Transformation Groups and the Method of Darboux

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TRANSFORMATION GROUPS AND THE METHOD OF DARBOUX

by

Brandon P. Ashley

A dissertation submitted in partial fulfillment
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematical Sciences

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ABSTRACT

Transformation Groups and the Method of Darboux

by

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Utah State University, 2021

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The purpose of this dissertation study various hyperbolic Darboux integrable systems of partial differential equations using the quotient theory of hyperbolic Darboux integrable systems developed by Anderson, Fels, and Vassiliou. In doing so, we provide conditions for which the quotient construction results in a hyperbolic distribution defining a scalar second-order hyperbolic PDE in the plane of the form

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \] (1)

which is Darboux integrable at order \( k \). We then further refine these conditions to give an invariant characterization of hyperbolic PDE of the form (1) which are of generic, Goursat, Monge-Ampère, and \( f \)-Gordon type. Using this characterization, we demonstrate, in two instances, how the quotient theory of Darboux integrable systems can be used to solve equivalence problems for Darboux integrable equations. We then utilize our invariant characterization of Darboux integrable \( f \)-Gordon equations to construct vector field representations of several new examples of \( f \)-Gordon equations which are Darboux integrable at order three.

(353 pages)
In the study of partial differential equations (PDE), one is often concerned as to whether or not explicit solutions can be obtained via various integration techniques. One such technique, known as the method of Darboux, has had particular success in solving nonlinear problems as demonstrated by the classical works of Goursat. Recently, Anderson, Fels, and Vassiliou provided a far-reaching generalization of Vessiot's group-theoretic interpretation of the method of Darboux. This generalization allows for the characterization of Darboux integrable systems in terms of fundamental geometric invariants as well as the construction of Darboux integrable systems in general.

In this work, we refine the theory of Anderson, Fels, and Vassiliou by providing conditions for which their construction gives rise to various classes of second-order PDE in the plane of the form

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \]

We use this refinement to completely characterize all linear Darboux integrable PDE in the plane and provide a simple proof concerning the classification of all PDE equivalent to the wave equation \( u_{xy} = 0 \). We then study the fundamental invariants associated to several classes of Darboux integrable equations, in particular, \( f \)-Gordon equations of the form

\[ u_{xy} = f(x, y, u, u_x, u_y). \]

In doing so, we construct several new examples of Darboux integrable \( f \)-Gordon equations with interesting geometric structure.
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Brandon P. Ashley
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PART I
PRELIMINARIES AND GENERALITIES
CHAPTER 1
INTRODUCTION

In this dissertation, we study Darboux integrable partial differential equations using the quotient theory of hyperbolic Darboux integrable systems developed by Anderson, Fels, and Vassiliou in [5]. Our main result uses this theory to construct vector field representations of several, forty-six to be precise, new examples of scalar $f$-Gordon equations of the form

$$u_{xy} = f(x, y, u, u_x, u_y)$$

which are Darboux integrable at order three. Along with this result, we also present several secondary results on the invariant classification of Darboux integrable hyperbolic PDE in the plane including solving the equivalence problem for linear hyperbolic PDE in the plane which are Darboux integrable at any order.

The subject matter of this dissertation lies within the general setting of the geometric theory of differential equations, the foundations of which were originally laid out by Sophus Lie and Gaston Darboux throughout the second half of the nineteenth century. Their broad goal focused on understanding the various properties of differential equations by studying their geometric invariants under suitable groups of transformations (point, fibre-preserving, and contact). The theory was then extensively developed by several of their students and contemporaries including, Engel, Goursat, Janet, Requier, Vessiot, Cartan, and many others. Of particular interest was the development of integration methods for partial differential equations by which explicit solutions could be found. Such methods included those of mathematicians Monge, Laplace, Lagrange, Charpit, Ampère, Boole, and Darboux. The so-called method of Darboux was by far the most powerful and has had remarkable success in solving nonlinear partial differential equations, as repeatedly demonstrated in the classical works of Goursat [19].
The method of Darboux hinges upon the existence of a sufficient number of auxiliary equations, called *intermediate integrals*, which are compatible with the original system of interest. Here, compatibility means that solutions to the system of intermediate integrals directly correspond with solutions of the original system. A system which is not Darboux integrable may become Darboux integrable after a certain number of prolongations. An equation is said to be *Darboux integrable at order* $k$ if it becomes Darboux integrable after $k - 2$ prolongations.

For $f$-Gordon equations, the intermediate integrals are given by the level sets of functions $I = I(x, y, u, u_y, u_{yy}, \ldots)$ and $J = J(x, y, u, u_x, u_{xx}, \ldots)$ satisfying $D_x(I) = D_y(J) = 0$ where $D_x$ and $D_y$ are total differential operators.

A classical example of an equation which is Darboux integrable at order two is the Liouville equation

$$u_{xy} = e^u.$$  \hfill (1.1)

The general solution to the Liouville equation can be obtained by recognizing that it has intermediate integrals

$$u_{xx} - \frac{1}{2}u_x^2 = \varphi(x) \quad \text{and} \quad u_{yy} - \frac{1}{2}u_y^2 = \psi(y). \hfill (1.2)$$

Together, equations (1.1) and (1.2) form a Frobenius system, and in this way, the integration of (1.1) can be accomplished using only ODE methods. Indeed, the equations (1.2) are ordinary differential equations of Ricatti-type for $u_x$ and $u_y$, respectively, and solving this system for $u_x$ and $u_y$ gives the solution

$$u_x = \frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi + \Psi} \quad \text{and} \quad u_y = \frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Phi + \Psi}. \hfill (1.3)$$
where the functions $\Phi = \Phi(x)$ and $\Psi = \Psi(y)$ satisfy
\[
\varphi(x) = \frac{2\Phi'\Phi'' - 3(\Phi'')^2}{2(\Phi')^2} \quad \text{and} \quad \psi(x) = \frac{2\Psi'\Psi'' - 3(\Psi'')^2}{2(\Psi')^2}.
\]

Integrating (1.3) gives the general solution to the Liouville equation,
\[
u = \ln \left( \frac{2\Phi'\Psi'}{(\Phi + \Psi)^2} \right).
\]

Though this method proved to be an effective tool in the study of nonlinear PDE, the geometric mechanisms behind the method remained a mystery for some time. Motivated by hopes for the development of a differential Galois theory, mathematicians including Picard, Vessiot, and Drach initiated research programs aimed at better understanding integrable differential equations in terms of their associated geometric and algebraic structures. Drach was the first to give a formal attempt at developing a differential Galois theory for first-order linear PDE in his dissertation [16], however, the validity of his theory received heavy criticism by many of the leading geometers of the time, particularly Vessiot and Cartan who jointly noted that

“Les vues de Drach étaient aussi originales que fécondes et d’une extrême importance, mais ses énoncés et ses démonstrations contenaient de graves lacunes et inexactitudes.”

Drach’s work was subsequently made rigorous by Vessiot himself in [34] resulting in Vessiot being awarded the 1902 Grand Prix des Sciences Mathématiques with many extensions of this work to follow.

In studying Darboux integrable systems, Vessiot [35, 36] went on to realize that the intermediate integrals associated with Darboux integrable systems are always of Lie type, meaning they satisfy a certain system of ODEs corresponding to an associated Lie group, and thus he was able to give a group-theoretic interpretation of the method of Darboux. In doing so, he associated to each Darboux integrable equation a symmetry group of it’s characteristic systems, now called the Vessiot group of the equation.
Much later, Anderson, Fels, and Vassiliou [5] gave a far-reaching generalization of the original theory of Vessiot. Their generalization not only provided an algorithm for calculating the fundamental invariants for a Darboux integrable system, consisting of the Vessiot group and a pair of associated exterior differential systems which constitute what they call a *Darboux pair*, but also showed how Darboux integrable systems can be constructed in general via quotients of exterior differential systems.

It is important to note that in this theory, one rarely works with Darboux integrable equations explicitly. Instead, each PDE system is encoded as an associated distribution (or in the most general case, an exterior differential system). For example, the Liouville equation (1.1), can be encoded as the rank 4 distribution

\[
\Delta = \{\partial_x + p\partial_u + r\partial_p + e^u\partial_q, \partial_y + q\partial_u + e^u\partial_p + t\partial_{\tau}, \partial_r, \partial_t\}
\]

defined on a 7-dimensional manifold with coordinates \(x, y, u, p, q, r, t\). This encoding has the benefit of readily allowing for the use of differential geometric techniques to study Darboux integrable systems. We discuss how this encoding can be done under a far more general setting in Chapter 2.

The primary goal of this dissertation is to use the theory developed by Anderson, Fels, and Vassiliou, which we will often refer to as the quotient theory of Darboux integrable systems, to further study scalar second order PDE in the plane which are Darboux integrable at various orders. We divide this work into four main parts. In Part I, we review the preliminary theory on distributions and hyperbolic systems relevant to later chapters. In particular, we highlight in Section 2.4 the following theorem given in [30] which gives necessary and sufficient conditions for a rank 4 hyperbolic distribution to define a hyperbolic PDE in the plane.

**Theorem 1.** Let \(\Delta\) a rank 4 hyperbolic distribution defined on a 7-dimensional manifold \(M\). Then \(\Delta\) is associated to a hyperbolic PDE in the plane if and only if

\[[i]\] \(\text{rank}(\Delta') = 6\) and \(\text{rank}(\Delta'') = 7\),
[ii] $\Delta$ has no Cauchy characteristics, and

[iii] the derived distribution $\Delta'$ admits two Cauchy characteristics.

In Section 2.5, we define what it means for a hyperbolic second-order PDE in the plane to be of generic, Goursat, or Monge-Ampère type, as done by Kamran and Gardner in [17], and reinterpret their invariant characterization of equations of these types in terms of associated characteristic distributions. This results in the following theorem.

**Theorem 2.** A second order hyperbolic PDE in the plane with characteristic distributions $\hat{\Delta}$ and $\tilde{\Delta}$ is

[i] an equation is of generic type if and only if

$$\text{rank}(\hat{\Delta}') = \text{rank}(\tilde{\Delta}') = 3 \text{ and } \text{rank}(\hat{\Delta}'') = \text{rank}(\tilde{\Delta}'') = 5,$$

[ii] an equation is of Goursat type if and only if

$$\text{rank}(\hat{\Delta}') = 3, \text{ rank}(\hat{\Delta}'') = 5 \text{ and } \text{rank}(\tilde{\Delta}') = 3, \text{ rank}(\tilde{\Delta}'') = 4,$$

or vice versa, and

[iii] an equation is of Monge-Ampère type if and only if

$$\text{rank}(\hat{\Delta}') = \text{rank}(\tilde{\Delta}') = 3 \text{ and } \text{rank}(\hat{\Delta}'') = \text{rank}(\tilde{\Delta}'') = 3.$$
group can be explicitly calculated along with a pair of rank two distributions, which we call *Vessiot distributions*, directly associated with the Darboux pairs described in [5].

In Chapter 5, we discuss the quotient theory of Darboux integrable systems developed by Anderson, Fels, and Vassiliou. This quotient theory gives a way to construct Darboux integrable systems in general, however, it does not specify when the differential system arising from the quotient will correspond systems of particular types. In Section 5.2, we refine the general quotient theory by providing conditions for which the quotient construction corresponds to a scalar second-order Darboux integrable PDE in the plane of the form

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \]

resulting in the following theorem.

**Theorem 3.** Let \( \mathcal{V} \) and \( \mathcal{V}' \) be two rank 2 bracket-generating distributions on manifolds \( \mathcal{M} \) and \( \mathcal{M}' \) of dimensions \( \tilde{n} \) and \( \tilde{n}' \), respectively. Let \( G \) be a common symmetry group of \( \mathcal{V} \) and \( \mathcal{V}' \) of dimension \( \tilde{n} + \tilde{n}' - 7 \). Denote the diagonal action of \( G \) by \( \Gamma_{\text{diag}} \), and suppose

[1] \( G \) acts freely on \( \mathcal{M} \) and \( \mathcal{M}' \);

[2] \( \Gamma_{\text{diag}} \cap (\mathcal{V} \oplus \mathcal{V}')' = \{0\} \).

Then the quotient distribution

\[ \Delta = (\mathcal{V} \oplus \mathcal{V}')/G_{\text{diag}} \]

is a rank 4 hyperbolic distribution corresponding to a Darboux integrable second-order hyperbolic PDE in the plane.

We then further specify when the quotient construction corresponds to equations with various geometric structure as highlighted by Kamran and Gardner in [17]. In particular,
we prove the following theorem concerning when the quotient construction gives a Darboux integrable equation of Monge-Ampère type; that is, an equation of the form

\[ Au_{xx} + 2Bu_{xy} + Cu_{yy} + D + E(u_{xx}u_{yy} - u_{xy}^2) = 0 \]

where \( A, B, C, D, E \) are functions of \( x, y, u, u_x, u_y \).

**Theorem 4.** Let \( \tilde{\mathcal{V}} \) and \( \tilde{\mathcal{V}}' \) be two rank two distributions satisfying the hypotheses of Theorem 5.2.1. Let \( G \) be a common symmetry group of \( \tilde{\mathcal{V}} \) and \( \tilde{\mathcal{V}}' \) of dimension \( \tilde{n} + \tilde{n} - 7 \), and denote the diagonal action of \( G \) by \( \Gamma_{\text{diag}} \). If \( \Gamma_{\text{diag}} \cap (\tilde{\mathcal{V}} \oplus \tilde{\mathcal{V}})' \) is rank 1, then the quotient distribution is a rank 4 hyperbolic distribution corresponding to a hyperbolic PDE in the plane of Monge-Ampère type.

In Part II, we demonstrate how the quotient theory of Darboux integrable systems can be used to solve equivalence problems for Darboux integrable equations. In Chapter 6, we give a simple proof of the well-known theorem of Lie on the (contact) equivalence of Darboux integrable equations to the wave equation \( u_{xy} = 0 \).

**Theorem 5.** Every equation whose characteristic systems each admit three functionally independent intermediate integrals is contact equivalent to the wave equation \( u_{xy} = 0 \).

Under the quotient theory of Darboux integrable systems, this theorem becomes a simple consequence of a well-known theorem of Engel.

In Chapter 7, we consider second-order linear hyperbolic PDE in the plane of the form

\[ u_{xy} = a(x, y)u_x + b(x, y)u_y + c(x, y)u, \]

and give the following invariant characterization of such equations.

**Theorem 6.** A second-order hyperbolic partial differential equation in the plane is linear and Darboux integrable at order \( k \geq 2 \) if and only if its fundamental invariants consist of an abelian Vessiot group of dimension \( 2k + 3 - r - s \) and Vessiot distributions equivalent to standard contact distributions on \( J^{k+3-r}(\mathbb{R}, \mathbb{R}) \) and \( J^{k+3-s}(\mathbb{R}, \mathbb{R}) \) for \( 2 \leq r, s \leq k + 1 \).
In Part III, we focus on Darboux integrable \( f \)-Gordon equations of the form

\[
    u_{xy} = f(x, y, u, u_x, u_y).
\]

Darboux integrable equations of this type have been extensively studied over the years with Goursat [19] being the first to give a classification, up to complex-valued contact equivalence, of all \( f \)-Gordon equations which are Darboux integrable at order two. In his classification, he presents eleven classes of \( f \)-Gordon equations, but was unable to find explicit solutions for all of the equations. Vessiot [35, 36] later improved upon Goursat’s classification using his group-theoretic methods and was able to write closed-form solutions to each of the equations in Goursat’s list. In his dissertation [9], Biesecker then further improved upon the results of Vessiot by giving a complete classification of \( f \)-Gordon equations Darboux integrable at order two up to real-valued contact equivalence through the analysis of the so-called generalized Laplace invariants associated to Darboux integrable equations (see [6, 9] for details).

In Chapter 8, we give an invariant characterization of all \( f \)-Gordon equations which are Darboux integrable at order two in terms of their fundamental invariants. This results in the following theorem.

**Theorem 7.** Let \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) be a hyperbolic distribution defined by an \( f \)-Gordon equation which is Darboux integrable at order two. Then the corresponding Vessiot distributions of \( \Delta \) are always locally equivalent to the standard contact distributions on either \( J^2(\mathbb{R}, \mathbb{R}) \) or \( J^3(\mathbb{R}, \mathbb{R}) \), and the Vessiot group of \( \Delta \) is the prolongation of a contact symmetry group of dimension 1, 2, or 3 acting on \( J^1(\mathbb{R}, \mathbb{R}) \).

The fact that the Vessiot group of these equations is always the prolongation of a contact symmetry group allows us to give a correspondence between the infinitesimal action of the Vessiot group and the Lie algebras of vector fields in the plane originally classified by Lie [23] and then later completed by Olver, Kamran, and González-López [18]. These Lie algebras (up to dimension five) are listed in Appendix A.
We then prove the converse of Theorem 7.

**Theorem 8.** Let \( \hat{\mathcal{V}} \) and \( \check{\mathcal{V}} \) be the standard contact distributions on manifolds \( \hat{M} = J^{r-2}(\mathbb{R}, \mathbb{R}) \) and \( \check{M} = J^{s-2}(\mathbb{R}, \mathbb{R}) \) where \( r, s \geq 4 \). Let \( G \) be a common symmetry group of dimension \( r + s - 7 \) of \( \hat{\mathcal{V}}, \check{\mathcal{V}} \). Suppose that in addition to the hypothesis of Theorem 4, that \( r, s \leq 5 \). Then the quotient distribution

\[
\Delta = (\hat{\mathcal{V}} \oplus \check{\mathcal{V}})/G_{\text{diag}}
\]

is a rank 4 distribution defining an \( f \)-Gordon equation which is Darboux integrable at order two.

We conclude Chapter 8 by calculating the fundamental invariants for each of the equations listed in the Biesecker classification. These results are summarized in Table 8.3.

In [38], Zhiber and Sokolov summarize their efforts to classify all Darboux integrable \( f \)-Gordon equations, and in doing so, list two new examples of \( f \)-Gordon equations which are Darboux integrable at order three. Their first example is given by

\[
\frac{u_{xy}}{u} = \frac{P_1(u_x)Q_1(u_y)}{u} \tag{1.4}
\]

where \( P_1 = P_1(u_x) \) and \( Q_1 = Q_1(u_y) \) are defined implicitly by

\[
P_1P_1' + P_1 = 2u_x, \quad Q_1Q_1' + Q_1 = 2u_y,
\]

and the second example is given by

\[
\frac{u_{xy}}{u} = \frac{P_1^2(P_1 - 1)Q_1(Q_1 - 1)^2}{6u + y} + \frac{Q_1^2(Q_1 - 1)P_1(P_1 - 1)^2}{6u + x} \tag{1.5}
\]

where \( P_1 = P_1(u_x) \) and \( Q_1 = Q_1(u_y) \) are defined implicitly by

\[
\frac{1}{3}P_1^3 - \frac{1}{2}P_1^2 = u_x, \quad \frac{1}{3}Q_1^3 - \frac{1}{2}Q_2^2 = u_y.
\]
In Chapter 9, we calculate the fundamental invariants for (1.4) and (1.5) as well as for the classical equation of Goursat

\[ u_{xy} = \frac{4\sqrt{u_x u_y}}{x + y}. \]  

(1.6)

In particular, we find that the Vessiot distributions for (1.5) are each the standard contact distribution on \( J^4(\mathbb{R}, \mathbb{R}) \) while the Vessiot distributions for (1.4) and (1.6) are the locally equivalent to the first prolongation of the (2,3,5)-distribution corresponding to the Hilbert-Cartan equation \( z' = (y'')^2 \). This leads us to search for other equations with this structure. It is interesting to note as well that the action of the Vessiot group for (1.5) is given by the prolongation of the unique action which pseudo-stabilizes on \( J^3(\mathbb{R}, \mathbb{R}) \).

The equations (1.4), (1.5), and (1.6) together constitute a list of all previously known nonlinear \( f \)-Gordon equations which are Darboux integrable at order three. We then show how these equations can be reconstructed using the quotient theory of Darboux integrable systems.

In Chapter 10, we characterize the fundamental invariants of \( f \)-Gordon equations which are Darboux integrable at order three. We restrict ourselves to the most interesting case where each of the prolonged characteristic distributions admit exactly two first integrals and prove the following analogs of Theorems 7 and 8.

**Theorem 9.** Let \( \Delta \) be a hyperbolic distribution defined by an \( f \)-Gordon equation which is Darboux integrable at order three and whose prolonged characteristic distributions each admit exactly two first integrals. Then the corresponding Vessiot distributions of \( \Delta \) are either contact distributions on \( J^4(\mathbb{R}, \mathbb{R}) \) or the first prolongation of (2,3,5)-distributions. The Vessiot group of \( \Delta \) is either the prolongation of a 5-dimensional contact symmetry group on \( J^1(\mathbb{R}, \mathbb{R}) \) to \( J^4(\mathbb{R}, \mathbb{R}) \) acting with codimension 1 orbits on \( J^3(\mathbb{R}, \mathbb{R}) \) or the prolongation of a 5-dimensional symmetry group of a (2,3,5)-distribution with codimension 1 orbits.

**Theorem 10.** Let \( \mathcal{V} \) and \( \mathcal{V}' \) be two rank 2 distributions on 5-manifolds \( \tilde{M} \) and \( \tilde{M}' \) with nonintegrable first-derived systems and common 5-dimensional symmetry group \( G \). Let \( \mathcal{V}(1), \mathcal{V}'(1) \)
be the prolongations of \( \mathcal{V}, \mathcal{V} \) to the 6-manifolds, \( \mathcal{M}^{(1)}, \mathcal{M}^{(1)} \), and denote the infinitesimal generators of the diagonal action of \( G \) by \( \Gamma_{\text{diag}} \) and the prolongation of these infinitesimal generators by \( \Gamma_{\text{diag}}^{(1)} \). Suppose that

[i] \( G \) acts freely on \( \mathcal{M}^{(1)} \) and \( \mathcal{M}^{(1)} \),

[ii] \( \Gamma_{\text{diag}} \cap (\mathcal{V} \oplus \mathcal{V})' = \{0\} \),

[iii] \( \Gamma_{\text{diag}}^{(1)} \cap (\mathcal{V}^{(1)} \oplus \mathcal{V}^{(1)})'' \) is 1-dimensional, and

[iv] \( G \) acts on \( \mathcal{M} \) and \( \mathcal{M} \) with codimension 1 orbits.

Then the quotient distribution

\[
\Delta = (\mathcal{V}^{(1)} \oplus \mathcal{V}^{(1)}) / G_{\text{diag}}^{(1)}
\]

is a rank 4 distribution defining an \( f \)-Gordon equation which is Darboux integrable at order three and not at order two.

Theorem 10 allows for the construction of \( f \)-Gordon equations which are Darboux integrable at order three provided two rank 2 distributions and a corresponding 5-dimensional symmetry group satisfying conditions [i]–[iv]. When the distributions are locally equivalent to the standard contact distribution on \( J^3(\mathbb{R}, \mathbb{R}) \), all possible actions of their 5-dimensional symmetry groups are listed in Appendix A. Alternatively, these distributions can be \( (2,3,5) \)-distributions. Cartan famously classified these distributions in his 1910 “five variables” paper [12]. His classification is based upon the root types of an associated quartic polynomial in two variables, called the Cartan quartic.

When the Cartan quartic has an infinite number of roots, or is of root type \([\infty]\), then the distribution is locally equivalent to the distribution corresponding to the Hilbert-Cartan equation \( z' = (y'')^2 \). This distribution has 14-dimensional symmetry algebra given by the split-real form of the exceptional Lie algebra \( g_2 \). Doubrov gave a classification of all subalgebras of \( g_2 \) in [14]. We list the subalgebras of \( g_2 \) with dimension greater than or equal to three in Appendix B.
There are several other root types to consider, however, for the purpose of this dissertation, we restrict ourselves to the case above along with the case where the Cartan quartic admits a single root of multiplicity four. When this is the case, we say the distribution is of root type \([4]\). In sections 9.3 – 9.6, we give a classification of all 5-dimensional subalgebras of the symmetry algebras for distributions of root type \([4]\).

After finding those 5-dimensional symmetry algebras corresponding to each type of rank 2 distribution described above, we present our main result concerning \(f\)-Gordon equations which are Darboux integrable at order three.

**Theorem 11.** Let \(\tilde{\mathcal{V}}, \check{\mathcal{V}}\) be two copies of the same rank 2-distribution \(\mathcal{V}\) defined on 5-dimensional manifolds \(\tilde{\mathcal{M}}, \check{\mathcal{M}}\) with common symmetry group \(G\), and let \(G^{(1)}_{\text{diag}}\) denote the first prolongation of diagonal action of \(G\). Then the quotient distribution

\[
\Delta = (\tilde{\mathcal{V}}^{(1)} \oplus \check{\mathcal{V}}^{(1)}) / G^{(1)}_{\text{diag}}
\]

defines a nonlinear \(f\)-Gordon equation which is Darboux integrable after one prolongation in the following cases:

[I] \(\mathcal{V}\) has derived dimensions \((2,3,4,5)\) and the action of \(G\) has infinitesimal generators

\[
\Gamma = \{ \partial_x, x\partial_x + 3y\partial_y + 2y_1\partial_{y_1} + y_2\partial_{y_2}, \partial_y, x\partial_y + \partial_{y_1}, x^2\partial_y + 2x\partial_{y_1} + 2\partial_{y_2} \}.
\]

[II] \(\mathcal{V}\) has derived dimensions \((2,3,5)\), the Cartan tensor vanishes identically, and the action of \(G\) has infinitesimal generators given by

\[
[N, 23], \ [S, 14], \ [S, 46]_{a=2}, \ [S, 49]_{a=2}, \text{ or } [S, 51]
\]

in Table B.3.

[III] \(\mathcal{V}\) has derived dimensions \((2,3,5)\), the Cartan tensor has a single root of multiplicity four with constant fundamental invariant \(I^2 \neq -1, \frac{9}{16}\), and the action of \(G\) has infinitesimal generators given by \(a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, \text{ or } a_{18}\).
in Table 10.1.

[IV] \( V \) has derived dimensions \((2,3,5)\), the Cartan tensor has a single root of multiplicity four with constant fundamental invariant \( I^2 = -1 \), and the action of \( G \) has infinitesimal generators given by \( b_1, b_6, b_7, b_{10}, b_{11}, b_{13}, b_{14} \), or \( b_{15} \) in Table 10.2.

[V] \( V \) has derived dimensions \((2,3,5)\), the Cartan tensor has a single root of multiplicity four with constant fundamental invariant \( I^2 = \frac{9}{16} \), and the action of \( G \) has infinitesimal generators given by \( c_1, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17} \), or \( c_{18} \) in Table 10.3.

[VI] \( V \) has derived dimensions \((2,3,5)\), the Cartan tensor has a single root of multiplicity four with non-constant fundamental invariant \( I \), and the action of \( G \) has infinitesimal generators given by \( d_1, d_2, d_3, d_4, d_5 \) in Table 10.5.

We note that Zhiber and Sokolov conjectured that with the addition of (1.4) and (1.5) their list constitutes a complete classification of all Darboux integrable \( f \)-Gordon equations, however, they do remark that

“the computations are extremely tedious, and the complete proof of the classification theorem is about 200 pages long, so there is some small chance of error in the computations.”

Not only does Theorem 11 recover the previously known equations (1.4), (1.5), and (1.6), but it also greatly expands the list of inequivalent nonlinear \( f \)-Gordon equations which are Darboux integrable at order three. We conclude Chapter 10 by explicitly writing a new Darboux integrable \( f \)-Gordon equation coming from this theorem as well as several other examples in terms of their representative distributions.

In Part IV, we present various results obtained in addition to those of Parts II and III. In Chapter 11, we note that every Monge-Ampère equation given by Goursat which is equivalent to an \( f \)-Gordon equation. We then give the representative distributions for two equations of strict Monge-Ampère type in the sense that they are not equivalent to
an $f$-Gordon equation. The first is Darboux integrable at order four, and the second is Darboux integrable at order five. The construction of these equations naturally leads to a family of strict Monge-Ampère equations which are Darboux integrable at order $k \geq 2$.

In Chapter 12, we give the fundamental invariants for two Darboux integrable equations of Goursat type. The first equation is given by

\[ u_{xx} = f(u_{xy}) \quad \text{with} \quad f_s \neq 0, \]

and has a particularly interesting Vessiot distribution whose root type varies depending on the forms of $f$. The second equation is given by

\[ u_{xx}u_{xy} = u_x, \]

and has a Vessiot distribution corresponding to the equation $z'' = (y'')^2$. This distribution has growth vector $(2,3,5,6)$ and motivates a study of the equivalence problem for $(2,3,5,6)$-distributions on 6-manifolds.

In Chapter 13, we give the fundamental invariants and quotient construction of two Darboux integrable equations of generic type. The first is the classical equation

\[ 3u_{xx}u_{yy}^3 + 1 = 0 \quad (1.7) \]

which was shown by The [32] to be one of two hyperbolic equations of generic type, Darboux integrable at order two, with maximal 9-dimensional symmetry algebra. The second equation is given by

\[ 3u_{xx}u_{yy}^3 = u_{xy}^3 \quad (1.8) \]

and is an example of a Darboux integrable hyperbolic equation of generic type with 7-dimensional symmetry algebra. The Vessiot distributions for each of these equations are locally equivalent to the Hilbert-Cartan distribution. The action of the Vessiot group for
(1.7) is given by $[N, 11]$ in Table B.1 while the action of Vessiot group for (1.8) is given by $[N, 12]$ in Table B.1.

Finally, in Chapter 14, we present several projects inspired by this work which we have partially completed and hope to finish in the near future.
CHAPTER 2
GEOMETRY OF DISTRIBUTIONS

In this chapter, we give a brief summary of the theory of general vector field distributions and distribution of hyperbolic type. In Section 2.2, we describe how hyperbolic distributions can be encoded using dual Pfaffian systems. In Section 2.3, we address the necessary and sufficient conditions for a hyperbolic distribution to define a scalar second-order hyperbolic partial differential equation in one dependent and two independent variables. We then review the conditions for when a hyperbolic distribution defines a PDE in the plane of generic, Goursat, or Monge-Ampère type as defined by Kamran and Gardner [17]. In Section 2.4, we further specify conditions for a hyperbolic distribution to define an equation of $f$-Gordon type. Finally, since our study of the fundamental invariants of Darboux integrable PDE in the plane relies heavily on the ability to characterize rank 2 distributions as either contact distributions, (2,3,5)-distributions, or prolongations thereof, we conclude this chapter by discussing some normal forms for distributions of these types.

2.1 Distributions and Hyperbolic Structures

In this section, we review the basic theory of vector field distributions and hyperbolic structures. Further details on this theory can be found in [30].

Given a smooth, nonvanishing vector field $X$ defined on a manifold $M$, an integral curve of $X$ is a differentiable curve $\gamma : (a, b) \subseteq \mathbb{R} \rightarrow M$ whose tangent vector at each point $p$ on $\gamma$ coincides with $X$ at $p$; that is,

$$\gamma'(t) = X_{\gamma(t)} \quad \text{for all } t \in (a, b).$$

The image of an integral curve $\gamma$ of $X$ forms an immersed submanifold $N \subseteq M$ with the property that for each point $p \in N$, the tangent space, $T_pN$, of $N$ at $p$ is spanned by $X_p$. We can discuss an important generalization of this idea in which for each point $p$ in
there exist \( k \) linearly independent vectors such that if \( N \) is a \( k \)-dimensional, immersed submanifold of \( M \), the tangent space \( T_pN \) is precisely the space spanned by these vectors.

To formalize this viewpoint, we first introduce the concept of a distribution on a manifold \( M \).

**Definition 2.1.1.** A rank \( k \) distribution \( \Delta \) on an \( m \)-dimensional manifold \( M \) is an assignment to each point \( p \in M \) a \( k \)-dimensional subspace \( \Delta_p \) of \( T_pM \). Such a distribution is said to be smooth if for each \( p \in M \), there is a neighborhood \( U \) of \( p \) and a local basis consisting of \( k \) smooth vector fields \( X_1, \ldots, X_k \) on \( U \) which span \( \Delta_q \) for each point \( q \in U \).

**Definition 2.1.2.** Let \( \Delta \) be a distribution on a manifold \( M \). An immersed submanifold \( (N, \psi : N \to M) \) of \( M \) is an integral manifold of a distribution \( \Delta \) if

\[
\psi_*(T_qN) = \Delta_{\psi(q)} \quad \text{for all} \ q \in N.
\]

Often, one says that a vector field \( X \) on \( M \) belongs to (or takes values in) the distribution \( \Delta \) if \( X_p \in \Delta_p \) for each \( p \in M \), and we will occasionally write \( X \in \Delta \) for brevity. We say \( \Delta \) is involutive if for any two vector fields \( X, Y \) on \( M \) taking values in \( \Delta \), the vector field \( Z = [X, Y] \) also takes values in \( \Delta \). The following theorem, due to Frobenius, states that if \( \Delta \) is to have an integral manifold through every point on \( M \), it is both necessary and sufficient that \( \Delta \) be involutive.

**Theorem 2.1.3** (Frobenius). Let \( \Delta \) be a rank \( k \), smooth distribution on \( M \). Then there exists an integral manifold of \( \Delta \) passing through each point \( p \in M \) if and only if \( \Delta \) is involutive. Moreover, there exists a cubical coordinate system \( (U, \varphi) \), centered at each \( p \), with coordinate functions \( x^1 \ldots x^m \) such that the slices \( x^j = a^j \) are integral manifolds of \( \Delta \) for constants \( a^j \) with \( k + 1 \leq j \leq m \).

A function \( f \in C^\infty(M) \) is called a first integral of \( \Delta \) if \( X(f) = 0 \) for all \( X \in \Delta \). A consequence of the Frobenius theorem is that if \( \Delta \) is involutive, then \( \Delta \) has a local basis
consisting entirely of coordinate vector fields such that

\[ \Delta_p = \text{span} \left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \ldots, \left. \frac{\partial}{\partial x^k} \right|_p \right\}, \]

and there are precisely \( m - k \) first integrals of \( \Delta \) given by the coordinate functions \( x^{m+1}, \ldots, x^k \).

We note that the first integrals of a distribution can be computed using the command \texttt{FirstIntegrals} in the \texttt{DifferentialGeometry} Maple package.

An important subdistribution of \( \Delta \) is generated by the set of \textit{Cauchy characteristic vector fields} (or simply Cauchy characteristics) of \( \Delta \).

\textbf{Definition 2.1.4.} Let \( \Delta \) be a distribution on a manifold \( M \). Then the set of Cauchy characteristics of \( \Delta \) is given by

\[ \mathcal{A}(\Delta) = \{ Z \in \Delta \mid [Z, X] \in \Delta \text{ for all } X \in \Delta \}. \]

If we let \( \mathcal{I} = \Delta^\perp \) be the dual Pfaffian system generated by 1-forms \( \{ \theta^i \} \), then we can alternatively define the set of Cauchy characteristics of \( \Delta \) as

\[ \mathcal{A}(\Delta) = \{ Z \in \Delta \mid Z \perp \theta = 0, Z \perp d\theta \in \mathcal{I} \text{ for all } \theta \in \mathcal{I} \}. \]

The set of Cauchy characteristics for a distribution can be computed using the command \texttt{CauchyCharacteristics} in the \texttt{DifferentialGeometry} Maple package.

\textbf{Theorem 2.1.5.} \( \mathcal{A}(\Delta) \) is involutive.

\textit{Proof.} Let \( Z_1, Z_2 \in \mathcal{A}(\Delta) \), and let \( X \in \Delta \). Then by the Jacobi identity,

\[ [[Z_1, Z_2], X] = -[[Z_2, X], Z_1] - [[X, Z_1], Z_2]. \]

Since \( Z_1, Z_2 \in \mathcal{A}(\Delta) \), the brackets \([Z_2, X]\) and \([X, Z_1]\) remain in \( \Delta \). It then follows that \([ [Z_2, X], Z_1] \) and \([ [X, Z_1], Z_2] \) are in \( \Delta \) as well, and we conclude that \([ [Z_1, Z_2], X] \in \Delta \). Therefore \([ Z_1, Z_2 ] \in \mathcal{A}(\Delta) \). \( \square \)
Suppose $\Delta$ is a rank $k$ distribution generated by $X_1, \ldots, X_k$. Then the first derived distribution $\Delta' = \Delta^{(1)}$ of $\Delta$ is a distribution generated by $X_i$ and $[X_i, X_j]$ for $1 \leq i, j \leq k$, or more succinctly,

$$\Delta' = [\Delta, \Delta] + \Delta.$$

The $k^{\text{th}}$ derived distribution of $\Delta$ is then defined recursively as

$$\Delta^{(k)} = \left[\Delta^{(k-1)}, \Delta^{(k-1)}\right] + \Delta^{(k-1)}, \quad k \geq 1,$$

where $\Delta^{(0)} = \Delta$. Similarly, we can also recursively define the $k^{\text{th}}$ weak derived distribution of a distribution $\Delta$ to be

$$\Delta_{(k)} = \left[\Delta_{(0)}, \Delta_{(k-1)}\right] + \Delta_{(k-1)}, \quad k \geq 1,$$

where $\Delta_{(0)} = \Delta$.

The nested sequence of derived and weak derived distributions of $\Delta$ are respectively called the derived and weak derived flags of $\Delta$. The derived flag of $\Delta$ can be computed using the command $\text{DerivedFlag}$ in the $\text{DifferentialGeometry}$ Maple package. The weak derived flag can be computed using the $\text{DerivedFlag}$ command with the optional keyword argument $\text{flagtype} = '\text{WeakDerivedFlag}'$.

Throughout this dissertation, we assume the derived flag is regular in that the rank of each derived distribution remains constant on all of $M$. If there exists an integer $\ell$ such that $\Delta^{(\ell+1)} = \Delta^{(\ell)}$, then we write $\Delta^{(\infty)} = \Delta^{(\ell+1)} = \Delta^{(\ell)}$. A distribution on $M$ is called bracket generating $\Delta^{(\infty)} = TM$. At the other extreme, a distribution $\Delta$ is involutive if $\Delta^{(\infty)} = \Delta$. With this definition, we can make the following theorem concerning the number of first integrals associated to a distribution on a manifold.

**Theorem 2.1.6.** The number of first integrals of a distribution $\Delta$ on an $m$-dimensional manifold $M$ is given by $m - \text{rank} \Delta^{(\infty)}$. 
In particular, a bracket-generating distribution has no first integrals.

An ordered set of linearly independent vector fields on an \( m \)-dimensional manifold \( M \) which span \( T_pM \) for all \( p \in M \) is called a frame on \( M \). If \( \Delta \) is a rank \( k \) distribution with local basis \( \{X_1, \ldots, X_k\} \) on \( M \), and \( \mathcal{F} = \{X_1, \ldots, X_k, Z_{k+1}, \ldots Z_m\} \) is a frame on \( M \), then we say \( \mathcal{F} \) is a frame adapted to \( \Delta \). Additionally, one can further adapt a frame on \( M \) to the derived distributions of \( \Delta \). For example, if \( \Delta' = \{X_1, \ldots, X_k; Y_{k+1}, \ldots Y_{\ell}\} \), then we say that the frame \( \mathcal{F} = \{X_1, \ldots, X_k; Y_{k+1}, \ldots Y_{\ell}; Z_{\ell+1}, \ldots Z_m\} \) is adapted to \( \Delta' \). Continuing in this manner, one can adapt the frame to \( \Delta^{(\infty)} \).

We will primarily be interested in distributions of hyperbolic type.

**Definition 2.1.7.** A rank \( k \), bracket generating distribution \( \Delta \) is hyperbolic if there exists two subdistributions \( \hat{\Delta} \) and \( \bar{\Delta} \) of rank \( r \) and \( s \), respectively, such that \( k = r + s \) and

\[
\Delta = \hat{\Delta} \oplus \bar{\Delta} \quad \text{with} \quad [\hat{\Delta}, \bar{\Delta}] \subseteq \Delta.
\]

We note that, in general, determining if a distribution is hyperbolic can be a difficult task. However, Stormark [30] gives a way to determine if a distribution is hyperbolic using so-called singular vector fields which we now define. The proof of the subsequent theorem can found in [30] as well.

**Definition 2.1.8.** Let \( \Delta = \{X_1, \ldots, X_k\} \) be a rank \( k \) distribution on \( M \) whose derived system is of the form \( \Delta' = \{X_1, \ldots, X_k; Z_1, \ldots, Z_\ell\} \) where \( [X_i, X_j] = c^b_{ij}Z_b \mod \Delta \). Let \( Y = a^kX_k \) be a nonzero vector in \( \Delta \), and define the involution matrix

\[
\mathfrak{S}(Y) = \begin{bmatrix}
\sum_{i=1}^k c^1_{i1}a^i & \cdots & \sum_{i=1}^k c^1_{i\ell}a^i \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^k c^\ell_{i1}a^i & \cdots & \sum_{i=1}^k c^\ell_{i\ell}a^i
\end{bmatrix}.
\]

The character of \( \Delta \) is defined by

\[
\rho = \max_{Y \in \Delta} \text{rank } \mathfrak{S}(Y),
\]
and the vector field $Y$ is said to be *singular* if rank $\mathfrak{S}(Y) < \rho$ or rank $\mathfrak{S}(Y) = 0$. In the case where rank $\mathfrak{S}(Y) = 0$, the vector field $Y$ is a Cauchy characteristic of $\Delta$.

