Massive Graviton Spectra in Supergravity

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MASSIVE GRAVITON SPECTRA IN SUPERGRAVITY

by

Kevin Dimmitt

A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Physics

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ABSTRACT

Massive Graviton Spectra in Supergravity

by

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Utah State University, 2020

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The study of supergravity is currently an important focus in attempting to answer a major open question in physics: how can the theory of general relativity be combined with quantum theory? This paper opens with some remarks on key concepts used in (gauged) supergravity. Then, the spectrum of massive gravitons in some supergravities of interest is computed. Specifically, vacua of different gaugings of $D = 4 \, \mathcal{N} = 8$ supergravity that preserve the same supersymmetries and bosonic symmetry tend to exhibit the same universal mass spectrum within their respective supergravities. For AdS$_4$ vacua in gauged supergravities that arise upon consistent truncation of string/M-theory, it is shown here that this universality is lost at higher Kaluza-Klein levels. However, universality is still maintained in a milder form, at least in the graviton sector: certain sums over a finite number of states remain universal. Further, a mass matrix for Kaluza-Klein gravitons is derived which is valid for all the AdS$_4$ vacua in string/M-theory that uplift from the gaugings of $D = 4 \, \mathcal{N} = 8$ supergravity that we consider. The mild universality of graviton mass sums is related to the trace of this mass matrix.

(67 pages)
Development of a unified theory of physics would pave the way for new research and technology development for many years to come. Unfortunately, the two best current theories explaining nature, the Standard Model of particle physics and general relativity, do not seem to be compatible, requiring the development of more complicated models which contains both of these at their respective limits. Supergravities are one set of theories which may, at least in part, provide hints as to how it may be possible to unify physics into a single model.

This research project follows from a line of investigation which the primary investigator and collaborators have been pursuing for the past several years exploring the properties of supergravity. The main goal of this project was attempting to confirm whether or not the mass spectra of gravitons are common among different theories of supergravity, and if not to find any properties which are universal, in order to improve understanding of supergravities as a whole. Our research group examined 8 sectors of symmetry within 3 different supergravity theories and found that these mass spectra are not universal, but that there do exist relations which are. We then found a way to write all the mass spectra we investigated and these relations we found in a universal way using the language of general relativity.
ACKNOWLEDGMENTS

This research was supported in part by the NSF grant PHY-1720364. Many thanks to Oscar Varela, Praxitelis Ntokos, and Gabriel Larios, with whom I collaborated on in this research and who helped mentor me in the process. Special thanks to my family, friends, and girlfriend, Kim, who have supported me throughout the years up to this point.

Kevin Dimmitt
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INTRODUCTION

The standard model combined with general relativity has been the best theory of physics for several decades in describing known particles and the four fundamental interactions: strong nuclear forces, weak nuclear forces, electromagnetic forces, and gravitational forces. The standard model consists of a quantum field theory describing the strong, weak, and electromagnetic interactions, while general relativity is still the best theory for gravity over 100 years after its conception. Unfortunately, general relativity is still a classical theory and does not explain the gravitational interaction on a quantum scale. Thus, a major open question in physics today concerns how it might be possible to incorporate gravitational interactions with the other three into a theory of everything.

One of the top candidates for addressing this problem is string theory, which has been a topic of interest for theorists for the last several decades. Unfortunately, there are many different types of string theory, and they are all incredibly complicated theories to study. One means of making this more tractable, though, is by looking at the low-energy limit of a string theory, which gives a respective theory of supergravity. The focus of this thesis is to explore a particular aspect of several theories of supergravity: properties of massive graviton spectra that may show some universality between the different theories.

This thesis opens with a brief overview of supergravity, providing some background concepts that motivate and develop these theories. The section begins with a description of the classification of particles as bosons and fermions. This leads into a brief discussion on symmetry used in physics and the concept of supersymmetry, which forms the foundation for supergravity. Following this is the research that was conducted, opening with an overview of the project itself, where we discuss more specifically the question we aim to understand: do lower dimensional theories of supergravity with the same mass spectra still have the same mass spectra when we uplift them to a higher dimensional theory? We then perform these calculations in the next section which show that this is not the case, but that there are relations related to the weighted traces of these mass spectra that do remain universal.
We also show that it is possible to construct a block-diagonal Kaluza-Klein matrix which can be reduced to reproduce all of the graviton mass spectra which we calculated. This is followed by a discussion of these results, and then concluded with some final thoughts. This thesis is based on the recent work by [1].
BACKGROUND

Before outlining the details of the project itself, it is worthwhile to review some relevant topics leading up to research in supergravity. This section will focus primarily on key concepts relevant to the project itself, but some additional topics in geometry and algebra will be included in appendix for further review.

Fermions and bosons

When it comes to describing the nature of particles, there are many ways to differentiate them. From a quantum perspective, the best way to do so is through quantum numbers: sets of discrete-valued (i.e. quantized) numbers that give allowed solutions to the relevant equations required to be satisfied, such as the Schrödinger equation. These are often used to describe the current state of a particle, such as energy and momentum levels, but they can also contain information on the particle itself. One such means of classifying particles is through their spin, which reflects the intrinsic angular momentum of the particle. While some quantum numbers depend on the state of the particle, the spin is fixed for each particle, and can be either integer-valued or half-integer-valued. Interestingly, particles divided into these two classes share some important and rather distinct properties, and thus has become a key means of categorizing particles.

Particles with odd half-integer spin ($\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$) are called fermions, and comprise much of what we know as matter, particles with mass that bond and form atoms, to include quarks and electrons. As a consequence of their wavefunctions being antisymmetric, one intriguing property which fermions obey is the Pauli exclusion principle [2]: no two fermions can share the same set of quantum numbers simultaneously. That such a unique characteristic is shared by particles with half-integer spin values would alone make it a worthwhile distinction, let alone their statistical behavior following Fermi-Dirac statistics. The Lagrangian of a free spin-$\frac{1}{2}$ particle [3],

$$\mathcal{L} = -\bar{\psi} \gamma^\mu \partial_\mu \psi + m \bar{\psi} \psi,$$ (1)
can be written using an object called a (Dirac) spinor $\psi$, with $\gamma^\mu$ an element of the Clifford algebra.

On the other hand, particles with integer spin (0, 1, 2, ...), referred to as bosons, tend to consist more of force-mediating particles, such as photons, gluons, the W and Z bosons, and gravitons. In contrast to fermions, bosons have symmetric wavefunctions and, thereby, are not restricted by Pauli exclusion [2], meaning multiple bosons can occupy the same quantum state simultaneously. Interestingly, special groups of fermions can collectively behave like a boson (e.g. a helium nucleus), but here we shall be concerned particularly with fundamental particles. This classification of particles as bosons and fermions sets the groundwork for building up the next main concept: supersymmetry.

**Symmetry and supersymmetry**

Throughout the development of physics, it has become clear that an understanding of how systems or quantities change when undergoing a particular transformation is critical. This is further cemented by the notion that often the best way to describe the nature of a system is by understanding when it does not change under such transformations. This naturally leads to the idea of symmetry as playing a central role in describing the fundamental nature of a theory of physics. For even more justification, one need not look further than Noether’s theorem, which found a correspondence between conservation laws, one of the cornerstone concepts of physics, and continuous symmetries of an action.

A symmetry is simply the idea that making a particular change, or transformation, to a system results in the system remaining in the same state as before the transformation was made. A classic example of this is how rotating a square by $90^\circ$ increments leaves the square in the same position as before, while rotating a circle about its center by any angle leaves the circle entirely unchanged. In the case of the circle, another way to say this is that the circle remains invariant under 2D rotations. We can take this even further by introducing group theory, which is essentially the mathematical language for discussing symmetry (see appendix for further discussion). The fact that a general element of the
Lie group SO(2) has the same form as a rotation of an object in the 2D plane allows us to now describe the circle as an object which is SO(2)-invariant.

One way in which it will be important to distinguish symmetries going forward is to understand the difference between a global symmetry and a local symmetry. Consider as an example the spin-$\frac{1}{2}$ particle $\psi$ from before, which undergoes a U(1) transformation $\psi \rightarrow e^{ia}\psi$ with $a$ being constant. Under this transformation, the Lagrangian in equation (1) remains invariant, i.e. the rotation by a factor that is everywhere constant is a global spacetime transformation, and that the system is left unchanged for a constant of that form means there is a global symmetry under such rotations. Thus, a global symmetry is one in which a transformation is applied in the same way at each point \[3\]. Such a symmetry can be promoted from a global to a local symmetry when the symmetry transformations are allowed to act at each point in spacetime independently. In practical terms, this means that the Lie group parameters are promoted as $a \rightarrow a(x)$. Under this change, equation (1) is no longer invariant, so it does have a global U(1) symmetry, but not a local U(1) symmetry.

While it is desirable to be able to classify our theory by symmetries, an algebraic type of description, it is natural to seek a means of writing a Langrangian, which is useful in a more geometric setting. This leads to the implementation of Kaluza-Klein theory, which allows one to geometrize internal symmetry groups by adding more coordinates of an appropriate type \[4\]. In other words, one can describe a higher-dimensional geometry containing both internal and external spacetime coordinates which can then be reduced to a 4D spacetime that contains all the requisite field content that we observe, such as potentials. Likewise, we can start with a 4D theory with content and then uplift it to a higher-dimensional theory that is a purely geometric description encoding all of the desired internal symmetries.

Lastly, a discussion of supergravity requires a description of supersymmetry, the idea which proposes that each particle has a partner differing only by a half-integer spin; in other words, every known boson would have a corresponding fermion superpartner with
the same mass and quantum numbers except spin, and vice versa \[2\]. A relevant example for this thesis is the graviton, a spin-2 boson, which has a spin-$\frac{3}{2}$ fermionic partner called a gravitino. Algebraically, this can be understood by the introduction of supercharges, spinor generators of the superalgebra. Extended supergravities include more supercharges, so a measure of the amount of supersymmetry in a theory is commonly given by one of the supercharge indices $N$, which with the number of dimensions gives information on how many supercharges there are in the theory. This becomes relevant in the sense that supersymmetry provides explanations for many open questions in physics, including the renormalization problem of general relativity as well as providing natural candidates for dark matter \[5\].

Theories of supergravity

Having built up the idea of supersymmetry, we can now discuss what is meant by supergravity. Supergravity is a theory which is invariant under local supersymmetry transformations. Supergravity theories usually start as being ungauged, meaning the vector fields are not coupled to any other field, although this presents a number of phenomenological problems as described by \[5\]. The bosonic part of the Lagrangian in supergravity is given by \[5, 6\],

$$L_{\text{bos}} = -\frac{1}{2}\sqrt{-g} \left( R + G_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \frac{1}{2} M_{MN} F_{\mu\nu}^M F_{\mu\nu}^N + \cdots \right),$$  \hspace{1cm} (2)

while the fermionic part can be written,

$$L_{\text{fer}} = -\frac{1}{2}\sqrt{-g} \left( \bar{\psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho + \cdots \right),$$  \hspace{1cm} (3)

with $R$ the Ricci scalar, $G_{ij}$ and $M_{MN}$ the scalar and vector kinetic matrices, respectively, and $F_{\mu\nu}$ the electromagnetic field strength. The Lagrangian can also contain higher-rank forms or additional topological terms.

Fortunately, the issues presented by such a theory can be addressed through a well-
defined process called gauging, whereby one can take a suitable subgroup of the global symmetry group of the theory and promote that subgroup to a local symmetry group, as well as introducing additional terms into the Lagrangian. It is, in fact, the only way that a scalar potential or fermion mass-terms can be added without breaking supersymmetry in an extended supergravity \[5\]. In \(D = 4\), the global symmetry group is \(E_{7(7)}\), the split real form of the exceptional Lie group \(E_7\) \[6\], so the group used to gauge the theory must be a subgroup of \(E_{7(7)}\).

Consideration of various gaugings of \(D = 4\) \(\mathcal{N} = 8\) supergravity is a key focus in this project. In \(D = 4\) spacetime dimensions, it is certainly worth noting that one can only construct theories up to \(\mathcal{N} = 8\) while restricting to particles of spin-2 or lower, as spin-\(\frac{5}{2}\) particles and above begin introducing inconsistencies \[3\]. Thus, the \(D = 4\) \(\mathcal{N} = 8\) case has become a theory of great interest, and is referred to as maximal.

An additional caveat worth mention here is that this project focuses on particular \(D = 4\) \(\mathcal{N} = 8\) gaugings that have some origin in a higher dimensional theory. Type IIA and type IIB supergravity are two of several 10D theories into which we are able to uplift these gauged 4D theories. We also consider the standard SO(8)-gauged supergravity which uplifts into the 11-dimensional M-theory, a theory which can be reduced to any of the various 10D supergravities. Reducing from 11-dimensions admits an \(E_7\) global bosonic symmetry, which in the maximal supergravity has \(SU(8)\) as the largest compact subgroup \[6\]. This deserves noting, as this project is ultimately framed in the embedding of these theories in \(E_{7(7)}\).

