the following diagram commute, where $M'_{ij} = \mathrm{Hom}_T(T \otimes_R Re_{ii}, T \otimes_R Re_{jj})$ and the multiplication maps in F' are given by composition.

$$\begin{tikzcd}
\mathcal{R}(M,F) \arrow[r, "\mu"] \arrow[d, "\nu"'] & T \arrow[d, "\cong"', "\gamma"] \\
\mathcal{R}(M',F') \arrow[r, "\cong", "\phi"'] & \mathrm{End}_T({}_TT)
\end{tikzcd}$$

By assumption, since μ is σ-inverting, $\tau = \mathbb{1}_T \otimes \sigma$ is an isomorphism of T-modules. Denoting $T_A = M'_{22}$, $T_B = M'_{23}$, $T_C = M'_{34}$, $T_D = M'_{33}$, $T_G = M'_{13}$, $T_H = M'_{35}$, $T_U = M'_{12}$, $T_V = M'_{25}$, $T_W = M'_{15}$ and $T_{ij} = M'_{ij}$ for the remaining positions, we obtain an isomorphism of matrix 5-rings

$$\eta = \mathcal{R}(\eta_{ij})\colon \mathcal{R}(M',F') \overset{\cong}{\to} \mathcal{R}(\widehat{T},\widehat{G}) = \begin{pmatrix} T_{11} & T_U & T_G & T_U & T_W \\ T_{21} & T_A & T_B & T_A & T_V \\ T_{31} & T_C & T_D & T_C & T_H \\ T_{41} & T_A & T_B & T_A & T_V \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} \end{pmatrix}$$

where $\eta_{i4}(f) = f \circ \tau^{-1}$ $(i \in \{1,2\})$, $\eta_{4j}(f) = \tau \circ f$ $(j \in \{2,3,5\})$, $\eta_{32}(f) = f \circ \tau$, $\eta_{44}(f) = \tau \circ f \circ \tau^{-1}$ and the remaining η_{ij} are the respective identity morphisms. The multiplication maps in $\widehat{G} = \{g_{ijk}\}$ are given by the obvious modifications of the maps in F' to compensate for the change of labels. In particular, they satisfy the relations

$$g_{ijk} = g_{rst} \qquad (1 \le i,j,k,r,s,t \le 5;\ i,r < 5;\ k,t > 1)$$

whenever any two indices occupying the same position are equal or both belong to $\{2,4\}$. It follows that $\omega = \gamma \circ \phi^{-1} \circ \eta\colon T \to \mathcal{R}(\widehat{T},\widehat{G})$ is a ring isomorphism and $\psi = \nu \circ \eta\colon R \to \mathcal{R}(\widehat{T},\widehat{G})$ is a morphism of matrix 5-rings such that $\mu = \psi \circ \omega^{-1}$. Thus $\psi = \mathcal{R}(\psi_{ij})$ with $\psi_{ij} = \nu_{ij} \circ \eta_{ij}$ $(1 \le i \le j \le 5)$.

Let $\zeta\colon Ze_{22} \to Ze_{44}$ be the Z-module homomorphism determined by $\zeta(e_{22}) = \xi$. We note that the following morphism obtained by restricting ψ

$$\widetilde{\psi} = \mathcal{R}(\psi_{ij})\colon Z = \begin{pmatrix} A_2 & M_{23} & M_{24} \\ 0 & A_3 & M_{34} \\ 0 & 0 & A_4 \end{pmatrix} \longrightarrow \mathcal{R}(\widetilde{T}, \widetilde{G}) = \begin{pmatrix} T_A & T_B & T_A \\ T_C & T_D & T_C \\ T_A & T_B & T_A \end{pmatrix}$$

is ζ-inverting as there exist matrix units in $\mathcal{R}(\widetilde{T}, \widetilde{G})$ in positions $(2,4)$ and $(4,2)$ and $\psi_{24}(\xi) = 1 \in T_A$. Furthermore, the restriction of λ to Z, denoted $\lambda'\colon Z \to Z'$, is the universal ζ-inverting localization of Z by Theorem 7.1.1. Thus there is a morphism of matrix 3-rings

$$\widetilde{\psi}' = \mathcal{R}(\psi'_{ij})\colon Z' = \begin{pmatrix} A & B & A \\ C & D & C \\ A & B & A \end{pmatrix} \longrightarrow \mathcal{R}(\widetilde{T}, \widetilde{G}) = \begin{pmatrix} T_A & T_B & T_A \\ T_C & T_D & T_C \\ T_A & T_B & T_A \end{pmatrix}$$

such that $\lambda_{ij} \circ \psi'_{ij} = \psi_{ij}$ for all $2 \le i \le j \le 4$. We proceed to define a morphism

$$\psi' = \mathcal{R}(\psi'_{ij})\colon R' = \begin{pmatrix} A_1 & U & G & U & W \\ 0 & A & B & A & V \\ 0 & C & D & C & H \\ 0 & A & B & A & V \\ 0 & 0 & 0 & 0 & A_5 \end{pmatrix} \longrightarrow \mathcal{R}(\widehat{T}, \widehat{G}) = \begin{pmatrix} T_{11} & T_U & T_G & T_U & T_W \\ T_{21} & T_A & T_B & T_A & T_V \\ T_{31} & T_C & T_D & T_C & T_H \\ T_{41} & T_A & T_B & T_A & T_V \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} \end{pmatrix}$$

by extending $\widetilde{\psi}'$ as follows. Let

$$\begin{aligned} \psi'_{ii}(a_i) &= \psi_{ii}(a_i) && (i \in \{1,5\}), \\ \psi'_{12}(x \otimes a + I) &= \psi'_{14}(x \otimes a + I) = \psi_{1j}(x)\psi'_{22}(a) && (x \in M_{1j}, j \in \{2,4\}), \\ \psi'_{25}(a \otimes x + J) &= \psi'_{45}(a \otimes x + J) = \psi'_{22}(a)\psi_{i5}(x) && (x \in M_{i5}, i \in \{2,4\}), \\ \psi'_{13}(x + u \otimes b + K) &= \psi_{13}(x) + g_{123}(\psi'_{12}(u), \psi'_{23}(b)), \\ \psi'_{35}(x + c \otimes v + L) &= \psi_{35}(x) + g_{345}(\psi'_{34}(c), \psi'_{45}(v)), \\ \psi'_{15}(x + u \otimes v + N) &= \psi_{15}(x) + g_{125}(\psi'_{12}(u), \psi'_{25}(v)). \end{aligned} \tag{7.21}$$

These are well-defined bimodule homomorphisms and, since ψ is a ring homomorphism, they satisfy all equations of type (7.15), so ψ' is indeed a morphism of matrix 5-rings. It follows from equations (7.17) and (7.21) that $\lambda \circ \psi' = \psi$.