**Theorem 2.1.9.** If $\Delta$ is a hyperbolic distribution, then there exists a local basis of singular vector fields for $\Delta$.

We conclude this section by presenting several examples of hyperbolic distributions. In particular, we show how hyperbolic first order systems of PDE in two independent and two dependent variables as well as hyperbolic second order scalar PDE in the plane can be encoded as rank 4 hyperbolic distributions.

**Example 2.1.10.** As a trivial example, let $\hat{\Delta}$ and $\check{\Delta}$ be two smooth, Frobenius distributions on manifolds $\hat{M}$ and $\check{M}$ of rank 2 and 3, respectively, and suppose that $\hat{M}$ has local coordinates $x^1, x^2$ and $M_2$ has local coordinates $y^1, y^2, y^3$. Then,

\[
\hat{\Delta} = T\hat{M} = \text{span}\{\partial_{x^1}, \partial_{x^2}\} \quad \text{and} \quad \check{\Delta} = T\check{M} = \text{span}\{\partial_{y^1}, \partial_{y^2}, \partial_{y^3}\},
\]

and we can define the rank 5 distribution

\[
\Delta = \hat{\Delta} \oplus \check{\Delta} = \text{span}\{\partial_{y^1}, \partial_{y^2}, \partial_{y^3}, \partial_{x^1}, \partial_{x^2}\}.
\]

Immediately one sees that rank $\mathfrak{S}(X) = 0$ for all $X \in \Delta$, so that $\Delta$ has a basis of (singular) Cauchy characteristic vector fields. Moreover, $[\hat{\Delta}, \check{\Delta}] = \{0\} \subset \Delta$, and $\Delta$ is immediately bracket generating, so $\Delta$ forms a hyperbolic distribution on $\hat{M} \times \check{M}$.

**Example 2.1.11 (First Order Systems).** As an example, consider the hyperbolic system of first order PDE in the plane discussed in [11],

\[
u_x = f(x, y, u, v, u_y, v_x), \quad v_y = g(x, y, u, v, u_y, v_x),
\]

where, in this instance, hyperbolic means

\[
4f_{v_x}g_{u_y} - (1 + f_{u_y}g_{v_x} - f_{v_x}g_{u_y})^2 < 0.
\]
The usual coding of (2.1) as a Pfaffian system is

\[ \theta^1 = du - f dx - u_y dy = 0, \]
\[ \theta^2 = dv - v_x dx - g dy = 0 \]

on a 6-dimensional manifold with coordinates \( x, y, u, v, u_y, \) and \( v_x. \) The distribution formulation of (2.1) is then

\[ \Delta = \{ X_1, X_2, X_3, X_4 \} \]

where \( \theta^i(X_j) = 0 \) for \( 1 \leq i \leq 2. \) For the sake of this example, let us take

\[ X_1 = \partial_x + f \partial_u + v_x \partial_v, \quad X_2 = \partial_y + u_y \partial_u + g \partial_v, \quad X_3 = \partial_{u_y}, \quad X_4 = \partial_{v_x}, \]
as a basis for \( \Delta. \) The singular vector fields \( Y = a^i X_i \) of \( \Delta \) are determined by requiring all \( 2 \times 2 \) minors of the involution matrix

\[ \mathfrak{S}(Y) = \begin{bmatrix} D_y(f)a_2 + f_{uy}a_3 + f_{vx}a_4 & -D_y(f)a_1 + a_3 & -f_{uy}a_1 - a_2 & -f_{vx}a_1 \\ -D_x(g)a_2 + a_4 & D_x(g)a_1 + g_{uy}a_3 + g_{vx}a_4 & -g_{uy}a_2 & -a_1 - g_{vx}a_2 \end{bmatrix} \]

vanish.

The minor formed by the last two columns of \( \mathfrak{S}(Y) \) gives the quadratic condition

\[ f_{uy}a_1^2 + (1 + f_{uy}g_{vx} - f_{vx}g_{uy})a_1a_2 + g_{vx}a_2^2 = 0. \] (2.2)

The discriminant of (2.2) is given by

\[ \mathcal{D} = (1 + f_{uy}g_{vx} - f_{vx}g_{uy})^2 - 4 f_{uy}g_{vx} = (1 + f_{uy}g_{vx} - f_{vx}g_{uy})^2 - 4 f_{vx}g_{uy} > 0, \]
and (2.2) guaranteed two distinct, non-proportional real roots \((a_1, a_2) = (\mu, \lambda)\) and \((a_1, a_2) = (\tilde{\mu}, \tilde{\lambda})\). These roots give the factorization
\[
(\lambda a_1 - \tilde{\mu} a_2)(\lambda a_1 - \mu a_2) = \kappa \left( f_{uv} a_1^2 + (1 + f_{uy} g_{uv} - f_{vx} g_{uy}) a_1 a_2 + g_{vx} a_2^2 \right),
\]
where \(\kappa\) is some nonvanishing function satisfying the relations
\[
\lambda \tilde{\lambda} = \kappa f_{uv}, \quad \lambda \tilde{\mu} + \lambda \mu = -\kappa (1 + f_{uy} g_{uv} - f_{vx} g_{uy}), \quad \mu \mu = \kappa g_{vx}.
\]

In the first branch, where \((a_1, a_2) = (\mu, \lambda)\), requiring the remaining \(2 \times 2\) minors of \(\mathfrak{X}(Y)\) vanish gives singular vector fields
\[
\begin{align*}
\hat{Y}_1 &= \tilde{\mu} X_1 + \tilde{\lambda} X_2 + \hat{A} X_3, \\
\hat{Y}_2 &= \tilde{B} X_3 + 2 f_{uv} g_{uy} X_4,
\end{align*}
\]
and similarly, the second branch yields singular vector fields
\[
\begin{align*}
\tilde{Y}_1 &= \mu X_1 + \lambda X_2 + \tilde{A} X_4, \\
\tilde{Y}_2 &= 2 f_{vx} g_{uy} X_3 + \tilde{B} X_4.
\end{align*}
\]
where
\[
\begin{align*}
\hat{A} &= -\frac{1}{g_{uy}} (\tilde{\lambda} + 1) X_1 (g) - 2 f_{vx} g_{uv} X_2 (f), \\
\tilde{A} &= -\frac{1}{f_{vx}} (\tilde{\mu} + 2 f_{vx} g_{uv}) X_2 (f) - 2 X_1 (g), \\
\hat{B} &= \tilde{B} = 1 - f_{vx} g_{uy} - f_{uy} g_{vx} + \sqrt{(1 + f_{uv} g_{vx} - f_{vx} g_{uv})^2 - 4 f_{uv} g_{vx}}.
\end{align*}
\]

Since these four singular vector fields are linearly independent, they define a basis for \(\Delta\), and \(\Delta\) decomposes as the direct sum \(\Delta = \hat{\Delta} \oplus \tilde{\Delta}\) where
\[
\hat{\Delta} = \{\hat{Y}_1, \hat{Y}_2\} \quad \text{and} \quad \tilde{\Delta} = \{\tilde{Y}_1, \tilde{Y}_2\}.
\]
It can also be shown that $\Delta$ is bracket generating, and the bracket relations satisfy

$$[\hat{Y}_i, \hat{Y}_j] \in \Delta, \quad 1 \leq i, j \leq 2.$$ 

Therefore, $\Delta$ is a hyperbolic distribution.

**Example 2.1.12 (Second Order Scalar PDE in the Plane).** Given a second order PDE in the plane

$$F(x, y, u, p, q, r, s, t) = 0, \quad (2.3)$$

where here we have used the classical Monge notation $p = u_x, q = u_y, r = u_{xx}, s = u_{xy}$, and $t = u_{yy}$, we say (2.3) is $\text{hyperbolic}$, $\text{parabolic}$, or $\text{elliptic}$ at some point $p \in \mathbb{R}^8$ depending on whether the discriminant of (2.3)

$$\mathcal{D}_F = F_s^2 - 4F_rF_t$$

is positive, zero, or negative at $p$. If (2.3) is hyperbolic on some open neighborhood $U$ for which

$$(F_r, F_s, F_t) \neq 0,$$

then at least one of the second order derivatives may be solved for explicitly, and (2.3) can be encoded as a hyperbolic distribution on a 7-dimensional manifold. We will show how this can be done in two cases. In the first, we suppose $r$ may explicitly be solved for in (2.3) so that $F = r - f(x, y, u, p, q, s, t) = 0$, and in the second, we suppose that we can solve for $s$ so that $F = s - f(x, y, u, p, q, r, t) = 0$, noting that the two cases are actually equivalent up to a linear change of independent variables. The hyperbolic distribution for a scalar second order PDE in the plane can be calculated using the command `HyperbolicDistribution` in the `DifferentialGeometry` Maple package.
**Case 1:** \( r = f(x, y, u, p, q, s, t) \)

Consider the case where

\[
F = r - f(x, y, u, p, q, s, t) = 0.
\]  

The usual coding of (2.4) is as a Pfaffian system generated by the 1-forms

\[
\begin{align*}
\theta^0 &= du - p\,dx - q\,dy, \\
\theta^1 &= dp - f\,dx - s\,dy, \\
\theta^2 &= dq - s\,dx - t\,dy,
\end{align*}
\]

on a 7-dimensional manifold \( M \) with coordinates \( x, y, u, p, q, s, \) and \( t \). The annihilator of this Pfaffian system forms a rank 4 distribution

\[
\Delta = \{X_1, X_2, X_3, X_4\}
\]

on \( M \) which equivalently encodes (2.4). Let us take for a basis of \( \Delta \),

\[
X_1 = \partial_x + p\partial_u + f\partial_p + s\partial_q, \quad X_2 = \partial_y + q\partial_u + s\partial_p + t\partial_q, \quad X_3 = \partial_s, \quad X_4 = \partial_t.
\]

The singular vector fields \( Y = a^i X_i \) of \( \Delta \) are determined by requiring all 2 \( \times 2 \) minors of the involution matrix

\[
\mathfrak{S}(Y) = \begin{bmatrix}
D_y(f) a^2 + f_s a^3 + f_t a^4 & -D_y(f) a^1 + a^3 & -f_s a^1 - a^2 & -f_t a^1 \\
 a^3 & a^4 & -a^1 & -a^2
\end{bmatrix}
\]

vanish. Setting the minor formed by the last two columns of \( \mathfrak{S}(Y) \) to zero gives the quadratic condition

\[
(a^2)^2 + f_s a^1 a^2 - f_t (a^1)^2 = 0.
\]
If we make the substitution $a^2 = \lambda a^1$, then we get the equation
\[
\lambda^2 + f_s \lambda - f_t = 0,
\]
and since we assume (2.3) is hyperbolic,
\[
f_s^2 + 4f_t > 0,
\]
(2.5) is guaranteed two distinct real roots $\hat{\lambda}$ and $\check{\lambda}$. Moreover, we note that $\hat{\lambda} + \check{\lambda} = -f_s$ and $\hat{\lambda} \cdot \check{\lambda} = -f_t$.

The minor formed by the second and third column vanishes provided
\[
D_y(f)(a^1)^2 - a^1 a^3 + (\lambda + f_s) a^1 a^4 = 0,
\]
and we may solve for $a^3$ giving
\[
a^3 = D_y(f) a^1 + (\lambda + f_s) a^4.
\]

Upon substituting
\[
(a^1, a^2, a^3, a^4) = (\alpha, \lambda \alpha, D_y(f) \alpha + (\lambda + f_s) \beta, \beta)
\]
into $\mathfrak{g}(Y)$, where $\alpha$ and $\gamma$ are arbitrary functions on $M$, we see that all of the remaining $2 \times 2$ minors vanish. Taking $\lambda = \hat{\lambda}$, we have the singular vector field
\[
\hat{Y} = \alpha (X_1 + \hat{\lambda} X_2 + D_y(f) X_3) + \beta (X_4 - \check{\lambda} X_3),
\]
and similarly, when $\lambda = \check{\lambda}$, we have the singular vector field,
\[
\check{Y} = \alpha (X_1 + \check{\lambda} X_2 + D_y(f) X_3) + \beta (X_4 - \hat{\lambda} X_3).
Since the set of singular vector fields is closed (add proposition), we obtain two singular subdistributions by setting $\alpha$ or $\beta$ to either one or zero in both $\tilde{Y}$ and $\tilde{Y}$. This gives,

$$\hat{\Delta} = \{X_1 + \lambda X_2 + D_y(f)X_3, X_4 - \lambda X_3\},$$
$$\check{\Delta} = \{X_1 + \lambda X_2 + D_y(f)X_3, X_4 - \lambda X_3\}.$$ 

One can then easily verify that $\Delta$ is bracket-generating, decomposes as the direct sum $\Delta = \hat{\Delta} \oplus \check{\Delta}$, and its subdistributions $\hat{\Delta}$ and $\check{\Delta}$ satisfy $[\hat{\Delta}, \check{\Delta}] \subseteq \Delta$, meaning $\Delta$ is hyperbolic.

**Case 2: $s = f(x, y, u, p, q, r, t)$**

Now consider the case where $s$ can explicitly be solved for, so that (2.3) takes the form

$$F = s - f(x, y, u, p, q, r, t) = 0. \quad (2.7)$$

As in Case 1, (2.7) is typically encoded as a Pfaffian system generated by 1-forms

$$\begin{align*}
\theta^0 &= du - px - qdy, \\
\theta^1 &= dp - rx - fdy, \\
\theta^2 &= dq - fx - td, \\
\theta^3 &= dx - tdy,
\end{align*}$$

on a 7-dimensional manifold with coordinates $x, y, u, p, q, r, t$. The annihilator of this Pfaffian system forms a rank 4 distribution

$$\Delta = \{X_1, X_2, X_3, X_4\}$$

which equivalently encodes (2.7) where we have taken

$$X_1 = \partial_x + p\partial_u + r\partial_p + f\partial_q, \quad X_2 = \partial_y + q\partial_u + f\partial_p + t\partial_t, \quad X_3 = \partial_r, \quad X_4 = \partial_t.$$
From here, one can again find singular vector fields \( Y = a^i X_i \), and therefore show that \( \Delta \) is a hyperbolic distribution, by requiring that all \( 2 \times 2 \) minors of the involution matrix

\[
\mathcal{I}(Y) = \begin{bmatrix}
-D_x f a^2 + a^3 & D_x f a^1 + f_r a^3 + f_t a^4 & -a^1 - f_r a^2 & -f_t a^2 \\
D_y f a^2 + f_r a^3 + f_t a^4 & -D_y f a^1 + a^4 & -f_r a^1 & -f_t a^1 - a^2
\end{bmatrix}
\]

vanish. But in doing so, we find that the formulas for these singular vector fields are quite unsavory. Instead, we propose introducing a linear change of independent variables \( \Phi \) such that

\[
\bar{x} = \alpha x + \beta y \quad \text{and} \quad \bar{y} = \gamma x + \delta y
\]

with \( \alpha \gamma - \beta \delta \neq 0 \). After applying this transformation, (2.7) becomes

\[
\bar{F} = \bar{r} - \bar{f}(\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}, \bar{s}, \bar{t}) = 0,
\]

so that we again arrive at an equation of the form (2.4), and as previously shown, the singular subdistributions associated with (2.8) are of the form

\[
\hat{\Delta} = \{ \hat{Y}_1 = \hat{X}_1 + \lambda \hat{X}_2 + D_y f \hat{X}_3, \hat{Y}_2 = \hat{X}_4 - \lambda \hat{X}_3 \},
\]

\[
\check{\Delta} = \{ \check{Y}_1 = \check{X}_1 + \check{\lambda} \check{X}_2 + D_y \check{f} \check{X}_3, \check{Y}_2 = \check{X}_4 - \check{\lambda} \check{X}_3 \},
\]

where here we have set

\[
\check{X}_1 = \partial_x + \bar{p} \partial_u + \bar{f} \partial_p + \bar{s} \partial_q, \quad \check{X}_2 = \partial_y + \bar{q} \partial_u + \bar{s} \partial_p + \bar{t} \partial_q, \quad \check{X}_3 = \partial_s, \quad \check{X}_4 = \partial_t.
\]

Since the transformation \( \Phi \) is a diffeomorphism, we obtain well-defined singular subdistributions for (2.7)

\[
\hat{\Delta} = \{ \Phi^{-1}_r \hat{Y}_1, \Phi^{-1}_r \hat{Y}_2 \} \quad \text{and} \quad \check{\Delta} = \{ \Phi^{-1}_r \check{Y}_1, \Phi^{-1}_r \check{Y}_2 \}
\]
such that $\Delta = \tilde{\Delta} \oplus \check{\Delta}$ is again hyperbolic.

**Remark.** More classically, one can also encode second order PDE in the plane as hyperbolic distributions by decomposing their so-called *characteristic vector fields* into pairs of singular vector fields; the disadvantage being that one must find (or more often guess) an initial pair of singular vector fields in the process.

The *characteristic vector fields* of (2.3) are, in general, given by

$$\hat{X} = m_x D_x + m_y D_y, \quad \text{and} \quad \check{X} = n_x D_x + n_y D_y$$

where $D_x$ and $D_y$ are total differential operators

$$D_x = \partial_x + p\partial_u + r\partial_p + s\partial_q + u_{xxx}\partial_r + u_{xxy}\partial_s + u_{xyy}\partial_t + \cdots, \quad D_y = \partial_y + q\partial_u + t\partial_q + u_{xxy}\partial_r + u_{xyy}\partial_s + u_{yyy}\partial_t + \cdots,$$

and the pairs $(\mu, \lambda) = (m_x, m_y)$ and $(\mu, \lambda) = (n_x, n_y)$ are distinct (non-proportional) real roots of the *characteristic equation* associated to (2.3),

$$F_r \lambda^2 - F_s \lambda \mu + F_t \mu^2 = 0. \quad (2.9)$$

If we again consider the case where $r = f(x,y,u,p,q,s,t)$, the characteristic equation for (2.4) becomes the quadratic,

$$\lambda^2 + f_s \lambda \mu - f_t \mu^2 = 0,$$

which has roots

$$(\mu, \lambda) = (1, \tilde{\lambda}) \quad \text{and} \quad (\mu, \lambda) = \left(1, \check{\lambda}\right)$$
where
\[
\tilde{\lambda} = \frac{f_s + \sqrt{f_s^2 + 4f_t}}{2} \quad \text{and} \quad \tilde{\lambda} = \frac{f_s - \sqrt{f_s^2 + 4f_t}}{2}.
\]

We may then explicitly write the characteristic vector fields of (2.4) as
\[
\begin{align*}
\hat{X} &= D_x^{(0)} + \hat{\lambda}D_y^{(0)} + D_y^{(0)}(f)\partial_s + (u_{xyy}f_s + u_{yyy}f_t)\partial_s + u_{xyy}(\partial_t + \hat{\lambda}\partial_s) + \hat{\lambda}u_{yyy}\partial_t, \\
\check{X} &= D_x^{(0)} + \check{\lambda}D_y^{(0)} + D_y^{(0)}(f)\partial_s + (u_{xyy}f_s + u_{yyy}f_t)\partial_s + u_{xyy}(\partial_t + \check{\lambda}\partial_s) + \check{\lambda}u_{yyy}\partial_t,
\end{align*}
\]
where $D_x^{(0)}$ and $D_y^{(0)}$ are truncated total differential operators,
\[
D_x^{(0)} = \partial_x + p\partial_u + r\partial_p + s\partial_q, \quad \text{and} \quad D_y^{(0)} = \partial_y + q\partial_u + s\partial_p + t\partial_q.
\]

From here, we can introduce vector fields
\[
\hat{X}_2 = \partial_t - \hat{\lambda}\partial_s \quad \text{and} \quad \check{X}_2 = \partial_t - \check{\lambda}\partial_s
\]
which allow us to completely eliminate third order terms from $\hat{X}$ and $\check{X}$. In doing so, we obtain the vector fields
\[
\begin{align*}
\hat{X}_1 &= \hat{X} - (u_{xyy} + \hat{\lambda}u_{yyy})\hat{X}_2 = D_x^{(0)} + \hat{\lambda}D_y^{(0)} + D_y^{(0)}(f)\partial_s, \\
\check{X}_1 &= \check{X} - (u_{xyy} + \check{\lambda}u_{yyy})\check{X}_2 = D_x^{(0)} + \check{\lambda}D_y^{(0)} + D_y^{(0)}(f)\partial_s.
\end{align*}
\]

We can then form the rank 4 distribution $\Delta = \{\hat{X}_1, \hat{X}_2, \check{X}_1, \check{X}_2\}$ such that solutions to the equation $r = f(x, y, u, p, q, s, t)$ are in direct correspondence with the integral manifolds of $\Delta$.

We can then see that if $\hat{\Delta} = \{\hat{X}_1, \hat{X}_2\}$ and $\check{\Delta} = \{\check{X}_1, \check{X}_2\}$, $\Delta$ can then be decomposed as the direct sum
\[
\Delta = \hat{\Delta} \oplus \check{\Delta}.
\]
By calculating the bracket relations, we find that

\[ [\hat{X}_i, \hat{X}_j] \in \Delta, \quad 1 \leq i, j \leq 2, \]

so that \([\hat{\Delta}, \hat{\Delta}] \equiv 0 \mod \Delta\), thus showing \(\Delta\) is a hyperbolic distribution as desired. In this context, we call the subdistributions \(\hat{\Delta}, \hat{\Delta}\) the characteristic distributions for \(\Delta\).

It should be noted that simply choosing vector fields that allow us to eliminate all third order terms does not necessarily lead to a hyperbolic distribution. For example, one could have just as well taken \(\hat{\hat{X}}_2\) and \(\hat{\hat{X}}_2\) to be the coordinate vector fields \(\partial_s\) and \(\partial_t\). Again, by eliminating all third order terms, we would arrive at the same vector fields \(\hat{X}_1\) and \(\hat{X}_1\) as above, but

\[ [\partial_s, \hat{X}_1] = -\frac{1}{D^{(0)}_y(f)} Z_1 + \left( \frac{f_s^2 + f_s \sqrt{f_s^2 + 4f_t^2} + 4f_t}{2f_t \sqrt{f_s^2 + 4f_t^2}} \right) Z_2 \mod \Delta \]

where \(Z_1 = -D_y(f) \partial_p\) and \(Z_2 = -f_s(\partial_p + \partial_q)\), showing that the hyperbolic structure equations are indeed violated.

**Example 2.1.13 \((f\text{-Gordon Equations})\).** We can see from Case 2 of the previous example that \(f\)-Gordon equations of the form

\[ s = f(x, y, u, p, q) \]

have hyperbolic distribution \(\Delta = \hat{\Delta} \oplus \hat{\Delta}\) where

\[ \hat{\Delta} = \{ \partial_x + p \partial_u + r \partial_p + f \partial_q + D_y(f) \partial_t, \partial_r \}, \]

\[ \hat{\Delta} = \{ \partial_y + q \partial_u + f \partial_p + t \partial_q + D_x(f) \partial_r, \partial_t \}, \]

**Example 2.1.14 \((Prolongation of PDE in the Plane)\).** The prolongation of a hyperbolic PDE in the plane of the form (2.3) will also admit a hyperbolic distribution. To illustrate, though we do so somewhat informally, we consider the first prolongation of (2.4). As a
Pfaffian system, the prolongation of (2.4) has the coding

\[
\begin{cases}
\theta^0 = du - p\,dx - q\,dy, & \theta^1 = dp - f\,dx - s\,dy, & \theta^2 = dq - s\,dx - t\,dy, \\
\theta^3 = ds - D_y(f)\,dx - \sigma\,dy, & \\
\theta^4 = dt - \sigma\,dx - \tau\,dy,
\end{cases}
\]

on a 9-dimensional manifold \( M^{(1)} \) with coordinates \( x, y, u, p, q, s, t, \sigma, \tau \). Taking the annihilator of this Pfaffian system, we obtain the rank 4 distribution \( \Delta = \{ X_1, X_2, X_3, X_4 \} \), where

\[
\begin{align*}
X_1 &= \partial_x + p\,\partial_u + f\,\partial_p + s\,\partial_q + D_y(f)\,\partial_s + \sigma\,\partial_t, \\
X_2 &= \partial_y + q\,\partial_u + s\,\partial_p + t\,\partial_q + \sigma\,\partial_s + \tau\,\partial_t, \\
X_3 &= \partial_\sigma, \\
X_4 &= \partial_\tau.
\end{align*}
\]

The singular vector fields \( Y = a^i X_i \) of \( \Delta \) are obtained by requiring that all \( 2 \times 2 \) minors of the involution matrix

\[
\mathbb{S}(Y) = \begin{bmatrix}
D_y^2(f)\,a^2 & f_s\,a^3 & f_t\,a^1 & -D_y^2(f)\,a^1 + a^3 & -f_s\,a^1 - a^2 & -f_t\,a^1 \\
& a^3 & a^4 & -a^1 & -a^2
\end{bmatrix}
\]

vanish. A similar calculation as in Case 1 of Example 2.1.12 yields the singular vector fields

\[
\begin{align*}
\hat{Y} &= \alpha \left( X_1 + \tilde{\lambda} X_2 + D_y^2(f)\ X_3 \right) + \beta \left( X_4 - \tilde{\lambda} X_3 \right), \\
\bar{Y} &= \alpha \left( X_1 + \tilde{\lambda} X_2 + D_y^2(f)\ X_3 \right) + \beta \left( X_4 - \tilde{\lambda} X_3 \right),
\end{align*}
\]
from which we can obtain the singular subdistributions

\[ \hat{\Delta} = \{ X_1 + \lambda X_2 + D^2_y(f) X_3, \ X_4 - \lambda X_3 \}, \]
\[ \tilde{\Delta} = \{ X_1 + \lambda X_2 + D^2_y(f) X_3, \ X_4 - \lambda X_3 \}. \]

From here, we see that \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) is bracket-generating and its subdistributions \( \hat{\Delta} \) and \( \tilde{\Delta} \) satisfy the structure equations

\[ [\hat{\Delta}, \tilde{\Delta}] \equiv 0 \mod \Delta. \]

### 2.2 The Pfaffian System Approach

Dually, we can describe vector field distributions in terms of Pfaffian systems. In this section, we describe hyperbolic structures in the context of exterior differential systems, a generalization of Pfaffian systems. Further details on hyperbolic exterior differential systems can be found in [10].

**Definition 2.2.1.** A hyperbolic system of class \( s \) is given by an exterior differential system \((M, I)\) where \( M \) is a manifold of dimension \( s + 4 \) and \( I \) is a differential ideal such that for each point \( p \in M \) there exists a local coframe

\[ (\theta; \bar{\omega}, \tilde{\omega}) = (\theta^1, \ldots, \theta^s; \bar{\omega}^1, \bar{\omega}^2, \tilde{\omega}^1, \tilde{\omega}^2) \]

such that \( I \) is algebraically generated by 1-forms and 2-forms

\[ I = \{ \theta^1, \ldots, \theta^s; \bar{\omega}^1 \wedge \bar{\omega}^2, \tilde{\omega}^1 \wedge \tilde{\omega}^2 \}. \]

**Remark.** For \( s > 1 \), the distribution \( \Delta = (I)^\perp \) is hyperbolic and can be written as \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \).
The coframing \((\theta; \bar{\omega}, \tilde{\omega})\) is often referred to as an admissable local coframing for \(\mathcal{I}\). Such an admissable coframing will statisfy the structure equations

\[
d\theta^i \equiv A^i \omega^1 \wedge \omega^2 + B^i \bar{\omega}^1 \wedge \bar{\omega}^2 \mod \{\theta^i\},
\]

where \(A^i\) and \(B^i\) are smooth functions on \(M\).

A hyperbolic EDS \(\mathcal{I}\) admits two subsystems \(\hat{\Xi}\) and \(\bar{\Xi}\) of rank \(s + 2\) known as the characteristic systems of \(\mathcal{I}\) which are algebraically generated by

\[
\hat{\Xi} = \{\theta^1, \ldots, \theta^s, \omega^1, \bar{\omega}^2\} \quad \text{and} \quad \bar{\Xi} = \{\theta^1, \ldots, \theta^s, \bar{\omega}^1, \omega^2\}.
\]

The characteristic systems of \(\mathcal{I}\) are related the the characteristic distributions \(\hat{\Delta}, \bar{\Delta}\) of \(\Delta\) by \(\hat{\Xi} = \hat{\Delta}^\perp\) and \(\bar{\Xi} = \bar{\Delta}^\perp\).

Given the characteristic systems for a hyperbolic system \(\mathcal{I}\), we may choose independent tuples of 1-forms \(\hat{\eta}\) and \(\bar{\eta}\) which satisfy

\[
\hat{\Xi}^{(\infty)} \cap \bar{\Xi} = \text{span}\{\hat{\eta}\} \quad \text{and} \quad \hat{\Xi} \cap \bar{\Xi}^{(\infty)} = \text{span}\{\bar{\eta}\},
\]

and then choose tuples of 1-forms \(\theta, \hat{\sigma},\) and \(\bar{\sigma}\) such that

\[
\hat{\Xi}^{(\infty)} = \text{span}\{\hat{\sigma}, \hat{\eta}\}, \quad \bar{\Xi}^{(\infty)} = \text{span}\{\bar{\sigma}, \bar{\eta}\}, \quad \hat{\Xi} \cap \bar{\Xi} = \text{span}\{\theta, \hat{\eta}, \bar{\eta}\},
\]

and such that the sets of 1-forms \(\{\hat{\sigma}, \hat{\eta}\}, \{\bar{\sigma}, \bar{\eta}\}\), and \(\{\theta, \hat{\eta}, \bar{\eta}\}\) are linearly independent. In doing so, the 1-forms \(\{\theta, \hat{\sigma}, \hat{\eta}, \bar{\sigma}, \bar{\eta}\}\) form a local coframe on \(M\),

\[
\hat{\Xi} = \text{span}\{\theta, \hat{\sigma}, \hat{\eta}, \bar{\eta}\}, \quad \text{and} \quad \bar{\Xi} = \text{span}\{\theta, \hat{\eta}, \bar{\sigma}, \bar{\eta}\}.
\]

Any local coframe satisfying (2.2) is called a \(\theta\)-adapted coframe for the characteristic systems \(\hat{\Xi}\) and \(\bar{\Xi}\).

It is convenient to prove following lemma in the context of Pfaffian systems.
Lemma 2.2.2. Let $\mathcal{J}$ and $\mathcal{J}'$ be Pfaffian systems and suppose $\mathcal{W} = \mathcal{J} + \mathcal{J}'$. Then $\mathcal{W}' = \mathcal{J}' + \mathcal{J}'$. Dually, if $\Delta = \Delta + \Delta'$, then $\Delta' = \Delta + \Delta'$.

Proof. Let $\mathcal{W} = \mathcal{J} + \mathcal{J}'$ where locally $\mathcal{J} = \{\hat{\theta}^1, \ldots, \hat{\theta}^k\}$ and $\mathcal{J}' = \{\tilde{\theta}^1, \ldots, \tilde{\theta}^\ell\}$. Suppose $\alpha \in \mathcal{W}$. Then we may write $\alpha = \lambda_i \hat{\theta}^i + \mu_j \tilde{\theta}^j$ for smooth functions $\lambda_i$ and $\mu_j$ and differentiating gives

$$d\alpha = d\lambda_i \wedge \hat{\theta}^i + \lambda_i d\hat{\theta}^i + d\mu_j \wedge \tilde{\theta}^j + \mu_j d\tilde{\theta}^j = \lambda_i d\hat{\theta}^i + \mu_j d\tilde{\theta}^j \mod \mathcal{W}.$$ 

Suppose $\alpha \in \mathcal{W}'$. Then by definition, $d\alpha \equiv 0 \mod \mathcal{W}$, so

$$\lambda_i d\hat{\theta}^i + \mu_j d\tilde{\theta}^j \equiv 0 \mod \mathcal{W}.$$ 

We then conclude $\lambda_i d\hat{\theta}^i = 0 \mod \mathcal{J}$ and $\mu_j d\tilde{\theta}^j = 0 \mod \mathcal{J}'$. But this implies $\lambda_i \hat{\theta}^i \in \hat{\mathcal{J}}^{(1)}$ and $\mu_j \tilde{\theta}^j \in \tilde{\mathcal{J}}^{(1)}$ since

$$d(\lambda_i \hat{\theta}^i) = d\lambda_i \wedge \hat{\theta}^i + \lambda_i d\hat{\theta}^i = 0 \mod \mathcal{J}, \text{ and}$$

$$d(\mu_j \tilde{\theta}^j) = d\mu_j \wedge \tilde{\theta}^j + \mu_j d\tilde{\theta}^j = 0 \mod \mathcal{J}'.$$

Therefore, every $\alpha \in \mathcal{W}'$ is of the form $\alpha = \hat{\beta} + \tilde{\beta}$ where $\hat{\beta} \in \hat{\mathcal{J}}'$ and $\tilde{\beta} \in \tilde{\mathcal{J}}'$. That is, $\mathcal{W}' \subseteq \hat{\mathcal{J}}' + \tilde{\mathcal{J}}'$.

Now suppose $\alpha \in \hat{\mathcal{J}}' + \tilde{\mathcal{J}}'$. Then $\alpha = \hat{\beta} + \tilde{\beta}$ for $\hat{\beta} \in \hat{\mathcal{J}}'$ and $\tilde{\beta} \in \tilde{\mathcal{J}}'$. But then $d\hat{\beta} \equiv 0 \mod \mathcal{J}$ and $d\tilde{\beta} \equiv 0 \mod \mathcal{J}$, so that

$$d\alpha = d\hat{\beta} + d\tilde{\beta} \equiv 0 \mod \mathcal{J} + \mathcal{J}.$$ 

Therefore, $\alpha \in \mathcal{W}'$, so $\hat{\mathcal{J}}' + \tilde{\mathcal{J}}' \subseteq \mathcal{W}'^{(1)}$, and we conclude $\mathcal{W}' = \hat{\mathcal{J}}' + \tilde{\mathcal{J}}'$.
2.3 Hyperbolic Systems and PDE in the Plane

In Section 2.1, we showed how a PDE in the plane of the form

\[ F(x, y, u, p, q, r, s, t) = 0 \]

can be encoded as a rank 4 hyperbolic distribution. The following theorem of Stormark [30] addresses the issue of when a rank 4 hyperbolic distribution can be associated with a PDE in the plane.

**Theorem 2.3.1.** Let \( \Delta \) a rank 4 hyperbolic distribution defined on a 7-dimensional manifold \( M \). Then \( \Delta \) is associated to a hyperbolic PDE in the plane, that is there exist coordinates on the 7-dimensional manifold such that \( \Delta \) can be written in the form given in Example 2.1.12, if (and only if)

1. the derived distributions satisfy \( \text{rank}(\Delta') = 6 \) and \( \text{rank}(\Delta'') = 7 \),
2. the set of Cauchy characteristics of \( \Delta \) is trivial, that is \( \mathcal{A}(\Delta) = \{0\} \), and
3. \( \mathcal{A}(\Delta') \) forms a rank 2 subdistribution of \( \Delta \).

**Remark.** The key step to the proof of this theorem is to show that there exist coordinates \( x, y, u, p, q \) such that \( (\Delta')^\perp = \{du - pdx - qdy\} \).

**Example 2.3.2.** As an example, let \( M \) be a 7-dimensional manifold with coordinates \( z_1, z_2, z_3, z_4, z_5, z_6, z_7 \), and let \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) be a rank 4 hyperbolic distribution given by

\[
\hat{\Delta} = \left\{ \partial_{z_1} - \left( \frac{z_2}{2} + \frac{z_4^2}{4} \right) \partial_{z_4} - 2z_4z_5 \partial_{z_5} - \left( 2z_4z_6 - \frac{z_5^2}{2} \right) \partial_{z_6} - \left( 2z_4z_7 - \frac{3z_5z_6}{2} \right) \partial_{z_7}, \partial_{z_2} \right\},
\]

\[
\tilde{\Delta} = \left\{ \partial_{z_3} - \frac{z_5}{4} \partial_{z_4} + z_6 \partial_{z_5} + z_7 \partial_{z_6}, \partial_{z_7} \right\}.
\]

The first and second derived distributions of \( \Delta \) are

\[
\Delta' = \Delta \oplus \{\partial_{z_4}, \partial_{z_6}\} \quad \text{and} \quad \Delta'' = \Delta' \oplus \{z_4 \partial_{z_4} + z_5 \partial_{z_5} + z_6 \partial_{z_6}\}.
\]
It then becomes clear that $\Delta$ satisfies [i]–[iii] above with $A(\Delta') = \{\partial_{z_2}, \partial_{z_7}\}$, so despite its appearance $\Delta$ defines a second order hyperbolic PDE in the plane. Indeed, if we introduce the transformation

$$z_1 = x, \quad z_2 = r - \frac{p^2}{2}, \quad z_3 = y, \quad z_4 = -\frac{p^2}{2}, \quad z_5 = 2e^u, \quad z_6 = 2qe^u, \quad z_7 = 2e^u(t + q^2),$$

then the characteristic distributions become

$$\hat{\Delta} = \{\partial_x + p\partial_u + r\partial_p + e^u\partial_q + qe^u\partial_t, \partial_r\},$$  \hspace{1cm} (2.10a)

$$\tilde{\Delta} = \{\partial_y + q\partial_u + e^u\partial_p + t\partial_q + pe^u\partial_r, \partial_t\}. \hspace{1cm} (2.10b)$$

This is precisely the hyperbolic distribution defined by the hyperbolic Liouville equation, $s = e^u$.

We now refine the relationship between hyperbolic systems and hyperbolic PDE in the plane using the classification of hyperbolic second order PDE in the plane given by Kamran and Gardner [17]. We rephrase their definitions using distributions instead of Pfaffian systems.

Central to their classification is the notion of the class of a distribution $\Delta$ which is the dimension of its Cartan system

$$\mathcal{C}(\Delta) = A(\Delta)^\perp = \{Z \in \Delta \mid [Z, \Delta] \subset \Delta\}^\perp.$$  

They then make the following definitions.

**Definition 2.3.3.** A second order hyperbolic PDE in the plane with hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ is called

[i] an equation is of generic type if $\text{class}(\hat{\Delta}') = \text{class}(\tilde{\Delta}') = 7$,

[ii] an equation is of Goursat type if $\text{class}(\hat{\Delta}') = 6$ and $\text{class}(\tilde{\Delta}') = 7$ or vice versa,

[iii] an equation is of Monge-Ampère type if $\text{class}(\hat{\Delta}') = 6 = \text{class}(\tilde{\Delta}') = 6$. 
Remark. We can reinterpret these conditions in terms of the number of Cauchy characteristics of the derived characteristic distributions. Indeed,

[i] if \( \text{class}(\Delta') = \text{class}(\Delta') = 7 \), then neither \( \Delta' \) nor \( \Delta' \) admit any Cauchy characteristics;

[ii] if \( \text{class}(\Delta') = 6 \) and \( \text{class}(\Delta') = 7 \) (or vice versa), then either \( \Delta' \) or \( \Delta' \) admit a single Cauchy characteristic;

[iii] if \( \text{class}(\Delta') = \text{class}(\Delta') = 6 \), then both \( \Delta' \) and \( \Delta' \) admit a single Cauchy characteristic.

We note that the terms generic, Goursat, and Monge-Ampère match with the classical terminology [19]. In particular, Kamran and Gardner provide the following theorem which relates their definition of Monge-Ampère equations to the classical form.

**Theorem 2.3.4.** A second order hyperbolic PDE in the plane \( F(x,y,u,p,q,r,s,t) = 0 \) satisfies \( \text{class}(\Delta') = \text{class}(\Delta') = 6 \) if and only if it is equivalent to an equation of the form

\[
Ar + 2Bs + Ct + D + E(rt - s^2) = 0
\]

where \( A, B, C, D, E \) are functions of \( x, y, u, p, q \).

They then further characterize the above equations of each type based on the dimension of the derived flags of their characteristic systems.