The gauging process allows us to select a subgroup of the global symmetry group \(E_{7(7)}\) for promotion to local symmetry. This entails introducing covariant derivatives, a gauge coupling constant \(g\), and an embedding tensor \(\Theta_{\alpha}^M\), where \(M\) is an index of the fundamental representation and \(\alpha\) is an index of the adjoint representation \[6\]. In this project, we then look at the possible (super)symmetries of AdS vacua that are at least \(SU(3)\). As we are concerned here with gauged supergravities that have higher dimensional origins, some symmetries can be embedded differently in \(E_{7(7)}\) under different gauge groups. When we consider different embeddings in the gauge group, what changes is the embedding
tensor. Framing the theories in this way then allows us to write a covariant mass matrix which reduces to all solutions obtained herein.
OVERVIEW OF THE PROJECT

It is now well established that maximal gauged supergravities in four dimensions typically come in one-parameter families \[7, 8, 9\]. All members in a given family share the same gauge group. The parameter, discrete or continuous, that characterises the family leaks in as a remnant of the freedom of the ungauged theory to select an electric/magnetic duality frame before the gauging is introduced. It also happens that different such gaugings usually have vacua, of anti-de Sitter (AdS), de Sitter or Minkowski type, that exhibit the same residual supersymmetries \(0 \leq N \leq 8\) and the same residual bosonic symmetry \[7, 10, 11, 12, 13, 14\]. Often (though not always), vacua with the same (super)symmetries have identical mass spectra within their corresponding gauged supergravities. An interesting question therefore arises for \(D = 4\) \(N = 8\) gaugings with a higher dimensional origin: for vacua of different such \(N = 8\) gaugings with the same symmetry and the same spectrum within their corresponding \(N = 8\) theories, are the masses still the same up the corresponding Kaluza-Klein (KK) towers?

In this paper, we will address this question by focusing on three \(N = 8\) gaugings with AdS vacua: the purely electric SO(8) gauging \[15\], the dyonic ISO(7) gauging \[16\] and the dyonic \((\text{SO}(6) \times \text{SO}(1, 1)) \rtimes \mathbb{R}^{12}\) gauging as described in \[10\]. All three gaugings enjoy higher dimensional origins in, respectively, \(D = 11\) supergravity \[17\] on \(S^7\) \[18\], massive type IIA supergravity \[19\] on \(S^6\) \[20, 21\] and type IIB supergravity \[22\] on an S-fold geometry \[23\]. In order to narrow down and simplify the problem, we will further focus on the SU(3)–invariant sector of these gaugings. This sector has been explicitly uplifted to the respective higher-dimensional supergravities \[24, 25, 26\], and the corresponding vacua have been charted in \[27, 16, 26\]. The list of possible symmetries of AdS vacua in this sector across all three gaugings of interest has been summarised for convenience in table \[1\]. Only the symmetries are indicated in the table, regardless of their actual embedding into the relevant gauge groups and ultimately \(E_{7(7)}\). For example, the same table row accounts for the \(\text{SU}(3) \times U(1)_c\) and \(\text{SU}(3) \times U(1)_v\) solutions of SO(8) and ISO(7) supergravity,
Table 1: Possible residual (super)symmetries, regardless of their $E_{7(7)}$ embedding, of AdS vacua in the SU(3)-invariant sector of the three different gaugings that we consider.

<table>
<thead>
<tr>
<th>(Super)symmetry</th>
<th>SO(8)</th>
<th>ISO(7)</th>
<th>$(SO(6) \times SO(1,1)) \ltimes \mathbb{R}^{12}$</th>
<th>same spectrum?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N} = 8$, SO(8)</td>
<td>✓</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathcal{N} = 2$, SU(3) × U(1)</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathcal{N} = 1$, $G_2$</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathcal{N} = 1$, SU(3)</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathcal{N} = 0$, SO(7)</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathcal{N} = 0$, SO(6)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathcal{N} = 0$, $G_2$</td>
<td>×</td>
<td>✓</td>
<td>×</td>
<td>–</td>
</tr>
<tr>
<td>$\mathcal{N} = 0$, SU(3)</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
</tr>
</tbody>
</table>

which differ in their embeddings into $E_{7(7)}$. The table also shows whether vacua with the same (super)symmetries in different theories exhibit the same mass spectrum within their corresponding $D = 4$ $\mathcal{N} = 8$ theories. All of them do, except for the solution with $\mathcal{N} = 0$ SU(3) symmetry. For this reason, we will not be concerned with the latter in this paper.

It was already shown in [28] that the $\mathcal{N} = 2$ SU(3) × U(1)-invariant solutions of the SO(8) [27] and ISO(7) gaugings [20] do cease to have the same KK spectrum upon uplift to $D = 11$ [29] and type IIA [20], thus answering in the negative the question posed above. Fortunately, it was not necessary to compute the entire KK spectrum to elucidate that question in [28]. Computing the spectrum of gravitons following [30] was enough to give an answer. In this paper, we will extend this statement to all solutions with at least SU(3) symmetry. Any pair of such solutions in table 1 with the same (super)symmetry and the same spectrum within their $\mathcal{N} = 8$ supergravities fails to have the same spectrum of KK gravitons. This is so regardless of the embedding of the residual symmetry group into the gauge group, and ultimately into the duality group $E_{7(7)}$ of the ungauged $\mathcal{N} = 8$ theory. For example, both the SO(7)$_v$ and SO(7)$_c$ solutions of the SO(8) gauging have the same spectrum within the $\mathcal{N} = 8$ theory [31]. But, as we will show in this paper, the spectrum of gravitons for both solutions differ.

It was also observed in [28] that, despite their different KK mode structure, certain sums of masses up the Kaluza-Klein tower did still remain universal for the $\mathcal{N} = 2$ SU(3) × U(1) solutions of the SO(8) and ISO(7) gaugings. In other words, while the eigenvalue-by-
eigenvalue equivalence of both spectra at the lowest KK level was lost up the KK tower, certain combinations of higher KK modes did still remain universal. In section , we will find the same behaviour for solutions of the SO(8) and ISO(7) gauging within the SU(3)-invariant sector. The solutions of the \((\text{SO}(6) \times \text{SO}(1,1)) \rtimes \mathbb{R}^{12}\) gauging behave similarly but slightly differently. For that reason, we relegate their discussion to the concluding section .

In \([28]\), the relevant combinations of KK modes were identified as traces of a KK graviton mass matrix. In section , we formalise this notion and introduce an SL(8)-covariant KK graviton mass matrix whose form is qualitatively similar to the bosonic mass matrices of \(D = 4 \, \mathcal{N} = 8\) gauged supergravity (see \([5]\)). Prior to discussing this, we complete in section the KK graviton spectrum for all known solutions of the \((\text{SO}(6) \times \text{SO}(1,1)) \rtimes \mathbb{R}^{12}\) gauging by computing the spin-2 spectrum of a solution with \(\mathcal{N} = 4\) supersymmetry and SO(4) symmetry \([23]\), which lies outside the SU(3) sector of section .

In this paper, we want to compute the spectrum of massive gravitons about the AdS\(_4\) solutions of the ten- and eleven-dimensional supergravities specified above. For later reference, the relevant geometries are of the form

\[ d\bar{s}^2_{d+4} = e^{2A(y)} \left[ (\bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x,y)) dx^\mu dx^\nu + d\bar{s}_d^2(y) \right], \tag{4} \]

where \(d = 7\) in M-theory and \(d = 6\) in type II. The Einstein frame is used in the latter case. The external and internal coordinates have been collectively denoted \((x,y)\). The metrics \(\bar{g}_{\mu\nu} dx^\mu dx^\nu \equiv ds^2(\text{AdS}_4)\) and \(d\bar{s}_d^2(y)\) respectively denote the unit-radius four-dimensional anti-de Sitter metric, and a background metric on the internal \(d\)-dimensional space that will be specified below on a case-by-case basis. The warp factor \(e^{2A(y)}\) takes values on the internal space. Finally, \(h_{\mu\nu}\) is taken to be a spin-2 perturbation over AdS\(_4\) that also depends on the internal coordinates. More concretely, the perturbation is assumed to take on the factorised form

\[ h_{\mu\nu}(x,y) = h^{[tr]}_{\mu\nu}(x) \mathcal{Y}(y), \tag{5} \]

with \(\mathcal{Y}(y)\) a function on the internal space only, and \(h^{[tr]}\) transverse \((\nabla^\mu h^{[tr]}_{\mu\nu} = 0)\) with
respect to the Levi-Civita connection corresponding to $\bar{g}_{\mu\nu}$, traceless ($\bar{g}^{\mu\nu} h^{[t]}_{\mu\nu} = 0$), and subject to the Fierz-Pauli equation

$$\Box h^{[t]}_{\mu\nu} = (M^2 L^2 - 2) h^{[t]}_{\mu\nu},$$

for a graviton of squared mass $M^2$. Here, $L$ is the effective AdS$_4$ radius (introduced in our context by the warping $e^{2A(y)}$), such that the combination $M^2 L^2$ is dimensionless.

While the computation of supergravity spectra in general is a very complicated problem, the calculation of massive graviton spectra is comparatively much simpler. The reason is that the linearised spin-2 equations decouple from the supergravity fluxes and become a boundary value problem involving only the warp factor and the internal background metric. Indeed, with the above assumptions, the linearised ten- and eleven-dimensional Einstein equations reduce to the eigenvalue problem [30]

$$-e^{-(d+2)A} \sqrt{\bar{g}} \partial_M (e^{(d+2)A} \sqrt{\bar{g}} \bar{g}^{MN} \partial_N \mathcal{Y}) = M^2 L^2 \mathcal{Y},$$

with $\bar{g}^{MN}$, $M, N = 1, \ldots, d$, the inverse metric and $\bar{g}$ the determinant of the internal metric $d\bar{s}_d^2$ in [4]. For unwarped geometries, $A = 0$, (7) reduces to the spectral problem for the Laplacian on the internal space. Previous calculations of KK graviton spectra in related contexts include [30, 32, 33, 34, 35, 36, 37, 38, 39].
We now compute the KK graviton spectrum about the AdS\(_4\) solutions of \(D = 11\) supergravity, massive type IIA supergravity and type IIB supergravity that uplift from critical points of SO(8) supergravity, dyonic ISO(7) supergravity and \((\text{SO}(6) \times \text{SO}(1, 1)) \ltimes \mathbb{R}^{12}\) supergravity with at least SU(3) symmetry. For convenience, some features of the SU(3)-invariant sector of these \(D = 4\) \(\mathcal{N} = 8\) gaugings are summarised in appendix \(\alpha\).

M-theory

The class of AdS\(_4\) solutions of \(D = 11\) supergravity \([17]\) that we are interested in arises upon consistent uplift on \(S^7\) \([18]\) of critical points of \(D = 4\) \(\mathcal{N} = 8\) SO(8)-gauged supergravity \([15]\). For definiteness, we will restrict to critical points that preserve at least the SU(3) subgroup of SO(8). There are six such vacua \([27]\). The corresponding uplifts are given by the \(D = 11\) solutions first found in \([40, 29, 41, 42, 43, 44]\). These solutions are invariant, both in \(D = 4\) and in \(D = 11\), under a number of subgroups of SO(8) larger than SU(3), and display supersymmetries \(\mathcal{N} = 0, 1, 2, 8\). See table \(1\) for a summary. The entire spectrum about the Freund-Rubin \(\mathcal{N} = 8\) SO(8)-invariant AdS\(_4\) solution \([40]\) has long been known \([45, 46, 47]\) (see also \([48]\) for a review). The spectrum of gravitons about the \(\mathcal{N} = 2\) SU(3) \(\times\) U(1)\(_c\)-invariant solution \([29]\) is also known \([32]\). The graviton spectra that we will give below for the four other AdS\(_4\) solutions in this sector are new.

A convenient starting point for our analysis is the local geometries recently presented in \([24]\). In that reference, the full, dynamical SU(3)-invariant sector of \(D = 4\) \(\mathcal{N} = 8\) SO(8)-gauged supergravity \([15]\) was uplifted to \(D = 11\) using the consistent truncation formulae of \([49]\). In particular, the results of \([24]\) provide a unified treatment for all the \(D = 11\) AdS\(_4\) solutions that uplift from critical points of \(D = 4\) \(\mathcal{N} = 8\) SO(8)-gauged supergravity with at least SU(3) symmetry. In order to simplify the calculations, we will focus on two disjoint further subsectors with symmetries G\(_2\) and SU(4)\(_c\) larger than SU(3). We will obtain the
graviton spectra for arbitrary constant values of the $D = 4$ scalars in those sectors. Finding the actual spectra about each individual solution will simply entail an evaluation of those formulae at the corresponding scalar vevs.

**Massive gravitons with at least $G_2$ symmetry**

The $G_2$-invariant sector of the $D = 4$ SO(8) supergravity contains an $\text{SL}(2, \mathbb{R})/\text{SO(2)}$ dilaton-axion pair $(\varphi, \chi)$, in the notation of appendix . The $D = 11$ uplift of this sector was given in section 3.2.3 of [24]. The warp factor and internal $d = 7$ geometry that feature in (4) are given by

$$e^{2A} = e^{-\varphi} X^{1/3} \Delta_1^{2/3} L^2, \quad ds_7^2 = g^{-2} L^{-2} \left( e^{3\varphi} X^{-3} d\beta^2 + e^{\varphi} \Delta_1^{-1} \sin^2 \beta ds^2(S^6) \right). \quad (8)$$

Here $\beta$ is an angle on $S^7$, with $0 \leq \beta \leq \pi$, and $ds^2(S^6)$ is the round Einstein metric on the unit radius $S^6$. The dilaton $\varphi$ appears explicitly in (8) and the axion $\chi$ appears through the combinations $X$ defined in equation (100) and

$$\Delta_1 = X \left( e^{2\varphi} \sin^2 \beta + e^{-2\varphi} X^2 \cos^2 \beta \right). \quad (9)$$

Finally, $g$ and $L$ are constants. The former is the gauge coupling of the $D = 4$ supergravity and the latter is related via (96) to the $G_2$–invariant potential $V$, given by (97) with the identifications (101). The geometry (8) is in fact invariant under the $\text{SO}(7)_v$ that rotates the round $S^6$. When $\chi \neq 0$, the symmetry of the full $D = 11$ configuration is broken to $G_2$ by the supergravity four-form field strength.