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[d, "\mu"'] \arrow[dr, "\psi"] & R' \arrow[d, "\psi'"] \\ T \arrow[r, "\cong", "\omega"'] & \mathcal{R}(\widehat{T}, \widehat{G}) \end{tikzcd}$$

Suppose now that $\chi = \mathcal{R}(\chi_{ij})\colon R' \to \mathcal{R}(\widehat{T}, \widehat{G})$ is an alternative morphism such that $\lambda \circ \chi = \psi$. We already know that $\chi_{ij} = \psi'_{ij}$ for all $2 \leq i, j \leq 4$ by Theorem 7.1.1. Since λ is the identity on A_1 and A_5, we must also have $\chi_{ii} = \psi'_{ii}$ for $i \in \{1, 5\}$. It follows further, by Morita equivalence, that $\chi_{12} = \chi_{14}$ and $\chi_{25} = \chi_{45}$. Let us examine next the effect of χ_{12} on the image of either summand $M_{1j} \otimes_{A_j} A$ in U. Using the bimodule homomorphism property of χ_{12}, we have

$$\begin{aligned}
\chi_{12}(x \otimes a + I) &= \chi_{12}(x \otimes 1 + I)\chi_{22}(a) \\
&= \chi_{12}(\lambda_{1j}(x))\chi_{22}(a) \\
&= \psi_{1j}(x)\psi'_{22}(a) \\
&= \psi'_{12}(x \otimes a + I).
\end{aligned}$$

Thus $\chi_{12} = \psi'_{12}$ which also implies $\chi_{14} = \psi'_{14}$. By similar calculations, $\chi_{25} = \psi'_{25}$ and $\chi_{45} = \psi'_{45}$. Furthermore, by the ring homomorphism property of χ,

$$\begin{aligned}
\chi_{13}(x + u \otimes b + K) &= \chi_{13}(x + K) + \chi_{13}(u \otimes b + K) \\
&= \chi_{13}(\lambda_{13}(x)) + g_{123}(\chi_{12}(u), \chi_{23}(b)) \\
&= \psi_{13}(x) + g_{123}(\psi'_{12}(u), \psi'_{23}(b)) \\
&= \psi'_{13}(x + u \otimes b + K).
\end{aligned}$$

This proves that $\chi_{13} = \psi'_{13}$ and a similar calculation shows $\chi_{35} = \psi'_{35}$. Finally,

$$\begin{aligned}
\chi_{15}(x + u \otimes v + N) &= \chi_{15}(x + N) + \chi_{15}(u \otimes v + N) \\
&= \chi_{15}(\lambda_{15}(x)) + g_{125}(\chi_{12}(u), \chi_{25}(v)) \\
&= \psi_{15}(x) + g_{125}(\psi'_{12}(u), \psi'_{25}(v)) \\
&= \psi'_{15}(x + u \otimes v + N)
\end{aligned}$$

which shows that $\chi_{15} = \psi'_{15}$. We can conclude therefore that $\chi = \psi'$. As in the proof of Theorem 7.1.1, this also shows that any ring homomorphism $\delta\colon R' \to T$, satisfying $\lambda \circ \delta = \mu$, must be equal to $\psi' \circ \omega^{-1}$. $\square$

We provide a concrete example of this theorem that builds off of the examples in Section 7.1.

Example 7.2.2. Let K be a ring with a fixed central element $b \in K$ and

$$R = \begin{pmatrix} K & 0 & K & K & K \\ 0 & K & K[x] & K^2 & K \\ 0 & 0 & K & K & K \\ 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 & K \end{pmatrix}$$

with multiplication maps $g_{234}(p, q) = (p(b)q, 0) \in K^2$, $g_{235}(p, q) = p(b)q \in K$. If $\sigma(e_{22}) = (1, 0) \in K^2$, then

$$\sigma^{-1}R = \begin{pmatrix} K & K[y] & K \oplus K[x, y] & K[y] & K \oplus K[y] \\ 0 & K[y] & K[x, y] & K[y] & K[y] \\ 0 & K[y] & K\langle X \times Y\rangle/W & K[y] & K \oplus K[y] \\ 0 & K[y] & K[x, y] & K[y] & K[y] \\ 0 & 0 & 0 & 0 & K \end{pmatrix}$$

where $X = \{x^i \mid i \geq 0\}$, $Y = \{y^j \mid j \geq 0\}$, and W is the ideal in

$$K\langle X \times Y\rangle = K\langle K[x, y]\rangle = K\langle K[y] \otimes_{K[y]} K[x, y]\rangle$$

generated by the elements $b^i w - vu$ for $u = (x^i, y^j), v = (x^s, y^t), w = (x^s, y^{j+t}) \in X \times Y$. The quotient ring $K\langle X \times Y\rangle/W$ acts on $K[x, y]$ and $K[y]$ by the equations $p(q \otimes p' + W) = (p|_{x=b})qp'$ and $(q \otimes p + W)q' = q(p|_{x=b})q'$ for all $p, p' \in K[x, y]$ and $q, q' \in K[y]$. The nonobvious multiplication maps are given as follows

$$\begin{aligned} f_{232}(p, q) = f_{234}(p, q) = f_{432}(p, q) = f_{434}(p, q) &= (p|_{x=b})q, \\ f_{323}(q, p) = f_{343}(q, p) &= qp + W, \\ f_{123}(q, p) = f_{143}(q, p) &= (0, qp), \\ f_{125}(q, q') = f_{145}(q, q') = f_{325}(q, q') = f_{345}(q, q') &= (0, qq'), \\ f_{132}((g, p), q) = f_{134}((g, p), q) &= gq + (p|_{x=b})q, \\ f_{135}((g, p), (h, q)) &= (gh, gq + (p|_{x=b})(h + q)), \\ f_{235}(p, (h, q)) = f_{435}(p, (h, q)) &= (p|_{x=b})(h + q). \end{aligned}$$

The morphism $\lambda = \mathcal{R}(\lambda_{ij})\colon R \to \sigma^{-1}R$ is given by $\lambda_{24}(g, h) = g + hy$ while the remaining λ_{ij} are inclusions into the appropriate first summands.

7.3 Universal Localization of Triangular Matrix ℓ-Rings

We now turn our attention to the general case of localizing a triangular matrix ℓ-ring $R = \mathcal{R}(M, F)$ with respect to a morphism $\sigma\colon Re_{pp} \to Re_{qq}$. By viewing M as a particular block matrix, the resulting matrix 5-ring structure on R satisfies the hypotheses of Theorem 7.2.1, hence the localization can be identified.

More precisely, let $R = \mathcal{R}(M, F)$ be a triangular matrix ℓ-ring and suppose $\sigma\colon Re_{pp} \to Re_{qq}$ is an R-module homomorphism $(1 \le p < q \le \ell)$. Then R can be viewed as a triangular matrix 5-ring

$$\begin{pmatrix} A'_1 & M'_{1p} & M'_{13} & M'_{1q} & M'_{15} \\ 0 & A_p & M'_{p3} & M_{pq} & M'_{p5} \\ 0 & 0 & A'_3 & M'_{3q} & M'_{35} \\ 0 & 0 & 0 & A_q & M'_{q5} \\ 0 & 0 & 0 & 0 & A'_5 \end{pmatrix}$$

with the entry bimodules given by the submatrices[1] of $M = (M_{ij})$ listed below.

$$A'_1 = (M_{ij})_{1\le i,j<p} \qquad M'_{1p} = (M_{ip})_{1\le i<p} \qquad M'_{13} = (M_{ij})^{1\le i<p}_{p<j<q} \qquad M'_{1q} = (M_{iq})_{1\le i<p}$$

$$A'_3 = (M_{ij})_{p<i,j<q} \qquad M'_{p3} = (M_{pj})_{p<j<q} \qquad M'_{p5} = (M_{pj})_{q<j\le\ell} \qquad M'_{15} = (M_{ij})^{1\le i<p}_{q<j\le\ell}$$

$$A'_5 = (M_{ij})_{q<i,j\le\ell} \qquad M'_{3q} = (M_{iq})_{p<i<q} \qquad M'_{35} = (M_{ij})^{p<i<q}_{q<j\le\ell} \qquad M'_{q5} = (M_{qj})_{q<j\le\ell}$$

We note the following two points:

(i) Each A'_i is a matrix ring and a subrng of R.

