5-2021

Studies of Classical Analysis after Whittaker and Watson

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STUDIES OF CLASSICAL ANALYSIS AFTER WHITTAKER AND WATSON

by

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A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

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UTAH STATE UNIVERSITY
Logan, Utah

2021

ABSTRACT

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by

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Utah State University, 2021

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The goal of this thesis is to solve problems from the first four chapters of the book, titled *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions with an Account of the Principal Transcendental Functions* by E.T. Whittaker and G.N. Watson [13]. The titles of the first four chapters are "Complex Numbers," "The Theory of Convergence," "Continuous Functions and Uniform Convergence," and "The Theory of Riemann Integration," respectively. This book is a classic mathematical analysis textbook that contains some challenging end-of-chapter exercises and some details within each chapter are often left to the readers. Many exercises are results of famous mathematicians or problems from an older era of the Cambridge Mathematical Tripos. The purpose of this thesis is to provide solutions to the exercises in Chapters 1-4 and give insight to the readers of the book.

(115 pages)

PUBLIC ABSTRACT

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The goal of this thesis is to solve problems from the first four chapters of the book, titled *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions with an Account of the Principal Transcendental Functions* by E.T. Whittaker and G.N. Watson [13]. The titles of the first four chapters are "Complex Numbers," "The Theory of Convergence," "Continuous Functions and Uniform Convergence," and "The Theory of Riemann Integration," respectively. This book is a classic mathematical analysis textbook that contains some challenging end-of-chapter exercises and some details within each chapter are often left to the readers. Many exercises are results of famous mathematicians or problems from an older era of the Cambridge Mathematical Tripos. The purpose of this thesis is to provide solutions to the exercises in Chapters 1-4 and give insight to the readers of the book.

ACKNOWLEDGMENTS

I received a great deal of assistance and guidance throughout my graduate studies and I would like to express my gratitude to those individuals who helped me succeed and overcome challenges that I had.

First, I would like to thank Dr. Zhaohu Nie for his guidance and support for this project. Moreover, I would like to acknowledge the support of my committee members, Dr. Dave Brown and Dr. Ian Anderson. Furthermore, I would like to thank the Mathematics Stack Exchange community for answering questions that I had and providing solutions for some exercises.

Second, I would like to extend my thanks to the faculty and staff in the USU Math and Stats Department for the work they've done for graduate students, especially during this pandemic. In particular, I am thankful for the graduate coordinator, Gary Tanner.

In addition, I am grateful to my professors from my undergraduate for encouraging me to always do my best and to advance my education. Finally, I would like to extend my gratitude to my friends and family for their support and love.

Ting-Yao Lee

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CHAPTER 1

INTRODUCTION

1.1 Overview

A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions with an Account of the Principal Transcendental Functions is a classic mathematical analysis textbook written by E.T. Whittaker and G.N. Watson. It was first published in 1902 by Cambridge University Press and written by Whittaker alone. Editions two through four were co-authored with Watson. It is still in print today. One of the features of this book is that it contains a large number of challenging exercises such as problems from an older era of Cambridge Mathematical Tripos exams.

This book is divided into two parts. The first part is called "The Processes of Analysis," which covers a lot in classical analysis. The first part includes Riemann integrals, infinite series, analytic functions, as well as a brief discussion of Fourier series, differential, and integral equations. The first part is a preparation for the second part, titled "The Transcendental Functions." The second part talks about special functions in mathematical physics such as the gamma, elliptical, and Riemann Zeta functions [12]. We will only focus on "The Processes of Analysis" for this thesis.

As mentioned previously, this book is known for having a lot of challenging exercises. The goal of this thesis is to provide solutions to those challenging exercises from the first four chapters of the book. We will use the fourth edition. When we provide solutions, we will often refer to some page numbers. It is understood that those page numbers are from the fourth edition of the book.

1.2 Summary of the First Four Chapters

The first chapter is called "Complex Numbers," which is the shortest chapter among the four. It briefly talks about the basics of complex numbers such as the modulus and the Argand diagram. There are only three exercises from this chapter.

The second chapter is titled "The Theory of Convergence." The focus is the convergence of infinite series. Later it briefly talks about the convergence of double series, power series, infinite products, and infinite determinants.

The third chapter is called "Continuous Functions and Uniform Convergence." This chapter talks about the continuity of functions and uniform convergence of infinite series, as well as the relationship between the two. It also includes the Weierstrass M-test for uniform convergence, uniform convergence of infinite products and power series, uniform continuity, and so on.

The fourth chapter is titled "The Theory of Riemann Integration." This is the longest chapter among the four. It also provides quite a number of exercises. It covers the definition of Riemann integration, the mean value theorems for integrals, infinite and improper integrals, principal values, complex integration, and the integration of infinite series.

CHAPTER 2

COMPLEX NUMBERS

2.1 Solutions to End-of-Chapter Exercises

Example 2.1.1. Shew that the representative points of the complex numbers $1+4i, 2+7i, 3+10i$, are colinear.

Proof. Recall that two or more points are *colinear* if they lie on the same line. The line $y = 3x + 1$ passes through the representative points of $1+4i, 2+7i, 3+10i$. This proves that $1+4i, 2+7i, 3+10i$ are colinear. □

Example 2.1.2. Shew that a parabola can be drawn to pass through the representative points of the complex numbers

$$2+i, 4+4i, 6+9i, 8+16i, 10+25i.$$

Proof. The parabola $y = \frac{1}{4}x^2$ passes through the representative points of $2+i, 4+4i, 6+9i, 8+16i$, and $10+25i$. □

Example 2.1.3. Determine the nth roots of unity by aid of the Argand diagram; and shew that the number of primitive roots (roots the powers of each of which give all the roots) is the number of integers (including unity) less than n and prime to it.

Prove that if $\theta_1, \theta_2, \theta_3, \cdots$ be the arguments of the primitive roots, $\sum \cos(p\theta) = 0$ when p is a positive integer less than $\frac{n}{abc\cdots k}$, where $a, b, c, \cdots k$ are the different constituent primes of n; and that, when $p = \frac{n}{abc\cdots k}$, $\sum \cos(p\theta) = \frac{(-1)^\mu n}{abc\cdots k}$, where μ is the number of the constituent primes.

First, We will prove the first part of the Example 2.1.3.

Proof. We want to solve the equation $x^n = 1$. Recall that the polar form of 1 is $e^{i(2\pi k)}$, where $k \in \mathbb{Z}$. It follows that

$$x^n = 1 \Rightarrow x^n = e^{i(2\pi k)}$$
$$\Rightarrow x = e^{i\frac{2\pi k}{n}}.$$

Therefore, the nth roots of unity are $e^{i\frac{2\pi k}{n}}$ for $k = 0, 1, 2, \cdots, n-1$.

Recall that the nth root of unity $e^{i\frac{2\pi k}{n}}$ is primitive if and only if its first nth powers are all distinct. We claim that $e^{i\frac{2\pi k}{n}}$ is primitive if and only if k and n are relatively prime.

Let's prove the forward direction by contrapositive. Assume that $d = \gcd(n,k) > 1$. Then $\frac{n}{d} < n$ and $\frac{k}{d} < n$. This implies that

$$\left(e^{i\frac{2\pi k}{n}}\right)^{n/d} = e^{i2\pi\frac{k}{d}} = 1,$$

where $\frac{k}{d} \in \mathbb{Z}$ since d divides k. Therefore, if $d = \gcd\,(n,k) > 1$, then the first n powers of $e^{i\frac{2\pi k}{n}}$ are not distinct which implies that it is not primitive.

Now, assume that $\gcd(n,k) = 1$. If $\left(e^{i\frac{2\pi k}{n}}\right)^a = 1$, then n divides ka. But since $\gcd(k,n) = 1$, we have n divides a and $n \leq a$. This implies that the first n powers of $e^{i\frac{2\pi k}{n}}$ are distinct. Hence, it is primitive.

This proves that the number of primitive roots is the number of integers less than n and prime to it, as desired. □

Before we prove the second part of the Example 2.1.3, we will introduce some background.

Definition 2.1.1. The *elementary symmetric polynomials* $\sigma_k(x_1, \cdots, x_n)$ on n variables $x_1, \cdots, x_n$ are defined by

$$\sigma_1(x_1, \cdots, x_n) = \sum_{1 \le i \le n} x_i$$
$$\sigma_2(x_1, \cdots, x_n) = \sum_{1 \le i < j \le n} x_i x_j$$
$$\sigma_3(x_1, \cdots, x_n) = \sum_{1 \le i < j < k \le n} x_i x_j x_k$$
$$\vdots$$
$$\sigma_n(x_1, \cdots, x_n) = \prod_{1 \le i \le n} x_i.$$

Definition 2.1.2. The *Newton functions* of $x_1, \cdots, x_n$ are

$$S_1(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i$$
$$S_2(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^2$$
$$\vdots$$
$$S_n(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^n.$$

Theorem 2.1.1. The *Newton's identities* are

$$S_k + \sum_{i=1}^{k-1} (-1)^i S_{k-i} \sigma_i + (-1)^k k \sigma_k = 0 \quad \text{if } 1 \le k \le n,$$

and

$$S_k + \sum_{i=1}^{n} (-1)^i S_{k-i} \sigma_i = 0 \quad \text{if } k > n \text{ [9]}.$$

Lemma 2.1.2. Let $e^{i\theta_0}, \cdots, e^{i\theta_{n-1}}$ be the nth roots of unity. Then if $1 \le i \le n-1$, we have

$$\sigma_i(e^{i\theta_0}, \cdots, e^{i\theta_{n-1}}) = 0 \quad \text{and} \quad \sigma_n(e^{i\theta_0}, \cdots, e^{i\theta_{n-1}}) = (-1)^{n-1}.$$

Proof. Recall that each $e^{i\theta_k}$ satisfies the equation $z^n - 1 = 0$ for $k = 0, 1, \cdots, n-1$. Then

$$
\begin{aligned}
& z^n - 1 = 0 \\
& \iff (z - e^{i\theta_0})(z - e^{i\theta_1}) \cdots (z - e^{i\theta_{n-1}}) = 0 \\
& \iff z^n - \left(\sum_{k=0}^{n-1} e^{i\theta_k} \right) z^{n-1} \pm \left(\sum_{0 \le a < b \le n-1} e^{i\theta_a} e^{i\theta_b} \right) z^{n-2} + \cdots + (-1)^n \prod_{k=0}^{n-1} e^{i\theta_k} = 0 \\
& \iff z^n - \sigma_1 z^{n-1} \pm \sigma_2 z^{n-2} + \cdots + (-1)^n \sigma_n = 0.
\end{aligned}
$$

Thus, we can conclude that for $1 \le i \le n-1$, we have

$$\sigma_i = 0 \quad \text{and} \quad \sigma_n = \prod_{k=0}^{n-1} e^{i\theta_k} = (-1)(-1)^{-n} = (-1)^{1-n} = (-1)^{n-1}.$$

□

Lemma 2.1.3. Let $e^{i\theta_k}$ be the nth root of unity for $k = 0, 1, \cdots, n-1$. Then the Newton functions of $e^{i\theta_0}, e^{i\theta_1}, \cdots, e^{i\theta_{n-1}}$ are

$$S_i(e^{i\theta_0}, \cdots, e^{i\theta_{n-1}}) = 0 \quad \text{and} \quad S_n(e^{i\theta_0}, \cdots, e^{i\theta_{n-1}}) = n$$

for $1 \le i \le n-1$.

Proof. By the Newtons' identities and Lemma 2.1.2, we get

$$\begin{aligned}
S_1 &= \sigma_1 = 0 \\
S_2 &= \sigma_1^2 - 2\sigma_2 = 0 \\
&\vdots \\
S_{n-1} &= -\sum_{i=1}^{n-2} (-1)^i S_{n-1-i}\sigma_i - (-1)^{n-1}(n-1)\sigma_{n-1} = 0 \\
S_n &= -\sum_{i=1}^{n-1} (-1)^i S_{n-i}\sigma_i - (-1)^n n\sigma_n \\
&= 0 - (-1)^n n\sigma_n \\
&= (-1)^{n+1} n(-1)^{n-1} \\
&= (-1)^{2n} n \\
&= n.
\end{aligned}$$

Also note that

$$S_n = \sum_{k=0}^{n-1} \left(e^{i\theta_k}\right)^n = \sum_{k=0}^{n-1} 1 = n.$$

□

Now, we will begin to prove the second part of the Example 2.1.3.

Proof. Recall that $\left\{e^{i\frac{2\pi k}{n}} : 0 \le k \le n-1\right\}$ is the set of nth roots of unity. By the first part of the Example 2.1.3, we know that the set of primitive roots of unity is defined as $\left\{e^{i\frac{2\pi k}{n}} : 0 \le k \le n-1 \text{ and } \gcd(n,k) = 1\right\}$. By Lemma 2.1.2, we have $\sum_{j=0}^{n-1} e^{i\theta_j} = 0$, where $\theta_j = \frac{2\pi j}{n}$. By Lemma 2.1.3, the Newton functions of $e^{i\theta_0}, e^{i\theta_1}, \cdots, e^{i\theta_{n-1}}$ are

$$S_i = \sum_{j=0}^{n-1} \left(e^{i\theta_j}\right)^i = 0 \quad \text{and} \quad S_n = \sum_{j=0}^{n-1} \left(e^{i\theta_j}\right)^n = n$$

for $1 \le i \le n-1$.

Let $a, b, c, \cdots, t$ be distinct prime factors of n and μ be the number of those distinct prime factors. Then by the Principle of Inclusion-Exclusion, we get $\sum\limits_{\theta:\text{primitive}} e^{ip\theta}$ is equal to

$$\sum_{j=0}^{n-1} e^{ip\theta_j} - \Bigg[\sum_{j\in\{a,b,c,\cdots,t\}} \sum_{j|k} e^{ip\theta_k} - \sum_{\substack{j,l\in\{a,b,c,\cdots,t\}\\ j\neq l}} \sum_{jl|k} e^{ip\theta_k}$$
$$+ \sum_{\substack{j,l,g\in\{a,b,c,\cdots,t\}\\ j\neq l\\ j\neq g\\ g\neq j}} \sum_{jlg|k} e^{ip\theta_k} - \cdots + (-1)^{\mu-1} \sum_{abc\cdots t|n} e^{ip\theta_k} \Bigg],$$

where

$$\sum_{j=0}^{n-1} e^{ip\theta_j} = S_p = \begin{cases} 0, & p < n, \\ n, & p = n, \end{cases}$$
$$\sum_{j|k} e^{ip\theta_k} = \begin{cases} 0, & p < \frac{n}{j}, \\ \frac{n}{j}, & p = \frac{n}{j}, \end{cases}$$
$$\sum_{jl|k} e^{ip\theta_k} = \begin{cases} 0, & p < \frac{n}{jl}, \\ \frac{n}{jl}, & p = \frac{n}{jl}, \end{cases}$$
$$\vdots$$
$$\sum_{abc\cdots t|n} e^{ip\theta_k} = \begin{cases} 0, & p < \frac{n}{abc\cdots t}, \\ \frac{n}{abc\cdots t}, & p = \frac{n}{abc\cdots t}. \end{cases}$$

It follows that if $p < \frac{n}{abc\cdots t}$, then

$$\sum_{\theta:\text{primitive}} e^{ip\theta} = 0, \text{ which implies that } \sum_{\theta:\text{primitive}} \cos(p\theta) = 0$$

since θ is primitive if and only if $2\pi - \theta$ is primitive.

If $p = \frac{n}{abc\cdots t}$, then

$$\sum_{\theta:\text{primitive}} e^{ip\theta} = -(-1)^{\mu-1}\frac{n}{abc\cdots t} = (-1)^{\mu}\frac{n}{abc\cdots t} = \sum_{\theta:\text{primitive}} \cos(p\theta).$$

□

CHAPTER 3

THE THEORY OF CONVERGENCE

3.1 Solutions to Exercises in the Chapter

Example 3.1.1 (P12)**.** Let $\lim z_m = l$ and $\lim z_m^{'} = l'$. Prove that $\lim(z_m - z_m^{'}) = l - l^{'}$, $\lim(z_m z_m^{'}) = ll^{'}$, and, if $l^{'} \neq 0$, $\lim \frac{z_m}{z_m^{'}} = \frac{l}{l^{'}}$.

Proof. Let $\epsilon > 0$. Since $\lim z_m = l$, there exists $N_1 \in \mathbb{N}$ such that if $m \geq N_1$, we have

$$|z_m - l| < \frac{\varepsilon}{2}.$$

Similarly, since $\lim z_m^{'} = l'$, there exists $N_2 \in \mathbb{N}$ such that if $m \geq N_2$, we have

$$|z_m^{'} - l'| < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. If $m \geq N$, we have

$$\begin{aligned} |(z_m - z_m^{'}) - (l - l')| &\leq |z_m - l| + |z_m^{'} - l'| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

This proves $\lim(z_m - z_m^{'}) = l - l^{'}$.

Since $\lim z_m = l$, $\{z_m\}$ is bounded by some $M > 0$. Since $\lim z_m = l$, there exists $N_1 \in \mathbb{N}$ such that if $m \geq N_1$, we have

$$|z_m - l| < \frac{\epsilon}{2(|l'| + 1)}.$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that if $m \geq N_2$, we have

$$|z_m^{'} - l^{'}| < \frac{\varepsilon}{2M}.$$

Choose $N = \max\{N_1, N_2\}$. If $m \geq N$, we have

$$\begin{aligned} |z_m z_m^{'} - ll^{'}| &= |z_m z_m^{'} - z_m l^{'} + z_m l^{'} - ll^{'}| \\ &\leq |z_m||z_m^{'} - l^{'}| + |l^{'}||z_m - l| \\ &\leq M|z_m^{'} - l^{'}| + |l^{'}||z_m - l| \\ &< M\frac{\varepsilon}{2M} + |l^{'}|\frac{\epsilon}{2(|l^{'}|+1)} \\ &< \varepsilon. \end{aligned}$$

This proves $\lim(z_m z_m^{'}) = ll^{'}$.

Now, assume $l' \neq 0$. By the product rule of limits, it suffices to show $\lim \frac{1}{z_m^{'}} = \frac{1}{l'}$. Since $\lim z_m^{'} = l'$, there exists $N_1 \in \mathbb{N}$ such that if $m \geq N_1$, we have

$$|z_m^{'} - l^{'}| < \frac{|l^{'}|}{2}.$$

This implies that if $m \geq N_1$, we have

$$\begin{aligned} |l^{'}| &= |l^{'} - z_m^{'} + z_m^{'}| \\ &\leq |l^{'} - z_m^{'}| + |z_m^{'}| \\ &< \frac{|l^{'}|}{2} + |z_m^{'}|. \end{aligned}$$

Thus, $\frac{|l^{'}|}{2} < |z_m^{'}|$ if $m \geq N_1$. Since $\lim z_m^{'} = l'$, there exists $N_2 \in \mathbb{N}$ such that if $m \geq N_2$, we have

$$|z_m^{'} - l^{'}| < \frac{|l^{'}|^2 \varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. If $m \geq N$, we have

$$\begin{aligned}\left|\frac{1}{z_m'} - \frac{1}{l'}\right| &= \frac{|z_m' - l'|}{|z_m'||l'|} \\ &< \frac{|l'|^2\varepsilon}{2}\frac{2}{|l'|}\frac{1}{|l'|} \\ &= \varepsilon.\end{aligned}$$

□

Example 3.1.2 (P18)**.** Shew that if $0 < \theta < 2\pi$, $|\sum_{n=1}^{p} \sin(n\theta)| < \csc\left(\frac{1}{2}\theta\right)$; and deduce that, if $f_n \to 0$ steadily, $\sum_{n=0}^{\infty} f_n \sin(n\theta)$ converges for all real values of θ, and that $\sum_{n=1}^{\infty} f_n \cos(n\theta)$ converges if θ is not an even multiple of π.

Before we proceed to prove Example 3.1.2, we will derive *Lagrange's Trigonometric Identities.*

Lemma 3.1.1 (Lagrange's Trigonometric Identities)**.** For $0 < \theta < 2\pi$, we have

$$1 + \sum_{k=1}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)},$$

and

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \frac{\cos\left(n + \frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)}.$$

Proof. Take $z = e^{i\theta}$, where $0 < \theta < 2\pi$. Then $z \neq 1$. Hence,

$$\begin{aligned}
1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \\
&= \frac{1 - e^{i(n+1)\theta}}{-e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)} \\
&= \frac{-e^{-i\frac{\theta}{2}}\left(1 - e^{i(n+1)\theta}\right)}{i2\sin\left(\frac{\theta}{2}\right)}\left(\frac{i}{i}\right) \\
&= \frac{i\left(e^{-i\frac{\theta}{2}} - e^{i\theta\left(n+\frac{1}{2}\right)}\right)}{2\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{i\left(\cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right) - \cos\left(n+\frac{1}{2}\right)\theta - i\sin\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)} + i\left(\frac{\cos\left(\frac{\theta}{2}\right) - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)}\right).
\end{aligned}$$

Equating real and imaginary parts, we get

$$1 + \sum_{k=1}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)},$$

and

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \frac{\cos\left(n+\frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)}.$$

□

Now, we are ready to prove Example 3.1.2.

Proof. Let $0 < \theta < 2\pi$. Then by Lagrange's Trigonometric Identity, we have

$$\begin{aligned}
\left|\sum_{n=1}^{p} \sin(n\theta)\right| &= \left|\frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \frac{1}{2}\cos\left(p+\frac{1}{2}\right)\theta\csc\left(\frac{\theta}{2}\right)\right| \\
&\leq \frac{1}{2}\left(\left|\cot\left(\frac{\theta}{2}\right)\right| + \left|\cos\left(p+\frac{1}{2}\right)\theta\csc\left(\frac{\theta}{2}\right)\right|\right) \\
&\leq \frac{1}{2}\left(\left|\cot\left(\frac{\theta}{2}\right)\right| + \csc\left(\frac{\theta}{2}\right)\right) && \left(\csc\left(\frac{\theta}{2}\right) \geq 0\right) \\
&\leq \frac{1}{2}\left(\csc\left(\frac{\theta}{2}\right) + \csc\left(\frac{\theta}{2}\right)\right) && \left(\left|\cot\left(\frac{\theta}{2}\right)\right| \leq \left|\frac{1}{\sin\left(\frac{\theta}{2}\right)}\right|\right) \\
&= \csc\left(\frac{\theta}{2}\right).
\end{aligned}$$

Let $\{f_n\}$ be a decreasing sequence of positive real numbers that converges to 0, i.e $f_n \to 0$ steadily. We want to show that the series $\sum_{n=1}^{\infty} f_n \sin(n\theta)$ converges for all $\theta \in \mathbb{R}$.

It suffices to show that the series converges for $0 \leq \theta < 2\pi$. If $\theta = 0$, then the series $\sum_{n=1}^{\infty} f_n \sin(n\theta) = \sum_{n=1}^{\infty} f_n \cdot 0 = 0$. Now, suppose that $0 < \theta < 2\pi$. Then the sequence of partial sums of $\sum_{n=1}^{\infty} \sin(n\theta)$ is bounded by $\csc\left(\frac{\theta}{2}\right)$. We also know that $f_n \searrow 0$. By Dirichlet test, we know that the series $\sum_{n=1}^{\infty} f_n \sin(n\theta)$ converges.

Next, we want to show that the series $\sum_{n=1}^{\infty} f_n \cos(n\theta)$ converges if θ is not an even multiple of π. It suffices to show the series converges for $0 < \theta < 2\pi$. Let $0 < \theta < 2\pi$. By Lagrange's Trigonometric Identity, we have

$$\begin{aligned}
\left|\sum_{k=1}^{p} \cos(n\theta)\right| &= \left|-\frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)}\right| \\
&\leq \frac{1}{2} + \left|\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right| \\
&= \frac{1}{2} + \frac{1}{2}\csc\left(\frac{\theta}{2}\right).
\end{aligned}$$

Thus, the sequence of partial sums of $\sum_{n=1}^{\infty} \cos(n\theta)$ is bounded when $0 < \theta < 2\pi$. By the Dirichlet test, the series $\sum_{n=1}^{\infty} f_n \cos(n\theta)$ converges. □

Example 3.1.3 (P18)**.** Shew that if $f_n \to 0$ steadily, $\sum_{n=1}^{\infty}(-1)^n f_n \cos(n\theta)$ converges if θ is real and not an odd multiple of π and $\sum_{n=1}^{\infty}(-1)^n f_n \sin(n\theta)$ converges for all real values of θ.

Proof. Assume that θ is not an odd multiple of π. Then

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n f_n \cos(n\theta) &= \sum_{n=1}^{\infty} f_n \cos(n\pi)\cos(n\theta)\\ &= \sum_{n=1}^{\infty} f_n \left(\cos(n(\pi+\theta)) + \underbrace{\sin(n\pi)}_{=0}\sin(n\theta)\right)\\ &= \sum_{n=1}^{\infty} f_n \cos(n(\pi+\theta)),\end{aligned}$$

which converges by Example 3.1.2.

Now, let $\theta \in \mathbb{R}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n f_n \sin(n\theta) &= \sum_{n=1}^{\infty} f_n \cos(n\pi)\sin(n\theta)\\ &= \sum_{n=1}^{\infty} f_n \left(\sin(n(\pi+\theta)) - \cos(n\theta)\underbrace{\sin(n\pi)}_{=0}\right)\\ &= \sum_{n=1}^{\infty} f_n \sin(n(\pi+\theta)),\end{aligned}$$

which converges by Example 3.1.2. □

Example 3.1.4 (P24)**.** Investigate the convergence of $\sum_{n=1}^{\infty} n^r e^{-k\sum_{m=1}^{n}\frac{1}{m}}$, when $r > k$ and when $r < k$.

Proof. Notice that for $\log(n+1) \leq \sum_{m=1}^{n}\frac{1}{m} \leq 1+\log(n)$ for all $n \in \mathbb{N}$, which can be easily shown since

$$\int_1^{n+1}\frac{1}{x}dx \leq \sum_{m=1}^{n}\frac{1}{m} \leq 1 + \int_1^{n}\frac{1}{x}dx.$$

It follows that the series $\sum_{n=1}^{\infty} n^r e^{-k\sum_{m=1}^{n}\frac{1}{m}}$ is bounded between two series $\sum_{n=1}^{\infty} n^r(1+n)^{-k}$ and $\sum_{n=1}^{\infty} e^{-k}n^{r-k}$. Both $\sum_{n=1}^{\infty} n^r(1+n)^{-k}$ and $\sum_{n=1}^{\infty} e^{-k}n^{r-k}$ converge if $k-r > 1$ and diverge if $k-r \leq 1$. By the comparison test, we can conclude that the series $\sum_{n=1}^{\infty} n^r e^{-k\sum_{m=1}^{n}\frac{1}{m}}$ converges if $k-r > 1$ and diverges if $k-r \leq 1$. □

Example 3.1.5 (P25)**.** If in the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

the order of the terms be altered, so that the ratio of the number of positive terms to the number of negative terms in the first n terms is ultimately a^2, shew that the sum of the series will become $\log(2a)$.