For the class of geometries (8), the differential equation (7) becomes

$$\left[ e^{-3\varphi} X^3 (\partial_\beta^2 + 6 \cot \beta \partial_\beta) + e^{-\varphi} \Delta_1 \sin^{-2} \beta \Box_{S^6} \right] Y = -g^{-2} M^2 Y, \quad (10)$$

---

1This range of $\beta$ corrects a typo below (B.22) of [24].
where $\Box_{S^6}$ is the $S^6$ Laplacian. Using separation of variables,

$$\mathcal{Y} = f \mathcal{Y}_k , \quad (11)$$

where $f = f(\beta)$ depends only on $\beta$ and $\mathcal{Y}_k$ are the $S^6$ spherical harmonics,

$$\Box_{S^6} \mathcal{Y}_k = -k(k+5) \mathcal{Y}_k , \quad (12)$$

the PDE (10) reduces to an ODE for $f(\beta)$,

$$e^{-3\varphi}X^3(f''(\beta) + 6 \cot \beta f'(\beta)) - e^{-\varphi} \Delta_1 \sin^2 \beta k(k+5)f(\beta) = -g^{-2}M^2 f(\beta) , \quad (13)$$

where a prime denotes derivative with respect to $\beta$. Finally, it is convenient to introduce a further change of variables,

$$u = \cos^2 \beta , \quad f(u) = (1 - u)^{\frac{b}{2}} H(u) . \quad (14)$$

The independent variable $u$ now ranges in $0 \leq u \leq 1$, covering this interval twice given the range of $\beta$ below (8). In the variables (14), the differential equation (13) takes on the standard hypergeometric form

$$u(1 - u)H'' + (c - (1 + a_+ + a_-)u)H' - a_+a_- H = 0 , \quad (15)$$

with

$$a_\pm = \frac{1}{2}(k + 3) \mp \frac{1}{2}e^{3\varphi/2}X^{-3/2}\sqrt{g^{-2}M^2 + 9e^{-3\varphi}X^3 + k(k+5)(e^{-3\varphi}X^3 - e^\varphi X)} , \quad c = \frac{1}{2} . \quad (16)$$

The two linearly independent solutions to (15) are given by the hypergeometric
functions

\[ 2F_1(a_+, a_-, c; u) \quad \text{and} \quad u^{1-c} 2F_1(1 + a_+ - c, 1 + a_- - c, 2 - c; u). \] (17)

Both solutions are regular at \( u = 0 \) for all values of the parameters \((16)\). At \( u = 1 \), however, regularity imposes restrictions on the parameters. Regularity of the first solution in \((17)\) demands \( a_+ = -j \) with \( j \) a non-negative integer. Bringing this condition to \((16)\), we find a first tower of KK graviton squared masses:

\[ g^{-2} M_{(1) j, k}^2 = e^{-3\varphi} X^3 (2j + k)(2j + k + 6) + e^{-\varphi} X(e^{2\varphi} - e^{-2\varphi} X^2)k(k + 5). \] (18)

The corresponding eigenfunctions are given by \((11), (14)\), with \( H(u) \) given by the first choice in \((17)\), namely

\[ \mathcal{Y}_{(1) j, k} = \mathcal{Y}_k \sin^k \beta \sum_{s=0}^{j} (-1)^s \binom{j}{s} \binom{j+k+3}{s} \cos^{2s} \beta \] (19)

(no sum in \( k \)), where

\[ (x)_s = \begin{cases} 1, & \text{if } s = 0 \\ x(x+1)\cdots(x+s-1), & \text{if } s > 0 \end{cases} \] (20)

is the Pochhammer symbol. Regularity of the second solution in \((17)\) at \( u = 1 \) in turn requires \( 1 + a_+ - c = -j \), with \( j \) again a non-negative integer. Bringing this condition to \((16)\), we find a second tower of KK graviton squared masses:

\[ g^{-2} M_{(2) j, k}^2 = e^{-3\varphi} X^3 (2j + 1 + k)(2j + 1 + k + 6) + e^{-\varphi} X(e^{2\varphi} - e^{-2\varphi} X^2)k(k + 5). \] (21)

The associated eigenfunctions are now given by \((11), (14)\), with \( H(u) \) given by the second
choice in [17]:

\( Y_{(2),j,k} = y_k \sin^k \beta \sum_{s=0}^{j} (-1)^s \binom{j}{s} \frac{\sin^{s+1} \beta}{s!} . \) \hspace{1cm} (22)

The eigenvalues [18] and [21] actually correspond to a unique tower of KK graviton masses. This is made apparent by introducing a new quantum number \( n \) defined as

\[ n = \begin{cases} 
2j + k & \text{for the first branch} \\
2j + 1 + k & \text{for the second branch}.
\end{cases} \] \hspace{1cm} (23)

In terms of \((n,k)\), [18] and [21] can be combined into the single KK tower:

\( g^{-2} M_{n,k}^2 = e^{-3\varphi} X^3 n(n+6) + e^{-\varphi} X (e^{2\varphi} - e^{-2\varphi} X^2) k(k+5) , \) \hspace{1cm} (24)

which is our final result. The quantum numbers range here as

\[ n = 0, 1, 2, \ldots , \quad k = 0, 1, \ldots , n . \] \hspace{1cm} (25)

Only \( n \) ranges freely over the non-negative integers, due to its definition [23] in terms of the non-negative but otherwise unconstrained integer \( j \). The range of \( k \) is limited to \( k \leq n \) by (23). At fixed \( n \), the eigenvalue (24) occurs with degeneracy

\[ D_{k,7} \equiv \dim [k,0,0]_{SO(7)}, \] \hspace{1cm} (26)

where, more generally, \( D_{k,N} \) is the dimension of the symmetric traceless representation \([k,0,\ldots,0]_{SO(N)}\) of \( SO(N) \),

\[ D_{k,N} = \binom{k+N-1}{k} - \binom{k+N-3}{k-2} \] \hspace{1cm} (27)

\[ = \frac{1}{(N-2)!} (2k+N-2)(k+N-3)(k+N-4) \cdots (k+2)(k+1) , \]
for $k \geq 2$ and
\[ D_{0,N} = 1, \quad D_{1,N} = N, \quad \text{for all } N = 2, 3 \ldots \] (28)

It is also useful to note that
\[ D_{n,N-1} = D_{n,N} - D_{n-1,N}, \quad \text{for all } n = 1, 2, \ldots \text{ and all } N = 2, 3 \ldots \] (29)

The eigenfunctions (19), (22) can be similarly combined into
\[ \mathcal{Y}_{n,k} = Y_k \sin^k \beta \sum_{s=0}^{[n-k/2]} (-1)^s \left[ \frac{n-k}{2} \right] \left( \frac{n-k}{2} + k + 3 + h_{n,k} \right)_s \cos^{2s+h_{n,k}} \beta, \] (30)

where $[\cdot]$ means integer part and we define the symbol $h_{n,k}$ as
\[ h_{n,k} = n - k - 2 \left[ \frac{n-k}{2} \right] = \begin{cases} 0, & n - k \text{ even (for the first branch)} \\ 1, & n - k \text{ odd (for the second branch)} \end{cases}. \] (31)

At fixed $n$ and $k$, the eigenfunctions (30) span the $[k,0,0]$ representation of SO(7)$_v$. Moreover, it can be checked that these eigenfunctions at fixed $n$ actually span the full symmetric traceless representation $[n,0,0,0]$ of SO(8). In other words, the eigenfunctions (30) turn out to be simply the SO(8) spherical harmonics of $S^7$, branched out into SO(7)$_v$ representations through
\[ [n,0,0,0] \xrightarrow{\text{SO(7)$_v$}} \sum_{k=0}^{n} [k,0,0]. \] (32)

This is consistent with the quantum number ranges (25). This is also compatible with the internal geometry (8) being topologically $S^7$: it can be continuously deformed into the round SO(8)--invariant geometry by setting $\varphi = \chi = 0$. These arguments suggest that the spectrum (24), (30) is in fact complete. Thus, the quantum number $n$ can be regarded as the Kaluza-Klein level, as it coincides with the unique integer that characterises the KK spectrum of the $\mathcal{N} = 8$ SO(8)-invariant Freund-Rubin solution: see e.g. table 9 of [48].
Massive gravitons with at least SU(4)\(_c\) symmetry

The SU(4)\(_c\)-invariant sector of SO(8)-gauged supergravity contains three pseudoscalars: \(\chi, \zeta, \tilde{\zeta}\) in the notation of appendix. In the Iwasawa parametrisation of the appendix, the SU(3)-invariant dilatons \(\varphi, \phi\) become identified in terms of the pseudoscalars via equation (102). With the understanding that \(\varphi, \phi\) depend on the independent fields \(\chi, \zeta, \tilde{\zeta}\), the former can be conveniently used to parametrise the SU(4)\(_c\)-invariant sector, as the resulting expressions are more compact. The embedding of this sector into the \(D = 11\) warp factor and internal metric reads \([31]\), in the notation of \([24]\),

\[
e^{2A} = e^{4\phi + \varphi} L^2, \quad d\bar{s}_7^2 = g^{-2}L^{-2} \left[ e^{-2\phi - \varphi} ds^2(\mathbb{C}P^3) + e^{-3\varphi}(d\psi + \sigma)^2 \right].
\] (33)

Here, \(ds^2(\mathbb{C}P^3)\) is the Fubini-Study metric on the complex projective space, \(\sigma\) a one-form potential for the Kähler form on the latter, and \(0 \leq \psi \leq 2\pi\) a coordinate on the Hopf fibre of \(S^7\). The constant \(g\) is again the coupling of SO(8) supergravity and \(L\) is fixed through (96) in terms of the SU(4)\(_c\)-invariant potential \(V\), given by (97) with the identifications (102). Away from the SO(7)\(_c\)-invariant locus, (2.39) of \([24]\), where the symmetry is enhanced accordingly, the geometry (33) is invariant under SU(4)\(_c\) \(\times\) U(1), with U(1) generated by \(\partial_\psi\). This U(1) is broken by the \(D = 11\) supergravity four-form.

The \(D = 11\) embedding (33) of the SU(4)\(_c\)-invariant sector is homogeneous: the warp factor depends only on the \(D = 4\) scalars and not on the \(S^7\) coordinates, and the metric \(d\bar{s}_7^2\) corresponds to a homogeneous stretching of the \(S^7\) geometry along its Hopf fibre. Therefore, the differential equation (7) simplifies for this geometry as

\[
\left[ (e^{3\varphi} - e^{2\phi + \varphi}) \partial_\psi^2 + e^{2\phi + \varphi} \Box_{S^7} \right] \mathcal{Y} = -g^{-2}M^2 \mathcal{Y} ,
\] (34)

with \(\Box_{S^7}\) the Laplacian on the round, Einstein metric on \(S^7\). The solutions of (34) are accordingly given by the SO(8) spherical harmonics on \(S^7\), branched out into representations...
of the SU$(4)_c \times U(1)$ symmetry group of (33) and (34) via

$$[n, 0, 0, 0]_{SU(4)_c \times U(1)} \rightarrow \sum_{r=0}^{n} [r, 0, n-r]_{2r-n},$$

(35)

with the subindex indicating the U(1) charge. More concretely, the $S^7$ spherical harmonics, in the $[n, 0, 0]$ of SO(8), split according to (35) as

$$Y_{n,r}(z, \bar{z}) = c_{a_1\ldots a_r b_1\ldots b_{n-r}} z^{a_1} \ldots z^{a_r} \bar{z}^{b_1} \ldots \bar{z}^{b_{n-r}},$$

(36)

for

$$n = 0, 1, 2, \ldots, \quad r = 0, 1, \ldots, n.$$

(37)

In (36), $z^1 = \mu^1 + i\mu^2$, etc, are complexified embedding coordinates of $\mathbb{R}^8$ constrained as $\delta_{AB} \mu^A \mu^B = 1$, with $A, B = 1, \ldots, 8$, and $c_{a_1\ldots a_r b_1\ldots b_{n-r}}$ is a constant tensor in the $[n-r, 0, r]$ of SU(4). The functions (36) obey

$$\Box_{S^7} Y_{n,r} = -n(n+6)Y_{n,r}, \quad \partial_\psi^2 Y_{n,r} = -(n-2r)^2 Y_{n,r},$$

(38)

and thus satisfy the differential equation (34) with eigenvalue

$$g^{-2} M_{n,r}^2 = e^{2\phi + \varphi} n(n+6) + (e^{3\varphi} - e^{2\phi + \varphi})(n-2r)^2.$$

(39)

This occurs with multiplicity

$$d_{n,r} = \text{dim}[r, 0, n-r]_{SU(4)} = \frac{1}{12} (n+3)(r+1)(r+2)(n-r+1)(n-r+2).$$

(40)

To summarise, the complete spectrum of the eigenvalue equation (34) is (39), (36), with the quantum numbers ranging as in (37). The eigenvalues (39) have multiplicity (40) and the eigenfunctions (36) are simply the $S^7$ spherical harmonics split into SU$(4)_c \times U(1)$.
representations through (35). The eigenvalues have been given in terms of $D = 4$ scalars. The massive KK graviton spectra about $D = 11$ AdS$_4$ solutions in this sector are obtained by fixing the $D = 4$ scalars to the corresponding vevs. Like in the case discussed in section , the integer $n$ is identified with the KK level by an argument similar to that put forward below (32).

