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# Classification of Jacobian Elliptic Fibrations on a Special Family of K3 Surfaces of Picard Rank Sixteen

Thomas Hill
*Utah State University*

# CLASSIFICATION OF JACOBIAN ELLIPTIC FIBRATIONS ON A SPECIAL FAMILY OF K3 SURFACES OF PICARD RANK SIXTEEN

by

Thomas Hill

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Approved:

| | |
|---|---|
| Andreas Malmendier, Ph.D.<br>Major Professor | Ian Anderson, Ph.D.<br>Committee Member |
| Nathan Geer, Ph.D.<br>Committee Member | D. Richard Cutler, Ph.D.<br>Interim Vice Provost of Graduate Studies |

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# ABSTRACT

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by

Thomas Hill, Master of Science

Utah State University, 2020

Major Professor: Andreas Malmendier, Ph.D.
Department: Mathematics and Statistics

We study a special family of K3 surfaces polarized by the rank-sixteen lattice $\mathrm{N} = H \oplus E_7(-1) \oplus E_7(-1)$. Isomorphism classes of Jacobian elliptic fibrations on such a N-polarized K3 surface $\mathcal{X}$ are in one-to-one correspondence with primitive lattice embeddings $H \hookrightarrow \mathrm{N}$. Our lattice theoretic analysis proves there are exactly four (non isomorphic) primitive lattice embeddings. Therefore, $\mathcal{X}$ carries (up to automorphism) four Jacobian elliptic fibrations. We describe each of these fibrations via geometric pencils on the quartic surface defining $\mathcal{X}$, and construct explicit Weierstrass models for these fibrations. A coarse moduli space for N-polarized K3 surfaces is given by the modular four fold $\Gamma_T^+\backslash\mathbf{H}_2$, where the period domain $\mathbf{H}_2$ is a bounded symmetric domain of type $I_{2,2}$, and $\Gamma_T^+$ is a discrete arithmetic group acting on $\mathbf{H}_2$. Based on an exceptional analytic equivalence between bounded symmetric domains of type $I_{2,2}$ and $IV_4$, we describe the Weierstrass coefficients for these four special fibrations in terms of modular forms relative to $\Gamma_T^+$.

We also apply our construction to string theory. In fact, our construction provides a geometric interpretation for the F-theory/heterotic string duality in eight dimensions with two non-trivial Wilson lines. The moduli space of the family N-polarized K3 surfaces provides a new example where the partial higgsing of the heterotic gauge algebra $\mathfrak{g} = \mathfrak{e}_8 \oplus \mathfrak{e}_8$

or $\mathfrak{g} = \mathfrak{so}(32)$ for the associated low energy effective eight-dimensional supergravity theory has, in each case, inequivalent Coulomb branches and no charged matter fields for the corresponding F-theory model.

(69 pages)

# PUBLIC ABSTRACT

Classification of Jacobian Elliptic Fibrations on a Special Family of K3 Surfaces of Picard Rank Sixteen

Thomas Hill

K3 surfaces are an important tool used to understand the symmetries in physics that link different string theories, called string dualities. For example, heterotic string theory compactified on an elliptic curve describes a theory physically equivalent to (dual to) F-theory compactified on a K3 surface. In fact, M-theory, the type IIA string, the type IIB string, the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string, and the $E_8 \times E_8$ heterotic string are all related by compactification on Calabi-Yau manifolds.

We study a a special family of K3 surfaces, namely a family of rank sixteen K3 surfaces polarized by the lattice $H \oplus E_7(-1) \oplus E_7(-1)$. A generic member of this family is a K3 surface defined by resolving the singularities of a specific quartic surface. Intersecting this quartic with a pencil of planes containing a particular line or conic corresponds with a Jacobian elliptic fibration on the resulting K3 surface. We show that a generic member of this family of K3 surfaces admits exactly four inequivalent Jacobian elliptic fibrations; i.e., there are four non-isomorphic ways a pencil of planes containing a line or conic can intersect the quartic surface defining a member of this special family. We construct explicit Weierstrass models for these Jacobian elliptic fibration whose coefficients are modular forms on a suitable bounded symmetric domain of type $IV$. Finally, we explain how this construction provides a geometric interpretation for the F-theory/heterotic string duality in eight dimensions with two Wilson lines.

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Thomas Hill

# CONTENTS

Page

# LIST OF TABLES

# LIST OF FIGURES

Figure Page

# CHAPTER 1

## Introduction

K3 surfaces are an important tool used to understand the symmetries in physics that link different string theories, called string dualities. For example heterotic string theory compactified on an elliptic curve describes a theory physically equivalent to (dual to) F-theory compactified on a K3 surface. In fact, M-theory, the type IIA string, the type IIB string, the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string, and the $E_8 \times E_8$ heterotic string are all related by compactification on Calabi-Yau manifolds.

We study a special family of rank sixteen K3 surfaces polarized by the lattice $H \oplus E_7(-1) \oplus E_7(-1)$, first introduced by Clingher and Doran in [8] and subsequently studied further by Clinger, Malmendier and Shaska [5]. A generic member of this family is a K3 surface defined by resolving a specific quartic surface. Intersecting this quartic with a pencil of planes corresponds with a Jacobian elliptic fibration on the resulting K3 surface. We show that a generic member of this family of K3 surfaces admits exactly four inequivalent Jacobian elliptic fibrations; i.e., there are four non-isomorphic ways a pencil of planes containing a line or conic can intersect the quartic surface defining a member of this special family. We construct explicit Weierstrass models for these Jacobian elliptic fibration with coefficients that are modular forms on a bounded symmetric domain of type $IV$. This construction provides a geometric interpretation for the F-theory/heterotic string duality in eight dimensions with two Wilson lines.

In Chapter 2, we present some foundational material. First we discuss some of the defining properties of a K3 surface and then explain how the lattice structure of the second integral cohomology group of a K3 surface can be used to distinguish algebraic K3 surfaces and describe their moduli spaces. Finally we discuss Jacobian elliptic fibrations on a K3 surface.

Chapter 3 consists of two papers I coauthored with Dr. Adrian Clingher and Dr.

Andreas Malmendier. In this chapter we study a special family of lattice polarized K3 surfaces of Picard rank sixteen. We classify all Jacobian elliptic fibrations on a generic member of this family and construct explicit Weierstrass models for each of these. We describe the coefficients of these models in terms of modular forms on a suitable bounded symmetric domain of type $IV$. Finally, we describe how these models provide a geometric interpretation for the F-theory/heterotic string duality in eight dimensions with two Wilson lines.

In Chapter 4, we give an outline for a future project studying the natural generalization of this situation in Picard rank fourteen.

CHAPTER 2

# Preliminaries

*"In the second part of my report, we deal with the Kähler varieties known as K3, named in honor of Kummer, Kähler, Kodaira and of the beautiful mountain K2 in Kashmir."*

– André Weil, describing the reason for the name K3 surface

## 2.1 K3 surfaces

**Definition 2.1.1.** *A K3 surface is a compact, complex smooth surface that is simply connected and has trivial canonical bundle.*

The top exterior power of the cotangent bundle is the canonical bundle. For a K3 surface this bundle required to be trivial. Some of the defining properties of K3 surfaces are listed in the proposition below.

**Proposition 2.1.2.**

*(i) All K3 surfaces are Kähler manifolds and admit a Ricci-flat metric.*

*(ii) All K3 surfaces are diffeomorphic.*

*(iii) The Hodge diamond of a K3 surface is given by*

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

*the Betti numbers are* $B_0 = B_4 = 1$, $B_1 = B_3 = 0$, $B_2 = 22$, *and the Euler characteristic is 24.*

(i) A *complex manifold* is a manifold equipped with an integrable almost complex structure $J$. That is, an endomorphism $J_p : T_pX \to T_pX$ such that $J_p^2 = -\mathbb{1}$. If the Nijenhuis tensor vanishes $N_J = 0$ (if certain sub-bundles of the complexified cotangent bundle are closed under Lie brackets of vector fields) , then $J$ is integrable. A *symplectic manifold* is a smooth manifold $X$, equipped with a closed non-degenerate differential 2-form $\omega$. A *Kähler manifold* is a symplectic manifold $(X, \omega)$ equipped with an integrable almost complex structure $J$ that is compatible with the symplectic form $\omega$ in that the bilinear form $g_p(u, v) = \omega(u, Jv)$ on $T_pX$ is symmetric and positive definite for each $p \in X$. The proof that all K3 surfaces are Kähler manifolds is due to Siu [55]. That every K3 surface admits a Ricci flat metric was famously conjectured by Calabi and proven by Yau [62].

(ii) The proof that all K3 surfaces are diffeomorphic is due to Kodaira [26].

(iii) *Hodge numbers* are defined as $h^{p,q} = \dim H^{p,q}(X)$, where $H^{p,q}(X)$ is the $(p, q)$-th Hodge cohomology group. We have symmetries $h^{p,q} = h^{q,p}$ (complex conjugation), and $h^{p,q} = h^{n-p,n-q}$ (Serre duality) where $n = \dim X$. There is one connected component so $h^{0,0} = 0$. Since $X$ is simply connected it follows that $h^{1,0} = 0$. Since the canonical bundle is trivial $h^{2,0} = 1$ (this means $X$ admits a non-vanishing holomorphic 2-form $dz \wedge dw$). The Euler characteristic $\chi$ is a diffeomorphism invariant, so it suffices to compute this for a particular K3 surface (say for the K3 surface in the example below). It turns out that $\chi = 24$. Because the alternating sum of the rows of the Hodge diamond is the Euler characteristic, it turns out that $h^{1,1} = 20$.

**Example 2.1.3.** Let $(a, b, c, d) \in \mathbb{C}^4$ be parameters of a quartic surface $Q_1 \subset \mathbb{P}^3$ given by

$$\begin{aligned}\mathbf{X}^4 + \mathbf{Y}^4 + \mathbf{Z}^4 + \mathbf{W}^4 + a(\mathbf{X}^2\mathbf{W}^2 + \mathbf{Y}^2\mathbf{Z}^2) + b(\mathbf{Y}^2\mathbf{W}^2 + \mathbf{Z}^2\mathbf{X}^2)&\\ +c(\mathbf{X}^2\mathbf{Y}^2 + \mathbf{Z}^2\mathbf{W}^2) + d\mathbf{XYZW} = 0.&\end{aligned}$$

If $4 - a^2 - b^2 - c^2 + abc + d^2 = 0$, the quartic defines a nodal surface with 16 double points. Given a double point $p = [\mathbf{X}_0 : \mathbf{Y}_0 : \mathbf{Z}_0 : \mathbf{W}_0]$, then one can list all double points as below:

$$\begin{array}{cccc}
[\mathbf{X}_0 : \mathbf{Y}_0 : \mathbf{Z}_0 : \mathbf{W}_0] & [-\mathbf{X}_0 : -\mathbf{Y}_0 : \mathbf{Z}_0 : \mathbf{W}_0] & [-\mathbf{X}_0 : \mathbf{Y}_0 : -\mathbf{Z}_0 : \mathbf{W}_0] & [-\mathbf{X}_0 : \mathbf{Y}_0 : \mathbf{Z}_0 : -\mathbf{W}_0] \\
[\mathbf{Y}_0 : \mathbf{X}_0 : \mathbf{W}_0 : \mathbf{Z}_0] & [-\mathbf{Y}_0 : -\mathbf{X}_0 : \mathbf{W}_0 : \mathbf{Z}_0] & [-\mathbf{Y}_0 : \mathbf{X}_0 : -\mathbf{W}_0 : \mathbf{Z}_0] & [-\mathbf{Y}_0 : \mathbf{X}_0 : \mathbf{W}_0 : -\mathbf{Z}_0] \\
[\mathbf{Z}_0 : \mathbf{W}_0 : \mathbf{X}_0 : \mathbf{Y}_0] & [-\mathbf{Z}_0 : -\mathbf{W}_0 : \mathbf{X}_0 : \mathbf{Y}_0] & [-\mathbf{Z}_0 : \mathbf{W}_0 : -\mathbf{X}_0 : \mathbf{Y}_0] & [-\mathbf{Z}_0 : \mathbf{W}_0 : \mathbf{X}_0 : -\mathbf{Y}_0] \\
[\mathbf{W}_0 : \mathbf{Z}_0 : \mathbf{Y}_0 : \mathbf{X}_0] & [-\mathbf{W}_0 : -\mathbf{Z}_0 : \mathbf{Y}_0 : \mathbf{X}_0] & [-\mathbf{W}_0 : \mathbf{Z}_0 : -\mathbf{Y}_0 : \mathbf{X}_0] & [-\mathbf{W}_0 : \mathbf{Z}_0 : \mathbf{Y}_0 : -\mathbf{X}_0]
\end{array}$$

Note that every smooth projective quartic surface $S \subset \mathbb{P}^3$ defines a K3 surface. To see why, consider a surface $S \subset \mathbb{P}^3$ a smooth surface of degree $d$. The canonical bundle of $\mathbb{P}^3$ is $-4H$, where $H$ is the hyperplane class. By the adjunction formula, the restriction of $(d-4)H$ to $S$ is equal to the canonical bundle of $X$. In the case that $d = 4$ (that is, when $S$ is a smooth projective quartic surface), the restriction of $(4-4)H = 0$ to $S$ is equal to the canonical bundle of $X$, hence the canonical bundle is trivial. Thus in this case $S$ is a K3 surface.

Returning to our example, it follows that the minimal resolution of the surface $Q_1$ is a K3 surface known as a *Kummer surface*. Kummer surfaces are an important example of K3 surface which are algebraic and have a high amount of symmetry. We see this in the example above: it is given by an algebraic equation that is invariant under changing signs of two coordinates, and permutation of $(\mathbf{XYZW})$ to $(\mathbf{YXWZ}), (\mathbf{ZWXY}), (\mathbf{WZYX})$. Applying these changes of sign to any pair of coordinates or permuting the variables as described has the effect of permuting the order of the singular points listed above.

Let $p_0$ be an ordinary double point on the quartic surface $Q_1 \subset \mathbb{P}^3$ (one of those listed above). A projective line through the point $p_0$ intersects the quartic $Q_1$ with a multiplicity of two, thus it will meet the $Q_1$ in just two other points. By identifying the projective lines through the point $p_0$ in $\mathbb{P}^3$ with $\mathbb{P}^2$, we obtain a double cover of $\mathbb{P}^2$, given by mapping any point $p_i \neq p_0$ to the line containing $p_0$ and $p_i$, and any line in the tangent cone of $p_0$ in $Q_1$ mapped to itself. The ramification locus of this double cover is a plane sextic curve in which the nodes of $Q_1$ are mapped to the nodes of the sextic (except $p_0$). The maximal number nodes on a sextic curve occurs when the curve is a union of six lines, (this follows from the genus-degree formula). In this case, we obtain the remaining nodes, $p_i$, apart from $p_0$. Because $p_0$ is a simple node, the tangent cone at this point is mapped to a conic under this double cover. The sextic curve (the union of six lines) is everywhere tangent to this conic. To summarize, the ramification locus of the double cover of $Q_1$ to $\mathbb{P}^2$ is a sextic curve, specifically the union of six lines all tangent to a particular conic.

## 2.2 The K3 lattice

There is a lattice structure on the endowed on the cohomology of a K3 surface. In this section we will discuss this lattice structure, following the exposition outlined in [63]. The following definition reviews some basic elements of lattice theory.

**Definition 2.2.1.** *A rank $n$* **lattice** *is a free $\mathbb{Z}$-module $L$ of finite rank $n$ equipped with an integer valued symmetric bilinear form*

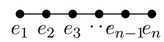
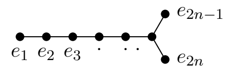
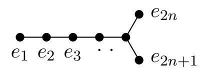
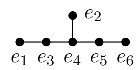
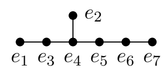
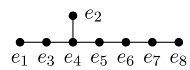
$$\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}.$$

- *A module isomorphism preserving the bilinear form is called an* **isometry.**
- *A lattice is* **even** *if $\langle x, x\rangle = 0 \pmod 2$ for all $x \in L$, otherwise it is odd.*
- *The* **discriminant** *$d(L)$ is the determinant of a matrix associated to $\langle\ ,\ \rangle$.*
- *A lattice $L$ is said to be* **unimodular** *if $d(L) = \pm 1$, and* **indefinite** *if the associated matrix is indefinite.*
- *If $L$ is non-degenerate (if $d(L) \neq 0$) then the* **signature** *of $L$ is a pair $(r_+, r_-)$ where $r_\pm$ is the multiplicity of the the eigenvalue $\pm 1$ for the quadratic form associated with $\langle\ ,\ \rangle$ on $L \otimes \mathbb{R}$.*
- *$L$ is called* **positive (negative) definite** *if the quadratic form takes only positive (negative) values.*
- *An* **embedding** *of lattices $j\colon M \hookrightarrow L$ is a $\mathbb{Z}$-module homomorphism which preserves the bilinear forms. If the quotient $L/M$ is torsion free, we say that the embedding is* **primitive**.

**Example 2.2.2.**

(a) The hyperbolic lattice $H$ is a lattice of rank two with bilinear form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $H$ is a unimodular, even lattice with signature $(1, 1)$.

(b) The lattices $A_n$, $D_n$, $E_6$, $E_7$, and $E_8$ are the positive definite lattices associated to the Dynkin diagrams in the table below.

| Lattice | rank | Dynkin diagram |
|---|---|---|
| $A_n$ | $n$ |  |
| $D_{2n}$ | $2n$ |  |
| $D_{2n+1}$ | $2n+1$ |  |
| $E_6$ | 6 |  |
| $E_7$ | 7 |  |
| $E_8$ | 8 |  |

The bilinear form for these lattices is defined on the vertices $e_i$ by the rule

$$\langle e_i, e_j \rangle = \begin{cases} 2, & \text{if } e_i = e_j \\ -1 & \text{if } e_i \neq e_j \text{ and } e_i \text{ and } e_j \text{ are joined by an edge} \\ 0 & \text{otherwise.} \end{cases}$$

The lattice $E_8$ is the unique positive definite, even, unimodular lattice of rank 8. This lattice is significant in the classification theorem of all indefinite, even, unimodular lattices given below.

**Theorem 2.2.3** (Milnor [38])**.** *Let $L$ be an indefinite, even, unimodular lattice with signature $(r_+, r_-)$. Then*

*(i)* $r_+ - r_- = 0 (\text{mod } 8)$,

*(ii)* $L \simeq H^n \oplus (E_8(-1))^m$, *where* $n = \min\{r_+, r_-\}$ *and* $m = |r_+ - r_-|/8$.

Now, we will discuss the lattice structure on the second integral cohomology group of a K3 surface, $X$. The second integral cohomology group $H^2(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22 [63]. Equipped with the intersection pairing induced by the cup product, $H^2(X, \mathbb{Z})$ is a lattice with the properties:

(a) unimodular (by Poincaré duality),

(b) even (by Wu's formula),

(c) indefinite and,

(d) has signature $(3, 19)$ (by Hodge index theorem).