Proof. Let $A(m,n)$ be the rearrangement of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ consisting m positive terms followed by n negative terms. We will first consider the partial sum of the first N terms of $A(m,n)$ where $m+n$ divides N. We denote O_N the sum of the first N odd terms and E_N the sum of the first N even terms of the series $\sum_{n=1}^{\infty}\frac{1}{n}$. We also denote H_N to be the Nth partial sum of the series $\sum_{n=1}^{\infty}\frac{1}{n}$.

We will show that $O_N+E_N=H_{2N}$ by induction. Clearly, it is true when $N=1$. Assume $O_N+E_N=H_{2N}$. Then

$$O_{N+1}+E_{N+1}=O_N+E_N+\frac{1}{2N+1}+\frac{1}{2N+2}=H_{2N}+\frac{1}{2N+1}+\frac{1}{2N+2}=H_{2N+2}.$$

We also want to show $2E_N=H_N$ by induction. Clearly, it is true when $N=1$. Assume $2E_N=H_N$. Then $2E_{N+1}=2\left(E_n+\frac{1}{2N+2}\right)=2E_N+\frac{1}{N+1}=H_N+\frac{1}{N+1}=H_{N+1}$.

Let S_N be the Nth partial sum of $A(m,n)$, where $N=(m+n)k$ for some $k\in\mathbb{N}$.

Collecting the positive and negative terms together, we have

$$\begin{aligned}
S_N &= S_{(m+n)k} \\
&= O_{mk} - E_{nk} \\
&= O_{mk} + E_{mk} - E_{mk} - E_{nk} \\
&= H_{2mk} - \frac{1}{2}H_{mk} - \frac{1}{2}H_{nk} \\
&= (H_{2mk} - \log(2mk)) - \frac{1}{2}(H_{mk} - \log(mk)) - \frac{1}{2}(H_{nk} - \log(nk)) \\
&+ \log(2mk) - \frac{1}{2}\log(mk) - \frac{1}{2}\log(nk) \\
&= (H_{2mk} - \log(2mk)) - \frac{1}{2}(H_{mk} - \log(mk)) - \frac{1}{2}(H_{nk} - \log(nk)) \\
&+ \log(2) + \frac{1}{2}\log\left(\frac{m}{n}\right).
\end{aligned}$$

Taking the limit of $S_{(m+n)k}$ as $k \to \infty$, we get

$$\begin{aligned}
\lim_{k\to\infty} S_{(m+n)k} &= \gamma - \frac{1}{2}\gamma - \frac{1}{2}\gamma + \log(2) + \frac{1}{2}\left(\frac{m}{n}\right) \\
&= \log\left(2\sqrt{\frac{m}{n}}\right),
\end{aligned}$$

where γ is the Euler's constant and $a = \sqrt{\frac{m}{n}}$. Now, fix $r \in \{1, 2, \cdots, m+n-1\}$, we have $S_{(m+n)k+r} = S_{(m+n)k} + \{r \text{ terms of } A(m,n)\}$. Since the terms of $A(m,n)$ approach 0, $\lim_{k\to\infty} S_{(m+n)k+r} = \log(2) + \frac{1}{2}\log(\frac{m}{n})$ for each r. This proves that $\lim_{N\to\infty} S_N = \log(2) + \frac{1}{2}\log(\frac{m}{n})$ even if $(m+n)$ does not divide N [2]. □

Example 3.1.6 (P29)**.** Shew from first principles that if the terms of an absolutely convergent double series be arranged in the order

$$u_{1,1} + (u_{2,1} + u_{1,2}) + (u_{3,1} + u_{2,2} + u_{1,3}) + (u_{4,1} + \cdots + u_{1,4}) + \cdots,$$

this series converges to S.

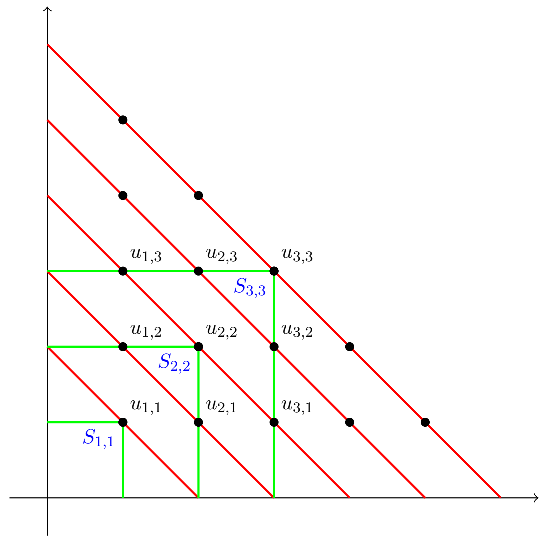

FIGURE 3.1. How N is determined.

Proof. Let the double series $\sum_{\mu,\nu} u_{\mu,\nu}$ be absolutely convergent. Then $\sum_{\mu,\nu} u_{\mu,\nu}$ converges to a limit, say S. Also, we have $\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} u_{\mu,\nu} = S$ by Section 2.52 on P28. Let $S_{\mu,\nu}$ and $\sigma_{\mu,\nu}$ be the sums of the rectangle of μ rows and ν columns of $\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} u_{\mu,\nu}$ and $\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} |u_{\mu,\nu}|$, respectively. Since the double series converges absolutely, $\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} |u_{\mu,\nu}|$ converges to a limit, say σ.

Let $\varepsilon > 0$. Since $\sigma = \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} |u_{\mu,\nu}|$, there exists $m \in \mathbb{N}$ such that if $\mu, \nu \geq m$, we have

$$|\sigma_{\mu,\nu} - \sigma| = \sigma - \sigma_{\mu,\nu} < \frac{\varepsilon}{2}.$$

Now, let $M \in \mathbb{N}$ be such that $M \geq m$. We need to take $N = \sum_{i=1}^{2M-1} i = M(2M-1)$ terms of the double series

$$u_{1,1} + (u_{2,1} + u_{1,2}) + (u_{3,1} + u_{2,2} + u_{1,3}) + (u_{4,1} + \cdots + u_{1,4}) + \cdots,$$

(in the order in which the terms are taken) in order to include all the terms of $S_{M,M}$ (See Figure 3.1) and let the sum of these terms be t_N.

Then $t_N - S_{M,M}$ consists of a sum of a finite number of terms of the type $u_{p,q}$, where

$p+q \in \{M+2, M+3, \cdots, 2M\}$ and ($p > M$ or $q > M$). Note that $|u_{p,q}|$ where p, q with the above properties is one of the terms in $\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}|u_{\mu,\nu}| - \sigma_{M,M} = \sigma - \sigma_{M,M}$. Thus,

$$\begin{aligned}
|t_N - S_{M,M}| &= \left|\sum_{\substack{p+q\in\{M+2,M+3,\cdots,2M\}\\ p>M \text{ or } q>M}} u_{p,q}\right| \\
&\leq \sum_{\substack{p+q\in\{M+2,M+3,\cdots,2M\}\\ p>M \text{ or } q>M}} |u_{p,q}| \\
&\leq \sigma - \sigma_{M,M} \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|S - S_{M,M}| &= \left|\sum_{p>M \text{ or } q>M} u_{p,q}\right| \\
&\leq \sum_{p>M \text{ or } q>M} |u_{p,q}| \\
&= \sigma - \sigma_{M,M} \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

If $n \geq N$, we have

$$\begin{aligned}
|t_n - S| &= |t_n - S_{M,M} + S_{M,M} - S| \\
&\leq |t_n - S_{M,M}| + |S_{M,M} - S| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

This proves $\lim_{n\to} t_n = S$, as desired. □

Example 3.1.7 (P29)**.** Shew that the series obtained by multiplying the two series

$$1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\frac{z^4}{2^2}+\cdots,\quad 1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots,$$

and rearranging according to powers of z, converges as long as the representative point of z lies in the ring shaped region bounded by the circles $|z|=1$ and $|z|=2$.

Proof. The series $1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\frac{z^4}{2^2}+\cdots$ converges absolutely if $\left|\frac{z}{2}\right|<1$ or $|z|<2$. The series $1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots$ converges absolutely if $\left|\frac{1}{z}\right|<1$ or $1<|z|$. Thus, the series obtained by multiplying these two series, written in any order, converges absolutely if $1<|z|<2$ by Cauchy's theorem on the multiplication of absolutely convergent series on P29.

Now, we will check the case when $|z|=1$ or $|z|=2$. Rearrange the product of two series according to powers of z, we obtain

$$\begin{aligned}
&\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\frac{z^4}{2^2}+\cdots\right)\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)\\
&=1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots+\frac{z}{2}+\frac{1}{2}+\frac{1}{2z}+\frac{1}{2z^2}+\cdots\\
&+\frac{z^2}{2^2}+\frac{z}{2^2}+\frac{1}{2^2}+\frac{1}{2^2z}+\cdots+\frac{z^3}{2^3}+\frac{z^2}{3^2}+\frac{z}{2^3}+\frac{1}{2^3}+\cdots\\
&=\cdots+\frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n+\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n+\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n+z\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n+z^2\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n+\cdots.
\end{aligned}$$

If $|z|=1$, then

$$\begin{aligned}
&\cdots+\frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n+\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n+\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\\
&=2\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n
\end{aligned}$$

diverges. On the other hand, if $|z| = 1$,

$$
\begin{aligned}
&|z| \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + |z^2| \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \cdots \\
&= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \left(\frac{1}{2}\right)^n \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} < \infty.
\end{aligned}
$$

Thus, if $|z| = 1$, then the product of two series rearranged according to powers of z is divergent.

Now, consider the case when $|z| = 2$. If $|z| = 2$, then $2 \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$ converges. Whereas, if $|z| = 2$,

$$
\begin{aligned}
&z \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + z^2 \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \cdots \\
&= \sum_{k=1}^{\infty} z^k \sum_{n=k}^{\infty} \left(\frac{1}{2}\right)^n \\
&= \sum_{k=1}^{\infty} z^k \left(\frac{1}{2}\right)^{k-1} \\
&= 2 \sum_{k=1}^{\infty} \left(\frac{z}{2}\right)^k
\end{aligned}
$$

diverges. Thus, we can conclude that if $|z| = 2$, then the product of two series rearranged according to powers of z is divergent. □

Example 3.1.8 (P33)**.** Shew that if $\prod_{n=1}^{\infty}(1 + a_n)$ converges, so does $\sum_{n=1}^{\infty} \log(1 + a_n)$ if the logarithms have their principal values.

Proof. Since the a_n are complex, we must agree on a definite branch of the logarithms, and we will choose the principal branch in each term, written as Log. Let the partial sum and the partial product be given by

$$S_n = \text{Log}\ (1 + a_1) + \text{Log}\ (1 + a_2) + \cdots + \text{Log}\ (1 + a_n)$$

(assuming the logarithms have their principal values), and

$$P_n = (1 + a_1)(1 + a_2) \cdots (1 + a_n),$$

respectively. Assume that $\prod_{n=1}^{\infty}(1 + a_n)$ converges to $P \neq 0$. That is, $\lim_{n\to\infty} P_n = P$. In general it is not true that the series $\sum_{n=1}^{\infty} \text{Log } (1 + a_n)$ formed with the principal values converges to the principal value Log (P). However, we will show that it converges to some value of $\log(P)$. We will denote the principal value of the logarithm by Log and its imaginary part by Arg.

Since $\lim_{n\to\infty} \frac{P_n}{P} = 1$, we have $\lim_{n\to\infty} \text{Log}(\frac{P_n}{P}) = 0$. For each $n \in \mathbb{N}$ there exists an integer h_n such that

$$\text{Log}\left(\frac{P_n}{P}\right) = S_n - \text{Log}(P) + h_n \cdot 2\pi i.$$

Taking the difference, we obtain

$$(h_{n+1} - h_n)2\pi i = \text{Log}\left(\frac{P_{n+1}}{P}\right) - \text{Log}\left(\frac{P_n}{P}\right) - \text{Log}(1 + a_n).$$

It follows that

$$(h_{n+1} - h_n)2\pi = \text{Arg}\left(\frac{P_{n+1}}{P}\right) - \text{Arg}\left(\frac{P_n}{P}\right) - \text{Arg}(1 + a_n).$$

By definition, $|\text{Arg}(1 + a_n)| \leq \pi$, and we know that $\lim_{n\to\infty} \text{Arg}\left(\frac{P_{n+1}}{P}\right) - \text{Arg}\left(\frac{P_n}{P}\right) = 0$. For large n this is incompatible with the previous equation unless $h_{n+1} = h_n$. Hence, h_n is ultimately equal to a fixed integer h, and it follows from $\text{Log}\left(\frac{P_n}{P}\right) = S_n - \text{Log}(P) + h_n \cdot 2\pi i$ that $\lim_{n\to\infty} S_n = \text{Log}(P) - h \cdot 2\pi i$ [1]. □

Example 3.1.9 (P37)**.** Shew that the necessary and sufficient condition for the absolute convergence of the infinite determinant

$$\lim_{m\to\infty}\begin{vmatrix} 1 & a_1 & 0 & 0 & \cdots & 0 \\ \beta_1 & 1 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \beta_2 & 1 & \alpha_3 & \cdots & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & \cdots & 0 & \beta_m & & 1 \end{vmatrix}$$

is that the series

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \cdots$$

shall be absolutely convergent.

Proof. Let

$$f(m) = \begin{vmatrix} 1 & a_1 & 0 & 0 & \cdots & 0 \\ \beta_1 & 1 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \beta_2 & 1 & \alpha_3 & \cdots & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & \cdots & 0 & \beta_m & & 1 \end{vmatrix}.$$

The article “Note on Infinite Determinants” by Eugene H. Roberts [10] suggests that the absolute convergence of the infinite determinant means that if we replace each term in the expansion of $f(m)$ by its absolute value, the determinant will still converge.

Let $g(m)$ be the function obtained by replacing each term in the expansion of $f(m)$ with its absolute value. Then

$$g(m) = g(m-1) + c_m g(m-2) \geq g(m-1) + c_m,$$

where $c_m = |\alpha_m \beta_m|$. Hence,

$$g(m) \geq 1 + |\alpha_1\beta_1| + |\alpha_2\beta_2| + \cdots + |\alpha_m\beta_m| \geq 1.$$

The sufficient condition can be proved as follows. If the series

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \cdots$$

converges absolutely, then the infinite product

$$(1 + |\alpha_1\beta_1|)(1 + |\alpha_2\beta_2|)(1 + |\alpha_3\beta_3|)\cdots > g(m) \geq 1$$

converges. Taking $m \to \infty$, we can conclude that $g(m)$ converges.

The necessary condition can be proved as follows. If $g(m)$ converges as m goes to infinity, then

$$g(m) \geq 1 + |\alpha_1\beta_1| + |\alpha_2\beta_2| + \cdots + |\alpha_m\beta_m| \geq 1$$

implies that the series

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \cdots$$

converges absolutely. [1] □

3.2 Solutions to End-of-Chapter Exercises

Example 3.2.1. Evaluate $\lim_{n\to\infty}\left(e^{-na}n^b\right)$, $\lim_{n\to\infty}\left(n^{-a}\log(n)\right)$ when $a > 0$, $b > 0$.

Proof. Observe that

$$\begin{aligned}\lim_{n\to\infty} e^{-na}n^b &= \lim_{n\to\infty} \frac{n^b}{1 + na + \frac{(na)^2}{2!} + \frac{(na)^3}{3!} + \cdots} \\ &= \lim_{n\to\infty} \frac{1}{\frac{1}{n^b} + \frac{na}{n^b} + \frac{n^2a^2}{2!n^b} + \frac{n^3a^3}{3!n^b} + \cdots} \\ &= 0.\end{aligned}$$

[1] I would like to thank user2249675, a user of Mathematics Stackexchange, for providing a solution to this problem.

Now, let $t = \log(n)$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log(n)}{n^a} &= \lim_{t\to\infty} \frac{t}{e^{at}} \\
&= \lim_{t\to\infty} \frac{t}{1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots} \\
&= \lim_{t\to\infty} \frac{1}{\frac{1}{t} + a + \frac{a^2 t}{2!} + \frac{a^3 t^2}{3!} + \cdots} \\
&= 0.
\end{aligned}$$

□

Example 3.2.2. Investigate the convergence of

$$\sum_{n=1}^{\infty} \left\{ 1 - n \log\left(\frac{2n+1}{2n-1}\right) \right\}.$$

Proof. We will show that the series $\sum_{n=1}^{\infty} \left\{ 1 - n \log\left(\frac{2n+1}{2n-1}\right) \right\}$ converges absolutely. Consider the series $\sum_{n=1}^{\infty} \left| 1 - n \log\left(\frac{2n+1}{2n-1}\right) \right|$. We want to compare this series with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\left| 1 - n \log\left(\frac{2n+1}{2n-1}\right) \right|}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \left| n^2 - n^3 \log\left(\frac{1 + \frac{1}{2n}}{1 - \frac{1}{2n}}\right) \right| \\
&= \lim_{n\to\infty} \left| n^2 - n^3 \left(\log\left(1 + \frac{1}{2n}\right) - \log\left(1 - \frac{1}{2n}\right) \right) \right| \\
&= \lim_{n\to\infty} \left| n^2 - n^3 \left[\left(\frac{1}{2n} - \frac{1}{2(2n)^2} + \frac{1}{3(2n)^3} - \cdots \right) - \left(-\frac{1}{2n} - \frac{1}{2(2n)^2} - \frac{1}{3(2n)^3} - \cdots \right) \right] \right| \\
&= \lim_{n\to\infty} \left| n^2 - 2n^3 \left(\frac{1}{2n} + \frac{1}{3(2n)^3} + \frac{1}{5(2n)^5} + \cdots \right) \right| \\
&= \lim_{n\to\infty} \left| n^2 - n^2 - \frac{1}{12} - \frac{1}{5 \cdot 2^4 n^2} - \cdots \right| \\
&= \frac{1}{12},
\end{aligned}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we can conclude that the series $\sum_{n=1}^{\infty} \left| 1 - n \log\left(\frac{2n+1}{2n-1}\right) \right|$ converges by the limit comparison test. This proves $\sum_{n=1}^{\infty}\left\{1-n\log\left(\frac{2n+1}{2n-1}\right)\right\}$ converges absolutely. [2]

$\square$

Example 3.2.3. Investigate the convergence of

$$\sum_{n=1}^{\infty}\left\{\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{4n+3}{2n+2}\right\}^2.$$

Proof. Let $a_n=\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}$. We will prove that $\frac{1}{\sqrt{4n}}\le a_n$ by induction. Clearly, it is true when $n=1$. Assume that $\frac{1}{\sqrt{4n}}\le a_n$. Then $a_{n+1}=\frac{2n+1}{2n+2}a_n\ge\frac{2n+1}{2n+2}\left(\frac{1}{\sqrt{4n}}\right)$. We show that $\frac{2n+1}{2n+2}\left(\frac{1}{\sqrt{4n}}\right)\ge\frac{1}{\sqrt{4n+4}}$ as follows:

$$\begin{aligned}&\frac{2n+1}{2n+2}\left(\frac{1}{\sqrt{4n}}\right)\ge\frac{1}{\sqrt{4n+4}}\\&\iff (2n+1)^2(\sqrt{4n+4})^2\ge(2n+2)^2(\sqrt{4n})^2\\&\iff n+1\ge 0.\end{aligned}$$

But $n+1\ge 0$ is always true for positive integer n. This shows that $\frac{2n+1}{2n+2}\left(\frac{1}{\sqrt{4n}}\right)\ge\frac{1}{\sqrt{4n+4}}$ for all $n\in\mathbb{N}$. Therefore, we can conclude that $a_n\ge\frac{1}{\sqrt{4n}}$ for all $n\in\mathbb{N}$. Hence, we have

$$\begin{aligned}&\frac{1}{\sqrt{4n}}\cdot\frac{4n+3}{2n+2}\le\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{4n+3}{2n+2}\\&\Rightarrow\frac{1}{4n}\left(\frac{4n+3}{2n+2}\right)^2\le\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{4n+3}{2n+2}\right)^2.\end{aligned}$$

Since $\sum_{n=1}^{\infty}\frac{1}{4n}\left(\frac{4n+3}{2n+2}\right)^2$ diverges, we can conclude that $\sum_{n=1}^{\infty}\left\{\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{4n+3}{2n+2}\right\}^2$ diverges by the comparison test. $\square$

Example 3.2.4. Find the range of values of z for which the series

$$2\sin^2(z)-4\sin^4(z)+8\sin^6(z)-\cdots+(-1)^{n+1}2^n\sin^{2n}(z)+\cdots$$

[2] I would like to thank Beni Bogosel, a user of Mathematics Stackexchange, for providing a solution to this problem.

is convergent.

Proof. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{2^{n+1}\sin^{2n+2}(z)}{2^n\sin^{2n}(z)}\right| \\ &= \lim_{n\to\infty}\left|2\sin^2(z)\right| \\ &= 2\left|\sin^2(z)\right|,\end{aligned}$$

we can conclude that the series converges when $|\sin^2(z)| < \frac{1}{2}$ by the ratio test. Consider the case when $|\sin^2(z)| = \frac{1}{2}$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|(-1)^{n+1}2^n\sin^{2n}(z)\right| &= \lim_{n\to\infty} 2^n\left|\sin^2(z)\right|^n \\ &= \lim_{n\to\infty} 2^n\frac{1}{2^n} \\ &= 1 \neq 0,\end{aligned}$$

the series diverges by the divergence test. Thus, we can conclude that the series converges for all $z \in \mathbb{C}$ such that $|\sin^2(z)| < \frac{1}{2}$. If $z = x + iy$, then the condition is

$$\sin^2(x) + \sinh^2(y) < \frac{1}{2}.$$

□

Example 3.2.5. Shew that the series

$$\frac{1}{z} - \frac{1}{z+1} + \frac{1}{z+2} - \frac{1}{z+3} + \cdots$$

is conditionally convergent, except for certain exceptional values of z; but that the series

$$\frac{1}{z} + \frac{1}{z+1} + \cdots + \frac{1}{z+p-1} - \frac{1}{z+p} - \frac{1}{z+p+1} - \cdots - \frac{1}{z+2p+q-1} + \frac{1}{z+2p+q} + \cdots,$$

in which $(p+q)$ negative terms always follow p positive terms, is divergent.

Proof. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z-1+n}$, where $z-1+n \neq 0$, i.e. $z \notin \{\cdots, -2, -1, 0\}$. First, we will show that the series $\sum_{n=1}^{\infty} \left|\frac{1}{z-1+n}\right|$ diverges. Since

$$\lim_{n\to\infty} \frac{\frac{1}{|z-1+n|}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{|z-1+n|} = 1,$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we can conclude that the series $\sum_{n=1}^{\infty} \left|\frac{1}{z-1+n}\right|$ diverges by the limit comparison test.

Next, we will show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z-1+n}$ converges. Applying summation by parts, we have

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{z-1+k} = \sum_{k=1}^{n} \left(\sum_{j=1}^{k} (-1)^{j+1}\right) \left(\frac{1}{z-1+k} - \frac{1}{z+k}\right) + \left(\sum_{k=1}^{n} (-1)^{k+1}\right) \frac{1}{z+n}.$$

The last term converges to 0 as $n \to \infty$ since $\sum_{k=1}^{n} (-1)^{k+1}$ is bounded and $\lim_{n\to\infty} \frac{1}{z+n} = 0$.

Observe that

$$\limsup_{n\to\infty} \frac{\left|\left(\sum_{j=1}^{k} (-1)^{j+1}\right)\left(\frac{1}{z-1+k} - \frac{1}{z+k}\right)\right|}{\frac{1}{k^2}} = \lim_{n\to\infty} \frac{k^2}{|(z-1+k)(z+k)|} = 1,$$

and $\sum_{n=n_0}^{\infty} \frac{1}{k^2}$ converges, we can conclude that the series

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} (-1)^{j+1}\right) \left(\frac{1}{z-1+k} - \frac{1}{z+k}\right)$$

converges absolutely by the limit comparison test. Therefore, we can conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z-1+n}$ converges conditionally.

Finally, we will show that the series

$$\underbrace{\frac{1}{z}+\frac{1}{z+1}+\cdots+\frac{1}{z+p-1}}_{p\text{ positive terms}}\underbrace{-\frac{1}{z+p}-\frac{1}{z+p+1}-\cdots-\frac{1}{z+2p+q-1}}_{p+q\text{ negative terms}}+\frac{1}{z+2p+q}$$

$$+\cdots,$$

diverges, assuming p and q are any fixed positive integers.

We may start from $m = 1$ since getting rid of the term $\frac{1}{z}$ since it will not affect the divergence of the original series. This also allows us to assume $z = 0$. Let $\sum_{n=1}^{\infty} a_m(z)$ be defined by

$$\underbrace{\frac{1}{z+1}+\cdots+\frac{1}{z+p}}_{p\text{ positive terms}}\underbrace{-\frac{1}{z+p+1}-\cdots-\frac{1}{z+2p}-\frac{1}{z+2p+1}-\cdots\frac{1}{z+2p+q}}_{p+q\text{ negative terms}}+\cdots$$

Then

$$\sum_{m=1}^{\infty} a_m(0) = 1+\frac{1}{2}+\cdots+\frac{1}{p}-\frac{1}{p+1}-\cdots-\frac{1}{2p}-\frac{1}{2p+1}-\cdots-\frac{1}{2p+q}+\cdots.$$

We will show that both $\sum_{m=1}^{\infty} a_m(z)$ and $\sum_{m=1}^{\infty} a_m(0)$ have the same behavior, i.e. both converge or both diverge, by showing that $\sum_{m=1}^{\infty} (a_m(z) - a_m(0))$ converges absolutely.