Type IIB

We now move on to compute the graviton spectrum about the AdS$_4$ solutions of type IIB supergravity recently obtained in [26]. These geometries arise upon consistent uplift [23] on an S-fold geometry of AdS$_4$ vacua of $D = 4$ $\mathcal{N} = 8$ gauged supergravity with dyonic $[\text{SO}(6) \times \text{SO}(1, 1)] \ltimes \mathbb{R}^{1, 2}$ gauging [8, 10] (see also [54]). The resulting type IIB uplifts correspond to limiting Janus-type solutions [51, 52, 53, 54]. As in section , we will focus in this section on solutions that preserve at least SU(3) symmetry. These were classified in [26]. We will compute the generic graviton spectra for arbitrary constant values of the SU(3)-invariant scalars of the $D = 4$ supergravity.

The type IIB geometries under consideration are of the form (4) with $d = 6$ and $\eta$.

$$
e^{2A} = \sqrt{Y} e^\varphi L^2, \quad ds_6^2 = \frac{e^{-\varphi}}{\sqrt{Y}} g^{-2} L^{-2} \left[ \sqrt{Y} e^{-2\varphi} \, d\eta^2 + \frac{1}{\sqrt{Y}} (ds^2(\mathbb{C}P^2) + Y (d\tau + \sigma')^2) \right].$$

The geometry inside the last parenthesis extends globally over a topological $S^5$, with $ds^2(\mathbb{C}P^2)$ the Fubini-Study metric on the complex projective plane within $S^5$ and $0 \leq \tau < 2\pi$ the Hopf fibre angle. The local one-form $\sigma'$ is a potential for the Kähler form on $\mathbb{C}P^2$. The sixth internal coordinate $\eta$ will be taken to be periodic, $\eta \sim \eta + T$, with $T$ a positive number. The ten-dimensional geometry (41) also depends on the SU(3)-invariant scalars of appendix both explicitly and through the combination $Y$ defined in (111). Finally, $g$ is the gauge coupling constant of the $D = 4$ supergravity, and $L$ is fixed through

$^2$We have conveniently rescaled the metric and warp factor with respect to [26].
in terms of the scalar potential \( V \) given by (99) with \( \chi = 0 \). For general values of the scalars, the geometry (41) displays an isometry group \( \text{SU}(3) \times \text{U}(1)_\tau \times \text{U}(1)_\eta \), with the \( \text{U}(1)_\eta \) factor broken by the type IIB fluxes. In particular, the type IIB fields charged under the S-duality group \( \text{SL}(2, \mathbb{R}) \) undergo a monodromy transformation as \( \eta \) crosses through different periods [23]. The type IIB metric is neutral under S-duality and thus insensitive to this transformation.

Like in section \( \text{96} \), the type IIB embedding (41) is homogeneous. Accordingly, the differential equation (7) reduces for this geometry to

\[
\left[ e^{3\varphi} \partial^2_\eta + e^{\varphi} (1 - Y) \partial^2_\tau + e^{\varphi} Y \Box_{S^5} \right] Y = -g^{-2} M^2 Y ,
\]

where \( \Box_{S^5} \) is the Laplacian on the round, Einstein metric on \( S^5 \). The complete set of eigenfunctions \( Y \equiv Y_{\ell,p,j} \) that solve (42) can be taken to satisfy

\[
\Box_{S^5} Y_{\ell,p,j} = -\ell(\ell + 4) Y_{\ell,p,j}, \quad \partial^2_\tau Y_{\ell,p,j} = -(\ell - 2p)^2 Y_{\ell,p,j}, \quad \partial^2_\eta Y_{\ell,p,j} = -\left( \frac{2\pi}{T_j} \right)^2 Y_{\ell,p,j} ,
\]

for

\[
\ell = 0, 1, 2, \ldots , \quad p = 0, 1, \ldots , \ell , \quad j = 0, \pm 1, \pm 2, \ldots ,
\]

with \( \ell \) and \( j \) unconstrained and \( p \) constrained by \( \ell \) through \( p \leq \ell \). In other words, the eigenfunctions \( Y_{\ell,p,j} \) come in representations of \( \text{SU}(3) \times \text{U}(1)_\tau \times \text{U}(1)_\eta \), and are explicitly given by products of harmonics on the \( S^1 \) generated by \( \partial_\eta \) and spherical harmonics \([\ell,0,0]_{\text{SO}(6)} \) on \( S^5 \) branched out into representations of \( \text{SU}(3) \times \text{U}(1)_\tau \) via

\[
[\ell,0,0]_{\text{SU}(3) \times \text{U}(1)_\tau} \sum_{p=0}^{\ell} [p, \ell-p]_{\ell-2p} .
\]
Bringing (43) to (42), we find the eigenvalues

\[ g^{-2}M_{\ell,p,j}^2 = e^{\phi}Y\ell(\ell + 4) + e^{\phi}(1 - Y)(\ell - 2p)^2 + e^{3\phi}\left(\frac{2\pi}{T}j\right)^2, \]

(46)

occurring with degeneracy

\[ d_{\ell,p,j} = \begin{cases} 
\dim[p, \ell - p]_{SU(3)} = \frac{1}{2}(p + 1)(\ell - p + 1)(\ell + 2), & \text{if } j = 0 \\
2\dim[p, \ell - p]_{SU(3)} = (p + 1)(\ell - p + 1)(\ell + 2), & \text{if } j \neq 0.
\end{cases} \]

(47)

In summary, the complete eigenvalue spectrum of equation (42) is (46) with the eigenfunctions \(\mathcal{Y}_{\ell,p,j}\) described above and with the quantum numbers ranging as in (44). The eigenvalues (46) have multiplicity (47), and have been given in terms of \(D = 4\) scalars. The massive KK graviton spectra about \(D = 11\) AdS\(_4\) solutions in this sector are obtained by fixing the \(D = 4\) scalars to the corresponding vevs, as we will see next.

Individual spectra in M-theory, type IIA and type IIB

Using the results of sections and as well as [28, 32, 48], we can write down the KK graviton spectra about the AdS\(_4\) solutions of the ten and eleven-dimensional supergravities that uplift from critical points with at least SU(3) symmetry of the three \(D = 4\) \(\mathcal{N} = 8\) gauged supergravities that we are considering in this paper.

In M-theory, the spectrum above the AdS\(_4\) solutions with at least G\(_2\) symmetry and at least SU(4)\(_c\) symmetry can be obtained by particularising (24) and (39), respectively, to the scalar vevs given in [24]. We have brought these results to table 2. In order to exhaust the KK graviton spectra of AdS\(_4\) solutions of \(D = 11\) supergravity that uplift from critical points of \(D = 4\) \(\mathcal{N} = 8\) SO(8)-gauged supergravity with at least SU(3) symmetry, the table also includes the spectrum [48] about the \(\mathcal{N} = 8\) Freund-Rubin solution [40] and the spectrum [32] about the SU(3)\(\times\)U(1)\(_c\)-invariant AdS\(_4\) solution [27, 29]. The latter is given as in [28], with \(n_{\text{here}} = n_{\text{there}}, r_{\text{here}} = r_{\text{there}}, p_{\text{here}} = p_{\text{there}}\) and \(\ell_{\text{here}} = p_{\text{there}} + q_{\text{there}}.\) The
Table 2: The KK graviton spectra of AdS$_4$ solutions of $D = 11$ supergravity that uplift from critical points of $D = 4$, $\mathcal{N} = 8$ SO(8)-gauged supergravity with at least SU(3) symmetry. See (27) for the notation $D_{k,N}$. The quantum numbers range as in (48).

corresponding multiplicities are also given in the table, and the quantum numbers range as

$$n = 0, 1, 2, \ldots, \quad r, k = 0, 1, \ldots, n, \quad \ell = p, \ldots, p + r, \quad p = 0, 1, \ldots, n - r. \quad (48)$$

The only quantum number that is free to range unrestricted over the non-negative integers is $n$, all the others being bound by it. This is consistent with the interpretation of $n$ as the SO(8) KK level, see below (32). At fixed KK level $n$, the degeneracy of the $\mathcal{N} = 8$ SO(8)–symmetric spectrum is broken into representations of the isometry group of the internal metric. This may be larger than the symmetry of each solution, as the fluxes will further break the isometry to the actual symmetry quoted in the table. Similarly, the eigenfunctions corresponding to each solution are simply the $S^7$ spherical harmonics branched out into the representations of the relevant group.

For convenience, table 3 imports from [28] the KK graviton spectra of AdS$_4$ solutions of massive IIA supergravity that uplift from critical points of $D = 4$, $\mathcal{N} = 8$ dyonic ISO(7)-gauged supergravity with at least SU(3) symmetry. The table includes the squared masses in units of the corresponding AdS radius $L$, as well as the multiplicities. In this case, the quantum numbers' ranges are

$$k = 0, 1, 2, \ldots, \quad \ell = 0, 1, \ldots, k, \quad p = 0, 1, \ldots, \ell, \quad (49)$$
Solution & Mass & Degeneracy \\
\hline
\mathcal{N} = 2, \text{ SU}(3) \times \text{U}(1)_v & L^2 M^2_{k,\ell,p} = \frac{2}{9} k(k+5) - \frac{2}{3} \ell(\ell+4) + \frac{1}{3} (\ell-2p)^2 & d_{k,\ell,p} = \frac{1}{2} (p+1)(\ell-p+1)(\ell+2) \\
\mathcal{N} = 1, \text{ G}_2 & L^2 M^2_k = \frac{1}{9} k(k+5) & d_k = D_{k,7} \\
\mathcal{N} = 1, \text{ SU}(3) & L^2 M^2_{k,\ell,p} = \frac{5}{6} k(k+5) - \frac{5}{12} \ell(\ell+4) - \frac{5}{36} (\ell-2p)^2 & d_{k,\ell,p} = \frac{1}{2} (p+1)(\ell-p+1)(\ell+2) \\
\mathcal{N} = 0, \text{ SO}(7)_v & L^2 M^2_k = \frac{5}{5} k(k+5) & d_k = D_{k,7} \\
\mathcal{N} = 0, \text{ SO}(6)_v & L^2 M^2_{k,\ell} = k(k+5) - \frac{1}{3} \ell(\ell+4) & d_{\ell} = D_{\ell,6} \\
\mathcal{N} = 0, \text{ G}_2 & L^2 M^2_k = \frac{1}{2} k(k+5) & d_k = D_{k,7} \\
\hline

Table 3: The KK graviton spectra of AdS$_4$ solutions of massive IIA supergravity that uplift from critical points of $D = 4$ $\mathcal{N} = 8$ dyonic ISO(7)-gauged supergravity with at least SU(3) symmetry, taken from [28]. See (27) for the notation $D_{k,N}$. The quantum numbers range as in (49).

with $k_{\text{here}} = n_{\text{in}}$ [28]. Again, $k$ is the only quantum number that is unrestricted. For this reason, $k$ can be interpreted in this case as the SO(7) KK level. The eigenfunctions are now the $S^6$ spherical harmonics split into representations of the internal isometry group. This again may be larger than the symmetry of each solution given in table 3 because the fluxes may further break the isometry to the actual symmetry of the metric and fluxes.

Finally, we turn to the spectrum of gravitons corresponding to the type IIB AdS$_4$ S-fold solutions that uplift from critical points with at least SU(3) symmetry [26] of $D = 4$ $\mathcal{N} = 8$ supergravity with $(\text{SO}(6) \times \text{SO}(1,1)) \ltimes \mathbb{R}^{12}$ gauging. These are found by bringing the corresponding vevs, collected in our conventions in table 6 in appendix , to equation (46). The results are summarised in table 4. In order to derive the generic scalar-dependent spectra in section , we assumed that the S-fold direction $\eta$ is compactified to a U(1)$_\eta$ with period $T$. The KK graviton spectra are sensitive to this period. The eigenfunctions are products of $S^5$ harmonics, possibly branched out into SU(3) $\times$ U(1)$_\tau$ representations, and U(1)$_\eta$ harmonics. This U(1)$_\eta$ is broken by the IIB fluxes.

Universality of traces

When regarded as vacua of their corresponding $D = 4$ $\mathcal{N} = 8$ gauged supergravities, the AdS solutions under consideration with at least SU(3) symmetry tend to exhibit the same mass spectrum of scalars, vectors and fermions within their $D = 4$ supergravities.
\[ N = 1, \text{SU}(3) \quad L^2 M^2_{\ell,p,j} = \frac{5}{6} \ell (\ell + 4) - \frac{5}{36} (\ell - 2p)^2 + \frac{5\pi^2}{T^2 J^2} d_{\ell,p,j} \]

\[ N = 0, \text{SO}(6) \quad L^2 M^2_{\ell,j} = \frac{3}{4} \ell (\ell + 4) + \frac{6\pi^2}{T^2 J^2} d_{\ell,j} = (2 - \delta_{j,0}) D_{\ell,6} \]

\[ N = 0, \text{SU}(3) \quad L^2 M^2_{\ell,j} = \frac{3}{4} \ell (\ell + 4) + \frac{6\pi^2}{T^2 J^2} d_{\ell,j} = (2 - \delta_{j,0}) D_{\ell,6} \]

Table 4: The KK graviton spectra of AdS\(_4\) S-fold solutions of type IIB supergravity that uplift from critical points of \(D = 4\) \(\mathcal{N} = 8\) \((\text{SO}(6) \times \text{SO}(1,1)) \ltimes \mathbb{R}^{12}\)-gauged supergravity with at least SU(3) symmetry. See (27) for the notation \(D_{e,N}\) and (47) for \(d_{\ell,p,j}\). The quantum numbers range as in (44).