By Theorem 2.2.3, there is a unique lattice with these properties, namely

$$\Lambda_{K3} := H^3 \oplus (E_8(-1))^2.$$

$\Lambda_{K3}$ is called the **K3 lattice**. When $X$ is an algebraic K3 surface, there is an isometry $\varphi \colon H^2(X, \mathbb{Z}) \simeq \Lambda_{K3}$ called a *marking*. The pairing $(X, \varphi)$ is called a *marked K3 surface.*

**Definition 2.2.4.** *Let $X$ be an algebraic K3 surface. Let $\omega_X$ be a nowhere vanishing holomorphic 2-form on $X$.*

*(i) The* **Nerón-Severi lattice** $\mathrm{NS}(X)$ *is the sublattice of $H^2(X, \mathbb{Z})$ given by*

$$\mathrm{NS}(X) = \{x \in H^2(X, \mathbb{Z}) \mid \langle \omega_X, x \rangle = 0\}.$$

*This is the sublattice of divisors in $H^2(X, \mathbb{Z})$ modulo algebraic equivalence. Intuitively, this sublattice corresponds to the cohomology classes of $X$ given by an algebraic equation.*

*(ii) The* **transcendental lattice** $T(X)$ *is the orthogonal complement of* $\mathrm{NS}(X)$ *in $H^2(X, \mathbb{Z})$, that is*

$$T(X) = \{x \in H^2(X, \mathbb{Z}) \mid \langle x, y \rangle = 0, \, \forall y \in \mathrm{NS}(X)\}.$$

The following proposition gives some of the defining properties of $\mathrm{NS}(X)$ and $T(X)$.

**Proposition 2.2.5** ( [63]).

*(i)* $\mathrm{NS}(X)$ *and* $T(X)$ *are primitive sublattices of $H^2(X, \mathbb{Z})$, and are independent of the choice of $\omega_X$.*

*(ii)* $\mathrm{NS}(X)$ *is a lattice of rank $\rho(X)$ and signature $(1, \rho(X)-1)$, where $1 \leq \rho(X) \leq 20$ is the Picard rank of $X$. $T(X)$ is a lattice of rank $22-\rho(X)$ and signature $(2, 19-(\rho(X)-1))$.*

*(iii)* $T(X) = \mathrm{NS}(X)^{\perp}$ *in* $H^2(X,\mathbb{Z})$ *with respect to the bilinear form* $\langle\ ,\ \rangle$.

**Example 2.2.6.** Let us return to the example of the K3 surface $X$ obtained as the minimal resolution of the quartic $Q_1$ given in Example 2.1.3. For generic parameters $(a,b,c,d) \in \mathbb{C}^4$

- $\mathrm{NS}(X) = \Lambda_{(16,6)}$ (here $\Lambda_{(16,6)}$ is the so called Kummer lattice),
- $T(X) = H(2) \oplus H(2) \oplus \langle -4 \rangle$.

So we see that the K3 surface $X$ is this case is of Picard rank 17.

**Example 2.2.7.** Let $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \in \mathbb{C}^6$, $(\gamma,\delta) \neq (0,0)$ and $(\varepsilon,\zeta) \neq (0,0)$ (this condition on the parameters ensures that the resulting singularities are rational double points). The minimal resolution of the projective quartic surface $Q_2 \subset \mathbb{P}^3$:

$$\mathbf{Y}^2\mathbf{Z}\mathbf{W} - 4\mathbf{X}^3\mathbf{Z} + 3\alpha\mathbf{X}\mathbf{Z}\mathbf{W}^2 + \beta\mathbf{Z}\mathbf{W}^3 + \gamma\mathbf{X}\mathbf{Z}^2\mathbf{W} - \frac{1}{2}\left(\delta\mathbf{Z}^2\mathbf{W}^2 + \zeta\mathbf{W}^4\right) + \varepsilon\mathbf{X}\mathbf{W}^3 = 0$$

is a K3 surface $\mathcal{X}$ of Picard rank 16. For generic parameters $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$:

- $\mathrm{NS}(\mathcal{X}) = H \oplus E_7(-1) \oplus E_7(-1)$,
- $T(\mathcal{X}) = H \oplus H \oplus \langle -1 \rangle \oplus \langle -1 \rangle$.

This example will be an important object of study in Chapter 3.

In both examples above, the Nerón-Severi lattices were listed for *generic* parameters. It is possible that for certain choices of the parameters that the Nerón-Severi lattice jumps up in rank. One would like to find a lattice which can always be injected into the Nerón-Severi lattice regardless of conditions on the parameters. This is one motivation for studying lattice polarized K3 surfaces.

## 2.3 Lattice polarized K3 surfaces

In this section we review the definition of a lattice polarized K3 surface, as it is presented in [13, 15, 63]. Let $H^{1,1}(X)_{\mathbb{R}} = H^{1,1}(X) \cap H^2(X,\mathbb{R})$. The subspace $H^{1,1}(X)_{\mathbb{R}}$ has signature $(1,19)$ and the cone $\left\{x \in H^{1,1}(X)_{\mathbb{R}} \mid \langle x,x \rangle > 0\right\}$ consists of two connected components. Let $V(X)^+$ denote the connected component that contains the class of $(1,1)$-forms associated to Kähler metrics. Let $\Delta(X) := \{\delta \in \mathrm{NS}(X) \mid \langle \delta, \delta \rangle = -2\}$ denote the set of roots of $\mathrm{NS}(X)$. By the Riemann-Roch theorem,

$$\Delta(X) = \Delta^+(X) \coprod -\Delta^+(X),$$

where $\Delta^+(X)$ is the set of effective divisor classes. Let,

$$C^+(X) := \left\{ v \in V^+(X) \mid \langle x, \delta \rangle > 0, \, \forall \delta \in \Delta^+(X) \right\}$$

denote the **Kähler cone** of $X$ and $C(X)$ denote its closure in $H^{1,1}(X)_{\mathbb{R}}$. The elements of $\mathrm{NS}^+(X) := C(X) \cap H^2(X, \mathbb{Z})$, are **pseudo-ample divisor classes**; that is, numerically effective divisor classes of positive self intersection. The elements of $\mathrm{NS}^{++} := C^+(X) \cap H^2(X, \mathbb{Z})$ are **ample divisor classes**.

**Definition 2.3.1.** *Let $X$ be a smooth complex algebraic K3 surface. Let $N$ be a an even, non-degenerate lattice with signature $(1, r)$ with $0 \leq r \leq 19$. An $N$-**polarized pseudo-ample K3 surface** is a pair $(X, i)$, where $i : N \hookrightarrow \mathrm{NS}(X)$ is a primitive lattice embedding, the image of which contains a pseudo-ample divisor class.*

*Two $N$-polarized K3 surfaces $(X, i)$ and $(X', i')$ are said to be isomorphic if there exists an analytic isomorphism $\alpha \colon X \to X'$, such that $\alpha^* \circ i' = i$, where $\alpha^*$ is the appropriate morphism at cohomology level.*

*If $N$ is a sublattice of $\Lambda_{K3}$, a* **marked** *$N$-polarized K3 surface is a triple $(X, \varphi, i_\varphi)$, where $\varphi : H^2(X, \mathbb{Z}) \to \Lambda_{K3}$ is a lattice isometry and $j_\varphi := \varphi^{-1}_{|N} \colon N \to \mathrm{NS}(X)$ is a primitive embedding.*

**Example 2.3.2.** Returning to Example 2.2.7 in which we investigated the K3 surface $\mathcal{X}$ obtained as minimal resolution of the projective quartic surface $Q_2 \subset \mathbb{P}^3$. Recall that generically $\mathrm{NS}(\mathcal{X}) = H \oplus E_7(-1) \oplus E_7(-1)$. Thus $\mathcal{X}$ is polarized by $N = H \oplus E_7(-1) \oplus E_7(-1)$.

The moduli spaces of lattice polarized K3 surfaces were described by Dolgachev [13]. Let $X$ be a K3 surface and let $\omega_X = dw \wedge dz$ be a holomorphic 2-form spanning $H^{2,0}(X)$. Fix a $\mathbb{Z}$-basis $\{\gamma_1, \gamma_2, \ldots, \gamma_{22}\}$ for $H_2(X, \mathbb{Z})$.

If the two-cycle $\gamma_i \in \mathrm{NS}(X)$, then $\gamma_i$ corresponds to a divisor defined by the locus $f(z, w) = 0$, where $f$ is some nonzero meromorphic function. Taking the total differential of $f$, we see that $f_z dz + f_w dw = 0$, so $f_z dz \wedge dw = 0$. Assuming $f_z \not\equiv 0$, we see that restricted to the 2-cycle $\gamma_i$, $dz \wedge dw = 0$. So $\int_{\gamma_i} \omega_X = 0$ for $\gamma_i \in \mathrm{NS}(X)$. If $\gamma_i \in T(X)$, then $\int_{\gamma_i} \omega_X$ is generically nonzero.

Up to reindexing, let $\left\{\gamma_1, \gamma_2, \ldots, \gamma_{22-\rho(X)}\right\}$ be a generating set for $T(X)$.

**Definition 2.3.3.** *The* period map *of $X$ is*

$$p(X) := \left[\int_{\gamma_1} \omega_X \;:\; \int_{\gamma_2} \omega_X \;:\; \ldots \;:\; \int_{\gamma_{22-\rho(X)}} \omega_X\right] \in \mathbb{P}^{22-\rho(X)-1}.$$

*The* period domain, $\Omega$, *is the set of images of the period map.*

The period map is surjective, i.e., every point of $\Omega$ occurs as the period of a marked K3 surface. The moduli behavior of K3 surfaces is closely related to the period domain: If $X$ is a family of K3 surfaces parameterized by a manifold $\mathcal{M}$, then there is a local isomorphism $\mathcal{M} \to \Omega$.

**Theorem 2.3.4** ( Dolgachev [13], [15])**.** *Let $N \subseteq \Lambda_{K3}$ be a primitive sublattice with signature $(1, r)$. The moduli space $\mathfrak{M}_N$ of $N$-polarized K3 surface is the quotient of*

$$\Omega(N) = \left\{\omega \in \mathbb{P}(N^{\perp_{\Lambda_{K3}}}) \;\middle|\; \omega \wedge \omega = 0 \text{ and } \omega \wedge \bar{\omega} > 0\right\}$$

*by an arithmetic subgroup the group of isometries of $N$. The dimension of this space is $20 - (r+1) = 19 - r$.*

### 2.4 Jacobian elliptic fibrations on a K3 surface

Explicitly determining a lattice polarization for a given K3 surface in general proves to be a difficult computation. Studying Jacobian elliptic fibrations on a K3 surface proves to be a useful way to rephrase the problem. In this section we will review the idea of a Jacobian elliptic fibration on a K3 surface, following the discussion outlined in [7]. Let $X$ be an algebraic K3 surface throughout this section.

**Definition 2.4.1.** *A* **Jacobian elliptic fibration** *(or elliptic fibration with section) on $X$ is a pair $(\pi, \sigma)$ where $\pi \colon X \to \mathbb{P}^1$ is a proper map of analytic spaces whose generic fiber is a smooth genus-one curve, and is a section $\sigma \colon \mathbb{P}^1 \to X$ in the fibration $\pi$.*

If $\sigma'$ is another section of the Jacobian fibration $(\pi, \sigma)$, then there is automorphism of $X$ which preserves $\pi$ and maps $\sigma$ to $\sigma'$ [14]. In this way, the set of sections can be identified with the group of automorphism of $X$ preserving $\pi$. Thus the set of sections form a group, known as the **Mordell-Weil group** of the Jacobian elliptic fibration, denoted by $\mathrm{MW}(\pi, \sigma)$.

When $X$ carries a Jacobian elliptic fibration, the classes of elliptic fiber and a global section generate a hyperbolic lattice $H$ inside the Néron-Severi lattice. In fact, the converse holds: if $H \hookrightarrow \mathrm{NS}(X)$ is a primitive lattice embedding whose image contains a pseudo-ample divisor class, there is a Jacobian elliptic fibration on $X$ whose classes of fiber and section span $H$ [29].

**Proposition 2.4.2** (Kondo [7], [29] )**.**

*(i) A Jacobian elliptic fibration is associated with a primitive lattice embedding $i \colon H \hookrightarrow \mathrm{NS}(X)$ if and only if $i(H)$ contains a pseudo-ample divisor class.*

*(ii) For any primitive lattice embedding $i \colon H \hookrightarrow \mathrm{NS}(X)$, there is an $\alpha \in \Gamma_X$, (where $\Gamma_X$ is the group of isometries of $H^2(X, \mathbb{Z})$ that preserve the Hodge decomposition), such that $\alpha \circ j(H)$ contains a pseudo-ample class.*

*(iii) There is a one-to-one correspondence between isomorphism classes of Jacobian elliptic fibrations on $X$ and isomorphism classes of primitive lattice embeddings $H \hookrightarrow \mathrm{NS}(X)$ modulo the action of isometries of $H^2(X, \mathbb{Z})$ preserving the Hodge decomposition:*

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{Jacobian elliptic fibrations on } X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{primitive lattice embeddings} \\ H \hookrightarrow \mathrm{NS}(X) \end{array} \right\} \Big/ \Gamma_X$$

An elliptic fibration can always be expressed in Weierstrass normal form. Singular fibers are genus-one fibers that degenerate over the discriminant locus, i.e., the codimension 1 locus in the base space. Such a fiber either degenerates to a $\mathbb{P}^1$ with a single singularity, or a sum of smooth $\mathbb{P}^1$. The types of singular fibers can be determined from the vanishing order of the coefficients of the Weierstrass equation, and have been completely classified by Kodaira [23], [25].

Again, let $j\colon H \to \mathrm{NS}(X)$ be a primitive lattice embedding. Let $K = j(H)^{\perp}$ be the orthogonal complement of $H$ in $\mathrm{NS}(X)$. Then $\mathrm{NS}(X) = j(H) \oplus K$. Let $\Sigma \subset \mathbb{P}^1$ be the set of points on the base of the elliptic fibration $\pi$ that correspond with singular fibers. For each singular point $p \in \Sigma$, let $T_p$ be the sublattice of $K$ spanned by the classes of the irreducible components of the singular fiber over $p$ that are disjoint from the section $\sigma$ of the elliptic fibration. The following is a standard result from Kodaira's classification of singular fibers.

**Proposition 2.4.3.**

*(i) For each $p \in \Sigma$ the lattice $T_p$ is a negative definite lattice of ADE-type, i.e. of type $A_m, D_n, E_\ell$.*

*(ii) Denote by $K^{\mathrm{root}}$ the sublattice spanned by the roots of $K$ (a root of $\mathrm{NS}(X)$ is an algebraic class of self-intersection $-2$ in $K$). Then the decomposition*

$$K^{\mathrm{root}} = \bigoplus_{p \in \Sigma} T_p$$

*is unique up to permutation of the summands.*

*(iii) There is a canonical group isomorphism $K/K^{\mathrm{root}} \to \mathrm{MW}(\pi, \sigma)$*

Parts (i) and (ii) are standard results from Kodaira's classification of singular fibers [23], [25]. Part (iii) was proven by Shioda [54].

**Example 2.4.4.** Return now to Example 2.2.7. Making the substitutions:

$$\mathbf{X} = tx^3\,, \qquad \mathbf{Y} = \sqrt{2}x^2y\,, \qquad \mathbf{W} = 2x^3\,, \qquad \mathbf{Z} = 2x^2(-\varepsilon t + \zeta)$$

into the quartic equation $Q_2 \subset \mathbb{P}^3$, results in Jacobian elliptic fibration $\pi_{\mathrm{alt}} \colon \mathcal{X} \to \mathbb{P}^1$ with fiber $\mathcal{X}_t$, given by the equation

$$\mathcal{X}_t \colon \quad y^2 = x(x^2 + B(t)x + A(t))$$

with discriminant $\Delta_{\mathcal{X}}(t) = A(t)^2(B(t)^2 - 4A(t))$, where $A(t) = (\gamma t - \delta)(\varepsilon t - \zeta)$ and $B(t) = t^3 - 3\alpha t - 2\beta$.

The section $\sigma$ of this fibration is given by the point at infinity in each fiber $\mathcal{X}_t$, We have singular fibers of Kodaira type $I_8^* + 2I_2 + 6I_1$. The Mordell-Weil group is given by $\mathrm{MW}(\pi_{alt}, \sigma) = \mathbb{Z}/2\mathbb{Z}$. This fibration will be an object of study in the following chapter.

## 2.5 Nikulin involutions.

In this section we will take a brief aside to motivate our interest in the K3 surface discussed in Example 2.2.7.

**Definition 2.5.1.** *A* **Nikulin involution***, $\iota$, is an involution on K3 surface $X$ which leaves the holomorphic 2-form $\omega_X$ unchanged, $\iota^*(\omega_X) = \omega_X$.*

It was proven by Nikulin that every such involution has sixteen isolated fixed points, and that the minimal resolution of the quotient of $X$ by this involution $\widehat{X/\langle\iota\rangle}$ is a K3 surface.

**Definition 2.5.2.** *A K3 surface $X$ admits a* **Shioda-Inose structure** *if there is an abelian surface $A$ and rational maps of degree 2 as in the diagram*

$$\begin{tikzcd}
A = Jac(C) \arrow[dr, "2"'] & & X \arrow[dl, "2"] \arrow[loop right, "\iota"] \\
& Y = \mathrm{Kum}(A) &
\end{tikzcd}$$

*such that $T(A) = T(X)$.* $\mathrm{Kum}(A)$ *denotes a Kummer surface as in Example 2.1.3.*

Morrison showed that K3 surfaces of Picard rank larger than seventeen admit a Shioda-Inose structure.

**Theorem 2.5.3** (Morrison [40]). *If $X$ is a K3 surface of Picard rank 18 or 19 then $X$ admits a Shioda-Inose structure. If the Picard rank is 17, then some additional conditions on the transcendental lattice guarantee there is a Shioda-Inose structure.*

Thus the K3 surface given in Example 2.1.3 admits a Shioda-Inose structure, however, the K3 surface given in Example 2.2.7 admits no such structure. However, there is a generalization of this structure in Picard rank 16.

Let $\mathcal{Y}$ be a family of K3 surfaces given by the double cover of $\mathbb{P}^2$ branched along 6 lines. Let $\mathcal{X}$ = the family of K3 in Example 2.2.7. A Van Geeman-Sarti involution, $\iota$, is given by translation by the two-torsion section $y = 0$, $x = 0$ in $(\pi_{alt}, \sigma)$ the fibration given in Example 2.4.4. Explicitly, $(x, y) \overset{\iota}{\mapsto} (\frac{A(t)}{x}, -\frac{A(t)}{x^2}y)$. It can be shown that the Weierstrass model for this fibration $y^2 = x(x^2 + B(t)x + A(t))$, and the holomorphic 2-form $\omega_X = dt \wedge \frac{dx}{y}$ are invariant under this involution.

This generalization of the Shioda-Inose structure motivates our interest in the K3 surface described in Example 2.2.7, and will be an object of study in the following chapter.