**Theorem 2.3.5.** A second order hyperbolic PDE in the plane with hyperbolic distribution \( \Delta = \Delta \oplus \Delta \) is

[i] an equation of generic type if and only if

\[
\text{rank}(\Delta') = \text{rank}(\Delta') = 3 \quad \text{and} \quad \text{rank}(\Delta'') = \text{rank}(\Delta'') = 5,
\]

[ii] an equation of Goursat type if and only if

\[
\text{rank}(\Delta') = 3, \quad \text{rank}(\Delta'') = 5 \quad \text{and} \quad \text{rank}(\Delta') = 3, \quad \text{rank}(\Delta'') = 4,
\]
or vice-versa, and

[iii] an equation of Monge-Ampère type if and only if

\[
\text{rank}(\hat{\Delta}') = \text{rank}(\hat{\Delta}') = 3 \quad \text{and} \quad \text{rank}(\hat{\Delta}'') = \text{rank}(\hat{\Delta}'') = 4.
\]

**Example 2.3.6.** The equation

\[3rt^3 + 1 = 0 \quad (2.11)\]

is an equation of generic type. The hyperbolic distribution \(\Delta = \hat{\Delta} \oplus \tilde{\Delta}\) for (2.11) is given by

\[
\hat{\Delta} = \left\{ \partial_x + \frac{1}{t^2} \partial_y + \frac{pt^2 + q}{t^2} \partial_u + \frac{3st - 1}{3t^3} \partial_p + \frac{st + 1}{t} \partial_q, \partial_s + t^2 \partial_t \right\},
\]

\[
\tilde{\Delta} = \left\{ \partial_x - \frac{1}{t^2} \partial_y + \frac{pt^2 - q}{t^2} \partial_u - \frac{3st + 1}{3t^3} \partial_p + \frac{st - 1}{t} \partial_q, \partial_s - t^2 \partial_t \right\},
\]

and a simple computation shows that the derived systems have dimensions

\[
\text{rank}(\hat{\Delta}') = \text{rank}(\hat{\Delta}') = 3 \quad \text{and} \quad \text{rank}(\hat{\Delta}'') = \text{rank}(\hat{\Delta}'') = 5.
\]

**Example 2.3.7.** The equation

\[rs = p \quad (2.12)\]

is an equation of Goursat type. The hyperbolic distribution \(\Delta = \hat{\Delta} \oplus \tilde{\Delta}\) for (2.11) is given by

\[
\hat{\Delta} = \left\{ \partial_x + p \partial_u + \frac{p}{s} \partial_p + s \partial_q + \frac{s^2}{p} \partial_t, \partial_s - \frac{s^2}{p} \partial_t \right\},
\]

\[
\tilde{\Delta} = \left\{ \partial_x + \frac{p}{s^2} \partial_y + \frac{p(q + s^2)}{s^2} \partial_u + \frac{2p}{s} \partial_p + \frac{p(ts)^3}{s^2} \partial_q, \partial_s, \partial_t \right\},
\]
and a simple computation shows that the derived systems have dimensions

\[ \text{rank}(\Delta') = 3, \quad \text{rank}(\Delta'') = 4 \quad \text{and} \quad \text{rank}(\tilde{\Delta}') = 3, \quad \text{rank}(\tilde{\Delta}'') = 5. \]

**Example 2.3.8.** We have seen that, in terms of the \( z_i \)-coordinates, the hyperbolic Liouville equation \( s = e^u \) has characteristic distributions given by

\[
\Delta = \left\{ \partial_{z_1} - \left( \frac{z_2}{2} + z_1^2 \right) \partial_{z_4} - 2z_4z_5 \partial_{z_5} - \left( 2z_4z_6 - \frac{z_5^2}{2} \right) \partial_{z_6} - \left( 2z_4z_7 - \frac{3z_5z_6}{2} \right) \partial_{z_7}, \partial_{z_2} \right\},
\]

\[
\tilde{\Delta} = \left\{ \partial_{z_3} - \frac{z_5}{4} \partial_{z_4} + z_6 \partial_{z_5} + z_7 \partial_{z_6}, \partial_{z_7} \right\}.
\]

By calculating the derived systems, we see that

\[
\Delta' = \Delta \oplus \{ \partial_{z_4} \} \quad \text{and} \quad \Delta'' = \Delta' \oplus \{ z_4 \partial_{z_4} + z_5 \partial_{z_5} + z_6 \partial_{z_6} + z_7 \partial_{z_7} \},
\]

\[
\tilde{\Delta}' = \tilde{\Delta} \oplus \{ \partial_{z_4} \} \quad \text{and} \quad \tilde{\Delta}'' = \tilde{\Delta}' \oplus \{ \partial_{z_3} \},
\]

so that

\[
\text{rank}(\Delta') = \text{rank}(\tilde{\Delta}') = 3 \quad \text{and} \quad \text{rank}(\Delta'') = \text{rank}(\tilde{\Delta}'') = 4.
\]

From this, we conclude that the Liouville equation is of Monge-Ampère type. Note, this can be shown using the characteristic distributions in the standard coordinates \( x, y, u, p, q, r, t \) as well.

### 2.4 Hyperbolic Systems and \( f \)-Gordon Equations

We now give necessary and sufficient conditions for a hyperbolic distribution to be associated with an \( f \)-Gordon equation of the form

\[
s = f(x, y, u, p, q).
\]

We first note that by inspection every \( f \)-Gordon equation is necessarily an equation of
Monge-Ampère type and therefore must have associated characteristic systems $\hat{\Delta}$ and $\tilde{\Delta}$ satisfying
\[
\text{rank}(\hat{\Delta}') = \text{rank}(\tilde{\Delta}') = 3 \quad \text{and} \quad \text{rank}(\hat{\Delta}'') = \text{rank}(\tilde{\Delta}'') = 3.
\]

In [21], Juráš proves that a second-order hyperbolic equation with hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ defined on a 7-dimensional manifold $M$ is contact-equivalent to an $f$-Gordon equation if and only if each of its characteristic distributions admit at least one first integral of order less than or equal to one. More geometrically, this means that each characteristic distribution admits a first integral on $M$ which exists on the 5-dimensional quotient manifold $\bar{M} = M/\mathcal{A}(\Delta')$. This gives the following geometric characterization of hyperbolic $f$-Gordon equations.

**Theorem 2.4.1.** Let $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ be a rank 4 hyperbolic distribution defined on a 7-dimensional manifold corresponding to a PDE of Monge-Ampère type. Then $\Delta$ is associated to an $f$-Gordon equation if and only if each of its characteristic distributions $\hat{\Delta}, \tilde{\Delta}$ admit at least one first integral which are invariants of $\mathcal{A}(\Delta')$.

**Remark.** This theorem allows us to detect when a hyperbolic distribution defines an $f$-Gordon equation without explicitly writing the equation in standard coordinates. This will be indispensable when showing our new examples of Darboux integrable equations are, in fact, $f$-Gordon equations.

**Example 2.4.2.** In standard coordinates, it is obvious that the hyperbolic distribution (2.10) for the Liouville equation corresponds to $f$-Gordon equation. However, if one was presented the hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ where
\[
\hat{\Delta} = \left\{ \partial_{z_1} - \left( \frac{z_2}{2} + z_4^2 \right) \partial_{z_4} - 2z_4z_5 \partial_{z_5} - \left( 2z_4z_6 - \frac{z_6^2}{2} \right) \partial_{z_6} - \left( 2z_4z_7 - \frac{3z_5z_6}{2} \right) \partial_{z_7}, \partial_{z_2} \right\},
\]
\[
\tilde{\Delta} = \left\{ \partial_{z_3} - \frac{z_5}{4} \partial_{z_4} + z_6 \partial_{z_5} + z_7 \partial_{z_6}, \partial_{z_7} \right\},
\]
this would be much less evident.
To utilize Theorem 2.4.1, we first calculate the first integrals of $\Delta$ and $\Delta$ to be

$$I_1 = z_3, \quad I_2 = \frac{2z_5z_7 - 3z_6^3}{2z_5^2}, \quad \tilde{I}_1 = z_1, \quad \tilde{I}_2 = z_2,$$

respectively. The Cauchy characteristics for $\Delta'$ are given by $\mathcal{A}(\Delta') = \{\partial_{z_2}, \partial_{z_7}\}$, and it can easily be verified that the first integrals $\tilde{I}_1$ and $\tilde{I}_1$ are $\mathcal{A}(\Delta')$-invariant functions, meaning they will descend to the quotient whereas $\tilde{I}_2$ and $\tilde{I}_2$ will not. Theorem 2.4.1 then implies that the distribution $\Delta$ defines an $f$-Gordon equation.

**Example 2.4.3.** In [19], Goursat analyzes the equation

$$r - qs + pt = 0 \quad (2.14)$$

which is manifestly a equation of Monge-Ampère type. Indeed, (2.14) has hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ which satisfies the hypotheses of Theorem 2.3.5 for an equation of Monge-Ampère type (though we refrain from writing the distribution here due to its length).

The first integrals of $\hat{\Delta}$ are

$$\hat{I}_1 = q + \sqrt{q^2 - 4p}, \quad \hat{I}_2 = tq - 2s + t\sqrt{q^2 - 4p},$$

and the first integrals of $\tilde{\Delta}$ are

$$\tilde{I}_1 = q - \sqrt{q^2 - 4p}, \quad \tilde{I}_2 = tq - 2s - t\sqrt{q^2 - 4p}.$$

The set of Cauchy characteristics of $\Delta'$ are

$$\mathcal{A}(\Delta') = \left\{\partial_s + \frac{2}{\hat{I}_1} \partial_t, \partial_s + \frac{2}{\tilde{I}_1} \partial_t\right\},$$

and we see that $\hat{I}_1$ and $\tilde{I}_1$ are $\mathcal{A}(\Delta')$-invariant functions. Theorem 2.4.1 then implies that the distribution $\Delta$ defines an $f$-Gordon equation. In fact, we show in Example 7.3.3 that
(2.14) can actually be transformed to the linear equation

\[ s + \frac{p - q}{x - y} = 0. \]

### 2.5 Classification of Contact Distributions on \( J^k(\mathbb{R}, \mathbb{R}) \)

As the hyperbolic distributions corresponding to second-order PDE in the plane decompose as the direct sum of two nonintegrable rank 2 subdistributions, the study of nonintegrable rank 2 distributions will play an essential role in our study of hyperbolic second-order PDE in the plane. In this section, we give necessary and sufficient conditions for a rank 2 distribution to be locally equivalent to the standard contact distribution

\[ \{ \partial_x + y' \partial_y + y'' \partial_{y'} + \cdots + y^{(k)} \partial_{y^{(k-1)}}, \partial_{y^{(k)}} \} \]

on \( J^k(\mathbb{R}, \mathbb{R}) \). The following theorem of Engel gives a local normal form for rank 2 distributions on 4-dimensional manifolds (see [11]).

**Theorem 2.5.1** (Engel’s Theorem). Let \( \Delta \) be a rank 2 distribution on a 4-dimensional manifold \( M \). Then the derived distributions of \( \Delta \) satisfy

\[ \text{rank } \Delta' = 3 \quad \text{and} \quad \text{rank } \Delta'' = 4 \]

if and only if there exist local coordinates \( x, y, y', y'' \) such that

\[ \Delta = \{ \partial_x + y' \partial_y + y'' \partial_{y'}, \partial_{y''} \}. \]

More generally, the following theorem of Murray [26] gives a characterization of the standard contact distribution on the jet space \( J^k(\mathbb{R}, \mathbb{R}) \).
Theorem 2.5.2. Let $\Delta$ be a rank 2 distribution on a $m$-dimensional manifold $M$. Then the derived and weak derived distributions of $\Delta$ satisfy

$$\text{rank } \Delta^{(i)} = \text{rank } \Delta_{(i)} = i + 2, \quad 0 \leq i \leq m - 2$$

if and only if there exist local coordinates $x, y, y', y'', \ldots, y^{(k)}$ such that

$$\Delta = \{ \partial_x + y' \partial_y + y'' \partial_y' + \cdots + y^{(k)} \partial_y^{(k-1)}, \partial_y^{(k)} \}.$$ 

If $\mathcal{I} = \Delta^\perp$ is the dual Pfafian system to a rank 2 distribution $\Delta$, then we can use the following theorems to characterize the standard contact system on $J^k(\mathbb{R}, \mathbb{R})$.

Theorem 2.5.3 (Goursat normal form). Let

$$\mathcal{I} = \{ \alpha^1, \ldots, \alpha^s \}$$

be a Pfaffian system of codimension two in a space of dimension $n = s + 2$. Suppose there exists a Pfaffian form $\pi \neq 0 \mod \mathcal{I}$, satisfying

$$d\alpha^i \equiv -\alpha^{i+1} \wedge \pi \mod \alpha^1, \ldots, \alpha^i, \quad 1 \leq i \leq s - 1, \quad d\alpha^s \neq 0 \mod \mathcal{I}.$$ 

There there is a local coordinate system $x, y, y', \ldots, y^{(s)}$, such that

$$\mathcal{I} = \{ dy - y' dx, \ldots, dy^{(s-1)} - y^{(s)} dx \}.$$ 

Remark. In [33], Vassiliou provides an algorithm for systematically calculating these co-

Theorem 2.5.4 (Modified Goursat Normal Form). Let $\mathcal{I} = \{ \alpha^1, \ldots, \alpha^s \}$ be a Pfaffian system of codimension two in a space of dimension $n = s + 2$. Suppose there exists a
Pfaffian form $\pi \not\equiv 0 \mod I$ satisfying

$$d\alpha^i \equiv f_{i+1} \pi \wedge \alpha^{i+1} \mod \{\alpha^1, \ldots, \alpha^i\}, \quad 1 \leq i \leq s - 1, \quad d\alpha^s \not\equiv 0 \mod I$$

for nonzero functions $f_i$. Then there is a local coordinate system $x, y, y', \ldots, y^{(s)}$ such that

$$I = \left\{dy - y' \, dx, \ldots, dy^{(s-1)} - y^{(s)} \, dx\right\}.$$

Proof. Suppose $I = \{\alpha^1, \ldots, \alpha^{(s)}\}$ is a Pfaffian system satisfying

$$d\alpha^i \equiv f_{i+1} \pi \wedge \alpha^{i+1} \mod \{\alpha^1, \ldots, \alpha^i\}$$

with $d\alpha^{(s)} \not\equiv 0 \mod I$. Then we can introduce the Pfaffian system

$$\tilde{I} = \{\beta^1 = g_1 \alpha^1, \beta^2 = g_2 \alpha^2, \ldots, \beta^s = g_s \alpha^s\}$$

where the $g_i$ are nonzero functions. Clearly $\tilde{I} = I$, as differential systems, and it follows that

$$d\beta^i = d \left(g_i \alpha^i\right)$$

$$= g_i d\alpha^i + dg_i \wedge \alpha^i$$

$$\equiv g_i f_{i+1} \pi \wedge \alpha^{i+1} \mod \{\alpha^1, \ldots, \alpha^i\}$$

$$\equiv \frac{g_i f_{i+1}}{g_{i+1}} \pi \wedge \beta^{i+1} \mod \{\beta^1, \ldots, \beta^i\}$$

We may then choose the functions $g_i$ such that they satisfy

$$g_{i+1} = g_i f_{i+1}, \quad 1 \leq i \leq s - 1.$$
In particular, we may always choose $g_1 = 1$, so that

$$g_2 = f_2, \ g_3 = f_2f_3, \ g_4 = f_2f_3f_4, \ \ldots, \ g_s = \prod_{i=2}^{s} f_i.$$ 

In doing so, the $\beta^i$ satisfy

$$d\beta^i = \pi \wedge \beta^{i+1} \mod \{\beta^1, \ldots, \beta^i\}, \ 1 \leq i \leq s - 1$$

with $d\beta^s \equiv g_s d\alpha^s \neq 0 \mod \tilde{I}$. That is, $\tilde{I}$ can be put into Goursat normal form. It follows that $I$ can be put into Goursat normal form as well.

The set of transformations which preserve the contact distribution $\Delta$ on $J^k(\mathbb{R}, \mathbb{R})$ are called contact transformations or contact symmetries. The following theorem due to Bäcklund relates higher order contact symmetries to those of first order.

**Theorem 2.5.5.** Every contact symmetry on $J^k(\mathbb{R}, \mathbb{R})$ for $k > 1$ is the prolongation of a contact symmetry on $J^1(\mathbb{R}, \mathbb{R})$.

In [24], Lie classified all finite-dimensional Lie groups of contact transformations, including point transformations, acting on $\mathbb{C}^2$. Later, González-López, Kamran, and Olver [18] classified all finite-dimensional Lie algebras of vector fields acting on $\mathbb{R}^2$ which includes all possible finite dimensional Lie groups of contact transformations. For our purposes, we only require those Lie algebras of vector fields acting on $\mathbb{R}^2$ up to dimension five. We list these in Appendix A. The complete list can, of course, be found in their original paper [18] as well as in [27].

### 2.6 Classification of (2,3,5)-Distributions and Their Prolongations

We now turn our attention to rank 2 distributions on 5-manifolds whose derived distributions have rank (2,3,5). Goursat [20] shows that these distributions, sometimes called
Monge distributions, can always be written in the form

\[
\{ \partial_x + F(x, z, y, y', y'') \partial_z + y' \partial_y + y'' \partial_{y'} + \partial_{y''} \}
\]

for some coordinates \( x, z, y, y', y'' \) where \( F_{y'y''} \neq 0 \).

**Remark.** Goursat provides an alternative normal form for these distributions given by

\[
\{ \partial_{x^1} + x^3 \partial_{x^2} + f_{x^5} \partial_{x^3} + (x^5 f_{x^5} - f) \partial_{x^4}, \partial_{x^5} \},
\]

for coordinates \( x^1, \ldots, x^5 \) and where \( f = f(x^1, x^2, x^3, x^4, x^5) \) with \( f_{x^5x^5} \neq 0 \). Strazzullo [31] provides an algorithm for obtaining these coordinates.

In [12], Cartan famously solved the equivalence problem (over \( \mathbb{R} \)) for all \((2,3,5)\)-distributions. In doing so, he showed that every \((2,3,5)\)-distribution has two fundamental invariants. The first, known as the Cartan quartic \( F \), is a fourth degree homogeneous polynomial in two variables, which is given by

\[
F(x, y) = A_1 x^4 + 4A_2 x^3 y + 6A_3 x^2 y^2 + 4A_4 xy^3 + A_5 y^4
\]

where the functions \( A_1, A_2, A_3, A_4, A_5 \) are functions of all five variables coming from the base manifold as well as nine additional auxiliary variables. The second invariant is a fourth degree homogeneous polynomial \( G \) in three variables, given by

\[
G(x, y, z) = F(x, y) + 4(B_1 x^3 + 3B_2 x^2 y + 3B_3 xy^2 + B_4 y^3) z
+ 6(C_1 x^2 + 2C_2 xy + C_3 y^2) z^2 + 4(D_1 x + D_2 y) z^3 + E z^4
\]

where the coefficients \( B_1, B_2, B_3, B_4, C_1, C_2, C_3, D_1, D_2, E \) are again all functions of the five base variables and nine additional auxiliary variables.

We note that \( G(x, y, 0) = F(x, y) \). The explicit formulas for these invariants are, in general, far too complicated to write down. However, Anderson and Struzzullo [31] were able
to implement a package in Maple, called \texttt{FiveVariables}, which allows to the computation of both $F(x, y)$ and $G(x, y, z)$.

A rough classification of these distributions is given by the following cases:

1. $F$ has root type $[\infty]$ if $F$ has infinitely many roots, that is, if $F \equiv 0$;
2. $F$ has root type $[4]$ if $F$ has a single root of multiplicity four;
3. $F$ has root type $[3, 1]$ if $F$ has one triple root and one simple root;
4. $F$ has root type $[2, 2]$ if $F$ has two double roots;
5. $F$ has root type $[2, 1, 1]$ if $F$ has one double root and two simple roots;
6. $F$ has root type $[1, 1, 1, 1]$ if $F$ has four simple roots.

Cartan gave a detailed analysis of these cases where $F$ has root type $[\infty]$, $[4]$, and $[2, 2]$ when viewed as a polynomial over $\mathbb{C}$.

**Remark.** As far as we can tell, Cartan does not analyze the root type $[2, 1, 1]$. An analysis of this root type is missing from Stormark’s presentation as well.

Cartan shows that all systems of root type $[\infty]$ must be equivalent to the rank 2 distribution corresponding to the so-called \textit{Hilbert-Cartan equation}, $z' = (y'')^2$; however, this is the only instance in which the root type completely characterizes the $(2,3,5)$-distribution. That is to say, two $(2,3,5)$-distributions with the same root type (excluding those with root type $[\infty]$) need not be equivalent. Cartan’s analysis does provide some normal forms for distributions of certain root types though, as well as information about their symmetry algebras which we will further discuss in Chapter 10.

We now present normal forms for the dual Pfaffian systems of $(2,3,5)$-distributions and their prolongations to higher-dimensional manifolds.

**Theorem 2.6.1** (Monge Normal Form). Let $\mathcal{I} = \{\theta^1, \theta^2, \theta^3\}$ be a rank 3 Pfaffian system defined on a 5-dimensional manifold such that

$$\text{rank}(\mathcal{I}') = 2 \quad \text{and} \quad \text{rank}(\mathcal{I}'') = 0.$$
Then there exist Pfaffian forms \( \pi^1 \neq 0 \mod I \) and \( \pi^2 \neq 0 \mod I \) satisfying

\[
\begin{align*}
  d\theta^1 &\equiv \theta^3 \wedge \pi^1 \mod \{\theta^1, \theta^2\}, \\
  d\theta^2 &\equiv \theta^3 \wedge \pi^2 \mod \{\theta^1, \theta^2\}, \\
  d\theta^3 &\equiv \pi^1 \wedge \pi^2 \mod \{\theta^1, \theta^2, \theta^3\}.
\end{align*}
\]

if and only if there exist local coordinate system \( x, z, y, y_1, y_2 \) such that

\[
I = \{dy - y_1 dx, dy_1 - y_2 dx, dz - F(x, z, y, y_1, y_2) dx\}, \quad F_{y_2 y_2} \neq 0.
\]

It will often be beneficial to know when a given Pfaffian system is the prolongation of a Monge system. We note that Stormark lays out the general prolongation theory for vector field distributions (and their dual Pfaffian systems) in [30]. For our purposes, however, we will only need to know how to prolong non-integrable rank 2 distributions which we show now. Suppose \( \Delta = \{X_1, X_2\} \) is a non-integrable rank 2 distribution on a manifold \( M \) with preferred vertical section \( X_2 \). Let \( M^{(1)} = M \times \mathbb{R} \) with coordinate \( \lambda \) on \( \mathbb{R} \). Then the prolongation of \( \Delta \) to \( M^{(1)} \) is given by

\[
\Delta^{(1)} = \{Y_1 = X_1 + \lambda X_2, Y_2 = \partial_\lambda\}.
\]

The following two theorems give conditions for a differential system to be either the first or second prolongation of a Monge system.

**Theorem 2.6.2.** Let \( \mathcal{J} = \{\theta^1, \theta^2, \theta^3, \theta^4\} \) be a rank 4 Pfaffian system defined on a 6-manifold such that

\[
\begin{align*}
  \text{rank}(\mathcal{J}') &= 3, \\
  \text{rank}(\mathcal{J}'') &= 2, \\
  \text{rank}(\mathcal{J}''') &= 0.
\end{align*}
\]
If there exist Pfaffian forms $\pi^1, \pi^2$ both nonzero mod $\mathcal{J}$ satisfying

\[
\begin{align*}
d\theta^1 &\equiv \theta^3 \wedge \pi^1 \mod \{\theta^1, \theta^2\}, \\
d\theta^2 &\equiv \theta^3 \wedge \theta^4 \mod \{\theta^1, \theta^2\}, \\
d\theta^3 &\equiv \theta^4 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3\}, \\
d\theta^4 &\equiv \pi^1 \wedge \pi^2 \mod \{\theta^1, \theta^2, \theta^3, \theta^4\},
\end{align*}
\]

then $\mathcal{J}$ is the prolongation of a rank 3 Monge system, $\mathcal{J}' = \{\theta^1, \theta^2, \theta^3\}$. We note that $\partial_{\pi^2}$ is a Cauchy characteristic $\mathcal{J}'$.

**Proof.** Suppose $\mathcal{I}$ is a rank 3 Pfaffian system on a 5-manifold satisfying the Monge congruences above. In order to prolong $\mathcal{I}$, we construct the new differential system

\[
\mathcal{I}^{(1)} = \{\theta^1, \theta^2, \theta^3, \theta^4\}
\]

where $\theta^4 = \pi^2 + \lambda \pi^1$. The structure equations for $\mathcal{I}^{(1)}$ then become

\[
\begin{align*}
d\theta^1 &\equiv \theta^3 \wedge \pi^1 \mod \{\theta^1, \theta^2\}, \\
d\theta^2 &\equiv \theta^3 \wedge (\theta^4 - \lambda \pi^1) \equiv \theta^3 \wedge \theta^4 - \lambda \theta^3 \wedge \pi^1 \mod \{\theta^1, \theta^2\}, \\
d\theta^3 &\equiv \pi^1 \wedge (\theta^4 - \lambda \pi^1) \equiv \pi^1 \wedge \theta^4 \mod \{\theta^1, \theta^2, \theta^3\}, \\
d\theta^4 &\equiv d\pi^2 + d(\lambda \pi^1) \equiv d\lambda \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3, \theta^4\}.
\end{align*}
\]

Applying the change of coframe

\[
\begin{align*}
\theta^1 &\mapsto \tilde{\theta}^1, \quad \theta^2 &\mapsto -\lambda \tilde{\theta}^1 - \tilde{\theta}^2, \quad \theta^3 &\mapsto \tilde{\theta}^3, \quad \theta^4 &\mapsto -\tilde{\theta}^4, \quad \pi^1 &\mapsto \tilde{\pi}^1, \quad \pi^2 &\mapsto \tilde{\pi}^2 = d\lambda.
\end{align*}
\]
Theorem 2.6.3. Let $J = \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\}$ be a rank 5 Pfaffian system defined on a 7-manifold such that

$$\text{rank}(J') = 4, \quad \text{rank}(J'') = 3, \quad \text{rank}(J''') = 2, \quad \text{rank}(J^{(4)}) = 0.$$ 

If there exist Pfaffian forms $\pi^1, \pi^2$ both nonzero mod $J$ satisfying

$$d\theta^1 \equiv \theta^3 \wedge \pi^1 \mod \{\theta^1, \theta^2\},$$
$$d\theta^2 \equiv -d(\lambda \theta^1) - d\theta^2 \equiv -\lambda d\theta^1 - \theta^3 \wedge \theta^4 + \lambda \theta^3 \wedge \pi^1 \equiv \theta^3 \wedge \theta^4 \mod \{\theta^1, \theta^2\},$$
$$d\theta^3 \equiv \theta^4 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3\},$$
$$d\theta^4 \equiv \theta^5 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3, \theta^4\},$$
$$d\theta^5 \equiv \pi^1 \wedge \pi^2 \mod \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\},$$

then $J$ is the second prolongation of a rank 3 Monge system.

Proof. Similar to the proof of the previous theorem, we suppose $I^{(1)} = \{\theta^1, \theta^2, \theta^3, \theta^4\}$ is the prolongation of a rank 3 Monge system. To prolong the system once more, we again construct a new differential system

$$I^{(2)} = \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\}.$$
where $\theta^5 = -\pi^2 - \lambda \pi^1$. The structure equations for $\mathcal{I}^{(2)}$ then become

\[ d\theta^1 \equiv \theta^3 \wedge \pi^1 \mod \{\theta^1, \theta^2\}, \]
\[ d\theta^2 \equiv \theta^3 \wedge \theta^4 \mod \{\theta^1, \theta^2\}, \]
\[ d\theta^3 \equiv \theta^4 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3\}, \]
\[ d\theta^4 \equiv \pi^1 \wedge (-\lambda \pi^1 - \theta^5) \equiv \theta^5 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3, \theta^4\}, \]
\[ d\theta^5 \equiv -d\pi^2 - d(\lambda \pi^1) \equiv \pi^1 \wedge d\lambda \mod \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\}. \]

Finally, by renaming $\pi^2 = d\lambda$, we obtain the required congruences.
CHAPTER 3
DISTRIBUTIONS WITH SYMMETRY

In this chapter, we review the theory of transformation groups and their actions on smooth manifolds. We then define symmetries of distributions and show how given a set of symmetries for a distribution, one can construct a new distribution called the quotient distribution.

3.1 Transformation Groups

We begin with an introduction to transformation groups and their actions on smooth manifolds. Much of the classical theory of transformation groups was laid out by Lie in his appropriately titled treatise, *Theorie der Transformationsgruppen*, [24] (see [25] for an English translation). Here, we summarize the more modern treatment by Olver, [27].

**Definition 3.1.1.** A *transformation group* acting on a smooth manifold $M$ is determined by a Lie group $G$ and a smooth map $\mu : G \times M \to M$, denoted by $\mu(g, p) = \mu_g(p) = g \cdot p$, which, for all $p \in M$, satisfies

(i) $\mu(e, p) = p$ for some $e \in G$, and

(ii) $\mu(g_1, \mu(g_2, p)) = \mu(g_1 \cdot g_2, p)$ for any $g_1, g_2 \in G$.

Such a function $\mu$ is often called a (global) $G$-action on $M$. If the action is only defined on an open subset of $G \times M$, then $\mu$ is called a local $G$-action on $M$.

When $G$ is a transformation group acting on $M$, we say that a subset $S$ of a manifold $M$ is *$G$-invariant* if for each $p \in S$, $\mu(g, p) \in S$ for all $g \in G$. An *orbit* of a transformation group $G$ through a point $p \in M$ is a minimal (nonempty), $G$-invariant subset of $M$ denoted by

$$O_p = \{\mu(g, p) \in M \mid g \in G\}.$$
A transformation group $G$ is said to act \emph{transitively} if it only has one orbit and \emph{intransitively} otherwise. The action is called \emph{regular} if all of its orbits have the same dimension and about each point $p \in M$ there exist arbitrarily small neighborhoods whose intersection with each orbit is a connected subset of the corresponding orbit.

The \emph{isotropy group} of a point $p \in M$ is given by

$$G_p = \{ g \in G \mid \mu(g, p) = p \}.$$ 

If all of the isotropy groups are trivial, so that $G_p = \{e\}$ for all $p \in M$, we say that the transformation group acts \emph{freely}, and we say that a transformation group acts \emph{effectively} if different group elements have different actions, so that $\mu(g_1, p) = \mu(g_2, p)$ for all $p \in M$ if and only if $g_1 = g_2$. Alternatively, one would say a transformation group acts effectively if the only element satisfying $\mu(g, p) = p$ is the identity element. The effectiveness of a group action is measured by its \emph{global isotropy group}

$$G_M = \bigcap_{p \in M} G_p = \{ g \in G \mid \mu(g, p) = p \text{ for all } p \in M \},$$

so that $G$ acts effectively if and only if $G_M = \{e\}$.

The action of a transformation group on $M$ is generates a set of vector fields on $M$ called the \emph{infinitesimal generators} of the action given by

$$\Gamma = \{ X \in \mathfrak{X}(M) \mid X_p = \mu_{p*}(\xi|_e) \text{ for all } p \in M \text{ and } \xi \in \mathfrak{g} \}$$

where $\mu_{p} : G \to M$ is the function defined by $\mu_{p}(g) = \mu(g, p)$ and $\mathfrak{g}$ is the Lie algebra of right invariant vector fields on $G$. The infinitesimal generators of the action $\mu$ can also be computed by differentiating the components of the map $\mu$ with respect to the group parameters $g$ and evaluating the result at the identity.
A vector field $Y$ on $M$ is called $G$-invariant if

$$
\mu_{g*}(Y|_p) = Y|_{\mu(g,p)}
$$

for all $g \in G$ and $p \in M$, or infinitesimally if

$$
\mathcal{L}_X(Y) = [X,Y] = 0
$$

for all infinitesimal generators $X$ of $\mu$. Generically, an action of a Lie group $G$ need not admit any invariant vector fields. However, a definitive count of $G$-invariant vector fields associated to the action can be given when $G$ acts freely.

**Theorem 3.1.2.** Let $G$ be an $n$-dimensional Lie group acting freely and effectively on the $m$-dimensional manifold $M$. Then, locally, there exist $m$ pointwise linearly independent $G$-invariant vector fields $X_1, \ldots, X_m$ on $M$.

Infinitesimally, we say that a group acts transitively on a connected manifold $M$ if and only if the set of infinitesimal generators for the action of $G$, denoted by $\Gamma$, satisfies $\Gamma|_p = T_pM$ for all $p \in M$.

Finally, if $G$ is an $n$-dimensional transformation group acting on the $m$-dimensional manifold $M$, there is an induced equivalence relation on the points of $M$ where two points are equivalent if they lie in the same orbit of $G$. Let $M/G$, called the quotient space of $M$ by $G$, denote the set of such equivalence classes, or equivalently, the set of orbits of $G$. The projection $\pi : M \to M/G$ associates to each point $p \in M$ its equivalence class $\pi(p) \in M/G$, which can be identified with the orbit of $G$ passing through $p$. We note that $\pi(gp) = \pi(p)$ for all $g \in G$ and $p \in M$. We will assume $G$ acts regularly on $M$ so that the quotient space $M/G$, when equipped with the standard quotient topology,

[i] forms a smooth manifold of dimension $m - n$,

[ii] the projection map $\pi : M \to M/G$ is smooth,
[iii] smooth local cross-sections exist in a neighborhood of every point $q \in M/G$; that is, if $U$ is an open neighborhood of the point $q = \pi(p)$, then there exists a smooth mapping $\sigma : U \to M$ such that $\pi \circ \sigma$ is the identity on $U$. 
3.2 Symmetries of Distributions

In this section, we introduce the notion of a symmetry group for a distribution and discuss some properties of distributions with symmetry.

**Definition 3.2.1.** A symmetry of a distribution $\Delta$ defined on a manifold $M$ is a diffeomorphism $\Phi : M \to M$ such that $\Phi_* (\Delta_p) = \Delta_{\Phi(p)}$. A symmetry group of $\Delta$ is a subgroup $G \subset \text{Diff}(M)$ whose elements are symmetries of $\Delta$. For our purposes, we require $G$ be a Lie group, where the action $\mu : G \times M \to M$ naturally satisfies

$$\mu_{g*} X_p \in \Delta_{\mu_g(p)}$$

for all $p \in M$, $X_p \in \Delta_p$, and $g \in G$.

A fundamental property of symmetries of distributions is that they map integral manifolds to integral manifolds.

**Proposition 3.2.2.** If $\Phi$ is a symmetry of $\Delta$ and $\psi : N \to M$ is an integral manifold of $\Delta$, then $\Phi \circ \psi$ is an integral manifold of $\Delta$.

**Proof.** Let $q \in N$. Calculating gives

$$(\Phi \circ \psi)_* (T_q N) = \Phi_* \circ \psi_* (T_q N) = \Phi_* (\Delta_{\psi(q)}) = \Delta_{\Phi \circ \psi(q)},$$

and we note that $\Phi \circ \psi$ is an immersion. □

When $G$ is a symmetry group of $\Delta$, the set of infinitesimal generators $\Gamma$ of $G$ form a set of infinitesimal symmetries of $\Delta$ satisfying

$$\mathcal{L}_Y (X) \in \Delta \quad \text{for all } X \in \Delta \text{ and } Y \in \Gamma.$$  

We say that a symmetry group of $\Delta$ is transverse if $\Gamma \cap \Delta = \{0\}$.

It is not difficult to see that a symmetry of a $\Delta$ is also a symmetry of its derived.
Proposition 3.2.3. Let $\Delta$ be a rank $k$ distribution on an $m$-dimensional manifold $M$. If $Y$ is an infinitesimal symmetry of $\Delta$, then $Y$ is an infinitesimal symmetry of $\Delta'$. 

Proof. Let $Y$ be an infinitesimal symmetry of $\Delta$. Then $\mathcal{L}_Y(X) \in \Delta$ for all $X \in \Delta$. Let $W = [X_1, X_2] \in \Delta'$. Then, by the Jacobi identity,

$$\mathcal{L}_Y W = [Y, W] = [Y, [X_1, X_2]] = -[X_1, [X_2, Y]] - [X_2, [Y, X_1]] \in \Delta',$$

since both brackets $[X_2, Y]$ and $[Y, X_1]$ remain in $\Delta$ by virtue of $Y$ being a symmetry. \qed

We note that Proposition 3.2.3 is also true at the transformation group level since if $\Phi : M \to M$ is a symmetry of $\Delta$, then

$$\Phi_* [X_1, X_2] = [\Phi_* X_1, \Phi_* X_2] \in \Delta'.$$

3.3 Quotient Distributions

Symmetries can be used to construct a new distribution called the quotient distribution. To see this, let $G$ be a symmetry group of a distribution $\Delta$ on $M$ with action $\mu : G \times M \to M$. Let $x \in M/G$ and pick any point $p \in M$ such that $\pi(p) = x$. We can then define the subspace $\tilde{\Delta}_x = \pi_* (\Delta_p)$ of $T_x(M/G)$ where well-definition follows from the fact that for any points $p, q \in M$ lying in the same $G$-orbit, $\mu_g(p) = q$ for some $g \in G$, and $\mu_g(\Delta_p) = \Delta_q$. Indeed,

$$\pi_* (\Delta_q) = \pi_* (\Delta_{\mu_g(p)}) = (\pi \circ \mu_g)_* (\Delta_p) = \pi_* (\Delta_p).$$

Definition 3.3.1. The assignment to each point $x \in M/G$ the subspace $\Delta_x$ forms the quotient distribution of $\Delta$ by $G$, which we denote by $\tilde{\Delta} = \Delta/G = \pi_* (\Delta)$.

If an $m$-dimensional manifold $M$ has local coordinates given by $x^1, \ldots, x^m$, and $M/G$ has local coordinates $z^\alpha$, then the quotient map can be constructed by first calculating the
$G$-invariant functions $I^\alpha(x_1, \ldots, x_m)$ and explicitly defining

$$
\pi^\alpha(x^1, \ldots, x^m) = z^\alpha
$$

where $z^\alpha = I^\alpha(x^1, \ldots, x^m)$. To write the quotient distribution $\tilde{\Delta}$ in these coordinates, we let $\sigma : M/G \to M$ be a local cross-section defined by

$$
\sigma^i(z^\alpha) = x^i.
$$

Then, $\tilde{\Delta}_q = \pi_* (\Delta_{\sigma(q)})$ for all $q \in M/G$. In Maple, the invariant functions $I^\alpha(x^1, \ldots, x^m)$ can be computed using the command `InvariantGeometricObjectFields` in the `DifferentialGeometry` package.

In the special case where the action of $G$ is both free and transverse, we obtain the following theorem.

**Theorem 3.3.2.** Let $\Delta$ be a rank $k$ distribution on a manifold $M$ with symmetry group $G$ of dimension $n$, acting freely and transversely, then $\Delta$ has a basis of $G$-invariant vector fields and the quotient distribution $\Delta/G$ has rank $k$ on $M/G$.

**Remark.** This gives an alternative way to calculate $\tilde{\Delta}$ by simply calculating the pushforward of the invariant vector fields on $M$ by the quotient map.

**Proof.** Let $\Delta$ be a rank $k$ distribution on $M$, and let $G$ be an $n$-dimensional symmetry group of $\Delta$, acting freely and transversely. Suppose $G$ has infinitesimal generators $\Gamma = \text{span}\{Y_\alpha\}$, and let $X_i \in \Delta$. 