This is the case for all these solutions, except for the two \(N = 0\), SU(3)–invariant critical points of ISO(7) supergravity and the \(N = 0\), SU(3)–invariant critical locus of \((\text{SO}(6) \times \text{SO}(1,1)) \ltimes \mathbb{R}^{12}\) supergravity. The question that we would like to address in this section is whether this situation persists for higher KK modes. The spectrum of gravitons computed for these solutions in section shows that this universality is indeed lost at higher KK levels: the KK gravitons do have completely different masses for all the solutions considered.

However, as we will now show, universality is still maintained, though in a milder form that is not apparent from the results of section. It turns out that certain sums of KK graviton masses weighted with their multiplicities do remain universal. This is the case at least for solutions in the same or different \(\mathcal{N} = 8\) gaugings with the same symmetry and whose spectra within the \(D = 4\) supergravity are the same. Specifically, if two AdS\(_4\) solutions of \(D = 11\) supergravity or massive IIA uplift from critical points with the same supersymmetry \(\mathcal{N} \leq 8\), the same symmetry \(G \supset \text{SU}(3)\) (possibly embedded differently into the gauge group) and the same spectrum within the \(D = 4\) \(\mathcal{N} = 8\) SO(8) or ISO(7) supergravities, then there exist infinitely many discrete combinations \(L^2 \text{tr} M^2_{(n)}\), \(n = 1, 2, 3, \ldots\), of graviton masses weighted with their multiplicities that are the same for both solutions. This statement was proven for the \(\mathcal{N} = 2\) SU(3) \(\times\) U(1)-invariant solutions in [28]. Here we will extend that result to all other solutions with at least SU(3) symmetry in the SO(8) and ISO(7) gaugings, summarised in table 1 of the introduction. As discussed in [28] and further in section below, the notation \(L^2 \text{tr} M^2_{(n)}\) relates to the fact that the
combinations in question correspond to traces of the (infinite-dimensional) KK graviton mass matrix at fixed KK level \( n \).

More concretely, for the M-theory solutions we define \( L^2 \text{tr} M^2_{(n)} \) to be the sum of the squared masses in units of the corresponding AdS radius \( L \), weighted with the corresponding multiplicity as given in table \( \ref{table:mult} \). The sum is taken at fixed KK level \( n \) and over all other quantum numbers ranging as in \( \ref{eq:quantum_numbers} \). For example, using this prescription, one obtains for the \( \mathcal{N} = 8 \) SO(8) solution \[ \ref{eq:N8SO8} \],

\[
L^2 \text{tr} M^2_{(n)} = L^2 M_n^2 d_n = 14 \, D_{n-1,10}.
\] (50)

In the last step, we have made use of the definition \( \ref{eq:sum_of_squares} \) as a shorthand for the resulting 8th degree polynomial in \( n \). Similarly, for the \( \mathcal{N} = 2 \) SU(3) \( \times \) U(1)\(_c\) solution, we have \( \ref{eq:N2SU3U1} \)

\[
L^2 \text{tr} M^2_{(n)} = L^2 \sum_{r=0}^{n} \sum_{p=0}^{n-r} \sum_{\ell=0}^{p+r} M^2_{n,p,\ell,r} d_{n,p,\ell,r} = \frac{56}{3} \, D_{n-1,10}.
\] (51)

Proceeding similarly, we compute the quantities \( L^2 \text{tr} M^2_{(n)} \), \( n = 1, 2, \ldots \), for the KK graviton spectra summarised in table \( \ref{table:mult} \) for \( D = 11 \) AdS\(_4\) solutions that uplift from critical points of \( D = 4 \) \( \mathcal{N} = 8 \) SO(8) supergravity with at least SU(3) symmetry. We obtain:

\[
\begin{align*}
\mathcal{N} = 8 \; , \; \text{SO}(8) & : \; L^2 \text{tr} M^2_{(n)} = 14 \, D_{n-1,10} , \\
\mathcal{N} = 2 \; , \; \text{SU}(3) \times \text{U}(1)\_c & : \; L^2 \text{tr} M^2_{(n)} = \frac{56}{3} \, D_{n-1,10} , \\
\mathcal{N} = 1 \; , \; \text{G}_2 & : \; L^2 \text{tr} M^2_{(n)} = \frac{35}{2} \, D_{n-1,10} , \\
\mathcal{N} = 0 \; , \; \text{SO}(7)\_v & : \; L^2 \text{tr} M^2_{(n)} = \frac{84}{5} \, D_{n-1,10} , \\
\mathcal{N} = 0 \; , \; \text{SO}(7)\_c & : \; L^2 \text{tr} M^2_{(n)} = \frac{84}{5} \, D_{n-1,10} , \\
\mathcal{N} = 0 \; , \; \text{SU}(4)\_c & : \; L^2 \text{tr} M^2_{(n)} = \frac{20}{2} \, D_{n-1,10} .
\end{align*}
\] (52)

In particular, the two SO(7)-invariant solutions have their residual symmetry embedded
differently into the SO(8) gauge group as SO(7)_v and SO(7)_c. They have the same mass spectrum within \( D = 4 \mathcal{N} = 8 \) SO(8) supergravity, according to table 1. Their KK graviton spectra are different, though, according to table 2. But as can be seen from equation (52), the quantity \( L^2 \text{tr} M_{(n)}^2 \) is the same for both solutions for all \( n \).

The quantities \( L^2 \text{tr} M_{(k)}^2 \) for the KK gravitons of massive IIA solutions with at least SU(3) symmetry that uplift from critical points of dyonic ISO(7) supergravity were computed similarly, for \( k = 1, 2, \ldots, \) in \[28\]:

\[
\begin{align*}
\mathcal{N} &= 2, \; \text{SU}(3) \times \text{U}(1)_v : & L^2 \text{tr} M_{(1)}^2 &= \frac{56}{3} D_{k-1,9} , \\
\mathcal{N} &= 1, \; \text{G}_2 : & L^2 \text{tr} M_{(1)}^2 &= \frac{35}{2} D_{k-1,9} , \\
\mathcal{N} &= 1, \; \text{SU}(3) : & L^2 \text{tr} M_{(1)}^2 &= \frac{65}{3} D_{k-1,9} , \\
\mathcal{N} &= 0, \; \text{SO}(7)_v : & L^2 \text{tr} M_{(1)}^2 &= \frac{84}{5} D_{k-1,9} , \\
\mathcal{N} &= 0, \; \text{SO}(6)_v : & L^2 \text{tr} M_{(1)}^2 &= \frac{39}{2} D_{k-1,9} , \\
\mathcal{N} &= 0, \; \text{G}_2 : & L^2 \text{tr} M_{(1)}^2 &= 21 D_{k-1,9} .
\end{align*}
\]

Here, we have again made use of the notation \( D_{k,\mathcal{N}} \) defined in \[27\] as a shorthand for the degree-7 polynomial in \( k \) that appears in the r.h.s.’s. Now, recall from section that \( k \) and \( n \) can respectively be regarded as the KK levels in massive IIA and \( D = 11 \). At first KK level, the quantities \( L^2 \text{tr} M_{(n=1)}^2 \) in \[52\] and \( L^2 \text{tr} M_{(k=1)}^2 \) in \[53\] can be checked to match, by virtue of the first relation in \[28\], for solutions with the same symmetry group regardless of the embedding of the latter within the corresponding gauge group. For example, for the \( D = 11 \) SU(3) \( \times \) U(1)_c solution \[29\] and the massive IIA SU(3) \( \times \) U(1)_v solution \[20\], \( [L^2 \text{tr} M_{(1)}^2]_{11D} = [L^2 \text{tr} M_{(1)}^2]_{\text{IIA}} = \frac{56}{3} \), at \( n = k = 1 \), as already noted in \[28\]. Inspection of \[52\] and \[53\] confirms that similar matches occur at KK level one, \( n = k = 1 \), for the \( D = 11 \) and massive IIA solutions with common (super)symmetry \( \mathcal{N} = 1, \; \text{G}_2, \) and \( \mathcal{N} = 0, \; \text{SO}(7), \) and \( \mathcal{N} = 0, \; \text{SU}(4) \sim \text{SO}(6) \).
Further, there is still matching at higher KK levels $n > 1$ in $D = 11$ and $k > 1$ in massive IIA, provided a prescription is adopted to relate $n$ and $k$. An argument will be given in section but, for now, these two quantum numbers can be thought of as being related as in (32), so that the $D = 11$ KK level $n$ formally contains all IIA KK levels $k = 0, 1, \ldots, n$. Using this prescription, it follows from (52) and (53) that

$$
\sum_{k=0}^{n} [L^2 \text{tr} M_{(k)}^2]_{\text{IIA}} = [L^2 \text{tr} M_{(n)}^2]_{11D}, \quad n = 0, 1, 2, \ldots ,
$$

(54)

for all the solutions that we are considering with the same symmetry and supersymmetry in massive IIA and $D = 11$. Here, $L^2 \text{tr} M_{(0)}^2 \equiv 0$ corresponds to the massless graviton, for both the $D = 11$ and type IIA cases, as well as for the IIB cases below. From (54) it immediately follows that

$$
\sum_{n=0}^{m} \sum_{k=0}^{n} [L^2 \text{tr} M_{(k)}^2]_{\text{IIA}} = \sum_{n=0}^{m} [L^2 \text{tr} M_{(n)}^2]_{11D}, \quad m = 0, 1, 2, \ldots ,
$$

(55)

again for all solutions with the same (super)symmetry. The sums in (55) obviously run over repeated number of states, both in IIA and in $D = 11$. In (54), there are no repeated $D = 11$ states on the r.h.s., but the sum in the l.h.s. does run as well over repeated states in IIA. These overcounting issues can be avoided by subtracting two adjacent KK levels in $D = 11$: formally, the difference between KK levels $n$ and $n - 1$ in $D = 11$ contains the same number of states as KK level $k = n$ in massive IIA. Using the identity (29), it follows from (52), (53) that

$$
[L^2 \text{tr} M_{(n)}^2]_{\text{IIA}} = [L^2 \text{tr} M_{(n)}^2]_{11D} - [L^2 \text{tr} M_{(n-1)}^2]_{11D}, \quad n = 1, 2, \ldots ,
$$

(56)

for solutions with the same (super)symmetry. This relation was already shown to hold in [28] for the $\mathcal{N} = 2$ SU(3) $\times$ U(1) invariant solutions. Here, we have extended this result to all other AdS solutions in the SU(3)-invariant sectors of SO(8) and ISO(7) gauged supergravities with the same symmetry and supersymmetry.
The situation is similar, though slightly different, for the type IIB AdS4 S-fold solutions that uplift from $D = 4 \mathcal{N} = 8 (\mathrm{SO}(6) \times \mathrm{SO}(1,1)) \times \mathbb{R}^{12}$-gauged supergravity. According to table 1, this supergravity also has critical points with the same symmetry $G \supset \mathrm{SU}(3)$ and supersymmetry as other critical points of the SO(8) and ISO(7) gauging: $\mathcal{N} = 0 \mathrm{SO}(6)$, $\mathcal{N} = 1 \mathrm{SU}(3)$ and $\mathcal{N} = 0 \mathrm{SU}(3)$. The former two have the same spectrum within their corresponding $D = 4$ supergravities, while the latter does not. For this reason, we will only be interested in the former two vacua. Both for the $\mathcal{N} = 1 \mathrm{SU}(3)$ and the $\mathcal{N} = 0 \mathrm{SO}(6) \sim \mathrm{SU}(4)$ solutions there are combinations $[L^2 \mathrm{tr} \tilde{M}^2_{(n)}]_{\mathrm{IIB}}$, of the eigenvalues in table 4 that match the quantities $[L^2 \mathrm{tr} M^2_{(n)}]_{\mathrm{IIA}}$ and $[L^2 \mathrm{tr} M^2_{(n)}]_{11\mathrm{D}}$ for the solutions with the same symmetry for a certain choice of the period $T$. The tilde in $[L^2 \mathrm{tr} \tilde{M}^2_{(n)}]_{\mathrm{IIB}}$ is taken to signify that, in this case, the combinations also involve subtraction of eigenvalues. More concretely, consider the following quantities for the type IIB solutions with the quantum numbers fixed as indicated:

\[
\mathcal{N} = 1 \ , \ \mathrm{SU}(3) \quad : \quad L^2 \mathrm{tr} \tilde{M}^2_{(1)} = L^2 \left[ \sum_{\ell=0}^{\ell} M^2_{\ell,p,j} d_{\ell,p,j} \right] |_{\ell=1, j=0} \\
- L^2 \left[ M^2_{\ell,p,j} d_{\ell,p,j} \right] |_{\ell=p=0, j=-1} - L^2 \left[ M^2_{\ell,p,j} d_{\ell,p,j} \right] |_{\ell=p=0, j=+1} ,
\]

\[
\mathcal{N} = 0 \ , \ \mathrm{SO}(6)_{v} \quad : \quad L^2 \mathrm{tr} \tilde{M}^2_{(1)} = L^2 \left[ M^2_{\ell,j} d_{\ell,j} \right] |_{\ell=1, j=0} \\
- L^2 \left[ M^2_{\ell,j} d_{\ell,j} \right] |_{\ell=0, j=-1} - L^2 \left[ M^2_{\ell,j} d_{\ell,j} \right] |_{\ell=0, j=+1} .
\]