(ii) If $A'_s = 0$, then the unitary A'_s-modules M'_{is} and M'_{sj} are trivial.

Suppose that $\sigma(e_{pp}) = \xi \in M_{pq}$ and let

$$\alpha^\xi = \mathcal{R}\begin{pmatrix} \alpha^\xi_{pp} & \alpha^\xi_{pq} \\ 0 & \alpha^\xi_{qq} \end{pmatrix} : S_{pq} = \begin{pmatrix} A_p & M_{pq} \\ 0 & A_q \end{pmatrix} \longrightarrow \mathcal{M}_2(A_\xi)$$

be the initial object in $\mathscr{C}(S_{pq}, \xi)$. Using Theorem 7.2.1, we can now describe the universal σ-inverting localization of R as the following morphism of matrix 5-rings

[1] We adopt the convention that $A'_1 = 0$, $A'_3 = 0$, or $A'_5 = 0$ whenever $p = 1$, $q = p + 1$, or $q = \ell$.

(alternatively, as a morphism of matrix ℓ-rings by expanding the rings A_1', D, and A_5' and the corresponding bimodules using the images of the idempotents $e_{ii} \in R$).

$$\lambda = \mathcal{R}(\lambda_{ij})\colon R = \begin{pmatrix} A_1' & M_{1p}' & M_{13}' & M_{1q}' & M_{15}' \\ 0 & A_p & M_{p3}' & M_{pq} & M_{p5}' \\ 0 & 0 & A_3' & M_{3q}' & M_{35}' \\ 0 & 0 & 0 & A_q & M_{q5}' \\ 0 & 0 & 0 & 0 & A_5' \end{pmatrix} \longrightarrow \sigma^{-1}R = \begin{pmatrix} A_1' & U & G & U & W \\ 0 & A & B & A & V \\ 0 & C & D & C & H \\ 0 & A & B & A & V \\ 0 & 0 & 0 & 0 & A_5' \end{pmatrix}$$

The entry bimodules of the $\sigma^{-1}R$ matrix and the corresponding λ-values are listed below.

$$\begin{array}{lll}
A = A_\xi & \lambda_{ij}(x) = \alpha_{ij}^{\xi}(x) & (i \le j,\ i,j \in \{p,q\}); \\
B = \sum_{p<j<q}^{\oplus} A \otimes_{A_p} M_{pj} & \lambda_{pj}(x) = 1 \otimes x; & \\
C = \sum_{p<i<q}^{\oplus} M_{iq} \otimes_{A_q} A & \lambda_{iq}(x) = x \otimes 1; & \\
D = A_3'\langle C \otimes_A B\rangle / E & \lambda_{ij}(x) = x + E & (p < i, j < q) \\
E = (c(bc') \otimes b' - (c \otimes b) \otimes (c' \otimes b')) & & (b, b' \in B, c, c' \in C); \\
A_1',\ A_5' & \lambda_{ij}(x) = x & (i, j < p \text{ or } i, j > q); \\
U = \left(\sum_{1 \le i < p}^{\oplus} U_i\right)/I & \lambda_{ij}(x) = x \otimes 1 + I & (j \in \{p,q\}) \\
U_i = M_{ip} \otimes_{A_p} A \oplus M_{iq} \otimes_{A_q} A & & \\
I = A_1'(x \otimes \alpha_{pq}^{\xi}(y) - xy \otimes 1)A & & (x \in M_{ip},\ y \in M_{pq}); \\
V = \left(\sum_{q < j \le \ell}^{\oplus} V_j\right)/J & \lambda_{ij}(x) = 1 \otimes x + J & (i \in \{p,q\}) \\
V_j = A \otimes_{A_p} M_{pj} \oplus A \otimes_{A_q} M_{qj} & & \\
J = A(1 \otimes xy - \alpha_{pq}^{\xi}(x) \otimes y)A_5' & & (x \in M_{pq},\ y \in M_{qj}); \\
G = (M_{13}' \oplus U \otimes_A B)/K & \lambda_{ij}(x) = x + K & \\
K = A_1'(xy - x \otimes 1 \otimes 1 \otimes y)A_3' & & (x \in M_{ip},\ y \in M_{pj}); \\
H = (M_{35}' \oplus C \otimes_A V)/L & \lambda_{ij}(x) = x + L & \\
L = A_3'(xy - x \otimes 1 \otimes 1 \otimes y)A_5' & & (x \in M_{iq},\ y \in M_{qj}); \\
W = (M_{15}' \oplus U \otimes_A V)/N & \lambda_{ij}(x) = x + N & \\
N = A_1'(xy - x \otimes 1 \otimes 1 \otimes y)A_5' & & (x \in M_{is}, y \in M_{sj}, \\
 & & \ s \in \{p,q\}).
\end{array}$$

The quotient ring, D, of the tensor algebra of the (A_3', A_3')-bimodule $C \otimes_A B$, has a right action on B and a left action on C defined by the equations

$$\begin{array}{ll}
b(c \otimes b' + E) = (bc)b' & (b \in B_{pj},\ c \in C_{iq},\ b' \in B_{pk}), \\
(c \otimes b + E)c' = c(bc') & (c \in C_{iq},\ b \in B_{pj},\ c' \in C_{kq}),
\end{array}$$

where $bc = a\alpha^{\xi}_{pq}(xy)a' \in A$ for $(b,c) = (a \otimes x, y \otimes a')$. Similarly, D has a right action on G and a left action on H given by

$$(x + u \otimes b + K)(c \otimes b' + E) = (xc + I) \otimes b' + u \otimes (bc)b' + K,$$
$$(c' \otimes b + E)(x + c \otimes v + L) = c \otimes (bx + J) + c'(bc) \otimes v + L.$$

These actions make B an (A,D)-bimodule, C a (D,A)-bimodule, G an (A'_1,D)-bimodule and H a (D,A'_5)-bimodule (all modules unitary).

7.4 Universal Localization of $\mathcal{U}_m(K)$

One important example of a triangular matrix ring is the ring of $m \times m$ upper triangular matrices over a ring K, denoted $\mathcal{U}_m(K)$. Applying Theorem 5.4.1 (or equivalently, Theorem 7.1.1 with $A_2 = 0$), we may see that for a morphism $\sigma\colon \mathcal{U}_2(K)e_{11} \to \mathcal{U}_2(K)e_{22}$ with $\sigma(e_{11}) = e_{12}$, the universal σ-inverting localization of $\mathcal{U}_2(K)$ is its inclusion into $\mathcal{M}_2(K)$. For larger values of m, the universal localization of $\mathcal{U}_m(K)$ with respect to $\sigma\colon \mathcal{U}_m(K)e_{11} \to \mathcal{U}_m(K)e_{mm}$ is not usually a full matrix ring. Nevertheless, Theorem 7.1.1 provides a full description of this localization.