Since $Y_\alpha$ is a symmetry of $\Delta$, $[Y_\alpha, X_i] = \lambda^j_{\alpha i} X_j$ for smooth functions $\lambda^j_{\alpha i}$ on $M$. Using this fact along with the Jacobi identity, we have

$$0 = [Y_\beta, [Y_\alpha, X_i]] + [Y_\alpha, [X_i, Y_\beta]] + [X_i, [Y_\beta, Y_\alpha]]$$

$$= [Y_\beta, \lambda^k_{\alpha i} X_k] + [Y_\alpha, -\lambda^k_{\beta i} X_k] + [X_i, c^\gamma_{\alpha \beta} Y_\gamma]$$

$$= Y_\beta (\lambda^k_{\alpha i}) X_k + \lambda^k_{\alpha i} Y_\beta (X_k) - \lambda^k_{\alpha i} X_k (Y_\beta) - Y_\alpha (\lambda^k_{\beta i}) X_k + \lambda^k_{\beta i} Y_\alpha (X_k) + \lambda^k_{\beta i} X_k (Y_\alpha) + c^\gamma_{\alpha \beta} [X_i, Y_\gamma]$$

$$= \left( Y_\beta (\lambda^k_{\alpha i}) - Y_\alpha (\lambda^k_{\beta i}) - c^\gamma_{\alpha \beta} \right) X_k + \lambda^k_{\alpha i} Y_\beta (X_k) - \lambda^k_{\beta i} Y_\alpha (X_k)$$

$$= \left( Y_\beta (\lambda^k_{\alpha i}) - Y_\alpha (\lambda^k_{\beta i}) - c^\gamma_{\alpha \beta} \lambda^k_{\gamma i} + \lambda^m_{\alpha i} \lambda^k_{\beta m} - \lambda^m_{\beta i} \lambda^k_{\alpha m} \right) X_k,$$

so the $\lambda^j_{\alpha i}$ must satisfy

$$Y_\alpha (\lambda^k_{\beta i}) - Y_\beta (\lambda^k_{\alpha i}) = \lambda^m_{\alpha i} \lambda^k_{\beta m} - \lambda^m_{\beta i} \lambda^k_{\alpha m} - c^\gamma_{\alpha \beta} \lambda^k_{\gamma i}.$$

A vector field $X^G = f^j X_j \in \Delta$ will be $G$-invariant if there exist functions $f^j \in C^\infty(M)$, not all zero, such that $\mathcal{L}_{Y_\alpha} (X^G_i) = 0$. In calculating, we have

$$0 = \mathcal{L}_{Y_\alpha} (X^G) = \mathcal{L}_{Y_\alpha} (f^j X_j) = [Y_\alpha, f^j X_j]$$

$$= Y_\alpha (f^j) X_j + f^j Y_\alpha (X_j) - f^j X_j (Y_\alpha)$$

$$= Y_\alpha (f^j) X_j + f^j [Y_\alpha, X_j]$$

$$= \left( Y_\alpha (f^j) + f^k \lambda^j_{\alpha k} \right) X_j.$$

Therefore, the functions $f^j$ must satisfy $Y_\alpha (f^j) = -f^k \lambda^j_{\alpha k}$. But, since $G$ acts freely, the vector fields $Y_\alpha$ are all pointwise linearly independent, and so the system is Frobenius; meaning the system will be satisfied if and only if its integrability condition is satisfied. That is, if

$$Y_\beta (Y_\alpha f^j) - Y_\alpha (Y_\beta f^j) = [Y_\beta, Y_\alpha] (f^j).$$
After evaluating,

\[ Y_\beta(Y_\alpha f^j) = -Y_\beta(f^k \lambda^j_{\alpha k}) = -Y_\beta(f^k) \lambda^j_{\alpha k} - f^k Y_\beta(\lambda^j_{\alpha k}) = f^\ell \left( \lambda^k_{\beta \ell} \lambda^j_{\alpha k} - Y_\beta(\lambda^j_{\alpha \ell}) \right), \]

we see that this is precisely the condition given by the Jacobi identity since,

\[ c^{\gamma}_{\alpha \beta} \lambda^j_{\gamma \ell} f^\ell = -c^{\gamma}_{\alpha \beta} Y_\gamma(f^j) = [Y_\beta, Y_\alpha](f^j) = Y_\beta(Y_\alpha f^j) - Y_\alpha(Y_\beta f^j) \]

\[ = \left( \lambda^k_{\beta \ell} \lambda^j_{\alpha k} - Y_\beta(\lambda^j_{\alpha \ell}) - \lambda^k_{\alpha \ell} \lambda^j_{\beta k} + Y_\alpha(\lambda^j_{\beta \ell}) \right) f^\ell. \]

Therefore, there always exist nonzero functions \( f^j \) such that \( X^G = f^j X_j \) is \( G \)-invariant.

The set \( \{ X^G_1, X^G_2, \ldots, X^G_k \} \) will be linearly independent if

\[ X^G_1 \wedge X^G_2 \wedge \cdots \wedge X^G_k = \pm \det(f^j_i) X_1 \wedge X_2 \wedge \cdots \wedge X_k \neq 0, \]

meaning \( \Delta \) will have a local basis of \( G \)-invariant vector fields provided \( \det(f^j_i) \neq 0 \).

**Theorem 3.3.3.** Let \( \Delta \) be a distribution on manifold \( M \) with Cauchy characteristics \( \mathcal{A}(\Delta) \) and let \( G \) be a symmetry group of \( \Delta \). Then \( \mathcal{A}(\Delta) \) is a \( G \)-invariant distribution; that is, if \( G \) has infinitesimal generators \( \Gamma \), then \( [\Gamma, \mathcal{A}(\Delta)] \subseteq \mathcal{A}(\Delta) \), and there exists a well-defined set of Cauchy characteristics \( \mathcal{A}(\Delta/G) \) on \( M/G \).

**Remark.** It is important to note that \( \text{rank}(\mathcal{A}(\Delta/G)) \geq \text{rank}(\mathcal{A}(\Delta)) \).

**Proof.** Let \( \Delta \) be a distribution on manifold \( M \) with Cauchy characteristics \( \mathcal{A}(\Delta) \) and let \( G \) be a symmetry group of \( \Delta \). Denote the infinitesimal generators of \( G \) by \( \Gamma \), and let \( X \in \Delta \), \( Y \in \mathcal{A}(\Delta) \), and \( Z \in \Gamma \). Then by the Jacobi identity,

\[ [X, [Z, Y]] = -[Z, [Y, X]] - [Y, [X, Z]]. \]

But since \( Y \in \mathcal{A}(\Delta) \) and \( Z \in \Gamma \), the brackets \([X, Y]\) and \([Z, X]\) are each in \( \Delta \), and it follows that \([Z, Y]\) is a Cauchy characteristic of \( \Delta \). We then conclude that \([\Gamma, \mathcal{A}(\Delta)] \subseteq \mathcal{A}(\Delta)\).
At the group level, if the action of $G$ is given by $\mu_g$, then

$$[[\mu_g Y, X]] = [[\mu_g Y, \mu_g^{-1} X]] = \mu_g [[Y, \mu_g^{-1} X]] \in \Delta,$$

and we again see that $\mathcal{A}(\Delta)$ is $G$-invariant.

It will be essential for us to know precisely when a set of vector fields in a distribution $\Delta$ quotient to become Cauchy characteristics of the quotient distribution $\bar{\Delta}$. We call vector fields with this property general Cauchy characteristics.

**Definition 3.3.4.** Let $\Delta$ be a distribution on manifold $M$, and let $G$ be a symmetry group of $\Delta$. Denote the set of infinitesimal generators of $G$ by $\Gamma$ and the distribution generated by the vector fields of $\Gamma$ by $\mathcal{L} = \text{span}_{C^\infty(M)} \Gamma$. A **generalized Cauchy characteristic** is a vector field $Y \in \Delta$ such that $[Y, \Delta] \subseteq \Delta \oplus \Gamma$. We denote the set of generalized Cauchy characteristics of $\Delta$ with respect to $\Gamma$ by $\mathcal{A}(\Delta, \Gamma)$.

**Theorem 3.3.5.** Let $\Delta$ be a distribution on a manifold $M$. Let $G$ be a symmetry group of $\Delta$ on $M$ and with infinitesimal generators $\Gamma$. Then the set of generalized Cauchy characteristics $\mathcal{A}(\Delta, \Gamma)$ is $G$-invariant, and if in addition $G$ acts freely and transversely on $M$, then

$$\mathcal{A}(\Delta, \Gamma)/G = \mathcal{A}(\Delta/G).$$

*Proof.* Let $X \in \Delta$, $Y \in \mathcal{A}(\Delta, \Gamma)$, and $Z \in \Gamma$. Then again by the Jacobi identity,

$$[[Z, Y], X] = -[[Y, X], Z] - [[X, Z], Y].$$

Since $[Y, X] \in \Delta \oplus \Gamma$, we can write $[Y, X] = W + U$ for some $W \in \Delta$ and $U \in \Gamma$. But then

$$[[Y, X], Z] = [W + U, Z] = [W, Z] + [U, Z] \in \Delta \oplus \Gamma.$$
Additionally, \([X, Z] \in \Delta\), and we conclude that the bracket \([[[X, Z], Y]]\) is in \(\Delta \oplus \Gamma\) as well. We then see that \([[[Z, Y], X]] \in \Delta \oplus \Gamma\), and conclude that \([Z, Y] \in \mathcal{A}(\Delta, \Gamma)\), meaning \(\mathcal{A}(\Delta, \Gamma)\) is \(G\)-invariant.

At the group level, if the action of \(G\) is given by \(\mu_g\), then

\[
\left[\mu_g Y, X\right] = \left[\mu_g Y, \mu_g (\mu_g^{-1} X)\right] = \mu_g [Y, \mu_g^{-1} X] \in \Delta \oplus \Gamma,
\]

and we again see that \(\mathcal{A}(\Delta, \Gamma)\) is \(G\)-invariant.

We now further suppose that the action of \(G\) is free and transverse on \(M\), so that \(\Delta\) has a basis of \(G\)-invariant vector fields. When this is the case, we immediately see that every generalized Cauchy characteristic quotients to a Cauchy characteristic of \(\Delta/G\), meaning \(\mathcal{A}(\Delta, \Gamma)/G \subseteq \mathcal{A}(\Delta/G)\), since

\[
\pi_*[\mathcal{A}(\Delta, \Gamma), \Delta] \subseteq \pi_* (\Delta \oplus \Gamma) = \pi_* (\Delta) = \Delta/G.
\]

Now let \(\bar{Y} \in \Delta/G\). We show that there exists a vector field \(Y \in \mathcal{A}(\Delta, \Gamma)\) such that \(\pi_* Y = \bar{Y}\). To do so, let \(X\) be any vector field in \(\Delta\), let \(\bar{X} = \pi_* X\), and let

\[
\bar{W} = [\bar{Y}, \bar{X}] \in \Delta/G.
\]

We may then pick \(W \in \Delta\) such that \(\pi_* W = \bar{W}\). It then follows that,

\[
[Y, X] = [\pi_* Y, \pi_* X] = \pi_* [Y, X] = \pi_* Z,
\]

which implies

\[
\pi_* Z - \pi_* [Y, X] = \pi_* (Z - [Y, X]) = 0.
\]

From this, we conclude that \(Z - [Y, X] \in \Gamma\), but since \(Z \in \Delta\), the vector field \([Y, X]\) must be in \(\Delta \oplus \Gamma\). Therefore, \(Y \in \mathcal{A}(\Delta, \Gamma)\), and \(\bar{Y} \in \mathcal{A}(\Delta, \Gamma)/G\). \(\square\)
When $G$ acts transversely to the derived distribution $\hat{\Delta}'$, we obtain the following theorem.

**Theorem 3.3.6.** Let $G$ be a symmetry group of the distribution $\Delta$ acting freely and transversely on the manifold $M$. Suppose in addition that $G$ is transverse to the derived distribution $\Delta'$. Then

$$(\Delta/G)' = \Delta'/G.$$  

**Proof.** Let $\Delta$ be a rank $k$ distribution on $M$, and let $G$ be a symmetry group of $\Delta$ which acts freely and transversely. Suppose in addition that $G$ acts transversely to $\Delta'$. Then by Theorem 3.3.2, there exists a $G$-invariant basis for $\Delta$ and $\Delta'$ such that $\Delta = \{X^G_1, \ldots, X^G_k\}$ and $\Delta' = \{X^G_1, \ldots, X^G_k; Y^G_1, \ldots, Y^G_\ell\}$. It then follows that

$$\Delta/G = \{\bar{X}^G_1, \ldots, \bar{X}^G_k\} \quad \text{and} \quad \Delta'/G = \{\bar{X}^G_1, \ldots, \bar{X}^G_k; \bar{Y}^G_1, \ldots, \bar{Y}^G_\ell\}$$

where $\bar{X}^G_i = \pi_*(X^G_i)$, $\bar{Y}^G_a = \pi_*(Y^G_a)$ and $\pi : M \to M/G$ is the canonical projection map. Computing $(\Delta/G)'$ gives,

$$(\Delta/G)' = \{\bar{X}^G_1, \ldots, \bar{X}^G_k; \bar{Z}_a = \lambda^{ij}_a [\bar{X}^G_i, \bar{X}^G_j]\}$$

for smooth functions $\lambda^{ij}_a$, but since bracket relations for $G$-invariant vector fields are preserved by the quotient map, we find that $\bar{Z}_a = A^b_a \bar{Y}^G_b$ for some invertible matrix $A$, and conclude that $(\Delta/G)' = \Delta'/G$. \hfill \Box
CHAPTER 4
HYPERBOLIC DARBOUX INTEGRABLE SYSTEMS

Classically, a of hyperbolic second order partial differential equations in the plane is said to be integrable by the method of Darboux, or Darboux integrable, if there exists an of auxiliary system of compatible equations, or intermediate integrals, such that when combined with the original system, form a system of total differential equations which can then be integrated using ODE methods to yield the general solution to the original system. It can happen that an equation which is not Darboux integrable can become Darboux integrable after some finite number of prolongations. If a scalar second-order PDE in the plane becomes Darboux integrable after \( k - 2 \) prolongations, then we say that the original equation is Darboux integrable at order \( k \geq 2 \). In this chapter, we give an interpretation of Darboux integrable hyperbolic partial differential equations in the plane in terms of hyperbolic distributions. We then give several examples of Darboux integrable systems.

4.1 Basic Definitions

In this section, we define what it means for a hyperbolic distribution to be Darboux integrable. Along with this definition, we also introduce an initial frame adaptation which can always be performed for Darboux integrable hyperbolic distributions.

**Definition 4.1.1.** A hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \check{\Delta} \) on an \( m \)-dimensional manifold \( M \) is said to be Darboux integrable if the pair of singular subdistributions \( \hat{\Delta}, \check{\Delta} \) satisfy

\[
\hat{\Delta} \cap \hat{\Delta}^{(\infty)} = \{0\} \quad \text{and} \quad \check{\Delta}^{(\infty)} \cap \check{\Delta} = \{0\}.
\]

Condition (4.1) gives lower bounds on the number of first integrals for \( \hat{\Delta} \) and \( \check{\Delta} \). Indeed, if we suppose \( \text{rank}(\hat{\Delta}) = \hat{r} \) and \( \text{rank}(\check{\Delta}) = \check{r} \) and that \( \hat{\Delta} \) and \( \check{\Delta} \) admit \( \hat{s} \) and \( \check{s} \) first integrals,
then we see that condition (4.1) guarantees that

\[ \hat{r} + \text{rank}(\hat{\Delta}^{(\infty)}) \leq m \quad \text{and} \quad \hat{s} + \text{rank}(\hat{\Delta}^{(\infty)}) \leq m. \]

But since \( \text{rank}(\hat{\Delta}^{(\infty)}) = m - \hat{s} \) and \( \text{rank}(\hat{\Delta}^{(\infty)}) = m - \hat{r} \), this implies \( \hat{s} \geq \hat{r} \) and \( \hat{s} \geq \hat{r} \).

**Remark.** When \( \Delta \) defines a hyperbolic PDE in the plane, condition (4.1) guarantees that \( \hat{\Delta} \) and \( \check{\Delta} \) each have at least two first integrals. These first integrals directly correspond to the intermediate integrals of the corresponding PDE in the plane, so that the equation defined by \( \Delta \) is Darboux integrable classical sense. We say that a hyperbolic distribution which defines a hyperbolic PDE in the plane is *Darboux integrable at order k* if the first integrals of \( \hat{\Delta} \) and \( \check{\Delta} \) are of differential order less than or equal to \( k \).

If \( \hat{\Delta}, \check{\Delta} \) are singular subdistributions of the hyperbolic distribution \( \Delta \) on \( M \) satisfying the hypotheses of Definition 4.1.1 and the derived distributions \( \hat{\Delta}^{(\infty)} \) and \( \check{\Delta}^{(\infty)} \), as well as the sums \( \hat{\Delta}^{(\infty)} + \check{\Delta} \) and \( \hat{\Delta} + \check{\Delta}^{(\infty)} \), are constant rank subbundles of \( TM \), then we may construct a frame on \( TM \) adapted to \( \Delta \) by choosing tuples of independent vector fields \( \hat{X}, \check{X}, \) and \( Z \) such that

\[ \hat{\Delta} = \text{span}\{\hat{X}\}, \quad \check{\Delta} = \text{span}\{\check{X}\}, \quad \text{and} \quad \hat{\Delta}^{(\infty)} \cap \check{\Delta}^{(\infty)} = \text{span}\{Z\}. \tag{4.2} \]

Property (4.1) guarantees the set of vector fields \( \{\hat{X}, \check{X}, Z\} \) is linearly independent. To see this, suppose

\[ A\hat{X} + B\check{X} + C\check{Z} = 0. \]

Then, \( A\hat{X} = -B\check{X} - C\check{Z} \in \hat{\Delta}^{(\infty)} \), and \( A\hat{X} \in \hat{\Delta} \cap \check{\Delta}^{(\infty)} = \{0\} \). Since the \( \hat{X} \) are independent, \( A = 0 \). We then see that \( B\check{X} = -C\check{Z} \in \hat{\Delta}^{(\infty)} \), and by slightly adjusting the previous argument, if follows that \( B = C = 0 \), proving the claim.
We can then further choose tuples of vector fields \( \hat{Y} \) and \( \tilde{Y} \) satisfying

\[
\hat{\Delta}^{(\infty)} = \text{span}\{ \hat{X}, \hat{Y}, \bar{Z} \} \quad \text{and} \quad \tilde{\Delta}^{(\infty)} = \text{span}\{ \bar{X}, \tilde{Y}, \bar{Z} \}.
\]  

(4.3)

In doing so, the set of vector fields \( \mathcal{F} = \{ \hat{X}, \hat{Y}, \bar{X}, \tilde{Y}, \bar{Z} \} \) forms a frame for \( TM \) adapted to \( \Delta \). Since \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) is bracket-generating, \( \mathcal{F} \) clearly spans \( TM \). To show \( \mathcal{F} \) is a linearly independent set of vector fields, let \( V = A\hat{X} + B\bar{X} + C\bar{Z} + E\tilde{Y} + F\hat{Y} \), and first notice that if \( V \in \hat{\Delta} \), then \( F\hat{Y} \in \hat{\Delta} \cap \hat{\Delta}^{(\infty)} = \{ 0 \} \), so that if \( V \in \hat{\Delta} \), then \( F = 0 \). Now if \( V = 0 \), we immediately see that \( B\bar{X} = -A\hat{X} - C\bar{Z} - E\tilde{Y} \in \hat{\Delta}^{(\infty)} \), and as before \( B\bar{X} \in \hat{\Delta}^{(\infty)} \cap \tilde{\Delta} \), from which we conclude \( B = 0 \). Then, since \( \hat{\Delta}^{(\infty)} \) is a linearly independent set of vector fields, it follows that \( A = C = E = 0 \). Therefore, \( \mathcal{F} \) forms a frame for \( TM \) adapted to \( \Delta \).

Any local frame satisfying (4.2) and (4.3) is called a \( 0 \)-adapted frame of \( \Delta \).

### 4.2 The Fundamental Theorem of Darboux Integrable Systems

Suppose \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) is a hyperbolic Darboux integrable distribution on \( M \) and that \( \text{rank}(\hat{\Delta}) = \hat{r} \) and \( \text{rank}(\tilde{\Delta}) = \tilde{r} \). Let \( \{ \hat{I}^a \} \) and \( \{ \tilde{I}^\alpha \} \) be sets of functionally independent first integrals of \( \hat{\Delta} \) and \( \tilde{\Delta} \), respectively, with \( 1 \leq a \leq \hat{s} \) and \( 1 \leq \alpha \leq \tilde{s} \).

**Lemma 4.2.1.** The matrices \( \hat{A}^a_i = \hat{X}_i(\hat{I}^a) \) with \( 1 \leq i \leq \hat{r} \) and \( 1 \leq a \leq \hat{s} \), and \( \tilde{B}^a_h = \tilde{X}_h(\tilde{I}^a) \) with \( 1 \leq h \leq \tilde{r} \) and \( 1 \leq a \leq \tilde{s} \), have full rank.

**Proof.** Let \( P = \xi^i \hat{X}_i \), and suppose \( P(\hat{I}^a) = \xi^i \hat{A}^a_i = 0 \) for all \( i, a \). Then \( P \in \hat{\Delta}^{(\infty)} \), and consequently, \( P \in \hat{\Delta} \cap \hat{\Delta}^{(\infty)} = \{ 0 \} \). Similarly, if \( Q = \tilde{\xi}^h \tilde{X}_h \), and we suppose \( Q(\tilde{I}^a) = \tilde{\xi}^h \tilde{B}^a_h = 0 \) for all \( h, a \), then \( Q \in \tilde{\Delta}^{(\infty)} \cap \tilde{\Delta} = \{ 0 \} \). Therefore, the nullspaces of \( \hat{A} \) and \( \tilde{B} \) are trivial, and both must have full rank. \( \square \)

Since \( \hat{s} \geq \hat{r} \) and \( \tilde{s} \geq \tilde{r} \), Lemma 4.2.1 implies that there exist nonsingular \( \hat{r} \times \hat{r} \) and \( \tilde{r} \times \tilde{r} \) matrices

\[
\hat{A}^i_j = [\hat{X}_j(\hat{I}^i)] \quad \text{and} \quad \tilde{B}^h_k = [\tilde{X}_k(\tilde{I}^h)],
\]
with \(1 \leq i, j \leq \tilde{r}\) and \(1 \leq h, k \leq \tilde{r}\), and vector fields

\[
\hat{U}_i = (A^{-1})^j_i \hat{X}_j \quad \text{and} \quad \tilde{U}_h = (B^{-1})^k_h \tilde{X}_k.
\]

**Theorem 4.2.2** (The Fundamental Theorem of Darboux Integrable Systems).

Let \(\Delta = \tilde{\Delta} \oplus \tilde{\Delta}\) be a hyperbolic Darboux integrable system. Then there exists local bases \(\{\hat{U}_i\}\) and \(\{\tilde{U}_h\}\) of \(\tilde{\Delta}\) and \(\tilde{\Delta}\), respectively, which satisfy \([\hat{U}_i, \tilde{U}_h] = 0\) for all \(i, h\).

**Proof.** We first note that for \(1 \leq i, j, p \leq \tilde{r}\),

\[
\hat{U}_i(\bar{I}^p) = (A^{-1})^j_i \hat{X}_j(\bar{I}^p) = (A^{-1})^j_i A^p_j = \delta^p_i,
\]

and similarly for \(1 \leq h, k, q \leq \tilde{r}\),

\[
\tilde{U}_h(\bar{I}^q) = (B^{-1})^k_h \tilde{X}_k(\bar{I}^q) = (B^{-1})^k_h B^q_k = \delta^q_h.
\]

Then,

\[
[\hat{U}_i, \tilde{U}_h](\bar{I}^q) = \hat{U}_i(\tilde{U}_h(\bar{I}^q)) - \tilde{U}_h(\hat{U}_i(\bar{I}^q)) = \hat{U}_i(\delta^q_h) - \tilde{U}_h(0) = 0,
\]

and a similar computation shows \([\hat{U}_i, \tilde{U}_h](\bar{I}^p) = 0\). But by hyperbolicity, we conclude the bracket must be of the form

\[
[\hat{U}_i, \tilde{U}_h] = M^j_{ih} \hat{U}_j + N^k_{ih} \tilde{U}_k,
\]

from which we see,

\[
0 = [\hat{U}_i, \tilde{U}_h](\bar{I}^q) = M^j_{ih} \hat{U}_j(\bar{I}^q) + N^k_{ih} \tilde{U}_k(\bar{I}^q) = N^k_{ih} \delta^q_k = N^q_{ih},
\]
and

\[ 0 = [\hat{U}_i, \hat{U}_h](\hat{I}^p) = M^j_{ih} \hat{U}_j(\hat{I}^p) + N^j_{ih} \hat{U}_k(\hat{I}^p) = M^j_{ih} \delta^p_j = M^p_{ih} \]

for all \( i, h, p, q \). Therefore, \([\hat{U}_i, \hat{U}_h] \equiv 0\) for all \( i, h \).

The vector fields \( \{\hat{U}_i\} \) and \( \{\hat{U}_h\} \) can be computed in Maple using the command `CommutingCharacteristicDistributions` in the `DifferentialGeometry` package.

### 4.3 Fundamental Invariants of Darboux Integrable Systems

In this section, we define the fundamental invariants associated to Darboux integrable systems: the Vessiot algebra and the Vessiot distributions. The full details associated with the construction of these invariants are quite involved and can be found in [5].

We begin with the Vessiot algebra associated to a Darboux integrable system. Let

\( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \)

be a hyperbolic Darboux integrable distribution on manifold \( M \). By Theorem 4.2.2, we may assume that \( \hat{\Delta} = \{\hat{U}_i\} \) and \( \tilde{\Delta} = \{\hat{U}_h\} \) where \([\hat{U}_i, \hat{U}_h] = 0\) for \( 1 \leq i \leq \hat{r} \) and \( 1 \leq h \leq \tilde{r} \). Let \( K = \hat{\Delta}^{(\infty)} \cap \tilde{\Delta}^{(\infty)} \), and denote the rank of \( K \) by

\[ \rho = \text{rank}(\hat{\Delta}^{(\infty)}) + \text{rank}(\tilde{\Delta}^{(\infty)}) - \dim M. \]

In [5], Anderson, Fels, and Vassiliou show that you can take iterated brackets of vector fields \( \{\hat{U}_i\} \) and \( \{\hat{U}_h\} \), respectively, to obtain two sets of local bases \( \{\hat{S}_i\} \) and \( \{\hat{S}_j\} \) for \( K \), with \( 1 \leq i, j \leq \rho \), satisfying

[i] \( \hat{S}_i(\hat{I}^a) = \hat{S}_i(\hat{I}^\alpha) = 0 \) and \( \tilde{S}_i(\tilde{I}^a) = \tilde{S}_i(\tilde{I}^\alpha) = 0 \) for \( 1 \leq a \leq \hat{s} \) and \( 1 \leq \alpha \leq \tilde{s} \);

[ii] \([\hat{S}_i, \tilde{S}_j] = 0\);

[iii] \([\hat{S}_i, \tilde{S}_j] = C^k_{ij} \hat{S}_k \) and \([\tilde{S}_i, \tilde{S}_j] = C^k_{ij} \tilde{S}_k \)

where the \( \hat{I}^a \) and \( \tilde{I}^\alpha \) are the \( \hat{s} \) and \( \tilde{s} \) first integrals of \( \hat{\Delta} \) and \( \tilde{\Delta} \), respectively, and the \( C^k_{ij} \) are structure constants for an abstract \( \rho \)-dimensional Lie algebra associated to \( \Delta \), called the Vessiot algebra of \( \Delta \).
**Definition 4.3.1.** Let $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ be a hyperbolic Darboux integrable distribution on manifold $M$, and calculate the vector fields $\{\hat{S}_i\}$ and $\{\tilde{S}_j\}$ described above. Then the structure constants $C_{ij}^k$ given by $[\hat{S}_i, \hat{S}_j] = C_{ij}^k \hat{S}_k$ and $[\tilde{S}_i, \tilde{S}_j] = C_{ij}^k \tilde{S}_k$ define an abstract Lie algebra of dimension $\text{rank}(\hat{\Delta}^{(\infty)}) + \text{rank}(\tilde{\Delta}^{(\infty)}) - \dim M$, called the Vessiot algebra of $\Delta$. Strictly speaking, it is the action of the Vessiot algebra generated by the vector fields $\hat{S}_i$ and $\tilde{S}_j$ which serve as the fundamental invariant.

**Remark.** Anderson, Fels, and Vassiliou prove that the vector fields $\hat{S}_i$ and $\tilde{S}_j$ can always be chosen to satisfy the structure equations

$$[\hat{S}_i, \hat{S}_j] = C_{ij}^k \hat{S}_k \quad \text{and} \quad [\tilde{S}_i, \tilde{S}_j] = C_{ij}^k \tilde{S}_k$$

by performing an algorithmic series of (co)frame adaptations. In practice, one first finds bases for $K$ coming from $\hat{\Delta}^{(\infty)}$ and $\tilde{\Delta}^{(\infty)}$ which instead satisfy

$$[\hat{S}_i, \hat{S}_j] = \hat{H}_{ij}(\hat{I}_a)\hat{S}_k \quad \text{and} \quad [\tilde{S}_i, \tilde{S}_j] = \tilde{H}_{ij}(\tilde{I}_a)\tilde{S}_k$$

where $\hat{I}^a$ and $\tilde{I}^a$ are the first integrals of $\hat{\Delta}$ and $\tilde{\Delta}$, respectively, and then finds a relatively simple change of basis involving functions of $\hat{I}^a$ and $\tilde{I}^a$, respectively, to arrive at the desired structure constants.

They then further show that the Vessiot algebra is indeed an invariant of a hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ via the following theorem.

**Theorem 4.3.2.** Let $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ be a hyperbolic distribution on a manifold $M$, and let $\nabla = \hat{\nabla} \oplus \tilde{\nabla}$ be another hyperbolic distribution on a manifold $N$. If $\Phi : M \to N$ is a diffeomorphism satisfying $\Phi_* (\Delta_p) = \nabla_{\Phi(p)}$, $\Phi_* (\hat{\Delta}_p) = \hat{\nabla}_{\Phi(p)}$, and $\Phi_* (\tilde{\Delta}_p) = \tilde{\nabla}_{\Phi(p)}$, then $\Phi_{p,*}$, at each point $p \in M$, induces a Lie algebra isomorphism from the Vessiot algebra of $\Delta$ to the Vessiot algebra of $\nabla$.

The second fundamental invariant associated to a hyperbolic distribution $\Delta$ is given by the pair of Vessiot distributions $\hat{\nabla}, \tilde{\nabla}$ given by the restrictions of the characteristic dis-
tributions $\tilde{\Delta}$ and $\tilde{\Delta}$ to their maximal, connected integral manifolds given by the level sets of their first integrals.

**Definition 4.3.3.** Let $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ be a Darboux integrable hyperbolic distribution, and suppose that $\hat{\Delta}$ admits the set of first integrals $\{\hat{I}_a\}$ and $\tilde{\Delta}$ admits the set of first integrals $\{\tilde{I}_a\}$. Denote by $\hat{M}$ and $\tilde{M}$ the maximal, connected integral manifolds of $\hat{\Delta}$ and $\tilde{\Delta}$ given by the level set of their first integrals. Then the Vessiot distributions $\hat{V}, \tilde{V}$ are given by the restriction of $\hat{\Delta}$ and $\tilde{\Delta}$ to $\hat{M}$ and $\tilde{M}$, respectively. We will often denote these restrictions by $\hat{V} = \hat{\Delta}|_{\hat{M}}$ and $\tilde{V} = \tilde{\Delta}|_{\tilde{M}}$.

Together, these fundamental invariants completely characterize hyperbolic Darboux integrable distributions, giving the following theorem.

**Theorem 4.3.4.** Let $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ be a hyperbolic distribution on a manifold $M$, and let $\nabla = \hat{\nabla} \oplus \tilde{\nabla}$ be another hyperbolic distribution on a manifold $N$. Then $\Delta$ is locally equivalent to $\nabla$ if and only if

[i] The Vessiot algebras of $\Delta$ and $\nabla$ are equivalent with identical actions, and

[ii] the Vessiot distributions of $\Delta$ and $\nabla$ are equivalent.

When these distributions define hyperbolic Darboux integrable PDE in the plane, Theorem 5.1 of [17] implies that we can reinterpret the local equivalence of the distributions $\Delta$ and $\nabla$ as a contact equivalence between their corresponding PDE. It should be noted however, that in general the equivalence described in Theorem 4.3.4 is an *intrinsic* equivalence between the distributions and may not correspond to an (*extrinsic*) contact equivalence.

### 4.4 Examples of Darboux Integrable Systems

We now provide several examples of hyperbolic Darboux integrable systems. For each, we show that the system is Darboux integrable and calculate its fundamental invariants.
Example 4.4.1 (Trivial Systems). Let $\Delta$ be the hyperbolic distribution given in Example 2.1.10. Then $\Delta$ has characteristic systems $\tilde{\Delta} = \{\partial_{x^1}, \partial_{x^2}\}$ and $\tilde{\Delta} = \{\partial_{y^1}, \partial_{y^2}, \partial_{y^3}\}$, and

$$\tilde{\Delta}^{(\infty)} = \tilde{\Delta} \quad \text{and} \quad \tilde{\Delta}^{(\infty)} = \tilde{\Delta}.$$ 

The first integrals for $\tilde{\Delta}$ are $\{y^1, y^2, y^3\}$, and the invariants of $\tilde{\Delta}$ are $\{x^1, x^2\}$. Since, each characteristic distribution admits at least two first integrals, $\Delta$ is Darboux integrable. However, $\tilde{\Delta}^{(\infty)} \cap \tilde{\Delta}^{(\infty)} = \{0\}$ which immediately implies that the Vessiot algebra is trivial. The Vessiot distributions are $\hat{V} = \tilde{\Delta}_{|y^1=y^2=y^3=0} = \{\partial_{x^1}, \partial_{x^2}\}$ and $\hat{V} = \tilde{\Delta}_{|x^1=x^2=0} = \{\partial_{y^1}, \partial_{y^2}, \partial_{y^3}\}$.

Example 4.4.2 (First-Order Systems). Consider the hyperbolic first-order system

$$u_x = e^v, \quad v_y = e^u.$$ \hspace{1cm} (4.4)

On the 6-dimensional manifold $M$ with coordinates $x, y, u, v, u_y, v_x$, the hyperbolic distribution is given by $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ where

$$\hat{\Delta} = \{\partial_x + e^v \partial_u + v_x \partial_v + e^{u+v} \partial_{u_y}, \partial_{v_x}\},$$

$$\tilde{\Delta} = \{\partial_y + u_y \partial_u + e^u \partial_u + e^{u+v} \partial_{v_x}, \partial_{u_y}\}.$$ 

The first integrals of $\hat{\Delta}$ and $\tilde{\Delta}$ are

$$\hat{I}_1 = y, \quad \hat{I}_2 = u_y - e^u, \quad \text{and} \quad \tilde{I}_1 = x, \quad \tilde{I}_2 = v_x - e^v,$$

respectively. Since each characteristic distribution admits two first integrals of order less than or equal to one, (4.4) is Darboux integrable at order one. Since

$$\hat{\Delta}^{(\infty)} \cap \tilde{\Delta}^{(\infty)} = \{\partial_u + e^u \partial_{u_y}, \partial_v + e^v \partial_{v_x}\},$$
we see that the Vessiot algebra must be 2-dimensional. Following Theorem 4.2.2, we construct commuting bases of vector fields

\[
\hat{\Delta} = \{ \hat{U}_1 = \partial_x + e^v \partial_u + v_x \partial_v, \hat{U}_2 = \partial_{v_x} \}, \\
\bar{\Delta} = \{ \bar{U}_1 = \partial_y + u_y \partial_u + v_y \partial_v, \bar{U}_2 = \partial_{u_y} \}.
\]

We then calculate the vector fields

\[
\hat{S}_1 = [\hat{U}_1, \hat{U}_2] = -\partial_v - e^v \partial_{v_x}, \\
\hat{S}_2 = [\hat{U}_1, \hat{S}_1] = e^v \partial_u + e^v \partial_v + e^{u+v} \partial_{u_y} + e^{2v} \partial_{v_x},
\]
and

\[
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2] = -\partial_u - e^u \partial_{u_y}, \\
\tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1] = e^u \partial_u + e^u \partial_v + e^{2u} \partial_{u_y} + e^{u+v} \partial_{v_x},
\]

and take \( \mathfrak{H} = \{-\hat{S}_1, \hat{S}_2\} \) and \( \mathfrak{G} = \{-\tilde{S}_1, \tilde{S}_2\} \) as representations for Vessiot algebra of (4.4). In particular, we see that these vector fields satisfy the structure equations

\[
[\hat{S}_1, \hat{S}_2] = \hat{S}_2 \quad \text{and} \quad [\tilde{S}_1, \tilde{S}_2] = \tilde{S}_2,
\]

and we can identify the abstract Vessiot algebra as the 2-dimensional nilpotent Lie algebra \( \mathfrak{s}_{2,1} \) in [28].

The restriction of \( \hat{\Delta} \) and \( \bar{\Delta} \) to the integral manifolds \( \hat{\mathcal{M}}, \bar{\mathcal{M}} \) given by \( \hat{I}_1 = \hat{I}_2 = 0 \) and \( \bar{I}_1 = \bar{I}_2 = 0 \), respectively, are

\[
\hat{\mathcal{V}} = \hat{\Delta}|_{\hat{\mathcal{M}}} = \{ \partial_x + e^v \partial_u + v_x \partial_v, \partial_{v_x} \} \quad \text{and} \quad \tilde{\mathcal{V}} = \bar{\Delta}|_{\bar{\mathcal{M}}} = \{ \partial_y + u_y \partial_u + e^u \partial_v, \partial_{u_y} \}.
\]

Each of these are rank 2 distributions on 4-dimensional manifolds, so Engel’s theorem (see Theorem 2.5.1) immediately implies that \( \hat{\mathcal{V}} \) and \( \tilde{\mathcal{V}} \) are each locally equivalent to the standard
Remark. We have shown (4.4) to be the only nonlinear system of this type which is Darboux integrable at order one. This system is completely characterized as the system whose actions of its Vessiot algebra are generated by second prolongation of $p_{2,2}$ in Table A.1 and whose Vessiot distributions are both the standard contact distributions on $J^2(\mathbb{R}, \mathbb{R})$.

Example 4.4.3. In Example 2.3.2, we gave the hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ for the hyperbolic Liouville equation, $s = e^u$, in terms of the characteristic distributions

$$\hat{\Delta} = \{ \partial_x + p\partial_u + r\partial_p + e^u\partial_q + qe^u\partial_t, \partial_r \},$$
$$\tilde{\Delta} = \{ \partial_y + q\partial_u + e^u\partial_p + t\partial_q + pe^u\partial_r, \partial_t \}.$$  

The first integrals of $\hat{\Delta}$ and $\tilde{\Delta}$ are

$$\hat{I}_1 = y, \quad \hat{I}_2 = t - \frac{q^2}{2}, \quad \text{and} \quad \tilde{I}_1 = x, \quad \tilde{I}_2 = r - \frac{p^2}{2},$$

respectively. Since each characteristic distribution admits two first integrals of order less than or equal to two, the hyperbolic Liouville equation is Darboux integrable at order two.

In utilizing Theorem 4.2.2, we construct the commuting bases $\{ \hat{U}_i \}$ and $\{ \tilde{U}_j \}$ for $\hat{\Delta}$ and $\tilde{\Delta}$, as

$$\hat{\Delta} = \{ \hat{U}_1 = \hat{X}_1 + r p \hat{X}_2, \hat{U}_2 = \hat{X}_2 \} \quad \text{and} \quad \tilde{\Delta} = \{ \tilde{U}_1 = \tilde{X}_1 + t q \tilde{X}_2, \tilde{U}_2 = \tilde{X}_2 \}.$$  

We then compute the sequences of vector fields

$$\hat{S}_1 = [\hat{U}_1, \hat{U}_2] = -\partial_p - p\partial_r,$$
$$\hat{S}_2 = [\hat{U}_1, \hat{S}_1] = \partial_u + p\partial_p + p^2\partial_r,$$
$$\hat{S}_3 = [\hat{U}_1, \hat{S}_2] = -p\partial_u + (r - p^2)\partial_p - e^u\partial_q + p(r - p^2)\partial_r - qe^u\partial_t,$$
and

\[
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2] = -\partial_q - q\partial_t,
\]
\[
\tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1] = \partial_u + q\partial_q + q^2\partial_t,
\]
\[
\tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2] = -q\partial_u - e^u\partial_p + (t - q^2)\partial_q - pep^u\partial_r + q(t - q^2)\partial_t.
\]

We also note that \([\tilde{U}_2, \tilde{S}_i] = 0, [\tilde{U}_2, \tilde{S}_i] = 0\) for \(1 \leq i \leq 2\), \([\tilde{U}_2, \tilde{S}_3] = -\tilde{S}_1\), and \([\tilde{U}_2, \tilde{S}_3] = -\tilde{S}_1\).

Since \(\tilde{\Delta}(\infty) \cap \tilde{\Delta}(\infty) = \{\partial_u, \partial_p, \partial_q\}\), these vector fields form bases \(\tilde{\mathfrak{G}} = \{\tilde{S}_i\}_{i=1}^3\) and \(\tilde{\mathfrak{H}} = \{\tilde{S}_j\}_{j=1}^3\) for the 3-dimensional Vessiot algebra of Liouville’s equation. In particular, the structure equations for \(\tilde{\mathfrak{G}}\) are

\[
[\tilde{S}_1, \tilde{S}_2] = \tilde{S}_1, \quad [\tilde{S}_1, \tilde{S}_3] = \tilde{S}_2, \quad [\tilde{S}_2, \tilde{S}_3] = \tilde{S}_3,
\]

and the structure equations for \(\tilde{\mathfrak{H}}\) are

\[
[\tilde{S}_1, \tilde{S}_2] = \tilde{S}_1, \quad [\tilde{S}_1, \tilde{S}_3] = \tilde{S}_2, \quad [\tilde{S}_2, \tilde{S}_3] = \tilde{S}_3.
\]

As abstract Lie algebras, both \(\tilde{\mathfrak{G}}\) and \(\tilde{\mathfrak{H}}\) are equivalent to \(\mathfrak{sl}(2)\) in [28].