These quantities involve sums of mass eigenvalues, weighted with their degeneracies as given in table 4 and affected by a $+$ or a $-$ sign depending on whether $j = 0$ or $j \neq 0$. Plugging in the expressions given in the table, the quantity $L^2 \mathrm{tr} \tilde{M}^2_{(1)}$ for the $\mathcal{N} = 1$, SU(3) solution evaluates to $\frac{65}{3}$ if $T = 2\pi$, matching the quantity $L^2 \mathrm{tr} M^2_{(k=1)}$ for its counterpart type IIA solution at KK level $k = 1$, given in (53). Similarly, $L^2 \mathrm{tr} \tilde{M}^2_{(1)}$ for the $\mathcal{N} = 0$, SO(6)$_{v}$ solution evaluated using the expressions given in table 4 gives $\frac{39}{7}$ for $T = 2\pi$. This again matches the quantity $L^2 \mathrm{tr} M^2_{(k=1)}$ at KK level $k = 1$ given in (53) for the $\mathcal{N} = 0$, SO(6)$_{v}$ solution of massive IIA. It also matches $L^2 \mathrm{tr} M^2_{(n=1)}$ at KK level $n = 1$ given in
for the $D = 11 \mathcal{N} = 0$, SU(4)$_c$ solution. Although it is not as clear cut in the type IIB case, it will be argued in section 5 that the states that enter the sums in (57) also belong to KK level $m = 1$ in an SL(8)-covariant sense. The formal analytic continuation $j' = ij$, with $i^2 = -1$, removes the minus signs in (57). Under this analytic continuation, relations similar to (54) and (55) relate these formal sums at higher KK levels for these type IIB solutions to their $D = 11$ and type IIA counterparts. We will return to this point in section 6. In the next section, we will turn to compute the KK graviton spectrum of a different type IIB S-fold solution of particular interest.
GRAVITON SPECTRUM ON THE
$\mathcal{N} = 4$ SO(4) TYPE IIB S-FOLD

The spin-2 spectrum about the AdS$_4$ solution \[23\] of type IIB supergravity that uplifts on a six-dimensional S-fold from the $\mathcal{N} = 4$ SO(4)-invariant critical point \[14\] of $D = 4$ $\mathcal{N} = 8$ [SO(6) $\times$ SO(1, 1)] $\times$ $\mathbb{R}^{12}$-gauged supergravity \[10, 8\] can be computed as in section . This solution has concrete field theory duals for specific choices of the period $T$ of the S-fold coordinate $\eta$ \[55\] (see also \[56, 57, 58, 59, 60\]). For this reason, it is particularly interesting to go in some detail about the corresponding spin-2 spectrum.

The relevant ten-dimensional geometry is \[41\] with $d = 6$ and \[23\]

$$
\begin{align*}
&d s_6^2 = g^{-2} L^{-2} \left[ d \eta^2 + \frac{dr^2}{1-r^2} + \frac{r^2}{1+2r^2} ds^2(S_1^2) + \frac{1-r^2}{3-2r^2} ds^2(S_2^2) \right], \\
e^{2A} = L^2 \left[ (1+2r^2)(3-2r^2) \right]^{1/4},
\end{align*}
$$

(58)

with $L^2 = \frac{1}{2} g^{-2}$. The coordinate $r$ ranges as $0 \leq r \leq 1$ and $\eta$ is taken to be periodic, $\eta \sim \eta + T$, for some $T > 0$. Also, $ds^2(S_i^2)$, $i = 1, 2$, is the round, Einstein metric on each of two spheres $S_i^2$. These are rotated by the SO(4) = SO(3)$_1$ $\times$ SO(3)$_2$ isometry of the geometry \[58\]. This SO(4) isometry is also respected by the type IIB forms and is thus a symmetry, in fact the R-symmetry, of the full $\mathcal{N} = 4$ ten-dimensional solution. In contrast, the U(1)$_\eta$ isometry generated by $\partial_\eta$ is broken by the supergravity forms. Topologically, the metric \[58\] extends over $S^5 \times S^1_\eta$ \[23\], with the $S^5$ directions corresponding to $r$ and $S_i^2$, $i = 1, 2$.

Next, we plug the geometry \[58\] into the PDE \[7\]. It is natural to separate the eigenfunction $\mathcal{Y}$ following the SO(3)$_1$ $\times$ SO(3)$_2$ $\times$ U(1)$_\eta$ isometry as

$$
\mathcal{Y} = f(r) \mathcal{Y}(1) \mathcal{Y}(2) \mathcal{Y}_\eta,
$$

(59)
where $\mathcal{Y}_1$, $\mathcal{Y}_2$, and $\mathcal{Y}_\eta$ are the spherical harmonics on $S^2_1$, $S^2_2$, and $S^1_\eta$,

\[
\Box_{S^2_1} \mathcal{Y}_1 = -\ell_1(\ell_1 + 1) \mathcal{Y}_1, \quad \Box_{S^2_2} \mathcal{Y}_2 = -\ell_2(\ell_2 + 1) \mathcal{Y}_2, \quad \partial^2_{\eta} \mathcal{Y}_\eta = -\left(\frac{2\pi}{\ell^2}\right)^2 \mathcal{Y}_\eta, \quad (60)
\]

with

\[
\ell_1 = 0, 1, 2, \ldots, \quad \ell_2 = 0, 1, 2, \ldots, \quad j = 0, \pm 1, \pm 2, \ldots \quad (61)
\]

The separation of variables (59) turns the PDE (7) into the following ODE in $r$:

\[
(1-r^2)f'' + \left(\frac{2}{r} - 5r\right)f' + \left[2M^2 L^2 - \frac{4\pi^2}{T^2} j^2 - \frac{1 + 2r^2}{r^2} \ell_1(\ell_1 + 1) - \frac{3 - 2r^2}{1 - r^2} \ell_2(\ell_2 + 1)\right]f = 0, \quad (62)
\]

with a prime denoting derivative with respect to $r$. Finally, the change of variables

\[
r^2 = u, \quad f = u^{\ell_1/2}(1 - u)^{\ell_2/2} H(u), \quad (63)
\]

where now $0 \leq u \leq 1$, brings the ODE (62) into hypergeometric form (15) with

\[
a_\pm = \frac{1}{2}(\ell_1 + \ell_2 + 2) \pm \frac{1}{2} \sqrt{4 + 2M^2 L^2 - 2\ell_1(\ell_1 + 1) - 2\ell_2(\ell_2 + 1) - \frac{4\pi^2}{T^2} j^2}, \quad c = \ell_1 + \frac{3}{2}. \quad (64)
\]

The two linearly independent solutions to (15) are written in (17). In the present case, the second solution therein blows up at $u = 0$ since $\ell_1 \geq 0$ by (61), and is therefore excluded. The first solution in (17) is regular at $u = 0$ for all values of the parameters (64), and is also regular at $u = 1$ if $a_- = -h$ for a non-negative integer $h$. For this choice, the eigenfunction becomes a polynomial in $u$. Defining a new quantum number

\[
\ell = 2h + \ell_1 + \ell_2, \quad (65)
\]
the graviton masses then follow from (64) as

\[ L^2 M_{\ell,\ell_1,\ell_2,j}^2 = \frac{1}{2} \ell(\ell + 4) + \ell_1(\ell_1 + 1) + \ell_2(\ell_2 + 1) + \frac{2\pi^2}{T^2} j^2, \]  

(66)

with the quantum numbers now ranging as

\[ \ell = 0, 1, 2, \ldots, \quad \ell_1 = 0, 1, \ldots, \ell, \quad \ell_2 = 0, 1, \ldots, \ell, \quad j = 0, \pm 1, \pm 2, \ldots, \]  

(67)

by (61), (65) and the fact that \( h \geq 0 \). The number of KK gravitons with mass (66) is

\[ d_{\ell,\ell_1,\ell_2,j} = \begin{cases} 
(2\ell_1 + 1)(2\ell_2 + 1), & \text{if } j = 0 \\
2(2\ell_1 + 1)(2\ell_2 + 1), & \text{if } j \neq 0.
\end{cases} \]

(68)

To summarise, the KK gravitons about the \( \mathcal{N} = 4 \) SO(4)-invariant AdS\(_4\) S-fold solution of type IIB supergravity reported in [23] have masses (66), with quantum numbers ranging as in (67) and degeneracy (68). The corresponding eigenfunctions are given by (59) with (63) and the first hypergeometric function in (17) for \( H(u) \), which now becomes a polynomial. More precisely, the eigenfunctions (59) are products of \( S^5 \) spherical harmonics branched out in \( \text{SO}(4) = \text{SO}(3)_1 \times \text{SO}(3)_2 \) representations, and harmonics on \( S^1_\eta \). A few of these modes for low values of the quantum numbers have been tabulated in table 5. The table includes the dimension \( \Delta \) of the corresponding operators in the dual field theory [55], where

\[ M^2 L^2 = \Delta(\Delta - 3). \]

(69)

The field theory is defined on a stack of D3-branes wrapped on \( S^1_\eta \). The form of the dual single-trace spin-2 operators can be inferred from the supergravity eigenfunctions. They correspond to bound states of the energy-momentum tensor \( T_{\mu\nu} \), the six scalars \( X_{(1)}^a, X_{(2)}^a, a = 1, 2, 3 \), corresponding to the directions transverse to the D3-branes, and a complex coordinate \( Z \) on the D3-brane. In the table, all these have been promoted to superfields. The trace in the adjoint of the gauge group is understood.
The supergroup OSp(4|4) also admits massive short representations. Some gravitons
in the spectrum can be identified to belong to such representations, particularly $A_2$ in the notation of [61]. Spin-2 states in these short multiplets arise as $Q^4$ descendants of the superconformal primary therein, and thus saturate the unitarity bound

$$\Delta = \ell_1 + \ell_2 + 3.$$  \quad (70)

Via (69), these states have masses

$$L^2M^2 = (\ell_1 + \ell_2)(\ell_1 + \ell_2 + 3).$$  \quad (71)

In the spectrum (66), short states of this type do indeed arise for any period $T$ and all $\ell_1 = 0, 1, 2, \ldots$, whenever $j = 0$ and the other quantum numbers $\ell, \ell_1, \ell_2$ are related through $2\ell_1 = 2\ell_2 = \ell$ (so that $h = 0$ in (65) and $\ell$ is even). From (66) and (69), the masses and conformal dimensions of states with these quantum numbers are

$$L^2M_{\ell_1}^2 = 2\ell_1(2\ell_1 + 3), \quad \Delta_{\ell_1} = 2\ell_1 + 3, \quad \ell_1 = 0, 1, 2, \ldots,$$  \quad (72)

which are indeed of the form (71), (70) and thus short.

All other gravitons belong to long multiplets. It can be checked that, away from the shortening relations among the quantum numbers, their $\Delta$’s computed through (69) from (71) are always above the unitarity bound (70). We conclude, however, with the following tantalising observation. Consider the analytical continuation of the quantum number $j$ into $j^\prime = ij$, with $i^2 = -1$, as at the end of section , and fix the S-fold coordinate period to $T = 2\pi$. Then, it follows from (66), that states with $j^\prime = \pm 1$, and $\ell_2 = \ell_1 + 1$, and $\ell = 2\ell_1 + 1$ (so that $h = 0$ in (65) and $\ell$ is odd) are also short with conformal dimensions $\Delta_{\ell_1} = 2(\ell_1 + 2)$ and masses $L^2M_{\ell_1}^2 = 4\ell_1(\ell_1 + 2)$, for all $\ell_1 = 0, 1, 2, \ldots$
KALUZA-KLEIN GRAVITON MASS MATRIX

We now switch gears to obtain a covariant expression for the infinite-dimensional KK graviton mass matrix and its associated trace formulae.

The mass matrix

We would like to determine the KK graviton mass matrix corresponding to string/M-theory AdS$_4$ solutions that uplift, at least, on the relevant spheres from the SO(8) and ISO(7) gaugings. In appendix A of [28], an SO(7)-covariant mass matrix was derived for KK gravitons about solutions that uplift from the ISO(7) gauging. Here, we would like to extend those results into a mass matrix that is formally SL(8) covariant, in agreement with the formal, manifest covariance that the SO(8) and ISO(7) gauged supergravitites take on using the embedding tensor formalism [65] particularised to gaugings contained in SL(8) $\subset$ E$_{7(7)}$.