Notice that

$$|a_m(z) - a_m(0)| = \left|\frac{1}{z+m} - \frac{1}{m}\right| = \frac{|z|}{m\,|m+z|}.$$

We want to show that if $2|z| \le m$, then $\frac{|z|}{m|m+z|} \le \frac{2|z|}{m^2}$. Assume $2|z| \le m$, then

$$\begin{aligned} &m \le 2(m-|z|) \le 2|m+z| \\ &\Rightarrow \frac{1}{|m+z|} \le \frac{2}{m} \\ &\Rightarrow \frac{|z|}{m|m+z|} \le \frac{2|z|}{m^2}. \end{aligned}$$

By comparison test, $\sum |a_m(z) - a_m(0)|$ converges since $\sum \frac{2|z|}{m^2}$ converges and discarding a finite number of terms will not affect the behavior of the series. Thus, we must conclude that either both series $\sum a_m(z)$ and $\sum a_m(0)$ converge or both diverge.

To show $\sum a_m(z)$ diverges, it suffices to show the series $\sum a_m(0)$ diverges. Let S_m be the mth partial sum of the series $\sum a_m(0)$. Observe that

$$\begin{aligned}
S_{2p+q} &= \underbrace{\left[1+\frac{1}{2}+\cdots+\frac{1}{p}-\frac{1}{p+1}-\cdots-\frac{1}{2p}\right]}_{\text{the first } p \text{ positive and } p \text{ negative terms}}-\frac{1}{2p+1}-\cdots-\frac{1}{2p+q}.\\
&= \left[\left(1-\frac{1}{p+1}\right)+\left(\frac{1}{2}-\frac{1}{p+2}\right)+\cdots+\left(\frac{1}{p}-\frac{1}{2p}\right)\right]-\sum_{m=1}^{q}\frac{1}{2p+m},\\
&= \sum_{m=1}^{p}\frac{p}{m(m+p)}-\sum_{m=1}^{q}\frac{1}{2p+m}\\
S_{(2p+q)2} &= \sum_{m=1}^{p}\frac{p}{m(m+p)}-\sum_{m=1}^{q}\frac{1}{2p+m}+\left[\frac{1}{2p+q+1}+\cdots+\frac{1}{2p+q+p}\right.\\
&\quad \left.-\frac{1}{2p+q+p+1}-\cdots-\frac{1}{2p+q+2p}\right]-\frac{1}{2p+q+2p+1}-\cdots-\frac{1}{2(2p+q)}\\
&= \sum_{m=1}^{p}\frac{p}{m(m+p)}-\sum_{m=1}^{q}\frac{1}{2p+m}+\sum_{m=2p+q+1}^{2p+q+p}\frac{p}{m(m+p)}-\sum_{m=2p+q+1}^{2p+2q}\frac{1}{2p+m}\\
&= \sum_{m=1}^{p}\frac{p}{m(m+p)}+\sum_{m=2p+q+1}^{2p+q+p}\frac{p}{m(m+p)}-\left(\sum_{m=1}^{q}\frac{1}{2p+m}+\sum_{m=2p+q+1}^{2p+2q}\frac{1}{2p+m}\right)\\
&\ \vdots\\
S_{(2p+q)n} &= \left(\sum_{m=1}^{p}\frac{p}{m(m+p)}+\sum_{m=2p+q+1}^{2p+q+p}\frac{p}{m(m+p)}+\cdots\sum_{m=(2p+q)(n-1)+1}^{(2p+q)(n-1)+p}\frac{p}{m(m+p)}\right)\\
&\quad -\left(\sum_{m=1}^{q}\frac{1}{2p+m}+\sum_{m=2p+q+1}^{2p+2q}\frac{1}{2p+m}+\cdots+\sum_{m=(2p+q)(n-1)+1}^{(2p+q)(n-1)+q}\frac{1}{2p+m}\right).
\end{aligned}$$

Taking $n\to\infty$, we have $S_{(2p+q)n}\to-\infty$ since

$$\sum_{m=1}^{p}\frac{p}{m(m+p)}+\sum_{m=2p+q+1}^{2p+q+p}\frac{p}{m(m+p)}+\cdots+\sum_{m=(2p+q)(n-1)+1}^{(2p+q)(n-1)+p}\frac{p}{m(m+p)}$$

converges being a subseries of the convergent positive series $\sum_{m=1}^{\infty} \frac{p}{m(m+p)}$, and

$$\sum_{m=1}^{q} \frac{1}{2p+m} + \sum_{m=2p+q+1}^{2p+2q} \frac{1}{2p+m} + \cdots + \sum_{m=(2p+q)(n-1)+1}^{(2p+q)(n-1)+q} \frac{1}{2p+m}$$

goes to ∞ because the series is bigger than the divergent positive series $\sum_{n=0}^{\infty} \frac{1}{2p+1+n(2p+q)}$. This proves that $\sum a_m(0)$ diverges, which implies that $\sum a_m(z)$ diverges. [3] □

Example 3.2.6. Shew that

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots = \frac{1}{2}\log(2).$$

Proof. The ratio of the number of positive terms to the number of negative terms in the first n terms is ultimately $\left(\frac{1}{\sqrt{2}}\right)^2$. By Example 3.1.5, the sum of the series is $\log(2 \cdot \frac{1}{\sqrt{2}}) = \frac{1}{2}\log(2)$. □

Example 3.2.7. Shew that the series

$$\frac{1}{1^\alpha} + \frac{1}{2^\beta} + \frac{1}{3^\alpha} + \frac{1}{4^\beta} + \cdots \qquad (1 < \alpha < \beta)$$

is convergent, although

$$\frac{u_{2n+1}}{u_{2n}} \to \infty.$$

Proof. Notice that since

$$0 \le \frac{1}{1^\alpha} + \frac{1}{2^\beta} + \frac{1}{3^\alpha} + \frac{1}{4^\beta} + \cdots \le \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges, we can conclude that

$$\frac{1}{1^\alpha} + \frac{1}{2^\beta} + \frac{1}{3^\alpha} + \frac{1}{4^\beta} + \cdots$$

[3] I would like to thank reuns and Conrad, users of Mathematics Stackexchange, for providing a hint for this problem.

converges by the comparison test.

However, since $1 < \alpha < \beta$, we have

$$\lim_{n\to\infty} \frac{u_{2n+1}}{u_{2n}} = \lim_{n\to\infty} \frac{\frac{1}{(2n+1)^{\alpha}}}{\frac{1}{(2n)^{\beta}}}$$

$$= \lim_{n\to\infty} \frac{(2n)^{\beta}}{(2n+1)^{\alpha}}$$

$$= \infty.$$

□

Example 3.2.8. Shew that the series

$$\alpha + \beta^2 + \alpha^3 + \beta^4 + \cdots \qquad (0 < \alpha < \beta < 1)$$

is convergent although

$$\frac{u_{2n}}{u_{2n-1}} \to \infty.$$

Proof. Notice that since

$$0 \leq \alpha + \beta^2 + \alpha^3 + \beta^4 + \cdots \leq \sum_{n=1}^{\infty} \beta^n$$

and $\sum_{n=1}^{\infty} \beta^n$ is converges by the geometric series test ($|\beta| < 1$), we can conclude that the series

$$\alpha + \beta^2 + \alpha^3 + \beta^4 + \cdots$$

converges by the comparison test.

However, since $\frac{\beta}{\alpha} > 1$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{u_{2n}}{u_{2n-1}} &= \lim_{n\to\infty} \frac{\beta^{2n}}{\alpha^{2n-1}} \\ &= \lim_{n\to\infty} \beta \left(\frac{\beta}{\alpha}\right)^{2n-1} \\ &= \infty. \end{aligned}$$

□

Example 3.2.9. Shew that the series

$$\sum_{n=1}^{\infty} \frac{nz^{n-1}\left\{(1+n^{-1})^n - 1\right\}}{(z^n-1)\left\{z^n - (1+n^{-1})^n\right\}}$$

converges absolutely for all values of z, except the values

$$z = \left(1 + \frac{a}{m}\right) e^{2k\pi i/m}$$

$$(a = 0, 1; k = 0, 1, \cdots, m-1; m = 1, 2, 3, \cdots).$$

Proof. Observe that this series is not defined when $z^n = 1$ or $z^n = \left(1 + \frac{1}{n}\right)^n$. So, we must exclude $z = \left(1 + \frac{1}{m}\right) e^{\frac{2k\pi i}{m}}$ and $z = e^{\frac{2k\pi i}{m}}, k = 0, 1, \cdots, m-1$ for all $m \in \mathbb{N}$. If $|z| < 1$, then

$$\begin{aligned} &\lim_{n\to\infty} \left| \frac{(n+1)z^n\left\{(1+(n+1)^{-1})^{n+1} - 1\right\}}{(z^{n+1}-1)\left\{z^{n+1} - (1+(n+1)^{-1})^{n+1}\right\}} \cdot \frac{(z^n-1)\{z^n - (1+n^{-1})^n\}}{nz^{n-1}\{(1+n^{-1})^n - 1\}} \right| \\ &= \left| \frac{z(e-1)}{(-1)(-e)} \left(\frac{(-1)(-e)}{e-1} \right) \right| \\ &= |z| < 1. \end{aligned}$$

Hence, the series converges absolutely when $|z| < 1$ by the ratio test.

If $|z| > 1$, then

$$\lim_{n\to\infty}\left|\frac{(n+1)z^n\left\{(1+(n+1)^{-1})^{n+1}-1\right\}}{(z^{n+1}-1)\left\{z^{n+1}-(1+(n+1)^{-1})^{n+1}\right\}}\cdot\frac{(z^n-1)\{z^n-(1+n^{-1})^n\}}{nz^{n-1}\{(1+n^{-1})^n-1\}}\right|$$
$$=\frac{1}{|z|}<1.$$

Thus, the series converges by the ratio test when $|z| > 1$.

Now, assume that $|z| = 1$ and z is not an nth root of unity so that each term in the series is defined. Then for n large enough, we have

$$\left|\frac{nz^{n-1}\left\{(1+n^{-1})^n-1\right\}}{(z^n-1)\left\{z^n-(1+n^{-1})^n\right\}}\right|\geq\frac{\frac{e}{2}n}{2(1+e)}\to\infty.$$

By the divergence test, the series diverges when $|z| = 1$ and z is not an nth root of unity. [4] □

Example 3.2.10. Shew that, when $s > 1$,

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{s-1}+\sum_{n=1}^{\infty}\left[\frac{1}{n^s}+\frac{1}{s-1}\left\{\frac{1}{(n+1)^{s-1}}-\frac{1}{n^{s-1}}\right\}\right],$$

and shew that the series on the right converges when $0 < s < 1$.

Proof. If $s > 1$, then

$$\frac{1}{s-1}+\sum_{n=1}^{\infty}\left[\frac{1}{n^s}+\frac{1}{s-1}\left\{\frac{1}{(n+1)^{s-1}}-\frac{1}{n^{s-1}}\right\}\right]$$
$$=\frac{1}{s-1}+\frac{1}{1^s}+\frac{1}{s-1}\left(\frac{1}{2^{s-1}}\right)-\frac{1}{s-1}\left(\frac{1}{1^{s-1}}\right)+\frac{1}{2^s}+\frac{1}{s-1}\left(\frac{1}{3^{s-1}}\right)-\frac{1}{s-1}\left(\frac{1}{2^{s-1}}\right)$$
$$+\frac{1}{3^s}+\frac{1}{s-1}\left(\frac{1}{4^{s-1}}\right)-\frac{1}{s-1}\left(\frac{1}{3^{s-1}}\right)+\cdots$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

[4] This result is different from the book's assertion that the series converges absolutely for all $|z| = 1$, where z is not an nth root of unity.

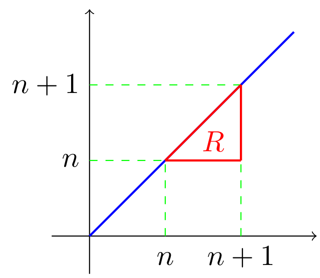

FIGURE 3.2. Region R.

Now, we will show that the series

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^s}+\frac{1}{s-1}\left\{\frac{1}{(n+1)^{s-1}}-\frac{1}{n^{s-1}}\right\}\right]$$

converges when $0<s<1$. Let $u_n=\frac{1}{n^s}+\frac{1}{s-1}\left\{\frac{1}{(n+1)^{s-1}}-\frac{1}{n^{s-1}}\right\}$. Then it can be easily checked that

$$0\le u_n=\frac{1}{n^s}-\int_n^{n+1}x^{-s}dx=\int_n^{n+1}\int_n^x sy^{-s-1}dydx.$$

Our goal is to show that $u_n=\int_n^{n+1}\int_n^x sy^{-s-1}dydx\le\frac{1}{n^{s+1}}$ for all $n\in\mathbb{N}$. Since $y^{-s-1}\ge 0$ over the region R (see Figure 3.2), the double integral $\int_n^{n+1}\int_n^x sy^{-s-1}dydx$ describes the volume of the solid below the surface sy^{-s-1} and above the region R. Note that the area of the region R is $\frac{1}{2}$ and the highest height for the solid is $s\frac{1}{n^{s+1}}$. Hence, the volume of the solid is less than $s\frac{1}{n^{s+1}}$. Thus, we can conclude that $\int_n^{n+1}\int_n^x sy^{-s-1}dydx\le s\frac{1}{n^{s+1}}\le\frac{1}{n^{s+1}}$ for all $n\in\mathbb{N}$.

Since $\sum_{n=1}^{\infty}\frac{1}{n^{s+1}}$ converges when $0<s<1$, we can conclude that the series

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^s}+\frac{1}{s-1}\left\{\frac{1}{(n+1)^{s-1}}-\frac{1}{n^{s-1}}\right\}\right]$$

converges by the comparison test. [5] □

[5] I would like to thank reuns, a user of Mathematics Stackexchange, for providing a hint for this problem.

Example 3.2.11. In the series whose general term is

$$u_n = q^{n-\nu} x^{\frac{1}{2}\nu(\nu+1)}, \qquad (0 < q < 1 < x)$$

where ν denotes the number of digits in the expression of n in the ordinary decimal scale of notation, shew that

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = q,$$

and that the series is convergent, although $\overline{\lim}_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty$.

Proof. First, we claim that $\lfloor \log_{10}(n) \rfloor + 1$ gives the number of digits of a positive integer n. Suppose a fixed positive integer n has ν digits. Then $10^{\nu-1} \le n < 10^{\nu}$. Taking log base 10, we obtain

$$\nu - 1 \le \log_{10}(n) < \nu.$$

Observe that $\lfloor \log_{10}(n) \rfloor = \nu - 1$ and it follows that $\nu = \lfloor \log_{10}(n) \rfloor + 1$.

Since

$$1 \le \nu \le \log_{10}(n) + 1 \le n \quad \text{for all } n \in \mathbb{N},$$

we have

$$q^n x \le q^{n-\nu} x^{\frac{1}{2}\nu(\nu+1)} \le q^{n-\log_{10}(n)-1} x^{\frac{1}{2}(\log_{10}(n)+1)(\log_{10}(n)+2)},$$

whenever $0 < q < 1 < x$ and $n \in \mathbb{N}$. Therefore, we have

$$(q^n x)^{\frac{1}{n}} \le \left(q^{n-\nu} x^{\frac{1}{2}\nu(\nu+1)}\right)^{\frac{1}{n}} \le \left(q^{n-\log_{10}(n)-1} x^{\frac{1}{2}(\log_{10}(n)+1)(\log_{10}(n)+2)}\right)^{\frac{1}{n}}.$$

Now, since

$$\lim_{n\to\infty} (q^n x)^{\frac{1}{n}} = \lim_{n\to\infty} q x^{\frac{1}{n}} = q,$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\left(q^{n-\log_{10}(n)-1}x^{\frac{1}{2}(\log_{10}(n)+1)(\log_{10}(n)+2)}\right)^{\frac{1}{n}}\\
&=\lim_{n\to\infty} q^{1-\frac{\log_{10}(n)}{n}-\frac{1}{n}}x^{\frac{\frac{1}{2}(\log_{10}(n)+1)(\log_{10}(n)+2)}{n}}\\
&=q,
\end{aligned}$$

we can conclude that $\lim_{n\to\infty} u_n^{\frac{1}{n}} = q$ by the squeeze theorem. It follows that the series $\sum_{n=1}^{\infty} u_n$ converges by the root test.

Finally, we will show that $\overline{\lim}_{n\to\infty}\frac{u_{n+1}}{u_n}=\infty$. Take a subsequence of $\frac{u_{n+1}}{u_n}$, call it $\frac{u_{n_k+1}}{u_{n_k}}$, such that $\nu(n_k+1)=\nu(n_k)+1$. [6] That is, $n_k = 10^k - 1$. Then

$$\begin{aligned}
\lim_{k\to\infty}\frac{u_{n_k+1}}{u_{n_k}} &= \lim_{k\to\infty}\frac{q^{n_k+1-\nu(n_k+1)}x^{\frac{1}{2}\nu(n_k+1)(\nu(n_k+1)+1)}}{q^{n_k-\nu(n_k)}x^{\frac{1}{2}\nu(n_k)(\nu(n_k)+1)}}\\
&= \lim_{k\to\infty}\frac{q^{n_k+1-\nu(n_k)-1}x^{\frac{1}{2}(\nu(n_k)+1)(\nu(n_k)+2)}}{q^{n_k-\nu(n_k)}x^{\frac{1}{2}\nu(n_k)(\nu(n_k)+1)}}\\
&= \lim_{k\to\infty} x^{\nu(n_k)+1}\\
&= \infty.
\end{aligned}$$

This proves that $\overline{\lim}_{n\to\infty}\frac{u_{n+1}}{u_n}=\infty$. □

Example 3.2.12. Shew that the series

$$q_1 + q_1^2 + q_2^3 + q_1^4 + q_2^5 + q_3^6 + q_1^7 + \cdots,$$

where

$$q_n = q^{1+(4/n)}, \qquad (0<q<1)$$

is convergent, although the ratio of the $(n+1)$th term to the nth is greater than unity when n is not a triangular number.

[6] $\nu(n_k)$ should be understood as the number of digits of n_k.

Proof. Since $0 < q < 1$,

$$0 < q^{1+\frac{4}{n}} < q < 1 \quad \text{for all } n \in \mathbb{N}.$$

Hence,

$$0 \leq q_1 + q_1^2 + q_2^3 + q_1^4 + q_2^5 + q_3^6 + q_1^7 + \cdots \leq \sum_{n=1}^{\infty} q^n,$$

where the series $\sum_{n=1}^{\infty} q^n$ converges by the geometric series test. Therefore, the series

$$q_1 + q_1^2 + q_2^3 + q_1^4 + q_2^5 + q_3^6 + q_1^7 + \cdots ,$$

converges by the comparison test.

Suppose that n is not a triangular number, i.e. $n \notin \left\{1, 3, 6, 10, \cdots , \frac{m(m+1)}{2}, \cdots \right\}$. Then $\frac{m(m+1)}{2} < n < \frac{(m+1)(m+2)}{2}$ for some positive integer m and

$$\begin{aligned} \frac{q_{k+1}^{n+1}}{q_k^n} &= \frac{\left(q^{1+\frac{4}{k+1}}\right)^{n+1}}{\left(q^{1+\frac{4}{k}}\right)^{n}} \\ &= q^{1+\frac{4(k-n)}{k(k+1)}}, \end{aligned}$$

where $1 \leq k \leq m \leq n$. If $m \geq 2$, then

$$\begin{aligned} \frac{4(k-n)}{k(k+1)} &\leq \frac{4(m-n)}{2} \\ &\leq 2\left(m - \frac{m(m+1)}{2}\right) \\ &= m(1-m) \\ &\leq -2. \end{aligned}$$

This implies that $\frac{q_{k+1}^{n+1}}{q_k^n} = q^{1+\frac{4(k-n)}{k(k+1)}} > 1$ since $1 + \frac{4(k-n)}{k(k+1)} < 0$ for $m \geq 2$. It is easy to verify that $\frac{q_{k+1}^{n+1}}{q_k^n} > 1$ when $1 < n < 3$.

□

Example 3.2.13. Shew that the series

$$\sum_{n=0}^{\infty} \frac{e^{2n\pi i x}}{(w+n)^s},$$

where w is real, and where $(w+n)^s$ is understood to mean $e^{s\log(w+n)}$, the logarithm being taken in its arithmetic sense, is convergent for all values of s, when $\mathrm{Im}(x)$ is positive, and is convergent for all values of s whose real part is positive, when x is real and not an integer.

Proof. First, we will show that the series converges for all $s \in \mathbb{C}$ and $x = a + bi$, where $b > 0$. Observe that

$$\begin{aligned}\frac{e^{2n\pi i x}}{(w+n)^s} &= \frac{e^{2n\pi i x}}{e^{s\log(w+n)}} \\ &= e^{2n\pi i x - s\log(w+n)} \\ &= e^{2n\pi i(a+bi) - s\log(w+n)} \\ &= e^{n(-2\pi b + 2\pi a i) - s\log(w+n)}.\end{aligned}$$

Since $b > 0$,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{e^{(n+1)(-2\pi b+2\pi a i)-s\log(w+n+1)}}{e^{n(-2\pi b+2\pi a i)-s\log(w+n)}}\right| &= \lim_{n\to\infty}\left|e^{-2\pi b+2\pi a i - s\log\left(\frac{w+n+1}{w+n}\right)}\right| \\ &= \left|e^{-2\pi b + 2\pi a i - s\log(1)}\right| \\ &= \left|e^{-2\pi b}e^{2\pi a i}\right| \\ &= e^{-2\pi b} < 1.\end{aligned}$$

Thus, we can conclude that the series converges by the ratio test.

Next, we will show that the series converges for all $s = \sigma + ti$ such that $\sigma > 0$ and $x \in \mathbb{R}\backslash\mathbb{Z}$. Without loss of generality, we can start the series from $n = 1$. Applying summation by parts, we get $\sum_{k=1}^{n} \frac{e^{2k\pi ix}}{(w+k)^s}$ is equal to

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\left(e^{2\pi ix}\right)^j\right)\left(\frac{1}{(w+k)^s}-\frac{1}{(w+k+1)^s}\right)+\left(\sum_{k=1}^{n}\left(e^{2\pi ix}\right)^k\right)\frac{1}{(w+n+1)^s}.$$

We claim that $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\left(e^{2\pi ix}\right)^j\right)\left(\frac{1}{(w+k)^s}-\frac{1}{(w+k+1)^s}\right)$ exists. Notice that

$$\begin{aligned}\left|\frac{1}{(w+k)^s}-\frac{1}{(w+k+1)^s}\right| &= \left|\int_k^{k+1}\frac{s}{(w+u)^{s+1}}du\right| \\ &\leq \int_k^{k+1}\frac{|s|}{(w+u)^{\sigma+1}}du \\ &\leq |s|\cdot\frac{1}{(w+k)^{\sigma+1}} \\ &= O\left(\frac{1}{k^{\sigma+1}}\right),\end{aligned}$$

as $k\to\infty$. Thus, it suffices to show the series $\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}(e^{2\pi ix})^j\right)\frac{1}{n^{\sigma+1}}$ converges. Observe that the sequence of partial sums $\left\{\sum_{j=1}^{n}\left(e^{2\pi ix}\right)^j\right\}$ is uniformly bounded since for all $n\in\mathbb{N}$

$$\begin{aligned}\left|\sum_{j=1}^{n}\left(e^{2\pi ix}\right)^j\right| &= \left|\frac{e^{2\pi ix}(1-e^{2n\pi ix})}{1-e^{2\pi ix}}\right| \qquad (e^{2\pi ix}\neq 1 \text{ since } x\in\mathbb{R}\backslash\mathbb{Z}) \\ &\leq \frac{2}{|1-e^{2\pi xi}|}.\end{aligned}$$

Now, since $\sigma>0$, it follows that

$$0\leq\sum_{n=1}^{\infty}\left|\left(\sum_{j=1}^{n}\left(e^{2\pi ix}\right)^j\right)\frac{1}{n^{\sigma+1}}\right|\leq\frac{2}{|1-e^{2\pi xi}|}\sum_{n=1}^{\infty}\frac{1}{n^{\sigma+1}}<\infty.$$

Finally, $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(e^{2\pi ix}\right)^k\right)\frac{1}{(w+n+1)^s}=0$ since $\sum_{k=1}^{n}\left(e^{2\pi ix}\right)^k$ is uniformly bounded and $\lim_{n\to\infty}\frac{1}{(w+n+1)^s}=0$. This proves that the series $\sum_{n=0}^{\infty}\frac{e^{2n\pi ix}}{(w+n)^s}$ converges when real part of s is positive and $x\in\mathbb{R}\backslash\mathbb{Z}$. □

Example 3.2.14. If $u_n > 0$, shew that if $\sum u_n$ converges, then $\underline{\lim}_{n\to\infty}(nu_n) = 0$, and that, if in addition $u_n \geq u_{n+1}$, then $\lim_{n\to\infty}(nu_n) = 0$.

Proof. Let $\sum u_n$ be a convergent series with $u_n > 0$ for all $n \in \mathbb{N}$. Assume, on the contrary, that $\underline{\lim}_{n\to\infty}(nu_n) \neq 0$. So, we must have $\underline{\lim}_{n\to\infty}(nu_n) = r_1 > 0$. Let $0 < r < r_1$. Then there exists $N \in \mathbb{N}$ such that $nu_n \geq r$ for all $n \geq N$. Therefore, we get

$$\begin{aligned}\sum_{n=1}^{\infty} u_n &= \sum_{n=1}^{N-1} u_n + \sum_{n=N}^{\infty} u_n \\ &\geq \sum_{n=1}^{N-1} u_n + \sum_{n=N}^{\infty} \frac{r}{n} \\ &= \infty,\end{aligned}$$

contradicting $\sum u_n$ is a convergent series. Thus, we must have $\underline{\lim}_{n\to\infty}(nu_n) = 0$.