Upon restricting to the integral manifold \(\tilde{M}\) given by \(\tilde{I}_1 = \tilde{I}_2 = 0\), we see that the distribution \(\tilde{\Delta}\) becomes,

\[
\hat{\mathfrak{V}} = \tilde{\Delta}|_{\tilde{M}} = \{\partial_x + p\partial_u + r\partial_p + e^u\partial_q, \partial_t\},
\]

and similarly, after restricting to the integral manifold \(\tilde{M}\) given by \(\tilde{I}_1 = \tilde{I}_2 = 0\), we see that the distribution \(\tilde{\Delta}\) becomes

\[
\hat{\mathfrak{V}} = \tilde{\Delta}|_{\tilde{M}} = \{\partial_y + q\partial_u + e^u\partial_p + t\partial_q, \partial_t\}.
\]

Both \(\hat{\mathfrak{V}}\) and \(\tilde{\mathfrak{V}}\) are rank 2 distributions defined on 5-manifolds whose derived flags have dimension \((2,3,4,5)\). We further see that the weak derived flag for each distribution has
dimension (2,3,4,5) as well, and by Theorem 2.5.2, we conclude that both \( \tilde{V} \) and \( \bar{V} \) are equivalent to contact distributions on \( J^3(\mathbb{R}, \mathbb{R}) \).

Since \( \mathfrak{H} \) and \( \mathfrak{H} \) are symmetry algebras of \( \Delta \) and \( \bar{\Delta} \), respectively, their restrictions \( \mathfrak{H}|_{\tilde{M}} \) and \( \mathfrak{H}|_{\bar{M}} \) will be symmetry algebras of \( \tilde{V} \) and \( \bar{V} \).

There are three different actions of \( \mathfrak{sl}(2) \) on \( J^3(\mathbb{R}, \mathbb{R}) \). To distinguish which of these actions corresponds to \( \mathfrak{H} \), we calculate the restriction of the Killing form \( \kappa \) of \( \mathfrak{H} \) to contact distribution \( \tilde{V} \). In doing so, we see that

\[
\tilde{\kappa}|_{\tilde{M}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},
\]

and clearly \( \det \tilde{\kappa}|_{\tilde{M}} = 0 \). This implies that \( \mathfrak{H} \) corresponds to the third prolongation of the intransitive \( \mathfrak{sl}(2) \) action given by \( p_{3,9} \) in Table A.1. A similar calculation shows that \( \mathfrak{H} \) corresponds to the third prolongation \( p_{3,9} \) as well.

**Example 4.4.4.** The equation

\[
s = pu
\]

was shown to be Darboux integrable at order three by Goursat in [19]. Indeed, after calculating the prolonged characteristic distributions

\[
\hat{\Delta}^{(1)} = \{ \partial_x + p\partial_u + r\partial_p + pu\partial_q + \rho\partial_r + p(u^2 + q)\partial_t + p(u^3 + 3qu + t)\partial_r, \partial_p \},
\]

\[
\tilde{\Delta}^{(1)} = \{ \partial_y + q\partial_u + pu\partial_p + t\partial_q + (p^2 + ru)\partial_r + \tau\partial_t + (3rp + \rho u)\partial_\rho, \partial_\tau \}.
\]

on the 9-dimensional manifold \( M^{(1)} \) with coordinates \( x,y,u,p,q,r,t,\rho,\tau \), we see that the first integrals for \( \hat{\Delta}^{(1)} \) and \( \tilde{\Delta}^{(1)} \) are

\[
\hat{I}_1 = y, \quad \hat{I}_2 = q - \frac{u^2}{2}, \quad \hat{I}_3 = t - qu, \quad \hat{I}_4 = \tau - tu - qu^2 + \frac{u^4}{4},
\]

\[
\tilde{I}_1 = y, \quad \tilde{I}_2 = \frac{2\rho p - 3r^2}{2p^2}.
\]
Since both characteristic distributions have at least two first integrals of order less than or equal to three, we conclude that (4.5) is Darboux integrable at order three.

In utilizing Theorem 4.2.2, we construct the commuting bases \( \{\hat{U}_i\} \) and \( \{\tilde{U}_j\} \) for \( \hat{\Delta} \) and \( \tilde{\Delta} \), as

\[
\hat{\Delta}^{(1)} = \left\{ \hat{U}_1 = \hat{X}_1 + \frac{r(4\rho p - 3r^2)}{p^2} \hat{X}_2, \hat{U}_2 = p\hat{X}_2 \right\},
\]

\[
\tilde{\Delta}^{(1)} = \left\{ \tilde{U}_1 = \tilde{X}_1 + (\tau u + tq + 2q^2u - qu^3)\tilde{X}_2, \tilde{U}_2 = \tilde{X}_2 \right\}.
\]

By calculating \( (\hat{\Delta}^{(1)})^{(\infty)} \cap (\tilde{\Delta}^{(1)})^{(\infty)} \), we see that the Vessiot group is 3-dimensional, and we compute the sequences of vector fields

\[
\hat{S}_1 = [\hat{U}_1, \hat{U}_2], \quad \hat{S}_2 = [\hat{U}_1, \hat{S}_1], \quad \hat{S}_3 = [\hat{U}_1, \hat{S}_2], \\
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2], \quad \tilde{S}_4 = [\tilde{U}_1, \tilde{S}_3], \quad \tilde{S}_5 = [\tilde{U}_1, \tilde{S}_4].
\]

These vector fields form bases \( \hat{\mathfrak{H}} = \{\hat{S}_i\}_{i=1}^3 \) and \( \tilde{\mathfrak{H}} = \{\tilde{S}_j\}_{j=3}^5 \) for the 3-dimensional Vessiot algebra for (4.5). The structure equations for \( \hat{\mathfrak{H}} \) are

\[
[\hat{S}_1, \hat{S}_2] = \hat{S}_1, \quad [\hat{S}_1, \hat{S}_3] = \hat{S}_2, \quad [\hat{S}_2, \hat{S}_3] = 2\hat{I}_4\hat{S}_1 + \hat{S}_3.
\]

After making the change of basis,

\[
\hat{S}_1 \mapsto \tilde{S}_1, \quad \hat{S}_2 \mapsto \tilde{S}_2, \quad \hat{S}_3 \mapsto \tilde{S}_3 + \tilde{I}_4\tilde{S}_1,
\]

the structure equations for \( \tilde{\mathfrak{H}} \) become

\[
[\tilde{S}_1, \tilde{S}_2] = \tilde{S}_1, \quad [\tilde{S}_1, \tilde{S}_3] = \tilde{S}_2, \quad [\tilde{S}_2, \tilde{S}_3] = \tilde{S}_3.
\]
At the outset, the structure equations for \( \mathfrak{V} \) look much worse, however, after making the change of basis,

\[
\tilde{S}_3 \mapsto \tilde{S}_3 + 2\tilde{I}_2\tilde{S}_1, \quad \tilde{S}_4 \mapsto \tilde{S}_4 + 2\tilde{I}_3\tilde{S}_1 + 2\tilde{I}_2\tilde{S}_2, \quad \tilde{S}_5 \mapsto \tilde{S}_5 + 2\tilde{I}_4\tilde{S}_1 + 4\tilde{I}_3\tilde{S}_2 + 3\tilde{I}_2\tilde{S}_3
\]

the structure equations for \( \mathfrak{V} \) become

\[
[\tilde{S}_1, \tilde{S}_2] = \tilde{S}_1, \quad [\tilde{S}_1, \tilde{S}_3] = \tilde{S}_2, \quad [\tilde{S}_2, \tilde{S}_3] = \tilde{S}_3.
\]

As abstract Lie algebras, both \( \mathfrak{V} \) and \( \mathfrak{V} \) are equivalent to \( \mathfrak{sl}(2) \) in [28].

Upon restricting to the integral manifold \( \tilde{M} \) given by \( \tilde{I}_1 = \tilde{I}_2 = \tilde{I}_3 = \tilde{I}_4 = 0 \), we see that the distribution \( \tilde{\Delta}^{(1)} \) becomes,

\[
\tilde{\mathcal{V}}^{(1)} = \tilde{\Delta}^{(1)}|_{\tilde{M}^{(1)}} = \{ \partial_x + p\partial_u + r\partial_r + \rho\partial_\rho, \partial_\rho \},
\]

which is equivalent to the standard contact distribution on \( J^3(\mathbb{R}, \mathbb{R}) \). Similarly, the restriction of \( \Delta \) to the 7-dimensional integral manifold \( \tilde{M}^{(1)} \) given by \( \tilde{I}_1 = \tilde{I}_2 = 0 \) is

\[
\tilde{\mathcal{V}}^{(1)} = \tilde{\Delta}^{(1)}|_{\tilde{M}^{(1)}} = \{ \partial_y + q\partial_u + pu\partial_p + t\partial_q + (ru + p^2)\partial_r + \tau\partial_r, \partial_r \}
\]

The derived and weak derived flags of \( \tilde{\mathcal{V}}^{(1)} \) are both \( (2,3,4,5,6,7) \), and so we conclude by Theorem 2.5.2 that \( \tilde{\mathcal{V}}^{(1)} \) is locally equivalent to the standard contact distribution on \( J^5(\mathbb{R}, \mathbb{R}) \).

The distributions \( \hat{\mathcal{V}}^{(1)} \) and \( \tilde{\mathcal{V}}^{(1)} \) are the Vessiot distributions for the prolongation of (4.5). These distributions can naturally be deprolonged via reduction of their first derived distributions by their Cauchy characteristics to give the Vessiot distributions

\[
\hat{\mathcal{V}} = (\hat{\mathcal{V}}^{(1)})'/\{\partial_\rho\} = \{ \partial_x + p\partial_u + r\partial_r, \partial_r \},
\]

\[
\tilde{\mathcal{V}} = (\tilde{\mathcal{V}}^{(1)})'/\{\partial_r\} = \{ \partial_y + q\partial_u + pu\partial_p + t\partial_q + (ru + p^2)\partial_r, \partial_r \}
\]
for (4.5). We note that \( \hat{\mathcal{V}} \) is locally equivalent to the standard contact system on \( J^2(\mathbb{R}, \mathbb{R}) \) and \( \hat{\mathcal{V}} \) is locally equivalent to the standard contact system on \( J^4(\mathbb{R}, \mathbb{R}) \).

As in the previous example, we can show by calculating the restriction of the killing form of \( \hat{\mathcal{G}} \) and \( \hat{\mathcal{G}} \) to \( \hat{\mathcal{V}} \) and \( \hat{\mathcal{V}} \), respectively that the action of Vessiot algebra is given by the intransitive action of \( \mathfrak{sl}(2) \) generated by prolongations of the vector fields \( p_{3,9} \) in Table A.1.
CHAPTER 5
QUOTIENT CONSTRUCTION OF DARBOUX INTEGRABLE SYSTEMS

In this chapter, we summarize the main results of the quotient construction of hyperbolic Darboux integrable systems by Anderson, Fels, and Vassiliou [5]. Their quotient construction provides an algorithm for producing Darboux integrable distributions, however these distributions need not correspond to Darboux integrable PDE. In Section 5.2, we present Theorem 5.2.1 for the quotient construction of scalar second-order PDE in the plane which are Darboux integrable at some order. We then give a refinement of Theorem 5.2.1 for equations of generic, Goursat, and Monge-Ampère type as defined in Definition 2.3.3, and remark how this construction can be used in conjunction with the results of [7] on rank 2 distributions corresponding to maximally symmetric equations Monge of the form \( z^{(m)} = (y^{(n)})^2 \) with \( m \leq n \). In Section 4.3, we present examples in which the quotient construction gives a hyperbolic distribution corresponding to Darboux integrable equations of each of these types.

5.1 The Quotient Theory of Anderson, Fels, and Vassiliou

In this section, we summarize the main results of [5] on the quotient construction of hyperbolic Darboux integrable exterior differential systems. Though their theory applies to exterior differential systems in general, we restrict our interpretation to Pfaffian systems so that we may immediately reinterpret their results in the language of distributions.

We begin with the following theorem which, given two bracket-generating distributions, allows one to always construct a corresponding Darboux integrable distribution.

**Theorem 5.1.1.** Let \( \tilde{\mathcal{V}} \) and \( \tilde{\mathcal{V}}' \) be two bracket-generating distributions on manifolds \( \tilde{M} \) and \( \tilde{M}' \), respectively. Then the direct sum \( \tilde{\mathcal{V}} \oplus \tilde{\mathcal{V}}' \) on \( \tilde{M} \times \tilde{M}' \) is Darboux integrable.

Using this direct sum, we can then construct quotient distributions which are Darboux integrable.
Theorem 5.1.2. Let \( \hat{V} \) and \( \bar{V} \) be two bracket-generating distributions on manifolds \( \hat{M} \) and \( \bar{M} \). Suppose that

[i] a Lie group \( G \) acts freely on \( \hat{M} \) and \( \bar{M} \) and as symmetry groups of \( \hat{V} \) and \( \bar{V} \);

[ii] \( G \) is transverse to both \( \hat{V} \) and \( \bar{V} \); and

[iii] the diagonal action \( G_{\text{diag}} \) of \( G \) on \( \hat{M} \times \bar{M} \) is regular.

Then the quotient distribution

\[
(\hat{V} \oplus \bar{V})/G_{\text{diag}}
\]

is hyperbolic and Darboux integrable.

Finally, the following theorem allows us to realize any hyperbolic Darboux integrable distribution as the quotient of its Vessiot distributions by the action of its Vessiot group.

Theorem 5.1.3. Let \( \Delta = \hat{\Delta} \oplus \bar{\Delta} \) on \( M \) be a hyperbolic Darboux integrable distribution. Suppose \( \Delta \) has Vessiot distributions \( \hat{V} \) and \( \bar{V} \) on manifolds \( \hat{M} \) and \( \bar{M} \) and Vessiot group \( G \) with diagonal action \( G_{\text{diag}} \) on \( \hat{M} \times \bar{M} \). Then

\[
\Delta \cong (\hat{V} \oplus \bar{V})/G_{\text{diag}}.
\]

We call the quotient distribution \( (\hat{V} \oplus \bar{V})/G_{\text{diag}} \) the quotient representation of \( \Delta \).

Sketch of proof. The key steps to the proof are to first show that the distribution \( \Delta = \hat{\Delta} \oplus \bar{\Delta} \) is indeed hyperbolic by showing that the characteristic distributions can be obtained via \( \hat{\Delta} = \tilde{\iota}_*(\hat{V})/G_{\text{diag}} \) and \( \bar{\Delta} = \tilde{\iota}_*(\bar{V})/G_{\text{diag}} \), where \( \tilde{\iota} : \hat{M} \to \hat{M} \times \bar{M} \) and \( \tilde{\iota} : \bar{M} \to \hat{M} \times \bar{M} \) are inclusion maps, and then showing that \( \Delta \) is Darboux integrable by showing that

\[
\hat{\Delta}^{(\infty)} \cap \bar{\Delta} = (\hat{V}/G_{\text{diag}})^{(\infty)} \cap (\bar{V}/G_{\text{diag}}) = (T\hat{M}/G_{\text{diag}}) \cap (\bar{V}/G_{\text{diag}}) = \{0\},
\]

and vice versa. \( \square \)
These theorems together allow us to generate new examples of hyperbolic Darboux integrable distributions with ease. In particular, Theorem 5.1.3 shows that every hyperbolic Darboux integrable distribution can realized as a quotient of a direct sum of bracket-generating distributions. As a consequence of this, the study of Darboux integrable distributions, and therefore Darboux integrable PDE, becomes a study of distributions and their symmetry groups.

5.2 Quotient Construction of PDE in the Plane

In this section, we give a theorem for when the quotient construction gives rise to second-order hyperbolic PDE in the plane which are Darboux integrable at some order. We then refine this theorem to show when the quotient construction corresponds to equations of generic, Goursat, and Monge-Ampère types. We remark that by using the results of [7], we can construct several examples of equations of each of these types using distributions corresponding to maximally symmetric Monge equations of the form

$$z^{(m)} = (y^{(n)})^2.$$

**Theorem 5.2.1.** Let $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ be two rank 2 bracket-generating distributions on manifolds $\hat{M}$ and $\check{M}$ of dimensions $\hat{n}$ and $\check{n}$, respectively, with $\hat{n}, \check{n} \geq 4$. Let $G$ be a common symmetry group of $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ of dimension $\hat{n} + \check{n} - 7$. Denote the infinitesimal generators of the diagonal action, $G_{\text{diag}}$, of $G$ on $\hat{M} \times \check{M}$ by $\Gamma_{\text{diag}}$, and suppose

[i] $G$ acts freely on $\hat{M}$ and $\check{M}$, and

[ii] $\Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \check{\mathcal{V}})' = \{0\}$.

Then the quotient distribution

$$\Delta = (\hat{\mathcal{V}} \oplus \check{\mathcal{V}})/G_{\text{diag}}$$

is a rank 4 hyperbolic distribution corresponding to a second-order hyperbolic PDE in the plane which is Darboux integrable at some order.
Proof. Let \( \mathcal{V} \) and \( \mathcal{V} \) be two rank 2 bracket-generating distributions on \( \hat{M} \) and \( \hat{M} \) of dimensions \( \hat{n} \) and \( \hat{n} \), respectively, with \( \hat{n}, \hat{n} \geq 4 \). Construct the rank 4 distribution \( \mathcal{V} \oplus \mathcal{V} \).

Conditions [i] and [ii] imply that \( G_{\text{diag}} \) acts freely on \( \hat{M} \times \hat{M} \) and transversely to both \( (\mathcal{V} \oplus \mathcal{V})' \) and \( \mathcal{V} \oplus \mathcal{V} \). Theorem 3.3.2 implies that we can then construct \( G \)-invariant bases for \( \mathcal{V} \oplus \mathcal{V} \) and \( (\mathcal{V} \oplus \mathcal{V})' \) and that the quotient distributions \( \Delta = (\mathcal{V} \oplus \mathcal{V})/G_{\text{diag}} \) has rank 4 and \( \Delta' = (\mathcal{V} \oplus \mathcal{V})'/G_{\text{diag}} \) has rank 6 on the 7-dimensional manifold \( M = (\hat{M} \times \hat{M})/G_{\text{diag}} \).

Dimensional constraints allow us to deduce that \( \Delta'' = (\mathcal{V} \oplus \mathcal{V})''/G_{\text{diag}} \) must have rank 7. Theorem 5.1.2 immediately implies that \( \Delta \) is Darboux integrable.

The distributions \( \mathcal{V}, \mathcal{V} \) must have nonintegrable derived distributions, and therefore, admit no Cauchy characteristics. It then follows that \( \mathcal{V} \oplus \mathcal{V} \) admits no generalized Cauchy characteristics, and we therefore conclude the quotient distribution \( \Delta \) admits no Cauchy characteristics.

To show that \( \Delta' \) admits a two Cauchy characteristics, and therefore defines a PDE in the plane by Theorem 2.3.1, we show that \( (\mathcal{V} \oplus \mathcal{V})' \) admits two generalized Cauchy characteristics. As described above, we may choose \( G \)-invariant bases for \( \mathcal{V} \oplus \mathcal{V} \) and their derived systems such that,

\[
\mathcal{V} \oplus \mathcal{V} = \{\hat{X}_1, \hat{X}_2, \tilde{X}_1, \tilde{X}_2\},
\]

\[
(\mathcal{V} \oplus \mathcal{V})' = \{\hat{X}_1, \hat{X}_2, \hat{Y}, \hat{X}_1, \hat{X}_2, \hat{Y} = [\hat{Y}_1, \hat{Y}_2]\}.
\]

Denote the infinitesimal generators of \( G_{\text{diag}} \) by \( \Gamma_{\text{diag}} = \{\Gamma_i\} \) where \( 1 \leq i \leq \hat{n} + \hat{n} - 7 \). We first note that the vector field system,

\[
\mathcal{V}' = (\mathcal{V} \oplus \mathcal{V})' \oplus \Gamma_{\text{diag}} = \{\hat{X}_1, \hat{X}_2, \hat{Y}, \tilde{X}_1, \tilde{X}_2, \hat{Y}, \Gamma_i\}, \quad 1 \leq i \leq \hat{n} + \hat{n} - 7,
\]

has rank \( \hat{n} + \hat{n} - 1 \), and so we may complete \( \mathcal{V}' \) to a frame on \( \hat{M} \times \hat{M} \) by choosing a single vector field \( W \).
Suppose there exists a generalized Cauchy characteristic of \((\hat{\mathcal{V}} \oplus \check{\mathcal{V}})\)',

\[ Z = a^1 \hat{X}_1 + a^2 \hat{X}_2 + a^3 \check{Y} + b^1 \hat{X}_1 + b^2 \hat{X}_2 + b^3 \check{Y}, \]

where the \(a^i, b^i\) are smooth functions on \(\hat{M} \times \check{M}\). Calculating brackets of \(Z\) with \(\hat{X}_1, \hat{X}_2, \check{X}_1, \) and \(\check{X}_2\) gives

\[
\begin{align*}
[Z, \hat{X}_1] &= -a^2 \check{Y} + a^3 \hat{V}_1 \equiv a^3 \hat{V}_1 \mod \mathcal{V}', \\
[Z, \hat{X}_2] &= a^1 \check{Y} + a^3 \hat{V}_2 \equiv a^3 \hat{V}_2 \mod \mathcal{V}', \\
[Z, \check{X}_1] &= -b^2 \check{Y} + b^3 \check{V}_1 \equiv b^3 \check{V}_1 \mod \mathcal{V}', \\
[Z, \check{X}_2] &= b^1 \check{Y} + b^3 \check{V}_2 \equiv b^3 \check{V}_2 \mod \mathcal{V}',
\end{align*}
\]

where \(\hat{V}_1 = [\hat{X}_1, \check{Y}], \hat{V}_2 = [\hat{X}_2, \check{Y}], \check{V}_1 = [\check{X}_1, \check{Y}],\) and \(\check{V}_2 = [\check{X}_2, \check{Y}]\). After requiring that each of these brackets lie inside \(\mathcal{V}'\), we see that since the pairs \(\hat{V}_1, \hat{V}_2\) and \(\check{V}_1, \check{V}_2\) cannot both be in \(\mathcal{V}'\) due to \(\hat{\mathcal{V}}\) and \(\check{\mathcal{V}}\) being bracket-generating, that \(a^3 = b^3 = 0\). Moreover, by calculating brackets of \(Z\) with \(\hat{\mathcal{V}}\) and \(\check{\mathcal{V}}\), we see

\[
\begin{align*}
[Z, \hat{\mathcal{V}}] &= a^1 \hat{V}_1 + a^2 \hat{V}_2 \mod \mathcal{V}', \\
[Z, \check{\mathcal{V}}] &= b^1 \check{V}_1 + b^2 \check{V}_2 \mod \mathcal{V}'.
\end{align*}
\]

However, the vector fields \(\hat{V}_1, \hat{V}_2, \check{V}_1,\) and \(\check{V}_2\) can be written as

\[
\begin{align*}
\hat{V}_1 &\equiv \lambda_1 W \mod \mathcal{V}', \\
\hat{V}_2 &\equiv \lambda_2 W \mod \mathcal{V}', \\
\check{V}_1 &\equiv \lambda_3 W \mod \mathcal{V}', \\
\check{V}_2 &\equiv \lambda_4 W \mod \mathcal{V}',
\end{align*}
\]

where \(\lambda^i\) are smooth functions on \(\hat{M} \times \check{M}\) and \(\lambda_1, \lambda_2, \lambda_3, \lambda_4\) are not both zero since \(\hat{\mathcal{V}}\) and \(\check{\mathcal{V}}\) are bracket-generating. We then see that if \(Z\) is to be a generalized Cauchy characteristic
of $\tilde{V} \oplus \tilde{V}'$, the system

$$(\lambda_1 a^1 + \lambda_2 a^2)W \equiv 0 \mod \mathcal{V}'$$

$$(\lambda_3 b^1 + \lambda_4 b^2)W \equiv 0 \mod \mathcal{V}'$$

must be satisfied, or equivalently, $A\xi = 0$ where

$$A = \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{bmatrix}$$

and

$$\xi = \begin{bmatrix} a^1 & a^2 & b^1 & b^2 \end{bmatrix}^T.$$ 

Since the pairs $\lambda_1, \lambda_2$, and $\lambda_3, \lambda_4$ are not both zero, we see that the matrix $A$ has rank 2, and we conclude that there always exist two linearly independent generalized Cauchy characteristics of $(\tilde{V} \oplus \tilde{V})'$. These generalized Cauchy characteristics naturally project under the quotient to give two Cauchy characteristics of $\Delta'$. We therefore conclude by Theorem 2.3.1 that the quotient distribution $\Delta$ defines a rank 4 distribution which defines a second-order PDE in the plane. Lastly, we see that since $[\tilde{X}_i, \tilde{X}_j] = 0$ for $1 \leq i, j \leq 2$, the distributions $\tilde{\Delta} = \tilde{V}/G_{\text{diag}}$ and $\tilde{\Delta} = \tilde{V}/G_{\text{diag}}$ naturally define a hyperbolic distribution $\Delta = \tilde{\Delta} \oplus \tilde{\Delta}$.

We now refine Theorem 5.2.1 to give conditions on when the quotient construction results in a of Darboux integrable equation of generic, Goursat, or Monge-Ampère type. We begin with the following lemma.

**Lemma 5.2.2.** The rank of the distribution $\Gamma_{\text{diag}} \cap (\tilde{V} \oplus \tilde{V})''$ is either 1, 2, or 3.

Moreover, if

- $[i]$ $\Gamma_{\text{diag}} \cap (\tilde{V} \oplus \tilde{V})''$ has rank 1, then $\text{rank}(\tilde{V}'') = \text{rank}(\tilde{V}'') = 4$;

- $[ii]$ $\Gamma_{\text{diag}} \cap (\tilde{V} \oplus \tilde{V})''$ has rank 2, then $\text{rank}(\tilde{V}'') = 4$ and $\text{rank}(\tilde{V}'') = 5$, or vice versa;

- $[iii]$ $\Gamma_{\text{diag}} \cap (\tilde{V} \oplus \tilde{V})''$ has rank 3, then $\text{rank}(\tilde{V}'') = \text{rank}(\tilde{V}'') = 5$;

*Proof.* Denote the rank of second derived distributions $\tilde{V}''$ and $\tilde{V}''$ by $\tilde{k}$ and $\tilde{k}$, respectively. Since $\tilde{V}$ and $\tilde{V}$ are bracket-generating on manifolds $\tilde{M}$ and $\tilde{M}$ of dimension $\hat{n}, \check{n} \geq 4$, it
follows that \( \hat{k}, \bar{k} \) must be either 4 or 5. The set of infinitesimal generators \( \Gamma_{\text{diag}} \) has rank \( \hat{n} + \bar{n} - 7 \). We then see that

\[
\text{rank}(\Gamma_{\text{diag}}) + \text{rank}(\hat{\mathcal{V}}'') + \text{rank}(\bar{\mathcal{V}}''') = \hat{n} + \bar{n} + \hat{k} + \bar{k} - 7.
\]

But since the product manifold \( \hat{M} \times \bar{M} \) is of dimension \( \hat{n} + \bar{n} \), the intersection \( \Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \bar{\mathcal{V}})'' \) must have rank \( \hat{k} + \bar{k} - 7 \). However, since \( \hat{k}, \bar{k} \) can only be either 4 or 5, we see that \( \Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \bar{\mathcal{V}})'' \) must have rank 1 when \( \hat{k} = \bar{k} = 4 \), rank 2 when \( \hat{k} = 4 \) and \( \bar{k} = 5 \) (or vice versa), and rank 3 when \( \hat{k} = \bar{k} = 5 \).

**Theorem 5.2.3.** Let \( \hat{\mathcal{V}} \) and \( \bar{\mathcal{V}} \) be two rank two distributions satisfying the hypotheses of Theorem 5.2.1. Denote the set of infinitesimal generators of \( G_{\text{diag}} \) by \( \Gamma_{\text{diag}} \). If

\[
\begin{align*}
\text{[i]} & \quad \Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \bar{\mathcal{V}})'' \text{ is rank 3, then the quotient distribution is a rank 4 hyperbolic distribution corresponding to a hyperbolic PDE in the plane of generic type;} \\
\text{[ii]} & \quad \Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \bar{\mathcal{V}})'' \text{ is rank 2, then the quotient distribution is a rank 4 hyperbolic distribution corresponding to a hyperbolic PDE in the plane of Goursat type;} \\
\text{[iii]} & \quad \Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \bar{\mathcal{V}})'' \text{ is rank 1, then the quotient distribution is a rank 4 hyperbolic distribution corresponding to a hyperbolic PDE in the plane of Monge-Ampère type.}
\end{align*}
\]

**Proof.** Let \( \hat{\mathcal{V}} \) and \( \bar{\mathcal{V}} \) be rank 2 distributions satisfying the hypotheses of Theorem 5.2.1, so that the quotient distribution \( \Delta = (\hat{\mathcal{V}} \oplus \bar{\mathcal{V}})/G_{\text{diag}} \) defines a hyperbolic PDE in the plane. In particular, we assume that \( \hat{\mathcal{V}} \) and \( \bar{\mathcal{V}} \) and their derived systems are written in terms of \( G \)-invariant bases, so that all bracket relations are preserved under the quotient.

When \( \Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \bar{\mathcal{V}})'' \) has rank 3, Lemma 5.2.2 implies that \( \text{rank}(\hat{\mathcal{V}}'') = \text{rank}(\bar{\mathcal{V}}''') = 5 \). The distributions \( \hat{\Delta} = \hat{\mathcal{V}}/G_{\text{diag}} \) and \( \bar{\Delta} = \bar{\mathcal{V}}/G_{\text{diag}} \) each have derived dimensions \( (2, 3, 5, \ldots) \), and by Theorem 2.3.5, \( \Delta \) defines an equation of generic type.

When \( \Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \bar{\mathcal{V}})'' \) has rank 2, Lemma 5.2.2 implies that \( \text{rank}(\hat{\mathcal{V}}'') = 4 \) and \( \text{rank}(\bar{\mathcal{V}}''') = 5 \), or vice versa. The distribution \( \hat{\Delta} = \hat{\mathcal{V}}/G_{\text{diag}} \) has derived dimensions
(2, 3, 4, \cdots), and the distribution $\Delta = \tilde{\mathcal{V}}/G_{\text{diag}}$ has derived dimensions (2, 3, 5, \cdots), or vice versa. By Theorem 2.3.5, $\Delta$ defines an equation of Goursat type.

When $\Gamma_{\text{diag}} \cap (\mathcal{V} \oplus \mathcal{V})''$ has rank 1, Lemma 5.2.2 implies that rank($\mathcal{V}'') = \text{rank}(\mathcal{V}'') = 4$. The distributions $\tilde{\Delta} = \mathcal{V}/G_{\text{diag}}$ and $\tilde{\Delta} = \tilde{\mathcal{V}}/G_{\text{diag}}$ each have derived dimensions (2, 3, 4, \cdots), and by Theorem 2.3.5, $\Delta$ defines an equation of Monge-Ampère type.

Remark. In [7], Anderson and Kruglikov analyze rank 2 distributions corresponding to maximally symmetric Monge equations of the form

$$z^{(m)} = (y^{(n)})^2 \quad (5.1)$$

where $m \leq n$. In particular, they prove that the symmetry group $S$ of (5.1) has dimension

$$\dim S = \begin{cases} 2n + 5 & \text{when } m = 1, n > 2, \\ 2n + 4 & \text{when } m > 1. \end{cases}$$

Here, we consider the symmetric case where the rank 2 distributions $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are defined on manifolds $\hat{M}$ and $\tilde{M}$ of the same dimension $\ell$. Theorem 5.2.1 implies that if the quotient distribution $\Delta = (\mathcal{V} \oplus \tilde{\mathcal{V}})/G_{\text{diag}}$ is to correspond to a PDE in the plane, then the symmetry group $G$ must have dimension

$$\dim G = 2\ell - 7.$$  

Since $\dim G \leq \dim S$, we see that the values of $\ell, m,$ and $n$ must be related, significantly reducing the number of admissible cases to consider. We further analyze the relationship between $\dim G$ and $\dim S$ in the following cases.

Case 1: $m = 1, n > 2$

When $m = 1$ and $n > 2$, the value of $\ell$ is given by $\ell = n + 3$, and $\dim G = 2n - 1$. The gap between the dimensions of $S$ and $G$ is given by $\dim S - \dim G = 6$.

Case 2: $m > 1$
When \( m > 1 \), the value of \( \ell \) is given by \( \ell = m + n + 1 \), and \( \dim G = 2m + 2n - 5 \). The condition that \( \dim G \) be less than or equal to \( \dim S \) further implies that \( 2m \leq 9 \) and that the gap between \( \dim S \) and \( \dim G \) must be \( 9 - 2m \). We conclude that the only admissible values for \( m \) in this case are \( m = 2, 3 \) or \( 4 \), and it follows that

(i) when \( m = 2 \), the gap between \( \dim S \) and \( \dim G \) is 5;

(ii) when \( m = 3 \), the gap between \( \dim S \) and \( \dim G \) is 3;

(iii) and when \( m = 4 \), the gap between \( \dim S \) and \( \dim G \) is 1.

If we allow for prolongations of the distributions corresponding to (5.1), then the quotient construction will yield PDE in the plane which are Darboux integrable at higher orders. With each level of prolongation, the value of \( \ell \) increases by two, and consequently, the dimension of the group \( G \) must decrease by two for the quotient to correspond to a PDE in the plane.

In Case 1, after one prolongation, the gap between \( S \) and \( G \) becomes 4; after two prolongations, the gap between \( S \) and \( G \) becomes 2; and after three prolongations, there is no gap between \( S \) and \( G \), meaning \( G \cong S \). Quotients of all subsequent prolongations will not yield PDE in the plane.

Similarly, in Case 2(i), the gaps between \( \dim S \) and \( \dim G \) at each level of prolongation are 5, 3, 1, and quotients of all subsequent prolongations will not yield PDE in the plane.

In Case 2(ii), the gaps between \( \dim S \) and \( \dim G \) at each level of prolongation are 3 and 1, and quotients of all subsequent prolongations will not yield PDE in the plane. Quotients of all prolongations of distributions corresponding to Case 2(iii) will not yield PDE in the plane.

Finally, we note that in Cases 1 and 2, the derived dimensions of the distributions corresponding to (5.1) are always \((2, 3, 5, \ldots)\). Theorem 5.2.3 and Lemma 5.2.2 imply that quotients by \( G \) will always yield equations of generic type. Prolongations of these distributions have derived dimensions \((2, 3, 4, \ldots)\), so we conclude that in the cases where the quotients correspond to PDE in the plane, quotients of a direct sum of a prolonged
distribution of these types with a non-prolonged distribution of these types will correspond to equations of Goursat type while quotients of two prolonged distributions of these types will correspond to equations of Monge-Ampère type.

5.3 Examples of Quotient Constructions

In this section, we present examples of the quotient construction for equations of generic, Goursat, and Monge-Ampère types.

Example 5.3.1. Let \( \tilde{V} \) and \( \check{V} \) be copies of the Hilbert-Cartan distribution

\[
\tilde{V} = \{ \partial_x + \phi_2 \partial_z + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1}, \partial_z \} \quad \text{and} \quad \check{V} = \{ \partial_y + \psi_2 \partial_w + \psi_1 \partial_\psi + \psi_2 \partial_{\psi_1}, \partial_w \},
\]

defined on 5-dimensional manifolds \( \tilde{M} \) and \( \check{M} \) with coordinates \( x, z, \phi, \phi_1, \phi_2 \) and \( y, w, \psi, \psi_1, \psi_2 \), respectively. Let \( G \) be the 3-dimensional abelian group generated by vector fields \( \tilde{\Gamma} = \{ \partial_\phi, x \partial_\phi + \partial_{\phi_1}, \partial_z \} \) and \( \check{\Gamma} = \{ \partial_\psi, y \partial_\psi + \partial_{\psi_1}, \partial_w \} \). It is easily shown that \( \tilde{\Gamma} \) is a symmetry algebra of \( \tilde{V} \) and \( \check{\Gamma} \) is a symmetry algebra of \( \check{V} \). Let \( \Gamma_{\text{diag}} \) denote the diagonal action of \( G \) generated by

\[
Z_1 = \partial_\phi + \partial_\psi, \quad Z_2 = x \partial_\phi + \partial_{\phi_1} + y \partial_\psi + \partial_{\psi_1}, \quad Z_3 = \partial_z + \partial_w.
\]

A simple computation shows that conditions [i] and [ii] of Theorem 5.2.1 are satisfied with

\[
\Gamma_{\text{diag}} \cap (\tilde{V} \oplus \check{V})'' = \{ Z_1, Z_2, Z_3 \}.
\]

The quotient of \( \tilde{V} \oplus \check{V} \) by \( \Gamma_{\text{diag}} \) is a rank 4 distribution defined on a 7-manifold \( M \) with coordinates \( z_1, z_2, z_3, z_4, z_5, z_6, z_7 \). The explicit formula for the quotient map \( q : \tilde{M} \times \check{M} \to M \) is given by

\[
z_1 = x, \quad z_2 = y, \quad z_3 = z - w, \quad z_4 = \phi - \psi - (x - y) \phi_1, \quad z_5 = \phi_1 - \psi_1, \quad z_6 = \phi_2, \quad z_7 = \psi_2.
\]
Calculating the pushforward by $q$ of $\tilde{\nabla} \oplus \tilde{\nabla}$ gives the hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ where

$$
\hat{\Delta} = \{ \partial z_1 + z_6^2 \partial z_3 - (z_1 - z_2) \partial z_4 + z_6 \partial z_5, \partial z_6 \},
$$

$$
\tilde{\Delta} = \{ \partial z_2 - z_7^2 \partial z_3 + z_5 \partial z_4 - z_7 \partial z_5, \partial z_7 \}.
$$

One can then verify that $\Delta$ satisfies Theorem 2.3.1, and therefore, defines a hyperbolic PDE in the plane. Calculating the derived flags of $\hat{\Delta}$ and $\tilde{\Delta}$ shows that they each have growth vector $(2,3,5)$, and by Theorem 2.3.5, $\Delta$ defines an equation of generic type. Moreover, the first integrals of $\hat{\Delta}$ are given by $\hat{I}_1 = z_2$ and $\hat{I}_2 = z_7$ and the first integrals of $\tilde{\Delta}$ are $\tilde{I}_1 = z_1$, and $\tilde{I}_2 = z_6$, so $\Delta$ is Darboux integrable.

**Remark.** In Section 12.1, we again calculate the quotient and show that the distribution $\Delta$ corresponds to the classical equation $3rt^3 + 1 = 0$ which, as we showed in Example 2.3.6, is of generic type.

**Example 5.3.2.** Let $\hat{\nabla}$ be the Hilbert-Cartan distribution

$$
\hat{\nabla} = \{ \partial_x + \phi_2^2 \partial_x + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1}, \partial_z \},
$$

on the 5-dimensional manifold $\hat{M}$ with coordinates $x, z, \phi, \phi_1, \phi_2$, and let $\tilde{\nabla}$ be the standard contact distribution

$$
\tilde{\nabla} = \{ \partial_y + \psi_1 \partial_\psi + \psi_2 \partial_{\psi_1} + \psi_3 \partial_{\psi_2}, \partial_{\psi_3} \}
$$

on $\tilde{M} = J^3(\mathbb{R}, \mathbb{R})$ with coordinates $y, \psi, \psi_1, \psi_2, \psi_3$. Let $G$ be the 3-dimensional group generated by vector fields

$$
\hat{\Gamma} = \{ \partial_\phi, x \partial_\phi + \partial_{\phi_1}, \partial_z \} \quad \text{and} \quad \tilde{\Gamma} = \{ \partial_\psi, y \partial_\psi + \partial_{\psi_1}, y^2 \partial_\psi + 2y \partial_{\psi_1} + 2 \partial_{\psi_2} \}.$$
on \( \hat{M} \) and \( \tilde{M} \), respectively. Let \( \Gamma_{\text{diag}} \) denote the diagonal action of \( G \) generated by

\[
Z_1 = \partial_\phi + \partial_\psi, \quad Z_2 = x\partial_\phi + \partial_{\psi_1} + y\partial_\psi + \partial_{\psi_1}, \quad Z_3 = \partial_z + y^2\partial_\psi + 2y\partial_{\psi_1} + 2\partial_{\psi_2}.
\]

A simple computation shows that conditions [i] and [ii] of Theorem 5.2.1 are satisfied with

\[
\Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \tilde{\mathcal{V}}) = \{Z_3 - y^2Z_1, Z_2 - yZ_1\}.
\]

The quotient of \( \hat{\mathcal{V}} \oplus \tilde{\mathcal{V}} \) by \( \Gamma_{\text{diag}} \) is a rank 4 distribution defined on a 7-manifold \( M \) with coordinates \( z_1, z_2, z_3, z_4, z_5, z_6, z_7 \). The explicit formula for the quotient map \( q : \hat{M} \times \tilde{M} \to M \) is given by

\[
\begin{align*}
z_1 &= x, \quad z_2 = y, \quad z_3 = \phi - \psi + y^2z - (x - y)\phi_1, \quad z_4 = \phi_1 - \psi_1 + 2yz, \\
z_5 &= \phi_2, \quad z_6 = 2z - \psi_2, \quad z_7 = \psi_3.
\end{align*}
\]

Calculating the pushforward by \( q \) of \( \hat{\mathcal{V}} \oplus \tilde{\mathcal{V}} \) gives the hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) where

\[
\begin{align*}
\hat{\Delta} &= \{\partial_{z_1} - (z_1z_5 - z_2z_5 - z_2^2z_6)\partial_{z_3} + (2z_2z_5^2 + z_5)\partial_{z_4} + 2z_2^2\partial_{z_6}, \partial_{z_5}\}, \\
\tilde{\Delta} &= \{\partial_{z_2} + z_4\partial_{z_3} + z_6\partial_{z_4} - z_7\partial_{z_6}, \partial_{z_7}\}.
\end{align*}
\]

One can verify that \( \Delta \) satisfies Theorem 2.3.1, and therefore, defines a hyperbolic PDE in the plane. The growth vector for \( \hat{\Delta} \) is \((2,3,5)\), and the growth vector for \( \tilde{\Delta} \) is \((2,3,4,5)\), so by Theorem 2.3.5, \( \Delta \) defines an equation of Goursat type. Moreover, the first integrals of \( \hat{\Delta} \) are given by \( \hat{I}_1 = \hat{z}_2 \) and \( \hat{I}_2 = \hat{z}_7 \) and the first integrals of \( \tilde{\Delta} \) are \( \tilde{I}_1 = \tilde{z}_1 \), and \( \tilde{I}_2 = \tilde{z}_5 \), so \( \Delta \) is Darboux integrable.