In order to do this, we start by assuming that the mass eigenfunctions,

$$\gamma^{A_1\ldots A_m} = \mu^{(A_1\ldots A_m)} , \quad m = 0, 1, 2, \ldots ,$$

are symmetric polynomials of the $\mathbb{R}^8$ coordinates $\mu^A$, $A = 1, \ldots, 8$. The latter are formally in the fundamental of SL(8) and constrained as

$$\theta_{AB} \mu^A \mu^B = 1 ,$$

with $\theta_{AB} = \delta_{AB}$ for the SO(8) gauging and $\theta_{AB} = \text{diag}(1,1,1,1,1,1,1,0)$ for the ISO(7) gauging. We then pose the KK graviton mass equation [7] for the consistent embedding metrics on the $S^7$ and $S^6$ as given in [20, 49]. With these assumptions, we follow similar steps to those in [28] to transform the PDE [7] into an algebraic eigenvalue problem by
reading off an infinite-dimensional block-diagonal mass matrix,

\[ M^2 = \text{diag}(M^2_{(0)}, M^2_{(1)}, M^2_{(2)}, \ldots, M^2_{(m)}, \ldots) . \]  

(75)

Now, each block is an SL(8)–covariant square matrix of size

\[ \text{dim } M^2_{(m)} \equiv [m, 0, 0, 0, 0, 0, 0]_{\text{SL}(8)} = \binom{m+7}{m} . \]  

(76)

These blocks are given explicitly by the following expressions: for \( m = 0 \),

\[ M^2_{(0)} = 0 , \]  

(77)

for \( m = 1 \),

\[ (M^2_{(1)})_{A}^{B} = -g^2 M^\text{MN}_{B}{}^{C} \Theta_{M}^{B} C \Theta_{N}^{C} A , \]  

(78)

for \( m = 2 \),

\[ (M^2_{(2)})_{A_1 A_2}^{B_1 B_2} = -2g^2 M^\text{MN}_{B_1}{}^{C} \Theta_{M}^{B_1} C \Theta_{N}^{C} (A_1 \delta_{A_2} | B_2) + \Theta_{M}^{B_1} (A_1 \Theta N B_2)_{A_2} \]  

(79)

and for \( m \geq 3 \),

\[ (M^2_{(m)})_{A_1 \ldots A_m}^{B_1 \ldots B_m} = -m g^2 M^\text{MN}_{B_1}{}^{C} \Theta_{M}^{B_1} C \Theta_{N}^{C} (A_1 \delta_{A_2} | B_2 \ldots \delta_{A_m}) | B_m \]  

\[ + (m - 1) \Theta_{M}^{B_1} (A_1 \Theta N B_2 A_2 \delta_{A_3} B_3 \ldots \delta_{A_m}) B_m \]  

(80)

Compared to [28], these expressions involve no trace removal within same-level indices. We have also restored the embedding tensors, \( \Theta^{M A B} \), with \( M = ([AB], [AB]) \) a fundamental \( E_{7(7)} \) index, and

\[ \Theta^{[AB]}_{[CD]} = 2 \delta^{C} [A] \theta_{B] D} , \quad \Theta^{[A][B]}_{C} = 2 \delta^{[A}_{D} \xi^{B]}_{C} , \]  

(81)

where \( \theta_{AB} \) was defined below (74), and \( \xi^{AB} = 0 \) for the SO(8) gauging while \( \xi^{AB} = \text{diag}(0, 0, 0, 0, 0, 0, 0, m/g) \) for ISO(7). Finally, \( g \) here and in (78)–(80) is the electric gauge
coupling, \( m \) the magnetic coupling, and \( \mathcal{M}^{MN} \) is the inverse \( \mathcal{N} = 8 \) scalar matrix. Like the bosonic mass matrices of \( D = 4 \mathcal{N} = 8 \) gauged supergravity (see [5]), the KK graviton mass matrices (77)–(80) are quadratic in the \( D = 4 \) embedding tensor.

Since we have refrained ourselves from removing traces on same-level indices of the KK graviton mass matrices (77)–(80), the latter are manifestly SL(8)–covariant. We can think of these as blocks in the diagonal of the infinite dimensional graviton mass matrix (75). The integer \( m \) can be thought of as an SL(8) KK level. Proceeding like this, though, the price one pays for the SO(8) gauging is that the spectrum at fixed SL(8) KK level \( m \geq 0 \) contains repeated physical modes: it includes modes of all SO(8) KK levels \( n \) (as defined below (32)) such that
\[
\begin{align*}
\sum_{s=0}^{[m/2]} \left[m - 2s, 0, 0, 0\right]_{SO(8)}.
\end{align*}
\] (82)

For the ISO(7) gauging, there is an even larger overcounting. Every SL(8) level \( m \geq 0 \) formally contains the SO(8) levels \( n \) specified in (82), and each of these, in turn, includes all SO(7) levels \( k = 0, 1, \ldots, n \) by (32). The repeated states can be projected out following (82) and (32), leaving only physical modes. We also remark that it is the full embedding tensor for the dyonic ISO(7) gauging, including the magnetic contributions from \( \xi^{AB} \), that enters (77)–(80) for this gauging.

Taking into account this overcounting, we have verified up to SL(8) KK level \( m = 3 \) that, particularising the KK graviton mass matrices (77)–(80) to the corresponding critical points of the SO(8) and ISO(7) gaugings, their eigenvalues correctly reproduce the spectra given in tables 2 and 3. Note that, in order to obtain matching, \( g^2 \) here has to be traded for \( L^2 \) there by making use of (95) with the appropriate scalar potential (97) or (98). See section for a discussion of the case corresponding to the \([SO(6) \times SO(1, 1)] \rtimes \mathbb{R}^{12} \) gauging.
Mass matrix traces

By (82), the first SL(8) KK level \( m = 1 \) contains only the first SO(8) KK level \( n = 1 \) (in our conventions, \( 8 \to 8_v \)). For the ISO(7) gauging, the first SL(8) KK level contains the SO(7) KK levels \( k = 0 \), which has zero mass by (77), and \( k = 1 \). At SL(8) KK level \( m = 1 \), the eigenvalues of the SL(8)-covariant mass matrix (78) thus reproduce the first KK-level eigenvalues with no overcounting for the SO(8) gauging. For the ISO(7) gauging, the first SL(8) level also reproduces the \( k = 1 \) eigenvalues, together with an extra zero eigenvalue corresponding to \( k = 0 \). The trace of the SL(8) level \( m = 1 \) mass matrix (78),

\[
\text{tr} \ M_2^{(1)} = -g^2 \mathcal{M}^{MN} \Theta_M^A \Theta_N^B \ ,
\]

must thus reproduce the KK level-one traces discussed in section , which indeed it does. Particularising (83) to each specific critical point with at least SU(3) symmetry of the SO(8) and ISO(7) gaugings, making use of the relevant embedding tensors, and again trading \( g^2 \) for \( L^2 \), all the r.h.s.’s of (52) and (53) with \( n = 1 \) and \( k = 1 \) are reproduced. For example, using the appropriate embedding tensors and vevs, we find that (83) evaluates to \( 56/3 \), both for the SU(3) \( \times \) U(1)\(_c\) point of the SO(8) gauging and for the SU(3) \( \times \) U(1)\(_v\) point of the ISO(7) gauging, once that \( g^2 \) is replaced with the relevant \( L^2 \). The trace relation (55) is a direct consequence of (83) and the overcounting feature mentioned in section .

In order to check consistency, it is also useful to evaluate (83) at generic invariant loci of the \( D = 4 \) \( \mathcal{N} = 8 \) scalar manifold. For example, on the SU(3)-invariant sector of the SO(8) gauging, (83) reduces to

\[
\text{tr} \ M_2^{(1)} = 2g^2 e^{-3\varphi - 2\phi} \left[ X^3 e^{2\phi} + 6 e^{2\varphi} X (Y^2 + Z^2) + 3(X + 4Y) e^{4\varphi + 2\phi} + 6X e^{2\varphi + 4\phi} \right] ,
\]

where the shorthand notations (100) have been employed. Further restricting (84) to the \( G_2 \)-invariant locus (101), we find

\[
\text{tr} \ M_2^{(1)} = 14g^2 (e^{-3\varphi} X^3 + 3 e^{\varphi} X) ,
\]
in agreement with (24). Alternatively, particularising \[\text{(84)}\] to the SU(4)\(_c\)-invariant sector through \[\text{(102)}\], we get
\[
\text{tr} \, M^2_{(1)} = 8g^2 \left( e^{3\phi} + 6e^{2\phi+\varphi} \right).
\] (86)
This expression can be retrieved from \[\text{(39)}\]. Similarly, for the dyonic ISO(7) gauging and SU(3)-invariant scalars, \[\text{(83)}\] reduces to
\[
\text{tr} \, M^2_{(1)} = 6g^2 \left[ e^\varphi(X + 4Y) + 2Xe^{2\phi-\varphi} \right],
\] (87)
in agreement with (2.34) of \[\text{(28)}\].

For the SO(8) and dyonic ISO(7) gaugings, only the electric embedding tensor \(\theta_{AB}\) actually participates in the trace formula \[\text{(83)}\], while its magnetic counterpart \(\xi^{AB}\) drops out. In order to see this, we expand \[\text{(83)}\] using the generic form \[\text{(81)}\] of the embedding tensor, to obtain
\[
\text{tr} \, M^2_{(1)} = -g^2 \left( \mathcal{M}^{ACDE} \theta_{CD} \theta_{EA} + \mathcal{M}_{ACDE} \xi^{CD} \xi^{EA} - 2\mathcal{M}^{CD} \theta_{DA} \xi^{FA} \right).
\] (88)
For the SO(8) gauging, the statement is immediate because \(\xi^{AB} = 0\), and only the first term is non-vanishing. For the dyonic ISO(7) gauging, the only relevant term in \[\text{(88)}\] is again the first one, because \(\mathcal{M}^{8888} = 0\) and \(\theta \xi = 0\).
DISCUSSION

We have verified that all solutions with the same residual supersymmetry, the same bosonic symmetry containing SU(3) and the same spectrum within the SO(8) [15], ISO(7) [16] and \((\text{SO}(6) \times \text{SO}(1,1)) \rtimes \mathbb{R}^{12}\) gaugings all have different spectra of KK gravitons. The universality of the lowest KK level spectra is thus lost at higher KK levels. However, following [28], we have found that this universality still persists up the different KK graviton towers in a weaker sense. Certain sums of KK graviton masses remain the same for all such solutions. We emphasise that this is the case if the gauged supergravity solutions with the same (super)symmetries also had the same spectrum at lowest KK level, namely, within their corresponding \(\mathcal{N} = 8\) supergravities.

For example, the \(\mathcal{N} = 0\) SU(3)-invariant points of dyonic ISO(7) [16] and \((\text{SO}(6) \times \text{SO}(1,1)) \rtimes \mathbb{R}^{12}\) supergravities [26] do not have the same spectrum within the \(D = 4\) supergravities. The relevant sums of KK graviton masses are also different. A similar observation holds for the U(1)-invariant \(\mathcal{N} = 1\) points of ISO(7) supergravity recently reported in [66]. Although they have the same residual symmetry, these points have different spectra within ISO(7) supergravity. Using the formula [83], we have verified at KK level one that the relevant KK graviton mass sums are also different for these two solutions. It is also worth stressing that the trace universality property works for solutions that check out the above requirements independently of how the common residual symmetry group is embedded into the corresponding gauge group and ultimately \(E_{7(7)}\). This was already noted in [28], where these observations were made for the \(\mathcal{N} = 2\) SU(3) \(\times\) U(1)-invariant points of SO(8) and ISO(7) supergravity. In the former case, the solution is embedded as SU(3) \(\times\) U(1)\(_c\) and in the latter as SU(3) \(\times\) U(1)\(_v\).

The relevant sums of KK graviton masses have been argued to be related to traces of the KK graviton mass matrix at fixed KK level. We have provided an SL(8)-covariant expression for the latter in section . This mass matrix has qualitatively the same form as the bosonic mass matrices of \(D = 4\) \(\mathcal{N} = 8\) gauged supergravity in that it is quadratic.
in the embedding tensor and depends on the $E_{7(7)}/SU(8)$ $\mathcal{N} = 8$ (inverse) scalar matrix $\mathcal{M}^{MN}$. The $SL(8)$-covariant KK graviton mass matrix \([77)-(80]\) reproduces the KK graviton spectra of tables 2 and 3 for the uplifts of SO(8) and ISO(7) gauged supergravity critical points, with redundancies introduced by the $SL(8)$ representations (versus branchings of SO(8) and SO(7) representations, as explained in section ). Interestingly, the mass matrix \((77)-(80)\) also reproduces the KK graviton spectrum of table 4 and equation (66) for type IIB S-folds with period $T = 2\pi$ that uplift from vacua of the $(SO(6) \times SO(1,1)) \rtimes \mathbb{R}^{12}$ gauging, *provided* the U(1)$_{\eta}$ quantum number $j$ is analytically continued as $j' = ij$, with $i^2 = -1$. The origin of this analytic continuation can be put down to the fact that the $SL(8)$-covariant graviton mass matrix formula \((77)-(80)\) actually sees the compactified U(1)$_{\eta}$ as the non-compact SO(1,1) factor in the $D = 4$ gauge group $(SO(6) \times SO(1,1)) \rtimes \mathbb{R}^{12}$. From a IIB perspective, this factor is associated to a hyperboloid uplift \([23]\). In any case, the (analytically continued) spectra of the type IIB S-folds can be also organised in $SL(8)$ KK levels $m = 0, 1, \ldots$ through the branching

\[ [m, 0, 0, 0, 0, 0]_{SL(8)} \xrightarrow{SO(6) \times SO(1,1)} \left[ \sum_{s=0}^{\lfloor m/2 \rfloor} \sum_{\ell=0}^{m-2s} \sum_{p=0}^{m-2s-\ell} \left[ \ell, 0, 0, m-2s-\ell-2p \right] \right]. \quad (89) \]

This approach contains redundant states that can be projected out as discussed below \([82]\). It would be interesting to determine if \((77)-(80)\) can be modified in such a way that the KK graviton spectra of vacua of the SO(8) and ISO(7) gauging are still obtained, and the physical spectra of the compactified S-folds is recovered as well.

The $SL(8)$-covariant KK graviton mass matrix should have an $E_{7(7)}$-covariant extension. From our analysis, it is not immediate to deduce what that extension should be. The trace formula \([83]\) does admit a naive straightforward $E_{7(7)}$-covariant extension, though:

\[ \text{tr} M^2_{(1)} = -g^2 \kappa_{\alpha\beta} \mathcal{M}^{MN} \Theta_M^\alpha \Theta_N^\beta, \quad (90) \]

with $\kappa_{\alpha\beta}$ the Killing-Cartan form of $E_{7(7)}$. This formula reduces to \((83)\) for gaugings
contained in SL(8). It would be interesting to test (90) for gaugings not contained in SL(8) that descend from higher dimensions. An example of such gaugings, which has Minkowski vacua though, is given in [67].
CONCLUSION

Developing an understanding of these gauged supergravities is worthwhile if it is possible to unify the standard model and general relativity in such a higher dimensional theory. While the universality of graviton mass spectra clearly fails upon uplift to higher dimensional theories, the existence of other relations that do remain universal provides some insight into the level of interconnection between these different gaugings of maximal $D = 4$ supergravity with origin in higher dimensional theories. We are still restricted here to looking at AdS$_4$ solutions with at least SU(3) symmetry, meaning this is still not a complete list of all possible (super)symmetries in maximal $D = 4$ supergravity. New directions are already being explored to approach this particular problem, such as the method presented subsequently by [68] using exceptional field theories following the submission of the research presented here. The study of supergravity is still rich with aspects yet to be explored or understood, and with that comes the hope that continued focus on these theories may ultimately lead to the development of a theory of everything.
REFERENCES


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Appendix A: Relevant geometry and algebra

The purpose of this appendix is to provide a brief review of some of the fundamental ideas from geometry and Lie groups used in this thesis, as well as some information on the Lie group $E_{7(7)}$, which features prominently in this thesis.