Let $\{u_n\}$ be a decreasing sequence of positive numbers and assume that $\sum_{n=1}^{\infty} u_n$ converges. It follows that the series $\sum_{n=1}^{\infty} 2^n u_{2^n}$ converges [7]. Hence, we have

$$\lim_{n\to\infty} 2^n u_{2^n} = 0.$$

Since $\{u_n\}$ is decreasing, for $2^k \leq n \leq 2^{k+1}$, we have

$$2^k u_{2^{k+1}} \leq nu_n \leq 2^{k+1} u_{2^k},$$

where both $2^k u_{2^{k+1}}$, $2^{k+1} u_{2^k}$ converge to 0 as $k \to \infty$. By the squeeze theorem, we can conclude that $\lim_{n\to\infty} nu_n = 0$. [7] □

Example 3.2.15. If

$$a_{m,n} = \frac{m-n}{2^{m+n}} \frac{(m+n-1)!}{m!n!}, \qquad (m, n > 0)$$

$$a_{m,0} = 2^{-m}, a_{0,n} = -2^{-n}, a_{0,0} = 0,$$

[7] I would like to thank Ragib Zaman, a user of Mathematics Stackexchange, for providing a hint for this problem.

shew that

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{m,n}\right) = -1, \quad \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{m,n}\right) = 1.$$

Proof. First, we will show that $\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{m,n}\right) = -1$. Notice that

$$\begin{aligned}
\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{m,n}\right) &= \sum_{m=1}^{\infty}\left(\sum_{n=0}^{\infty} a_{m,n}\right) - \sum_{n=1}^{\infty}\frac{1}{2^n} \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\left[\frac{m-n}{2^{m+n}}\frac{(m+n-1)!}{m!n!}\right] + \frac{1}{2^m}\right) - \sum_{n=1}^{\infty}\frac{1}{2^n} \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\left[\frac{m}{2^{m+n}}\frac{(m+n-1)!}{m!n!}\right] - \sum_{n=1}^{\infty}\left[\frac{n}{2^{m+n}}\frac{(m+n-1)!}{m!n!}\right] + \frac{1}{2^m}\right) \\
&- \sum_{n=1}^{\infty}\frac{1}{2^n}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{m}{2^{m+n}}\frac{(m+n-1)!}{m!n!} &= \frac{1}{2^m}\sum_{n=1}^{\infty}\frac{(m+n-1)!}{(m-1)!n!}\left(\frac{1}{2}\right)^n \\
&= \frac{1}{2^m}\sum_{n=1}^{\infty}\binom{m+n-1}{n}\left(\frac{1}{2}\right)^n \\
&= \frac{1}{2^m}\sum_{n=1}^{\infty}(-1)^n\binom{-m}{n}\left(\frac{1}{2}\right)^n \\
&= \frac{1}{2^m}\left[\left(1-\frac{1}{2}\right)^{-m} - 1\right] && \text{(binomial expansion)} \\
&= \frac{1}{2^m}\left[2^m - 1\right] \\
&= 1 - \frac{1}{2^m},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{2^{m+n}} \frac{(m+n-1)!}{m!n!} &= \frac{1}{2^m} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{(m+n-1)!}{m!(n-1)!} \\
&= \frac{1}{2^m} \sum_{n=1}^{\infty} \binom{m+n-1}{n-1} \frac{1}{2^n} \\
&= \frac{1}{2^m} \sum_{n=1}^{\infty} (-1)^{n-1} \binom{-(m+1)}{n-1} \frac{1}{2^{n-1}} \frac{1}{2} \\
&= \frac{1}{2^m} \frac{1}{2} \left(1 - \frac{1}{2}\right)^{-(m+1)} \\
&= \frac{1}{2^{m+1}} 2^{m+1} \\
&= 1,
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{m,n} \right) &= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left[\frac{m}{2^{m+n}} \frac{(m+n-1)!}{m!n!} \right] - \sum_{n=1}^{\infty} \left[\frac{n}{2^{m+n}} \frac{(m+n-1)!}{m!n!} \right] + \frac{1}{2^m} \right) \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= \sum_{m=1}^{\infty} \left(1 - \frac{1}{2^m} - 1 + \frac{1}{2^m} \right) - 1 \\
&= -1.
\end{aligned}$$

Second, we want to show that $\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} \right) = 1$. Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} \right) &= \sum_{n=1}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} \right) + \sum_{m=1}^{\infty} \frac{1}{2^m} \\
&= \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \left(\frac{m-n}{2^{m+n}} \frac{(m+n-1)!}{m!n!} \right) - \frac{1}{2^n} \right] + \sum_{m=1}^{\infty} \frac{1}{2^m} \\
&= \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{m}{2^{m+n}} \frac{(m+n-1)!}{m!n!} - \sum_{m=1}^{\infty} \frac{n}{2^{m+n}} \frac{(m+n-1)!}{m!n!} - \frac{1}{2^n} \right] \\
&\quad + \sum_{m=1}^{\infty} \frac{1}{2^m}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{m}{2^{m+n}} \frac{(m+n-1)!}{m!n!} &= \frac{1}{2^n} \sum_{m=1}^{\infty} \frac{(m+n-1)!}{(m-1)!n!} \frac{1}{2^m} \\
&= \frac{1}{2^n} \sum_{m=1}^{\infty} \binom{m+n-1}{m-1} \frac{1}{2^m} \\
&= \frac{1}{2^{n+1}} \sum_{m=1}^{\infty} (-1)^{m-1} \binom{-(n+1)}{m-1} \frac{1}{2^{m-1}} \\
&= \frac{1}{2^{n+1}} \sum_{m=1}^{\infty} \binom{-(n+1)}{m-1} \left(-\frac{1}{2}\right)^{m-1} \\
&= \frac{1}{2^{n+1}} \left(1-\frac{1}{2}\right)^{-(n+1)} \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{n}{2^{m+n}} \frac{(m+n-1)!}{m!n!} &= \frac{1}{2^n} \sum_{m=1}^{\infty} \frac{(m+n-1)!}{m!(n-1)!} \frac{1}{2^m} \\
&= \frac{1}{2^n} \sum_{m=1}^{\infty} \binom{m+n-1}{m} \frac{1}{2^m} \\
&= \frac{1}{2^n} \sum_{m=1}^{\infty} (-1)^m \binom{-n}{m} \frac{1}{2^m} \\
&= \frac{1}{2^n} \sum_{m=1}^{\infty} \binom{-n}{m} \left(-\frac{1}{2}\right)^m \\
&= \frac{1}{2^n} \left[\left(1-\frac{1}{2}\right)^{-n} - 1\right] \\
&= 1 - \frac{1}{2^n},
\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}a_{m,n}\right) &= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{m}{2^{m+n}}\frac{(m+n-1)!}{m!n!}-\sum_{m=1}^{\infty}\frac{n}{2^{m+n}}\frac{(m+n-1)!}{m!n!}-\frac{1}{2^n}\right]\\ &+\sum_{m=1}^{\infty}\frac{1}{2^m}\\ &=\sum_{n=1}^{\infty}\left(1-1+\frac{1}{2^n}-\frac{1}{2^n}\right)+1\\ &=1.\end{aligned}$$

□

Example 3.2.16. By converting the series

$$1+\frac{8q}{1-q}+\frac{16q^2}{1+q^2}+\frac{24q^3}{1-q^3}+\cdots,$$

(in which $|q|<1$), into a double series, shew that it is equal to

$$1+\frac{8q}{(1-q)^2}+\frac{8q^2}{(1+q^2)^2}+\frac{8q^3}{(1-q^3)^2}+\cdots.$$

Before we proceed with the proof of Example 3.2.16, we will introduce a theorem.

Theorem 3.2.1. Let $(u_{\mu,\nu})$ be a double sequence. If the sum by rows of the series $\sum_{\mu,\nu}|u_{\mu,\nu}|$ exists, then the sum by rows and the sum by columns of the series $\sum_{\mu,\nu}u_{\mu,\nu}$ exist and they are equal [7].

Now, we are ready to prove Example 3.2.16.

Proof. Notice that

$$
\begin{array}{ccccccccc}
\frac{8q}{1-q} & = & 8q & + & 8q^2 & + & 8q^3 & + & \cdots \\
 & & + & & + & & + & & + \\
\frac{16q^2}{1+q^2} & = & 16q^2 & - & 16q^4 & + & 16q^6 & - & \cdots \\
 & & + & & + & & + & & + \\
\frac{24q^3}{1-q^3} & = & 24q^3 & + & 24q^6 & + & 24q^9 & + & \cdots \\
 & & + & & + & & + & & + \\
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \parallel & & \parallel & & \parallel & & \\
 & & \frac{8q}{(1-q)^2} & & \frac{8q^2}{(1+q^2)^2} & & \frac{8q^3}{(1-q^3)^2} & &
\end{array}.
$$

From the equations above, we know that the series

$$1 + \frac{8q}{1-q} + \frac{16q^2}{1+q^2} + \frac{24q^3}{1-q^3} + \cdots$$

and

$$1 + \frac{8q}{(1-q)^2} + \frac{8q^2}{(1+q^2)^2} + \frac{8q^3}{(1-q^3)^2} + \cdots$$

represent the sum by rows and the sum by columns of the double series, respectively. These two series will be equal provided that the series

$$1 + \frac{8|q|}{1-|q|} + \frac{16|q|^2}{1-|q|^2} + \frac{24|q|^3}{1-|q|^3} + \cdots$$

converges by Theorem 3.2.1. Observe that

$$
\begin{aligned}
\lim_{n\to\infty} \left| \frac{8(n+1)|q|^{n+1}}{1-|q|^{n+1}} \cdot \frac{1-|q|^n}{8n|q|^n} \right| &= \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{|q|-|q|^{n+1}}{1-|q|^{n+1}} \right| \\
&= |q| \\
&< 1.
\end{aligned}
$$

Hence, the series

$$1 + \frac{8|q|}{1-|q|} + \frac{16|q|^2}{1-|q|^2} + \frac{24|q|^3}{1-|q|^3} + \cdots$$

converges by the ratio test. $\square$

Example 3.2.17. Assuming that

$$\sin(z) = z\prod_{r=1}^{\infty}\left(1-\frac{z^2}{r^2\pi^2}\right),$$

shew that if $m\to\infty$ and $n\to\infty$ in such a way that $\lim\left(\frac{m}{n}\right)=k$, where k is finite, then

$$\lim\prod_{r=-n}^{m}{}'\left(1+\frac{z}{r\pi}\right)=k^{\frac{z}{\pi}}\frac{\sin(z)}{z},$$

the prime indicating that the factor for which $r=0$ is omitted.

Proof. Let $m\colon\mathbb{N}\to\mathbb{N}$ be a function such that $\lim_{n\to\infty}\frac{m(n)}{n}=k$, where k is finite. Since $\prod_{r=1}^{\infty}\left(1-\frac{z}{r\pi}\right)e^{\frac{z}{r\pi}}$ and $\prod_{r=1}^{\infty}\left(1+\frac{z}{r\pi}\right)e^{-\frac{z}{r\pi}}$ are absolute convergent (P34), we have

$$\lim_{n\to\infty}\prod_{r=-n}^{m(n)}{}'\left(1+\frac{z}{r\pi}\right)e^{-\frac{z}{r\pi}}=\prod_{r=1}^{\infty}\left(1-\frac{z^2}{r^2\pi^2}\right)$$

by rearranging. Then it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\prod_{r=-n}^{m(n)}{}'\left(1+\frac{z}{r\pi}\right)\\
&=\lim_{n\to\infty}\prod_{r=-n}^{m(n)}{}'\left(1+\frac{z}{r\pi}\right)e^{-\frac{z}{r\pi}}e^{\frac{z}{r\pi}}\\
&=\lim_{n\to\infty}e^{\frac{z}{\pi}\left(\sum_{r=1}^{m(n)}\frac{1}{r}-\sum_{r=1}^{n}\frac{1}{r}\right)}\prod_{r=-n}^{m(n)}{}'\left(1+\frac{z}{r\pi}\right)e^{-\frac{z}{r\pi}}\\
&=e^{\frac{z}{\pi}\left(\lim_{n\to\infty}\left[\sum_{r=1}^{m(n)}\frac{1}{r}-\sum_{r=1}^{n}\frac{1}{r}\right]\right)}\lim_{n\to\infty}\prod_{r=-n}^{m(n)}{}'\left(1+\frac{z}{r\pi}\right)e^{-\frac{z}{r\pi}}\\
&=e^{\frac{z}{\pi}\lim_{n\to\infty}\left(\sum_{r=1}^{m(n)}\frac{1}{r}-\log(m(n))\right)-\left(\sum_{r=1}^{n}\frac{1}{r}-\log(n)\right)+\log\left(\frac{m(n)}{n}\right)}\prod_{r=1}^{\infty}\left(1+\frac{z}{r\pi}\right)e^{-\frac{z}{r\pi}}\left(1-\frac{z}{r\pi}\right)e^{\frac{z}{r\pi}}\\
&=e^{\frac{z}{\pi}(\gamma-\gamma+\log(k))}\prod_{r=1}^{\infty}\left(1-\frac{z^2}{r^2\pi^2}\right)\\
&=e^{\frac{z}{\pi}\log(k)}\frac{\sin(z)}{z}\\
&=k^{\frac{z}{\pi}}\frac{\sin(z)}{z},
\end{aligned}$$

where γ is Euler's constant. $\square$

Example 3.2.18. If $u_0 = u_1 = u_2 = 0$, and if, when $n > 1$,

$$u_{2n-1} = -\frac{1}{\sqrt{n}}, u_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n} + \frac{1}{n\sqrt{n}},$$

then $\prod_{n=0}^{\infty}(1+u_n)$ converges, though $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} u_n^2$ are divergent.

Proof. Observe that

$$\begin{aligned}(1+u_{2n-1})(1+u_{2n}) &= \left(1-\frac{1}{\sqrt{n}}\right)\left(1+\frac{1}{\sqrt{n}}+\frac{1}{n}+\frac{1}{n\sqrt{n}}\right)\\ &= 1-\frac{1}{n^2},\end{aligned}$$

and if we let P_n denote the nth partial product, then

$$\begin{aligned}P_{2m} &= \prod_{n=2}^{m}\left(1-\frac{1}{n^2}\right)\\ \Rightarrow P_{2m+1} &= P_{2m}\left(1-\frac{1}{\sqrt{m+1}}\right).\end{aligned}$$

Since the series $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges, the $2m$th partial product P_{2m} converges as m goes to infinity. Moreover,

$$\begin{aligned}\lim_{m\to\infty} P_{2m+1} &= \lim_{m\to\infty} P_{2m}\left(1-\frac{1}{\sqrt{m+1}}\right)\\ &= \lim_{m\to\infty} P_{2m}.\end{aligned}$$

Thus, both even and odd partial products converge to the same value. This proves that the infinite product $\prod_{n=0}^{\infty}(1+u_n)$ converges. [8]

[8] I would like to thank RRL, a user of Mathematics Stackexchange, for providing the proof of the convergence of $\prod_{n=0}^{\infty}(1+u_n)$.

We will proceed to prove $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} u_n^2$ diverge. Since

$$\begin{aligned}\sum_{n=0}^{\infty} u_n &= \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{3} + \frac{1}{3\sqrt{3}} + \frac{1}{4} + \frac{1}{4\sqrt{4}} \\ &> \sum_{n=2}^{\infty} \frac{1}{n},\end{aligned}$$

and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we can conclude that the series $\sum_{n=0}^{\infty} u_n$ diverges by the comparison test.

Similarly, since

$$\begin{aligned}\sum_{n=0}^{\infty} u_n^2 &= \sum_{n=2}^{\infty} \left(\frac{1}{n} + \left(\frac{1}{\sqrt{n}} + \frac{1}{n} + \frac{1}{n\sqrt{n}} \right)^2 \right) \\ &> \sum_{n=2}^{\infty} \frac{1}{n},\end{aligned}$$

and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we can conclude that the series $\sum_{n=0}^{\infty} u_n^2$ diverges by the comparison test. □

Example 3.2.19. Prove that

$$\prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{n}\right)^{n^k} \exp\left(\sum_{m=1}^{k+1} \frac{n^{k-m} z^m}{m} \right) \right\},$$

where k is any positive integer, converges absolutely for all values of z.

Proof. To show the infinite product

$$\prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{n}\right)^{n^k} \exp\left(\sum_{m=1}^{k+1} \frac{n^{k-m} z^m}{m} \right) \right\},$$

converges absolutely for all $z \in \mathbb{C}$, it suffices to show the series

$$\sum_{n=1}^{\infty} \log \left\{ \left(1 - \frac{z}{n}\right)^{n^k} \exp\left(\sum_{m=1}^{k+1} \frac{n^{k-m} z^m}{m} \right) \right\}$$

converges absolutely. Fix $z \in \mathbb{C}$. Then there exists $n_0 \in \mathbb{N}$ such that $|z| < n_0$. Hence, $\log\left(1 - \frac{z}{n}\right) = -\sum_{m=1}^{\infty} \frac{z^m}{n^m m}$ converges for all $n \geq n_0$. It follows that

$$\begin{aligned}
\sum_{n=n_0}^{\infty} \left| \log\left\{ \left(1 - \frac{z}{n}\right)^{n^k} \exp\left(\sum_{m=1}^{k+1} \frac{n^{k-m} z^m}{m} \right) \right\} \right| &= \sum_{n=n_0}^{\infty} \left| n^k \log\left(1 - \frac{z}{n}\right) + \sum_{m=1}^{k+1} \frac{n^{k-m} z^m}{m} \right| \\
&= \sum_{n=n_0}^{\infty} \left| -n^k \sum_{m=1}^{\infty} \frac{z^m}{n^m m} + \sum_{m=1}^{k+1} \frac{n^{k-m} z^m}{m} \right| \\
&= \sum_{n=n_0}^{\infty} \left| - \sum_{m=1}^{\infty} \frac{n^{k-m} z^m}{m} + \sum_{m=1}^{k+1} \frac{n^{k-m} z^m}{m} \right| \\
&= \sum_{n=n_0}^{\infty} \left(\sum_{m=k+2}^{\infty} \frac{n^{k-m} |z|^m}{m} \right) \\
&\leq \sum_{n=n_0}^{\infty} n^k \left(\sum_{m=k+2}^{\infty} \left(\frac{|z|}{n} \right)^m \right) \\
&= \sum_{n=n_0}^{\infty} n^k \frac{\left(\frac{|z|}{n} \right)^{k+2}}{1 - \frac{|z|}{n}} \\
&= \sum_{n=n_0}^{\infty} \frac{|z|^{k+2}}{n(n - |z|)} < \infty.
\end{aligned}$$

□

Example 3.2.20. If $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series of real terms, then $\prod_{n=1}^{\infty}(1 + a_n)$ converges (but not absolutely) or diverges to zero according as $\sum_{n=1}^{\infty} a_n^2$ converges or diverges.

Proof. Let the real series $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. We may assume that $a_n \neq 0$ for all $n \in \mathbb{N}$. If $a_n = 0$ for some n, we may discard 0 from the series $\sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} a_n$ is convergent, $\lim_{n \to \infty} a_n = 0$. Let $\varepsilon = \frac{1}{2}$. There exists $N \in \mathbb{N}$ such that if $n \geq N$, we have $|a_n| < \frac{1}{2}$. It follows that if $n \geq N$, we obtain

$$\begin{aligned}
\log(1 + a_n) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{a_n^k}{k} \\
&= a_n - \frac{a_n^2}{2} + O\left(a_n^3\right).
\end{aligned}$$

Observe that

$$\sum_{n=N}^{\infty} a_n = \sum_{n=N}^{\infty} a_n^2 \frac{a_n - \log(1+a_n)}{a_n^2} + \log(1+a_n).$$

Then we have

$$\begin{aligned} \frac{a_n - \log(1+a_n)}{a_n^2} &= \frac{a_n - a_n + \frac{a_n^2}{2} + O\left(a_n^3\right)}{a_n^2} \\ &= \frac{1}{2} + O(a_n) \\ &\to \frac{1}{2} \quad \text{as } n \to \infty. \end{aligned}$$

Assume $\sum_{n=N}^{\infty} a_n^2$ converges. Since $\lim_{n\to\infty} \frac{a_n - \log(1+a_n)}{a_n^2} = \frac{1}{2}$, then $\sum_{n=N}^{\infty} (a_n - \log(1+a_n))$ converges by the limit comparison test. It follows that

$$\sum_{n=N}^{\infty} \log(1+a_n) = \sum_{n=N}^{\infty} a_n - \sum_{n=N}^{\infty} (a_n - \log(1+a_n))$$

converges. This shows that $\prod_{n=1}^{\infty}(1+a_n)$ converges.

Now, assume that $\sum_{n=N}^{\infty} a_n^2$ diverges. Since $\lim_{n\to\infty} \frac{a_n - \log(1+a_n)}{a_n^2} = \frac{1}{2}$, we have

$$\sum_{n=N}^{\infty} a_n - \log(1+a_n)$$

diverges by the limit comparison test. Because $\sum_{n=N}^{\infty} a_n$ converges, we have

$$\sum_{n=N}^{\infty} \log(1+a_n)$$

diverges. This implies that $\prod_{n=1}^{\infty}(1+a_n)$ diverges. [9] □

[9] I would like to thank Robert Israel, a user of Mathematics Stackexchange, for providing a hint for this problem.

Example 3.2.21. Let $\sum_{n=1}^{\infty} \theta_n$ be an absolutely convergent series. Shew that the infinite determinant

$$\Delta(c) = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{(c-4)^2-\theta_0}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_4}{4^2-\theta_0} & \cdots \\ \cdots & \frac{-\theta_1}{2^2-\theta_0} & \frac{(c-2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_3}{2^2-\theta_0} & \cdots \\ \cdots & \frac{-\theta_2}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & \frac{c^2-\theta_0}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & \frac{-\theta_2}{0^2-\theta_0} & \cdots \\ \cdots & \frac{-\theta_3}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \frac{(c+2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \cdots \\ \cdots & \frac{-\theta_4}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & \frac{(c+4)^2-\theta_0}{4^2-\theta_0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

converges; and shew that the equation

$$\Delta(c) = 0$$

is equivalent to the equation

$$\sin^2\left(\frac{1}{2}\pi c\right) = \Delta(0)\sin^2\left(\frac{1}{2}\pi\theta_0^{\frac{1}{2}}\right).$$

Proof. Proof can be found in section 19.42, titled “The Evaluation of Hill’s Determinant,” of the book [13]. □

CHAPTER 4

CONTINUOUS FUNCTIONS AND UNIFORM CONVERGENCE

4.1 Solutions to Exercises in the Chapter

Example 4.1.1 (P50)**.** Shew that, if $\delta > 0$, the series

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n}, \quad \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$$

converge uniformly in the range $\delta \le \theta \le 2\pi - \delta$. Obtain the corresponding result for the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\theta)}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\theta)}{n}$$

by writing $\theta + \pi$ for θ.

Proof. We will apply the Dirichlet test for uniform convergence of series. Notice that $\frac{1}{n}$ decreases to 0 uniformly. We want to show that the sequences of partial sums $\left\{\sum_{n=1}^{N} \cos(n\theta)\right\}$ and $\left\{\sum_{n=1}^{N} \sin(n\theta)\right\}$ are uniformly bounded on $[\delta, 2\pi - \delta]$, where $\delta > 0$. Observe that

$$\begin{aligned}
\left|\sum_{n=1}^{N} \sin(n\theta)\right| &= \left|\operatorname{Im}\left(\sum_{n=1}^{N} e^{in\theta}\right)\right| \\
&= \left|\operatorname{Im}\left(\frac{e^{i\theta}(1 - e^{iN\theta})}{1 - e^{i\theta}}\right)\right| \\
&\le \left|\frac{e^{i\theta}(1 - e^{iN\theta})}{1 - e^{i\theta}}\right| \\
&\le \frac{2}{|1 - e^{i\theta}|}.
\end{aligned}$$

Since $0 < \delta \leq \theta \leq 2\pi - \delta$, there exists $\theta_0 \in [\delta, 2\pi - \delta]$ such that $0 < |1 - e^{i\theta_0}| \leq |1 - e^{i\theta}|$ for all $\theta \in [\delta, 2\pi - \delta]$. It follows that

$$\frac{2}{|1 - e^{i\theta}|} \leq \frac{2}{|1 - e^{i\theta_0}|} \quad \text{for all } \theta \in [\delta, 2\pi - \delta].$$

This proves that the sequence of partial sums $\left\{\sum_{n=1}^{N} \sin(n\theta)\right\}$ is uniformly bounded. Similarly, $\left\{\sum_{n=1}^{N} \cos(n\theta)\right\}$ is uniformly bounded. By the Dirichlet test, we can conclude that the series $\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n}$, $\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$ converges uniformly in the range $\delta \leq \theta \leq 2\pi - \delta$..

Now, replace θ with $\theta + \pi$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\cos(n(\theta+\pi))}{n} &= \sum_{n=1}^{\infty} (\cos(n\theta)\cos(n\pi) - \sin(n\theta)\sin(n\pi)) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\theta)}{n},\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sin(n(\theta+\pi))}{n} &= \sum_{n=1}^{\infty} \frac{1}{n} (\sin(n\theta)\cos(n\pi) + \sin(n\pi)\cos(n\theta)) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\theta)}{n}.\end{aligned}$$

□

Example 4.1.2 (P52)**.** Prove that the series

$$\sum \frac{1}{(m_1^2 + m_2^2 + \cdots + m_r^2)^\mu},$$

in which the summation extends over all positive and negative integral values and zero values of $m_1, m_2, \cdots, m_r$ except the set of simultaneous zero values, is absolutely convergent if $\mu > \frac{1}{2}r$.

Proof. First, notice that the convergence of a series over $\mathbb{Z}^r$ is the natural generalization of the convergence of double series on P27.

Let $\mu > \frac{1}{2}r$. Then we can rewrite the series as

$$\sum_{\mathbf{m}\in\mathbb{Z}^r\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{m}\|^{2\mu}} = \sum_{\mathbf{m}\in\mathbb{Z}^r\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{m}\|^{r+\varepsilon}}$$

for some $\varepsilon > 0$. Since $\max_{1\leq i\leq r} |m_i| \leq \|\mathbf{m}\|$, we have

$$0 \leq \sum_{\mathbf{m}\in\mathbb{Z}^r\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{m}\|^{r+\varepsilon}} \leq \sum_{\mathbf{m}\in\mathbb{Z}^r\setminus\{\mathbf{0}\}} \frac{1}{\max_{1\leq i\leq r} |\mathbf{m}_i|^{r+\varepsilon}}.$$

Since the set of all $\mathbf{m}$ with $k-1 \leq \max_{1\leq i\leq r} |\mathbf{m}_i| \leq k$ has size that is less than or equal to Ck^{r-1} for some constant C, we have

$$\sum_{\mathbf{m}\in\mathbb{Z}^r\setminus\{\mathbf{0}\}} \frac{1}{\max_{1\leq i\leq r} |\mathbf{m}_i|^{r+\varepsilon}} \leq \sum_{k=1}^{\infty} C\frac{k^{r-1}}{(k-1)^{r+\varepsilon}} < \infty.$$

By the comparison test, we can conclude that the series

$$\sum \frac{1}{\left(m_1^2 + m_2^2 + \cdots + m_r^2\right)^{\mu}}$$

converges absolutely. [1] □

Example 4.1.3 (P57)**.** If $f(x)$ is monotonic in the range $[a, b]$, its total fluctuation in the range is $|f(a) - f(b)|$. [2]

Proof. Let $f\colon [a, b] \to \mathbb{R}$ be a bounded increasing function. Then for all $u, v \in [a, b]$ such that $u < v$, we have $f(v) - f(u) \geq 0$. Hence, for any fluctuation

$$|f(a) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_n) - f(b)|,$$

where

$$a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b,$$

[1] I would like to thank Eric Naslund, a user of Mathematics Stackexchange, for providing the solution to this problem.