Let K be a ring and $R = \mathcal{U}_m(K)$ with $m \geq 3$. Suppose additionally that $\sigma\colon Re_{11} \to Re_{mm}$ satisfies $\sigma(e_{11}) = e_{1m}$. Let us apply Theorem 7.1.1 to the triangular matrix 3-ring structure on R given by $\mathcal{U}_m(K) = \left(\begin{smallmatrix} K & K^n & K \\ 0 & \mathcal{U}_n(K) & {}^nK \\ 0 & 0 & K \end{smallmatrix}\right)$ with $n = m-2$. In the notation of Theorem 7.1.1, we find that

$$\begin{aligned}
&A = A_\xi = K,\\
&B = A \otimes_{A_1} M_{12} = K \otimes_K \begin{pmatrix} K & K & \cdots & K \end{pmatrix} = K^n,\\
&C = M_{23} \otimes_{A_3} A = \begin{pmatrix} K & K & \cdots & K \end{pmatrix}^\top \otimes_K K = {}^nK,\\
&D = A_2\langle C \otimes_A B\rangle/E = \mathcal{U}_n(K)\langle {}^nK \otimes_K K^n\rangle/E \cong Q,\\
&E = (c(bc') \otimes b' - (c \otimes b) \otimes (c' \otimes b)),
\end{aligned}$$

where $Q = \mathcal{R}(N, G)$ is a matrix n-ring, for which $N = (N_{ij})_{2 \leq i,j \leq n+1}$, $N_{ij} = L = K[t]/(t - t^2)$ $(i \leq j)$, and $N_{ij} = K$ $(i > j)$.

To justify the isomorphism $D \cong Q$, we first observe that the correspondence ${}^nK \otimes_K K^n \to \mathcal{M}_n(K)$, initially defined on tensors by the rule

$$\begin{pmatrix} c_2 & c_3 & \cdots & c_{n+1} \end{pmatrix}^\top \otimes \begin{pmatrix} b_2 & b_3 & \cdots & b_{n+1} \end{pmatrix} \mapsto (c_i b_j),$$

is a well-defined isomorphism of $(\mathcal{U}_n(K),\mathcal{U}_n(K))$-bimodules (see Lemma 2.3.6) which can then be extended to a $(\mathcal{U}_n(K),\mathcal{U}_n(K))$-algebra isomorphism $\mathcal{U}_n(K)\langle {}^nK \otimes_K K^n\rangle \cong \mathcal{U}_n(K)\langle \mathcal{M}_n(K)\rangle$ by the universal property of the tensor algebra. By Proposition 2.3.5, we also have an isomorphism of $\mathcal{U}_n(K), \mathcal{M}_n(K)$-bimodules and, by restriction, of $\mathcal{U}_n(K),\mathcal{U}_n(K)$-bimodules

$$\mathcal{M}_n(K) \cong \mathcal{U}_n(K) \otimes_{\mathcal{U}_n(K)} \mathcal{M}_n(K) \cong \mathcal{M}_n(K) \otimes_{\mathcal{U}_n(K)} \mathcal{M}_n(K).$$

Inductively, we then obtain a ring isomorphism

$$\mathcal{U}_n(K)\langle \mathcal{M}_n(K)\rangle \cong \mathcal{U}_n(K) \oplus \textstyle\sum_{j\geq 1}^{\oplus} \mathcal{M}_n(K) = \mathcal{M}_n(K)_{\mathcal{U}_n(K)}[t]$$

with the ring of polynomials in a central indeterminate t with upper triangular constant terms. Under this isomorphism, E corresponds to the ideal $(t - t^2)$. It follows that

$$D \cong \mathcal{M}_n(K)_{\mathcal{U}_n(K)}[t]/(t - t^2) \cong Q.$$

The multiplication and action maps can also be traced via these isomorphisms. We summarize the result as follows.

Theorem 7.4.1. *Let $R = \mathcal{U}_m(K)$ for any ring K and $m \geq 3$. Suppose $\sigma\colon Re_{11} \to Re_{mm}$ satisfies $\sigma(e_{11}) = e_{1m}$. Then the universal σ-inverting localization of R is the inclusion*

$$\lambda\colon R \xrightarrow{\subset} \sigma^{-1}R = \begin{pmatrix} K & \cdots & K \\ \vdots & Q & \vdots \\ K & \cdots & K \end{pmatrix} = \begin{pmatrix} K & K & K & \cdots & K & K & K \\ K & L & L & \cdots & L & L & K \\ K & K & L & \cdots & L & L & K \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ K & K & \ddots & \ddots & L & L & K \\ K & K & K & \cdots & K & L & K \\ K & K & K & \cdots & K & K & K \end{pmatrix},$$

where $Q = \mathcal{R}(N, G)$ is a matrix n-ring with $n = m - 2$, for which $N = (N_{ij})_{2 \le i,j \le n+1}$, $N_{ij} = L = K[t]/(t - t^2)$ $(i \le j)$, and $N_{ij} = K$ $(i > j)$. The right and left actions of L on K are given by $kp = kp(1)$ and $pk = p(1)k$. The multiplication maps, other than the module actions and the obvious choice of multiplication, are defined as follows

$$\begin{array}{ll} K \times K \to L, & (c, b) \mapsto cbt; \\ K \times L \to L, & (c, p) \mapsto cpt; \\ L \times K \to L, & (p, b) \mapsto pbt. \end{array}$$

□

We may similarly consider, for $R = \mathcal{U}_m(K)$, a morphism $\sigma \colon Re_{pp} \to Re_{qq}$ satisfying $\sigma(e_{pp}) = e_{pq}$ where $1 \le p < q \le m$ and $n = q - p - 1 \ge 1$. Following in the manner of Theorem 7.4.1, we find that

$$\sigma^{-1}R = \begin{pmatrix} K & & & \cdots & & & K \\ 0 & \ddots & & & & \iddots & \\ & & K & \cdots & K & & \\ \vdots & & \vdots & Q & \vdots & & \vdots \\ & & K & \cdots & K & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & 0 & K \end{pmatrix}$$

where $Q = \mathcal{R}(N, G) \cong \mathcal{M}_n(K)_{\mathcal{U}_n(K)}[t](t - t^2)$, with $N = (N_{ij})_{p<i,j<q}$, $N_{ij} = L = K[t]/(t - t^2)$ $(i \le j)$, and $N_{ij} = K$ $(i > j)$.

7.5 Localization to a Full Matrix Ring

The results of Theorem 7.1.1 and Theorem 7.2.1 demonstrate that the localization of a matrix ring R with respect to a morphism $\sigma \colon Re_{pp} \to Re_{qq}$ yields a matrix ring with an increased degree of symmetry. Our proofs of these results utilize the fact that in any σ-inverting ring, $\mathbb{1} \otimes \sigma$ and its inverse can be used to build isomorphisms between some of the entry bimodules in the matrix decomposition of the endomorphism ring, ultimately revealing the aforementioned symmetry. From another point of view, however, we might recognize that the idempotents e_{pp} and e_{qq}, along with

$\xi = \sigma(e_{pp})$, nearly form a set of matrix units for a 2×2 matrix subrng of R; the σ-inverting localization essentially appends an element ξ' to the R subject to the relations $\xi\xi' = e_{pp}$ and $\xi'\xi = e_{qq}$. The elements e_{11}, e_{22}, ξ, and ξ' of the localization are then a full set of matrix units for a subrng of the localization $\sigma^{-1}R$, showing that $\sigma^{-1}R$ contains a full 2×2 matrix ring, $\mathcal{M}_2(A)$. Many of the entry bimodules in the localization are then part of some unitary module over $\mathcal{M}_2(A)$ and so additional symmetry can be seen as a consequence of Morita equivalence.

The results of Schofield [27] and Sheiham [29] showed that the localization of a triangular 2-ring is a full matrix ring; the observation that localizations of matrix rings increase the amount of symmetry explains this, as the only possible increase in symmetry for a 2×2 upper triangular matrix ring is full symmetry.