# CHAPTER 3

## Classification of Jacobian elliptic fibrations

The following chapter is a direct copy of two papers that I coauthored with Dr. Adrian Clingher and Dr. Andreas Malmendier. We study a special family of K3 surfaces $\mathcal{X}$ polarized by the rank-sixteen lattice $H \oplus E_7(-1) \oplus E_7(-1)$. The main result of this paper consists of two parts: (1) a generic member of the family $\mathcal{X}$ admits exactly four inequivalent Jacobian elliptic fibrations; (2) Explicit Weierstrass models for these fibrations are constructed using modular forms on a suitable bounded symmetric domain of type $IV$. This construction provides a geometric interpretation for the F-theory/heterotic string duality in eight dimensions with two Wilson lines. My main contributions to these papers were the computation of the explicit Weierstrass models for these Jacobian elliptic fibrations (see Sections 3.3.1-3.3.4), the expression of these models in terms of modular forms on a suitable bounded symmetric domain of type $IV$ (see Theorem 3.4.3), and the coalescence of singular fibers for each fibration (see Fig 3.5). Throughout this chapter "the author" refers to the authors of the papers, namely myself, Dr. Adrian Clingher, and Dr. Andreas Malmendier.

### 3.1 Introduction and Summary of results

Let $\mathcal{X}$ be a smooth complex algebraic K3 surface. Denote by $\mathrm{NS}(\mathcal{X})$ the Néron-Severi lattice of $\mathcal{X}$. This is known to be an even lattice of signature $(1, p_{\mathcal{X}} - 1)$, where $p_{\mathcal{X}}$ being the Picard rank of $\mathcal{X}$, with $1 \leq p_{\mathcal{X}} \leq 20$. In this context, a *lattice polarization* [12, 44–47]

on $\mathcal{X}$ is then given by a primitive lattice embedding $i \colon \mathrm{N} \hookrightarrow \mathrm{NS}(\mathcal{X})$, the image of which is to contain a pseudo-ample class. Here, N is a choice of even lattice of signature $(1, r)$, with $0 \leq r \leq 19$. Two N-polarized K3 surfaces $(\mathcal{X}, i)$ and $(\mathcal{X}', i')$ are said to be isomorphic, if there exists an analytic isomorphism $\alpha \colon \mathcal{X} \to \mathcal{X}'$ such that $\alpha^* \circ i' = i$ where $\alpha^*$ is the appropriate morphism at cohomology level.

The present paper focuses on a special class of such objects - K3 surfaces polarized by the rank sixteen lattice:

$$\mathrm{N} \ = \ H \oplus E_7(-1) \oplus E_7(-1)\,. \tag{3.1.1}$$

Here $H$ stands for the standard hyperbolic lattice of rank two, and $E_7(-1)$ is the negative definite even lattice associated with the analogous root system. This notation will be used throughout this article.

Our interest in this class of K3 surfaces has multiple motivations. First, as observed in earlier work [10] by the authors, K3 surfaces of this type are explicitly constructible. In fact, they fit into a six-parameter family of quartic *normal forms*:

**Theorem 3.1.1** ( [10]). *Let $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \in \mathbb{C}^6$. Consider the projective quartic surface in $\mathbb{P}^3(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})$ defined by the homogeneous equation:*

$$\mathbf{Y}^2\mathbf{Z}\mathbf{W} - 4\mathbf{X}^3\mathbf{Z} + 3\alpha\mathbf{X}\mathbf{Z}\mathbf{W}^2 + \beta\mathbf{Z}\mathbf{W}^3 + \gamma\mathbf{X}\mathbf{Z}^2\mathbf{W} - \frac{1}{2}\left(\delta\mathbf{Z}^2\mathbf{W}^2 + \zeta\mathbf{W}^4\right) + \varepsilon\mathbf{X}\mathbf{W}^3 = 0\,. \tag{3.1.2}$$

*Assume that $(\gamma,\delta) \neq (0,0)$ and $(\varepsilon,\zeta) \neq (0,0)$. Then, the surface $\mathrm{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ obtained as the minimal resolution of (3.1.2) is a K3 surface endowed with a canonical $\mathrm{N}$-polarization.*

All N-polarized K3 surfaces, up to isomorphism, are in fact realized in this way. Moreover, one can tell precisely when two members of the above family are isomorphic. Let $\mathfrak{G}$ be the subgroup of $\mathrm{Aut}(\mathbb{C}^6)$ generated by the set of transformations given below:

$$(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \ \longrightarrow \ (t^2\alpha,\ t^3\beta,\ t^5\gamma,\ t^6\delta,\ t^{-1}\varepsilon,\ \zeta), \ \text{with } t \in \mathbb{C}^*$$

$$(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \ \longrightarrow \ (\alpha,\beta,\varepsilon,\zeta,\gamma,\delta)\,. \tag{3.1.3}$$

It follows then that two K3 surfaces in the above family are isomorphic if and only if their six-parameter coefficient sets belong to the same orbit of $\mathbb{C}^6$ under $\mathfrak{G}$. This fact leads one to define the following set of invariants associated to the K3 surfaces in the family.

$$J_2 = \alpha, \quad J_3 = \beta, \quad J_4 = \gamma\cdot\varepsilon, \quad J_5 = \gamma\cdot\zeta + \delta\cdot\varepsilon, \quad J_6 = \delta\cdot\zeta \tag{3.1.4}$$

which then allows one to prove:

**Theorem 3.1.2** ( [10]). *The four-dimensional open analytic space*

$$\mathfrak{M}_{\mathrm{N}} \;=\; \Big\{\; [\; J_2,\; J_3,\; J_4,\; J_5,\; J_6\; ] \in \mathbb{WP}(2,3,4,5,6) \mid (J_3,\; J_4,\; J_5) \neq (0,0,0) \;\Big\} \tag{3.1.5}$$

*forms a coarse moduli space for* N*-polarized K3 surfaces.*

The results above fit extremely well with the Hodge theory (periods) classification of these these K3 objects. Hodge-theoretically, lattice polarized K3 surfaces are well understood, by classical work of Dolgachev [13]. For N-polarized K3 surfaces, appropriate Torelli type arguments give a Hodge-theoretic coarse moduli space given by a modular four-fold:

$$\Gamma_T^+ \backslash \mathbf{H}_2\,, \tag{3.1.6}$$

where the period domain $\mathbf{H}_2$ is a four-dimensional open domain of type $I_{2,2} \cong IV_4$ and $\Gamma_T^+$ is a discrete arithmetic group acting on $\mathbf{H}_2$. This can be made very precise, via work by Matsumoto [34]. Namely, we set

$$\mathbf{H}_2 = \left\{ \varpi = \begin{pmatrix} \tau_1 & z_1 \\ z_2 & \tau_2 \end{pmatrix} \in \mathrm{Mat}(2,2;\mathbb{C}) \;\Big|\; \mathrm{Im}\,(\tau_1)\cdot \mathrm{Im}\,(\tau_2) > \frac{1}{4}|z_1 - \bar{z}_2|^2,\ \mathrm{Im}\,\tau_1 > 0 \right\},$$

while $\Gamma_T^+$ is a certain index-two subgroup of the group

$$\Gamma_T = \left\{ G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}\left(4, \mathbb{Z}[i]\right) \Big| \, G^\dagger \cdot J \cdot G = J \right\} \rtimes \mathbb{Z}_2\,.$$

Here, $J$ denotes the standard symplectic matrix, the action of an element $G \in \mathrm{GL}(4, \mathbb{Z}[i])$ on $\mathbb{H}_2$ is the classical:

$$G \cdot \varpi \;=\; (C \cdot \varpi + D)^{-1} (A \cdot \varpi + B)\,,$$

and the generator of the right $\mathbb{Z}_2$ factor acts by transposition, i.e., $\varpi \mapsto T \cdot \varpi = \varpi^t$.

In the above context, the five invariants of (3.1.4) can be computed in terms of theta functions on $\mathbf{H}_2$. The result is a modular interpretation:

**Theorem 3.1.3** ( [10]). *Invariants $J_2, J_3, J_4, J_5, J_6$ are modular forms of respective weights 4, 6, 8, 10, and 12, relative to the group $\Gamma_T^+$.*

Invariants of this type were independently obtained by Vinberg [58]. Moreover, by Vinberg's work, one also obtains that $\{J_k\}_{k=2}^{6}$ are algebraically independent and generate, over $\mathbb{C}$, the graded ring of even modular forms relative to the group $\Gamma_T^+$.

The goal of the present work is to classify the Jacobian elliptic fibrations on the generic N-polarized K3 surface $\mathcal{X}$. Our results include the following: Proposition 3.2.3 shows that there are exactly four inequivalent Jacobian elliptic fibrations on $\mathcal{X}$. Each of the fibrations can be described via geometric pencils on the quartic surface defining $\mathcal{X}$. We identify the pencils associated to these fibrations in the context of the quartic normal form (3.1.2) in Theorem 3.3.6. Explicit Weierstrass models are then computed, and finally we express the Weierstrass coefficients in terms of modular forms relative to the group $\Gamma_T^+$; this will be the content of Theorem 3.4.3.

This article is structured as follows: Isomorphism classes of Jacobian elliptic fibrations on $\mathcal{X}$ are in one-to-one correspondence to primitive lattice embeddings $H \hookrightarrow \mathrm{N}$. In Section 3.2, a lattice theoretic analysis of this problem reveals that there are exactly four such (non-isomorphic) primitive lattice embeddings. A N-polarized K3 surface carries therefore, up to automorphisms, four special Jacobian elliptic fibrations, the only ones existing in the generic case. We geometrically identify the fibrations in the context of the normal forms (3.1.2) in Section 3.3. In Section 3.4 we produce explicit Weierstrass forms with coefficients in terms of the modular forms. Finally, in Section 3.5 we show that our construction provides a geometric interpretation for the F-theory/heterotic string duality in eight dimensions with two non-trivial Wilson lines. From a physics point of view, the moduli space $\mathfrak{M}_{\mathrm{N}}$ provides the interesting example where the partial higgsing of the heterotic gauge algebra $\mathfrak{g} = \mathfrak{e}_8 \oplus \mathfrak{e}_8$ or $\mathfrak{g} = \mathfrak{so}(32)$ for the associated low energy effective eight-dimensional supergravity theory has, in each case, two inequivalent Coulomb branches and no charged matter fields for the corresponding F-theory model.

### 3.2 Lattice theoretic considerations for the K3 surfaces

Let $\mathcal{X}$ be a generic N-polarized K3 surface with $\mathrm{N} = H \oplus E_7(-1) \oplus E_7(-1)$. We start with a brief lattice-theoretic investigation regarding the possible Jacobian elliptic fibration structures appearing on the surface $\mathcal{X}$. Recall that a *Jacobian elliptic fibration* on $\mathcal{X}$ is a pair $(\pi, \sigma)$ consisting of a proper map of analytic spaces $\pi : \mathcal{X} \to \mathbb{P}^1$, whose generic fiber is a smooth curve of genus one, and a section $\sigma : \mathbb{P}^1 \to \mathcal{X}$ in the elliptic fibration $\pi$. If $\sigma'$ is another section of the Jacobian fibration $(\pi, \sigma)$, then there exists an automorphism of $\mathcal{X}$ preserving $\pi$ and mapping $\sigma$ to $\sigma'$. One can then realize an identification between the set of sections of $\pi$ and the group of automorphisms of $\mathcal{X}$ preserving $\pi$. This is the *Mordell-Weil group* $\mathrm{MW}(\pi, \sigma)$ of the Jacobian fibration.We have the following:

**Lemma 3.2.1.** *Let $\mathcal{X}$ be a generic* $\mathrm{N}$*-polarized K3 surface and $(\pi, \sigma)$ a Jacobian elliptic fibration on $\mathcal{X}$. Then, the Mordell-Weil group has finite order. In particular, we have*

$$\operatorname{rank} \mathrm{MW}(\pi, \sigma) = 0\,. \tag{3.2.1}$$

*Proof.* For $\mathrm{NS}(\mathcal{X}) = \mathrm{N}$, it follows, via work of Nikulin [46, 48, 49] and Kondo [28], that the group of automorphisms of $\mathcal{X}$ is finite. In fact, $\mathrm{Aut}(\mathcal{X}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, any Jacobian elliptic fibration on $\mathcal{X}$ must have a Mordell-Weil group of finite order and cannot admit any infinite-order section. □

Given a Jacobian elliptic fibration $(\pi, \sigma)$ on $\mathcal{X}$, the classes of fiber and section span a rank-two primitive sub-lattice of $\mathrm{NS}(\mathcal{X})$ which is isomorphic to the standard rank-two hyperbolic lattice $H$. The converse also holds: given a primitive lattice embedding $H \hookrightarrow \mathrm{NS}(\mathcal{X})$ whose image contains a pseudo-ample class, it is known from [6]*Thm. 2.3 that there exists a Jacobian elliptic fibration on the surface $\mathcal{X}$, whose fiber and section classes span $H$. Moreover, one has a one-to-one correspondence between isomorphism classes of Jacobian elliptic fibrations on $\mathcal{X}$ and isomorphism classes of primitive lattice embeddings $H \hookrightarrow \mathrm{NS}(\mathcal{X})$ modulo the action of isometries of $H^2(\mathcal{X}, \mathbb{Z})$ preserving the Hodge decomposition [7]*Lemma 3.8. These are standard and well-known results; see also the overall discussion in [29, 51].

Let us then investigate the possible primitive lattice embeddings $H \hookrightarrow \mathrm{N}$. Assume $j\colon H \hookrightarrow \mathrm{N}$ is such an embedding. Denote by $\mathrm{K} = j(H)^\perp$ the orthogonal complement in $\mathrm{N}$.

It follows that $\mathrm{N} = j(H) \oplus \mathrm{K}$. The lattice K is negative-definite of rank fourteen and has discriminant group and form

$$\Big(D(\mathrm{K}),\ q_{\mathrm{K}}\Big) \ \simeq\ \Big(D(\mathrm{N}), q_{\mathrm{N}}\Big) \ \simeq\ \Big(\mathbb{Z}_2 \oplus \mathbb{Z}_2,\ (1/2) \otimes (1/2)\Big)\,. \tag{3.2.2}$$

Using Nikulin's classification theory [46, 48], isomorphism classes of embeddings $H \hookrightarrow \mathrm{N}$ are in one-to-one correspondence with even, negative-definite, rank-fourteen lattices K, satisfying condition (3.2.2).

Going back to a choice of embedding $j\colon H \hookrightarrow \mathrm{N}$, we denote by $\mathrm{K}^{\text{root}}$ the sub-lattice spanned by the roots of K, i.e., the algebraic class of self-intersection $-2$ in K. Let $\Sigma \subset \mathbb{P}^1$ be the set of points on the base of the elliptic fibration $\pi$ that correspond to singular fibers. For each singular point $p \in \Sigma$, we denote by $\mathrm{T}_p$ the sub-lattice spanned by the classes of the irreducible components of the singular fiber over $p$ that are disjoint from the section $\sigma$ of the elliptic fibration. Standard K3 geometry arguments tell us that $\mathrm{K}^{\text{root}}$ is of ADE-type, meaning for each $p \in \Sigma$ the lattice $\mathrm{T}_p$ is a negative definite lattice of type $A_m$, $D_m$ and $E_l$, and we have

$$\mathrm{K}^{\text{root}} = \bigoplus_{p\in\Sigma} \mathrm{T}_p\,. \tag{3.2.3}$$

We also introduce

$$\mathcal{W} = \mathrm{K}/\mathrm{K}^{\text{root}}\,. \tag{3.2.4}$$

Shioda [54] proved that there is a canonical group isomorphism $\mathcal{W} \simeq \mathrm{MW}(\pi, \sigma)$, identifying $\mathcal{W}$ with the Mordell-Weil group of the corresponding Jacobian elliptic fibration $(\pi, \sigma)$. We have the following:

**Lemma 3.2.2.** *In the situation described, the only possible choices for* K *are*

$$\mathrm{K} = E_7(-1) \oplus E_7(-1)\,, \quad E_8(-1) \oplus D_6(-1)\,, \quad D_{14}(-1) \quad \textit{if } \mathcal{W} = \{\mathbb{I}\},$$

*or* $\mathrm{K} = D_{12}(-1) \oplus A_1(-1) \oplus A_1(-1)$ *if* $\mathcal{W} = \mathbb{Z}/2\mathbb{Z}$.

*Proof.* Because of Lemma 3.2.1 it follows that $\mathcal{W}$ must be finite. Moreover, via

$$\mathrm{K}/\mathrm{K}^{\text{root}} \hookrightarrow \big(\mathrm{K}^{\text{root}}\big)^*/\mathrm{K}^{\text{root}}\,, \tag{3.2.5}$$

the group $\mathcal{W}$ can be identified as a subgroup of the discriminant group $D(\mathrm{K}^{\mathrm{root}})$, isotropic with respect to the discriminant form $q_{\mathrm{K}^{\mathrm{root}}}$. Probing for the possible choices of lattice K, we explore therefore two cases:

Case I: $\mathcal{W} = \{\mathbb{I}\}$. This corresponds to $\mathrm{K} = \mathrm{K}^{\mathrm{root}}$. We are then searching for rank-fourteen ADE-type lattices K with

$$\Big(D(\mathrm{K}),\ q_{\mathrm{K}}\Big) \ \simeq \ \Big(\mathbb{Z}_2 \oplus \mathbb{Z}_2,\ (1/2) \otimes (1/2)\Big).$$

A look at the classical ADE-type lattice discriminant form list reveals that only three lattices fit the bill. These are $\mathrm{K} = E_7(-1) \oplus E_7(-1)$, $E_8(-1) \oplus D_6(-1)$, and $D_{14}(-1)$.