**Example 5.3.3.** Let \( \hat{\mathcal{V}} \) and \( \tilde{\mathcal{V}} \) be copies to the standard contact distribution on \( \hat{M} = J^3(\mathbb{R}, \mathbb{R}) \) and \( \tilde{M} = J^3(\mathbb{R}, \mathbb{R}) \) with coordinates \( \phi, \phi_1, \phi_2, \phi_3 \) and \( y, \psi, \psi_1, \psi_2, \psi_3 \). Let \( G \) be
the 3-dimensional group generated by vector fields

\[ \hat{Z}_1 = \partial_{\phi}, \]
\[ \hat{Z}_2 = \phi \partial_{\psi} + \phi_1 \partial_{\phi_1} + \phi_2 \partial_{\phi_2} + \phi_3 \partial_{\phi_3}, \]
\[ \hat{Z}_3 = \phi^2 \partial_{\phi} + 2\phi \phi_1 \partial_{\phi_1} + (2\phi \phi_2 + 2\phi_1^2) \partial_{\phi_2} + (2\phi \phi_3 + 6\phi_1 \phi_2) \partial_{\phi_3}, \]

and

\[ \tilde{Z}_1 = \partial_{\psi}, \]
\[ \tilde{Z}_2 = \psi \partial_{\psi} + \psi_1 \partial_{\psi_1} + \psi_2 \partial_{\psi_2} + \psi_3 \partial_{\psi_3}, \]
\[ \tilde{Z}_3 = \psi^2 \partial_{\psi} + 2\psi \psi_1 \partial_{\psi_1} + (2\psi \psi_2 + 2\psi_1^2) \partial_{\psi_2} + (2\psi \psi_3 + 6\psi_1 \psi_2) \partial_{\psi_3}, \]

on manifolds \( \hat{M} \) and \( \tilde{M} \), respectively. Let \( \Gamma_{\text{diag}} \) denote the diagonal action of \( G \) generated by

\[ Z_1 = \hat{Z}_1 - \tilde{Z}_1, \quad Z_2 = \hat{Z}_2 + \tilde{Z}_2, \quad Z_3 = \hat{Z}_3 - \tilde{Z}_3. \]

A simple computation shows that conditions [i] and [ii] of Theorem 5.2.1 are satisfied with

\[ \Gamma_{\text{diag}} \cap (\hat{\mathcal{V}} \oplus \tilde{\mathcal{V}})' = \{ Z_3 - (\phi + \psi)Z_2 - \phi \psi Z_1 \}. \]

The quotient of \( \hat{\mathcal{V}} \oplus \tilde{\mathcal{V}} \) by \( \Gamma_{\text{diag}} \) is a rank 4 distribution defined on a 7-manifold \( M \) with coordinates \( x, y, u, p, q, r, t \). The explicit formula for the quotient map \( q : \hat{M} \times \tilde{M} \to M \) is given by

\[ x = x, \quad y = y, \quad u = \ln \left( \frac{2\phi_1 \psi_1}{(\phi + \psi)^2} \right), \]
\[ p = D_x(u), \quad q = D_y(u), \quad r = D_x(p), \quad t = D_y(q). \]
Calculating the pushforward by \( q \) of \( \mathcal{V} \oplus \mathcal{V} \) gives the hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \check{\Delta} \) given by

\[
\hat{\Delta} = \{ \partial_x + p \partial_u + r \partial_p + e^u \partial_q + q e^u \partial_t, \partial_r \},
\]

\[
\check{\Delta} = \{ \partial_y + q \partial_u + e^u \partial_p + t \partial_q + p e^u \partial_r, \partial_t \}.
\]

Thus, we have given the explicit quotient construction of the hyperbolic Liouville equation \( s = e^u \) in terms of its Vessiot distributions and the action of its Vessiot group.
Part II

Equivalence Problems for Darboux Integrable Equations
In 1881, Sophus Lie characterized equations as being contact equivalent to the wave equation \( u_{xy} = 0 \) with the following theorem [22].

**Theorem 6.0.1.** Every equation whose characteristic systems each admit three functionally independent intermediate integrals is contact equivalent to the wave equation \( u_{xy} = 0 \).

This theorem was later proved by Goursat [19] and later by Kamran and Gardner [17] and Stormark [30]. Under the quotient theory, we see that this theorem becomes a simple consequence of the well-known theorem of Engel.

**Proof.** Let \( \mathcal{I} = (\hat{\mathcal{I}} + \tilde{\mathcal{I}})/G \) be a hyperbolic Darboux integrable system on the 7-manifold \( M = (\hat{M} \times \tilde{M})/G \), such that the characteristic systems \( \hat{\mathcal{I}} \) and \( \tilde{\mathcal{I}} \) each admit three intermediate integrals. Since the dimension of the Vessiot group is given by

\[
\dim G = \dim M - (\# \text{ intermediate integrals for } \hat{\mathcal{I}}) - (\# \text{ intermediate integrals for } \tilde{\mathcal{I}}) = 1,
\]

the set of infinitesimal generators is given by the single vector field \( X \). Moreover, since

\[
\text{rank}(\mathcal{I}) = \text{rank}(\hat{\mathcal{I}}) + \text{rank}(\tilde{\mathcal{I}}) - \dim G,
\]

we have that \( \text{rank}(\hat{\mathcal{I}}) + \text{rank}(\tilde{\mathcal{I}}) = 4 \). Therefore, we must consider the following cases:

1. \( \text{rank}(\hat{\mathcal{I}}) = 3, \text{rank}(\tilde{\mathcal{I}}) = 1 \),
2. \( \text{rank}(\hat{\mathcal{I}}) = \text{rank}(\tilde{\mathcal{I}}) = 2 \).
where $\hat{I}$ and $\tilde{I}$ are defined on 4-manifolds. The dual version of Theorem 2.3.1 states that if $I$ is to correspond to a PDE in the plane, then
\[
\text{rank}(I) = 3, \quad \text{rank}(I') = 1, \quad \text{rank}(I'') = 0
\]

However, due to the fact that a Pfaffian system of codimension one is Frobenius [10],
\[
\text{rank}(\hat{I}') = \text{rank}(\hat{I}) = 3.
\]

But then,
\[
\text{rank}(I') = \text{rank}\left( (\hat{I}' + \tilde{I}')/G \right) \geq 2,
\]

and we see Case (i) will not give a PDE in the plane.

This leaves us with Case (ii). We assume neither $\hat{I}$ nor $\tilde{I}$ are Frobenius systems, so as to satisfy the hypothesis of Theorem 2.3.1. Under this assumption (along with the requirements that the $G$-action be regular, free, transverse), it follows that
\[
\text{rank}(\hat{I}') = \text{rank}(\tilde{I}') = 1 \quad \text{and} \quad \text{rank}(\hat{I}'') = \text{rank}(\tilde{I}'') = 0.
\]

Therefore, by Engel’s theorem, $\hat{I}$ and $\tilde{I}$ must be locally equivalent to the standard contact system $C^2(\mathbb{R}, \mathbb{R})$ on $J^2(\mathbb{R}, \mathbb{R})$. That is,
\[
I = \left( C^2(\mathbb{R}, \mathbb{R}) + C^2(\mathbb{R}, \mathbb{R}) \right)/G
\]

where $G$ is 1-dimensional. But this is precisely the quotient representation of $u_{xy} = 0$. \qed
We have seen that the action of the Vessiot group and the Vessiot distributions serve as the fundamental invariants for Darboux integrable systems. Throughout this chapter, we consider linear partial differential equations,

\[ u_{xy} = A(x, y)u_x + B(x, y)u_y + C(x, y)u, \]  

(7.1)

which are Darboux integrable at order \( k \geq 2 \). In particular, we give a complete characterization of these equations in terms of their fundamental invariants as described by the following two theorems.

**Theorem 7.0.1.** Let \( \Delta = \Delta \oplus \tilde{\Delta} \) be the hyperbolic distribution on a 7-dimensional manifold for the linear equation (7.1). Suppose \( \Delta \) is Darboux integrable at order \( k \) and that the characteristic systems \( \Delta \) and \( \tilde{\Delta} \) admit \( r \) and \( s \) intermediate integrals, respectively. Then the Vessiot group \( G \) of \( \Delta \) is an abelian group of dimension \( 2k + 3 - r - s \), and the Vessiot distributions of the prolonged differential system \( L^{(k-2)} \) are locally equivalent to the standard contact distribution on \( J^{2k+3-r}(\mathbb{R}, \mathbb{R}) \) and \( J^{2k+3-s}(\mathbb{R}, \mathbb{R}) \), respectively. The hyperbolic distribution \( \Delta \) is therefore a quotient of the direct sum of the standard contact distributions on \( J^{k+3-r}(\mathbb{R}, \mathbb{R}) \) and \( J^{k+3-s}(\mathbb{R}, \mathbb{R}) \) by \( G \).

**Theorem 7.0.2.** Let \( \tilde{\mathcal{V}} \) and \( \tilde{\mathcal{V}} \) be rank 2 distributions which are locally equivalent to the standard contact distributions on \( J^{k+3-r}(\mathbb{R}, \mathbb{R}) \) and \( J^{k+3-s}(\mathbb{R}, \mathbb{R}) \), and let \( G \) be an abelian group of dimension \( 2k + 3 - r - s \) with \( 2 \leq r, s \leq k + 1 \) acting freely on the product manifold \( J^{k+3-r}(\mathbb{R}, \mathbb{R}) \times J^{k+3-s}(\mathbb{R}, \mathbb{R}) \). Then the quotient distribution

\[ \Delta = (\tilde{\mathcal{V}} \oplus \tilde{\mathcal{V}})/G \]
defines a linear equation which is Darboux integrable at order \( k \).

Together these theorems imply that a second order hyperbolic partial differential equation in the plane is linear and Darboux integrable at order \( k \geq 2 \) if and only if its fundamental invariants consist of an abelian Vessiot group of dimension \( 2k + 3 - r - s \) and Vessiot distributions equivalent to standard contact distributions on \( J^{k+3-r}(\mathbb{R}, \mathbb{R}) \) and \( J^{k+3-s}(\mathbb{R}, \mathbb{R}) \) for \( 2 \leq r, s \leq k + 1 \).

7.1 Fundamental Invariants of Linear Equations

We begin the proof of Theorem 7.0.1 by showing the Vessiot group for any Darboux integrable linear equation must be abelian. Suppose (7.1) is Darboux integrable at order \( k \geq 2 \). Then its hyperbolic distribution \( \Delta = \Delta_0 \oplus \Delta_0 \), defined on the prolonged equation manifold \( M^{(k-2)} \) of dimension \( 2k + 3 \), is given by

\[
\Delta_0 = \left\{ \begin{array}{l}
\hat{X}_1 = D^{(k)}_x + (D^{(k)}_y)^{k-1}f \partial_{q_k}, \quad \hat{X}_2 = \partial_{p_k} \\
\hat{X}_1 = D^{(k)}_y + (D^{(k)}_x)^{k-1}f \partial_{p_k}, \quad \hat{X}_2 = \partial_{q_k}
\end{array} \right.,
\]

where \( f = Ap_1 + Bq_1 + Cu \). Here, \( p_i \) and \( q_j \) denote the \( i \)th and \( j \)th order \( x \) and \( y \) derivatives of \( u \), and \( D^{(k)}_x \) and \( D^{(k)}_y \) are defined recursively by

\[
D^{(i+1)}_x = D^{(i)}_x + p_{i+1} \partial_{p_i} + \left(D^{(i)}_y\right)^{i-1}f \partial_{q_j} \quad \text{and} \quad D^{(i+1)}_y = D^{(i)}_y + \left(D^{(i)}_x\right)^{i-1}f \partial_{p_j} + q_{i+1} \partial_{q_i},
\]

for \( i = 2, 3, \ldots, k - 1 \), with

\[
D^{(2)}_x = \partial_x + p_1 \partial_u + p_2 \partial_{p_1} + f \partial_{q_1} \quad \text{and} \quad D^{(2)}_y = \partial_y + q_1 \partial_u + f \partial_{p_1} + q_2 \partial_{q_1}.
\]

Immediately, we see that \( \hat{I}_0 = y \) and \( \hat{I}_0 = x \) are first integrals of \( \hat{\Delta} \) and \( \hat{\Delta} \), respectively. Under the assumption that (7.1) is Darboux integrable at order \( k \), there must exist \( k \)th order first integrals \( \hat{I}_k \) and \( \hat{I}_k \). The following lemma gives a preliminary form for these first integrals.
Lemma 7.1.1. The kth order first integrals $\tilde{I}_k$ and $\bar{I}_k$ for (7.1) are of the form

$$\tilde{I}_k = \alpha_0(x, y)u + \sum_{i=1}^{k} \alpha_i(x, y)q_i \quad \text{and} \quad \bar{I}_k = \alpha_0(x, y)u + \sum_{i=1}^{k} \tilde{\alpha}_i(x, y)p_i,$$

with $\tilde{\alpha}_k(x, y) \neq 0$ and $\tilde{\alpha}_k(x, y) \neq 0$.

With these first integrals, we can construct the nonsingular matrices

$$[\tilde{P}_j^i] = [\tilde{X}_i(\tilde{I}_j)] = \begin{pmatrix} 1 & \tilde{X}_1(\tilde{I}_j) \\ 0 & \tilde{\alpha}_k \end{pmatrix} \quad \text{and} \quad [\bar{P}_j^i] = [\bar{X}_i(\bar{I}_j)] = \begin{pmatrix} 1 & \bar{X}_1(\bar{I}_j) \\ 0 & \bar{\alpha}_k \end{pmatrix}.$$

The inverses of these matrices are given by

$$\tilde{P}^{-1} = \begin{pmatrix} 1 & -\tilde{X}_1(\tilde{I}_k)/\tilde{\alpha}_k \\ 0 & 1/\tilde{\alpha}_k \end{pmatrix} \quad \text{and} \quad \bar{P}^{-1} = \begin{pmatrix} 1 & -\bar{X}_1(\bar{I}_k)/\bar{\alpha}_k \\ 0 & 1/\bar{\alpha}_k \end{pmatrix}.$$

It then follows from Theorem 4.2.2, that we can construct commuting bases of vector fields \{\tilde{U}_i = (\tilde{P}^{-1})^j_i \tilde{X}_j\} and \{\bar{U}_i = (\bar{P}^{-1})^j_i \bar{X}_j\} for $\tilde{\Delta}$ and $\bar{\Delta}$. These vector fields are explicitly given by

$$\tilde{U}_1 = \tilde{X}_1 - \frac{\tilde{X}_1(\tilde{I}_2)}{\tilde{\alpha}_k} \tilde{X}_2, \quad \tilde{U}_2 = \frac{1}{\tilde{\alpha}_k} \tilde{X}_2, \quad \bar{U}_1 = \bar{X}_1 - \frac{\bar{X}_1(\bar{I}_2)}{\bar{\alpha}_k} \bar{X}_2, \quad \bar{U}_2 = \frac{1}{\bar{\alpha}_k} \bar{X}_2.$$

We then define sequences of iterated brackets \(\tilde{S}_i = [\tilde{U}_1, \tilde{S}_{i-1}]\) and \(\bar{S}_j = [\bar{U}_1, \bar{S}_{j-1}]\) where \(\tilde{S}_2 = [\tilde{U}_1, \tilde{U}_2]\) and \(\bar{S}_2 = [\bar{U}_1, \bar{U}_2]\). In doing so, we will show that the vector fields \{\tilde{U}_1, \tilde{U}_2, \tilde{S}_i\} and \{\bar{U}_1, \bar{U}_2, \bar{S}_j\} will span $\tilde{\Delta}^{(\infty)}$ and $\bar{\Delta}^{(\infty)}$, respectively. Therefore, we may choose bases for $K = \tilde{\Delta}^{(\infty)} \cap \bar{\Delta}^{(\infty)}$ consisting only of vector fields $\tilde{V}_i \in \{\tilde{S}_i\}$ or $\bar{V}_j \in \{\bar{S}_j\}$. The vector fields $V_i$ and $V_j$ will then form bases for the Vessiot algebra of (7.1). We claim that the Vessiot algebra must be abelian. To see this, we begin by explicitly calculating the vector field

$$\tilde{S}_2 = [\tilde{U}_1, \tilde{U}_2] = \left[ \tilde{X}_1 - \frac{\tilde{X}_1(\tilde{I}_2)}{\tilde{\alpha}_k} \tilde{X}_2, \frac{1}{\tilde{\alpha}_k} \tilde{X}_2 \right] = -\frac{1}{\tilde{\alpha}_k} \partial_{p_{k-1}} + \frac{\tilde{\alpha}_{k-1}}{\tilde{\alpha}_k^2} \partial_{p_k}.$$
Knowing the explicit form of \( \mathcal{S}_2 \) allows us to argue the following lemma.

**Lemma 7.1.2.** The vector fields \( \mathcal{S}_i \) must be of the form

\begin{equation}
\star \partial_u + \star \partial_{p_1} + \star \partial_{q_1} + \star \partial_{p_2} + \star \partial_{q_2} + \cdots + \star \partial_{p_k} + \star \partial_{q_k}
\end{equation}

(7.2)

where each \( \star \) denotes an arbitrary function of only \( x \) and \( y \).

**Proof.** Since \( \mathcal{U}_1 \partial_x = 1, \mathcal{U}_1 \partial_y = 0 \), and \( \mathcal{S}_2 \) is given as it is above, we easily conclude the subsequent brackets \( \mathcal{S}_i = [\mathcal{U}_1, \mathcal{S}_{i-1}] \) will never have nonzero \( \partial_x \) or \( \partial_y \) components. Furthermore, since the components of \( \mathcal{U}_1 \) are at most linear in \( u, p_1, q_1, \ldots, p_k, q_k \), we see that after each iteration of bracketing, the vector fields \( \mathcal{S}_i \) must be of the form (7.2).

From this lemma, we immediately see that \( [\mathcal{S}_i, \mathcal{S}_j] \equiv 0 \) for all \( i \) and \( j \) as again, all components of \( \mathcal{S}_i \) are strictly functions of \( x \) and \( y \), and the components of \( \partial_x \) and \( \partial_y \) always vanish. This allows us to further observe that \( [\mathcal{U}_2, \mathcal{S}_i] \equiv 0 \). To see this, consider the bracket

\[
[\mathcal{U}_2, \mathcal{S}_2] = \left[ -\frac{1}{\alpha_k} \partial_{p_k}, -\frac{1}{\alpha_k} \partial_{p_{k-1}} + \frac{\alpha_{k-1}}{\alpha_k^2} \partial_{p_k} \right] = 0,
\]

and proceed by induction giving

\[
[\mathcal{U}_2, \mathcal{S}_{i+1}] = [\mathcal{U}_2, [\mathcal{U}_1, \mathcal{S}_i]] = -[\mathcal{U}_1, [\mathcal{S}_i, \mathcal{U}_2]] - [\mathcal{S}_i, [\mathcal{U}_2, \mathcal{U}_1]] = [\mathcal{S}_i, \mathcal{S}_2] = 0.
\]

This means that by our construction, the vector fields \( \mathcal{V}_i \) must be of the form (7.2) as well. But then we immediately see that \( [\mathcal{V}_i, \mathcal{V}_j] \equiv 0 \) for all \( i, j \). Using a similar argument, we also conclude that \( [\mathcal{V}_i, \mathcal{V}_j] \equiv 0 \) for all \( i, j \). Therefore, the Vessiot algebra corresponding to any hyperbolic Darboux integrable equation of the form (7.1) must be abelian.

We now show that the Vessiot differential systems must be canonical contact systems on jet space. We have seen that the characteristic distributions \( \hat{\Delta} = \{\mathcal{U}_1, \mathcal{U}_2\} \) and \( \bar{\Delta} = \{\mathcal{U}_1, \mathcal{U}_2\} \) have derived flags satisfying

\[
\hat{\Delta}^{(i)} = \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{S}_i\}_{i=1}^{2k+3-r} \quad \text{and} \quad \bar{\Delta}^{(j)} = \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{S}_j\}_{j=1}^{2k+3-s}
\]
where again, \(r, s \geq 2\) denote the number of first integrals of \(\hat{\Delta}\) and \(\bar{\Delta}\), respectively. Since 
\[
[U_2, S_i] = [U_2, S_j] = 0 \quad \text{and} \quad [S_i, S_a] = [S_j, S_b] = 0 \quad \text{for all} \quad i, j, a, b,
\]
the weak derived flags of \(\hat{\Delta}\) and \(\bar{\Delta}\) must be exactly the same as the derived flags of \(\hat{\Delta}\) and \(\bar{\Delta}\).

Let \(\hat{\Delta}\) be an integral manifold of \(\hat{\Delta}\) given by a level set of the first integrals of \(\hat{\Delta}\), and let \(\iota : \hat{\Delta} \to M^{(k-2)}\) be an inclusion. Then the restriction \(V = \hat{\Delta}|_{\hat{\Delta}}\) of \(\hat{\Delta}\) to \(\hat{\Delta}\) given by 
\[
\hat{\Delta} = \iota^*(V)
\]
has derived flag and weak derived flag dimensions \((2, 3, \ldots, 2k + 3 - r)\). From this, we conclude by Theorem 2.5.2 that \(V\) must be locally equivalent to the canonical contact distribution on \(J^{2k+1-r}(\mathbb{R}, \mathbb{R})\). Similarly, let \(\bar{\Delta}\) be an integral manifold of \(\bar{\Delta}\) given by a level set of the first integrals of \(\bar{\Delta}\), and let \(\bar{\iota} : \bar{\Delta} \to M^{(k-2)}\) be an inclusion. Then by repeating the argument above, we see that the restriction \(V = \bar{\Delta}|_{\bar{\Delta}}\) is locally equivalent to the canonical contact distribution on \(J^{2k+1-s}(\mathbb{R}, \mathbb{R})\).

We therefore conclude that the Vessiot distributions of the \((k-2)\)-prolongation of (7.1) are the canonical contact distributions on \(J^{k+3-r}(\mathbb{R}, \mathbb{R})\) and \(J^{k+3-s}(\mathbb{R}, \mathbb{R})\).

Moreover, since \(\hat{\Delta}, \bar{\Delta}\) are locally equivalent to the canonical contact distributions on \(J^{2k+1-r}(\mathbb{R}, \mathbb{R})\) and \(J^{2k+1-s}(\mathbb{R}, \mathbb{R})\), it follows that \(\hat{\Delta}\) must be the \((k-2)\)-prolongation of a distribution which is locally equivalent to the canonical contact distribution on \(J^{k+3-r}(\mathbb{R}, \mathbb{R})\), and \(\bar{\Delta}\) must be the \((k-2)\)-prolongation of a distribution which is locally equivalent to the canonical contact distribution on \(J^{k+3-s}(\mathbb{R}, \mathbb{R})\).

### 7.2 Quotient Construction of Linear Equations

In [2], Anderson and Fels prove the following theorem.

**Theorem 7.2.1.** Let \(\hat{\mathcal{I}}\) and \(\bar{\mathcal{I}}\) be the standard contact systems on \(J^{m-2}(\mathbb{R}, \mathbb{R})\) and \(J^{n-2}(\mathbb{R}, \mathbb{R})\), respectively, and let \(G\) be an abelian contact symmetry group of dimension \(m + n - 3\) acting freely on the product manifold \(J^{m-2}(\mathbb{R}, \mathbb{R}) \times J^{n-2}(\mathbb{R}, \mathbb{R})\). Then the quotient differential systems
\[
\mathcal{I} = (\hat{\mathcal{I}} + \bar{\mathcal{I}})/G
\]
is the standard rank 3 Pfaffian system, defined on a 7-manifold, for a linear Darboux integrable equation of the form (7.1).

If we let \( m = k + 3 - r \) and \( n = k + 3 - s \), then this statement matches that of Theorem 7.0.2 without the assertion that the equation will be Darboux integrable at order \( k \). To show this, we begin by prolonging the differential system, \( I \), \( (k - 2) \) times, so that the corresponding prolonged equation is of order \( k \), giving

\[
I^{(k-2)} = \left( \hat{I}^{(k-2)} + \tilde{I}^{(k-2)} \right)/G^{(k-2)}
\]

where \( G^{(k-2)} \) denotes the \((k - 2)\)-fold prolongation of the action of \( G \).

The prolonged differential systems \( \hat{I}^{(k-2)} \) and \( \tilde{I}^{(k-2)} \) are defined on manifolds \( \hat{M}^{(k-2)} = J^{2k+1-r}(\mathbb{R}, \mathbb{R}) \) and \( \tilde{M}^{(k-2)} = J^{2k+1-s}(\mathbb{R}, \mathbb{R}) \), respectively. Since \( G \) acts freely, the number of differential invariants of \( G^{(k-2)} \) on \( \hat{M}^{(k-2)} \) is given by

\[
\dim \hat{M}^{(k-2)} - \dim G^{(k-2)} = s,
\]

and the number of differential invariants of \( G^{(k-2)} \) on \( \tilde{M}^{(k-2)} \) is given by

\[
\dim \tilde{M}^{(k-2)} - \dim G^{(k-2)} = r.
\]

Each of these differential invariants project to become the first integrals of the characteristic systems of \( I^{(k-2)} \) and all must be of order less than or equal to \( k \). Moreover, since \( 2 \leq r, s \leq k + 1 \), we see that there are sufficient numbers of invariants for the prolonged differential system to be Darboux integrable, proving Theorem 7.0.2.

### 7.3 Examples

We now present several examples of linear equations. In particular, we give the corresponding hyperbolic distribution, characteristic systems, their first integrals, and the fundamental invariants which, as stated in Theorem 7.0.1 always consist of contact distributions
and an abelian Vessiot group.

**Example 7.3.1.** For our first example, we consider the equation

\[ u_{xy} = \frac{u_x}{x + y}. \]  

(7.3)

The hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \bar{\Delta} \) is given by

\[
\hat{\Delta} = \left\{ \hat{X}_1 = \partial_x + p \partial_u + r \partial_p + \frac{p}{x + y} \partial_q, \hat{X}_2 = \partial_r \right\},
\]

\[
\bar{\Delta} = \left\{ \bar{X}_1 = \partial_y + q \partial_u + \frac{p}{x + y} \partial_p + t \partial_q + \frac{(x + y)r - p}{(x + y)^2} \partial_r, \bar{X}_2 = \partial_t \right\}.
\]

The first integrals are

\[
\hat{I}_1 = y, \quad \hat{I}_2 = t, \quad \hat{I}_1 = x, \quad \hat{I}_2 = \frac{p}{x + y}, \quad \hat{I}_3 = \frac{r}{x + y} - \frac{p}{(x + y)^2}.
\]

Using Theorem 4.2.2, we may construct commuting local bases of vector fields for \( \hat{\Delta} \) and \( \bar{\Delta} \) which are explicitly given by

\[
\hat{\Delta} = \left\{ \hat{U}_1 = \hat{X}_1 - 2 \left( \frac{r}{x + y} + \frac{p}{(x + y)^2} \right) \hat{X}_2, \hat{U}_2 = (x + y) \partial_r \right\},
\]

\[
\bar{\Delta} = \left\{ \bar{U}_1 = \bar{X}_1, \bar{U}_2 = \bar{X}_2 \right\}.
\]

We then construct the sequences of vector fields

\[
\hat{S}_1 = [\hat{U}_1, \hat{U}_2] = -(x + y) \partial_p - \partial_r,
\]

\[
\hat{S}_2 = [\hat{U}_1, \hat{S}_1] = (x + y) \partial_u + \partial_q,
\]

\[
\hat{S}_3 = [\hat{U}_1, \hat{S}_2] = \partial_u,
\]
\[ \mathcal{S}_1 = [\mathcal{U}_1, \mathcal{U}_2] = -\partial_q, \]
\[ \mathcal{S}_2 = [\mathcal{U}_1, \mathcal{S}_1] = \partial_u. \]

A simple calculation shows \( \hat{\Delta}^{(\infty)} \cap \tilde{\Delta}^{(\infty)} = \{ \partial_u, \partial_q \} \), and so \( \mathfrak{H} = \{ \mathcal{S}_2, \mathcal{S}_3 \} \) and \( \mathfrak{H} = \{ \mathcal{S}_1, \mathcal{S}_2 \} \) both form bases for the Vessiot algebra which is clearly abelian.

Upon restricting to the integral manifold \( \hat{M} \) given by \( \hat{I}_1 = \hat{I}_2 = 0 \), we see that the distribution \( \hat{\Delta} \) becomes,
\[ \mathcal{V} = \hat{\Delta}|_{\hat{M}} = \left\{ \partial_x + p\partial_u + r\partial_p + \frac{p}{x} \partial_q, \partial_r \right\}, \]
and similarly, after restricting to the integral manifold \( \tilde{M} \) given by \( \tilde{I}_1 = \tilde{I}_2 = \tilde{I}_3 = 0 \), we see that the distribution \( \tilde{\Delta} \) becomes
\[ \mathcal{V} = \tilde{\Delta}|_{\tilde{M}} = \left\{ \partial_y + q\partial_u + t\partial_q, \partial_t \right\}. \]

The derived and weak derived flags of \( \mathcal{V} \) are both \( (2, 3, 4, 5) \), and so we conclude by Theorem 2.5.2 that \( \mathcal{V} \) is locally equivalent to the standard contact distribution on \( J^3(\mathbb{R}, \mathbb{R}) \). The derived and weak derived flags for \( \mathcal{V} \) are both \( (2, 3, 4) \), so we conclude \( \mathcal{V} \) is locally equivalent to the standard contact distributions on \( J^2(\mathbb{R}, \mathbb{R}) \).

**Example 7.3.2.** We now consider the equation
\[ u_{xy} + \frac{u_x - u_y}{x - y} = 0. \] (7.4)

The hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) is given by
\[ \hat{\Delta} = \left\{ \hat{X}_1 = \partial_x + p\partial_u + r\partial_p - \frac{p - q}{x - y} \partial_q + \frac{t}{x - y} \partial_t, \hat{X}_2 = \partial_r \right\}, \]
\[ \tilde{\Delta} = \left\{ \tilde{X}_1 = \partial_y + q\partial_u - \frac{p - q}{x - y} \partial_p + t\partial_q - \frac{r}{x - y} \partial_r, \tilde{X}_2 = \partial_t \right\}. \]
The first integrals for \( \Delta \) and \( \bar{\Delta} \) are, respectively,

\[
\hat{I}_1 = y, \quad \hat{I}_2 = -\frac{t}{x-y}, \quad \bar{I}_1 = x, \quad \bar{I}_2 = \frac{r}{x-y}.
\]

These first integrals allow us to construct commuting bases for \( \Delta \) and \( \bar{\Delta} \) which are explicitly given by

\[
\Delta = \begin{cases} 
U_1 = X_1 + \frac{r}{x-y} \partial_r, & U_2 = (x-y) \partial_r \\
U_1 = X_1 - \frac{t}{x-y} \partial_t, & U_2 = (y-x) \partial_t
\end{cases},
\]

\[
\bar{\Delta} = \begin{cases} 
\bar{U}_1 = \bar{X}_1 - \frac{t}{x-y} \partial_t, & \bar{U}_2 = (y-x) \partial_t \\
\bar{U}_1 = \bar{X}_1 + \frac{r}{x-y} \partial_r, & \bar{U}_2 = (x-y) \partial_r
\end{cases}.
\]

We then construct the sequences of vector fields

\[
\hat{S}_1 = [\hat{U}_1, \hat{U}_2] = -(x-y) \partial_p, \quad \hat{S}_2 = [\hat{U}_1, \hat{S}_1] = (x-y) \partial_u - \partial_p - \partial_q, \quad \hat{S}_3 = [\hat{U}_1, \hat{S}_2] = 2 \partial_u,
\]

and

\[
\bar{S}_1 = [\bar{U}_1, \bar{U}_2] = (x-y) \partial_q, \quad \bar{S}_2 = [\bar{U}_1, \bar{S}_1] = -(x-y) \partial_u - \partial_p - \partial_q, \quad \bar{S}_3 = [\bar{U}_1, \bar{S}_2] = 2 \partial_u.
\]

A simple calculation shows \( \hat{\Delta}^{(\infty)} \cap \bar{\Delta}^{(\infty)} = \{ \partial_u, \partial_p, \partial_q \} \), and so \( \hat{\mathcal{V}} = \{ \hat{S}_1, \hat{S}_2, \hat{S}_3 \} \) and \( \bar{\mathcal{V}} = \{ \bar{S}_1, \bar{S}_2, \bar{S}_3 \} \) both form bases for the Vessiot algebra which is clearly abelian.

Upon restricting to the integral manifold \( \hat{M} \) given by \( \hat{I}_1 = \hat{I}_2 = 0 \), we see that the distribution \( \hat{\Delta} \) becomes,

\[
\hat{\mathcal{V}} = \hat{\Delta}|_{\hat{M}} = \left\{ \partial_x + p \partial_u + r \partial_p - \frac{p-q}{x} \partial_q, \partial_r \right\},
\]

and similarly, after restricting to the integral manifold \( \bar{M} \) given by \( \bar{I}_1 = \bar{I}_2 = 0 \), we see that the distribution \( \bar{\Delta} \) becomes

\[
\bar{\mathcal{V}} = \bar{\Delta}|_{\bar{M}} = \left\{ \partial_y + q \partial_u + \frac{p-q}{y} \partial_p + t \partial_q, \partial_t \right\}.
\]
Both $\mathcal{V}$ and $\mathcal{V}$ are rank 2 distributions defined on 5-dimensional manifolds whose derived and weak derived flags are $(2,3,4,5)$, and by Theorem 2.5.2, we conclude that $\mathcal{V}$ and $\mathcal{V}$ must both be locally equivalent to the standard contact distributions on $J^3(\mathbb{R}, \mathbb{R})$.

**Example 7.3.3.** In [19], Goursat shows that the nonlinear equation

$$u_{xx} - u_y u_{xy} + u_x u_{yy} = 0$$

is Darboux integrable at order two. Interestingly, we find that the Vessiot group for (7.5) is abelian and the Vessiot distributions are each standard contact distributions on $J^3(\mathbb{R}, \mathbb{R})$. Therefore, (7.5) is necessarily equivalent to a linear equation. In fact, (7.5) is contact equivalent to (7.4) via the Legendre transformation

$$\tilde{x} = -u_x, \quad \tilde{y} = -u_y, \quad \tilde{u} = u - xu_x - yu_y, \quad \tilde{u}_x = x, \quad \tilde{u}_y = y$$

composed with the transformation

$$\tilde{x} = \frac{\tilde{J}_0 \tilde{J}_0}{4}, \quad \tilde{y} = \frac{\tilde{J}_0 + \tilde{J}_0}{2}, \quad \tilde{u} = \tilde{u},$$

where $\tilde{J}_0$ and $\tilde{J}_0$ are given by

$$\tilde{J}_0 = y + \sqrt{y^2 - x} \quad \text{and} \quad \tilde{J}_0 = y - \sqrt{y^2 - x}.$$  

The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ associated to (7.5) is given by the vector fields

$$\hat{\Delta} = \left\{ \hat{X}_1 = D_x + \hat{I}_1 D_y + D_y(f) \partial_s, \hat{X}_2 = \partial_t - \hat{I}_1 \partial_s \right\},$$

$$\tilde{\Delta} = \left\{ \tilde{X}_1 = D_x + \tilde{I}_1 D_y + D_y(f) \partial_s, \tilde{X}_2 = \partial_t - \tilde{I}_1 \partial_s \right\},$$

where $\hat{I}_1$ and $\tilde{I}_1$ are given by

$$\hat{I}_1 = y + \sqrt{y^2 - x} \quad \text{and} \quad \tilde{I}_1 = y - \sqrt{y^2 - x}.$$
where

\[ f = qs - pt, \quad D_x = \partial_x + p \partial_u + f \partial_p + s \partial_q, \quad D_y = \partial_y + q \partial_u + s \partial_p + t \partial_q, \]

and the functions \( \hat{I}_1, \hat{I}_2, \tilde{I}_1, \tilde{I}_2 \) are first integrals of \( \Delta \) and \( \tilde{\Delta} \) given by

\[
\hat{I}_1 = \frac{-q - \sqrt{q^2 - 4p}}{2} \quad \text{and} \quad \hat{I}_2 = t + \frac{(-q + \sqrt{q^2 - 4p})s}{2p},
\]

and

\[
\tilde{I}_1 = \frac{-q + \sqrt{q^2 - 4p}}{2} \quad \text{and} \quad \tilde{I}_2 = t - \frac{(q + \sqrt{q^2 - 4p})s}{2p}.
\]

If we introduce the transformation to invariant coordinates given by

\[
x = x, \quad y = y, \quad u = u, \quad p = \hat{I}_1 \tilde{I}_1, \quad q = -(\hat{I}_1 + \tilde{I}_1),
\]

\[
s = \frac{\hat{I}_1 \tilde{I}_1 (\tilde{I}_2 - \hat{I}_2)}{\tilde{I}_1 - \hat{I}_1}, \quad t = \frac{\hat{I}_1 \tilde{I}_2 - \hat{I}_2 \tilde{I}_1}{\tilde{I}_1 - \hat{I}_1},
\]

then the characteristic distributions become

\[
\hat{\Delta} = \left\{ \hat{U}_1 = -\frac{1}{\hat{I}_1 \tilde{I}_2} \partial_x - \frac{1}{\tilde{I}_2} \partial_y + \frac{\hat{I}_1}{\tilde{I}_2} \partial_u + \partial_{\hat{I}_1}, \hat{U}_2 = \partial_{\tilde{I}_2} \right\}
\]

and

\[
\tilde{\Delta} = \left\{ \tilde{U}_1 = -\frac{1}{\hat{I}_1 \tilde{I}_2} \partial_x - \frac{1}{\tilde{I}_2} \partial_y + \frac{\hat{I}_1}{\tilde{I}_2} \partial_u + \hat{I}_1 \tilde{I}_2 \partial_{\hat{I}_1}, \tilde{U}_2 = \partial_{\tilde{I}_2} \right\}.
\]

These vector fields automatically form commuting bases for \( \hat{\Delta} \) and \( \tilde{\Delta} \). We can see that \( \hat{\Delta}^{(\infty)} \cap \tilde{\Delta}^{(\infty)} = \{ \partial_x, \partial_y, \partial_u \} \) and conclude that the Vessiot algebra for (7.5) is 3-dimensional.
We then calculate the sequence of vector fields,

\[ S_1 = [\tilde{U}_1, \tilde{U}_2] = -\frac{1}{I_1 I_2} \partial_x - \frac{1}{I_2} \partial_y + \frac{\tilde{I}_1}{I_2} \partial_u, \]

\[ S_2 = [\tilde{U}_1, \tilde{S}_1] = \frac{1}{I_1 I_2} \partial_x + \frac{\tilde{I}_1}{I_2} \partial_u, \]

\[ S_3 = [\tilde{U}_1, \tilde{S}_2] = \frac{2}{I_1 I_2} \partial_x, \]

from which we form bases \( \mathfrak{G} = \{ \tilde{S}_1, \tilde{S}_2, \tilde{S}_3 \} \) and \( \mathfrak{G} = \{ \tilde{S}_1, \tilde{S}_2, \tilde{S}_3 \} \) for the Vessiot algebra of (7.5) which is clearly abelian.