[2] The original problem assumes that $f(x)$ is monotonic in the range (a, b). But the statement is not true if we have removable discontinuities at the endpoints.

we have

$$|f(a) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_n) - f(b)| = |f(a) - f(b)|.$$

This proves that $|f(a) - f(b)|$ is the total fluctuation in the range $[a, b]$. The proof for decreasing functions is similar. □

Example 4.1.4 (P57)**.** A function with limited total fluctuation can be expressed as the difference of two positive increasing monotonic functions.

Proof. Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function with limited total fluctuation. For each $x \in [a, b]$, we have the total fluctuation F_a^x in the range $[a, x]$, defined as the least upper bound of the fluctuation, independent of n, for all choices of $a \le x_1 \le x_2 \le x_n \le x$.

Now, since f is a bounded function, $\sup_{t\in[a,b]} |f(t)|$ exists. Define

$$G_a^x = F_a^x + \sup_{t\in[a,b]} |f(t)| + 1,$$

then we have $f(x) = \frac{1}{2}\left(G_a^x + f(x)\right) - \frac{1}{2}\left(G_a^x - f(x)\right).$

Let $u, v \in [a, b]$ be such that $u < v$ and let $\varepsilon > 0$. Choose

$$a \le x_1 \le x_2 \le \cdots \le x_n \le u$$

such that

$$|f(a) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_n) - f(u)| > F_a^u - \varepsilon$$

by the definition of F_a^u. Then

$$|f(a) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_n) - f(u)| + |f(u) - f(v)| \le F_a^v,$$

and it follows that

$$|f(u) - f(v)| < F_a^v - F_a^u + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $|f(u) - f(v)| \leq F_a^v - F_a^u$. This implies that

$$\begin{aligned}(F_a^v + f(v)) - (F_a^u + f(u)) &= (F_a^v - F_a^u) - (f(u) - f(v)) \\ &\geq 0,\end{aligned}$$

and

$$\begin{aligned}(F_a^v - f(v)) - (F_a^u - f(u)) &= (F_a^v - F_a^u) - (f(v) - f(u)) \\ &\geq 0.\end{aligned}$$

This proves both functions $\frac{1}{2}(F_a^x + f(x))$ and $\frac{1}{2}(F_a^x - f(x))$ are increasing, so are $\frac{1}{2}(G_a^x + f(x))$ and $\frac{1}{2}(G_a^x - f(x))$.

Since we define $G_a^x > |f(x)|$ for all $x \in [a, b]$, both functions $\frac{1}{2}(G_a^x + f(x))$ and $\frac{1}{2}(G_a^x - f(x))$ are positive. □

Example 4.1.5 (P57)**.** If $f(x)$ have limited total fluctuation in the range (a, b), then the limit $f(x \pm 0)$ exist at all points in the interior of the range.

Proof. Since $f(x)$ has limited total fluctuation, $f(x) = f_1(x) - f_2(x)$ for some increasing functions f_1 and f_2 by Example 4.1.4. It follows that the limits $f_1(x \pm 0)$ and $f_2(x \pm 0)$ exist at all points in the interior of the range (a, b) by Example on P43. Thus, we have

$$f(x \pm 0) = f_1(x \pm 0) - f_2(x \pm 0)$$

for all $x \in (a, b)$. □

4.2 Solutions to End-of-Chapter Exercises

Example 4.2.1. Shew that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

is equal to $\frac{1}{(1-z)^2}$ when $|z| < 1$ and is equal to $\frac{1}{z(1-z)^2}$ when $|z| > 1$. Is this fact connected with the theory of uniform convergence?

Proof. We can rewrite the nth term of the series as

$$\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{(z-1)z(z^n-1)} - \frac{1}{(z-1)z(z^{n+1}-1)},$$

assuming $z \in \mathbb{Z}\backslash\{0\}$ satisfying z is not an nth root nor an $n+1$th root of unity. Computing the nth partial sum, we obtain

$$\begin{aligned} S_n(z) &= \frac{1}{z(z-1)^2} - \frac{1}{(z-1)z(z^2-1)} + \frac{1}{(z-1)z(z^2-1)} - \frac{1}{(z-1)z(z^3-1)} \\ &+ \cdots + \frac{1}{(z-1)z(z^n-1)} - \frac{1}{(z-1)z(z^{n+1}-1)} \\ &= \frac{1}{z(z-1)^2} - \frac{1}{(z-1)z(z^{n+1}-1)}. \end{aligned}$$

If $|z| > 1$, then $\lim_{n\to\infty} S_n(z) = \frac{1}{z(z-1)^2}$. On the other hand, if $0 < |z| < 1$, then

$$\begin{aligned} \lim_{n\to\infty} S_n(z) &= \frac{1}{z(z-1)^2} + \frac{1}{z(z-1)} \\ &= \frac{1}{(z-1)^2}. \end{aligned}$$

Thus, the series converges pointwisely to the function

$$S(z) = \begin{cases} \frac{1}{z(z-1)^2}, & |z| > 1, \\ \frac{1}{(z-1)^2}, & 0 < |z| < 1. \end{cases}$$

We will show that the convergence $S_n(z) \to S(z)$ is not uniform. Let $\varepsilon_0 = \frac{1}{e-1}$. For all $n \in \mathbb{N}$, choose $z_n = 1 + \frac{1}{n+1}$. Then

$$\begin{aligned}
|S_n(z_n) - S(z_n)| &= \left| \left(\frac{1}{z_n(z_n-1)^2} - \frac{1}{(z_n-1)z_n(z_n^{n+1}-1)} \right) - \frac{1}{z_n(z_n-1)^2} \right| \\
&= \frac{1}{\left(\frac{1}{n+1}\right)\left(1+\frac{1}{n+1}\right)\left(\left(1+\frac{1}{n+1}\right)^{n+1} - 1\right)} \\
&= \frac{(n+1)^2}{n+2} \cdot \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1} - 1} \\
&\geq \frac{1}{e-1} = \varepsilon_0
\end{aligned}$$

since $\left(1 + \frac{1}{n+1}\right)^{n+1}$ increases to e. This proves the convergence is not uniform. The limit of this series is not surprising since the the convergence is not uniform. □

Example 4.2.2. Shew that the series

$$2\sin\left(\frac{1}{3z}\right) + 4\sin\left(\frac{1}{9z}\right) + \cdots + 2^n \sin\left(\frac{1}{3^n z}\right) + \cdots$$

converges absolutely for all values of z ($z = 0$ excepted), but does not converge uniformly near $z = 0$.

Proof. We want to show that the series

$$\sum_{n=1}^{\infty} \left| 2^n \sin\left(\frac{1}{3^n z}\right) \right|$$

converges. Let $z = x + iy$. Then

$$\frac{1}{3^n z} = \frac{x - iy}{3^n x^2 + 3^n y^2}.$$

Applying facts such as $|\sin(u+iv)| = \sqrt{\sin^2(u)+\sinh^2(v)}$, $\sin(x) \approx x$ and $\sinh(x) \approx x$ as $x \to 0$, we obtain

$$\begin{aligned}
\left|2^n \sin\left(\frac{1}{3^n z}\right)\right| &= 2^n \left|\sin\left(\frac{x-iy}{3^n x^2 + 3^n y^2}\right)\right| \\
&= 2^n \sqrt{\sin^2\left(\frac{x}{3^n x^2 + 3^n y^2}\right) + \sinh^2\left(\frac{-y}{3^n x^2 + 3^n y^2}\right)} \\
&\approx 2^n \sqrt{\left(\frac{x}{3^n x^2 + 3^n y^2}\right)^2 + \left(\frac{-y}{3^n x^2 + 3^n y^2}\right)^2} \\
&= O\left(\left(\frac{2}{3}\right)^n\right)
\end{aligned}$$

for large n. Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we can conclude that $\sum_{n=1}^{\infty} \left|2^n \sin\left(\frac{1}{3^n z}\right)\right|$, as desired.

Now, we will show that the convergence is not uniform near 0. Let $\varepsilon = 1$. For all $N \in \mathbb{N}$, choose $z_0 = \frac{1}{3^{N+1}}$. Then we have

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} 2^n \sin\left(\frac{1}{3^n z_0}\right) - \sum_{n=1}^{N} 2^n \sin\left(\frac{1}{3^n z_0}\right)\right| &= \left|\sum_{n=N+1}^{\infty} 2^n \sin\left(\frac{1}{3^n z_0}\right)\right| \\
&= \left|\sum_{n=N+1}^{\infty} 2^n \sin\left(\frac{1}{3^n \frac{1}{3^{N+1}}}\right)\right| \\
&= \sum_{n=N+1}^{\infty} 2^n \underbrace{\sin\left(\frac{1}{3^{n-N-1}}\right)}_{\geq 0} \\
&\geq 1.
\end{aligned}$$

This proves that the convergence is not uniform near 0. □

Example 4.2.3. If

$$u_n(x) = -2(n-1)^2 x e^{-(n-1)^2 x^2} + 2n^2 x e^{-n^2 x^2},$$

shew that $\sum_{n=1}^{\infty} u_n(x)$ does not converge uniformly near $x = 0$.

Proof. The nth partial sum can be written as

$$\begin{aligned} S_n(x) &= \frac{2x}{e^{x^2}} + \frac{8x}{e^{4x^2}} - \frac{2x}{e^{x^2}} + \cdots + \frac{2n^2x}{e^{n^2x^2}} - \frac{2(n-1)^2x}{e^{(n-1)^2x^2}} \\ &= \frac{2n^2x}{e^{n^2x^2}}. \end{aligned}$$

For all $x \in \mathbb{R}$, we have $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = 0$. Thus, the series converges to the constant function $S(x) = 0$ pointwisely.

Now, we will show that the convergence is not uniform near 0. Let $\varepsilon = \frac{1}{e}$. For all $N \in \mathbb{N}$, choose $x_0 = \frac{1}{N}$. Then we have

$$\begin{aligned} |S_N(x_0) - S(x_0)| &= \left| \frac{2N^2x_0}{e^{N^2x_0^2}} \right| \\ &= \left| \frac{2N^2\frac{1}{N}}{e^{N^2\frac{1}{N^2}}} \right| \\ &= \frac{2N}{e} \\ &\geq \frac{1}{e}. \end{aligned}$$

This proves that the convergence is not uniform near $x = 0$. □

Example 4.2.4. Shew that the series $\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$ is convergent, but that its square (formed by Abel's rule)

$$\frac{1}{1} - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) + \cdots$$

is divergent.

Proof. Clearly, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges by the alternating series test. We will show the series

$$\frac{1}{1} - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) - \cdots$$

diverges. Let $a_i = \frac{1}{\sqrt{i}}$ for all $i \in \mathbb{N}$. Define the sequence

$$b_n = a_1 a_n + a_2 a_{n-1} + a_3 a_{n-2} + \cdots + a_n a_1.$$

Notice that

$$\begin{aligned}
b_1 &= 1 \cdot 1 = 1 \\
b_2 &= 1 \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot 1 = \frac{2}{\sqrt{2}} \\
b_3 &= 1 \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot 1 = \frac{2}{\sqrt{3}} + \frac{1}{2} \\
b_4 &= 1 \cdot \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} \cdot 1 = \frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}} \\
&\vdots
\end{aligned}$$

Hence, the series can be written as

$$\frac{1}{1} - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} b_n.$$

We claim that $b_n \geq 1$ for all $n \in \mathbb{N}$. Let $f\colon [0,k] \to \mathbb{R}$ be defined by $x(k-x)$. Then f reaches its maximum value of $\frac{k^2}{4}$ at $x = \frac{k}{2}$. It follows that for $1 \leq i \leq n$, we have

$$\begin{aligned}
& i(n+1-i) \leq \frac{(n+1)^2}{4} \\
& \Rightarrow \sqrt{i(n+1-i)} \leq \frac{n+1}{2} \\
& \Rightarrow a_i a_{n+1-i} = \frac{1}{\sqrt{i(n+1-i)}} \geq \frac{2}{n+1}.
\end{aligned}$$

This implies that $b_n \geq \frac{2n}{n+1} = \frac{n+1+n-1}{n+1} = 1 + \frac{n-1}{n} \geq 1$. Because b_n is bounded below by 1, we have $\lim_{n\to\infty} (-1)^{n+1} b_n \neq 0$. By the divergence test, we can conclude that the series

$$\frac{1}{1} - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) + \cdots$$

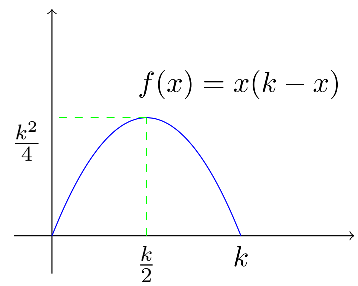

FIGURE 4.1. Maximum value of $x(k-x)$.

diverges. [3]

□

Example 4.2.5. If the convergent series $s = \frac{1}{1^r} - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \cdots \, (r > 0)$ be multiplied by itself the terms of the product being arranged as in Abel's result, shew that the resulting series diverges if $r \leq \frac{1}{2}$ but converges to the sum s^2 if $r > \frac{1}{2}$.

Proof. Proof can be found in section four from the article titled "On the Multiplication and Involution of Semi-Convergent Series" by Florian Cajori [3]. □

Example 4.2.6. If the two conditionally convergent series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^r} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},$$

where r and s lie between 0 and 1, be multiplied together, and the product arranged as in Abel's result, shew that the necessary and sufficient condition for the convergence of the resulting series is $r + s > 1$.

Proof. Proof can be found in section seven from the article titled "On the Multiplication and Involution of Semi-Convergent Series" by Florian Cajori [3]. □

[3] I would like to thank André Nicolas, a user of Mathematics Stackexchange, for providing a solution to this problem.

Example 4.2.7. Shew that if the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

be multiplied by itself any number of times, the terms of the product being arranged as in Abel's result, the resulting series converges.

Proof. Proof can be found in section six from the article titled "On the Multiplication and Involution of Semi-Convergent Series" by Florian Cajori [3]. □

Example 4.2.8. Shew that the qth power of the series

$$a_1 \sin(\theta) + a_2 \sin(2\theta) + \cdots + a_n \sin(n\theta) + \cdots$$

is convergent whenever $q(1-r) < 1$, r being the greatest number satisfying the relation

$$a_n \leq n^{-r}$$

for all values of n.

Proof. Proof can be found in section thirteen from the article titled "On the Multiplication and Involution of Semi-Convergent Series" by Florian Cajori [3]. □

Example 4.2.9. Shew that if θ is not equal to 0 or a multiple of 2π, and if $u_0, u_1, u_2, \cdots$ be a sequence such that $u_n \to 0$ steadily, then the series $\sum u_n \cos(n\theta + a)$ is convergent.

Shew also that, if the limit of u_n is not zero, but u_n is still monotonic, the sum of the series is oscillatory if $\frac{\theta}{\pi}$ is rational, but that, if $\frac{\theta}{\pi}$ is irrational, the sum may have any value between certain bounds whose difference is $a \csc\left(\frac{1}{2}\theta\right)$, where $a = \lim_{n\to\infty} u_n$.

Proof. Let the sequence $u_n \to 0$ steadily and assume that $\theta \neq 2\pi k$ for $k \in \mathbb{Z}$. Let $a \in \mathbb{R}$. We want to show that the sequence of partial sums $\left\{\sum_{n=1}^{N} \cos(n\theta + a)\right\}$ is bounded. Observe that

$$\begin{aligned}
\left|\sum_{n=1}^{N} \cos(n\theta + a)\right| &= \left|\operatorname{Re}\left(\sum_{n=1}^{N} e^{i(n\theta+a)}\right)\right| \\
&= \left|\operatorname{Re}\left(e^{ia}\sum_{n=1}^{N} e^{i(n\theta)}\right)\right| \\
&= \left|\operatorname{Re}\left(e^{ia}\frac{e^{i\theta}(1-e^{iN\theta})}{1-e^{i\theta}}\right)\right| \\
&\leq \left|e^{ia}\frac{e^{i\theta}(1-e^{iN\theta})}{1-e^{i\theta}}\right| \\
&\leq \frac{2}{|1-e^{i\theta}|}.
\end{aligned}$$

By the Dirichlet test, we can conclude that the series $\sum u_n \cos(n\theta + a)$ converges.

Now, assume that the sequence $\{u_n\}$ is monotone and $\lim_{n\to\infty} u_n = u \neq 0$. Then the sequence $\{u_n - u\}$ converges to 0 monotonically. Notice that we can rewrite the series as follows:

$$\sum_{n=1}^{\infty} u_n \cos(n\theta + a) = \sum_{n=1}^{\infty} (u_n - u)\cos(n\theta + a) + u\sum_{n=1}^{\infty} \cos(n\theta + a).$$

By the Dirichlet test, the series $\sum_{n=1}^{\infty}(u_n - u)\cos(n\theta + a)$ converges. Hence, we only need to consider the behavior of the series $\sum_{n=1}^{\infty} \cos(n\theta + a)$. We can rewrite the partial sums $\sum_{n=1}^{N} \cos(n\theta + a)$ as follows:

$$\begin{aligned}
S_N(\theta) = \sum_{n=1}^{N} \cos(n\theta + a) &= \sum_{n=1}^{N} \frac{\sin\left(n\theta + a + \frac{\theta}{2}\right) - \sin\left(n\theta + a - \frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\sin\left(N\theta + a + \frac{\theta}{2}\right) - \sin\left(\theta + a - \frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\sin\left((N + \frac{1}{2})\theta + a\right) - \sin\left(\frac{\theta}{2} + a\right)}{2\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\sin\left((N + \frac{1}{2})\theta + a\right) - \sin(\frac{\theta}{2})\cos(a) - \sin(a)\cos(\frac{\theta}{2})}{2\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{1}{2}\left(\csc\left(\frac{\theta}{2}\right)\sin\left(\left(N + \frac{1}{2}\right)\theta + a\right) - \cos(a) - \sin(a)\cot\left(\frac{\theta}{2}\right)\right).
\end{aligned}$$

Let θ be fixed. Then we only need to consider the term $\sin\left(\left(N+\frac{1}{2}\right)\theta+a\right)$ when we examine the partial sums $S_N(\theta)$.

Consider the case when $\frac{\theta}{\pi}\in\mathbb{Q}$. Then there exists $p,q\in\mathbb{Z}$, with $q\neq 0$ such that $\frac{\theta}{\pi}=\frac{p}{q}$. It follows that

$$\begin{aligned}\sin\left(\left(N+\frac{1}{2}\right)\theta+a\right)&=\sin\left(\left(N+\frac{1}{2}\right)\frac{p\pi}{q}+a\right)\\&=\sin\left(\frac{(2N+1)p\pi}{2q}+a\right),\end{aligned}$$

which is a periodic function of $N\in\mathbb{N}$. This proves that the series $\sum u_n\cos(n\theta+a)$ is oscillatory.

Finally, consider the case when $\frac{\theta}{\pi}$ is irrational. We claim that the sequence

$$\left\{\sin\left(\left(n+\frac{1}{2}\right)\theta+a\right)\right\}=\left\{\sin(n\theta+a')\right\},$$

where $a'=\frac{1}{2}\theta+a$, is dense in $[-1,1]$. It suffices to show every point $b\in[-1,1]$ is a limit point for the sequence $\{\sin(n\theta+a')\}$. Consider the case when $\theta>0$. The proof for the case $\theta<0$ is similar. Let $\varepsilon>0$. Choose $c\in[0,\frac{2\pi}{\theta}]$ such that $\sin(c\theta+a')=b$. Choose $\delta>0$ such that

$$|\sin(\theta x+a')-b|<\varepsilon\quad\text{ for }\ |x-c|<\delta.$$

Notice that if α is irrational and $\mu\in[0,1]$, then μ is a limit point for the sequence $\{n\alpha-[n\alpha]\}$, where $[n\alpha]$ is the greatest integer in $n\alpha$ [11]. Take $\alpha=\frac{\theta}{2\pi}$, which is irrational, and $\mu=\frac{c\theta}{2\pi}$, which is a point in $[0,1]$. Then there exists a natural number N such that

$$0<\frac{c\theta}{2\pi}-\left(\frac{N\theta}{2\pi}-\left[\frac{N\theta}{2\pi}\right]\right)<\frac{\delta\theta}{2\pi}.$$

Or, $0<c-\frac{2\pi}{\theta}\left(\frac{N\theta}{2\pi}-\left[\frac{N\theta}{2\pi}\right]\right)<\delta$. It follows that

$$\left|\sin\left(\theta\frac{2\pi}{\theta}\left(\frac{N\theta}{2\pi}-\left[\frac{N\theta}{2\pi}\right]\right)+a'\right)-b\right|<\varepsilon.$$

But

$$\frac{2\pi}{\theta}\left(\frac{N\theta}{2\pi}-\left[\frac{N\theta}{2\pi}\right]\right)=N-\frac{2\pi}{\theta}k,$$

for some integer k. Hence, we have $|\sin(\theta N+a')-b|<\varepsilon$. This proves that the sequence $\left\{\sin\left(\left(n+\frac{1}{2}\right)\theta+a\right)\right\}$ is dense in $[-1,1]$. Thus, the set of partial sums of the series $u\sum_{n=1}^{\infty}\cos(n\theta+a)$ is dense in the interval

$$\left[S_1-\frac{1}{2}u\csc\left(\frac{\theta}{2}\right), S_1+\frac{1}{2}u\csc\left(\frac{\theta}{2}\right)\right],$$

where $S_1=\frac{u}{2}\left(-\cos(a)-\sin(a)\cot\left(\frac{\theta}{2}\right)\right)$. It follows that the set of partial sums of the series $\sum_{n=1}^{\infty}u_n\cos(n\theta+a)$ is dense in the interval

$$\left[S-\frac{1}{2}u\csc\left(\frac{\theta}{2}\right), S+\frac{1}{2}u\csc\left(\frac{\theta}{2}\right)\right],$$

where $S=S_1+\sum_{n=1}^{\infty}(u_n-u)\cos(n\theta+a)$. [4]

□

[4] I would like to thank Julián Aguirre, a user of Mathematics Stackexchange, for providing a solution to this problem. Notice that for the case when $\frac{\theta}{\pi}$ is irrational, we are actually proving the set of partial sums of the series $\sum_{n=1}^{\infty}u_n\cos(n\theta+a)$ is "dense" in the interval $\left[S-\frac{1}{2}u\csc\left(\frac{\theta}{2}\right), S+\frac{1}{2}u\csc\left(\frac{\theta}{2}\right)\right]$. It seems to be an error in this exercise. The set of partial sums is countable, it can only be dense in an interval, but not equal to the interval.

CHAPTER 5

THE THEORY OF RIEMANN INTEGRATION

5.1 Solutions to Exercises in the Chapter

Example 5.1.1 (P62)**.** $\int_a^b \{f(x) + \phi(x)\}\, dx = \int_a^b f(x)dx + \int_a^b \phi(x)dx$.

Proof. Let f and ϕ be integrable on $[a, b]$. Divide the interval $[a, b]$ at the points $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$. Let

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}), & s_n &= \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) \\ T_n &= \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} \phi(x)(x_i - x_{i-1}), & t_n &= \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} \phi(x)(x_i - x_{i-1}) \\ M_n &= \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} (f(x) + \phi(x))(x_i - x_{i-1}), & m_n &= \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} (f(x) + \phi(x))(x_i - x_{i-1}). \end{aligned}$$

Since f and ϕ are integrable, $\lim_{n\to\infty} S_n - s_n = \lim_{n\to\infty} T_n - t_n = 0$. But then

$$s_n + t_n \leq m_n \leq M_n \leq S_n + T_n. \tag{5.1}$$

It follows that

$$M_n - m_n \leq (S_n - s_n) + (T_n - t_n) \to 0.$$

This proves that $f + \phi$ is integrable. Since

$$\begin{aligned} \lim_{n\to\infty} s_n + t_n &= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n \\ &= \int_a^b f(x)dx + \int_a^b \phi(x)dx \\ &= \lim_{n\to\infty} S_n + \lim_{n\to\infty} T_n \\ &= \lim_{n\to\infty} S_n + T_n. \end{aligned}$$

By (5.1) and the squeeze theorem, we have

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} M_n = \int_a^b \{f(x) + \phi(x)\}dx = \int_a^b f(x)dx + \int_a^b \phi(x)dx.$$

□

Example 5.1.2. By means of Example 5.1.1, define the integral of a continuous complex function of a real variable.

Proof. Let $h\colon [a,b] \to \mathbb{C}$ be a continuous complex function of a real variable defined by $h(x) = f(x) + i\phi(x)$. Then the integral of h on $[a,b]$ is defined by

$$\int_a^b h(x)dx = \int_a^b \{f(x) + i\phi(x)\}dx = \int_a^b f(x)dx + i\int_a^b \phi(x)dx.$$

□

Example 5.1.3 (P63)**.** The product of two integrable functions is an integrable function.

Proof. Let f and ϕ be integrable on $[a,b]$. We claim that f^2 is also integrable on $[a,b]$.

Since f is integrable, f is bounded. There exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Observe that

$$\begin{aligned} |(f(x))^2 - (f(y))^2| &= |f(x) + f(y)||f(x) - f(y)| \\ &\leq 2M|f(x) - f(y)| \quad \text{for all } \ x, y \in [a,b]. \end{aligned}$$

Let

$$\begin{aligned} S_n &= \sum_{i=1}^n \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}), & s_n &= \sum_{i=1}^n \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}), \\ T_n &= \sum_{i=1}^n \sup_{x\in[x_{i-1},x_i]} (f(x))^2(x_i - x_{i-1}), & t_n &= \sum_{i=1}^n \inf_{x\in[x_{i-1}x_i]} (f(x))^2(x_i - x_{i-1}). \end{aligned}$$

Then

$$\begin{aligned} T_n - t_n &= \sum_{i=1}^{n} \left(\sup_{x\in[x_{i-1},x_i]} (f(x))^2 - \inf_{x\in[x_{i-1},x_i]} (f(x))^2 \right) (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \sup_{x,y\in[x_{i-1},x_i]} |(f(x))^2 - (f(y))^2| (x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} \sup_{x,y\in[x_{i-1},x_i]} 2M|f(x) - f(y)|(x_i - x_{i-1}) \\ &= 2M(S_n - s_n) \to 0, \end{aligned}$$

as $n \to \infty$. This proves f^2 is integrable on $[a, b]$.