Case II: $\mathcal{W} \neq \{\mathbb{I}\}$. This corresponds to the case when $\mathrm{K}^{\mathrm{root}}$ is a sub-lattice of positive index in K. The following condition must be then satisfied:

$$|D(\mathrm{K}^{\mathrm{root}})| \ = \ |D(\mathrm{K})| \cdot \ |\mathcal{W}|^2 \ = \ 4|\mathcal{W}|^2 \,. \tag{3.2.6}$$

In [53] Shimada provides a complete list of pairs $(\mathrm{K}^{\mathrm{root}}, \mathcal{W})$ that occur in Jacobian elliptic K3 surfaces, with $\mathcal{W}$ finite. According to Shimada's list, there's 392 cases of ADE-type lattices in rank fourteen, together with the possible groups $\mathcal{W}$: one first eliminates all the cases with $\mathcal{W} = \{\mathbb{I}\}$, followed by eliminating all cases with fibers of type $A_n$ with $n \geq 2$. For the remaining cases, we computed the determinant of the discriminant form and checked whether it equals 4. Only one case remains which satisfies condition (3.2.6) for $\mathcal{W} \neq \{\mathbb{I}\}$. This is $\mathrm{K} = D_{12}(-1) \oplus A_1(-1) \oplus A_1(-1)$ with $\mathcal{W} = \mathbb{Z}/2\mathbb{Z}$. □

Lemma 3.2.2 immediately implies:

**Proposition 3.2.3.** *A generic* N*-polarized K3 surface $\mathcal{X}$ admits exactly four inequivalent Jacobian elliptic fibrations $(\pi, \sigma)$, up to isomorphism, with $(\mathrm{K}^{root}, \mathrm{MW}(\pi, \sigma))$ given by Lemma 3.2.2.*

### 3.3 Generalized Inose quartic and its elliptic fibrations

Let $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \in \mathbb{C}^6$ be a set of parameters. We consider the projective quartic surface $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ in $\mathbb{P}^3$ defined by the homogeneous equation

$$\mathbf{Y}^2\mathbf{ZW} - 4\mathbf{X}^3\mathbf{Z} + 3\alpha\mathbf{XZW}^2 + \beta\mathbf{ZW}^3 + \gamma\mathbf{XZ}^2\mathbf{W} - \frac{1}{2}\left(\delta\mathbf{Z}^2\mathbf{W}^2 + \zeta\mathbf{W}^4\right) + \varepsilon\mathbf{XW}^3 = 0. \quad (3.3.1)$$

The family (3.3.1) was first introduced by the first author and Doran in [8] as a generalization of the Inose quartic in [20]. We denote by $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ the smooth complex surface obtained as the minimal resolution of $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$. The quartic $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ has two special singularities at the following points:

$$\mathrm{P}_1 = [0, 1, 0, 0], \qquad \mathrm{P}_2 = [0, 0, 1, 0]\,. \quad (3.3.2)$$

One verifies that the singularity at $\mathrm{P}_1$ is a rational double point of type $\mathrm{A}_9$ if $\varepsilon \neq 0$, and of type $\mathrm{A}_{11}$ if $\varepsilon = 0$. The singularity at $\mathrm{P}_2$ is of type $\mathrm{A}_5$ if $\gamma \neq 0$, and of type $\mathrm{E}_6$ if $\gamma = 0$. For a generic sextuple $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$, the points $\mathrm{P}_1$ and $\mathrm{P}_2$ are the only singularities of Equation (3.3.1) and are rational double points. The two sets $a_1, a_2, \dots, a_9$ and $b_1, b_2, \dots, b_5$ will denote the curves appearing from resolving the rational double point singularities at $\mathrm{P}_1$ and $\mathrm{P}_2$, respectively.

In this section we assume that $\gamma\varepsilon \neq 0$. The specializations $\gamma = 0$ and $\varepsilon = 0$ were already considered in [10]. We introduce the following three special lines, denoted by $\mathrm{L}_1$, $\mathrm{L}_2$, $\mathrm{L}_3$ and given by

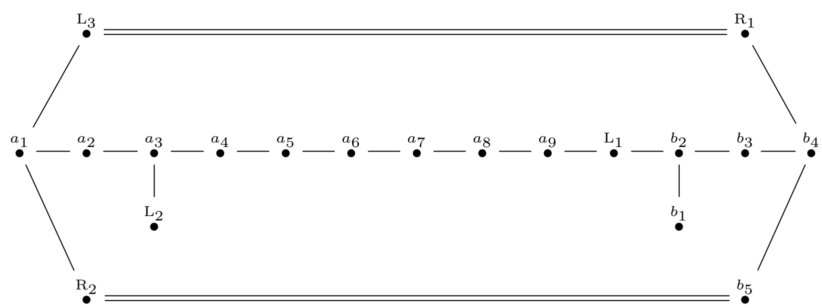
$$\mathbf{X} = \mathbf{W} = 0\,, \quad \mathbf{Z} = \mathbf{W} = 0\,, \quad 2\varepsilon\mathbf{X} - \zeta\mathbf{W} = \mathbf{Z} = 0\,.$$

Note that $\mathrm{L}_1$, $\mathrm{L}_2$, $\mathrm{L}_3$ lie on the quartic in Equation (3.3.1). Because of $\gamma\varepsilon \neq 0$, the lines $\mathrm{L}_1$, $\mathrm{L}_2$, $\mathrm{L}_3$ are distinct and concurrent, meeting at $\mathrm{P}_1$. Moreover, we consider the following complete intersections:

$$2\varepsilon\mathbf{X} - \zeta\mathbf{W} \;=\; \left(3\alpha\varepsilon^2\zeta + 2\beta\varepsilon^3 - \zeta^3\right)\mathbf{W}^2 - \varepsilon^2\left(\delta\varepsilon - \gamma\zeta\right)\mathbf{ZW} + 2\varepsilon^3\mathbf{Y}^2 \;=\; 0\,,$$

$$2\gamma\mathbf{X} - \delta\mathbf{W} \;=\; \left(3\alpha\gamma^2\delta + 2\beta\gamma^3 - \delta^3\right)\mathbf{ZW}^2 - \gamma^2\left(\gamma\zeta - \delta\varepsilon\right)\mathbf{W}^3 + 2\gamma^3\mathbf{Y}^2\mathbf{Z} \;=\; 0\,.$$

Assuming appropriate generic conditions, the above equations determine two projective curves $\mathrm{R}_1$, $\mathrm{R}_2$, of degrees two and three, respectively. The conic $\mathrm{R}_1$ is a (generically smooth) rational curve tangent to $\mathrm{L}_1$ at $\mathrm{P}_2$. The cubic $\mathrm{R}_2$ has a double point at $\mathrm{P}_2$, passes through $\mathrm{P}_1$ and is generically irreducible. When resolving the quartic surface (3.3.1), these two curves lift to smooth rational curves on $\mathrm{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$, which by a slight abuse of notation we shall denote by the same symbol. One obtains the following dual diagrams of rational curves:



(3.3.3)

We proved the theorem in [10]:

**Theorem 3.3.1.** *Assume that $(\gamma, \delta) \neq (0,0)$ and $(\varepsilon, \zeta) \neq (0,0)$. Then, the surface $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ obtained as the minimal resolution of $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ is a K3 surface endowed with a canonical $\mathrm{N}$-polarization.*

We also note:

**Remark 3.3.2.** *The degree-four polarization determined on $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$, assuming the case $\gamma\varepsilon \neq 0$, is given by the following polarizing divisor*

$$\mathcal{H} = \mathrm{L}_2 + (a_1 + 2a_2 + 3a_3 + 3a_4 + 3a_5 + \cdots 3a_9) + 3\mathrm{L}_1 + (2b_1 + 4b_2 + 3b_3 + 2b_4 + b_5)\ . \tag{3.3.4}$$

In [10] we proved:

**Lemma 3.3.3.** *Let $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \in \mathbb{C}^6$ with $(\gamma, \delta) \neq (0,0)$ and $(\varepsilon, \zeta) \neq (0,0)$. Then, one has the following isomorphisms of $\mathrm{N}$-polarized K3 surfaces:*

*(a)* $\mathcal{X}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \ \simeq\ \mathcal{X}(t^2\alpha,\ t^3\beta,\ t^5\gamma,\ t^6\delta,\ t^{-1}\varepsilon,\ \zeta)$, *for any $t \in \mathbb{C}^*$.*

*(b)* $\mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \simeq \mathcal{X}(\alpha,\beta,\varepsilon,\zeta,\gamma,\delta)$.

*Proof.* Let $q$ be a square root of $t$. Then, the projective automorphism, given by

$$\mathbb{P}^3 \longrightarrow \mathbb{P}^3, \quad [\mathbf{X}:\mathbf{Y}:\mathbf{Z}:\mathbf{W}] \mapsto [\, q^8\mathbf{X} : \, q^9\mathbf{Y} : \mathbf{Z} : \, q^6\mathbf{W}\,], \tag{3.3.5}$$

extends to an isomorphism $\mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \simeq \mathcal{X}(t^2\alpha, t^3\beta, t^5\gamma, t^6\delta, t^{-1}\varepsilon, \zeta)$ preserving the lattice polarization. Similarly, the birational involution

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^3, \quad [\mathbf{X}:\mathbf{Y}:\mathbf{Z}:\mathbf{W}] \mapsto [\, \mathbf{XZ} : \, \mathbf{YZ}, \mathbf{W}^2, \, \mathbf{ZW}\,] \tag{3.3.6}$$

extends to an isomorphism between $\mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ and $\mathcal{X}(\alpha,\beta,\gamma,\delta,\zeta,\varepsilon)$. $\square$

We also have the following:

**Proposition 3.3.4.** *Let* $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \in \mathbb{C}^6$ *as before. A Nikulin involution on the* N*-polarized K3 surfaces* $\mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ *is induced by the map*

$$\begin{aligned} \Psi : \qquad \mathbb{P}^3 &\longrightarrow \mathbb{P}^3, \\ [\mathbf{X}:\mathbf{Y}:\mathbf{Z}:\mathbf{W}] &\mapsto [\, (2\gamma\mathbf{X}-\delta\mathbf{W})\mathbf{XZ} \, : \, -(2\gamma\mathbf{X}-\delta\mathbf{W})\mathbf{YZ} : \\ &\qquad (2\varepsilon\mathbf{X}-\zeta\mathbf{W})\mathbf{W}^2 \, : \, (2\gamma\mathbf{X}-\delta\mathbf{W})\mathbf{Z}^2 \,]\,. \end{aligned} \tag{3.3.7}$$

*Proof.* One checks that $\Psi$ constitutes an involution of the projective quartic surface $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta) \subset \mathbb{P}^3(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})$. If we use the affine chart $\mathbf{W}=1$ then the unique holomorphic two-form is given by $d\mathbf{X}\wedge d\mathbf{Y}/F_{\mathbf{Z}}(\mathbf{X},\mathbf{Y},\mathbf{Z})$ where $F(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is the left side of Equation (3.3.1). One then checks that $\Psi$ in Equation (3.3.7) constitutes a symplectic involution after using $F(\mathbf{X},\mathbf{Y},\mathbf{Z})=0$. $\square$

**Remark 3.3.5.** *For the K3 surfaces* $\mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$*, quotienting by the involution and blowing up recovers a so-called double-sextic surface, i.e., the double cover of the projective plane branched along the union of six lines. The corresponding family of double-sextic surfaces was described in [10, 22].*

We now state a major result of this article:

**Theorem 3.3.6.** *For each Jacobian elliptic fibration $(\pi, \sigma)$ on the generic* $\mathrm{N}$*-polarized K3 surface* $\mathcal{X}$ *an associated pencil on the quartic normal form (3.3.1) can be identified:*

| *Name* | *Singular Fibers* | $\mathrm{MW}(\pi,\sigma)$ | *Pencil* |
|---|---|---|---|
| *standard* | $2III^* + 6I_1$ | *trivial* | *residual quadric surface intersection of* $\mathrm{L}_2(u,v) = 0$ *and* $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ |
| *alternate* | $I_8^* + 2I_2 + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | *residual quadric surface intersection of* $\mathrm{L}_1(u,v) = 0$ *and* $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ |
| *base-fiber dual* | $II^* + I_2^* + 6I_1$ | *trivial* | *residual quadric surface intersection of* $\mathrm{L}_3(u,v) = 0$ *and* $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ |
| *maximal* | $I_{10}^* + 8I_1$ | *trivial* | *residual cubic surface intersection of* $\mathrm{C}_3(u,v) = 0$ *and* $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ |

*Proof.* We construct the Weierstrass models for these four Jacobian elliptic fibrations explicitly in the Sections 3.3.1-3.3.4. The fact that these Jacobian elliptic fibrations are the *only* possible fibrations was already proven in Proposition 3.2.3. □

### 3.3.1 The standard fibration

There are exactly two ways of embedding two disjoint reducible fibers, each given by an extended Dynkin diagram $\hat{E}_7$, into the diagram (3.3.3). These two ways are depicted in Figure 3.1, where the green and the blue nodes indicate the two reducible fibers. In the case of Figure 3.1a, we have

$$\hat{\mathrm{E}}_7 = \langle \mathrm{L}_3, a_1, a_2, a_3, \mathrm{L}_2, a_4, a_5; a_6 \rangle \,, \quad \hat{\mathrm{E}}_7 = \langle b_5, b_4, b_3, b_2, b_1, \mathrm{L}_1, a_9; a_8 \rangle \,. \tag{3.3.8}$$

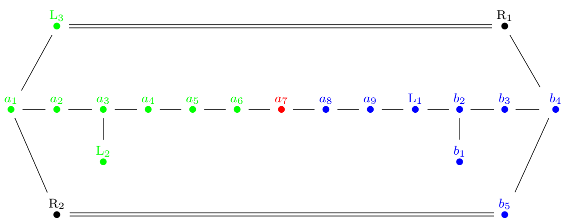
Thus, the smooth fiber class is given by

$$\mathrm{F}^{(a)}_{\mathrm{std}} = \mathrm{L}_3 + 2a_1 + 3a_2 + 4a_3 + 2\mathrm{L}_2 + 3a_4 + 2a_5 + a_6 \,, \tag{3.3.9}$$

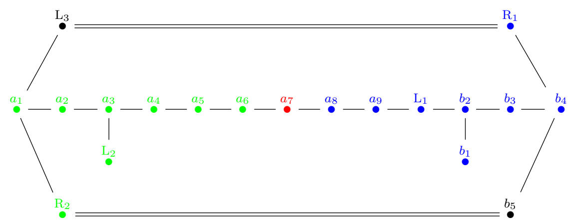
and the class of a section is $a_7$.

**Remark 3.3.7.** *Using the polarizing divisor* $\mathcal{H}$ *in Equation (3.3.4), one checks that*

$$\mathcal{H} - \mathrm{F}^{(a)}_{std} - \mathrm{L}_2 \equiv a_1 + 2a_2 + 3a_3 + \cdots + 3a_7 + 2a_8 + a_9 \,, \tag{3.3.10}$$




(a)




(b)

Fig. 3.1: The *standard* ways of fitting 2 fibers of type $\hat{E}_7$

*which shows that an elliptic fibration with section, called the* standard fibration, *is induced by intersecting the quartic surface* $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ *with the pencil of planes containing the line* $\mathrm{L}_2$*, denoted by* $\mathrm{L}_2(u,v) = u\mathbf{W} - v\mathbf{Z} = 0$ *for* $[u:v] \in \mathbb{P}^1$*.*

Making the substitutions

$$\mathbf{X} = uvx\,, \quad \mathbf{Y} = y\,, \quad \mathbf{Z} = 4u^4v^2z\,, \quad \mathbf{W} = 4u^3v^3z\,, \tag{3.3.11}$$

in Equation (3.3.1), compatible with $\mathrm{L}_2(u,v) = 0$, yields the Jacobian elliptic fibration $\pi_{\text{std}} : \mathcal{X} \to \mathbb{P}^1$ with fiber $\mathcal{X}_{[u:v]}$, given by the Weierstrass equation

$$\mathcal{X}_{[u:v]}: \quad y^2z = x^3 + f(u,v)\,xz^2 + g(u,v)\,z^3\,, \tag{3.3.12}$$

admitting the section $\sigma_{\text{std}} : [x:y:z] = [0:1:0]$, and with the discriminant

$$\Delta(u,v) = 4f^3 + 27g^2 = 64\,u^9v^9p(u,v)\,, \tag{3.3.13}$$

where

$$f(u,v) = -4u^3v^3\Big(\gamma u^2 + 3\alpha uv + \varepsilon v^2\Big)\,, \quad g(u,v) = 8u^5v^5\Big(\delta u^2 - 2\beta uv + \zeta v^2\Big)\,, \tag{3.3.14}$$

and $p(u,v) = 4\gamma^3u^6 + \cdots + 4\varepsilon^3v^6$ is an irreducible homogeneous polynomial of degree six. We have the following:

**Lemma 3.3.8.** *Equation (3.3.12) defines a Jacobian elliptic fibration with six singular fibers* *of Kodaira type $I_1$, two singular fibers of Kodaira type $III^*$ (ADE type $E_7$), and a trivial Mordell-Weil group of sections* $\mathrm{MW}(\pi_{std}, \sigma_{std}) = \{\mathbb{I}\}$.

*Proof.* The proof easily follows by checking the Kodaira type of the singular fibers at $p(u, v) = 0$ and $u = 0$ and $v = 0$. □

Applying the Nikulin involution in Proposition 3.3.4, we obtain the fiber configuration in Figure 3.1b with

$$\hat{\mathrm{E}}_7 = \langle \mathrm{R}_2, a_1, a_2, a_3, \mathrm{L}_2, a_4, a_5; a_6 \rangle \,, \quad \hat{\mathrm{E}}_7 = \langle \mathrm{R}_1, b_4, b_3, b_2, b_1, \mathrm{L}_1, a_9; a_8 \rangle \,. \tag{3.3.15}$$

The smooth fiber class is given by

$$\mathrm{F}^{(b)}_{\mathrm{std}} = \mathrm{R}_2 + 2a_1 + 3a_2 + 4a_3 + 2\mathrm{L}_2 + 3a_4 + 2a_5 + a_6 \,, \tag{3.3.16}$$

and the class of the section is $a_7$. Using the polarizing divisor $\mathcal{H}$ in Equation (3.3.4), one checks that

$$2\mathcal{H} - \mathrm{F}^{(b)}_{\mathrm{std}} - \mathrm{L}_1 - \mathrm{L}_2 - \mathrm{L}_3 \equiv 2a_1 + 3a_2 + 4a_3 + \dots + 4a_7 + 3a_8 + 2a_9 + b_1 + 2b_2 + \dots + 2b_5 \,, \tag{3.3.17}$$

which shows that the standard fibration can also be induced by intersecting the quartic $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ with the pencil of quadratic surfaces, denoted by $\mathrm{C}_1(u, v) = 0$ with $[u : v] \in \mathbb{P}^1$, containing the lines $\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3$. A computation yields

$$\mathrm{C}_1(u, v) = v\mathbf{W}\big(2\varepsilon\mathbf{X} - \zeta\mathbf{W}\big) - u\mathbf{Z}\big(2\gamma\mathbf{X} - \delta\mathbf{W}\big) = 0 \,. \tag{3.3.18}$$

### 3.3.2 The alternate fibration

There is exactly one way of embedding a reducible fiber given by the extended Dynkin diagram $\hat{D}_{12}$ and two reducible fibers of type $\hat{A}_1$ into the diagram (3.3.3). The configuration is invariant when applying the Nikulin involution in Proposition 3.3.4 and shown in Figure 3.2. We have

$$\hat{\mathrm{A}}_1 = \langle \mathrm{L}_3; \mathrm{R}_1 \rangle \,, \quad \hat{\mathrm{A}}_1 = \langle \mathrm{R}_2; b_5 \rangle \,, \quad \hat{\mathrm{D}}_{12} = \langle a_2, \mathrm{L}_1, \dots, b_1; b_3 \rangle \,, \tag{3.3.19}$$

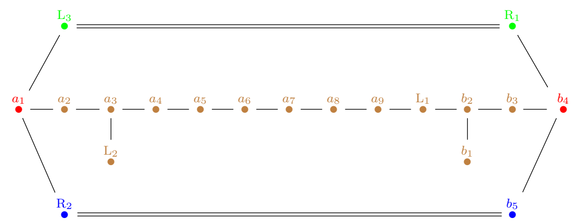



Fig. 3.2: The *alternate* way of fitting a fiber of type $\hat{D}_{12}$

Thus, the smooth fiber class is given by

$$\mathrm{F}_{\mathrm{alt}} = a_2 + 2a_3 + \mathrm{L}_2 + 2a_4 + \cdots + 2a_9 + 2\mathrm{L}_1 + b_1 + 2b_2 + b_3\,, \tag{3.3.20}$$

and the classes of a section and two-torsion section are $a_1, b_4$.