The restrictions of the characteristic distributions to the integral manifolds \( \hat{M} \) and \( \hat{M} \) given by \( \hat{I}_1 = 1, \hat{I}_2 = 0 \) and \( \hat{I}_1 = 1, \hat{I}_2 = 0 \) remain unchanged, so that the Vessiot distributions are

\[ \tilde{\mathcal{V}} = \hat{\Delta} |_{\hat{M}} = \{ \tilde{U}_1, \tilde{U}_2 \} \]

and

\[ \tilde{\mathcal{V}} = \hat{\Delta} |_{\hat{M}} = \{ \tilde{U}_1, \tilde{U}_2 \}. \]

Each of these distributions have derived and weak derived flags of dimension (2,3,4,5), and by Theorem 2.5.2 must be locally equivalent to the standard contact distribution on \( J^3(\mathbb{R}, \mathbb{R}) \).

By Theorem 7.0.1, the equation (7.5) must be equivalent to a linear equation of the form (7.1). To see this, we first introduce the Legendre transformation,

\[ \tilde{x} = -u_x, \quad \tilde{y} = -u_y, \quad \tilde{u} = u - xu_x - yu_y, \quad \tilde{x} \tilde{z} = x, \quad \tilde{u} \tilde{y} = y \]

which transforms (7.5) to the linear equation,

\[ xu_{xx} + yu_{xy} + u_{yy} = 0, \] (7.6)
where we have omitted tildes. Under this transformation, the first order invariants become

\[ \tilde{J}_0 = y + \sqrt{y^2 - x} \quad \text{and} \quad \tilde{J}_0 = y - \sqrt{y^2 - x}. \]

This leads us to make the further change of variables

\[ \tilde{x} = \frac{\tilde{J}_0 \tilde{J}_0}{4}, \quad \tilde{y} = \frac{\tilde{J}_0 + \tilde{J}_0}{2}, \quad \tilde{u} = \tilde{u}, \]

taking (7.6) to the linear \( f \)-Gordon equation (7.4).

**Example 7.3.4.** In [8], the authors describe a geometrical approach to the Born-Infeld scalar field model and its relation to the theory of the closed relativistic string. In doing so, they arrive at the so-called Born-Infeld equation

\[ (u_y^2 - 1)u_{xx} - 2u_x u_y u_{xy} + (u_x^2 + 1)u_{yy} = 0. \quad (7.7) \]

Similar to Example 7.3.3, equation (7.7) has a 3-dimensional abelian Vessiot group and Vessiot distributions which are locally equivalent to the standard contact distributions on \( J^3(\mathbb{R}, \mathbb{R}) \), so (7.7) can be transformed to a linear equation of the form (7.1).

The hyperbolic distribution \( \Delta = \tilde{\Delta} \oplus \tilde{\Delta} \) associated to (7.7) is generated by the vector fields

\[
\tilde{\Delta} = \begin{cases} 
\tilde{X}_1 = D_x + \tilde{I}_1 D_y + D_y (f) \partial_s, \\
\tilde{X}_2 = \partial_t - \tilde{I}_1 \partial_s,
\end{cases}
\]

where

\[ f = \frac{2pq - (p^2 + 1)t}{q^2 - 1}, \quad D_x = \partial_x + p \partial_u + f \partial_p + s \partial_q, \quad D_y = \partial_y + q \partial_u + s \partial_p + t \partial_q. \]
and

\[
\hat{I}_1 = \frac{-pq - \sqrt{1 + p^2 - q^2}}{q^2 - 1} \quad \text{and} \quad \hat{I}_2 = \frac{-pq + \sqrt{1 + p^2 - q^2}}{q^2 - 1}.
\]

The functions \(\hat{I}_1\) and \(\hat{I}_2\) are first integrals of \(\hat{\Delta}\) and \(\bar{\Delta}\), respectively. The characteristic distributions additionally admit the second-order first integrals

\[
\hat{I}_2 = \frac{t}{q^2 - 1} - \frac{s}{pq + \sqrt{1 + p^2 - q^2}} \quad \text{and} \quad \hat{I}_2 = \frac{t}{q^2 - 1} - \frac{s}{pq - \sqrt{1 + p^2 - q^2}}.
\]

It then becomes a routine calculation to show that the Vessiot algebra is 3-dimensional and abelian, however, we omit the generators due to their length.

The restriction of \(\hat{\Delta}\) to the integral manifold \(\hat{M}\) given by \(\hat{I}_1 = 1\) and \(\hat{I}_2 = 0\) gives the Vessiot distribution

\[
\hat{\mathcal{V}} = \hat{\Delta} |_{\hat{\mathcal{M}}} = \left\{ \partial_x + \frac{p^2 + 1}{p^2 - 1} \partial_y - \frac{2p}{p^2 - 1} \partial_u + \frac{2s}{p^2 - 1} \partial_p, \partial_s \right\}.
\]

This distribution has derived and weak derived flag dimensions \((2,3,4,5)\), so we conclude that it is locally equivalent to the standard contact distribution on \(J^3(\mathbb{R}, \mathbb{R})\). Similarly, the restriction of \(\bar{\Delta}\) to the integral manifold \(\bar{M}\) given by \(\bar{I}_1 = -1\) and \(\bar{I}_2 = 0\) gives the Vessiot distribution

\[
\bar{\mathcal{V}} = \bar{\Delta} |_{\bar{\mathcal{M}}} = \left\{ \partial_x - \frac{p^2 + 1}{p^2 - 1} \partial_y - \frac{2p}{p^2 - 1} \partial_u - \frac{2s}{p^2 - 1} \partial_p, \partial_s \right\}
\]

which again has derived and weak derived flag dimensions \((2,3,4,5)\), meaning \(\bar{\mathcal{V}}\) is locally equivalent to the standard contact distribution on \(J^3(\mathbb{R}, \mathbb{R})\) as well.

As in Example 7.3.3, we may linearize (7.7) by applying the Legendre transformation

\[
\bar{x} = -u_x, \quad \bar{y} = -u_y, \quad \bar{u} = u - xu_x - yu_y, \quad \bar{u}_x = x, \quad \bar{u}_y = y, \\
\bar{u}_{\bar{x}\bar{x}} = -\frac{u_{yy}}{u_{xx} u_{yy} - u_{xy}^2}, \quad \bar{u}_{\bar{x}\bar{y}} = \frac{u_{xy}}{u_{xx} u_{yy} - u_{xy}^2}, \quad \bar{u}_{\bar{y}\bar{y}} = -\frac{u_{xx}}{u_{xx} u_{yy} - u_{xy}^2}.
\]
This gives the equation

\[(x^2 + 1)u_{xx} + 2xyu_{xy} + (y^2 - 1)u_{yy} = 0 \quad (7.8)\]

where we have omitted any tildes. We notice that this equation has invariants

\[ \hat{J}_1 = -xy + \sqrt{1 + x^2 - y^2} \quad \text{and} \quad \hat{J}_1 = -xy - \sqrt{1 + x^2 - y^2} / (y^2 - 1) \]

which leads us to make the further change of variables,

\[ \tilde{x} = e^{\hat{J}_1}e^{\hat{J}_1} - 1, \quad \tilde{y} = e^{\hat{J}_1}e^{\hat{J}_1} + 1, \quad \tilde{u} = \bar{u}. \]

This takes (7.8) to the linear \( f \)-Gordon equation

\[ u_{xy} + 2 \left( \frac{xu_x - yu_y}{x^2 - y^2} \right) = 0. \]

**Example 7.3.5.** As a final example, we consider the linear equation

\[ u_{xy} = \frac{4u}{(x + y)^2} \quad (7.9) \]

which is Darboux integrable at order three. The hyperbolic distribution \( \Delta = \tilde{\Delta} \oplus \Delta \) for the prolonged equation is given by \( \tilde{\Delta} = \{ \tilde{X}_1, \tilde{X}_2 \} \) and \( \Delta = \{ \tilde{X}_1, \tilde{X}_2 \} \) where

\[ \tilde{X}_1 = \partial_x + p\partial_u + r\partial_p + \frac{6u}{(x + y)^2} \partial_q + \rho\partial_r + \frac{6q(x + y) - 12u}{(x + y)^3} \partial_t + A\partial_r, \quad \tilde{X}_2 = \partial_p, \]

\[ \tilde{X}_1 = \partial_y + q\partial_u + \frac{6u}{(x + y)^2} \partial_p + t\partial_q + \frac{6p(x + y) - 12u}{(x + y)^3} \partial_r + \tau\partial_t + B\partial_p, \quad \tilde{X}_2 = \partial_r, \]

and the coefficients \( A, B \) are given by

\[ A = \frac{6t(x + y)^2 - 24q(x + y) + 36u}{(x + y)^4} \quad \text{and} \quad B = \frac{6r(x + y)^2 - 24p(x + y) + 36u}{(x + y)^4}. \]
The first integrals for $\Delta$ and $\bar{\Delta}$ are, respectively,

$$I_1 = y, \quad I_2 = \tau + \frac{6(tx+ty+q)}{(x+y)^2}, \quad I_1 = x, \quad I_2 = \rho + \frac{6(rx+ry+p)}{(x+y)^2}.$$ 

Using Theorem 4.2.2, we construct commuting bases for $\Delta$ and $\bar{\Delta}$ which are explicitly given by

$$\Delta = \left\{ \bar{U}_1 = X_1 + \frac{12p - 6\rho(x+y)^2}{(x+y)^3}, \bar{U}_2 = \partial_x \right\},$$

$$\Delta = \left\{ \bar{U}_1 = X_1 + \frac{12p - 6\tau(x+y)^2}{(x+y)^3}, \bar{U}_2 = \partial_x \right\}.$$

A simple calculation shows that

$$\Delta^{(\infty)} \cap \bar{\Delta}^{(\infty)} = \left\{ \partial_u, \partial_p - \frac{6}{(x+y)^2} \partial_x, \partial_q - \frac{6}{(x+y)^2} \partial_{xu}, \partial_r - \frac{6}{x-y} \partial_x, \partial_t - \frac{6}{x-y} \partial_y \right\},$$

so that the Vessiot algebra is 5-dimensional. We can then take as bases for the Vessiot algebra $\mathfrak{V} = \{\mathcal{S}_i\}_{i=1}^5$ and $\{\mathcal{S}_j\}_{j=1}^5$ where

$$\mathcal{S}_1 = [\bar{U}_1, \bar{U}_2], \quad \mathcal{S}_2 = [\bar{U}_1, \bar{S}_1], \quad \mathcal{S}_3 = [\bar{U}_1, \bar{S}_2], \quad \mathcal{S}_4 = [\bar{U}_1, \bar{S}_3], \quad \mathcal{S}_5 = [\bar{U}_1, \bar{S}_4],$$

$$\mathcal{S}_1 = [\bar{U}_1, \bar{S}_1], \quad \mathcal{S}_2 = [\bar{U}_1, \bar{S}_2], \quad \mathcal{S}_3 = [\bar{U}_1, \bar{S}_3], \quad \mathcal{S}_4 = [\bar{U}_1, \bar{S}_4], \quad \mathcal{S}_5 = [\bar{U}_1, \bar{S}_4].$$

These vector fields satisfy the structure equations $[\mathcal{S}_i, \mathcal{S}_j] = 0$ and $[\bar{S}_i, \bar{S}_j] = 0$ for $1 \leq i, j \leq 5$, and we see that the Vessiot algebra is abelian.

Upon restricting $\Delta$ and $\bar{\Delta}$ to the integral manifolds $\mathcal{M}$ given by $\bar{I}_1 = \bar{I}_2 = 0$ and $\bar{\mathcal{M}}$ given by $\bar{I}_1 = \bar{I}_2 = 0$, we obtain the Vessiot distributions,

$$\hat{\mathcal{V}} = \Delta|_{\mathcal{M}} = \left\{ \partial_x + p \partial_u + r \partial_p + \frac{6u}{x^2} \partial_q + \rho \partial_r + \frac{6(xq-2u)}{x^3} \partial_t, \partial_x \right\},$$

$$\tilde{\mathcal{V}} = \bar{\Delta}|_{\bar{\mathcal{M}}} = \left\{ \partial_y + q \partial_u + \frac{6u}{y^2} \partial_p + t \partial_q + \frac{6(yp-2u)}{y^3} \partial_r + \tau \partial_t, \partial_r \right\}.$$ 

Each of these distributions have derived and weak derived flag dimensions (2,3,4,5,6,7), so
by 2.5.2, we conclude that both are locally equivalent to the standard contact distribution on $J^5(\mathbb{R}, \mathbb{R})$. 
Part III

Darboux Integrable $f$-Gordon Equations
CHAPTER 8
DARBOUX INTEGRABLE $f$-GORDON EQUATIONS – ORDER TWO

The classification of $f$-Gordon equations, up to complex-valued contact equivalence, of the form

$$u_{xy} = f(x, y, u, u_x, u_y)$$

which are Darboux integrable at order two was originally studied by Goursat [19]. In his classification, he presented eleven classes of equations but was not able to find explicit solutions for all of the equations. Later, Vessiot [35] introduced group-theoretic integration methods to the study of Darboux integrable equations, and in doing so, was able to improve upon Goursat’s classification as well as write closed-form solutions to all of the equations in Goursat’s list.

In his thesis [9], Biesecker further improved the results of Vessiot by giving a complete classification of $f$-Gordon equations Darboux integrable at order two up to real-valued contact equivalence. His analysis established the following thirteen classes of equations:

(I) \( s = 0, \)

(II) \( s = \frac{p}{x + y}, \)

(III) \( s = qe^u, \)

(IV) \( s = \frac{2u}{(x + y)^2}, \)

(V) \( s = a(x, y)p + b(x, y)q - a(x, y)b(x, y)u, \) where \( e^{ax}, e^{by} \) solve the \( A2 \) Toda lattice, and \( a_x \neq b_y, \)

(VI) \( s = e^u, \)

(VII) \( s = \frac{2u + x + y}{(u + x)(u + y)}pq, \)

(VIII) \( s = e^u \sqrt{p^2 + 1}, \quad s = e^u \sqrt{-1 - p^2} \)

(IX) \( s = \sqrt{\frac{(p^2 + 1)(q^2 + 1)}{u}}, \quad s = \sqrt{\frac{(p^2 + 1)(q^2 - 1)}{u}}, \quad s = \sqrt{\frac{(1 - p^2)(1 - q^2)}{u}}. \)
In this chapter, we characterize the fundamental invariants of \( f \)-Gordon equations which are Darboux integrable at order two and describe the quotient construction of these equations in terms of their fundamental invariants. We then calculate the fundamental invariants for each of representative equations listed above.

8.1 The Fundamental Invariants

The following theorem characterizes the fundamental invariants of \( f \)-Gordon equations which are Darboux integrable at order two and their properties.

**Theorem 8.1.1.** Let \( \Delta = \hat{\Delta} \oplus \bar{\Delta} \) be a hyperbolic distribution defined by an \( f \)-Gordon equation which is Darboux integrable at order two. Then the corresponding Vessiot distributions of \( \Delta \) are always locally equivalent to the standard contact distributions on either \( J^2(\mathbb{R}, \mathbb{R}) \) or \( J^3(\mathbb{R}, \mathbb{R}) \), and the Vessiot group of \( \Delta \) is the prolongation of a contact symmetry group of dimension 1, 2, or 3 acting on \( J^1(\mathbb{R}, \mathbb{R}) \).

**Proof.** The hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \bar{\Delta} \) for a general \( f \)-Gordon equation \( s = f(x, y, u, p, q) \) defined on a 7-manifold is given by

\[
\hat{\Delta} = \{ \partial_x + p \partial_u + r \partial_p + f \partial_q + D_y(f) \partial_t, \partial_t \},
\]

\[
\bar{\Delta} = \{ \partial_y + q \partial_u + f \partial_p + t \partial_q + D_x(f) \partial_t, \partial_t \}.
\]

A simple calculation shows that the derived flags of \( \hat{\Delta} \) and \( \bar{\Delta} \) are generically \((2, 3, 4, 6)\) and the weak derived flags are \((2, 3, 4, 5, 6)\). If we suppose that \( \Delta \) is Darboux integrable
at order two, then both $\hat{\Delta}$ and $\bar{\Delta}$ must admit at least two first integrals, but since the number of first integrals of a distribution $D$ on an $m$-dimensional manifold is given by $m - \text{rank}(D^{(\infty)})$, it follows that $4 \leq \text{rank}(\hat{\Delta}^{(\infty)}) \leq 5$ and $4 \leq \text{rank}(\bar{\Delta}^{(\infty)}) \leq 5$.

If a characteristic distribution has terminal derived of rank 4, then we see that it will have three first integrals, and the integral manifold given by the level sets of the first integrals will be 4-dimensional. Engel’s theorem then implies that the distribution will be equivalent to the canonical contact distribution on $J^2(\mathbb{R}, \mathbb{R})$.

If the terminal derived is of rank 5, then the characteristic distribution will have two first integrals, and the integral manifold given by the level sets of the first integrals will be 5-dimensional. We then see that both the derived flag and weak derived flag will have dimension $(2,3,4,5)$ and the characteristic distribution will be equivalent to the canonical contact distribution on $J^3(\mathbb{R}, \mathbb{R})$.

The dimension of the Vessiot group is given by the dimension of the equation manifold minus the total number of first integrals of $\hat{\Delta}$ and $\bar{\Delta}$. Since each of the characteristic distributions must have either 2 or 3 first integrals, we see that the dimension of the Vessiot group must be either 1, 2, or 3. Furthermore, both characteristic distributions are equivalent to contact distributions, and since the Vessiot group is always a symmetry group of $\hat{\Delta}$ and $\bar{\Delta}$, the Vessiot group for an $f$-Gordon equation which is Darboux integrable at order two must be a contact symmetry group, and by Backlund’s theorem, it’s action must be the prolongation of the action of a contact symmetry group on $J^1(\mathbb{R}, \mathbb{R})$. 

8.2 The Quotient Construction

In this section, we give a converse to the previous theorem. This gives the general quotient construction of $f$-Gordon equations which are Darboux integrable at order two.

**Theorem 8.2.1.** Let $\hat{\nabla}$ and $\bar{\nabla}$ be the standard contact distributions on manifolds $\hat{M} = J^{r-2}(\mathbb{R}, \mathbb{R})$ and $\bar{M} = J^{s-2}(\mathbb{R}, \mathbb{R})$ where $r, s \geq 4$. Let $G$ be a common symmetry group of dimension $r + s - 7$ of $\hat{\nabla}, \bar{\nabla}$, and denote the infinitesimal generators of the diagonal action of $G$ by $\Gamma_{\text{diag}}$. Suppose
[i] $G$ acts freely on $\tilde{M}$ and $\tilde{M}$,

[ii] $\Gamma_{\text{diag}} \cap (\tilde{V} \oplus \tilde{V})' = \{0\}$, and

[iii] $\Gamma_{\text{diag}} \cap (\tilde{V} \oplus \tilde{V})''$ is 1-dimensional.

Then the quotient distribution

$$\Delta = (\tilde{V} \oplus \tilde{V})/G_{\text{diag}}$$

is a rank 4 distribution defining an equation of Monge-Ampère type which is Darboux integrable at order $k = \max(r, s) - 3$.

If, in addition, $r, s \leq 5$, then $\Delta$ defines an $f$-Gordon equation which is Darboux integrable at order two.

Proof. The fact that the quotient distribution $\Delta$ defines a PDE of Monge-Ampère type follows immediately from Theorem 5.2.3. After prolonging $(k - 2) = \max(r, s) - 1$ times (so that the differential order of the corresponding PDE is $k$), the number of differential invariants on the prolonged manifolds $\tilde{M}^{(k-2)}$ and $\tilde{M}^{(k-2)}$ are given by

$$5 - s + k = 2 - s + \max(r, s) \quad \text{and} \quad 5 - r + k = 2 - r + \max(r, s),$$

respectively. From this, we see that when $r, s \geq 4$ there always exist at least two differential invariants of $G$ acting on $\tilde{M}^{(k-2)}$ and $\tilde{M}^{(k-2)}$. These differential invariants directly correspond to the first integrals of the characteristic systems of $\Delta^{(k-2)}$, and we see that $\Delta$ is Darboux integrable at order $k = \max(r, s) - 3$. Moreover, for purely dimensional reasons, when $r, s \leq 5$ the orbit dimension of the Vessiot group is either 1 on $J^1(\mathbb{R}, \mathbb{R})$ or 2 or 3 on $J^2(\mathbb{R}, \mathbb{R})$. In other words, the orbits will always be of codimension 1 or greater on $\tilde{M}$ and $\tilde{M}$. This guarantees the existence of differential invariants defined on $J^1(\mathbb{R}, \mathbb{R})$ or $J^2(\mathbb{R}, \mathbb{R})$ which quotient to first integrals of order 1, and therefore guarantees that $\Delta$ defines an $f$-Gordon equation.
Remark. We emphasize that when \( r, s > 5 \), the condition that the action of the Vessiot group be codimension 1 is not automatic.

Remark. Every classical Darboux integrable Monge-Ampère equation given by Goursat [19], which is not manifestly \( f \)-Gordon, has been shown to be equivalent to an \( f \)-Gordon equation. In Chapter 11, we use Theorem 8.2.1 to give new examples of Monge-Ampère equations, Darboux integrable at higher orders, which are not equivalent to \( f \)-Gordon equations.

8.3 Classification in Terms of Fundamental Invariants

In this section, we calculate the fundamental invariants for each of the equations in Biesecker list [9]. For classes (I)–(IV), we explicitly show that the Vessiot distributions are locally equivalent to standard contact distributions on \( J^2(\mathbb{R}, \mathbb{R}) \) or \( J^3(\mathbb{R}, \mathbb{R}) \). For classes (V)–(XIII), we note that the characteristic distributions each admit two first integrals, and as a consequence of Theorem 8.1.1, the corresponding Vessiot distributions must be locally equivalent to the standard contact distribution on \( J^3(\mathbb{R}, \mathbb{R}) \). For this reason, we will omit the explicit coordinate representations of the Vessiot distributions for these classes when they are not needed.

Class (I)

We first consider the class of equations equivalent to the wave equation

\[
s = 0. \tag{8.1}
\]

The hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \check{\Delta} \) is given by

\[
\hat{\Delta} = \{ \hat{X}_1 = \partial_x + p\partial_u + r\partial_p, \hat{X}_2 = \partial_r \} \quad \text{and} \quad \check{\Delta} = \{ \check{X}_1 = \partial_y + q\partial_u + t\partial_q, \check{X}_2 = \partial_t \}.
\]

The first integrals are

\[
\hat{I}_0 = y, \quad \hat{I}_1 = q, \quad \hat{I}_2 = t, \quad \check{I}_0 = x, \quad \check{I}_1 = p, \quad \check{I}_2 = r.
\]
In order to give the quotient representation of (8.1), we notice that the local bases for \( \hat{\Delta} \) and \( \tilde{\Delta} \) already commute, and so we may immediately calculate the sequences of vector fields

\[
\hat{S}_1 = [\hat{X}_1, \hat{X}_2] = -\partial_p, \quad \hat{S}_2 = [\hat{X}_1, \hat{S}_1] = \partial_u, \quad \hat{S}_1 = [\hat{X}_1, \hat{X}_2] = -\partial_q, \quad \hat{S}_2 = [\hat{X}_1, \hat{S}_1] = \partial_u.
\]

Now, since \( K = \hat{\Delta}^{(\infty)} \cap \tilde{\Delta}^{(\infty)} = \{\partial_u\} \), we see that we may choose bases for \( K \) consisting only of the vector fields \( \hat{S}_2 \) and \( \tilde{S}_2 \). This gives the realizations

\[
\hat{\mathcal{V}} = \{\partial_u\} \quad \text{and} \quad \tilde{\mathcal{V}} = \{\partial_u\}
\]

for the 1-dimensional Vessiot algebra of (8.1). This is the abstract Lie algebra \( n_{1,1} \) in [28].

Upon restricting \( \hat{\Delta} \) to the integral manifold \( \mathcal{M} \) given by \( \mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3 = 0 \), we obtain the Vessiot distribution

\[
\hat{\mathcal{V}} = \hat{\Delta}|_{\mathcal{M}} = \{\partial_x + p\partial_u + r\partial_p, \partial_r\},
\]

and similarly, upon restricting \( \tilde{\Delta} \) to the integral manifold \( \tilde{\mathcal{M}} \) given by \( \tilde{\mathcal{I}}_1 = \tilde{\mathcal{I}}_2 = \tilde{\mathcal{I}}_3 = 0 \), we obtain the Vessiot distribution

\[
\tilde{\mathcal{V}} = \tilde{\Delta}|_{\tilde{\mathcal{M}}} = \{\partial_y + q\partial_u + t\partial_q, \partial_t\}.
\]

Each of these distributions are equivalent to the standard contact distribution on \( J^2(\mathbb{R}, \mathbb{R}) \). The Vessiot algebra is therefore generated by the prolongation of \( p_{1,1} \) in Table A.1.

**Class (II)**

We showed in Example 7.3.1 that the equations equivalent to

\[
s = \frac{p}{x + y}
\]

have fundamental invariants consisting of Vessiot distributions which are locally equivalent
to the standard contact distributions on $J^2(\mathbb{R}, \mathbb{R})$ and $J^3(\mathbb{R}, \mathbb{R})$ and a 2-dimensional abelian Vessiot algebra whose generators are given by the prolongation of $p_{2,1}$ in Table A.1 to $J^2(\mathbb{R}, \mathbb{R})$ and $J^3(\mathbb{R}, \mathbb{R})$. As an abstract Lie algebra, this is $2n_{1,1}$ in [28].

Class (III)

The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \bar{\Delta}$ for the class of equations equivalent to

$$s = qe^u$$

(8.3)

is given by

$$\hat{\Delta} = \left\{ \hat{X}_1 = \partial_x + p\partial_u + qe^u\partial_q + e^u(q^2 + t)\partial_t, \hat{X}_2 = \partial_r \right\},$$

$$\bar{\Delta} = \left\{ \bar{X}_1 = \partial_y + q\partial_u + qe^u\partial_p + t\partial_q + qe^u(e^u + p)\partial_r, \bar{X}_2 = \partial_t \right\}.$$ 

The first integrals for $\hat{\Delta}$ and $\bar{\Delta}$ are

$$\hat{I}_0 = y, \quad \hat{I}_2 = \frac{t}{q} - q, \quad \bar{I}_0 = x, \quad \bar{I}_1 = p - e^u, \quad \bar{I}_2 = r - pe^u.$$ 

Using Theorem 4.2.2, we construct the commuting bases

$$\hat{\Delta} = \left\{ \hat{U}_1 = \hat{X}_1 + e^u(r + p^2)\hat{X}_2, \hat{U}_2 = \hat{X}_2 \right\},$$

$$\bar{\Delta} = \left\{ \bar{U}_1 = \bar{X}_1 + \frac{t(t + q^2)}{q}\bar{X}_2, \bar{U}_2 = \bar{X}_2 = q\bar{X}_2 \right\},$$

and then compute the sequences of vector fields

$$\hat{S}_1 = [\hat{U}_1, \hat{U}_2] = -\partial_p - e^u\partial_r,$$

$$\hat{S}_2 = [\hat{U}_1, \hat{S}_1] = \partial_u + e^u\partial_p + e^u(p + e^u)\partial_r,$$

$$\hat{S}_3 = [\hat{U}_1, \hat{S}_2] = -e^u\partial_u - e^{2u}\partial_p - qe^u\partial_q - e^{2u}(p + e^u)\partial_r - e^u(t + q^2)\partial_t.$$
\[
\begin{align*}
\tilde{S}_1 &= [\tilde{U}_1, \tilde{U}_2] = -q\partial_q - (t + q^2)\partial_t, \\
\tilde{S}_2 &= [\tilde{U}_1, \tilde{S}_1] = q\partial_u + q\epsilon u \partial_p + q^2 \partial_q + q\epsilon u (p + \epsilon u) \partial_r + q(t + q^2)\partial_t.
\end{align*}
\]

Now, since
\[
K = \tilde{\Delta}^{(\infty)} \cap \tilde{\Delta}^{(\infty)} = \begin{cases} 
q\partial_q + (t + q^2)\partial_t, \\
q\partial_u + q\epsilon u \partial_p + q^2 \partial_q + q\epsilon u (p + \epsilon u) \partial_r + q(t + q^2)\partial_t,
\end{cases}
\]
we may pick bases \(\tilde{\mathcal{H}} = \{\tilde{S}_2, \tilde{S}_3\}\) and \(\tilde{\mathcal{H}} = \{\tilde{S}_2, \tilde{S}_1\}\) for \(K\) which serve as realizations for the 2-dimensional, solvable Vessiot algebra of (8.3). In doing so, the abstract structure equations for each realization are
\[
[e_1, e_2] = e_2.
\]
This is the 2-dimensional Lie algebra \(\mathfrak{s}_{2,1}\) in [28].

Upon restricting \(\tilde{\Delta}\) to the integral manifold \(\tilde{M}\) given by \(\tilde{I}_0 = \tilde{I}_2 = 0\), we obtain the Vessiot distribution
\[
\hat{\mathcal{V}} = \tilde{\Delta}|_{\tilde{M}} = \{\partial_x + p\partial_u + r\partial_p + q\epsilon u \partial_q, \partial_r\},
\]
and similarly, restricting \(\tilde{\Delta}\) to the integral manifold \(\tilde{M}\) given by \(\tilde{I}_0 = \tilde{I}_1 = \tilde{I}_2 = 0\) gives the Vessiot distribution
\[
\check{\mathcal{V}} = \check{\Delta}|_{\check{M}} = \{\partial_y + q\partial_u + t\partial_q, \partial_t\}.
\]
The derived and weak derived flags of \(\hat{\mathcal{V}}\) each have dimension \((2,3,4,5)\), so we conclude by Theorem 2.5.2 that \(\hat{\mathcal{V}}\) is locally equivalent to the standard contact distribution on \(J^3(\mathbb{R}, \mathbb{R})\). The distribution \(\check{\mathcal{V}}\) is the standard contact distribution on \(J^2(\mathbb{R}, \mathbb{R})\).
Since the Vessiot algebra must be a symmetry algebra of the Vessiot distributions, we conclude that it must have generators given by the prolongation of $p_{2,2}$ in Table A.1.

Class (IV)

The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \bar{\Delta}$ for the class of equations equivalent to

$$s = \frac{2u}{(x+y)^2}$$

is given by

$$\Delta = \left\{ \bar{\mathbf{X}}_1 = \partial_x + p\partial_u + r\partial_p + \frac{2u}{(x+y)^2}\partial_y + \frac{2(qx+qy-2u)}{(x+y)^3}\partial_t, \bar{\mathbf{X}}_2 = \partial_r \right\},$$

$$\bar{\Delta} = \left\{ \bar{\mathbf{X}}_1 = \partial_y + q\partial_u + \frac{2u}{(x+y)^2}\partial_p + t\partial_q + \frac{2(px+py-2u)}{(x+y)^3}\partial_r, \bar{\mathbf{X}}_2 = \partial_t \right\}.$$

The first integrals for these systems are

$$\hat{I}_0 = x, \quad \hat{I}_2 = t + \frac{2q}{x+y}, \quad \bar{I}_0 = y, \quad \bar{I}_2 = r + \frac{2p}{x+y}.$$

Using these first integrals, we construct the commuting bases of vector fields,

$$\hat{\Delta} = \left\{ \mathbf{U}_1 = \hat{\mathbf{X}}_1 - \frac{2(rx+ry-p)}{(x+y)^2}\hat{\mathbf{X}}_2, \mathbf{U}_2 = \hat{\mathbf{X}}_2 \right\},$$

$$\mathbf{\bar{\Delta}} = \left\{ \bar{\mathbf{U}}_1 = \bar{\mathbf{X}}_1 - \frac{2(tx+ty-q)}{(x+y)^2}\bar{\mathbf{X}}_2, \bar{\mathbf{U}}_2 = \bar{\mathbf{X}}_2 \right\},$$

and the sequences of vector fields

$$\hat{\mathbf{S}}_1 = [\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2] = -\partial_p + \frac{2}{x+y}\partial_r,$$

$$\hat{\mathbf{S}}_2 = [\hat{\mathbf{U}}_1, \hat{\mathbf{S}}_1] = \partial_u - \frac{2}{x+y}\partial_p + \frac{4}{(x+y)^2}\partial_r,$$

$$\hat{\mathbf{S}}_3 = [\hat{\mathbf{U}}_1, \hat{\mathbf{S}}_2] = \frac{2}{(x+y)^2}\partial_u - \frac{2}{(x+y)^2}\partial_p - \frac{2}{(x+y)^2}\partial_q + \frac{4}{(x+y)^3}\partial_r + \frac{4}{(x+y)^3}\partial_t.$$
and

\[ S_1 = [\bar{U}_1, \bar{U}_2] = -\partial_u + \frac{2}{x+y} \partial_t, \]

\[ S_2 = [\bar{U}_1, \bar{S}_1] = \partial_u - \frac{2}{x+y} \partial_p + \frac{4}{(x+y)^2} \partial_r, \]

\[ S_3 = [\bar{U}_1, \bar{S}_2] = \frac{2}{x+y} \partial_u - \frac{2}{(x+y)^2} \partial_p - \frac{2}{(x+y)^2} \partial_q + \frac{4}{(x+y)^3} \partial_r + \frac{4}{(x+y)^3} \partial_t. \]

Since

\[ K = \hat{\Delta}^{(\infty)} \cap \bar{\Delta}^{(\infty)} = \left\{ \partial_u, \partial_p - \frac{2}{x+y} \partial_r, \partial_q - \frac{2}{x+y} \partial_t \right\}, \]

we may pick bases \( \mathcal{N} = \{ \bar{S}_1, \bar{S}_2, \bar{S}_3 \} \) and \( \bar{\mathcal{N}} = \{ \bar{S}_1, \bar{S}_2, \bar{S}_3 \} \) for \( K \) which serve as realizations for the 3-dimensional abelian Vessiot algebra of (8.4) which we immediately seen to be abelian. As an abstract Lie algebra, this is \( 3\mathfrak{n}_{1,1} \) in [28].

Furthermore, upon restricting \( \hat{\Delta} \) to the integral manifold \( \hat{M} \) given by \( \bar{I}_0 = \bar{I}_2 = 0 \), we obtain the Vessiot distribution,

\[ \hat{\mathcal{V}} = \hat{\Delta}_{|\hat{M}} = \left\{ \partial_x + p \partial_u + r \partial_p + \frac{2u}{x^2} \partial_q, \partial_t \right\}. \]

Similarly, upon restricting \( \bar{\Delta} \) to the integral manifold \( \bar{M} \) given by \( \bar{I}_0 = \bar{I}_2 = 0 \) we obtain the Vessiot distribution,

\[ \bar{\mathcal{V}} = \bar{\Delta}_{|\bar{M}} = \left\{ \partial_y + q \partial_u + \frac{2u}{x^2} \partial_p + t \partial_q, \partial_t \right\}. \]

Both \( \hat{\mathcal{V}} \) and \( \bar{\mathcal{V}} \) are rank 2 distributions defined on 5-manifolds whose derived and weak derived flags have dimension \((2,3,4,5)\), and by Theorem 2.5.2, are locally equivalent to contact distributions on \( J^3(\mathbb{R}, \mathbb{R}) \).

Finally, since \( \hat{\mathcal{N}} \) and \( \bar{\mathcal{N}} \) are symmetry algebras of \( \hat{\Delta} \) and \( \bar{\Delta} \), respectively, their restrictions \( \hat{\mathcal{N}}_{|\hat{M}} \) and \( \bar{\mathcal{N}}_{|\bar{M}} \) will be symmetry algebras of \( \hat{\Delta}_{|\hat{M}} \) and \( \bar{\Delta}_{|\bar{M}} \). The only 3-dimensional abelian symmetry algebra of the contact distribution on \( J^3(\mathbb{R}, \mathbb{R}) \) is given by the third
prolongation of $p_{3,7}$ in Table A.1. Due to the generators $\{\widehat{S}_i\}$ and $\{\tilde{S}_i\}$, $1 \leq i \leq 3$ being identical upon interchanging $x$ with $y$ (and their respective derivatives), we see that the actions of the Vessiot algebra must be equivalent.

**Class (V)**

In [9], Biesecker shows that the class of equations given by

$$s = a(x, y)p + b(x, y)q - a(x, y)b(x, y)u$$

is contact equivalent to (8.4) when $a_x = b_y$. When this is not the case, that is when $a_x \neq b_y$, he shows that in order for (8.5) to be Darboux integrable at order two, $a_x \neq 0$, $b_y \neq 0$, and $\ln(-a_x)$ and $\ln(-b_y)$ must satisfy the $A_2$ Toda lattice so that

$$(\ln -a_x)_{xy} = -2a_x + b_y \quad \text{and} \quad (\ln -b_y)_{xy} = a_x - 2b_y.$$  

Since we assume the equation is Darboux integrable at order two, the corresponding characteristic systems must admit two first integrals (and no more since this would force the equation to be in Class (I) or Class (II)). We can then conclude by Theorem 7.0.1, that the Vessiot distributions of (8.5) are locally equivalent to the standard contact distributions on $J^3(\mathbb{R}, \mathbb{R})$ and the Vessiot algebra the 3-dimensional abelian Lie algebra $3n_{1,1}$ in [28]. The action of the Vessiot algebra is generated by the prolongation of $p_{3,7}$ in Table A.1 to $J^3(\mathbb{R}, \mathbb{R})$ where the respective actions on each side are taken to be inequivalent to avoid Class (III).

**Remark.** Biesecker states that if (8.5) is Darboux integrable, the coefficients $a(x, y)$ and $b(x, y)$ are given by

$$a(x, y) = \frac{\partial}{\partial y} (\ln \det [\psi_i(y), \phi_i(x), \phi'_i(x)]) \quad \text{and} \quad b(x, y) = \frac{\partial}{\partial x} (\ln \det [\phi_i(x), \psi_i(y), \psi'_i(y)])$$

where $\phi_i, \psi_i, 1 \leq i \leq 3$ are arbitrary smooth functions. As part of our future work, we plan to show how the coefficients $a(x, y)$ and $b(x, y)$ depend on the functions $\zeta_i$ appearing in the
action $p_{3,7}$.

**Class (VI)**

We showed in Example 4.4.3 that the hyperbolic Liouville equation

$$s = e^u$$

has fundamental invariants consisting of Vessiot distributions which are locally equivalent to the standard contact distribution on $J^3(\mathbb{R}, \mathbb{R})$ and a 3-dimensional Vessiot algebra generated by prolongation of the intransitive action $p_{3,9}$ of $\mathfrak{sl}(2)$ to $J^3(\mathbb{R}, \mathbb{R})$.

**Class (VII)**

The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \check{\Delta}$ for the class of equations equivalent to the equation

$$s = \frac{2u + x + y}{(u + x)(u + y)}pq$$

is given by

$$\hat{\Delta} = \left\{ \hat{X}_1 = \partial_x + p\partial_u + r\partial_p + \frac{2u + x + y}{(u + x)(u + y)}pq\partial_y + A\partial_t, \hat{X}_2 = \partial_r \right\},$$

$$\check{\Delta} = \left\{ \check{X}_1 = \partial_y + q\partial_u + \frac{2u + x + y}{(u + x)(u + y)}pq\partial_p + t\partial_q + B\partial_r, \check{X}_2 = \partial_t \right\}$$

where

$$A = \frac{(u + y)(2u + x + y)pt + 2pq(qu + qy - u - x)}{(u + x)(u + y)^2},$$

$$B = \frac{(u + x)(2u + x + y)qr + 2pq(pu + qx - u - y)}{(u + x)^2(u + y)}.$$

The first integrals for these systems are

$$\tilde{I}_0 = x, \quad \tilde{I}_2 = \frac{t}{q} - \frac{2q + 1}{u + y}, \quad \tilde{I}_0 = y, \quad \tilde{I}_2 = \frac{r}{p} - \frac{2p + 1}{u + x}.$$
In utilizing Theorem 4.2.2, we construct the commuting bases \( \{ \tilde{U}_i \} \) and \( \{ \tilde{V}_j \} \) for \( \tilde{\Delta} \) and \( \tilde{\Delta} \), respectively, as

\[
\tilde{\Delta} = \left\{ \tilde{U}_1 = \tilde{X}_1 - p\tilde{X}_1(\tilde{I}_2)\tilde{X}_2, \quad \tilde{U}_2 = p\tilde{X}_2 \right\}
\]

\[
\tilde{\Delta} = \left\{ \tilde{U}_1 = \tilde{X}_1 - q\tilde{X}_1(\tilde{I}_2)\tilde{X}_2, \quad \tilde{U}_2 = q\tilde{X}_2 \right\}.
\]

We then compute the sequences of vector fields

\[
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2],
\]

\[
\check{S}_1 = [\check{U}_1, \check{U}_2], \quad \check{S}_2 = [\check{U}_1, \check{S}_1], \quad \check{S}_3 = [\check{U}_1, \check{S}_2].
\]

We note that \([\tilde{U}_2, \tilde{S}_i] = 0, [\check{U}_2, \check{S}_i] = 0\) for \(1 \leq i \leq 2\), \([\check{U}_2, \check{S}_3] = \check{S}_2\), and \([\tilde{U}_2, \check{S}_3] = \check{S}_2\).