Geometry

The central object of study in differential geometry is the manifold. An $n$-dimensional manifold is an object which at every point there is some scale that the neighborhood about the point that looks like $\mathbb{R}^n$. In general relativity, one usually considers manifolds that are differentiable at each point, allowing calculus on the manifolds to be well-defined everywhere. In other words, the manifolds used in physics are objects or spaces that are smooth without sharp edges or points.

Geometric objects or spaces are those for which we have a notion of distance between any two points. One means of defining such a notion is with the line element $ds$, which describes the total infinitesimal length based on the infinitesimal displacement of each coordinate. Another common way to define distances on a manifold is with a metric $g_{mn}$, a symmetric, bilinear, non-degenerate tensor field which encodes the inner products between coordinates at each point. Either of these can be used to define the geometry of a manifold, and contain within them information on the symmetries of the manifold.

The simplest $n$-manifold is simply $\mathbb{R}^n$ itself, a maximally symmetric space with no curvature anywhere. The setting for special relativity, Minkowski space, is similar in that it has no curvature, but one of the four coordinates is time-like, which gives a line element of the form,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$  \hspace{1cm} (91)

However, since the presence of matter induces curvature of spacetime, it is imperative to look at line elements with curvature in general relativity. The next level of generalization is to consider spaces that are maximally symmetric with constant curvature, de Sitter (dS)
and Anti-de Sitter (AdS) space, which respectively have positive and negative constant curvature. An example for Anti-de Sitter space is a metric taking the form,

\[ ds^2 = \frac{1}{z^2}(-dt^2 + dx^2 + dy^2 + dz^2) . \]  

(92)

Much of the focus on AdS space in the past few decades can be attributed in large part to the development of the AdS/CFT correspondence, a duality between a theory with gravity and a conformal field theory without gravity, which has led to much renewed interest in string theory and supergravity [69].

In this thesis, however, the manifolds discussed are often not just spacetime geometries with curvature that extend to infinity. Rather, there are additional coordinates that describe an internal space, a space in which the coordinates are compact at each point, and is independent of any external coordinates. In other words, these internal coordinates are periodic, such as adding a phase at each point that does not depend on the position in the larger spacetime. Since these internal coordinates are independent of the external coordinates, it is useful to divide the total metric into external and internal parts [4]. In higher dimensional theories, the internal metric is where all of the symmetries of the theory are encoded in a geometric way.

**Lie Groups**

As mentioned in section 2.2, symmetries are best understood using group theory. A group is a set equipped with an associative operation such that combining two elements with this operation gives another element of the set. Any group must also have an identity, and every element must have an inverse that takes that element to the identity [70]. The example SO(2) used in section 2.2 is the group of $2 \times 2$ orthogonal matrices ($A^T I A = I$) with unit determinant, which forms a group under the operation of matrix multiplication.
A general element of this group can be written as a matrix of the form,

\[
A = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix},
\]

(93)
a matrix which rotates a 2D vector by an angle \(0 < \theta < 2\pi\), making \(\text{SO}(2)\) the group of rotations in \(\mathbb{R}^2\).

An object of great interest in physics is a group which is also a manifold, known as a Lie group \([71]\). Lie groups are useful because they allow for the consideration of symmetry groups as geometric objects, which can then be written in the metric of a manifold. The general linear group \(\text{GL}(n)\), the orthogonal group \(\text{O}(n)\), and the unitary group \(\text{U}(n)\) are all Lie groups which are used frequently in physics. The standard model, as a practical example, is a theory with local \(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)\) gauge symmetry.

Throughout this thesis, we refer to the representations of groups, so it is worthwhile to discuss this here briefly. A representation \(\rho\) of a group \(G\) on an \(n\)-dimensional vector space \(V\) is a mapping from the group \(G\) to the general linear group of the vector space \(\text{GL}(V)\), \(\rho : G \rightarrow \text{GL}(V)\) \([71]\). In the context of matrices, \(\text{GL}(V)\) consists of non-degenerate \(n \times n\) matrices, which is precisely the set of all invertible transformations that can act on a vector in \(V\). Thus, a representation can map elements of a group to invertible matrices that act on vectors in an appropriate vector space. The adjoint representation of a group is a particular representation that maps \(G\) to automorphisms on the tangent space of \(G\) evaluated at its identity, \(\text{Ad} : G \rightarrow \text{Aut}(T_eG)\) \([71]\). A proper description of the fundamental representation is more difficult to give here, but in practice it is associated with the smallest faithful representation of a group. More precisely, it can be described in terms of weights, a generalization of the concept of an eigenvalue. The fundamental representation is then an irreducible representation with a non-zero weight space where the highest weight is a fundamental weight. Further reading on this and a more in depth discussion on weights and fundamental weights can be found in \([71]\).

Of great focus in this thesis is the exceptional Lie group \(E_7\), with its associated
Lie algebra of the same name [3, 71]. While there is a complex form of $E_7$, we focus in particular on the split real form $E_7(7)$, which has 133 real dimensions. The fundamental representation of this group has 56 dimensions, which along with the size of the adjoint representation determines the size of the embedding tensor $\Theta_{M^\alpha}$, so the fundamental index $M$ ranges from 1...56 and the adjoint index $\alpha$ from 1...133.
Appendix B: SU(3) invariance in $D = 4$ 

$\mathcal{N} = 8$ gauged supergravity

We find it useful to collect here some facts about the SU(3)-invariant sector of the three different gaugings of $D = 4 \mathcal{N} = 8$ supergravity considered in the main text. The field content is of course the same for all the gaugings considered but the interactions differ.

The SU(3)-invariant sector contains three scalars, $\varphi, \phi, a$, and three pseudoscalars $\chi, \zeta, \tilde{\zeta}$. All these are coordinates on a submanifold

$$\frac{SU(1, 1)}{U(1)} \times \frac{SU(2, 1)}{SU(2) \times U(1)}$$

of $E_7(7)/SU(8)$, with the first factor parametrised by $(\varphi, \chi)$, and the second by $(\phi, a, \zeta, \tilde{\zeta})$.

The Lagrangian in this sector is

$$L = R \text{vol}_4 + \frac{3}{2} (d\varphi)^2 + \frac{3}{2} e^{2\varphi} (d\chi)^2 + 2(D\phi)^2 + \frac{1}{2} e^{4\phi} (Da + \frac{1}{2} (\zeta D\tilde{\zeta} - \tilde{\zeta} D\zeta))^2$$

$$+ \frac{1}{2} e^{2\phi} (D\zeta)^2 + \frac{1}{2} e^{2\phi} (D\tilde{\zeta})^2 + \frac{1}{2} I_{\Lambda\Sigma} H_{(2)}^\Lambda \wedge *H_{(2)}^\Sigma + \frac{1}{2} R_{\Lambda\Sigma} H_{(2)}^\Lambda \wedge H_{(2)}^\Sigma - V \text{vol}_4.$$ (95)

The scalar kinetic terms are given by a standard metric on (94). The precise form of the minimal $(D\phi, \text{etc}.)$ and non-minimal $(R_{\Lambda\Sigma}, I_{\Lambda\Sigma})$ couplings of the scalars to the vectors are not needed in this paper. We do need the expression of the scalar potential $V$, which fixes the radius $L$ of its AdS$_4$ vacua (for which $V_0 < 0$ at a critical point) as

$$L^2 = -\frac{6}{V_0}.$$ (96)

The potential is different for each gauging. For the SU(3)-invariant sector of the purely electric SO(8) gauging [15], the potential is [27], in the conventions of [24],

$$g^{-2}V = -12e^\varphi - 6e^{-2\phi - \varphi} XY (e^{4\phi} + Y^2 + Z^2) - 12e^\varphi (Y - 1)(1 + Y - \frac{3}{2} XY)$$

$$+ 6e^{-2\phi - \varphi} (Y - 1)(e^{4\phi} + Y^2 + Z^2) X^2 + e^{-3\varphi} \left[ \frac{1}{2} e^{-4\phi} + a^2 - 1 + \frac{1}{2} e^{4\phi} (1 + a^2)^2 \right]$$

$$- 24e^{-3\phi} (Y - 1)(e^{4\phi} + Y^2 + Z^2) X^2 - 12e^{-\varphi} (Y - 1)(e^{4\phi} + Y^2 + Z^2) X^2 + 12e^{-3\varphi} (Y - 1)(e^{4\phi} + Y^2 + Z^2) X^2$$

$$+ e^{-2\phi - \varphi} (Y - 1)(e^{4\phi} + Y^2 + Z^2) X^2 + e^{-\varphi} (Y - 1)(e^{4\phi} + Y^2 + Z^2) X^2$$

$$+ 24e^{-\varphi} (Y - 1)(e^{4\phi} + Y^2 + Z^2) X^2 + e^{-\varphi} (Y - 1)(e^{4\phi} + Y^2 + Z^2) X^2.$$ (97)
\[ + \frac{1}{2} e^{-\phi}(Y - 1)(1 + 2Z^2 - 2e^{4\phi} + Y(1 + 2e^{4\phi} + 2Z^2) + Y^2 + Y^2) \right] X^3. \]

For the dyonic ISO(7) gauging the, SU(3)-invariant potential reads

\[ V = \frac{1}{2} g^2 \left[ X^3 e^{4\phi - 3\varphi} + 12e^{2\phi - \varphi} \left( X^2(Y - 1) - XY \right) + 12e^\varphi \left( 3XY(Y - 1) - 2Y^2 \right) \right] \]
\[ - g m \chi e^{3\varphi + 2\phi} \left[ 6(Y - 1) + e^{2\phi - 2\varphi}(X - 1) \right] + \frac{1}{2} m^2 e^{3\varphi + 4\phi}. \]

Finally, the SU(3)-invariant potential for the [SO(6) × SO(1, 1)] × \( \mathbb{R}^{12} \) gauging is

\[ V = 6 g^2 e^\varphi \left[ 3XY(Y - 1) - 2Y^2 \right] + 6 g m \chi e^{3\varphi + 2\phi} (Y - 1) \left[ 1 - e^{-4\phi} (Y^2 + Z^2) \right] \]
\[ + \frac{1}{2} m^2 e^{3\varphi} \left[ e^{4\phi} + e^{-4\phi} (Y^2 + Z^2)^2 - 2 \left( Y^2 - 2Y + Z^2 \right) \right], \]

as follows from \[26\]. In \[(97)–(99)\], \( g \) and \( m \) are the electric and magnetic gauge couplings of the parent \( \mathcal{N} = 8 \) supergravities. For the latter two gaugings at hand, these can be set equal, \( m = g \), without loss of generality \[8\]. This is in fact what we have done, following \[23, 26\], to write the type IIB uplifts of sections and . We have also employed the shorthand notations \[16\]

\[ X \equiv 1 + e^{2\varphi} \chi^2, \quad Y \equiv 1 + \frac{1}{4} e^{2\phi} (\tilde{\zeta}^2 + \tilde{\zeta}^2), \quad Z \equiv e^{2\phi} a. \]

The AdS vacua of the SO(8), ISO(7) and [SO(6) × SO(1, 1)] × \( \mathbb{R}^{12} \) \( \mathcal{N} = 8 \) gaugings that preserve at least the SU(3) subgroup of those gauge groups were respectively investigated in \[27, 16, 26\]. In our conventions, these correspond to extrema of the scalar potentials \[(97)–(99)\]. The location of these vacua in scalar space, in the notation that we are using, can be respectively found in table 2 of \[24\], table 3 of \[16\] (with labels + there replaced with labels \( v \) here), and table \[6\] below.

For the SO(8) and ISO(7) gaugings, the \( G_2 \)-invariant scalar sector (employed in the former context in section of the main text) is reached from the SU(3)-invariant sector
Table 6: All critical loci of $\mathcal{N} = 8$ $\left[\text{SO}(6) \times \text{SO}(1,1)\right] \rtimes \mathbb{R}^{12}$-gauged supergravity with at least SU(3) invariance. All of these are AdS. For each point we give the residual supersymmetry $\mathcal{N}$ and bosonic symmetry $G_0$ within the full $\mathcal{N} = 8$ theory, their location in the parametrisation that we are using and the cosmological constant $V_0$ and the masses of the scalars in units of the AdS radius. The $\mathcal{N} = 0$ SO(6)$_v$ vacuum is the $\chi = a = 0$ point of the $\mathcal{N} = 0$ SU(3) critical locus.

through the identifications $[24, 16]$

$$\phi = \varphi, \quad \tilde{\zeta} = -2\chi, \quad a = \zeta = 0.$$  

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The SU(4)$_c$-invariant sector of the SO(8) gauging is retrieved via $[24]$

$$e^{-2\phi} = 1 - \frac{1}{4}(\zeta^2 + \tilde{\zeta}^2), \quad a = 0, \quad e^{-2\varphi} = 1 - \chi^2.$$  

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