The inversion of a single morphism is not sufficient to achieve full symmetry in matrix rings of larger order, as we have seen. By inverting more morphisms, we might expect to see more symmetry. Indeed, for a sufficiently large set of morphisms (precisely, a maximal tree of morphisms), a triangular matrix ring does localize to a full matrix ring.

Before stating the result, we recall the category $\mathscr{C}(\mathcal{R}(M,F))$ for a ring $R = \mathcal{R}(M,F)$ consisting of morphisms from R into full matrix rings, as discussed in Section 7.1. This category has an initial object, $\alpha\colon R \to \mathcal{M}_\ell(A_*)$, where $A_* = \mathbb{Z}\langle\sum_{i,j}^{\oplus} M_{ij}\rangle/I$ and I is the ideal generated by the relations $1 - 1_{A_i}$ and $xy - x \otimes y$ for all $x \in M_{ij}$, $y \in M_{jk}$ $(1 \le i,j,k \le \ell)$.

Theorem 7.5.1. *Let $R = \mathcal{R}(M,F)$ be a triangular matrix ℓ-ring. Suppose Σ is a tree*[2] *of R-module homomorphisms $\sigma_{ij}\colon Re_{ii} \to Re_{jj}$ that includes all vertices*

[2] Let $\Sigma = \{\sigma_{ij}\colon Re_{ii} \to Re_{jj}\}$ be a family of morphisms. We may consider the graph obtained by taking the domains and codomains of the σ_{ij} as vertices and, for each $\sigma_{ij} \in \Sigma$, including an edge between Re_{ii} and Re_{jj}. The family Σ is called a *tree* of morphisms if this graph is a tree, *i.e.* connected and acyclic.

Re_{tt} $(1 \leq t \leq \ell)$. If $\sigma_{ij} \in \Sigma$ are given by $\sigma_{ij}(e_{ii}) = \xi_{ij} \in M_{ij}$ and $\pi\colon A_* \to A$ denotes the quotient ring homomorphism modulo the two-sided ideal generated by the elements $1 - \alpha_{ij}(\xi_{ij})$, then the morphism

$$\lambda = \alpha \circ \mathcal{M}_\ell(\pi)\colon R \to \Sigma^{-1}R = \mathcal{M}_\ell(A)$$

is the universal Σ-inverting localization of R.

Proof. The morphism λ is clearly Σ-inverting. Suppose $\mu\colon R \to T$ is a Σ-inverting ring homomorphism. Because Σ is a maximal tree, The maps $\mathbb{1}\otimes\sigma_{ij}$ and their inverses provide a unique path of bimodule and ring isomorphisms

$$\mathrm{Hom}_T(T \otimes_R Re_{ii}, T \otimes_R Re_{jj}) \cong \mathrm{Hom}_T(T \otimes_R Re_{11}, T \otimes_R Re_{11}) = S$$

for each $1 \leq i, j \leq \ell$. These isomorphisms, along with Theorem 6.2.2, can be used to construct a morphism $\psi = \mathcal{R}(\psi_{ij})$ satisfying $\psi_{ij}(\xi_{ij}) = 1_S$ for all $\sigma_{ij} \in \Sigma$ which makes the following diagram commute.

$$\begin{tikzcd}
\mathcal{R}(M,F) \arrow[r, "\mu"] \arrow[d, "\psi"'] & T \arrow[d, "\gamma", "\cong"'] \\
\mathcal{M}_\ell(S) \arrow[r, "\cong", "\phi"'] & \mathrm{End}_T({}_TT)
\end{tikzcd}$$

The morphism ψ can then be uniquely factored through λ by the universal property of α. Therefore, μ also has a unique factorization $R \xrightarrow{\lambda} \Sigma^{-1}R \to T$. $\square$

One nice example illustrating Theorem 7.5.1 is that of $R = \mathcal{U}_\ell(K)$. If $\sigma_i\colon Re_{ii} \to Re_{i,i+1}$ $(i \leq j < \ell)$ are R-module homomorphisms given by $\sigma_i(e_{ii}) = e_{i,i+1}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_{\ell-1}\}$, then the universal Σ-inverting localization of R is the inclusion $\mathcal{U}_\ell(K) \xrightarrow{\subset} \mathcal{M}_\ell(K)$.

We can similarly consider other trees of morphisms between left ideals of R. For any such family, Σ, the universal Σ-inverting localization will always contain $\mathcal{M}_k(A)$ as a subrng, where $k = |\Sigma|+1$, although the other entry bimodules could be somewhat

complicated depending on the tree in question. One case that is easy to analyze with our methodology is the case where the vertices of Σ are $Re_{pp}, \dots, Re_{qq}$.

Theorem 7.5.2. *Let $R = \mathcal{R}(M, F)$ be a triangular matrix ℓ-ring and $1 \le p < q \le \ell$. Suppose Σ is a tree of R-module homomorphisms $\sigma_{ij}\colon Re_{ii} \to Re_{jj}$ with vertices Re_{tt} for $p \le t \le q$. If $\sigma_{ij} \in \Sigma$ are given by $\sigma_{ij}(e_{ii}) = \xi_{ij} \in M_{ij}$, then the following morphism of matrix 3-rings*

$$\lambda = \mathcal{R}(\lambda_{ij})\colon R = \begin{pmatrix} A'_1 & M'_{12} & M'_{13} \\ 0 & A'_2 & M'_{23} \\ 0 & 0 & A'_3 \end{pmatrix} \longrightarrow \Sigma^{-1}R = \begin{pmatrix} A'_1 & U & W \\ 0 & \mathcal{M}_k(A) & V \\ 0 & 0 & A'_3 \end{pmatrix}$$

is the universal Σ-inverting localization of R, where[3]

$$A'_1 = (M_{ij})_{1\le i,j<p}\ , \qquad A'_2 = (M_{ij})_{p\le i,j\le q}\ , \qquad A'_3 = (M_{ij})_{q<i,j\le \ell}\ ,$$
$$M'_{12} = (M_{ij})^{1\le i<p}_{p\le j\le q}\ , \qquad M'_{13} = (M_{ij})^{1\le i<p}_{q<j\le \ell}\ , \qquad M'_{23} = (M_{ij})^{p\le i\le q}_{q<j\le \ell}\ ,$$

and $k = q - p + 1$. The entry bimodules of the $\Sigma^{-1}R$ matrix and the corresponding λ-values are listed below, denoting by $\alpha = \mathcal{R}(\alpha_{ij})\colon A'_2 \to \mathcal{M}_k(A_*)$ the initial object in the category $\mathscr{C}(A'_2)$ and by $\pi\colon A_* \to A$ the quotient ring homomorphism modulo the two-sided ideal generated by the elements $1 - \alpha_{ij}(\xi_{ij})$ for $\sigma_{ij} \in \Sigma$.

$A'_1,\ A'_3$	$\lambda_{ij}(x) = x$	$(1 \le i, j < p \text{ or } q < i, j \le \ell)$;
$\mathcal{M}_k(A)$	$\lambda_{ij}(x) = \pi(\alpha_{ij}(x))$	$(p \le i, j \le q)$;
$U = (U_j)_{p\le j\le q}$	$\lambda_{ij}(x) = x\otimes 1 + I$	
$U_j = \big(\sum^{\oplus}_{\substack{1\le i<p,\\ p\le s\le q}} M_{is} \otimes_{A_s} A\big)/I$		
$I = A'_1(x\otimes\pi(\alpha_{ps}(y)) - xy\otimes 1)A$		$(x \in M_{ip}, y \in M_{ps}, p < s \le q)$;
$V = (V_i)_{p\le i\le q}$	$\lambda_{ij}(x) = 1\otimes x + J$	
$V_i = \big(\sum^{\oplus}_{\substack{p\le t\le q\\ q<j\le \ell}} A \otimes_{A_t} M_{tj}\big)/J$		
$J = A(1\otimes xy - \pi(\alpha_{pt}(x))\otimes y)A'_3$		$(x \in M_{pt}, y \in M_{tj}, p < t \le q)$;
$W = (M'_{13} \oplus U_p \otimes_A V_p)/N$	$\lambda_{ij}(x) = x + N$	
$N = A'_1(xy - x\otimes 1\otimes 1\otimes y)A'_3$		$(x \in M_{is}, y \in M_{sj}, p \le s \le q)$.