Since the linear combination of two integrable functions is integrable and the square of an integrable function is still integrable, we can conclude that

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}$$

is integrable.

□

Example 5.1.4 (P64)**.** Shew that

$$\lim_{n\to\infty} \frac{1 + \cos\left(\frac{x}{n}\right) + \cos\left(\frac{2x}{n}\right) + \cdots + \cos\left(\frac{(n-1)x}{n}\right)}{n} = \frac{\sin(x)}{x}.$$

Proof. Let $f(t) = \cos(t)$. Then $f(t)$ is integrable and

$$\begin{aligned} \frac{1}{x}\int_0^x f(t)dt &= \frac{1}{x}\sin(t)\Big|_0^x \\ &= \frac{\sin(x)}{x}. \end{aligned}$$

Recall that a Riemann sum is given by

$$\sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}), \quad t_{i-1} \leq t_i^* \leq t_i, \quad t_i = \frac{ix}{n}.$$

Choose $t_i^* = \frac{(i-1)x}{n}$ for $i = 1, 2, \cdots, n$. It follows that

$$\begin{aligned}\frac{\sin(x)}{x} &= \frac{1}{x}\int_0^x f(t)dt \\ &= \frac{1}{x}\lim_{n=\infty}\sum_{i=1}^{n}\cos\left(\frac{(i-1)x}{n}\right)\frac{x}{n} \\ &= \lim_{n\to\infty}\frac{1+\cos(\frac{x}{n})+\cos(\frac{2x}{n})+\cdots+\cos(\frac{(n-1)x}{n})}{n}.\end{aligned}$$

□

Example 5.1.5 (P64)**.** If $f(x)$ has ordinary discontinuities at the points $a_1, a_2, \cdots, a_k$, then

$$\int_a^b f(x)dx = \lim\left\{\int_a^{a_1-\delta_1} + \int_{a_1+\varepsilon_1}^{a_2-\delta_2} + \cdots + \int_{a_k+\varepsilon_k}^{b} f(x)dx\right\},$$

where the limit is taken by making $\delta_1, \delta_2, \cdots \delta_k, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ tend to $+0$ independently.

Proof. By Example 2 on P63, we know that a function that is continuous except at a finite number of ordinary discontinuities is integrable. The idea here is that the proper integral $\int_a^b f(x)dx$ can be treated as an improper integral and this can be explained by Example 5.1.9. □

Example 5.1.6 (P65)**.** If $f(x)$ is integrable when $a_1 \le x \le b_1$ and if, when $a_1 \le a < b < b_1$, we write

$$\int_a^b f(x)dx = \phi(a, b),$$

and if $f(b+0)$ exists, then

$$\lim_{\delta\to+0}\frac{\phi(a, b+\delta)-\phi(a, b)}{\delta} = f(b+0).$$

Deduce that, if $f(x)$ is continuous at a and b,

$$\frac{d}{da}\int_a^b f(x)dx = -f(a), \quad \frac{d}{db}\int_a^b f(x)dx = f(b).$$

Proof. Fix $a \in [a_1, b_1]$ and assume that $f(b+0) = L$ exists. Then

$$\lim_{\delta \to +0} \frac{\phi(a, b+\delta) - \phi(a, b)}{\delta} = \lim_{\delta \to 0+} \frac{\int_a^{b+\delta} f(x)dx - \int_a^b f(x)dx}{\delta} = \lim_{\delta \to +0} \frac{\int_b^{b+\delta} f(x)dx}{\delta}.$$

Let $\varepsilon > 0$. Since $f(b+0) = L$, we can find a positive number η such that if $x \in (b, b+\eta)$, we have

$$L - \varepsilon < f(x) < L + \varepsilon.$$

If $0 < \delta < \eta$, we have

$$\underbrace{\delta(L-\varepsilon)}_{\int_b^{b+\delta}(L-\varepsilon)dx} < \int_b^{b+\delta} f(x)dx < \underbrace{\delta(L+\varepsilon)}_{\int_b^{b+\delta}(L+\varepsilon)dx}$$
$$\Rightarrow L - \varepsilon < \frac{\int_b^{b+\delta} f(x)dx}{\delta} < L + \varepsilon.$$

This proves that

$$\lim_{\delta \to +0} \frac{\int_b^{b+\delta} f(x)dx}{\delta} = L = f(b+0).$$

Now, assume that $f(x)$ is continuous at a and b. Notice that

$$\frac{d}{db} \int_a^b f(x)dx = \frac{d}{db}\phi(a, b) = \lim_{\delta \to 0} \frac{\phi(a, b+\delta) - \phi(a, b)}{\delta} = \lim_{\delta \to 0} \frac{\int_b^{b+\delta} f(x)dx}{\delta}.$$

Since f is continuous at b, there exists $h > 0$ such that if $x \in (b-h, b+h)$, then

$$f(b) - \varepsilon < f(x) < f(b) + \varepsilon.$$

If $0 < \delta < h$, we have

$$\begin{aligned}\delta(f(b)-\varepsilon) &< \int_b^{b+\delta} f(x)dx < \delta(f(b)+\varepsilon)\\ \Rightarrow f(b)-\varepsilon &< \frac{\int_b^{b+\delta} f(x)dx}{\delta} < f(b)+\varepsilon.\end{aligned}$$

If $-h < \delta < 0$, we have

$$\begin{aligned}\delta(f(b)+\varepsilon) &< \int_b^{b+\delta} f(x)dx = -\int_{b+\delta}^{b} f(x)dx < \delta(f(b)-\varepsilon)\\ \Rightarrow f(b)-\varepsilon &< \frac{\int_b^{b+\delta} f(x)dx}{\delta} < f(b)+\varepsilon.\end{aligned}$$

This proves that

$$\begin{aligned}\frac{d}{db}\int_a^b f(x)dx &= \lim_{\delta\to 0}\frac{\int_b^{b+\delta} f(x)dx}{\delta}\\ &= f(b).\end{aligned}$$

Finally,

$$\begin{aligned}\frac{d}{da}\int_a^b f(x)dx &= -\frac{d}{da}\int_b^a f(x)dx\\ &= -f(a).\end{aligned}$$

□

Example 5.1.7 (P65)**.** Prove by differentiation that, if $\phi(x)$ is continuous function of x and $\frac{dx}{dt}$ a continuous function of t, then

$$\int_{x_0}^{x_1} \phi(x)dx = \int_{t_0}^{t_1} \phi(x)\frac{dx}{dt}dt.$$

Proof. Let $\phi(x)$ be a continuous function of x and $x(t)$ be continuously differentiable. By the fundamental theorem of calculus part 1, the function $F(x) := \int_{x_0}^{x} \phi(t)dt$ satisfies $F'(x) =$

$\phi(x)$. Then

$$\frac{d}{dt}F(x(t)) = F'(x(t))\frac{dx}{dt} = \phi(x(t))\frac{dx}{dt}.$$

By the fundamental theorem of calculus part 1 and 2, we have

$$\begin{aligned}\int_{t_0}^{t_1} \phi(x(t))\frac{dx}{dt}dt &= \int_{t_0}^{t_1} F'(x(t))\frac{dx}{dt}dt \\ &= \int_{t_0}^{t_1} \frac{d}{dt}F(x(t))dt \\ &= F(x(t_1)) - F(x(t_0)) \\ &= F(x_1) - F(x_0) \\ &= \int_{x_0}^{x_1} \phi(x)dx.\end{aligned}$$

□

Example 5.1.8 (P65)**.** If $f'(x)$ and $\phi'(x)$ are continuous when $a \le x \le b$, shew from Example 5.1.6 that

$$\int_a^b f'(x)\phi(x)dx + \int_a^b \phi'(x)f(x)dx = f(b)\phi(b) - f(a)\phi(a).$$

Proof. The ideal for this problem is to apply the fundamental theorem of calculus part 2, which can be proved using Example 5.1.6. Let $h(x) = f(x)\phi(x)$, where f and g are continuous differentiable on $[a, b]$. Then $h'(x) = f'(x)\phi(x) + \phi'(x)f(x)$. That is, h is an antiderivative of the continuous functions $f'(x)\phi(x) + \phi'(x)f(x)$. By fundamental theorem of calculus part 2, we have

$$\begin{aligned}h(b) - h(a) &= f(b)\phi(b) - f(a)\phi(a) \\ &= \int_a^b \left(f'(x)\phi(x) + \phi'(x)f(x)\right)dx \\ &= \int_a^b f'(x)\phi(x)dx + \int_a^b \phi'(x)f(x)dx.\end{aligned}$$

□

Example 5.1.9 (P65)**.** If $f(x)$ is integrable in the range (a, c) and $a \leq b \leq c$, shew that $\int_a^b f(x)dx$ is a continuous function of b.

Proof. Let f be integrable on (a, c) and $b \in [a, c]$. Fix $b_0 \in [a, c]$ and let $\varepsilon > 0$. Since f is integrable, f is bounded. There exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, c]$. Choose $\delta = \frac{\varepsilon}{M}$. Then if $|b - b_0| < \delta$, we have

$$\begin{aligned}\left|\int_a^b f(x)dx - \int_a^{b_0} f(x)dx\right| &= \left|\int_{b_0}^b f(x)dx\right| \\ &\leq \left|\int_{b_0}^b |f(x)|dx\right| \\ &\leq \left|\int_{b_0}^b M dx\right| \\ &= M|b - b_0| \\ &< M\delta \\ &< M\frac{\varepsilon}{M} \\ &= \varepsilon.\end{aligned}$$

This proves that $\int_a^b f(x)dx$ is a continuous function of b. □

Example 5.1.10 (P65)**.** If $f(x)$ is continuous and $\phi(x) \geq 0$, shew that ξ can be found such that

$$\int_a^b f(x)\phi(x)dx = f(\xi)\int_a^b \phi(x)dx.$$

Proof. Since f is continuous on $[a, b]$, there exists $x_m, x_M \in [a, b]$ such that

$$f(x_m) = m = \inf_{x\in[a,b]} f(x) \quad \text{and} \quad f(x_M) = M = \sup_{x\in[a,b]} f(x)$$

by the extreme value theorem. Since $\phi(x) \geq 0$, we have

$$m\phi(x) \leq f(x)\phi(x) \leq M\phi(x) \quad \text{for } x \in [a, b].$$

It follows that

$$m\int_a^b \phi(x)dx \le \int_a^b f(x)\phi(x)dx \le M\int_a^b \phi(x)dx.$$

Since $\phi(x) \ge 0$ for all $x \in [a,b]$, $\int_a^b \phi(x)dx \ge 0$. If $\int_a^b \phi(x)dx = 0$, then any ξ will work. Assume $\int_a^b \phi(x)dx > 0$. Then

$$m \le \frac{\int_a^b f(x)\phi(x)dx}{\int_a^b \phi(x)dx} \le M.$$

By the intermediate value theorem, there exists ξ between x_m and x_M such that

$$f(\xi) = \frac{\int_a^b f(x)\phi(x)dx}{\int_a^b \phi(x)dx}.$$

□

Example 5.1.11 (P66)**.** By writing $|\phi(x) - \phi(b)|$ in place of $\phi(x)$ in Bonnet's form of the mean value theorem, shew that if $\phi(x)$ is a monotonic function, then a number ξ exists such that $a \le \xi \le b$ and

$$\int_a^b f(x)\phi(x)dx = \phi(a)\int_a^\xi f(x)dx + \phi(b)\int_\xi^b f(x)dx.$$

Proof. Assume that ϕ is monotone. Take $h(x) = |\phi(x) - \phi(b)|$. Then $h(x) \ge 0$ and h is decreasing. Then by the Bonnet's form of the second mean value theorem for integrals (P66), there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)|\phi(x) - \phi(b)|dx = |\phi(a) - \phi(b)|\int_a^\xi f(x)dx.$$

Assume ϕ is increasing. The proof for ϕ is decreasing is similar. Then $|\phi(x)-\phi(b)| = \phi(b)-\phi(x)$. It follows that

$$\begin{aligned}\int_a^b f(x)\phi(x)d &= -\int_a^b f(x)(\phi(b)-\phi(x))dx + \phi(b)\int_a^b f(x)dx \\ &= -(\phi(b)-\phi(a))\int_a^\xi f(x)dx + \phi(b)\int_a^b f(x)dx \\ &= \phi(a)\int_a^\xi f(x)dx - \phi(b)\int_a^\xi f(x)dx + \phi(b)\int_a^b f(x)dx \\ &= \phi(a)\int_a^\xi f(x)dx + \phi(b)\int_\xi^b f(x)dx.\end{aligned}$$

□

Example 5.1.12 (P67)**.** Assume that both $f(x,\alpha)$ and f_α are continuous functions of both variables x and α. If a, b be not constants but functions of α with continuous differential coefficients, shew that

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f}{\partial\alpha}dx.$$ [1]

Proof. Since f_α is a continuous function of both variables x and α, if a and b were constants, we have

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = \int_a^b \frac{\partial f}{\partial\alpha}dx.$$

Since $f(x,\alpha)$ is a continuous function of both variables x and α, $f(x,\alpha)$ is continuous at x. Hence, we have

$$\frac{d}{da}\int_a^b f(x,\alpha)dx = -f(a,\alpha) \quad \text{and} \quad \frac{d}{db}\int_a^b f(x,\alpha)dx = f(b,\alpha).$$

By the chain rule, we obtain

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f}{\partial\alpha}dx.$$

[1] This is also known as Leibniz integral rule.

□

Example 5.1.13 (P67). If $f(x,\alpha)$ is a continuous function of both variables, $\int_a^b f(x,\alpha)dx$ is a continuous function of α.

Proof. Assume that $f(x,\alpha)$ is a continuous function of both variables x and α. Let $\varepsilon > 0$ and fix α_0. Since f is continuous in α, for any sequence $\{\alpha_n\}$ in the domain of α converging to α_0, we have $f(x,\alpha_n) \to f(x,\alpha_0)$. We may restrict the domain f to $[a,b] \times (\{\alpha_n\} \cup \{\alpha_0\})$, which is compact. Hence, f is uniformly continuous on this restricted domain. By the uniform continuity, there exists $\delta > 0$ (independent of x) such that if $|\alpha - \alpha_0| < \delta$, we have

$$|f(x,\alpha) - f(x-\alpha_0)| < \frac{\varepsilon}{b-a}.$$

If $|\alpha - \alpha_0| < \delta$, we have

$$\begin{aligned}\left|\int_a^b f(x,\alpha)dx - \int_a^b f(x,\alpha_0)dx)\right| &\le \int_a^b |f(x,\alpha) - f(x,\alpha_0)|dx \\ &\le \int_a^b \frac{\varepsilon}{b-a}dx \\ &= \varepsilon.\end{aligned}$$

This proves that $\int_a^b f(x,\alpha)dx$ is a continuous function of α. □

Example 5.1.14 (P69). By integrating by parts, shew that $\int_0^\infty t^n e^{-t}dt = n!$.

Proof.

$$\begin{aligned}\int_0^\infty t^n e^{-t}dt &= \lim_{b\to\infty}\left(-e^{-t}\left(t^n + nt^{n-1} + n(n-1)t^{n-2} + \cdots + n!\right)\Big|_0^b\right) \\ &= \lim_{b\to\infty}\left(n! - \frac{b^n + nb^{n-1} + n(n-1)b^{n-2} + \cdots + n!}{e^b}\right) \\ &= n!.\end{aligned}$$

□

Example 5.1.15 (P71)**.** If $|f(x)| \leq g(x)$ and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges absolutely.

Proof. Let $\varepsilon > 0$. Since $\int_a^\infty g(x)dx$ converges, there exists $X > 0$ such that if $x'' \geq x' \geq X$, we have

$$\left| \int_{x'}^{x''} g(x)dx \right| = \int_{x'}^{x''} g(x)dx < \varepsilon.$$

If $x'' \geq x' \geq X$, we have

$$\left| \int_{x'}^{x''} |f(x)|dx \right| = \int_{x'}^{x''} |f(x)|dx \leq \int_{x'}^{x''} g(x)dx < \varepsilon.$$

This proves $\int_a^\infty |f(x)|dx$ converges by the Cauchy criterion. Hence, $\int_a^\infty f(x)dx$ converges absolutely. □

Example 5.1.16 (P72)**.** $\int_0^\infty \frac{\sin(x)}{x}dx$ converges.

Proof. Since the function $\frac{\sin(x)}{x}$ is integrable on $(0, 1)$, it suffices to show that $\int_1^\infty \frac{\sin(x)}{x}dx$ converges. Notice that $\frac{1}{x}$ converges to 0 steadily as $x \to \infty$ and $\left|\int_1^b \sin(x)dx\right| \leq 2$ for all $b \geq 1$. By Chartier's test for integrals involving periodic functions (P72), the integral $\int_1^\infty \frac{\sin(x)}{x}dx$ converges. □

Example 5.1.17 (P72)**.** $\int_0^\infty x^{-1}\sin(x^3 - \alpha x)dx$ converges.

Proof. Since the function $x^{-1}\sin(x^3 - \alpha x)$ is integrable on $(0, b)$ for any $b > 0$, it suffices to show that the integral $\int_b^\infty x^{-1}\sin(x^3 - \alpha x)dx$ converges.

Consider the case when $\alpha \geq 0$ and the integral

$$\int_{\sqrt{\frac{\alpha}{3}}+1}^\infty x^{-1}\sin(x^3 - \alpha x)dx.$$

For all $x > \sqrt{\frac{\alpha}{3}}$, we have $g(x) = x^3 - \alpha x$ is strictly increasing whose inverse function $h(x)$ behaves like $x^{\frac{1}{3}}$ for a large value of x. Hence,

$$h'(x) = \frac{1}{g'(h(x))} = \frac{1}{3(h(x))^2 - \alpha}.$$

It follows that for all $L \geq \sqrt{\frac{\alpha}{3}} + 1$

$$\int_{\sqrt{\frac{\alpha}{3}}+1}^{L} \sin(x^3 - \alpha x)dx = \int_{g(\sqrt{\frac{\alpha}{3}}+1)}^{g(L)} \frac{\sin(t)}{3(h(t))^2 - \alpha}dt.$$

Since the integral $\int_{g(\sqrt{\frac{\alpha}{3}}+1)}^{g(L)} \sin(t)dt$ is bounded and $\frac{1}{3(h(t))^2-\alpha}$ converges to 0 steadily, the integral

$$\int_{\sqrt{\frac{\alpha}{3}}+1}^{\infty} \sin(x^3 - \alpha x)dx = \int_{g(\sqrt{\frac{\alpha}{3}}+1)}^{\infty} \frac{\sin(t)}{3(h(t))^2 - \alpha}dt$$

converges by Chartier's test for integral involving periodic function (P72). Applying Chartier's test for integral involving periodic function again, we can conclude that the integral

$$\int_{\sqrt{\frac{\alpha}{3}}+1}^{\infty} x^{-1} \sin(x^3 - \alpha x)dx$$

converges. The proof for the case $\alpha < 0$ is similar. [2] □

Example 5.1.18 (P72)**.** $\int_0^\infty x^{\alpha-1}e^{-x}dx$ converges uniformly in any interval (A, B) such that $1 \leq A \leq B$.

Proof. The integral is proper at $x = 0$ in this case, so it suffices to show $\int_1^\infty x^{\alpha-1}e^{-x}dx$ converges uniformly on (A, B). Notice that for all $\alpha \in (A, B)$, we have

$$\left|x^{\alpha-1}e^{-x}\right| \leq x^{B-1}e^{-x}, \quad \forall x \geq 1$$

and the integral

$$\int_1^\infty x^{B-1}e^{-x}dx$$

[2] I would like to thank Jack D'Aurizio, a user of Mathematics Stackexchange, for providing a hint for this problem.

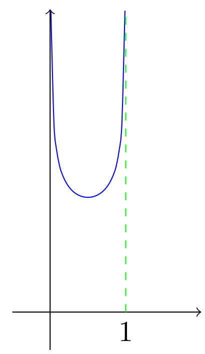

FIGURE 5.1. The function $x^{\lambda}(1-x)^{\mu-1}$.

converges. Thus, we can conclude that $\int_1^{\infty} x^{\alpha-1}e^{-x}dx$ converges uniformly on (A, B) by De la Vallée Poussin's test (P72). □

Example 5.1.19 (P75)**.** $\int_0^{\pi} x^{-\frac{1}{2}}\cos(x)dx$ is an improper integral.

Proof. Notice that for $x \in (0, \pi)$, we have

$$|x^{-\frac{1}{2}}\cos(x)| \leq x^{-\frac{1}{2}}$$

and the integral

$$\int_0^{\pi} x^{-\frac{1}{2}}dx$$

converges. Hence, $\int_0^{\pi} x^{-\frac{1}{2}}\cos(x)dx$ converges absolutely. □

Example 5.1.20 (P75)**.** $\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}dx$ [3] is an improper integral if $0 < \lambda < 1$, $0 < \mu < 1$. It does not converge for negative values of λ and μ.

Proof. Assume $0 < \lambda < 1$, $0 < \mu < 1$. Notice that the function $x^{\lambda}(1-x)^{\mu-1}$ goes to infinity as $x \to 0+0$ and $x \to 1-0$ (See Figure 5.1). We can write

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}dx = \int_0^{\frac{1}{2}} x^{\lambda-1}(1-x)^{\mu-1}dx + \int_{\frac{1}{2}}^1 x^{\lambda-1}(1-x)^{\mu-1}dx.$$

[3] This integral is called the beta function.

If $0 < x \leq \frac{1}{2}$, then $(1-x)^{\mu-1} \leq 2^{1-\mu}$. It follows that

$$\begin{aligned}
0 &\leq \int_0^{\frac{1}{2}} x^{\lambda-1}(1-x)^{\mu-1}dx \\
&\leq \int_0^{\frac{1}{2}} x^{\lambda-1}2^{1-\mu}dx \\
&= \lim_{\delta \to +0} 2^{1-\mu} \int_{0+\delta}^{\frac{1}{2}} x^{\lambda-1}dx \\
&= \lim_{\delta \to +0} \frac{2^{1-\mu}}{\lambda} x^{\lambda}\Big|_{0+\delta}^{\frac{1}{2}} \\
&= \frac{2^{1-\mu}}{\lambda}\left(\frac{1}{2^{\lambda}} - 0\right) < \infty.
\end{aligned}$$

Consider the integral

$$\int_{\frac{1}{2}}^{1} x^{\lambda-1}(1-x)^{\mu-1}dx.$$

Let $t = 1-x$. Then

$$\begin{aligned}
\int_{\frac{1}{2}}^{1} x^{\lambda-1}(1-x)^{\mu-1}dx &= -\int_{\frac{1}{2}}^{0} t^{\mu-1}(1-t)^{\lambda-1}dt \\
&= \int_0^{\frac{1}{2}} t^{\mu-1}(1-t)^{\lambda-1}dt \\
&< \infty.
\end{aligned}$$

Thus, we can conclude that

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}dx < \infty.$$

Assume that $\lambda, \mu < 0$. Then

$$\begin{aligned}
\int_0^1 \frac{1}{x^{1-\lambda}} \frac{1}{(1-x)^{1-\mu}} dx \geq \int_0^1 \frac{1}{x^{1-\lambda}} dx & \qquad \left(\frac{1}{(1-x)^{1-\mu}} \geq 1\right) \\
= \infty & \qquad (1-\lambda > 1).
\end{aligned}$$

□

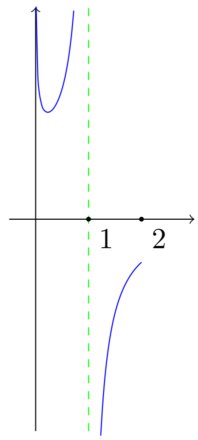

FIGURE 5.2. The function $\frac{x^{\alpha-1}}{1-x}$.

Example 5.1.21 (P75). $P\int_0^2 \frac{x^{\alpha-1}}{1-x}dx$ is the principal value of an improper integral when $0<\alpha<1$.

Proof. Assume $0<\alpha<1$. Let $t=2-x$. Then

$$\begin{aligned} P\int_0^2 \frac{x^{\alpha-1}}{1-x}dx &= \lim_{\delta\to+0}\left\{\int_0^{1-\delta}\frac{x^{\alpha-1}}{1-x}dx + \int_{1+\delta}^2 \frac{x^{\alpha-1}}{1-x}dx\right\} \\ &= \lim_{\delta\to+0}\left\{\int_{1+\delta}^2 \frac{(2-t)^{\alpha-1}}{t-1}dt + \int_{1+\delta}^2 \frac{x^{\alpha-1}}{1-x}dx\right\} \\ &= \lim_{\delta\to+0}\int_{1+\delta}^2 \frac{x^{\alpha-1}-(2-x)^{\alpha-1}}{1-x}dx \\ &< \infty, \end{aligned}$$

because

$$\lim_{x\to 1}\frac{x^{\alpha-1}-(2-x)^{\alpha-1}}{1-x} = 2(1-\alpha)$$

by L'Hôspital's rule.

□

Example 5.1.22 (P80). Discuss, in a similar manner, the series

$$\sum_{n=1}^{\infty}\frac{2e^n x\left\{1-n(e-1)+e^{n+1}x^2\right\}}{n(n+1)(1+e^n x^2)(1+e^{n+1}x^2)}$$

for real values of x.