**Remark 3.3.9.** *Using the polarizing divisor $\mathcal{H}$ in Equation (3.3.4), one checks that*

$$\mathcal{H} - \mathrm{F}_{alt} - \mathrm{L}_1 \equiv a_1 + \cdots + a_9 + b_1 + 2b_2 + b_3\,, \tag{3.3.21}$$

*which shows that an elliptic fibration with section, called the* alternate fibration, *is induced by intersecting the quartic surface $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ with the pencil of planes containing the line $\mathrm{L}_1$, denoted by $\mathrm{L}_1(u, v) = u\mathbf{W} - v\mathbf{X} = 0$ for $[u : v] \in \mathbb{P}^1$.*

Making the substitutions

$$\mathbf{X} = 2uvx\,, \quad \mathbf{Y} = y\,, \quad \mathbf{Z} = 4v^5(-2\varepsilon u + \zeta v)z\,, \quad \mathbf{W} = 2v^2x\,, \tag{3.3.22}$$

into Equation (3.3.1), compatible with $\mathrm{L}_1(u, v) = 0$, determines the Jacobian elliptic fibration $\pi_{\mathrm{alt}} : \mathcal{X} \to \mathbb{P}^1$ with fiber $\mathcal{X}_{[u:v]}$, given by the equation

$$\mathcal{X}_{[u:v]}: \quad y^2 z = x\Big(x^2 + A(u, v)\, xz + B(u, v)\, z^2\Big)\,, \tag{3.3.23}$$

admitting the section $\sigma_{\mathrm{alt}} : [x : y : z] = [0 : 1 : 0]$ and the two-torsion section $[x : y : z] = [0 : 0 : 1]$, and with the discriminant

$$\Delta(u, v) = B(u, v)^2 \Big(A(u, v)^2 - 4B(u, v)\Big)\,, \tag{3.3.24}$$

where

$$A(u,v) = 4v(4u^3 - 3\alpha uv^2 - \beta v^3)\,, \quad B(u,v) = 4v^6(2\gamma u - \delta v)(2\varepsilon u - \zeta v)\,. \tag{3.3.25}$$

We have the following:

**Lemma 3.3.10.** *Equation (3.3.23) defines a Jacobian elliptic fibration with six singular fibers of Kodaira type $I_1$, two singular fibers of Kodaira type $I_2$ (ADE type $A_1$), and a singular fiber of Kodaira type $I_8^*$ (ADE type $D_{12}$), and a Mordell-Weil group of sections $\mathrm{MW}(\pi_{alt}, \sigma_{alt}) = \mathbb{Z}/2\mathbb{Z}$. Moreover, the Nikulin involution in Proposition 3.3.4 acts on the Jacobian elliptic fibration (3.3.23) as a Van Geemen-Sarti involution [57].*

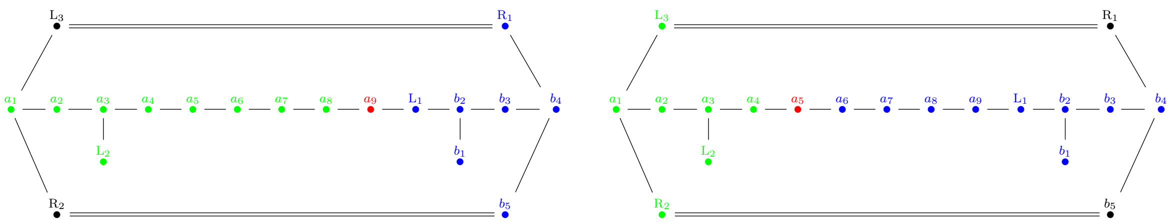
*Proof.* The proof easily follows by checking the Kodaira type of the singular fibers at $B(u,v) = 0$ and $A(u,v)^2 - 4B(u,v) = 0$. Applying the Nikulin involution in Proposition 3.3.4, we obtain the same configuration of singular fibers, only with the roles of the section and the two-torsion section interchanged. This means that the involution in Equation (3.3.7) acts on the Jacobian elliptic fibration (3.3.23) by fiberwise translation by two-torsion, i.e., by mapping

$$\Big[x : y : z\Big] \mapsto \Big[B(u,v)\, xz \; : -B(u,v)\, yz \; : \; x^2\Big] \tag{3.3.26}$$

for $[x : y : z] \neq [0 : 1 : 0], [0 : 0 : 1]$, and swapping $[0 : 1 : 0] \leftrightarrow [0 : 0 : 1]$. $\square$

### 3.3.3 The base-fiber dual fibration

It is easy to see that the K3 surfaces given by Equation (3.3.23) are double covers of the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ branched along a curve of type $(4,4)$, i.e., along a section in the line bundle $\mathcal{O}_{\mathbb{F}_0}(4,4)$. Every such cover has two natural elliptic fibrations corresponding to the two rulings of the quadric $\mathbb{F}_0$ coming from the two projections $\pi_i : \mathbb{F}_0 \to \mathbb{P}^1$ for $i = 1, 2$. The fibration $\pi_1 = \pi_{\mathrm{alt}}$ with two fibers of type $I_2^*$ is easily shown to correspond to double covers of $\mathbb{F}_0$ branched along curves of the form $\mathrm{F}_1 + \mathrm{F}_2 + \sigma$ where $\mathrm{F}_1, \mathrm{F}_2$ are fibers of $\pi_1$ and $\pi_2$ and $\sigma$ is a section of $\mathcal{O}_{\mathbb{F}_0}(2,4)$. A second elliptic fibration on the K3 surface then naturally arises from the second projection $\pi_2$. This second fibration arises




(a) (b)

Fig. 3.3: The *base-fiber dual* ways of fitting fibers of type $\hat{E}_8$ and $\hat{D}_6$

in a simple geometric manner, roughly speaking, by interchanging the roles of base and fiber coordinates in the affine model in Equation (3.3.23). We refer to this fibration as the *base-fiber dual fibration*. The fibration has reducible fibers of type $\hat{E}_8$ and $\hat{D}_6$.

There are exactly two ways of embedding the disjoint reducible fibers of type $\hat{E}_8$ and $\hat{D}_6$ into the diagram (3.3.3). These two ways are depicted in Figure 3.3. In the case of Figure 3.3a, we have

$$\hat{\mathrm{E}}_8 = \langle a_1, a_2, a_3, \mathrm{L}_2, a_4, a_5, a_6, a_7; a_8 \rangle \,, \quad \hat{\mathrm{D}}_6 = \langle \mathrm{R}_1, b_5, b_4, b_3, b_2, b_1; \mathrm{L}_1 \rangle \,. \tag{3.3.27}$$

Thus, the smooth fiber class is given by

$$\mathrm{F}^{(a)}_{\mathrm{bfd}} = \mathrm{L}_1 + b_1 + 2b_2 + 2b_3 + 2b_4 + b_5 + \mathrm{R}_1 \,, \tag{3.3.28}$$

and the class of a section is $a_9$.

**Remark 3.3.11.** *Using the polarizing divisor $\mathcal{H}$ in Equation (3.3.4), one checks that*

$$\mathcal{H} - \mathrm{F}^{(a)}_{bfd} - \mathrm{L}_3 \equiv a_1 + \cdots + a_9 \,, \tag{3.3.29}$$

*which shows that an elliptic fibration with section, called the* base-fiber dual fibration*, is induced by intersecting the quartic surface $\mathrm{Q}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ with the pencil of planes containing the line $\mathrm{L}_3$, denoted by $\mathrm{L}_3(u, v) = u\mathbf{Z} - v(2\varepsilon\mathbf{X} - \zeta\mathbf{W}) = 0$ for $[u : v] \in \mathbb{P}^1$.*

Making the substitutions

$$\begin{aligned} \mathbf{X} = 3uv(x + 6\gamma\varepsilon uv^3 z)\,, \quad \mathbf{Y} = y\,, \\ \mathbf{Z} = 6v^2(\varepsilon x - 6\gamma\varepsilon^2 uv^3 z - 18\zeta u^2 v^2 z)\,, \quad \mathbf{W} = 108 u^3 v^3 z\,, \end{aligned} \tag{3.3.30}$$

into Equation (3.3.1), compatible with $\mathrm{L}_3(u,v)=0$, determines a Jacobian elliptic fibration $\pi_{\text{bfd}} : \mathcal{X} \to \mathbb{P}^1$ with fiber $\mathcal{X}_{[u:v]}$, given by the equation

$$\mathcal{X}_{[u:v]}: \quad y^2 z = x^3 + F(u,v)\, xz^2 + G(u,v)\, z^3\,, \tag{3.3.31}$$

admitting the section $\sigma_{\text{bfd}} : [x:y:z] = [0:1:0]$, and with the discriminant

$$\Delta(u,v) = 4F^3 + 27G^2 = -2^6 3^{12} u^8 v^{10} P(u,v)\,, \tag{3.3.32}$$

where

$$\begin{aligned} F(u,v) = &\ -108\, u^2 v^4 \Big(9\alpha u^2 - 3(\gamma\zeta + \delta\varepsilon)uv + \gamma^2\varepsilon^2 v^2\Big)\,, \\ G(u,v) = &\ -216 u^3 v^5 \Big(27u^4 + 54\beta u^3 v + 27(\alpha\gamma\varepsilon + \delta\zeta)u^2 v^2 \\ &\ - 9\gamma\varepsilon(\gamma\zeta + \delta\varepsilon)uv^3 + 2\gamma^3\epsilon^3 v^4\Big)\,, \end{aligned} \tag{3.3.33}$$

and $P(u,v) = \gamma^2\varepsilon^2(\gamma\zeta - \delta\varepsilon)^2 v^6 + O(u)$ is an irreducible homogeneous polynomial of degree six. We have the following:

**Lemma 3.3.12.** *Equation (3.3.31) defines a Jacobian elliptic fibration with six singular fibers of Kodaira type $I_1$, one singular fibre of Kodaira type $I_2^*$ (ADE type $D_6$), and a singular fiber of Kodaira type $II^*$ (ADE type $E_8$), and a Mordell-Weil group of sections* $\mathrm{MW}(\pi_{bfd}, \sigma_{bfd}) = \{\mathbb{I}\}$.

*Proof.* The proof easily follows by checking the Kodaira type of the singular fibers at $P(u,v)=0$, $u=0$, and $v=0$. □

Applying the Nikulin involution in Proposition 3.3.4, we obtain the fiber configuration in Figure 3.3b with

$$\hat{\mathrm{D}}_6 = \langle \mathrm{L}_3, \mathrm{R}_2, a_1, a_2, a_3, \mathrm{L}_2; a_4 \rangle\,, \quad \hat{\mathrm{E}}_8 = \langle b_4, b_3, b_1, b_2, \mathrm{L}_1, a_9, a_8, a_7; a_6 \rangle\,. \tag{3.3.34}$$

The smooth fiber class is given by

$$\mathrm{F}^{(b)}_{\mathrm{bfd}} = \mathrm{R}_2 + \mathrm{L}_2 + \mathrm{L}_3 + 2a_1 + 2a_2 + 2a_3 + a_4 \,, \tag{3.3.35}$$

and the class of the section is $a_5$. Using the polarizing divisor $\mathcal{H}$ in Equation (3.3.4), one checks that

$$\begin{aligned} & 2\mathcal{H} - \mathrm{F}^{(b)}_{\mathrm{bfd}} - \mathrm{L}_1 - 2\mathrm{L}_2 \\ \equiv\ & 2a_1 + 4a_2 + 6a_3 + 6a_4 + 6a_5 + 5a_6 + 4a_7 + 3a_8 + 2a_9 + b_1 + 2b_2 + \dots + 2b_5 \,, \end{aligned} \tag{3.3.36}$$

which shows that the base-fiber dual fibration can also be induced by intersecting the quartic $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ with the pencil of quadratic surfaces, denoted by $\mathrm{C}_2(u,v)=0$ with $[u:v]\in\mathbb{P}^1$, containing the lines $\mathrm{L}_1, \mathrm{L}_2$ and also being tangent to $\mathrm{L}_2$. A computation yields

$$\mathrm{C}_2(u,v) = v\mathbf{Z}\big(2\gamma\mathbf{X} - \delta\mathbf{W}\big) - u\mathbf{W}^2 = 0 \,. \tag{3.3.37}$$

### 3.3.4 The maximal fibration

There are exactly two ways of embedding one reducible ADE-type fiber of the biggest possible rank, namely a fiber of type $\hat{D}_{14}$, into the diagram (3.3.3). These two ways are depicted in Figure 3.4. In the case of Figure 3.4a, we have

$$\hat{\mathrm{D}}_{14} = \langle \mathrm{R}_2, \mathrm{L}_3, a_1, \dots, a_9, \mathrm{L}_1, b_2, b_1; b_3 \rangle \,. \tag{3.3.38}$$

Thus, the smooth fiber class is given by

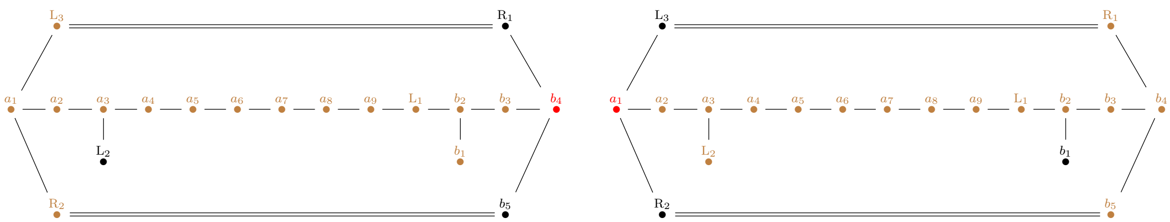
$$\mathrm{F}^{(a)}_{\mathrm{max}} = \mathrm{R}_2 + \mathrm{L}_3 + 2a_1 + \dots + 2a_9 + 2\mathrm{L}_1 + 2b_2 + b_1 + b_3 \,, \tag{3.3.39}$$

and the class of a section is $b_4$.

**Remark 3.3.13.** *Using the polarizing divisor $\mathcal{H}$ in Equation (3.3.4), one checks that*

$$2\mathcal{H} - \mathrm{F}^{(a)}_{max} - \mathrm{R}_1 \equiv b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 \,, \tag{3.3.40}$$

*This shows that an elliptic fibration with section, called the* maximal fibration*, is induced*




(a) (b)

Fig. 3.4: The *maximal* ways of fitting a fiber of type $\hat{D}_{14}$

*by intersecting the quartic surface* $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ *with a special pencil of quadric surfaces containing the curve* $\mathrm{R}_1$*, denoted by* $\mathrm{C}_3(u,v)=0$ *with* $[u:v]\in\mathbb{P}^1$*.*

Because of the identity

$$3\mathcal{H}-\mathrm{F}^{(a)}_{\text{max}}-\mathrm{R}_1-\mathrm{R}_2-\mathrm{L}_1\equiv a_1+\dots a_9+2b_1+4b_2+5b_3+6b_4+5b_5\,, \tag{3.3.41}$$

the quadric surfaces must also define a pencil of reducible cubic surfaces containing the curves $\mathrm{L}_1$, $\mathrm{R}_1$, and $\mathrm{R}_2$. This pencil turns out to be $(2\gamma\mathbf{X}-\delta\mathbf{W})\mathrm{C}_3(u,v)=0$ with

$$\begin{aligned}\mathrm{C}_3(u,v)=v\Big(&2\gamma^2\delta\varepsilon\zeta\mathbf{X}\mathbf{Z}+(6\alpha\gamma\delta\varepsilon\zeta+4\beta\gamma\delta\varepsilon^2+4\beta\gamma^2\varepsilon\zeta+2\delta^2\zeta^2)\mathbf{X}\mathbf{W}-\gamma\delta^2\varepsilon\zeta\mathbf{Z}\mathbf{W}\\&+2\gamma\delta\varepsilon\zeta\mathbf{Y}^2-(8\beta\gamma^2\varepsilon^2+4\delta^2\varepsilon\zeta+4\gamma\delta\zeta^2)\mathbf{X}^2\Big)+u(2\gamma\mathbf{X}-\delta\mathbf{W})(2\varepsilon\mathbf{X}-\zeta\mathbf{W})=0\,.\end{aligned} \tag{3.3.42}$$

Making the substitutions

$$\mathbf{X}=\delta\zeta v\big((2\beta\gamma\varepsilon v-u)x-2\gamma\delta^5\varepsilon\zeta^5v^5z\big)\,,\quad \mathbf{Y}=y\,,\quad \mathbf{W}=2\delta^2\zeta^2v^2x\,, \tag{3.3.43}$$

and $\mathbf{Z}=\mathbf{Z}(x,y,z,u,v)$, obtained by solving Equation (3.3.42) for $\mathbf{Z}$, determines a Jacobian elliptic fibration $\pi_{\text{max}}:\mathcal{X}\to\mathbb{P}^1$ with fiber $\mathcal{X}_{[u:v]}$, given by the equation

$$\mathcal{X}_{[u:v]}:\quad y^2z=x^3+a(u,v)\,x^2z+b(u,v)\,xz^2+c(u,v)\,z^3\,, \tag{3.3.44}$$

admitting the section $\sigma_{\text{max}}:[x:y:z]=[0:1:0]$, and with the discriminant

$$\Delta(u,v)=b^2\big(a^2-4b\big)-2ac\big(2a^2-9b\big)-27c^2=64\delta^{16}\zeta^{16}v^{16}d(u,v)\,, \tag{3.3.45}$$

where

$$
\begin{aligned}
a(u,v) &= -2\delta\zeta v\Big(u^3 - 6\beta\gamma\varepsilon u^2 v + 3(4\beta^2\gamma^2\varepsilon^2 - \alpha\delta^2\zeta^2)uv^2 \\
&\qquad - 2\beta(4\beta^2\gamma^3\varepsilon^3 - 3\alpha\gamma\delta^2\varepsilon\zeta^2 - \delta^3\zeta^3)v^3\Big)\,, \\
b(u,v) &= -4\delta^6\zeta^6 v^6\Big(2\gamma\varepsilon u^2 - (8\beta\gamma^2\varepsilon^2 + \gamma\delta\zeta^2 + \delta^2\varepsilon\zeta)uv \\
&\qquad + (8\beta^2\gamma^3\varepsilon^3 - 3\alpha\gamma\delta^2\varepsilon\zeta^2 + 2\beta\gamma^2\delta\varepsilon\zeta^2 + 2\beta\gamma\delta^2\varepsilon^2\zeta - \delta^3\zeta^3)v^2\Big)\,, \\
c(u,v) &= -8\gamma\delta^{11}\varepsilon\zeta^{11}v^{11}\Big(\gamma\varepsilon u - (2\beta\gamma^2\varepsilon^2 + \gamma\delta\zeta^2 + \delta^2\varepsilon\zeta)v\Big)\,,
\end{aligned}
\tag{3.3.46}
$$

and $d(u,v) = (\gamma\zeta - \delta\varepsilon)^2 u^8 + O(v)$ is an irreducible homogeneous polynomial of degree eight. We have the following:

**Lemma 3.3.14.** *Equation (3.3.44) defines a Jacobian elliptic fibration with eight singular fibers of Kodaira type $I_1$, one singular fibre of Kodaira type $I_{10}^*$ (ADE type $D_{14}$), and a Mordell-Weil group of sections* $\mathrm{MW}(\pi_{max}, \sigma_{max}) = \{\mathbb{I}\}$.