All but one of the multiplication maps of the ring $\Sigma^{-1}R$ are module action maps, the exception being the map $U \times V \to W$ given by $uv = u \otimes v + N$.

[3] We adopt the convention that $A'_1 = 0$ or $A'_3 = 0$ whenever $p = 1$ or $q = \ell$.

Proof Sketch. The proof of this result can be built on Theorem 7.5.1 in a similar way to which the proof of Theorem 7.2.1 was built upon Theorem 7.1.1. □

We conclude with generalizations of a few results of Schofield. These examples, which follow from the main theorems of this section, show that some interesting choices of matrix rings have localizations which are Morita equivalent to important constructions in ring theory, such as the amalgamated free product (see Definition 5.2.3) and bimodule amalgamated free product (whose definition will follow).

Example 7.5.3. This example generalizes a result of Schofield [27, Thm. 4.10]. Let C be a ring and suppose $\gamma_i \colon C \to A$ are ring homomorphisms. Let $R = \mathcal{R}(M, F)$ be a triangular matrix ℓ-ring, where $M_{ij} = A_i \otimes_C A_{i+1} \otimes_C \cdots \otimes_C A_j$ $(i \leq j)$ and $f_{ijk} \colon M_{ij} \times M_{jk} \to M_{ik}$ are given by

$$f_{ijk}(a_i \otimes a_{i+1} \otimes \cdots \otimes a_j, a'_j \otimes a'_{j+1} \otimes \cdots \otimes a'_k) = a_i \otimes a_{i+1} \otimes \cdots \otimes a_j a'_j \otimes a'_{j+1} \otimes \cdots \otimes a'_k.$$

If $\sigma_i \colon Re_{ii} \to Re_{i,i+1}$ $(1 \leq i < \ell)$ are defined by $\sigma_i(e_{ii}) = 1_{A_i} \otimes 1_{A_{i+1}} \in M_{i,i+1}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_{\ell-1}\}$, then $\Sigma^{-1}R = \mathcal{M}_\ell(\coprod_C A_i)$, where $\coprod_C A_i$ is the amalgamated free product of the rings A_i over C. This example is subsumed by the next example.

Definition 7.5.4. Let $\{A_s\}_{s \in S}$ be a family of rings and suppose $\{M_k\}_{k \in K}$ is a family of bimodules, where each $M_k = A_s x_k A_t$ is a cyclic unitary (A_s, A_t)-bimodule generated by x_k for some pair of indices $s, t \in S$. There is a category whose objects are rings B equipped with a family of ring homomorphisms $\{\beta_s \colon A_s \to B\}_{s \in S}$ that are *amalgamated* over the family of pointed bimodules[4] $\mathfrak{M} = \{(M_k, x_k)\}_{k \in K}$. This last condition means that, for each $k \in K$, the equations $\sum_v a_{s,v}(1_B)a_{t,v} = 0$ hold in B whenever $\sum_v a_{s,v} x_k a_{t,v} = 0 \in M_k$, where $a_{s,v} \in A_s$ and $a_{t,v} \in A_t$, thus allowing the assignment $x_k \mapsto 1_B$ to extend uniquely to a well-defined bimodule homomorphism

[4] A *pointed bimodule* is a bimodule with a distinguished element. A morphism of pointed bimodules $(M, x) \to (N, y)$ is a bimodule homomorphism $f \colon M \to N$ such that $f(x) = y$.

$M_k \to B$ for each $k \in K$. The initial object in this category will be denoted by $\coprod_{\mathfrak{M}} A_s$ and called the *bimodule amalgamated free product* of the rings A_s over the family $\mathfrak{M}$.

We remark that if $\beta_i \colon A_i \to B$ and $\gamma_i \colon C \to A_i$ are ring homomorphisms, then the maps β_i are amalgamated over C (that is, $\gamma_i \circ \beta_i = \gamma_j \circ \beta_j$ for all i, j) if and only if they are amalgamated over the family $\mathfrak{M} = \{(M_i, x_i)\}$ where $M_i = A_i \otimes_C A_{i+1}$ and $x_i = 1_{A_i} \otimes 1_{A_{i+1}}$. Therefore $\coprod_{\mathfrak{M}} A_i$ coincides, in this case, with the amalgamated free product of the A_i over C, that is, $\coprod_C A_i$.

Example 7.5.5. This example generalizes another result of Schofield [27, Thm. 13.1]. Let $A_1, A_2, \ldots, A_\ell$ be rings and $R = \mathcal{R}(M, F)$ be the triangular matrix ℓ-ring constructed using cyclic unitary bimodules $M_{i,i+1} = A_i \xi_i A_{i+1}$ $(1 \leq i < \ell)$ with $M_{ii} = A_i$ and

$$M_{ij} = M_{i,i+1} \otimes_{A_{i+1}} M_{i+1,i+2} \otimes_{A_{i+2}} \cdots \otimes_{A_{j-1}} M_{j-1,j}$$

for all $i < j$ and $M_{ij} = 0$ for all $i > j$. If $\sigma_i \colon Re_{ii} \to Re_{i,i+1}$ $(1 \leq i < \ell)$ is defined by $\sigma_i(e_{ii}) = \xi_i$, $\Sigma = \{\sigma_1, \ldots, \sigma_{\ell-1}\}$, and $\mathfrak{M} = \{(M_{i,i+1}, \xi_i)\}$, then $\Sigma^{-1}R = \mathcal{M}_\ell(\coprod_{\mathfrak{M}} A_i)$.

To justify this claim, we define $A = A_*/J$ as in Theorem 7.5.1 where J is the ideal generated by the elements $1 - \alpha_{i,i+1}(\xi_i)$. We note that for $j > i + 1$, we have $M_{ij} = M_{i,i+1} M_{i+1,i+2} \cdots M_{j-1,j}$ in R, so the relations $xy - x \otimes y$ imply that A_* is generated by the modules M_{ii} $(1 \leq i \leq \ell)$ and $M_{i,i+1}$ $(1 \leq i < \ell)$. The relation $1 - \alpha_{i,i+1}(\xi_i) \in J$ shows that, after passing to A, the image of $M_{i,i+1}$ is the cyclic (A_i, A_{i+1})-bimodule generated by 1_A; in particular, it is contained in the subring generated by the images of A_i and A_{i+1}. As a consequence, the ring A is generated by the images of the rings A_i. Furthermore, the image of $\sum_v a_{i,v} \xi_i a_{i+1,v} \in M_{i,i+1}$ in A is $\sum_v a_{i,v}(1_A) a_{i+1,v}$, so A is bimodule amalgamated over the family $\mathfrak{M} = \{(M_{i,i+1}, \xi_i)\}$.