Proof. It is easy to verify

$$\frac{2e^n x\left\{1-n(e-1)+e^{n+1}x^2\right\}}{n(n+1)(1+e^n x^2)(1+e^{n+1}x^2)} = \frac{2xe^n}{n(1+e^n x^2)} - \frac{2xe^{n+1}}{(n+1)(1+e^{n+1}x^2)}.$$

Then the nth partial sum of the series is

$$\frac{2xe}{1+ex^2} - \frac{2xe^{n+1}}{(n+1)(1+e^{n+1}x^2)}.$$

Hence, for all $x \in \mathbb{R}$, we have

$$\sum_{n=1}^{\infty} \frac{2e^n x\left\{1-n(e-1)+e^{n+1}x^2\right\}}{n(n+1)(1+e^n x^2)(1+e^{n+1}x^2)} = \frac{2xe}{1+ex^2}.$$

Notice that

$$\int_0^x \frac{2te}{1+et^2}dt = \log(1+ex^2).$$

On the other hand,

$$\begin{aligned}\int_0^x \left(\frac{2te}{1+et^2} - \frac{2te^{n+1}}{(n+1)(1+e^{n+1}t^2)}\right)dt &= \log(1+ex^2) - \frac{1}{n+1}\log(1+e^{n+1}x^2)\\ &\to \begin{cases}\log(1+ex^2), & x=0,\\ \log(1+ex^2)-1, & x\neq 0.\end{cases}\end{aligned}$$

Therefore, the integral of the sum of the series differs from the sum of the integrals of the terms by 1 if $x \neq 0$ and those two values are equal if $x = 0$. □

Example 5.1.23 (P80)**.** Compare the values of

$$\int_0^z \left\{\sum_{n=1}^{\infty} u_n\right\} dz \quad \text{and} \quad \sum_{n=1}^{\infty}\int_0^z u_n dz,$$

where

$$u_n = \frac{2n^2 z}{(1+n^2z^2)\log(n+1)} - \frac{2(n+1)^2 z}{\{1+(n+1)^2z^2\}\log(n+2)}.$$

Proof. Notice that the nth partial sum of the series is

$$\frac{2z}{(1+z^2)\log(2)} - \frac{2(n+1)^2 z}{\{1+(n+1)^2z^2\}\log(n+2)}.$$

Hence, for all $z \in \mathbb{R}$, we have

$$\sum_{n=1}^{\infty} u_n = \frac{2z}{(1+z^2)\log(2)}.$$

Notice that

$$\begin{aligned}\int_0^z \sum_{n=1}^{\infty} u_n dt &= \int_0^z \frac{2t}{(1+t^2)\log(2)} dt \\ &= \frac{\log(1+z^2)}{\log(2)},\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n} \int_0^z u_k dt &= \int_0^z \frac{2t}{(1+t^2)\log(2)} - \frac{2(n+1)^2 t}{\{1+(n+1)^2t^2\}\log(n+2)} dt \\ &= \frac{\log(1+z^2)}{\log(2)} - \frac{\log(1+(n+1)^2z^2)}{\log(n+2)} \\ &\to \begin{cases} \frac{\log(1+z^2)}{\log(2)}, & z = 0, \\ \frac{\log(1+z^2)}{\log(2)} - 2, & z \neq 0. \end{cases}\end{aligned}$$

□

5.2 Solutions to End-of-Chapter Exercises

Example 5.2.1. Shew that the integrals

$$\int_0^{\infty} \sin(x^2)dx, \quad \int_0^{\infty} \cos(x^2)dx, \quad \int_0^{\infty} x\exp(-x^6\sin^2(x))dx$$

converge.

Proof. Consider the integral $\int_0^\infty \sin(x^2)dx$. Since $\sin(x^2)$ is integrable on $[0,1]$, it suffices to show that the integral $\int_1^\infty \sin(x^2)dx$ converges. Applying integration by parts, we obtain

$$\begin{aligned}\int_1^\infty \sin(x^2)dx &= \int_1^\infty \frac{-1}{2x}\left(-2x\sin(x^2)\right)dx\\ &= \frac{-\cos(x^2)}{2x}\Big|_1^\infty - \int_1^\infty \frac{\cos(x^2)}{2x^2}dx\\ &< \infty,\end{aligned}$$

where $\int_1^\infty \frac{\cos(x^2)}{2x^2}dx$ converges absolutely.

To show $\int_0^\infty \cos(x^2)dx$ converges, it suffices to show $\int_1^\infty \cos(x^2)dx$ converges. Applying integration by parts, we obtain

$$\begin{aligned}\int_1^\infty \cos(x^2)dx &= \int_1^\infty \frac{1}{2x}\left(2x\cos(x^2)\right)dx\\ &= \frac{\sin(x^2)}{2x}\Big|_1^\infty + \int_1^\infty \frac{\sin(x^2)}{2x^2}dx\\ &< \infty.\end{aligned}$$

Finally, we will show that the integral $\int_0^\infty xe^{-x^6\sin^2(x)}dx$ converges. [4] Notice that

$$\begin{aligned}\int_0^\infty xe^{-x^6\sin^2(x)}dt &= \sum_{k=0}^\infty \int_{k\pi}^{(k+1)\pi} xe^{-x^6\sin^2(x)}dx\\ &= \sum_{k=0}^\infty \int_0^\pi (t+k\pi)e^{-(t+k\pi)^6\sin^2(t)}dt \qquad (x = t+k\pi).\end{aligned}$$

[4] I would like to thank Brian Sittinger, a user of Quora, for providing a solution for the convergence of this integral.

For $k \geq 1$, we have

$$\begin{aligned}
&\int_0^{\pi} (t + k\pi) e^{-(t+k\pi)^6 \sin^2(t)} dt \\
&\leq \int_0^{\pi} (\pi + k\pi) e^{-(0+k\pi)^6 \sin^2(t)} dt \\
&= \pi(1+k) \int_0^{\pi} e^{-k^6\pi^6 \sin^2(t)} dt \\
&= 2\pi(1+k) \int_0^{\frac{\pi}{2}} e^{-k^6\pi^6 \sin^2(t)} dt && \text{(by symmetry)} \\
&\leq 2\pi(1+k) \int_0^{\frac{\pi}{2}} e^{-k^6\pi^6 \left(\frac{2t}{\pi}\right)^2} dt && \left(\sin(t) \geq \frac{2t}{\pi}, \quad \forall\, t \in \left[0, \frac{\pi}{2}\right]\right) \\
&= 2\pi(1+k) \int_0^{\frac{\pi}{2}} e^{-4k^6\pi^4 t^2} dt \\
&\leq 2\pi(1+k) \int_0^{\infty} e^{-4k^6\pi^4 t^2} dt \\
&= \frac{1+k}{k^3\pi} \int_0^{\infty} e^{-u^2} du && (u = 2k^3\pi^2 t).
\end{aligned}$$

We will show that $\int_0^\infty e^{-u^2} du$ converges and it suffices to show that $\int_1^\infty e^{-u^2} du$ since e^{-u^2} is integrable on $[0, 1]$. It follows that

$$0 \leq \int_1^{\infty} e^{-u^2} du \leq \int_1^{\infty} e^{-u} du < \infty.$$

Denote $\int_0^\infty e^{-u^2} du = L$, and it's well-known that $L = \frac{\sqrt{\pi}}{2}$. Thus, we have

$$\int_0^{\infty} x e^{-x^6 \sin^2(x)} dx \leq \int_0^{\pi} x e^{-x^6 \sin^2(x)} dx + L \sum_{k=1}^{\infty} \frac{1+k}{k^3\pi} < \infty.$$

□

Example 5.2.2. If a is real, the integral

$$\int_0^{\infty} \frac{\cos(ax)}{1+x^2} dx$$

is a continuous function of a.

Proof. Let $\varepsilon > 0$ and fix $a_0 \in \mathbb{R}$. To show $\int_0^\infty \frac{\cos(ax)}{1+x^2} dx$ is continuous at a_0, we need to find $\delta > 0$ such that if $|a - a_0| < \delta$, we have

$$\left| \int_0^\infty \frac{\cos(ax)}{1+x^2} dx - \int_0^\infty \frac{\cos(a_0 x)}{1+x^2} dx \right| < \varepsilon.$$

Since $\int_0^\infty \frac{1}{1+x^2} dx$ converges, there exists $X > 0$ such that

$$\left| \int_X^\infty \frac{1}{1+x^2} dx \right| < \frac{\varepsilon}{4}.$$

We claim that $\cos(x)$ is uniformly continuous on $\mathbb{R}$. Choose $\delta' = \varepsilon$. If $|x - y| < \delta'$, we have

$$\begin{aligned}
|\cos(x) - \cos(y)| &= \left| -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \right| \\
&\leq 2 \left| \sin\left(\frac{x-y}{2}\right) \right| \\
&\leq 2 \left| \frac{x-y}{2} \right| \\
&= |x - y| \\
&< \delta' = \varepsilon.
\end{aligned}$$

This proves $\cos(x)$ is uniformly continuous on $\mathbb{R}$. Hence, we can find $\delta'' > 0$ such that if $|x - y| < \delta''$, we have

$$|\cos(x) - \cos(y)| < \frac{\varepsilon}{\pi}.$$

Choose $\delta = \frac{\delta''}{X}$. If $|a - a_0| < \delta$ ($|ax - a_0x| < \delta''$ for all $0 \le x \le X$), then

$$\begin{aligned}
\left|\int_0^\infty \frac{\cos(ax)}{1+x^2}dx - \int_0^\infty \frac{\cos(a_0x)}{1+x^2}dx\right| &= \left|\int_0^\infty \frac{\cos(ax)-\cos(a_0x)}{1+x^2}dx\right| \\
&\le \int_0^X \frac{|\cos(ax)-\cos(a_0x)|}{1+x^2}dx + \int_X^\infty \frac{2dx}{1+x^2} \\
&< \frac{\varepsilon}{\pi}\int_0^X \frac{dx}{1+x^2} + \frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{\pi}\arctan(X) + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{\pi}\cdot\frac{\pi}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

□

Example 5.2.3. Discuss the uniformity of the convergence of $\int_0^\infty x\sin(x^3 - ax)dx$ [5].

Proof. We claim that the integral $\int_0^\infty x\sin(x^3 - ax)dx$ is uniformly convergent for all a such that $|a| \le M$. Let $\varepsilon > 0$. Applying integration by parts, we obtain

$$\begin{aligned}
\left|\int_c^d x\sin(x^3 - ax)dx\right| &\le \left|\frac{1}{3}\left(\frac{1}{x} + \frac{a}{3x^3}\right)\cos(x^3 - ax)\right|_c^d + \left|\frac{1}{3}\int_c^d \left(\frac{1}{x^2} + \frac{a}{x^4}\cos(x^3 - ax)\right)dx\right| \\
&+ \left|\frac{1}{9}a^2\int_c^d \frac{\sin(x^3 - ax)}{x^3}dx\right| \\
&\le \frac{2}{3}\left(\frac{1}{c} + \frac{M}{3c^3}\right) + \frac{1}{3}\int_c^d\left(\frac{1}{x^2} + \frac{M}{x^4}\right)dx + \frac{M^2}{9}\int_c^d \frac{1}{x^3}dx.
\end{aligned}$$

Then we can choose $X > 0$ independent of a such that if $d \ge c \ge X$, we have

$$\frac{2}{3}\left(\frac{1}{c} + \frac{M}{3c^3}\right) + \frac{1}{3}\int_c^d\left(\frac{1}{x^2} + \frac{M}{x^4}\right)dx + \frac{M^2}{9}\int_c^d \frac{1}{x^3}dx < \varepsilon.$$

This proves that the integral $\int_0^\infty x\sin(x^3 - ax)dx$ is uniformly convergent in the bounded range $[-M, M]$ of a.

We are interested in knowing if this integral is uniformly convergent as $a \to \infty$. We will show that the integral $\int_0^\infty x\sin(x^3 - ax)dx$ in not uniformly convergent in the range

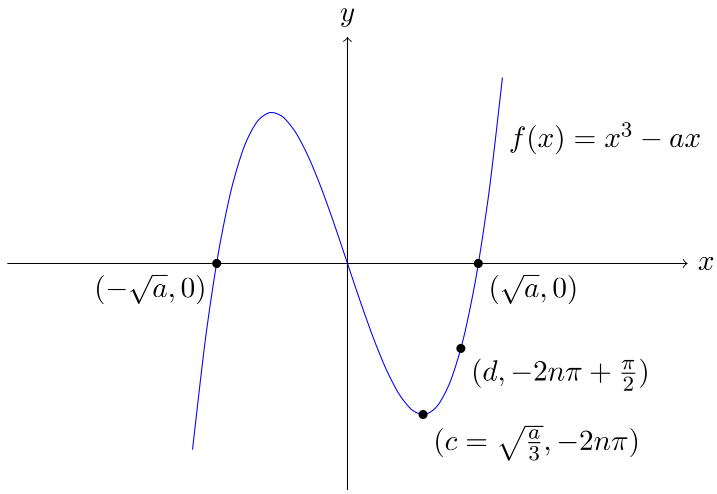

FIGURE 5.3. Visualization of $x^3 - ax$.

$[0, \infty)$ of a.

Notice that the lowest value of the function $f(x) = x^3 - ax$ $(x > 0)$ occurs at $x = \sqrt{\frac{a}{3}}$. Set $c = \sqrt{\frac{a}{3}}$.

We want $f(c) = -2n\pi$, where $n \in \mathbb{N}$. Take $a = 3x^2$. Then

$$
\begin{aligned}
& x^3 - 3x^3\big|_{x=\sqrt{\frac{a}{3}}} = -2n\pi \\
& \Rightarrow -2x^3\big|_{x=\sqrt{\frac{a}{3}}} = -2n\pi \\
& \Rightarrow x^3\big|_{x=\sqrt{\frac{a}{3}}} = n\pi \\
& \Rightarrow x\big|_{x=\sqrt{\frac{a}{3}}} = (n\pi)^{\frac{1}{3}} \\
& \Rightarrow a = 3(n\pi)^{\frac{2}{3}} \\
& \Rightarrow c = \sqrt{\frac{a}{3}} = (n\pi)^{\frac{1}{3}}.
\end{aligned}
$$

Let $d > c$ be the solution of

$$x^3 - ax = -2n\pi + \frac{\pi}{2}.$$

We wish to find an upper bound for d. Let $d = c + \delta$, $\delta > 0$. Then

$$\begin{aligned} f(d) - f(c) &= (-2n\pi + \frac{\pi}{2}) - (-2n\pi) \\ &= \frac{\pi}{2}, \end{aligned}$$

and

$$\begin{aligned} f(d) - f(c) &= (d^3 - ad) - (c^3 - ac) \\ &= \big((c+\delta)^3 - a(c+\delta)\big) - (c^3 - ac) \\ &= \delta^2(3c + \delta), \end{aligned}$$

by $a = 3c^2$. This implies that

$$\begin{aligned} \delta^2(3c+\delta) &= \frac{\pi}{2} \\ \Rightarrow \delta^2 &= \frac{\frac{\pi}{2}}{3c+\delta} < \frac{\pi}{6c} \\ \delta &< \sqrt{\frac{\pi}{6c}} = \sqrt{\frac{\pi}{6}}(n\pi)^{-\frac{1}{6}}. \end{aligned}$$

Hence, the upper bound of d is

$$c + \sqrt{\frac{\pi}{6}}(n\pi)^{-\frac{1}{6}} = (n\pi)^{\frac{1}{3}} + \sqrt{\frac{\pi}{6}}(n\pi)^{-\frac{1}{6}}.$$

Now, consider the integral $\int_c^d x \sin(x^3 - ax)dx$, where $-2n\pi \leq x^3 - ax \leq -2n\pi + \frac{\pi}{2}$. If $c \leq x \leq d$, we have

$$\sin(x^3 - ax) \geq \frac{2}{\pi}(x^3 - ax + 2n\pi) \quad \text{(see Figure 5.4)}.$$

Hence, we have

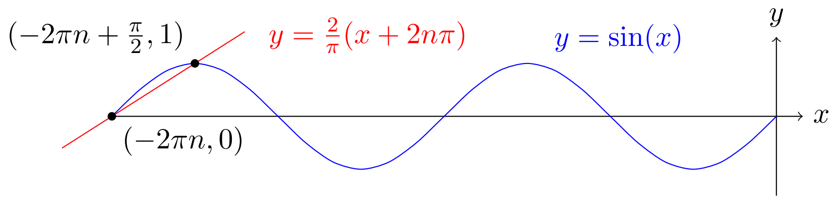

FIGURE 5.4. Visualization of $\sin(x)$ and $\frac{2}{\pi}(x + 2n\pi)$.

$$\begin{aligned}
\int_c^d x\sin(x^3 - ax)dx &\geq \int_c^d x\frac{2}{\pi}(x^3 - ax + 2n\pi)dx \\
&= \frac{2}{\pi}\left(\frac{x^5}{5} - (n\pi)^{\frac{2}{3}}x^3 + n\pi x^2\right)\Bigg|_{c=(n\pi)^{\frac{1}{3}}}^{d} \\
&= \frac{2}{\pi}\left(-\frac{(n\pi)^{\frac{5}{3}}}{5} + \frac{d^5}{5} - (n\pi)^{\frac{2}{3}}d^3 + n\pi d^2\right).
\end{aligned}$$

Since $d = c + \delta < (n\pi)^{\frac{1}{3}} + \sqrt{\frac{\pi}{6}}(n\pi)^{-\frac{1}{6}}$, $d \sim (n\pi)^{\frac{1}{3}}$ and the highest order of

$$-\frac{(n\pi)^{\frac{5}{3}}}{5} + \frac{d^5}{5} - (n\pi)^{\frac{2}{3}}d^3 + n\pi d^2$$

is $(n\pi)^{\frac{5}{3}}$. However, the coefficient for the highest order $(n\pi)^{\frac{5}{3}}$ is 0. We need to find a term of lower order. To do that, recall $d^3 = ad - 2n\pi + \frac{\pi}{2}$. Then we have

$$\begin{aligned}
&-\frac{(n\pi)^{\frac{5}{3}}}{5} + \frac{d^5}{5} - (n\pi)^{\frac{2}{3}}d^3 + n\pi d^2 \\
&= -\frac{(n\pi)^{\frac{5}{3}}}{5} + \frac{d^2}{5}\left(ad - 2n\pi + \frac{\pi}{2}\right) - (n\pi)^{\frac{2}{3}}d^3 + n\pi d^2 \\
&= -\frac{(n\pi)^{\frac{5}{3}}}{5} - \frac{2}{5}(n\pi)^{\frac{2}{3}}d^3 + \frac{3}{5}n\pi d^2 + \frac{\pi}{10}d^2 \\
&= -\frac{(n\pi)^{\frac{5}{3}}}{5} - \frac{2}{5}(n\pi)^{\frac{2}{3}}\left(ad - 2n\pi + \frac{\pi}{2}\right) + \frac{3}{5}n\pi d^2 + \frac{\pi}{10}d^2 \\
&= -\frac{(n\pi)^{\frac{5}{3}}}{5} - \frac{2}{5}(n\pi)^{\frac{2}{3}}\left(3(n\pi)^{\frac{2}{3}}((n\pi)^{\frac{1}{3}} + \delta) - 2n\pi + \frac{\pi}{2}\right) \\
&+ \frac{3}{5}n\pi\left((n\pi)^{\frac{2}{3}} + 2(n\pi)^{\frac{1}{3}}\delta + \delta^2\right) + \frac{\pi}{10}\left((n\pi)^{\frac{2}{3}} + 2(n\pi)^{\frac{1}{3}}\delta + \delta^2\right) \\
&= \frac{3}{5}n\pi\delta^2 - \frac{\pi}{10}(n\pi)^{\frac{2}{3}} + \frac{\pi}{5}(n\pi)^{\frac{1}{3}}\delta + \frac{\pi}{10}\delta^2,
\end{aligned}$$

where both the leading $(n\pi)^{\frac{5}{3}}$ and the next $(n\pi)^{\frac{4}{3}}\delta$ terms cancel out.

Recall that $\delta^2(3c+\delta) = \frac{\pi}{2}$ and $c = (n\pi)^{\frac{1}{3}}$. Then we have

$$\begin{aligned}\delta^2 &= \frac{\pi}{6c+2\delta}\\ &= \frac{\pi}{6c}\frac{1}{1+\frac{\delta}{3c}}\\ &= \frac{\pi}{6c}\left(1-\frac{\delta}{3c}+\frac{\frac{\delta^2}{9c^2}}{1+\frac{\delta}{3c}}\right).\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{\pi}{2}\int_c^d x\sin(x^3-ax)dx\\ &\geq \frac{3}{5}n\pi\delta^2 - \frac{\pi}{10}(n\pi)^{\frac{2}{3}} + \frac{\pi}{5}(n\pi)^{\frac{1}{3}}\delta + \frac{\pi}{10}\delta^2\\ &= \frac{3}{5}n\pi\frac{\pi}{6}(n\pi)^{-\frac{1}{3}}\left(1-\frac{\delta}{3c}+\frac{\frac{\delta^2}{9c^2}}{1+\frac{\delta}{3c}}\right) - \frac{\pi}{10}(n\pi)^{\frac{2}{3}} + \frac{\pi}{5}(n\pi)^{\frac{1}{3}}\delta + \frac{\pi}{10}\delta^2\\ &= \frac{\pi}{6}(n\pi)^{\frac{1}{3}}\delta + \frac{\pi}{10}\delta^2 + \frac{\frac{\pi}{90}\delta^2}{1+\frac{\delta}{3c}}.\end{aligned}$$

Now, we let n vary and denote the dependence on n by subscripts. Since $\delta_n < \sqrt{\frac{\pi}{6}}(n\pi)^{-\frac{1}{6}}$, the last two terms go to 0 as $n\to\infty$. We will show that $\lim_{n\to\infty}\delta_n(n\pi)^{\frac{1}{3}} = \infty$. Since $\delta_n^2 = \frac{\pi}{6c_n}\frac{1}{1+\frac{\delta_n}{3c_n}}$, we have

$$\begin{aligned}\lim_{n\to\infty}\delta_n^2(n\pi)^{\frac{1}{3}} &= \lim_{n\to\infty}\frac{\pi}{6}\frac{1}{1+\frac{\delta_n}{3c_n}}\\ &= \frac{\pi}{6}.\end{aligned}$$

This implies that $\lim_{n\to\infty}\delta_n(n\pi)^{\frac{1}{3}} = \infty$ since $\lim_{n\to\infty}\delta_n = 0$. Therefore, we have constructed a_n, c_n, d_n with $d_n > c_n$ for all n and $c_n \to \infty$ such that

$$\int_{c_n}^{d_n} x\sin(x^3-a_nx)dx \to \infty.$$

This proves that the integral is not uniformly convergent in the range $[0,\infty)$ of a. $\square$

Example 5.2.4. Shew that $\int_0^\infty \exp[-e^{ia}(x^3 - nx)]dx$ converges uniformly in the range $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ of the values of a.

Proof. Applying integration by parts, we obtain

$$\begin{aligned}\int_c^d e^{-e^{ia}(x^3-nx)}dx &= \int_c^d \frac{-e^{ia}(3x^2-n)}{-e^{ia}(3x^2-n)} e^{-e^{ia}(x^3-nx)}dx \\ &= -\frac{e^{-e^{ia}(x^3-nx)}}{e^{ia}(3x^2-n)}\bigg|_c^d - \int_c^d \frac{6xe^{-e^{ia}(x^3-nx)}}{e^{ia}(3x^2-n)^2}dx.\end{aligned}$$

Notice that

$$\begin{aligned}\left|e^{-e^{ia}(x^3-nx)}\right| &= \left|e^{-(\cos(a)+i\sin(a))(x^3-nx)}\right| \\ &= e^{-\cos(a)(x^3-nx)}.\end{aligned}$$

We want to find the maximum value of the function $e^{-\cos(a)(x^3-nx)}$ for $x \geq 0$. Since $a \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$, $\cos(a) \neq 0$. It follows that

$$\frac{d}{dx}e^{-\cos(a)(x^3-nx)} = -\cos(a)(3x^2-n)e^{-\cos(a)(x^3-nx)} = 0$$

implies that

$$3x^2 - n = 0 \implies x = \sqrt{\frac{n}{3}}.$$

It can be easily checked that the function attains its maximum at $x = \sqrt{\frac{n}{3}}$. Hence,

$$\begin{aligned}e^{-\cos(a)(x^3-nx)} &\leq e^{-\cos(a)\left(\left(\frac{n}{3}\right)^{\frac{3}{2}}-n\sqrt{\frac{n}{3}}\right)} \\ &\leq e^{-\left(\left(\frac{n}{3}\right)^{\frac{3}{2}}-n\sqrt{\frac{n}{3}}\right)} = M.\end{aligned}$$

Let $\varepsilon > 0$. We can choose a large $X > 0$ independent of a such that if $d \geq c \geq X$, we have

$$\left|\frac{M}{3d^2-n}\right| < \frac{\varepsilon}{3}, \quad \left|\frac{M}{3c^2-n}\right| < \frac{\varepsilon}{3}, \quad \text{and} \quad \int_c^d \frac{6Mx}{(3x^2-n)^2}dx < \frac{\varepsilon}{3}.$$

Then

$$\begin{aligned}\left|\int_c^d e^{-e^{ia}(x^3-nx)}dx\right| &\le \left|\frac{1}{e^{ia}(3x^2-n)}e^{-e^{ia}(x^3-nx)}\Big|_c^d\right| + \left|\int_c^d e^{-e^{ia}(x^3-nx)}\frac{6x}{e^{ia}(3x^2-n)^2}dx\right| \\ &\le \left|\frac{M}{3d^2-n}\right| + \left|\frac{M}{3c^2-n}\right| + \int_c^d \frac{6Mx}{(3x^2-n)^2}dx \\ &< \varepsilon.\end{aligned}$$

This proves that $\int_0^\infty e^{-e^{ia}(x^3-nx)}dx$ converges uniformly in the range $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ of the values of a. □

Example 5.2.5. Discuss the convergence of

$$\int_0^\infty \frac{x^\mu dx}{1+x^\nu|\sin(x)|^p}$$

when μ, ν, p are positive [6].

Proof. We will show that the integral $\int_0^\infty \frac{x^\mu dx}{1+x^\nu|\sin(x)|^p}$ $(\mu,\nu,p>0)$ converges if $\nu > (p+1)(\mu+1)$. We will apply the following method to show the sufficient condition for the convergence. We divide the range of integration into two sets of intervals

$$M_i = (a_i, b_i), \quad N_i = (b_i, a_{i+1})$$

for $i = 1, 2, \cdots$. In the intervals N_i, the integrand $f(x) = \frac{x^\mu}{1+x^\nu|\sin(x)|^p}$ is less than some function whose integral up to infinity converges; and that a upper bound of $f(x)$ in M_i is L_i. Then it is clear that it is enough to establish the convergence of

$$\sum_{i=1}^\infty |M_i| L_i,$$

where $|M_i| = b_i - a_i$.