*Proof.* The proof easily follows by checking the Kodaira type of the singular fibers at $d(u,v) = 0$ and $v = 0$. □

Applying the Nikulin involution in Proposition 3.3.4, we obtain the fiber configuration in Figure 3.4b with

$$
\hat{\mathrm{D}}_{14} = \langle b_5, \mathrm{R}_1, b_4, b_3, b_2, \mathrm{L}_1, a_9, \ldots, a_3, \mathrm{L}_2; a_2\rangle\,. \tag{3.3.47}
$$

The smooth fiber class is given by

$$
\mathrm{F}^{(b)}_{\mathrm{max}} = \mathrm{R}_1 + \mathrm{L}_2 + 2\mathrm{L}_1 + a_2 + 2a_3 + \cdots + 2a_9 + 2b_2 + 2b_3 + 2b_4 + b_5\,, \tag{3.3.48}
$$

and the class of the section is $a_1$. Using the polarizing divisor $\mathcal{H}$ in Equation (3.3.4), one checks that the elliptic fibration is also induced by intersecting the quartic surface $\mathrm{Q}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ with a special pencil of cubic surfaces containing the curves $\mathrm{L}_2$, $\mathrm{L}_3$, $\mathrm{R}_2$, denoted by $\mathrm{T}(u,v) = 0$ with $[u:v] \in \mathbb{P}^1$. This pencil of cubic surfaces is

$$
\mathrm{T}(u,v) = \mathrm{C}_3(u,v)\mathbf{Z} - \gamma\delta\varepsilon\zeta\Big(\mathrm{C}_2(u,v)\mathbf{Z} - \mathrm{L}_3(u,v)\mathbf{W}^2\Big)\,. \tag{3.3.49}
$$

### 3.4 Modular description of the parameters

Based on the work of Dolgachev [13], it is known that Torelli type theorems exist for lattice polarized K3 surfaces. Let us briefly describe the argument in our situation.

Let $L^{2,4}$ be the orthogonal complement $\mathrm{N}^{\perp} \subset \Lambda_{K3}$ in the K3 lattice $\Lambda_{K3} = H^{\oplus 3} \oplus E_8(-1) \oplus E_8(-1)$ with orthogonal transformations $O(L^{2,4})$. Let $\mathcal{D}_{2,4}$ be the Hermitian symmetric space, specifically the bounded symmetric domain of type $IV_4$, given as

$$\mathcal{D}_{2,4} = O^+(2,4)/\big(SO(2) \times O(4)\big)\,, \tag{3.4.1}$$

where $O^+(2,4)$ denotes the subgroup of index two of the pseudo-orthogonal group $O(2,4)$ consisting of the elements whose upper left minor of order two is positive. Let $O^+(2,4;\mathbb{Z}) = O(L^{2,4}) \cap O^+(2,4)$ be the arithmetic lattice of $O^+(2,4)$, i.e., the discrete cofinite group of holomorphic automorphisms on the bounded Hermitian symmetric domain $\mathcal{D}_{2,4}$. We also set $SO^+(2,4) = O^+(2,4) \cap SO(2,4)$ and $SO^+(2,4;\mathbb{Z}) = O(L^{2,4}) \cap SO^+(2,4)$.

An appropriate version of the Torelli theorem [13] gives rise to an analytic isomorphism between the moduli space $\mathfrak{M}_{\mathrm{N}}$ of N-polarized K3 surfaces and the quasi-projective four-dimensional algebraic variety $\mathcal{D}_{2,4}/O^+(2,4;\mathbb{Z})$. We consider the normal, finitely generated algebra $A(\mathcal{D}_{2,4},\mathcal{G}) = \oplus_{k\geq 0} A(\mathcal{D}_{2,4},\mathcal{G})_k$ of automorphic forms on $\mathcal{D}_{2,4}$ relative to a discrete subgroup $\mathcal{G}$ of finite covolume in $O^+(2,4)$, graded by the weight $k$ of the automorphic forms. For $\mathcal{G} = O^+(2,4;\mathbb{Z})$, the algebra $A(\mathcal{D}_{2,4},\mathcal{G})$ is freely generated by forms $J_k$ of weight $2k$ with $k = 2,3,4,5,6$; this is a special case of a general result proven by Vinberg in [58, 59]. A subtle point here is that one has to obtain $A(\mathcal{D}_{2,4},\mathcal{G})$ as the *even part*

$$A(\mathcal{D}_{2,4},\mathcal{G}) = \Big[A(\mathcal{D}_{2,4},\mathcal{G}_0)\Big]_{\mathrm{even}} \tag{3.4.2}$$

of the ring of automorphic forms with respect to the index-two subgroup $\mathcal{G}_0 = SO^+(2,4;\mathbb{Z})$.

Based on the exceptional analytic equivalence between the bounded symmetric domains of type $IV_4$ and of type $I_{2,2}$, an explicit description of the generators $\{J_k\}_{k=2}^{6}$ can be derived.

To start, we remark that $\mathcal{D}_{2,4} \cong \mathbf{H}_2$, where

$$\mathbf{H}_2 = \left\{ \begin{pmatrix} \tau_1 & z_1 \\ z_2 & \tau_2 \end{pmatrix} \in \mathrm{Mat}(2,2;\mathbb{C}) \;\Big|\; \mathrm{Im}\,(\tau_1) \cdot \mathrm{Im}\,(\tau_2) > \frac{1}{4}|z_1 - \bar{z}_2|^2 ,\ \mathrm{Im}\,\tau_1 > 0 \right\} . \tag{3.4.3}$$

The domain $\mathbf{H}_2$ is a generalization of the Siegel upper-half space $\mathbb{H}_2$ in the sense that

$$\mathbb{H}_2 = \left\{ \varpi \in \mathbf{H}_2 \;\middle|\; \varpi^t = \varpi \right\} . \tag{3.4.4}$$

A subgroup $\Gamma \subset \mathrm{U}(2,2)$, given by

$$\Gamma = \left\{ G \in \mathrm{GL}\left(4, \mathbb{Z}[i]\right) \;\Big|\; G^\dagger \cdot \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix} \cdot G = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix} \right\} , \tag{3.4.5}$$

acts on $\varpi \in \mathbf{H}_2$ by

$$\forall\, G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma : \quad G \cdot \varpi = (C \cdot \varpi + D)^{-1} (A \cdot \varpi + B) . \tag{3.4.6}$$

There is an involution $T$ acting on $\mathbf{H}_2$ by transposition, i.e., $\varpi \mapsto T \cdot \varpi = \varpi^t$, yielding an extended group as the semi-direct product $\Gamma_T = \Gamma \rtimes \langle T \rangle$. Moreover, the group $\Gamma_T$ has the index-two subgroup given by

$$\Gamma_T^+ = \left\{ g = G\, T^n \in \Gamma_T \;\Big|\; n \in \{0,1\},\ (-1)^n \det G = 1 \right\} . \tag{3.4.7}$$

The identification $\mathcal{D}_{2,4} \cong \mathbf{H}_2$ gives rise to an homomorphism $\mathrm{U}(2,2) \to SO^+(2,4)$ which identifies $\mathcal{G}_0 \cong \Gamma_T^+$ [58].

Let us consider the principal modular sub-group of $\Gamma$ of complex level $1+i$ over the Gaussian integers, denoted by $\Gamma(1+i)$, and the subgroup $\Gamma_T^+(1+i) = \Gamma_T^+ \cap \Gamma_T(1+i)$. The ring of modular forms relative to $\Gamma_T^+(1+i)$ is generated by any five out of ten standard theta functions $\theta_i^2(\varpi)$ for $1 \le i \le 10$ of weight two, transforming with a character $\chi(g) = \det G$ for $g = G\, T^n \in \Gamma_T^+(1+i)$, and a unique modular form $\Theta(\varpi)$ of weight four whose square is a polynomial of degree four in $\{\theta_i^2(\varpi)\}_{i=1}^{10}$. These theta function were described by Matsumoto in [34]. We recall that there is a well-known isomorphism $\Gamma/\Gamma(1+i) \cong S_6 -$

where $S_6$ is the permutation group of six elements that naturally acts on the theta functions $\theta_i^2(\varpi)$ – and that under the action $\varpi \mapsto T \cdot \varpi = \varpi^t$ we have

$$\Big(\theta_1(\varpi), \dots, \theta_{10}(\varpi), \Theta(\varpi)\Big) \ \mapsto \ \Big(\theta_1(\varpi), \dots, \theta_{10}(\varpi), -\Theta(\varpi)\Big) . \tag{3.4.8}$$

In [10] we described a set of explicit generators for the ring of modular forms relative to the subgroup $\mathcal{G}_0 \cong \Gamma_T^+$: five modular forms $J_k$ of weight $2k$ with $k = 2, 3, 4, 5, 6$ and even characteristic, i.e., $\chi(g) = \det(G)^k$, were constructed as $S_6$-invariant polynomials in the theta functions $\{\theta_i^2(\varpi)\}_{i=1}^{10}$ such that $J_4(\varpi) = (\Theta(\varpi)/15)^2$ and

$$J_5(\varpi)^2 - 4\, J_4(\varpi)\, J_6(\varpi) = 2^{-4} 3^{10} \prod_{i=1}^{10} \theta_i^2(\varpi) , \tag{3.4.9}$$

We also introduce the automorphic form $\mathfrak{a}(\varpi) = 2^{-2}3^5 \prod \theta_i(\varpi)$ of weight 10 with nontrivial automorphic factor satisfying $\mathfrak{a}^2 = J_5^2 - 4J_4J_6$, and the modular form $J_{30}$ of weight 60 (with canonical automorphic factor) given by

$$J_{30} = 3^{30}(\theta_2^2 - \theta_3^2)^2(\theta_3^2 - \theta_4^2)^2(\theta_4^2 - \theta_5^2)^2(\theta_5^2 - \theta_6^2)^2(\theta_2^2 - \theta_4^2)^2(\theta_4^2 - \theta_6^2)^2 \prod_{j=2}^{10}(\theta_1^2 - \theta_j^2)^2 . \tag{3.4.10}$$

Equation (3.4.10) can also be rewritten as a polynomial in terms of the generators $\{J_k\}_{k=2}^6$ with integer coefficients; see [10]. We have:

**Theorem 3.4.1** ( [10]). *The graded ring of modular forms relative to $\Gamma_T^+$ of even characteristic is generated over $\mathbb{C}$ by the five algebraically independent modular forms $J_k(\varpi)$ of weight $2k$ with $k = 2, \dots, 6$.*

We also have:

**Theorem 3.4.2** ( [10]). *The coarse moduli space of* $\mathrm{N}$*-polarized K3 surfaces is*

$$\mathfrak{M}_{\mathrm{N}} = \Big\{ \vec{J} = [J_1 : J_2 : J_3 : J_4 : J_5 : J_6] \in \mathbb{WP}(2,3,4,5,6) \ \mid \ (J_3, J_4, J_5) \neq (0,0,0) \Big\} . \tag{3.4.11}$$

*In particular, the parameters $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta) \in \mathbb{C}^6$ in Equation (3.3.1) are given by*

$$\vec{J} = [J_2 :\ J_3 :\ J_4 :\ J_5 :\ J_6] = [\,\alpha :\ \beta :\ \gamma \cdot \varepsilon :\ \gamma \cdot \zeta + \delta \cdot \varepsilon :\ \delta \cdot \zeta] , \tag{3.4.12}$$

*as points in the four dimensional weighted projective space* $\mathbb{WP}(2,3,4,5,6)$.

We now prove:

**Theorem 3.4.3.** *The four inequivalent Jacobian elliptic fibrations on the family of* $\mathrm{N}$-*polarized K3 surfaces* $\mathcal{X}(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ *admit Weierstrass models with coefficients in* $\mathbb{Q}[J_2,J_3,J_4,J_5,J_6]$ *(or* $\mathbb{Q}[J_2,J_3,\mathfrak{a},J_5,J_6]$ *in case of the standard fibration).*

*Proof.* We determine the Weierstrass models whose coefficients are modular forms:

**The standard fibration:** The Jacobian elliptic fibration $\pi_{\text{std}}:\mathcal{X}\to\mathbb{P}^1$ in Lemma 3.3.8 is written in a suitable affine coordinate chart as

$$Y^2 = X^3 + f(t)\,X + g(t)\,, \tag{3.4.13}$$

with

$$f(t) = -t^3\Big(\frac{J_5-\mathfrak{a}}{2\,J_6}t^2 + 3J_2t + \frac{J_5+\mathfrak{a}}{2}\Big)\,,\qquad g(t) = t^5\Big(t^2-2J_3t+J_6\Big)\,, \tag{3.4.14}$$

and a discriminant $\Delta = J_6^{-3}t^9p(t)$ where

$$p(t) = 2\Big(J_5^2(J_5+\mathfrak{a}) - J_4J_6(3J_5+\mathfrak{a})\Big)t^6 + \dots + 2J_6^3\Big(J_5^2(J_5-\mathfrak{a}) - J_4J_6(3J_5-\mathfrak{a})\Big)\,. \tag{3.4.15}$$

**The alternate fibration:** The Jacobian elliptic fibration $\pi_{\text{alt}}:\mathcal{X}\to\mathbb{P}^1$ in Lemma 3.3.10 is written in a suitable affine coordinate chart as

$$Y^2 = X\Big(X^2 + A(t)\,X + B(t)\Big)\,, \tag{3.4.16}$$

with

$$A(t) = t^3 - 3J_2\,t - 2J_3\,,\qquad B(t) = J_4\,t^2 - J_5\,t + J_6\,, \tag{3.4.17}$$

and a discriminant $\Delta = E(t)^2D(t)$ where $E(t) = J_4t^2 - J_5t + J_6$ and

$$D(t) = t^6 - 6J_2t^4 - 4J_3t^3 + (9J_2^2-4J_4)t^2 + (12J_2J_3+4J_5)t + 4(J_3^2-J_6)\,. \tag{3.4.18}$$

**The base-fiber dual fibration:** The Jacobian elliptic fibration $\pi_{\text{bfd}} : \mathcal{X} \to \mathbb{P}^1$ in Lemma 3.3.12 is written in a suitable affine coordinate chart as

$$Y^2 = X^3 + F(t)\,X + G(t)\,, \tag{3.4.19}$$

with

$$\begin{aligned} F(t) &= t^2\Big( - 3J_2 t^2 - J_5 t - \frac{1}{3} J_4^2 \Big)\,, \\ G(t) &= t^3 \Big( t^4 - 2J_3 t^3 + (J_2 J_4 + J_6) t^2 + \frac{1}{3} J_4 J_5 t + \frac{2}{27} J_4^3 \Big)\,, \end{aligned} \tag{3.4.20}$$

and a discriminant $\Delta = t^8 P(t)$ where $P(t) = -27t^6 + 108 J_3 t^5 + \dots + \mathfrak{a}^2 J_4^2$.

**The maximal fibration:** The Jacobian elliptic fibration $\pi_{\text{max}} : \mathcal{X} \to \mathbb{P}^1$ in Lemma 3.3.14 is written in a suitable affine coordinate chart as

$$Y^2 = X^3 + a(t)\,X^2 + b(t)\,X + c(t)\,, \tag{3.4.21}$$

with

$$\begin{aligned} a(t) &= J_6 \Big( t^3 + 6 J_3 J_4 t^2 + 3(4 J_3^2 J_4^2 - J_2 J_6^2) t - 2 J_3 (3 J_2 J_4 J_6^2 - 4 J_3^2 J_4^3 + J_6^3) \Big)\,, \\ b(t) &= - J_6^6 \Big( 2 J_4 t^2 + (8 J_3 J_4^2 + J_5 J_6) t + (8 J_3^2 J_4^3 - 3 J_2 J_4 J_6^2 + 2 J_3 J_4 J_5 J_6 - J_6^3) \Big)\,, \\ c(t) &= J_4 J_6^{11} \Big( J_4 t + (2 J_3 J_4^2 + J_5 J_6) \Big)\,, \end{aligned}$$

and a discriminant $\Delta = J_6^{16} d(t)$ where $d(t) = \mathfrak{a}^2 t^8 + \dots$ is an irreducible polynomial of degree eight. By an appropriate change of coordinates, one can write the fibration 3.4.21 in Weierstrass normal form

$$y^2 = x^3 + \alpha(t) x + \beta(t)$$

where $\alpha(t) = -\frac{1}{3}\, {J_6}^2 t^6 - 4\, J_4 t^5 J_3 {J_6}^2 + \dots$ and $\beta(t) = \frac{2\, t^9 {J_6}^3}{27} + \frac{4}{3}\, J_4 t^8 J_3 {J_6}^3 - \dots$ are irreducible polynomials of degree six and nine respectively. A simple computation shows that the various discriminants (denoted by $\text{Disc}_t$) and resultants (denoted by $\text{Res}_t$) are related to the modular form $J_{30}$ by

$$J_{30} = \text{Disc}_t\, D = \text{Disc}_t\, d = \frac{2^4}{3^{18} J_6^{30}} \frac{\text{Disc}_t\, p}{\text{Res}_t^3(t^{-3} f,\, t^{-5} g)} = - \frac{J_2^9}{3^{21}} \frac{\text{Disc}_t\, P}{\text{Res}_t^3(t^{-2} F,\, t^{-3} G)}\,. \tag{3.4.22}$$

□

In Table 3.5 we show what how the lattice polarization is extended for the four inequivalent Jacobian elliptic fibrations determined in Theorem 3.4.3. We used the following isomorphisms for the lattice $\Lambda$ with discriminant group $D(\Lambda)$:

| rank | isomorphisms for the lattice $\Lambda$ | | | $D(\Lambda)$ |
|---|---|---|---|---|
| 16 | $H \oplus E_7(-1) \oplus E_7(-1)$ | $\cong$ | $H \oplus D_{14}(-1)$ | $\mathbb{Z}_2^2$ |
| | | $\cong$ | $H \oplus E_8(-1) \oplus D_6(-1)$ | |
| 17 | $H \oplus E_8(-1) \oplus D_7(-1)$ | $\cong$ | $H \oplus D_{15}(-1)$ | $\mathbb{Z}_4$ |
| 17 | $H \oplus E_7(-1) \oplus E_7(-1) \oplus A_1(-1)$ | $\cong$ | $H \oplus E_8(-1) \oplus D_6(-1) \oplus A_1(-1)$ | $\mathbb{Z}_2^3$ |
| | | $\cong$ | $H \oplus D_{14}(-1) \oplus A_1(-1)$ | |
| 18 | $H \oplus E_8(-1) \oplus E_8(-1)$ | $\cong$ | $H \oplus D_{16}^+(-1)$ | 0 |

### 3.5 A string theory point of view

In a standard compactification of the type IIB string theory, the axio-dilaton field $\tau$ is constant and no D7-branes are present. Vafa's idea in proposing F-theory [56] was to simultaneously allow a variable axio-dilaton field $\tau$ and D7-brane sources, defining at a new class of models in which the string coupling is never weak. These compactifications of the type IIB string in which the axio-dilaton field varies over a base are referred to as *F-theory models*. They depend on the following key ingredients: an $\mathrm{SL}_2(\mathbb{Z})$ symmetry of the physical theory, a complex scalar field $\tau$ with positive imaginary part on which $\mathrm{SL}_2(\mathbb{Z})$ acts by fractional linear transformations, and D7-branes serving as the source for the multi-valuedness of $\tau$. In this way, F-theory models correspond geometrically to torus fibrations over some compact base manifold.