Suppose now that B is a ring and $\beta_i \colon A_i \to B$ are ring morphisms which are bimodule amalgamated over $\mathfrak{M}$. the function

$$g_i\big(\sum_v a_{i,v}\xi_i a_{i+1,v}\big) = \sum_v a_{i,v}(1_B)a_{i+1,v} = \sum_v \beta(a_{i,v})\beta_{i+1,v}(a_{i+1,v})$$

is then a well-defined (A_i, A_{i+1})-bimodule homomorphism due to the amalgamated property of the β_i. The assignment on tensors

$$h_{ij}(x_i \otimes x_{i+1} \otimes \cdots \otimes x_{j-1}) = g_i(x_i)g_{i+1}(x_{i+1}) \cdots g_{j-1}(x_{j-1})$$

defines then an (A_i, A_j)-bimodule homomorphism $h_{ij} \colon M_{ij} \to B$ for each $i < j$. Together with $h_{ii} = \beta_i \colon M_{ii} \to B$, the maps h_{ij} extend to an additive homomorphism $\sum_{i,j}^{\oplus} M_{ij} \to B$ and then to a ring homomorphism $h \colon \mathbb{Z}\langle \sum_{i,j}^{\oplus} M_{ij}\rangle \to B$. Since h respects the relations defining I and J, it factors through a ring homomorphism of the quotient ring A to B. Uniqueness of the latter morphism follows from the fact that A is generated by the images of the A_i. It follows that $A = \coprod_{\mathfrak{M}} A_i$ and by Theorem 7.5.1,

$$\lambda = \alpha \circ \mathcal{M}_\ell(\pi) \colon R \to \mathcal{M}_\ell(A) = \mathcal{M}_\ell(\textstyle\coprod_{\mathfrak{M}} A_i)$$

is the universal Σ-inverting localization of R.

Returning to Example 7.5.3, we note that if $\gamma_i \colon C \to A_i$ are ring homomorphisms and $M_{i,i+1} = A_i \otimes_C A_{i+1}$ with $\xi_i = 1_{A_i} \otimes 1_{A_{i+1}}$ then in the construction of Example 7.5.5, $M_{ij} \cong A_i \otimes_C A_{i+1} \otimes_C \cdots \otimes_C A_j$ $(i \le j)$ and the ring R becomes the ring $\mathcal{R}(M, F)$ of Example 7.5.3, proving the statement there.

Example 7.5.6. This example generalizes yet another part of Schofield's results in [27, Thm. 13.1]. Let $A_1, A_2, \ldots, A_\ell$ be rings. Suppose $\mathcal{R}(M, F)$ is a matrix ℓ-ring with $M_{ii} = A_i$, $M_{1j} = A_1\xi_j A_j \oplus N_{1j}$ $(2 \le j \le \ell)$ for some (A_1, A_j)-bimodules N_{1j} and $M_{ij} = 0$ otherwise. All multiplication maps f_{ijk} are either zero or module action maps in this case. If $\Sigma = \{\sigma_2, \ldots, \sigma_\ell\}$ with $\sigma_j(e_{11}) = (\xi_j, 0) \in M_{1j}$, then $\Sigma^{-1}R = \mathcal{M}_\ell(A\langle Q\rangle)$, where $A = \coprod_{\mathfrak{M}} A_i$, $\mathfrak{M} = \{(A_1\xi_j A_j, \xi_j)\}$, and $Q = \sum_j^{\oplus} A \otimes_{A_1} N_{1j} \otimes_{A_j} A$.

7.6 Module Localization

In any notion of localization of a ring $R \to R'$, it is often useful to apply the extension-of-scalars functor $R' \otimes_R -$ in order to make modules over R into modules over R'. This process is called *module localization.* Though its definition is straightforward, we might hope to clarify the localized module by providing a simpler model for it, perhaps using fewer generators and relations or using more easily understandable relations than those used in the general construction of the tensor product.

Definition 7.6.1. Let R be a ring and N an R-module. If Σ is a collection of R-module homomorphisms and $\lambda\colon R \to \Sigma^{-1}R$ is the universal Σ-inverting localization of R, then the $\Sigma^{-1}R$-module $\Sigma^{-1}N = \Sigma^{-1}R \otimes_R N$ is called the *localization left $\Sigma^{-1}R$-module.*

In Chapter 6, we saw that the modules over a generalized matrix ring $R = \mathcal{R}(M, F)$ are tuples of $A_i = M_{ii}$ modules satisfying some additional conditions. Given that $\sigma^{-1}R$ is a triangular matrix ring for any $\sigma\colon Re_{pp} \to Re_{qq}$, we may decompose a localization left module $\sigma^{-1}N$ into a tuple and describe its entries as quotients of certain tensor products over the entries of $\sigma^{-1}R$.

Theorem 7.6.2. *Let $R = \mathcal{R}(M, F)$ be a triangular matrix 5-ring. Suppose an R-module homomorphism $\sigma\colon Re_{22} \to Re_{44}$ is given by $\sigma(e_{22}) = \xi \in M_{24}$. For any unitary left R-module N, the localization left $\sigma^{-1}R$-module $\sigma^{-1}N$ is isomorphic to $(X_1\ P\ X_3\ P\ X_5)^\top$, where*

$$R' = \sigma^{-1}R = \begin{pmatrix} A_1 & U & G & U & W \\ 0 & A & B & A & V \\ 0 & C & D & C & H \\ 0 & A & B & A & V \\ 0 & 0 & 0 & 0 & A_5 \end{pmatrix}$$

is the ring defined in Theorem 7.2.1, the left R_{ii}-module X_i, for $R_{ii} = e'_{ii}R'e'_{ii}$ ($i \in \{1,3,5\}$), is the quotient of $e'_{ii}R' \otimes N$ modulo the relations

$$x \otimes yz - x\lambda(y) \otimes z \quad (x \in e'_{ii}R',\ y \in M_{ij},\ z \in e_{jj}N),$$

and the left A-module P is the quotient of $e'_{22}R' \otimes N$ modulo the relations

$$x \otimes yz - x\lambda(y) \otimes z \quad (x \in e'_{ii}R',\ y \in M_{ij},\ z \in e_{jj}N,\ i \in \{2,4\}).$$

The ring $\sigma^{-1}R$ acts on the left of $(X_1\ P\ X_3\ P\ X_5)^\top$ by matrix multiplication.

Proof. The relations defining the X_i and P imply that $X_i \cong e'_{ii}R' \otimes_R N$ and $P \cong e'_{22}R' \otimes_R N \cong e'_{44}R' \otimes_R N$. Denoting

$$S = \begin{pmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & A & 0 \\ 0 & 0 & D & 0 & 0 \\ 0 & A & 0 & A & 0 \\ 0 & 0 & 0 & 0 & A_5 \end{pmatrix},$$

we see that $Y = (X_1\ P\ X_3\ P\ X_5)^\top$ is a left S-module under matrix multiplication and there is a natural isomorphism of left S-modules $\eta\colon Y \cong R' \otimes_R N$. We can then extend the S-module structure on Y to an R'-module structure by defining $r'y = \eta^{-1}(r'\eta(y))$, so that Y and $R' \otimes_R N$ are isomorphic modules over R'. □

Both the triangular 3-ring and triangular 2-ring cases are then easy corollaries.