We take $M_i = (i\pi - \varepsilon_i, i\pi + \varepsilon_i)$, where ε_i is determined by the condition that in the remaining intervals N_i,

$$x^{1+\alpha} f(x) < C,$$

where C and α are positive constants. If $x = i\pi \pm \varepsilon$, where $\varepsilon > \varepsilon_i$, this condition is

$$\begin{aligned}
&(i\pi \pm \varepsilon)^{1+\alpha}\left(\frac{(i\pi \pm \varepsilon)^{\mu}}{1 + (i\pi \pm \varepsilon)^{\nu}|\sin(\varepsilon))|^{p}}\right) < C \\
&\Rightarrow \frac{(i\pi \pm \varepsilon)^{1+\alpha+\mu}}{1 + (i\pi \pm \varepsilon)^{\nu}|\sin(\varepsilon)|^{p}}\left(\frac{(i\pi \pm \varepsilon)^{-\nu}}{(i\pi \pm \varepsilon)^{-\nu}}\right) < C \\
&\Rightarrow \frac{(i\pi \pm \varepsilon)^{1+\alpha+\mu-\nu}}{(i\pi \pm \varepsilon)^{-\nu} + |\sin(\varepsilon)|^{p}} < C \\
&\Rightarrow \frac{(i\pi \pm \varepsilon)^{-\nu} + |\sin(\varepsilon)|^{p}}{(i\pi \pm \varepsilon)^{1+\alpha+\mu-\nu}} > \frac{1}{C}.
\end{aligned}$$

Now, it is obvious that this is substantially equivalent to

$$i^{\nu-\mu-1-\alpha} \sin^{p}(\varepsilon_i) > H$$

for if we can choose α, H to satisfy the latter, we can choose α, C to satisfy the former. If $\nu > \mu + 1 + \alpha$, ε_i may be supposed to tend to 0, when i tends to infinity. Hence, we may replace $\sin^{p}(\varepsilon_i)$ by ε_i^{p}.

Therefore, we have $x^{1+\alpha} f(x) < C$ in μ_i (for sufficiently large values of i) if we choose ε_i so that

$$i^{\nu-\mu-1-\alpha}\varepsilon_i^{p} > H.$$

Now, in M_i, $f(x) < x^{\mu}$, and so the integral will certainly converge if $i^{\nu-\mu-1-\alpha}\varepsilon_i^{p} > H$ can be satisfied, and at the same time

$$\sum \varepsilon_i (i\pi + \varepsilon_i)^{\mu}$$

converges. This latter condition will be satisfied if (for sufficiently large values of i)

$$\varepsilon_i < K i^{-\mu-1-\beta},$$

where K, β are positive constants. These two conditions imply

$$i^{-p(\mu+1+\beta)} > i^{-(\nu-\mu-1-\alpha)}\frac{H}{K^p},$$

or

$$\nu > (p+1)(\mu+1) + p\beta + \alpha,$$

and therefore

$$\nu > (p+1)(\mu+1).$$

Now, if $\nu > (p+1)(\mu+1)$, we can choose positive quantities α, β, so that $\nu > (p+1)(\mu+1) + p\alpha + \beta$, and then choose H, K, ε_i so that the conditions $i^{\nu-\mu-1-\alpha}\varepsilon_i^p > H$ and $\varepsilon_i < Ki^{-\mu-1-\beta}$ are satisfied. Hence, the integral is convergent if $\nu > (p+1)(\mu+1)$ [6]. □

Example 5.2.6. Examine the convergence of the integrals

$$\int_0^\infty \left(\frac{1}{x} - \frac{1}{2}e^{-x} + \frac{1}{1-e^x}\right)\frac{dx}{x}, \quad \int_0^\infty \frac{\sin(x+x^2)}{x^n}dx.$$

Proof. Notice that $\left(\frac{1}{x} - \frac{1}{2}e^{-x} + \frac{1}{1-e^x}\right)\frac{1}{x}$ is integrable on $[0,1]$, so it suffices to show that the integral $\int_1^\infty \left(\frac{1}{x} - \frac{1}{2}e^{-x} + \frac{1}{1-e^x}\right)\frac{dx}{x}$ converges. Notice that

$$\int_1^\infty \left(\frac{1}{x} - \frac{1}{2}e^{-x} + \frac{1}{1-e^x}\right)\frac{dx}{x} = \int_1^\infty \frac{1}{x^2}dx - \frac{1}{2}\int_1^\infty \frac{1}{xe^x}dx + \int_1^\infty \frac{1}{x(1-e^x)}dx.$$

Notice that $\int_1^\infty \frac{1}{xe^x}dx \le \int_1^\infty \frac{1}{x^2}dx < \infty$. We will show the integral $\int_1^\infty \frac{1}{x(1-e^x)}dx$ converges.

If $x \in [1,\infty)$, then

$$\begin{aligned} e^x - 1 \ge x &\Rightarrow x(e^x-1) \ge x^2 \\ &\Rightarrow x(1-e^x) \le -x^2 \\ &\Rightarrow -\frac{1}{x^2} \le \frac{1}{x(1-e^x)} \le 0 \\ &\Rightarrow -\infty < -\int_1^\infty \frac{1}{x^2}dx \le \int_1^\infty \frac{1}{x(1-e^x)}dx \le 0. \end{aligned}$$

Thus, we can conclude that the integral $\int_1^\infty \left(\frac{1}{x} - \frac{1}{2}e^{-x} + \frac{1}{1-e^x}\right)\frac{dx}{x}$ converges.

Consider the integral $\int_0^\infty \frac{\sin(x+x^2)}{x^n}dx$. We will show that this integral converges for $-1 < n < 2$ and diverges otherwise.

If $n < 2$, then the function

$$\frac{\sin(x+x^2)}{x^n} \sim \frac{1}{x^{n-1}} \quad \text{as } x \to 0^+$$

is integrable on $(0,1)$. It suffices to show $\int_1^\infty \frac{\sin(x+x^2)}{x^n}dx$ converges. Applying integration by parts, we have $\int_1^\infty \frac{\sin(x+x^2)}{x^n}dx$ is equal to

$$\begin{aligned}
&-\frac{\cos(x+x^2)}{(1+2x)x^n}\Big|_1^\infty - \int_1^\infty \frac{(2n+2)x+n}{(1+2x)^2x^{n+1}}\cos(x+x^2)dx \\
&= -\frac{\cos(x+x^2)}{(1+2x)x^n}\Big|_1^\infty - (2n+2)\int_1^\infty \frac{x\cos(x+x^2)}{(1+2x)^2x^{n+1}}dx - n\int_1^\infty \frac{\cos(x+x^2)}{(1+2x)^2x^{n+1}}dx.
\end{aligned}$$

Clearly, $\frac{\cos(x+x^2)}{(1+2x)x^n}\Big|_1^\infty$ converges when $-1 < n$. We claim that $\int_1^\infty \cos(x+x^2)dx$ converges. Applying integration by parts, we obtain

$$\begin{aligned}
\int_1^\infty \cos(x+x^2)dx &= \int_1^\infty \cos(x+x^2)\left(\frac{1+2x}{1+2x}\right)dx \\
&= \frac{\sin(x+x^2)}{1+2x}\Big|_1^\infty + 2\int_1^\infty \frac{\sin(x+x^2)}{(1+2x)^2}dx \\
&< \infty.
\end{aligned}$$

Since $\frac{x}{(1+2x)^2x^{n+1}}$ and $\frac{1}{(1+2x)^2x^{n+1}}$ decrease to 0 as long as $n > -2$ when x is sufficiently large and $\left|\int_1^\infty \cos(x+x^2)dx\right|$ is bounded, by Chartier's test for integrals involving periodic functions (P72), we can conclude that the integrals

$$\int_1^\infty \frac{x\cos(x+x^2)}{(1+2x)^2x^{n+1}}dx \quad \text{and} \quad \int_1^\infty \frac{\cos(x+x^2)}{(1+2x)^2x^{n+1}}dx$$

converge for $-1 < n < 2$.

Consider the case when $2 \leq n$. Notice that

$$\int_0^\infty \frac{\sin(x+x^2)}{x^n}dx = \int_0^{\frac{1}{2}} \frac{\sin(x+x^2)}{x^n}dx + \int_{\frac{1}{2}}^\infty \frac{\sin(x+x^2)}{x^n}dx,$$

where $\int_{\frac{1}{2}}^\infty \frac{\sin(x+x^2)}{x^n}dx$ converges absolutely. If $0 < x \leq \frac{1}{2}$, then

$$\begin{aligned}
&x^{n-1} \leq \sin(x+x^2) \\
&\Rightarrow x^n \leq x\sin(x+x^2) \\
&\Rightarrow \frac{1}{x} \leq \frac{\sin(x+x^2)}{x^n} \\
&\Rightarrow \infty = \int_0^{\frac{1}{2}} \frac{1}{x}dx \leq \int_0^{\frac{1}{2}} \frac{\sin(x+x^2)}{x^n}dx.
\end{aligned}$$

Thus, we can conclude that $\int_0^\infty \frac{\sin(x+x^2)}{x^n}dx$ diverges when $2 \leq n$.

Finally, we will show the integral diverges when $n \leq -1$. Denote $1 \leq -n = m$ and $f(x) = x^m \sin(x+x^2)$. Fix $m \geq 1$. Suppose, on the contrary, that

$$\lim_{b\to\infty} \int_0^b f(x)dx = L \in \mathbb{R}.$$

Define $F(b) = \int_0^b f(x)dx$. Let $\varepsilon = \frac{1}{4000}$. Then there exists $b_0 > 10$ such that if $b \geq b_0$, we have $|F(b) - L| < \varepsilon$.

Choose a large $k \in \mathbb{N}$ such that there exists $x_2 > b_0 + 100$ satisfying $x_2 + x_2^2 = 2k\pi + \frac{\pi}{2}$. Let $x_1 \in (0, x_2)$ be such that $x_1 + x_1^2 = 2k\pi + \frac{\pi}{4}$. Note that such x_1 exists and is unique as follows. Let $g(x) = x + x^2$. Observe that $g(0) < 2k\pi + \frac{\pi}{4} < g(x_2)$. By intermediate value theorem, such x_1 exists. Since g is strictly increasing on $[0, \infty)$, such x_1 is unique.

Notice that $\frac{\pi}{4} = (x_2 + x_2^2) - (x_1 + x_1^2) = (x_2 - x_1)(1 + x_1 + x_2)$. Hence, $x_2 - x_1 = \frac{\pi}{4(1+x_1+x_2)} < 1$. This implies that $x_1 > x_2 - 1 > b_0$. Moreover, $1 - \frac{x_1}{x_2} = \frac{\pi}{4(1+x_1+x_2)x_2}$ and hence $\frac{x_1}{x_2} = 1 - \frac{\pi}{4(1+x_1+x_2)} > \frac{1}{2}$.

Observe that $\sin(x+x^2)$ increases from $\frac{1}{\sqrt{2}}$ to 1 when x increases from x_1 to x_2, so

$f(x) \geq x_1^n \cdot \frac{1}{\sqrt{2}}$ for all $x \in [x_1, x_2]$. Now

$$
\begin{aligned}
F(x_2) - F(x_1) &= \int_{x_1}^{x_2} f(x)dx \\
&\geq \frac{1}{\sqrt{2}} x_1^n (x_2 - x_1) \\
&= \frac{\pi}{4\sqrt{2}} \cdot \frac{x_1^n}{1 + x_1 + x_2} \\
&\geq \frac{\pi}{4\sqrt{2}} \cdot \frac{x_1}{3x_2} \\
&> \frac{\pi}{4\sqrt{2}} \cdot \frac{1}{6} \\
&> \frac{1}{1000} \\
&= 4\varepsilon.
\end{aligned}
$$

On the other hand, since $x_1, x_2 \geq b_0$, we have

$$
\begin{aligned}
|F(x_2) - F(x_1)| &\leq |F(x_2) - L| + |F(x_1) - L| \\
&< 2\varepsilon,
\end{aligned}
$$

which is a contradiction. Thus, we must conclude that the integral $\int_0^\infty x^m \sin(x + x^2)dx$ diverges when $m \geq 1$. [5] □

Example 5.2.7. Shew that $\int_\pi^\infty \frac{dx}{x^2(\sin(x))^{\frac{2}{3}}}$ exists.

[5] I would like to thank Danny Pak-Keung Chan, a user of Math Stackexchange, for providing a solution for the divergence of the integral $\int_0^\infty x^m \sin(x + x^2)dx$.

Proof. Let $I = \int_{\pi}^{\infty} \frac{dx}{x^2(\sin(x))^{\frac{2}{3}}}$. Then

$$\begin{aligned} 0 \leq I &= \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{dx}{x^2(\sin(x))^{\frac{2}{3}}} \\ &\leq \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{k\pi}^{(k+1)\pi} \frac{dx}{(\sin(x))^{\frac{2}{3}}} \\ &= \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{0}^{\pi} \frac{dx}{(\sin(x))^{\frac{2}{3}}} && \text{(by symmetry)} \\ &= \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{0}^{\frac{\pi}{2}} \frac{dx}{(\sin(x))^{\frac{2}{3}}} && \text{(by symmetry).} \end{aligned}$$

We want to show that the integral $\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin(x))^{\frac{2}{3}}}$ converges. Since $\frac{1}{(\sin(x))^{\frac{2}{3}}}$ is integrable on $\left[\frac{1}{2}, \frac{\pi}{2}\right]$, it suffices to show

$$\int_0^{\frac{1}{2}} \frac{dx}{(\sin(x))^{\frac{2}{3}}}$$

converges. If $0 < x \leq \frac{1}{2}$, then

$$\begin{aligned} x^{\frac{2.5}{2}} \leq \sin(x) &\Rightarrow x^{\frac{2.5}{3}} \leq (\sin(x))^{\frac{2}{3}} \\ &\Rightarrow \frac{1}{(\sin(x))^{\frac{2}{3}}} \leq \frac{1}{x^{\frac{2.5}{3}}}. \end{aligned}$$

It follows that

$$0 \leq \int_0^{\frac{1}{2}} \frac{dx}{(\sin(x))^{\frac{2}{3}}} \leq \int_0^{\frac{1}{2}} \frac{dx}{x^{\frac{2.5}{3}}} < \infty.$$

This proves the integral $\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin(x))^{\frac{2}{3}}}$ converges. Thus, we can conclude that

$$I \leq \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\frac{\pi}{2}} \frac{dx}{(\sin(x))^{\frac{2}{3}}} < \infty.$$

□

Example 5.2.8. Shew that

$$\int_a^\infty x^{-n} e^{\sin(x)} \sin(2x) dx$$

converges if $a > 0, n > 0$.

Proof. Applying integration by part, we obtain

$$\int_a^\infty x^{-n} e^{\sin(x)} \sin(2x) dx = \left. \frac{2e^{\sin(x)}(\sin(x) - 1)}{x^n} \right|_a^\infty + \int_a^\infty \frac{2ne^{\sin(x)}(\sin(x) - 1)}{x^{n+1}} dx.$$

Clearly, $\left. \frac{2e^{\sin(x)}(\sin(x)-1)}{x^n} \right|_a^\infty$ converges and $\int_a^\infty \frac{2ne^{\sin(x)}(\sin(x)-1)}{x^{n+1}} dx$ converges absolutely since

$$\int_a^\infty \left| \frac{2ne^{\sin(x)}(\sin(x) - 1)}{x^{n+1}} \right| dx \leq 4ne \int_a^\infty \frac{1}{x^{n+1}} < \infty.$$

Thus, we can conclude that

$$\int_a^\infty x^{-n} e^{\sin(x)} \sin(2x) dx$$

converges for $a > 0, n > 0$. □

Example 5.2.9. If a series $g(z) = \sum_{\nu=0}^\infty (c_\nu - c_{\nu+1}) \sin[(2\nu + 1)\pi z]$, (in which $c_0 = 0$), converges uniformly in an interval, shew that $g(z)\frac{\pi}{\sin(\pi z)}$ is the derivative of the series $f(z) = \sum_{\nu=1}^\infty \frac{c_\nu}{\nu} \sin(2\nu\pi z)$ [8].

Proof. Let $g(z) = \sum_{\nu=0}^\infty (c_\nu - c_{\nu+1}) \sin[(2\nu + 1)\pi z]$ (in which $c_0 = 0$), be a uniformly convergent series in an interval. Now, let

$$h(z) = g(z) \frac{\pi}{\sin(\pi z)}.$$

Then

$$\frac{1}{\pi} h(z) = \lim_{n\to\infty} \sum_{\nu=0}^n (c_\nu - c_{\nu+1}) \frac{\sin[(2\nu + 1)\pi z]}{\sin(\pi z)}.$$

Applying summation by parts, we obtain

$$\sum_{\nu=0}^{n} a_\nu b_\nu = \sum_{\nu=0}^{n-1} A_\nu (b_\nu - b_{\nu+1}) + A_n b_n,$$

where $A_\nu = \sum_{k=0}^{\nu} a_k$, $a_\nu = c_\nu - c_{\nu+1}, b_\nu = \frac{\sin[(2\nu+1)\pi z]}{\sin(\pi z)}$. Notice that

$$\begin{aligned} A_\nu &= a_0 + a_1 + \cdots + a_\nu \\ &= c_0 - c_1 + c_1 - c_2 + c_2 - c_3 + \cdots + c_\nu - c_{\nu+1} \\ &= -c_{\nu+1}, \end{aligned}$$

and

$$\begin{aligned} b_\nu - b_{\nu+1} &= \frac{\sin[(2\nu+1)\pi z]}{\sin(\pi z)} - \frac{\sin[(2\nu+3)\pi z]}{\sin(\pi z)} \\ &= -2\cos[(2\nu+2)\pi z]. \end{aligned}$$

Hence, we have

$$\frac{1}{\pi} h(z) = \lim_{n\to\infty} \left[\sum_{\nu=0}^{n-1} c_{\nu+1} 2\cos[(2\nu+2)\pi z] - c_{n+1} \frac{\sin[(2n+1)\pi z]}{\sin(\pi z)} \right].$$

Now, let z_0, z_1 be in the interval, in which the series $h(z)$ is uniformly convergent. Then by the property of uniform convergence of $h(z)$, we have

$$\int_{z_0}^{z_1} h(z)dz = \lim_{n\to\infty} \left[\sum_{\nu=1}^{n} \frac{c_\nu}{\nu} \left(\sin(2\nu\pi z_1) - \sin(2\nu\pi z_0)\right) - R_n \right],$$

where

$$R_n = c_{n+1} \int_{z_0}^{z_1} \frac{\pi \sin[(2n+1)\pi z]}{\sin(\pi z)} dz.$$

Applying integration by parts, then R_n is equal to

$$-\frac{c_{n+1}}{2n+1} \left(\frac{\cos[(2n+1)\pi z_1]}{\sin(\pi z_1)} - \frac{\cos[(2n+1)\pi z_0]}{\sin(\pi z_0)} \right) - \frac{\pi c_{n+1}}{2n+1} \int_{z_0}^{z_1} \frac{\cos[(2n+1)\pi z]\cos(\pi z)}{\sin^2(\pi z)} dz.$$

The infinitely small quantity of $\frac{c_{n+1}}{2n+1}$ is multiplied by two quantities which remain finite as $n \to \infty$. Thus, we have

$$\lim_{n\to\infty} R_n = 0,$$

which shows that

$$\begin{aligned}\int_{z_0}^{z_1} h(z)dz &= \sum_{\nu=1}^{\infty} \frac{c_\nu}{\nu}(\sin(2\nu\pi z_1) - \sin(2\nu\pi z_0)) \\ &= f(z_1) - f(z_0).\end{aligned}$$

This proves that $h(z)$ is the derivative of the series $f(z)$. □

Example 5.2.10. Shew that

$$\int^{\infty}\int^{\infty}\cdots\int^{\infty} \frac{dx_1dx_2\cdots dx_n}{(x_1^2+x_2^2+\cdots+x_n^2)^\alpha} \quad \text{and} \quad \int^{\infty}\int^{\infty}\cdots\int^{\infty} \frac{dx_1dx_2\cdots dx_n}{x_1^\alpha+x_2^\beta+\cdots+x_n^\lambda}$$

converge when $\alpha > \frac{1}{2}n$ and $a^{-1}+\beta^{-1}+\cdots+\lambda^{-1} < 1$ respectively.

Proof. By the arithmetic mean and geometric mean inequality, we get

$$n(x_1^2x_2^2\cdots x_n^2)^{\frac{1}{n}} \le x_1^2+x_2^2+\cdots+x_n^2.$$

This implies that

$$\frac{1}{(x_1^2+x_2^2+\cdots x_n^2)^\alpha} \le \frac{1}{n^\alpha(x_1^2x_2^2\cdots x_n^2)^{\frac{\alpha}{n}}}.$$

It follows that if $\alpha > \frac{n}{2}$, we have

$$\begin{aligned}0 &\le \int_1^{\infty}\int_1^{\infty}\cdots\int_1^{\infty} \frac{dx_1dx_2\cdots dx_n}{(x_1^2+x_2^2+\cdots+x_n^2)^\alpha} \\ &\le \frac{1}{n^\alpha}\int_1^{\infty}\int_1^{\infty}\cdots\int_1^{\infty} \frac{dx_1dx_2\cdots dx_n}{x_1^{\frac{2\alpha}{n}}x_2^{\frac{2\alpha}{n}}\cdots x_n^{\frac{2\alpha}{n}}} \\ &< \infty.\end{aligned}$$

Next, we want to show that

$$\int_1^\infty \int_1^\infty \cdots \int_1^\infty \frac{dx_1 dx_2 \cdots dx_n}{x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n}}$$

converges when $\sum_{i=1}^n \frac{1}{\alpha_i} < 1$.

Recall that the weighted arithmetic mean and geometric mean inequality states that: If $0 \le x_i \in \mathbb{R}$ and $0 \le \lambda_i \in \mathbb{R}$ for $i = 1, 2, \cdots, n$ such that $\sum_{i=1}^n \lambda_i = 1$, then

$$\prod_{i=1}^n x_i^{\lambda_i} \le \sum_{i=1}^n \lambda_i x_i \text{ [4]}.$$

Assume that $\sum_{i=1}^n \frac{1}{\alpha_i} < 1$.[6] Denote $p = \frac{1}{\sum_{i=1}^n \frac{1}{\alpha_i}} > 1$, $y_i = \log(x_i^{\alpha_i})$, $\lambda_i = \frac{p}{\alpha_i} \in (0, 1]$ for $i = 1, 2, \cdots, n$. Since $\sum_{i=1}^n \lambda_i = 1$, by the weighted AM-GM inequality, we have

$$\begin{aligned} \sum_{i=1}^n x_i^{\alpha_i} &= \sum_{i=1}^n e^{\log(x_i^{\alpha_i})} \\ &\ge \sum_{i=1}^n \lambda_i e^{y_i} \\ &\ge \prod_{i=1}^n (e^{y_i})^{\lambda_i} \\ &= \prod_{i=1}^n x_i^p. \end{aligned}$$

It follows that

$$\begin{aligned} &\int_1^\infty \int_1^\infty \cdots \int_1^\infty \frac{dx_1 dx_2 \cdots dx_n}{x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n}} \\ &\le \int_1^\infty \int_1^\infty \cdots \int_1^\infty \frac{dx_1 dx_2 \cdots dx_n}{x_1^p x_2^p \cdots x_n^p} \qquad (p > 1) \\ &< \infty. \end{aligned}$$

□

Example 5.2.11. If $f(x, y)$ be a continuous function of both x and y in the ranges $(a \le$

[6]Each α_i is assumed to be positive.

$x \leq b$), ($a \leq y \leq b$) except that it has ordinary discontinuities at points on a finite number of curves, with continuously turning tangents, each of which meets any line parallel to the coordinate axes only a finite number of times, then $\int_a^b f(x,y)dx$ is a continuous function of y.

[Consider $\int_a^{a_1-\delta_1} + \int_{a_1+\varepsilon_1}^{a_2-\delta_2} + \cdots + \int_{a_n+\varepsilon_n}^{b} \{f(x,y+h) - f(x,y)\}\, dx$, where the numbers $\delta_1, \delta_2, \cdots, \varepsilon_1, \varepsilon_2, \cdots$ are chosen as to exclude the discontinuous of $f(x, y+h)$ from the range of integration; $a_1, a_2, \cdots$ being the discontinuities of $f(x,y)$.]

Proof. Let $\varepsilon > 0$. To show $\int_a^b f(x,y)dx$ is a continuous function of y, we need to find $t > 0$ such that if $|(y+h) - y| < t$, we have

$$\left| \int_a^b f(x+y+h)dx - \int_a^b f(x,y)dx \right| < \varepsilon.$$

Since f has only ordinary discontinuities at a finite number of curves, f is bounded by a bound $M > 0$ on $[a,b] \times [a,b]$. Let $a_1, a_2, \cdots, a_n$ be the discontinuities of f along the horizontal line at y. Choose $\varepsilon_1, \delta_1, \cdots, \varepsilon_n, \delta_n > 0$ small enough such that

$$\int_{a_1-\varepsilon_1}^{a_1+\delta_1} + \cdots + \int_{a_n-\varepsilon_n}^{a_n+\delta_n} |f(x,y+h) - f(x,y)|\, dx \leq 2M \sum_{i=1}^{n} (\delta_i + \varepsilon_i) < \frac{\varepsilon}{2}.$$

Now, consider $\int_a^{a_1-\delta_1} + \int_{a_1+\varepsilon_1}^{a_2-\delta_2} + \cdots + \int_{a_n+\varepsilon_n}^{b} \{f(x,y+h) - f(x,y)\}\, dx$. By Example 5.1.13, we know that $\int_a^{a_1-\delta_1}, \int_{a_1+\varepsilon_1}^{a_2-\delta_2}, \cdots, \int_{a_n+\varepsilon_n}^{b} f(x,y)dx$ are continuous functions of y. Hence, there exists $t_1 > 0$ such that if $|(y+h) - y| < t_1$, we have

$$\left| \int_a^{a_1-\delta} \{f(x,y+h) - f(x,y)\}\, dx \right| < \frac{\varepsilon}{2(n+1)}.$$

Likewise, there exist $t_2, t_3, \cdots, t_{n+1} > 0$ satisfy the similar property above.

Choose $t = \min\{t_1, t_2, \cdots t_{n+1}\}$. If $|(y+h) - y| < t$, then

$$\begin{aligned}
&\left|\int_a^b f(x, y+h) - f(x, y)dx\right| \\
&\leq \int_{a_1-\varepsilon_1}^{a_1+\delta_1} + \cdots + \int_{a_n-\varepsilon_n}^{a_n+\delta_n} |f(x, y+h) - f(x, y)|\, dx \\
&+ \left|\int_a^{a_1-\delta_1} + \int_{a_1+\varepsilon_1}^{a_2-\delta_2} + \cdots + \int_{a_n+\varepsilon_n}^{b} \{f(x, y+h) - f(x, y)\}\, dx\right| \\
&\leq \frac{\varepsilon}{2} + \left|\int_a^{a_1-\delta_1} \{f(x, y+h) - f(x, y)\}\, dx\right| + \cdots + \left|\int_{a_n+\varepsilon_n}^{b} \{f(x, y+h) - f(x, y)\}\, dx\right| \\
&< \frac{\varepsilon}{2} + \underbrace{\frac{\varepsilon}{2(n+1)} + \cdots + \frac{\varepsilon}{2(n+1)}}_{n+1 \text{ copies}} \\
&= \varepsilon.
\end{aligned}$$

□

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