A well-known duality in string theory asserts that compactifying M-theory on a torus $\mathbf{T}^2$ with complex structure parameter $\tau$ and area $A$ is dual to the type IIB string compactified on a circle of radius $A^{-3/4}$ with axio-dilaton field $\tau$ [1, 52]. This gives a connection between F-theory models and geometric compactifications of M-theory: after compactifying an F-theory model further on $S^1$ without breaking supersymmetry, one obtains a model that is dual to M-theory compactified on the total space of the torus fibration. The geometric M-theory model preserves supersymmetry exactly when the total space of the family is a Calabi-Yau manifold. In this way, we recover the familiar condition for supersymmetric F-theory models in eight dimensions: the total space of the fibration has to be a K3 surface.

In this article, we will focus on F-theory models associated with eight-dimensional compactifications that correspond to genus-one fibrations with a section, or Jacobian elliptic fibrations. As pointed out by Witten [60], this subclass of models is physically easier to treat since the existence of a section implies the absence of NS-NS and R-R fluxes in F-theory. Geometrically, the restriction to Jacobian elliptic fibrations facilitates model building with various non-Abelian gauge symmetries using the Tate algorithm [21, 30] where insertions of seven-branes in an F-theory model correspond to singular fibers in the M-theory model. Through work of Kodaira [24] and Néron [43], all possible singular fibers in one-parameter families of elliptic curves have been classified. The catalog and its physical interpretation is by now well-known; see [33].

### 3.5.1 Non-geometric heterotic models and string duality

An eight-dimensional effective theory for the heterotic string compactified on $\mathbf{T}^2$ has a complex scalar field which takes its values in the Narain space [41]

$$\mathcal{D}_{2,18}/O(\Lambda^{2,18}), \tag{3.5.1}$$

where $\mathcal{D}_{p,q}$ is the symmetric space for $O(p,q)$, i.e.,

$$\mathcal{D}_{p,q} = (O(p) \times O(q))\backslash O(p,q). \tag{3.5.2}$$

The *Narain space* is the quotient of the symmetric space for $O(2,18)$ by the automorphism group $O(\Lambda^{2,18})$ of the unique integral even unimodular lattice of signature $(2,18)$, i.e.,

$$\Lambda^{2,18} = H \oplus H \oplus E_8(-1) \oplus E_8(-1)\ . \tag{3.5.3}$$

In an appropriate limit, the Narain space decomposes as a product of spaces parameterizing the Kähler and complex structures on $\mathbf{T}^2$ as well as sixteen Wilson line expectation values around the two generators of $\pi_1(\mathbf{T}^2)$; see [42] for details. However, the decomposition is not preserved when the moduli vary arbitrarily. Families of heterotic models employing the full $O(\Lambda^{2,18})$ symmetry are therefore considered *non-geometric* compactifications, because the Kähler and complex structures on $\mathbf{T}^2$, and the Wilson line values, are not distinguished under the $O(\Lambda^{2,18})$-equivalences but instead are mingled together.

If we restrict ourselves to a certain index-two sub-group $O^+(\Lambda^{2,18}) \subset O(\Lambda^{2,18})$ in the construction above, the non-geometric models can be described by holomorphic modular forms. This is because the group $O^+(\Lambda^{2,18})$ is the maximal sub-group whose action preserves the complex structure on the symmetric space, and thus is the maximal sub-group for which modular forms are holomorphic. The statement of the *F-theory/heterotic string duality* in eight dimensions [56] is the statement that quotient space

$$\mathcal{D}_{2,18}/O^+(\Lambda^{2,18}) \tag{3.5.4}$$

coincides with the parameter space of elliptically fibered K3 surfaces with a section, i.e, the moduli space of F-theory models. This statement has been known in the mathematics literature as well; see, for example, [16]. However, to construct the duality map between F-theory model and heterotic sting vacua explicitly, one has to know the ring of modular forms relative to $O^+(\Lambda^{2,18})$ and their connection to the corresponding elliptically fibered K3 surfaces. However, this ring of modular forms is not known in general. In this article, we consider the restriction to a natural four-dimensional sub-space of the space in Equation (3.5.4).

Let $L^{2,4}$ be the lattice of signature $(2,4)$ which is the orthogonal complement of $E_7(-1) \oplus E_7(-1)$ in $\Lambda^{2,18}$. By insisting that the Wilson lines values associated to the $E_7(-1) \oplus E_7(-1)$ sub-lattice are trivial, we restrict to heterotic vacua parameterized by the sub-space

$$\mathcal{D}_{2,4}/O(L^{2,4})\,. \tag{3.5.5}$$

The corresponding degree-two cover is precisely the quotient space discussed above, namely

$$\mathcal{D}_{2,4}/O^+(L^{2,4})\,. \tag{3.5.6}$$

For this four-dimensional sub-space in the full eighteen dimensional moduli space we will determine the duality map (and thus the quantum-exact effective interactions) between a dual F-theory/heterotic string pair in eight space-time dimensions. As we will show, the restriction to this sub-space describes the partial higgsing of the corresponding heterotic gauge algebra $\mathfrak{g} = \mathfrak{e}_8 \oplus \mathfrak{e}_8$ to either $\mathfrak{e}_8 \oplus \mathfrak{so}(12)$ or $\mathfrak{e}_7 \oplus \mathfrak{e}_7$ for the associated low energy effective eight-dimensional supergravity theory, and similarly the higgsing of $\mathfrak{g} = \mathfrak{so}(32)$ to either $\mathfrak{so}(24) \oplus \mathfrak{su}(2)^{\oplus 2}$ or $\mathfrak{so}(28)$. Moreover, our results from Section 3.2 prove that for the dual F-theory models there are *no* Jacobian elliptic fibrations on the sub-space (3.5.6) with a Mordell-Weil group of positive rank. Non-torsion sections in a Weierstrass model are known to describe the charged matter fields of the corresponding F-theory model [11, 36]. Thus, for generic families of non-geometric heterotic compactifications sampling the moduli space (3.5.6) there cannot be any charged matter fields.

Since there is no microscopic description of the dual F-theory, the explicit form of the F-theory/heterotic string duality in this article provides new insights into the physics of F-theory compactifications and is also of critical importance for the understanding of non-perturbative aspects of the heterotic string, for example, as it relates to NS5-branes states and small instantons [50, 61]. For no or one non-trivial Wilson line parameters, an analogous approach was proven to provide a quantum-exact effective description of non-geometric heterotic models [17, 33, 37].

### 3.5.2 The $\mathfrak{e}_8 \oplus \mathfrak{e}_8$-string

As we have seen, the space in Equation (3.5.6) parameterizes pseudo-ample K3 surfaces with $H \oplus E_7(-1) \oplus E_7(-1)$ lattice polarization. Theorem 3.3.6 proves that these K3 surfaces admit an elliptic fibration with section which has one fiber of Kodaira type $I_2^*$ or worse and another fiber of type precisely $II^*$. Here, we have used the lattice isomorphism

$$H \oplus E_7(-1) \oplus E_7(-1) \cong H \oplus E_8(-1) \oplus D_6(-1)\,. \tag{3.5.7}$$

Because of the presence of a $II^*$ fiber, the Mordell-Weil group is always trivial, including all cases with gauge symmetry enhancement. From a physics point of view as argued in [33], assuming that one fiber is fixed and of Kodaira type $II^*$ will avoid "pointlike instantons" on the heterotic dual after further compactification to dimension six or below, at least for general moduli.

The key geometric fact for the construction of F-theory models is that Equation (3.4.19) defines an elliptically fibered K3 surface $\mathcal{X}$ with section whose periods determine a point $\varpi \in \mathbf{H}_2$, with the coefficients in the equation being modular forms relative to $\Gamma^+_{\mathcal{T}}$ of even characteristic. The explicit form of the F-theory/heterotic string duality on the moduli space in Equation (3.5.6) then has two parts: starting from $\varpi \in \mathbf{H}_2$, we always obtain the equation of a Jacobian elliptic fibration on the K3 surface given by Equation (3.4.19). Conversely, we can start with any Jacobian elliptic fibration given by the general equation

$$Y^2 = X^3 + a\,t^2 X + b\,t^3 + c\,t^3 X + c\,d\,t^4 + e\,t^4 X + (d\,e + f)\,t^5 + g\,t^6 + t^7\,. \tag{3.5.8}$$

If we then determine a point in $\varpi \in \mathbf{H}_2$ by calculating the periods of the holomorphic two-form $\omega_{\mathcal{X}} = dT \wedge dX/Y$ over a basis of the lattice $H \oplus E_7(-1) \oplus E_7(-1)$ in $H^2(\mathcal{X}, \mathbb{Z})$, it follows that for some non-vanishing scale factor $\lambda$ we have

$$c = -\lambda^{10} J_5(\varpi), \quad d = -\frac{\lambda^8}{3} J_4(\varpi), \quad e = -3\,\lambda^4 J_2(\varpi), \quad f = \lambda^{12} J_6(\varpi), \quad g = -2\,\lambda^6 J_3(\varpi),$$

and $a = -3\,d^2, \quad b = -2\,d^3$. Under the restriction of $\mathbf{H}_2$ to $\mathbb{H}_2$, we have $d = 0$ and

$$[J_2(\varpi) : J_3(\varpi) : J_4(\varpi) : J_5(\varpi) : J_6(\varpi)] = [\psi_4(\tau) : \psi_6(\tau) : 0 : 2^{12}3^5\chi_{10}(\tau) : 2^{12}3^6\chi_{12}(\tau)]\,, \tag{3.5.9}$$

as points in the four dimensional weighted projective space $\mathbb{WP}(2, 3, 4, 5, 6)$, where $\psi_4, \psi_6, \chi_{10}$, and $\chi_{12}$ are Siegel modular forms of respective weights $4, 6, 10$ and $12$ introduced by Igusa in [19]. Moreover, after a simple rescaling, Equation (3.4.19) reduces to

$$Y^2 = X^3 - t^3\Big(\frac{1}{48}\psi_4(\tau)t + 4\chi_{10}(\tau)\Big)X + t^5\Big(t^2 - \frac{1}{864}\psi_6(\tau)\,t + \chi_{12}(\tau)\Big)\,, \tag{3.5.10}$$

which is precisely the equation derived in [37] for the F-theory dual of a non-geometric heterotic theory with gauge algebra $\mathfrak{g} = \mathfrak{e}_8 \oplus \mathfrak{e}_7$.

### 3.5.3 Condition for five-branes and supersymmetry

The strategy for constructing *families* of non-geometric heterotic compactifications is the following: start with a compact manifold $\mathfrak{Z}$ as parameter space and a line bundle $\Lambda \to \mathfrak{Z}$. Choose sections $c(z)$, $d(z)$, $e(z)$, $f(z)$, and $g(z)$ of the bundles $\Lambda^{\otimes 10}$, $\Lambda^{\otimes 8}$, $\Lambda^{\otimes 4}$, $\Lambda^{\otimes 12}$, and $\Lambda^{\otimes 6}$, respectively; then, for each point $z \in \mathfrak{Z}$, there is a non-geometric heterotic compactification given by Equation (3.5.8) with $c = c(z)$, $d = d(z)$, etc., and $a = -3\,d(z)^2$, $b = -2\,d(z)^3$ and moduli $\varpi \in \mathbf{H}_2$ and $O^+(L^{2,4})$ symmetry such that Equations (3.5.2) hold.

Appropriate five-branes must still be inserted on $\mathfrak{Z}$ as dictated by the geometry of the corresponding family of K3 surfaces. The change in the singularities and the lattice polarization for the fibration (3.4.19) occur along three loci of co-dimension one, namely, $\mathfrak{a} = 0$, $J_{30} = 0$, and $J_4 = 0$. Each locus is the fixed locus of elements in $\Gamma_{\mathcal{T}} \setminus \Gamma^+_{\mathcal{T}}$. It is trivial to write down the reflections in $O^+(L^{2,4})\backslash SO^+(L^{2,4})$ corresponding to $\mathfrak{a} = 0$, $J_{30} = 0$, and $J_4 = 0$, respectively; see [10].

From the point of view of K3 geometry, given as a reflection in a lattice element $\delta$ of square $-2$ we have the following: if the periods are preserved by the reflection in $\delta$, then $\delta$ must belong to the Néron-Severi lattice of the K3 surface. That is, the Néron-Severi lattice is enlarged by adjoining $\delta$. We already showed that there are three ways an enlargement can happen: the lattice $H \oplus E_7(-1) \oplus E_7(-1)$ of rank sixteen can be extended to $H \oplus E_7(-1) \oplus E_7(-1) \oplus \langle -2 \rangle$, $H \oplus E_8(-1) \oplus E_7(-1)$, or $H \oplus E_8(-1) \oplus D_7(-1)$, each of rank seventeen.

On the heterotic side these five-brane solitons are easy to see: when $J_{30} = 0$, we have a gauge symmetry enhancement from $\mathfrak{e}_7 \oplus \mathfrak{e}_7$ to include an additional $\mathfrak{su}(2)$, and the parameters of the theory include a Coulomb branch for that gauge theory on which the Weyl group $W_{\mathfrak{su}(2)} = \mathbb{Z}_2$ acts. Thus, there is a five-brane solution in which the field has a $\mathbb{Z}_2$ ambiguity encircling the location in the moduli space of enhanced gauge symmetry. When $J_4 = 0$, we have an enhancement to $\mathfrak{e}_8 \oplus \mathfrak{e}_7$ gauge symmetry, and, when $\mathfrak{a} = 0$ an enhancement to $\mathfrak{e}_8 \oplus \mathfrak{so}(14)$. Further enhancement to $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ gauge symmetry occurs along $J_4 = J_5 = 0$.

To understand when such families of compactifications are supersymmetric, we mirror the discussion in [33]. A heterotic compactification on $\mathbf{T}^2$ with parameters given by $\varpi \in \mathbf{H}_2$ is dual to the F-theory compactification on the elliptically fibered K3 surface $\mathcal{X}(\varpi)$. For sections $c(z)$, $d(z)$, $e(z)$, $f(z)$, and $g(z)$ of line bundles over $\mathfrak{Z}$, we have a criterion for when F-theory compactified on the elliptically fibered manifold (3.5.8) is supersymmetric: this is the case if and only if the total space defined by Equation (3.5.8) – now considered as an elliptic fibration over a base space locally given by variables $t$ and $z$ – is itself a Calabi–Yau manifold. The base space of the elliptic fibration is a $\mathbb{P}^1$-bundle $\pi : \mathfrak{W} \to \mathfrak{Z}$ which takes the form $\mathfrak{W} = \mathbb{P}(\mathcal{O} \oplus \mathcal{M})$ where $\mathcal{M} \to \mathfrak{Z}$ is the normal bundle of $\Sigma_0 := \{t = 0\}$ in $\mathfrak{W}$. Monomials of the form $t^n$ are then considered sections of the line bundles $\mathcal{M}^{\otimes n}$. We also set $\Sigma_\infty := \{t = \infty\}$ such that $-K_{\mathfrak{W}} = \Sigma_0 + \Sigma_\infty + \pi^{-1}(-K_{\mathfrak{Z}})$.

When the elliptic fibration (3.5.8) is written in Weierstrass form, the coefficients of $X^1$ and $X^0$ must again be sections of $\mathcal{L}^{\otimes 4}$ and $\mathcal{L}^{\otimes 6}$, respectively, for a line bundle $\mathcal{L} \to \mathfrak{W}$. The condition for supersymmetry of the total space is $\mathcal{L} = \mathcal{O}_{\mathfrak{W}}(-K_{\mathfrak{W}})$. Restricting the various terms in Equation (3.5.8) to $\Sigma_0$, we find relations

$$\begin{aligned}
(\mathcal{L}|_{\Sigma_0})^{\otimes 4} &= \Lambda^{\otimes 4} \otimes \mathcal{M}^{\otimes 4} = \Lambda^{\otimes 10} \otimes \mathcal{M}^{\otimes 3} = \Lambda^{\otimes 16} \otimes \mathcal{M}^{\otimes 2}, \\
(\mathcal{L}|_{\Sigma_0})^{\otimes 6} &= \mathcal{M}^{\otimes 7} = \Lambda^{\otimes 6} \otimes \mathcal{M}^{\otimes 6} \\
&= \Lambda^{\otimes 12} \otimes \mathcal{M}^{\otimes 5} = \Lambda^{\otimes 18} \otimes \mathcal{M}^{\otimes 4} = \Lambda^{\otimes 24} \otimes \mathcal{M}^{\otimes 3}
\end{aligned} \tag{3.5.11}$$

Thus, it follows that $\mathcal{M} = \Lambda^{\otimes 6}$ and $\mathcal{L}|_{\Sigma_0} = \Lambda^{\otimes 7}$ (up to torsion) and the $\mathbb{P}^1$-bundle takes the form $\mathfrak{W} = \mathbb{P}(\mathcal{O} \oplus \Lambda^{\otimes 6})$. Since $\Sigma_0$ and $\Sigma_\infty$ are disjoint, the condition for supersymmetry is equivalent to $\Lambda = \mathcal{O}_{\mathfrak{Z}}(-K_{\mathfrak{Z}})$.

### 3.5.4 Double covers and pointlike instantons

To a reader familiar with elliptic fibrations, it might come as a surprise that the Weierstrass model we considered in Equation (3.5.8) did not simply have two fibers of Kodaira type $III^*$ and a trivial Mordell-Weil group. On each K3 surface endowed with a $H \oplus E_7(-1) \oplus E_7(-1)$ lattice polarization such a fibration exists, and we constructed it in Equation (3.4.13). However, it is not guaranteed that the fibration extends across any parameter space, and there might be anomalies present.

If we vary non-geometric heterotic vacua with $(J_3, J_4, J_5) \neq (0,0,0)$ over a parameter space $\mathfrak{Z}$ as in Section 3.5.3, the functions $J_k$ are again sections of line bundles $\Lambda^{\otimes 2k} \to \mathfrak{Z}$. For the coefficient of $t^5 X$ in Equation (3.4.13) to be well defined, a necessary condition is $J_6 \neq 0$ over $\mathfrak{Z}$ which implies that $J_6$ is a trivializing section for the bundle $\Lambda^{\otimes 12}$; in particular, we have $\Lambda^{\otimes 12} \cong \mathcal{O}_{\mathfrak{Z}}$. Thus, $\mathfrak{a}^2$ is a section of the line bundle $\Lambda^{\otimes 20}$. We want to take the square root of this line bundle $\Lambda^{\otimes 20}$, that is, construct a line bundle $\Lambda' \to \mathfrak{Z}$ with $(\Lambda')^{\otimes 2} = \Lambda^{\otimes 20}$ such that $\mathfrak{a}$ becomes a section of the new line bundle. The square root of a line bundle (if it exists) is not unique in general, and any two of them will differ by a two-torsion line bundle. If the Picard group of $\mathfrak{Z}$ is torsion free, then there is at most one square root. We already know that one square root exists, namely the line bundle $\Lambda^{\otimes 10} \to \mathfrak{Z}$. Therefore, setting $H^2(\mathfrak{Z}, \mathbb{Z}_2) = 0$ guarantees that the square root is isomorphic to $\Lambda^{\otimes 10}$.