Corollary 7.6.3. *Let $R = \mathcal{R}(M, F)$ be a triangular matrix 3-ring. Suppose an R-module homomorphism $\sigma\colon Re_{11} \to Re_{33}$ is given by $\sigma(e_{11}) = \xi \in M_{13}$. For any unitary left R-module N, the localization left $\sigma^{-1}R$-module $\sigma^{-1}N$ is isomorphic to $(P\ Q\ P)^\top$, where*

$$\sigma^{-1}R = \begin{pmatrix} A & B & A \\ C & D & C \\ A & B & A \end{pmatrix}$$

is the ring defined in Theorem 7.1.1, the left A-module P is the quotient of $(A \oplus B \oplus A) \otimes N$ modulo the relations

$$x \otimes yz - x\lambda(y) \otimes z \quad (x \in A \oplus B \oplus A,\ y \in M_{ij},\ z \in e_{jj}N,\ i \in \{1,3\}),$$

and the left D-module Q is the quotient of $(C \oplus D \oplus C) \otimes N$ modulo the relations

$$x \otimes yz - x\lambda(y) \otimes z \quad (x \in C \oplus D \oplus C,\ y \in M_{2j},\ z \in e_{jj}N).$$

The ring $\sigma^{-1}R$ acts on the left of $(P\ Q\ P)^{\top}$ by matrix multiplication. $\square$

Corollary 7.6.4 (Sheiham, 2006 [29, Thm. 2.12])**.** *Let $R = \mathcal{R}(M, F)$ be a triangular matrix 2-ring. Suppose an R-module homomorphism $\sigma\colon Re_{11} \to Re_{22}$ is given by $\sigma(e_{11}) = \xi \in M_{12}$. For any unitary left R-module N, the localization left $\sigma^{-1}R$-module $\sigma^{-1}N$ is isomorphic to $\left(\begin{smallmatrix} P \\ P \end{smallmatrix}\right)$, where $\sigma^{-1}R = \mathcal{M}_2(A)$, A is the initial object in the category $\mathscr{C}(\mathcal{R}(M, F), \xi)$, and the left A-module P is the quotient of $A^2 \otimes N$ modulo the relations*

$$x \otimes yz - x\lambda(y) \otimes z \quad (x \in A^2,\ y \in M_{ij},\ z \in e_{jj}N,\ i \in \{1, 2\}).$$

The ring $\mathcal{M}_2(A)$ acts on the left of $\left(\begin{smallmatrix} P \\ P \end{smallmatrix}\right)$ by matrix multiplication. $\square$

REFERENCES

[1] G. M. Bergman, *Coproducts and some universal ring constructions*, Trans. Amer. Math. Soc. **200** (1974), 33–88. MR 357503

[2] T. B. Bowles and D. M. Wilczyński, *Universal localization of certain matrix rings*, preprint (2020).

[3] P. M. Cohn, *The embedding of firs in skew fields*, Proc. London Math. Soc. **3** (1971), no. 23, 193–213. MR 297814

[4] ______, *Free rings and their relations*, 1st ed., London Mathematical Society Monographs, no. 2, Academic Press, Inc., London-New York, 1971, ISBN: 0-12-179150-5. MR 371938

[5] ______, *Rings of fractions*, Amer. Math. Monthly **78** (1971), no. 6, 596–615. MR 285561

[6] ______, *Skew fields of fractions, and the prime spectrum of a general ring*, Lectures on Rings and Modules, Lecture notes in Math., vol. 246, Springer, Berlin, 1972, pp. 1–71. MR 342560

[7] ______, *Universal skew fields of fractions*, Symposia Mathematica (London), vol. 8, Academic Press, Inc., 1972, pp. 135–148. MR 342543

[8] ______, *Free rings and their relations*, 2nd ed., London Mathematical Society Monographs, no. 19, Academic Press, Inc., London, 1985, ISBN: 0-12-179152-1. MR 800091

[9] ______, *From hermite rings to sylvester domains*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 1899–1904. MR 1646314

[10] ______, *Introduction to ring theory*, Springer Undergraduate Mathematics Series, Springer-Verlag London, London, 2000, ISBN: 1-85233-206-9. MR 1732101

[11] ______, *Free ideal rings and localization in general rings*, New Mathematical Monographs, vol. 3, Cambridge Univ. Press, Cambridge, 2006, ISBN: 978-0-521-85337-8. MR 2246388

[12] ______, *Localization in general rings, a historical survey*, Non-commutative localization in algebra and topology, London Mathematical Society Lecture Note Series, vol. 330, Cambridge Univ. Press, Cambridge, 2006, pp. 5–23. MR 2222479

[13] P. M. Cohn and W. Dicks, *Localization in semifirs ii*, J. London Math. Soc. **2** (1976), no. 3, 411–418. MR 424856

[14] P. M. Cohn and A. H. Schofield, *On the law of nullity*, Math. Proc. Cambridge Philos. Soc., vol. 91, Cambridge Univ. Press, 1982, pp. 357–374. MR 654083

[15] W. Dicks and E. D. Sontag, *Sylvester domains*, J. Pure Appl. Algebra **13** (1978), no. 3, 243–275. MR 509164

[16] A. W. Goldie, *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc. **3** (1958), no. 8, 589–608. MR 103206

[17] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, 2nd ed., London Mathematical Society Student Texts, vol. 61, Cambridge Univ. Press, Cambridge, 2004, ISBN: 0-521-83687-5. MR 2080008

[18] E. L. Green, *On the representation theory of rings in matrix form.*, Pacific J. Math. **100** (1982), no. 1, 123–138. MR 661444

[19] A. Haghany and K. Varadarajan, *Study of formal triangular matrix rings*, Comm. Algebra **27** (1999), no. 11, 5507–5525. MR 1713049

[20] T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York-Berlin, 1974, ISBN: 0-387-90518-9. MR 600654

[21] A. I. Malcev, *On the immersion of an algebraic ring into a field*, Mathematische Annalen **113** (1937), no. 1, 686–691. MR 1513116

[22] P. Malcolmson, *A prime matrix ideal yields a skew field*, J. London Math. Soc. **2** (1978), no. 2, 221–233. MR 509937

[23] ———, *Construction of universal matrix localizations*, Advances in Non-Commutative Ring Theory: Proc. 12th G. H. Hudson Symp., Plattsburgh, 1981 (Berlin), Lecture Notes in Math., Springer-Verlag, 1982, pp. 117–131.

[24] M. Nivat, *Séries rationnelles et algébriques en variables non commutatives*, (1969/70).

[25] Ø. Ore, *Linear equations in non-commutative fields*, Ann. of Math. (1931), 463–477. MR 1503010

[26] A. A. Ranicki, *Noncommutative localization in topology*, Non-commutative localization in algebra and topology, London Mathematical Society Lecture Note Series, vol. 330, Cambridge Univ. Press, Cambridge, 2006, pp. 81–102. MR 2222483

[27] A. H. Schofield, *Representations of rings over skew fields*, London Mathematical Society Lecture Note Series, vol. 92, Cambridge Univ. Press, Cambridge, 1985, ISBN: 0-521-27853-8. MR 800853

[28] M. Schützenberger, *On a theorem of R. Jungen*, Proc. Amer. Math. Soc. **13** (1962), 885–890. MR 142781

[29] D. Sheiham, *Universal localization of triangular matrix rings*, Proc. Amer. Math. Soc. **134** (2006), no. 12, 3465–3474. MR 2240657

[30] Z. Škoda, *Noncommutative localization in noncommutative geometry.*, Noncommutative localization in algebra and topology, London Mathematical Society Lecture Note Series, vol. 330, Cambridge Univ. Press, Cambridge, 2006, pp. 220–313. MR 2222486

[31] D. M. Wilczyński, *On the fundamental theorem of algebra for polynomial equations over real composition algebras*, J. Pure Appl. Algebra **218** (2014), no. 7, 1195–1205. MR 3168491