If we further assume that the line bundle is effective, i.e., $\Lambda^{\otimes 10} \cong \mathcal{O}_{\mathfrak{Z}}(D)$ for some effective divisor $D$ – which is equivalent to $\dim H^0(\mathfrak{Z}, \Lambda^{\otimes 10}) > 0$ – then the existence of the square root of $\Lambda^{\otimes 20}$ is equivalent to *the existence of a double cover* $\mathfrak{Y} \to \mathfrak{Z}$ branched along the zero locus of the holomorphic section given by $J_5 - \mathfrak{a} = 0$.

Using the condition for supersymmetry already established in Section 3.5.3, we will assume that

$$\begin{array}{llll} (1) & H^2(\mathfrak{Z}, \mathbb{Z}_2) = 0\,, & (2) & \Lambda = \mathcal{O}_{\mathfrak{Z}}(-K_{\mathfrak{Z}})\,, \\ (3) & \dim H^0(\mathfrak{Z}, \Lambda^{\otimes 10}) > 0\,, & (4) & \Lambda^{\otimes 12} \cong \mathcal{O}_{\mathfrak{Z}}\,. \end{array} \tag{3.5.12}$$

Then, we obtain a consistent and supersymmetric family of non-geometric heterotic vacua with gauge algebra $\mathfrak{e}_7 \oplus \mathfrak{e}_7$ over the parameter space $\mathfrak{Y}$ which is a double cover of $\mathfrak{Z}$ branched along the locus $J_5 - \mathfrak{a} = 0$. The conditions derived in Equation (3.5.12) are similar to conditions governing global and local anomaly cancellation [31, 32]. For $J_4 = 0$ it follows $J_5^2 = \mathfrak{a}^2$, and the choice of square root $\mathfrak{a} = \pm J_5$ determines which of the two fibers of type $III^*$ is extended to a fiber of type $II^*$; Equation (3.4.13) then again reduces to Equation (3.5.10).

### 3.5.5 The $\mathfrak{so}(32)$-string

We proved that a K3 surface $\mathcal{X}$ with lattice polarization $H \oplus E_7(-1) \oplus E_7(-1)$ also admits two other fibrations, which we called the *alternate* and the *maximal* Jacobian elliptic fibration. These turn out to be related to the $\mathfrak{so}(32)$ heterotic string. We will now establish the explicit form of the F-theory/heterotic string duality on the moduli space (3.5.6) for the $\mathfrak{so}(32)$ string. The intrinsic property of the elliptically fibered K3 surfaces which lead to the corresponding F-theory backgrounds is the requirement that there is one singular fiber in the fibration be of type $I_n^*$ for some $n \geq 8$, and that there *either* is *or* is not a two-torsion element in the Mordell–Weil group. Under theses assumptions, following the same argument as in [37], we can always choose coordinates so that its Weierstrass equation is given by either Equation (3.4.16) or Equation (3.4.21).

In the first case, we can start with any Jacobian elliptic fibration given by

$$Y^2 = X^3 + \Big(t^3 + e\,t + g\Big)\,X^2 + \Big(-3\,d\,t^2 + c\,t + f\Big)\,X\,. \tag{3.5.13}$$

We then determine a point in $\mathcal{D}_{2,4}$ by calculating the periods of the holomorphic two-form $\omega_{\mathcal{X}}$ over a basis of the period lattice $H \oplus E_7(-1) \oplus E_7(-1)$ in $H^2(\mathcal{X}, \mathbb{Z})$. The gauge algebra is enhanced to $\mathfrak{so}(24) \oplus \mathfrak{su}(2)^{\oplus 2}$. It follows as in [2, 3] that the gauge group of this model is $(\mathrm{Spin}(24) \times SU(2) \times SU(2))/\mathbb{Z}_2$. For $J_{30} = 0$, we have a gauge symmetry enhancement to include an additional $\mathfrak{su}(2)$. When $\mathfrak{a} = 0$ the gauge algebra is enhanced to $\mathfrak{so}(24) \oplus \mathfrak{su}(4)$.

In the second case, a similar argument can be carried out to obtain Equation (3.4.21). The fiber over $t = \infty$ is of Kodaira type $I_8^*$ so that the gauge algebra is enhanced to $\mathfrak{so}(28)$. It can be easily see from the proof of Theorem 3.4.3 that for $\mathfrak{a} = 0$, we have a gauge symmetry enhancement to $\mathfrak{so}(30)$.

For $J_4 = 0$, the two cases coincide. That is, after using Equation (3.5.9) and a simple rescaling, both Equation (3.4.16) and Equation (3.4.21) restrict to

$$Y^2 = X^3 + \Big(t^3 - \frac{1}{48}\,\psi_4(\tau)\,t - \frac{1}{864}\,\psi_6(\tau)\Big)X^2 - \Big(4\,\chi_{10}(\tau)\,t - \chi_{12}(\tau)\Big)X\,, \tag{3.5.14}$$

which is precisely the equation derived in [37] for the F-theory dual of a heterotic theory with gauge algebra $\mathfrak{g} = \mathfrak{so}(28) \oplus \mathfrak{su}(2)$ and one non-vanishing Wilson line. For $J_4 = J_5 = 0$ the gauge group is further enhanced to $\mathrm{Spin}(32)/\mathbb{Z}_2$.

| Fibration (3.4.13) | $p_{\mathcal{X}}$ | Singular Fibers | $\mathrm{MW}(\pi_{\mathrm{std}}, \sigma_{\mathrm{std}})$ | Lattice Polarization $\Lambda$ | $D(\Lambda)$ |
|---|---|---|---|---|---|
| generic | 16 | $2III^* + 6I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_7(-1) \oplus E_7(-1)$ | $\mathbb{Z}_2^2$ |
| $\mathrm{Res}_t(t^{-3}f,\, t^{-5}g) = 0$ | 16 | $2III^* + II + 4I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_7(-1) \oplus E_7(-1)$ | $\mathbb{Z}_2^2$ |
| $J_{30} = 0$ | 17 | $2III^* + I_2 + 4I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_7(-1) \oplus E_7(-1) \oplus A_1(-1)$ | $\mathbb{Z}_2^3$ |
| $J_4 = 0$ | 17 | $II^* + III^* + 5I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus E_7(-1)$ | $\mathbb{Z}_2$ |
| $J_4 = J_5 = 0$ | 18 | $2II^* + 4I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus E_8(-1)$ | 0 |
| $\vec{J} = [\frac{5\,(6^{2/3})}{18} : -\frac{7}{9} : \frac{3\,\sqrt[3]{6}}{8} : \frac{6^{2/3}}{2} : 1]$ | 20 | $2III^* + I_4 + I_2$ | $\{\mathbb{I}\}$ | $H \oplus E_7(-1)^2 \oplus A_3(-1) \oplus A_1(-1)$ | 0 |

(a) Extensions of lattice polarizations for the *standard* fibration

| Fibration (3.4.16) | $p_{\mathcal{X}}$ | Singular Fibers | $\mathrm{MW}(\pi_{\mathrm{alt}}, \sigma_{\mathrm{alt}})$ | Lattice Polarization $\Lambda$ | $D(\Lambda)$ |
|---|---|---|---|---|---|
| generic | 16 | $I_8^* + 2I_2 + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $H \oplus E_7(-1) \oplus E_7(-1)$ | $\mathbb{Z}_2^2$ |
| $\mathrm{Res}_t(D,\, E) = 0$ | 16 | $I_8^* + III + I_2 + 5I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $H \oplus E_7(-1) \oplus E_7(-1)$ | $\mathbb{Z}_2^2$ |
| $\mathfrak{a} = 0$ | 17 | $I_8^* + I_4 + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $H \oplus E_8(-1) \oplus D_7(-1)$ | $\mathbb{Z}_4$ |
| $J_{30} = 0$ | 17 | $I_8^* + 3I_2 + 4I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $H \oplus E_7(-1) \oplus E_7(-1) \oplus A_1(-1)$ | $\mathbb{Z}_2^3$ |
| $J_4 = 0$ | 17 | $I_{10}^* + I_2 + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $H \oplus E_8(-1) \oplus E_7(-1)$ | $\mathbb{Z}_2$ |
| $J_4 = J_5 = 0$ | 18 | $I_{12}^* + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $H \oplus E_8(-1) \oplus E_8(-1)$ | 0 |

(b) Extensions of lattice polarizations for the *alternate* fibration

| Fibration (3.4.19) | $p_{\mathcal{X}}$ | Singular Fibers | $\mathrm{MW}(\pi_{\mathrm{bfd}}, \sigma_{\mathrm{bfd}})$ | Lattice Polarization $\Lambda$ | $D(\Lambda)$ |
|---|---|---|---|---|---|
| generic | 16 | $II^* + I_2^* + 6I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus D_6(-1)$ | $\mathbb{Z}_2^2$ |
| $\mathrm{Res}_t(t^{-2}F,\, t^{-3}G) = 0$ | 16 | $II^* + I_2^* + II + 4I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus D_6(-1)$ | $\mathbb{Z}_2^2$ |
| $\mathfrak{a} = 0$ | 17 | $II^* + I_3^* + 5I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus D_7(-1)$ | $\mathbb{Z}_4$ |
| $J_{30} = 0$ | 17 | $II^* + I_2^* + I_2 + 4I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus D_6(-1) \oplus A_1(-1)$ | $\mathbb{Z}_2^3$ |
| $J_4 = 0$ | 17 | $II^* + III^* + 5I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus E_7(-1)$ | $\mathbb{Z}_2$ |
| $J_4 = J_5 = 0$ | 18 | $2II^* + 4I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus E_8(-1)$ | 0 |
| $\vec{J} = [\frac{10}{9} : -\frac{28}{27} : 1 : \frac{8}{3} : \frac{16}{9}]$ | 20 | $II^* + I_5^* + I_2 + I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus D_9(-1) \oplus A_1(-1)$ | 0 |
| $\vec{J} = [\frac{2}{3} : 0 : 1 : 0 : 0]$ | 20 | $II^* + I_6^* + 2I_1$ | $\{\mathbb{I}\}$ | $H \oplus E_8(-1) \oplus D_{10}(-1)$ | 0 |

(c) Extensions of lattice polarizations for the *base-fiber dual* fibration

| Fibration (3.4.21) | $p_{\mathcal{X}}$ | Singular Fibers | $\mathrm{MW}(\pi_{\max}, \sigma_{\max})$ | Lattice Polarization $\Lambda$ | $D(\Lambda)$ |
|---|---|---|---|---|---|
| generic | 16 | $I_{10}^* + 8I_1$ | $\{\mathbb{I}\}$ | $H \oplus D_{14}(-1)$ | $\mathbb{Z}_2^2$ |
| $\mathrm{Res}_t(\alpha, \beta) = 0,$ | 16 | $I_{10}^* + II + 6I_1$ | $\{\mathbb{I}\}$ | $H \oplus D_{14}(-1)$ | $\mathbb{Z}_2^2$ |
| $\mathfrak{a} = 0$ | 17 | $I_{11}^* + 7I_1$ | $\{\mathbb{I}\}$ | $H \oplus D_{15}(-1)$ | $\mathbb{Z}_4$ |
| $J_{30} = 0$ | 17 | $I_{10}^* + I_2 + 6I_1$ | $\{\mathbb{I}\}$ | $H \oplus D_{14}(-1) \oplus A_1(-1)$ | $\mathbb{Z}_2^3$ |
| $J_4 = 0$ | 17 | $I_{10}^* + I_2 + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $H \oplus E_8(-1) \oplus E_7(-1)$ | $\mathbb{Z}_2$ |
| $J_4 = J_5 = 0$ | 18 | $I_{12}^* + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $H \oplus E_8(-1) \oplus E_8(-1)$ | 0 |

(d) Extensions of lattice polarizations for the *maximal* fibration

Fig. 3.5: Extensions of lattice polarization

# CHAPTER 4

## Future Directions

Configurations of six lines in $\mathbb{P}^2$, no three of them coincident in a single point, have been studied extensively in the literature [35], [18], [5]. The double cover of $\mathbb{P}^2$ branched along the union of these six lines is associated with a family $\mathcal{Y}$ of K3 surfaces of Picard rank sixteen. In turns out that members of this family $\mathcal{Y}$ generically have the transcendental lattice $H(2) \oplus H(2) \oplus \langle -2 \rangle \oplus \langle -2 \rangle$, and admit a canonical Jacobian elliptic fibration with the singular fibers of Kodaira type $6I_1 + 2I_0^*$ and the Mordell-Weil group $(\mathbb{Z}/2\mathbb{Z})^2$.

Moreover, Kloosterman classified all inequivalent Jacobian elliptic fibrations with section on the family $\mathcal{Y}$ in [22]. Among those is the so-called "alternate fibration", which has the singular fibers $I_4^* + 6I_2 + 2I_1$, the Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$, and a unique two-torsion section. A Van Geemen-Sarti involution on $\mathcal{Y}$ is a special Nikulin involution obtained as the fiber-wise translation by this two-torsion section in the alternate fibration. This involution was constructed in [5]. Taking the quotient of $\mathcal{Y}$ by this Nikulin involution results in a second family, $\mathcal{X}$, of K3 surfaces, after blowing up the eight fixed points. Geometrically, the family $\mathcal{X}$ is a rational double cover of the double sextic surfaces described above. In [5] it was also proven that the members of the family $\mathcal{X}$ admit a Jacobian elliptic fibration with the singular fibers $I_8^* + I_1 + 2I_2$, have a canonical $H \oplus E_7(-1) \oplus E_7(-1)$-lattice polarization, and admit a normal form that generalizes the Inose quartic. Based on an exceptional isomorphism of the four-dimensional bounded symmetric domains $D_{IV}^4 \cong I_{2,2}$, the authors of [5] then described the families $\mathcal{X}$ and $\mathcal{Y}$ in terms of a ring of modular forms which generalizes the ring of Siegel modular forms for $I_{2,2}$.

As explained above, Clingher, Malmendier, and I succeeded in classifying *all* Jacobian elliptic fibrations on $\mathcal{X}$ [9]. In particular, we proved that $\mathcal{X}$ admits exactly four inequivalent elliptic fibrations with section, all of which have only torsion in their Mordell-Weil groups. The proof follows from results by Kondō [28]. Additionally, we constructed explicit

Weierstrass models for these elliptic fibrations and described their coefficients in terms of modular forms.

**Future Directions:** An interesting future research project would be to study the natural generalization of these structures in Picard rank fourteen. A generalization of the family $\mathcal{Y}$ is the family $\mathcal{F}$ arising as double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along four bi-degree (1,1) curves. This family was considered in [27], where it was shown to have Picard rank fourteen and a $D_4^3 \oplus \langle 2 \rangle \oplus \langle -2 \rangle$-lattice polarization. The family $\mathcal{F}$ carries a canonical Jacobian elliptic fibration with singular fibers $12I_2$ and has a Mordell-Weil group of $(\mathbb{Z}/2\mathbb{Z})^2$, which generalizes the canonical fibration on $\mathcal{Y}$ with singular fibers $6I_1 + 2I_0^*$.

Using results of Shimada [53], I have already proven that $\mathcal{F}$ admits a second Jacobian elliptic fibration with a Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$ and the singular fibers $I_0^* + 8I_2 + 2I_1$ (of ADE type $D_4 + 8A_1$). In fact, this fibration generalizes the alternate fibration on $\mathcal{Y}$ by "splitting-up" the $I_4^*$ into one $I_0^*$ and two $I_2$ fibers. From this fibration, I have constructed a Van Geemen-Sarti involution and a second family of K3 surfaces, $\mathcal{G}$, with lattice polarization $H \oplus E_8 \oplus A_1^4$, generalizing the family $\mathcal{X}$ from above.

Research objectives for this new project would include the following:

**(R1): Classify all elliptic fibrations on $\mathcal{G}$.** The lattice $H \oplus E_8 \oplus A_1^4$ appears in a list by Kondō as one of the polarizing lattices for algebraic K3 surfaces with finite automorphism classes [28]. Therefore, we can use the same methods as in [9] to classify all inequivalent Jacobian elliptic fibrations on $\mathcal{G}$. Our preliminary results show that the family $\mathcal{G}$ admits exactly six such inequivalent Jacobian elliptic fibrations with section.

**(R2): Determine a hypersurface normal form for $\mathcal{G}$ which further generalizes the Inose quartic and exhibits the canonical $H \oplus E_8 \oplus A_1^4$-lattice polarization.** Given the explicit description of the canonical polarization on the hypersurface $\mathcal{X}$ in [8] and [9], it should be straightforward to derive such normal form in Picard rank fourteen realizing the splitting of the singular fibers described above.

**(R3): Describe the parameters of the family $\mathcal{F}$ in terms of modular forms over the six dimensional bounded symmetric domain $D_{IV}^6$.** In the rank fourteen case, there is again an exceptional isomorphism of bounded symmetric domains $D^6_{IV} \cong D_{II}(4)$; see [27], [58]. The methods used in [5] to construct modular forms describing the coefficients of $\mathcal{Y}$ from certain geometric data called Satake roots should also generalize to this case.

**(R4): Classify *all* Jacobian elliptic fibrations on the family $\mathcal{F}$.** This represents a generalization of the work by Kloosterman in [22], now for K3 surfaces of Picard rank fourteen [39].

**Intellectual Merit:** As described above, a modular description of the family of K3 surfaces is possible due to the exceptional isomorphism of bounded symmetric domains $D^6_{IV} \cong D_{II}(4)$. This is the highest of such exceptional analytic isomorphisms [27]. Therefore, this is the "last case" where relatively simple modular forms can be expected to give an explicit description of the moduli of the corresponding lattice-polarized K3 surfaces. At the same time, this moduli space for the family $\mathcal{F}$ is the largest moduli space we could hope to describe in this way, providing us with a coarse moduli space of K3 surfaces with many interesting sub-loci of varying co-dimension that we can use to understand and test different conjectures.

This project also has important applications to physics, in particular to the F-theory/ heterotic string duality and mirror symmetry. In [9], we showed how our modular description of the family of elliptically fibered K3 surfaces $\mathcal{X}$ gave rise to new and *explicit* constructions of the duality map (and thus the quantum-exact effective interactions) between a dual F-theory/heterotic string theories in eight space-time dimensions. Moreover, families of pseudo-ample lattice-polarized K3 surfaces are also an important testing ground for mirror symmetry. The explicit modular description of the family of K3 surfaces $\mathcal{X}$ is conjectured to be particularly useful in generalizing work in [4], that is how the notion of stability conditions and stability manifolds ought to be extended on the mirror dual side.

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