These vector fields form bases \( \tilde{\mathfrak{N}} = \{ \tilde{S}_i \}_{i=1}^3 \) and \( \check{\mathfrak{N}} = \{ \check{S}_j \}_{j=1}^3 \) for the 3-dimensional Vessiot algebra of (8.7). In particular, the nonzero structure equations for \( \tilde{\mathfrak{N}} \) are

\[
[\tilde{S}_1, \tilde{S}_2] = -\tilde{S}_2, \quad [\tilde{S}_1, \tilde{S}_3] = -\tilde{S}_3,
\]

and the structure equations for \( \check{\mathfrak{N}} \) are

\[
[\check{S}_1, \check{S}_2] = -\check{S}_2, \quad [\check{S}_1, \check{S}_3] = -\check{S}_3.
\]

As abstract Lie algebras, both \( \tilde{\mathfrak{N}} \) and \( \check{\mathfrak{N}} \) are equivalent to \( s_{3,1} \) of [28] with \( a = 1 \). Moreover, since \( \tilde{\mathfrak{N}} \) and \( \check{\mathfrak{N}} \) must be contact symmetries, we can identify them as actions in Table A.1.

There are two different actions of \( s_{3,1} \) on \( \mathbb{R}^2 \) given by \( p_{3,5} \) and \( p_{3,8} \), the first of which is transitive on \( J^1(\mathbb{R}, \mathbb{R}) \) (except when \( \alpha = 1 \)), and the second of which is always intransitive on \( J^1(\mathbb{R}, \mathbb{R}) \). Upon deprolonging \( \tilde{\mathfrak{N}} \) and \( \check{\mathfrak{N}} \) to \( J^1(\mathbb{R}, \mathbb{R}) \), we find that

\[
\tilde{\mathfrak{N}} = \left\{ \frac{p}{u} \partial_u + \frac{p^2}{u^2} \partial_p, \frac{2p^2}{u} \partial_p, \frac{p}{u} \partial_u \right\}^{(2)} \quad \text{and} \quad \check{\mathfrak{N}} = \left\{ \frac{q}{u} \partial_u + \frac{q^2}{u^2} \partial_q, \frac{q}{u} \partial_u + \frac{2q^2}{u} \partial_q \right\}^{(2)}.
\]

Each of these actions are intransitive on \( J^1(\mathbb{R}, \mathbb{R}) \), so we conclude that the actions of the
Vessiot algebra each must correspond to the prolongation of either \( p_{3,5} \) with \( \alpha = 1 \) or \( p_{3,8} \), but these are equivalent.

**Class (VIII)**

In his thesis [9], Biesecker proves that every Darboux integrable equation of the form

\[
\frac{u_{xy}}{u_{xy}} = \frac{A(u_x)B(u_y)}{C(u)}, \tag{8.8}
\]

with \( A_{pp} \neq 0 \) and \( B_{qq} = 0 \) is contact equivalent to an equation of the form

\[
\begin{align*}
\text{(a)} & \quad s = e^u \sqrt{p^2 + 1}, \\
\text{(b)} & \quad s = e^u \sqrt{p^2 - 1}, \\
\text{(c)} & \quad s = s = e^u \sqrt{1 - p^2}.
\end{align*}
\]

In doing so, he first shows that a Darboux integrable equation of the form (8.8) is necessarily equivalent to an equation of the form

\[
u_{xy} = e^u A(p) \tag{8.9}
\]

where \( A(p) \) satisfies the relation,

\[
A' = \frac{\lambda p}{A} \tag{8.10}
\]

for some nonzero constant \( \lambda \). Here, we assume this relation holds and calculate the Vessiot algebra for all three classes simultaneously.

To begin, (8.9) has hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \check{\Delta} \) given by

\[
\begin{align*}
\hat{\Delta} &= \left\{ \begin{array}{l}
\hat{X}_1 = \partial_x + p \partial_u + r \partial_p + e^u A \partial_q + \left( q e^u A + \lambda p e^{2u} \right) \partial_t, \\
\hat{X}_2 = \partial_r,
\end{array} \right. \\
\check{\Delta} &= \left\{ \begin{array}{l}
\check{X}_1 = \partial_y + q \partial_u + e^u A \partial_p + t \partial_q + \left( p e^u A + \frac{\lambda r}{A} \right) \partial_t, \\
\check{X}_2 = \partial_t.
\end{array} \right.
\end{align*}
\]
The characteristic distributions $\hat{\Delta}$ and $\tilde{\Delta}$ admit first integrals

\[
\hat{I}_1 = y, \quad \hat{I}_2 = t - \frac{q^2}{2} - \frac{\lambda e^{2u}}{2}, \quad \tilde{I}_1 = x, \quad \tilde{I}_2 = \frac{r}{A} + \frac{A}{\lambda},
\]

respectively.

We can then use Theorem 4.2.2 to adjust the bases for $\hat{\Delta}$ and $\tilde{\Delta}$ so that they commute. Doing so gives,

\[
\hat{\Delta} = \{ \hat{U}_1 = \hat{X}_1 + \frac{rp(\lambda r + A^2)}{A^2} \hat{X}_2, \quad \hat{U}_2 = A \partial_r, \} \\
\tilde{\Delta} = \{ \tilde{U}_1 = \tilde{X}_1 + q(t + \lambda e^{2u}) \tilde{X}_2, \quad \tilde{U}_2 = \partial_t. \}
\]

The generators for the Vessiot algebra corresponding to each characteristic distributions are given by

\[
\hat{S}_1 = [\hat{U}_1, \hat{U}_2], \quad \hat{S}_2 = [\hat{U}_1, \hat{S}_1], \quad \hat{S}_3 = [\hat{U}_1, \hat{S}_2],
\]

and

\[
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2].
\]

The initial structure equations for $\tilde{\mathcal{V}}$ do not depend on the parameter $\lambda$, and so we list them first:

\[
[\tilde{S}_1, \tilde{S}_2] = \tilde{S}_1, \quad [\tilde{S}_1, \tilde{S}_3] = \tilde{S}_2, \quad [\tilde{S}_2, \tilde{S}_3] = 2\hat{I}_2 \tilde{S}_1 + \tilde{S}_3.
\]

We can then apply the change of basis

\[
\tilde{S}_3 = \hat{I}_2 \tilde{S}_1 + \tilde{S}_3,
\]
so that the structure equations become

\[
[S_1, S_2] = S_1, \quad [S_1, S_3] = S_2, \quad [S_2, S_3] = S_3.
\]

The Killing form for $\tilde{\mathfrak{g}}$ is given by

\[
\tilde{\kappa} = \begin{pmatrix}
0 & 0 & -2 \\
0 & 2 & 0 \\
-2 & 0 & 0
\end{pmatrix}.
\]

We see that $\det \tilde{\kappa} < 0$, and so the Vessiot algebra is $\mathfrak{sl}(2)$. Let $\tilde{\kappa}|_{\tilde{\mathcal{M}}}$ be the restriction of $\tilde{\kappa}$ to the contact distribution on the integral manifold $\tilde{\mathcal{M}}$ given by $\tilde{I}_1 = \tilde{I}_2 = 0$. Then,

\[
\tilde{\kappa}|_{\tilde{\mathcal{M}}} = \begin{pmatrix}
2 & 0 \\
0 & 0
\end{pmatrix}
\]

with $\det \tilde{\kappa}|_{\tilde{\mathcal{M}}} = 0$,

meaning that the action of $\mathfrak{sl}(2)$ generated by $\tilde{\mathfrak{g}}$ on $\tilde{\mathcal{M}}$ is intransitive and corresponds to $\mathfrak{p}_{3,9}$.

The initial structure equations for $\tilde{\mathfrak{g}}$ are

\[
[S_1, S_2] = \frac{\lambda p^2 - A^2}{\lambda \tilde{I}_2} \tilde{S}_1 - \frac{1}{\tilde{I}_2} \tilde{S}_3,
\]

\[
[S_1, S_3] = -\lambda \tilde{I}_2 \tilde{S}_2,
\]

\[
[S_2, S_3] = \frac{(\lambda p^2 - A^2)(A^2 - \lambda p^2 - \lambda^2 \tilde{I}_2^2)}{\lambda^2 \tilde{I}_2^2} \tilde{S}_1 + \frac{\lambda p^2 - A^2}{\lambda \tilde{I}_2} \tilde{S}_3,
\]

however upon substituting the first integral of (8.10),

\[
A^2 = \lambda p^2 + c,
\]
the structure equations become
\[
[\hat{S}_1, \hat{S}_2] = -\frac{c}{\lambda I_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3, \quad [\hat{S}_1, \hat{S}_3] = -\lambda \hat{I}_2 \hat{V}_2, \quad [\hat{S}_2, \hat{S}_3] = \frac{c(\lambda^2 \hat{I}_2^2 - c)}{\lambda^2 \hat{I}_2^2} \hat{S}_1 - \frac{c}{\lambda \hat{I}_2} \hat{S}_3.
\]

At this point, we can consider different cases depending on the values of \(c\) and \(\lambda\). We first note that we may exclude the case where \(c = 0\), as this forces \(A\) to be linear in \(p\). Secondly, we exclude the case where both \(\lambda\) and \(c\) are negative as we are only interested in equations with real-valued coefficients.

**Case 1.1:** Suppose \(\lambda, c > 0\). Then (8.9) is of the form
\[
s = e^u \sqrt{\lambda p^2 + c}.
\]

By introducing the transformation \(u \rightarrow u - \ln(\lambda)/2\), we obtain the equation
\[
s = e^u \sqrt{p^2 + \frac{c}{\lambda}},
\]
and if we further transform \(x \rightarrow x \sqrt{\frac{\lambda}{c}}\), then the equation becomes
\[
s = e^u \sqrt{p^2 + 1}.
\]

That is, without loss of generality, we can set \(\lambda = c = 1\). In doing so, the \(\hat{H}\) structure equations become
\[
[\hat{S}_1, \hat{S}_2] = -\frac{1}{I_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3, \quad [\hat{S}_1, \hat{S}_3] = -\hat{I}_2 \hat{S}_2, \quad [\hat{S}_2, \hat{S}_3] = \frac{\hat{I}_2^2 - 1}{\hat{I}_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3.
\]

Applying the change of basis
\[
\hat{S}_3 = -\frac{1}{I_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3,
\]
gives
\[ [\hat{S}_1, \hat{S}_2] = \hat{S}_3, \quad [\hat{S}_1, \hat{S}_2] = \hat{S}_2, \quad [\hat{S}_2, \hat{S}_3] = -\hat{S}_1. \]

The Killing form for \( \hat{\mathfrak{H}} \) is
\[
\hat{\kappa} = \begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -2
\end{pmatrix}.
\]

We see that \( \det \hat{\kappa} < 0 \), and so the Vessiot algebra is \( \mathfrak{sl}(2) \). Let \( \hat{\kappa}|_{\tilde{M}} \) be the restriction of \( \hat{\kappa} \) to the contact distribution on the integral manifold \( \tilde{M} \) given by \( \tilde{I}_1 = \tilde{I}_2 = 0 \). Then we see that,
\[
\hat{\kappa}|_{\tilde{M}} = \begin{pmatrix}
2 & 0 \\
0 & -2e^{-2u}/(p^2 + 1)
\end{pmatrix}
\text{ with } \det \hat{\kappa}|_{\tilde{M}} = -\frac{4e^{-2u}}{(p^2 + 1)} < 0,
\]
meaning that the action of \( \mathfrak{sl}(2) \) generated by \( \hat{\mathfrak{H}} \) on \( \tilde{M} \) is of Type I and corresponds to \( \mathfrak{p}_{3,1} \) in Table A.1.

Case 1.2: We now suppose \( \lambda > 0 \) and \( c < 0 \). Then (8.9) is of the form
\[
s = e^u \sqrt{p^2 - \frac{|c|}{\lambda}}
\]
where we have performed the same transformation \( u \rightarrow u - \ln(\lambda)/2 \) as in the previous case. We can then introduce the transformation \( x \rightarrow x \sqrt{\frac{\lambda}{|c|}} \) taking the equation to the form
\[
s = e^u \sqrt{p^2 - 1}
\]
so that we may set $\lambda = 1$ and $c = -1$ without loss of generality. In doing so, the $\mathfrak{H}$ structure equations become

$$[\hat{S}_1, \hat{S}_2] = \frac{1}{I_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3, \quad [\hat{S}_1, \hat{S}_2] = -\frac{1}{I_2} \hat{S}_2, \quad [\hat{S}_2, \hat{S}_3] = \frac{- (I_2^2 + 1)}{I_2^2} \hat{S}_1 + \frac{1}{I_2} \hat{S}_3.$$ 

Applying the change of basis

$$\hat{S}_3 = \frac{1}{I_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3,$$

gives

$$[\hat{S}_1, \hat{S}_2] = \hat{S}_3, \quad [\hat{S}_1, \hat{S}_2] = \hat{S}_2, \quad [\hat{S}_2, \hat{S}_3] = \hat{S}_1.$$ 

The Killing form for $\mathfrak{H}$ is

$$\hat{\kappa} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We see that $\det \hat{\kappa} < 0$, and so the Vessiot algebra is $\mathfrak{sl}(2)$. Let $\hat{\kappa} | \overset{\sim}{M}$ be the restriction of $\hat{\kappa}$ to the contact distribution on the integral manifold $\overset{\sim}{M}$ given by $\overset{\sim}{I}_1 = \overset{\sim}{I}_2 = 0$. Then we see that,

$$\hat{\kappa} | \overset{\sim}{M} = \begin{pmatrix} -2 & 0 \\ 0 & \frac{2 e^{-2u}}{p^2 - 1} \end{pmatrix} \quad \text{with} \quad \det \hat{\kappa} | \overset{\sim}{M} = -\frac{4 e^{-2u}}{p^2 - 1} < 0,$$

meaning that the action of $\mathfrak{sl}(2)$ generated by $\mathfrak{H}$ on $\overset{\sim}{M}$ is of Type I and corresponds to $p_{3,1}$ in Table A.1.

**Case 2:** Finally, suppose $\lambda < 0$ and $c > 0$. In this case, (8.9) is of the form

$$s = e^u \sqrt{c - |\lambda| p^2}.$$
Applying the transformation \( u \to u - \ln(|\lambda|)/2 \) takes the equation to the form

\[
s = e^u \sqrt{\frac{c}{|\lambda|} - p^2},
\]

and if we subsequently apply the transformation \( x \to x \sqrt{\frac{|\lambda|}{c}} \), then we arrive at the equation

\[
s = e^u \sqrt{1 - p^2};
\]

that is, we may set \( \lambda = -1 \) and \( c = 1 \) without loss of generality. The \( \hat{\mathfrak{g}} \) structure equations become

\[
[\hat{S}_1, \hat{S}_2] = \frac{p^2 + A^2}{I_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3,
\]

\[
[\hat{S}_1, \hat{S}_3] = \hat{I}_2 \hat{S}_2,
\]

\[
[\hat{S}_2, \hat{S}_3] = -\frac{(p^2 + A^2)(A^2 + p^2 - I_2^2)}{I_2^2} \hat{S}_1 + \frac{p^2 + A^2}{I_2} \hat{S}_3,
\]

Applying the change of basis

\[
\hat{S}_3 = \frac{1}{I_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3,
\]

gives

\[
[\hat{S}_1, \hat{S}_2] = \hat{S}_3, \quad [\hat{S}_1, \hat{S}_2] = -\hat{S}_2, \quad [\hat{S}_2, \hat{S}_3] = -\hat{S}_1.
\]

The Killing form for \( \hat{\mathfrak{g}} \) is

\[
\hat{\kappa} = \begin{pmatrix}
-2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

We see that \( \det \hat{\kappa} < 0 \), and so the Vessiot algebra is \( \mathfrak{sl}(2) \). Let \( \hat{\kappa}|_M \) be the restriction of \( \hat{\kappa} \).
to the contact distribution on the integral manifold $\tilde{M}$ given by $\tilde{I}_1 = \tilde{I}_2 = 0$. Then we see that,

$$\hat{\kappa}|_{\tilde{M}} = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{2e^{-2u}}{p^2 - 1} \end{pmatrix} \quad \text{with} \quad \det\hat{\kappa}|_{\tilde{M}} = \frac{4e^{-2u}}{1 - p^2} > 0,$$

meaning that the action of $\mathfrak{sl}(2)$ generated by $\hat{\mathfrak{G}}$ on $\tilde{M}$ is of Type II and corresponds to $p_{3,2}$ in Table A.1.

**Remark.** We saw that the actions given by $\hat{\mathfrak{G}}$ for $s = e^u\sqrt{p^2 + 1}$ and $s = e^u\sqrt{p^2 - 1}$ both correspond to the action given by $p_{3,1}$. A further study of the explicit generators is needed to distinguish the fundamental invariants for these equations.

**Class (IX)**

The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ for the equation

$$s = \frac{2\sqrt{pq}}{x + y} \quad (8.11)$$

is given by

$$\hat{\Delta} = \left\{ \hat{X}_1 = \partial_x + p\partial_u + r\partial_p + \frac{2\sqrt{pq}}{x + y}\partial_q + A\partial_t, \hat{X}_2 = \partial_t \right\}$$

$$\tilde{\Delta} = \left\{ \tilde{X}_1 = \partial_y + q\partial_u + \frac{2\sqrt{pq}}{x + y}\partial_p + t\partial_q + B\partial_r, \tilde{X}_2 = \partial_t \right\},$$

where

$$A = \frac{tp}{(x + y)\sqrt{pq}} + \frac{2q(\sqrt{pq} - p)}{(x + y)^2\sqrt{pq}} \quad \text{and} \quad B = \frac{rq}{(x + y)\sqrt{pq}} + \frac{2p(\sqrt{pq} - q)}{(x + y)^2\sqrt{pq}}.$$ 

The first integrals for these systems are

$$\hat{I}_0 = x, \quad \hat{I}_2 = \frac{t}{q} + \frac{2\sqrt{q}}{x + y}, \quad \tilde{I}_2 = y, \quad \tilde{I}_2 = \frac{r}{p} + \frac{2\sqrt{p}}{x + y}.$$
In utilizing Theorem 4.2.2, we construct the commuting bases \( \{ \tilde{U}_i \} \) and \( \{ \tilde{U}_j \} \) for \( \tilde{\Delta} \) and \( \tilde{\Delta} \), respectively, as

\[
\tilde{\Delta} = \left\{ \tilde{U}_1 = \tilde{X}_1 + \left( \frac{r^2}{2p} - \frac{r}{x+y} + \frac{2p}{(x+y)^2} \right) \tilde{X}_2, \tilde{U}_2 = \sqrt{p} \tilde{X}_2 \right\},
\]

\[
\tilde{\Delta} = \left\{ \tilde{U}_1 = \tilde{X}_1 + \left( \frac{t^2}{2q} - \frac{t}{x+y} + \frac{2q}{(x+y)^2} \right) \tilde{X}_2, \tilde{U}_2 = \sqrt{q} \tilde{X}_2 \right\}.
\]

We then compute the sequences of vector fields

\[
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2],
\]

\[
\bar{S}_1 = [\bar{U}_1, \bar{U}_2], \quad \bar{S}_2 = [\bar{U}_1, \bar{S}_1], \quad \bar{S}_3 = [\bar{U}_1, \bar{S}_2].
\]

These vector fields form bases \( \tilde{\mathcal{H}} = \{ \tilde{S}_i \}_{i=1}^{3} \) and \( \bar{\mathcal{H}} = \{ \bar{S}_i \}_{i=1}^{3} \) for the 3-dimensional Vessiot algebra of (8.11). After applying the change of basis,

\[
\tilde{S}_1 \mapsto \bar{I}_2 \tilde{S}_1, \quad \tilde{S}_2 \mapsto \bar{S}_2, \quad \tilde{S}_3 \mapsto \tilde{S}_3,
\]

the nonzero structure equations for \( \tilde{\mathcal{H}} \) become

\[
[\tilde{S}_1, \tilde{S}_2] = -\tilde{S}_3.
\]

Similarly, after applying the change of basis

\[
\bar{S}_1 \mapsto \bar{I}_2 \bar{S}_1, \quad \bar{S}_2 \mapsto \bar{S}_2, \quad \bar{S}_3 \mapsto \bar{S}_3,
\]

the nonzero structure equations for \( \bar{\mathcal{H}} \) become

\[
[\bar{S}_1, \bar{S}_2] = -\bar{S}_3.
\]

As abstract Lie algebras, both \( \tilde{\mathcal{H}} \) and \( \bar{\mathcal{H}} \) correspond to \( n_{3,1} \) in [28], and therefore, each must correspond to prolongation of the action \( p_{3,3} \) in Table A.1 to \( J^3(\mathbb{R}, \mathbb{R}) \).
Classes (X) and (XI)

In his thesis [9], Biesecker proves that every Darboux integrable equation of the form

\[ u_{xy} = h(x, y, u) \sqrt{\pm p^2 \pm 1} \sqrt{\pm q^2 \pm 1} \quad (8.12) \]

is contact equivalent to one of three classes of equations given by

(a) \[ s = \sqrt{\pm p^2 1} \sqrt{\pm q^2 1} C(u), \]
(b) \[ s = \sqrt{1 - p^2 \sqrt{1 - q^2}} C(u), \]
(c) \[ s = \sqrt{p^2 1} \sqrt{1 - q^2} C(u). \]

In doing so, he begins with the fact that the functions \( A, B, \) and \( C \) must satisfy the relations,

\[ A' = \frac{\lambda p}{A}, \quad B' = \frac{\mu q}{B}, \quad C'' = \frac{(C')^2 - \lambda \mu}{C}, \quad (8.13) \]

for nonzero constants \( \lambda \) and \( \mu \). Here, we assume these relations hold and calculate the Vessiot algebra for all three classes simultaneously.

To begin, (8.12) has hyperbolic distribution \( \Delta = \tilde{\Delta} \oplus \bar{\Delta} \) given by

\[ \tilde{\Delta} = \begin{cases} \tilde{X}_1 = \partial_x + p \partial_u + r \partial_p + \frac{AB}{C} \partial_q + \left( \frac{\mu qtA}{BC} + \frac{\lambda p B^2}{C^2} - \frac{qABC'}{C^2} \right) \partial_t, \\ \tilde{X}_2 = \partial_r, \end{cases} \]
\[ \bar{\Delta} = \begin{cases} \bar{X}_1 = \partial_y + q \partial_u + \frac{AB}{C} \partial_p + t \partial_q + \left( \frac{\lambda prB}{AC'} + \frac{\mu q A^2}{C^2} - \frac{pABC'}{C^2} \right) \partial_t, \\ \bar{X}_2 = \partial_t. \end{cases} \]

The subdistributions \( \tilde{\Delta} \) and \( \bar{\Delta} \) admit first integrals

\[ \tilde{I}_1 = y, \quad \tilde{I}_2 = t \frac{BC'}{\mu C'}, \quad \bar{I}_1 = x, \quad \bar{I}_2 = r \frac{AC'}{\lambda C'}. \]
Knowing these, we then adjust the local bases for $\Delta$ and $\bar{\Delta}$ so that they commute. Doing so gives,

$$
\Delta = \begin{cases} 
\hat{U}_1 = \hat{X}_1 + \frac{p(\lambda r^2 C^2 - r A^2 CC' + \mu A^4)}{A^2 C^2} \hat{X}_2, \\
\hat{U}_2 = A \partial_r,
\end{cases}
$$

$$
\bar{\Delta} = \begin{cases} 
\tilde{U}_1 = \tilde{X}_1 + \frac{q(\mu t^2 C^2 - t B^2 CC' + \lambda B^4)}{B^2 C^2} \tilde{X}_2, \\
\tilde{U}_2 = B \partial_t.
\end{cases}
$$

The generators for the Vessiot algebra corresponding to each characteristic distributions are given by

$$
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2],
$$

and

$$
\check{S}_1 = [\check{U}_1, \check{U}_2], \quad \check{S}_2 = [\check{U}_1, \check{S}_1], \quad \check{S}_3 = [\check{U}_1, \check{S}_2].
$$

The initial structure equations for $\tilde{\mathfrak{g}}$ are

$$
[\tilde{S}_1, \tilde{S}_2] = \frac{(\lambda p^2 - A^2)(C'' - \lambda \mu)}{\lambda I_2 C^2} \tilde{S}_1 - \frac{1}{I_2} \tilde{S}_2,
$$

$$
[\tilde{S}_1, \tilde{S}_2] = -I_2 \tilde{S}_2,
$$

$$
[\check{S}_2, \check{S}_3] = \frac{(\lambda p^2 - A^2)(C'' - \lambda \mu)(A^2(C')^2 - \lambda \mu A^2 - \lambda^2 I_2^2 C^2 - \lambda p^2(C')^2 + \lambda^2 \mu p^2)}{\lambda I_2 C^2} \check{S}_1
$$

$$
+ \frac{(\lambda p^2 - A^2)(C'' - \lambda \mu)}{\lambda I_2 C^2} \check{S}_3,
$$

however upon substituting the first integrals of (8.13)

$$
A^2 = \lambda p^2 + c_1, \quad B^2 = \mu q^2 + c_2, \quad \frac{(C')^2 - \lambda \mu}{C^2} = c_3
$$
and making the change of basis

\[ \hat{S}_3 = -\frac{c_1c_3}{\lambda I_2} \hat{S}_1 - \frac{1}{I_2} \hat{S}_3, \]

the structure equations simplify to

\[ [\hat{S}_1, \hat{S}_2] = \hat{S}_3, \quad [\hat{S}_1, \hat{S}_3] = \lambda \hat{S}_2, \quad [\hat{S}_2, \hat{S}_3] = -\varepsilon_1 \hat{S}_1. \] (8.14)

where \( \varepsilon_1 = c_1c_3 \). Similarly, we find the structure equations for \( \tilde{V} \) can be written as

\[ [\tilde{S}_1, \tilde{S}_2] = \tilde{S}_3, \quad [\tilde{S}_1, \tilde{S}_3] = \mu \tilde{S}_2, \quad [\tilde{S}_2, \tilde{S}_3] = -\varepsilon_2 \tilde{S}_1 \] (8.15)

where \( \varepsilon_2 = c_2c_3 \).

From here, we conclude the abstract Vessiot algebra is either \( s_{3,1}, s_{3,3}, sl(2) \), or \( so(3) \) depending on the values of \( \lambda, \mu, \varepsilon_1, \) and \( \varepsilon_2 \). The actions of the Vessiot algebra for each equation can be determined from the explicit generators \( \{\hat{S}_i\} \) and \( \{\tilde{S}_j\} \).

After restricting \( \tilde{V} \) and \( \hat{V} \) to the integral manifolds \( \hat{M} = J^3(\mathbb{R}, \mathbb{R}) \) and \( \tilde{M} = J^3(\mathbb{R}, \mathbb{R}) \) given by \( \hat{I}_1 = 0, \hat{I}_2 = 1 \) and \( \tilde{I}_1 = 0, \tilde{I}_2 = 1 \), respectively, the generators become

\[ \hat{S}_1 = -B \partial_q, \quad \hat{S}_2 = B \partial_u + \frac{pAq}{C} \partial_p - \frac{qBC'}{C} \partial_q + \frac{\mu(A^2B + \lambda qrt - A^2C'pq)}{AC^2} \partial_r, \]
\[ \hat{S}_3 = -\mu \partial_u - \frac{AB}{C} \partial_p + \frac{B((q^2\mu - B^2)(C')^2 + BCC'C\mu - \lambda \mu^2 pq + c_2c_3C^2 + \lambda \mu B^2)}{\mu C^2} \partial_q \]
\[ + \frac{\mu(pA^2BC' - \lambda qB) - \mu qA^3)}{AC^2} \partial_r, \]

and

\[ \tilde{S}_1 = -A \partial_p, \quad \tilde{S}_2 = A \partial_u - \frac{pAC'}{C} \partial_p + \frac{\lambda \mu B}{C} \partial_q + \frac{\lambda (AB^3 + \mu qrt - pqB^2C')}{BC^2} \partial_t, \]
\[ \tilde{S}_3 = -\mu \partial_u + \frac{A((\lambda q^2 - A^2)(C')^2 + ACC'(\lambda - \lambda^2 \mu p^2 + c_3c_3C^2 + \lambda \mu A^2))}{\lambda C^2} \partial_p - \frac{\lambda AB^3}{C} \partial_q \]
\[ + \frac{\lambda (qAB^2C' - \mu qrtAC + \lambda \mu B^3)}{BC^2} \partial_t. \]
Since \( V' \) has Cauchy characteristic \( \partial_r \), and \( \tilde{V}' \) has Cauchy characteristic \( \partial_t \), we can further calculate the reduction of these generators to \( J^2(\mathbb{R}, \mathbb{R}) \) as

\[
\hat{T}_1 = -B \partial_q, \quad \hat{T}_2 = B \partial_u + \frac{\mu A q}{C} \partial_p - \frac{q B C'}{C} \partial_q, \\
\hat{T}_3 = -\mu q \partial_u - \frac{\mu A B C}{C} \partial_p + \frac{B((q^2 \mu - B^2)(C')^2 + BCC' \mu - \lambda \mu^2 q^2 + c_2 c_3 C^2 + \lambda \mu B^2)}{\mu C^2} \partial_q,
\]

and

\[
\tilde{T}_1 = -A \partial_p, \quad \tilde{T}_2 = A \partial_u - \frac{p A C'}{C} \partial_p + \frac{\lambda p B}{C} \partial_q, \\
\tilde{T}_3 = -\lambda p \partial_u + \frac{A((\lambda p^2 - A^2)(C')^2 + A C' C^2 - \lambda^2 \mu p^2 + c_1 c_3 C^2 + \lambda \mu A^2)}{\lambda C^2} \partial_p - \frac{\lambda A B C}{C} \partial_q.
\]

These generators still satisfy the structure equations (8.14) and (8.15) above. Furthermore, the reduction of \( \hat{V}' \) and \( \tilde{V}' \) by these Cauchy characteristics have the simple form

\[
\hat{W} = \left\{ \partial_x + p \partial_u + \frac{AB}{C} \partial_q, \partial_p \right\} \quad \text{and} \quad \tilde{W} = \left\{ \partial_y + q \partial_u + \frac{AB}{C} \partial_p, \partial_q \right\}.
\]

The first-derived distributions, \( \hat{W}' \) and \( \tilde{W}' \) each admit a single Cauchy characteristic given by

\[
A(\hat{W}') = \left\{ \partial_x + p \partial_u - \frac{A^2 C'}{\lambda C} \partial_p + \frac{AB}{C} \partial_q \right\}, \\
A(\tilde{W}') = \left\{ \partial_y + q \partial_u + \frac{AB}{C} \partial_p - \frac{B^2 C'}{\mu C} \partial_q \right\}.
\]

Moreover, since \( x \) and \( y \) are differential invariants of \( \{ \hat{T}_i \}_{i=1}^3 \) and \( \{ \tilde{T}_i \}_{i=1}^3 \), respectively, and since these invariants do not survive the reduction to \( J^1(\mathbb{R}, \mathbb{R}) \) by the corresponding Cauchy characteristics, we conclude that the action of the Vessiot algebra must always be transitive on \( J^1(\mathbb{R}, \mathbb{R}) \). This implies that for equations (X.1) – (X.3) the action must be given by \( p_{3,5} \) in Table A.1. We list the abstract Vessiot algebra and the actions corresponding to each of the equations of Class (X) in the table below. We remark it remains to show which
values of $\alpha$ in the actions given by $p_{3,5}$ correspond to equations (X.1)–(X.3).

Table 8.2: Vessiot Algebra and Actions for Class (X)

<table>
<thead>
<tr>
<th>Class</th>
<th>Equation</th>
<th>$\mathcal{G}$</th>
<th>$\mathcal{H}$</th>
<th>Vessiot Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X.1)</td>
<td>$s = \frac{\sqrt{p^2 + 1} \sqrt{q^2 + 1}}{u}$</td>
<td>$p_{3,5}$</td>
<td>$p_{3,5}$</td>
<td>$\mathfrak{s}_{3,1}$</td>
</tr>
<tr>
<td>(X.2)</td>
<td>$s = \frac{\sqrt{p^2 + 1} \sqrt{q^2 - 1}}{u}$</td>
<td>$p_{3,5}$</td>
<td>$p_{3,5}$</td>
<td>$\mathfrak{s}_{3,1}$</td>
</tr>
<tr>
<td>(X.3)</td>
<td>$s = \frac{\sqrt{p^2 - 1} \sqrt{q^2 - 1}}{u}$</td>
<td>$p_{3,5}$</td>
<td>$p_{3,5}$</td>
<td>$\mathfrak{s}_{3,1}$</td>
</tr>
<tr>
<td>(X.4)</td>
<td>$s = \sqrt{1 - p^2} \sqrt{1 - q^2}$</td>
<td>$p_{3,10}$</td>
<td>$p_{3,10}$</td>
<td>$\mathfrak{s}_{3,3}$</td>
</tr>
</tbody>
</table>

The Vessiot algebra for each of the equations in Class (XI) is $\mathfrak{sl}(2)$ with the exception of the equation

$$s = \frac{\sqrt{1 - p^2} \sqrt{1 - q^2}}{\sin u}$$

whose Vessiot algebra is $\mathfrak{so}(3)$ and whose actions correspond to prolongations of $p_{3,12}$ to $J^3(\mathbb{R}, \mathbb{R})$. It remains to classify the specific $\mathfrak{sl}(2)$-actions for other equations in this class. A preliminary analysis has shown that all of the actions for these equations must be transitive on $J^4(\mathbb{R}, \mathbb{R})$, however, we have not yet been able to further classify these actions.

Class (XII)

We now consider equations of the form

$$s = \frac{\alpha(p)\beta(q)}{x + y}$$  \hspace{1cm} (8.16)
where $|\alpha(p)+1| = e^{p+\alpha(p)}$ and $|\beta(q)+1| = e^{q+\beta(q)}$. In Biesecker’s derivation of this equation, he shows that the functions $\alpha(p)$ and $\beta(q)$ must satisfy

$$\alpha_p = -\frac{1}{\alpha} - 1 \quad \text{and} \quad \beta_q = -\frac{1}{\beta} - 1 \quad \text{with} \quad \alpha_{pp} \neq 0, \beta_{qq} \neq 0.$$ 

The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \check{\Delta}$ for (8.16) is given by

$$\hat{\Delta} = \left\{ \hat{X}_1 = \partial_x + p\partial_u + r\partial_p + \frac{\alpha\beta}{x+y}\partial_q - A_1\partial_t, \hat{X}_2 = \partial_r \right\},$$

$$\check{\Delta} = \left\{ \check{X}_1 = \partial_y + q\partial_u + \frac{\alpha\beta}{x+y}\partial_p + t\partial_q - B_1\partial_r, \check{X}_2 = \partial_t \right\},$$

where

$$A_1 = \frac{\alpha\beta}{(x+y)^2} + \frac{(\alpha+1)\beta^2}{(x+y)^2} + \frac{t\alpha(\beta+1)}{\beta(x+y)} \quad \text{and} \quad B_1 = \frac{\alpha\beta}{(x+y)^2} + \frac{\alpha^2(\beta+1)}{(x+y)^2} + \frac{r(\alpha+1)\beta}{\alpha(x+y)}.$$ 

The first integrals for these systems are

$$\hat{I}_0 = y, \quad \hat{I}_2 = \frac{t}{\beta} - \frac{\beta}{x+y}, \quad \text{and} \quad \check{I}_0 = x, \quad \check{I}_2 = \frac{r}{\alpha} - \frac{\alpha}{x+y}.$$ 

In utilizing Theorem 4.2.2, we construct the commuting bases $\{\hat{U}_i\}$ and $\{\check{U}_j\}$ for $\hat{\Delta}$ and $\check{\Delta}$, respectively, as

$$\hat{\Delta} = \left\{ \hat{U}_1 = \hat{X}_1 - A_2\hat{X}_2, \hat{U}_2 = \alpha\partial_r \right\},$$

$$\check{\Delta} = \left\{ \check{U}_1 = \check{X}_1 - B_2\check{X}_2, \check{U}_2 = \beta\partial_t \right\}$$

where

$$A_2 = \frac{\alpha^4 + r(x+y)\alpha^3 + r(x+y)\alpha^2 + r(x+y)^2\alpha + r^2(x+y)^2}{\alpha^2(x+y)^2},$$

$$B_2 = \frac{\beta^4 + t(x+y)\beta^3 + t(x+y)\beta^2 + t^2(x+y)^2\beta + t^2(x+y)^2}{\beta^2(x+y)^2}.$$
We then compute the sequences of vector fields

\[ S_1 = [\tilde{U}_1, \tilde{U}_2], \quad S_2 = [\tilde{U}_1, \tilde{S}_1], \quad S_3 = [\tilde{U}_1, \tilde{S}_2], \]

\[ \tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2]. \]

We note that \([\tilde{U}_2, \tilde{S}_i] = 0, [\tilde{U}_2, \tilde{S}_i] = 0\) for \(1 \leq i \leq 2\), \([\tilde{U}_2, \tilde{S}_3] = 1/I_2\tilde{S}_3\), and \([\tilde{U}_2, \tilde{S}_3] = 1/I_2\tilde{S}_3\).

These vector fields form bases \(\mathfrak{H} = \{\tilde{S}_i\}_{i=1}^3\) and \(\mathfrak{F} = \{\tilde{S}_j\}_{j=1}^3\) for the 3-dimensional Vessiot algebra of (8.16). In particular, the nonzero structure equations for \(\mathfrak{H}\) are

\[ [\tilde{S}_1, \tilde{S}_2] = -\tilde{S}_3, \quad [\tilde{S}_1, \tilde{S}_3] = \tilde{S}_3, \]

and the structure equations for \(\mathfrak{F}\) are

\[ [\tilde{S}_1, \tilde{S}_2] = -\tilde{S}_3, \quad [\tilde{S}_1, \tilde{S}_3] = \tilde{S}_3. \]

As abstract Lie algebras, both of these structure equations define the Lie algebra \(n_{3,1}\) in [28]. There is only one action in Table A.1 which corresponds to \(n_{3,1}\), and so we conclude that the actions given by \(\mathfrak{H}\) and \(\mathfrak{F}\) must correspond to the prolongation of the action \(p_{3,3}\) to \(J^3(\mathbb{R}, \mathbb{R})\).

**Class (XIII)**

Finally, we consider equations of the form

\[ s + \frac{\epsilon \alpha(p)\beta(p)}{u} = 0 \]  \hspace{1cm} (8.17)

where

\[ \alpha_p = \epsilon \frac{p}{\alpha} + c, \quad \text{and} \quad \beta_q = \epsilon \frac{q}{\beta} + c, \quad c > 0. \]
The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \check{\Delta}$ for (8.17) is given by

\[
\hat{\Delta} = \left\{ \hat{X}_1 = \partial_x + p\partial_u + r\partial_p - \frac{\epsilon\alpha\beta}{u} \partial_q - A_1 \partial_t, \hat{X}_2 = \partial_r \right\},
\]

\[
\check{\Delta} = \left\{ \check{X}_1 = \partial_y + q\partial_u - \frac{\epsilon\alpha\beta}{u} \partial_p + t\partial_q - B_1 \partial_r, \check{X}_2 = \partial_t \right\},
\]

where

\[
A_1 = \frac{\epsilon t\alpha(ep + c\beta)}{u\beta} - \frac{\epsilon q\alpha\beta}{u^2} - \frac{\beta^2(ep + c\alpha)}{u^2},
\]

\[
B_1 = \frac{\epsilon r\beta(ep + c\alpha)}{u} - \frac{\epsilon p\alpha\beta}{u^2} - \frac{\alpha^2(ep + c\beta)}{u^2}.
\]

The first integrals for these systems are

\[
\hat{I}_0 = y, \quad \hat{I}_2 = \frac{t}{\beta} + \frac{\epsilon\beta}{u}, \quad \text{and} \quad \check{I}_0 = x, \quad \check{I}_2 = \frac{r}{\alpha} + \frac{\epsilon\alpha}{u}.
\]

In utilizing Theorem 4.2.2, we construct the commuting bases $\{\hat{U}_i\}$ and $\{\check{U}_j\}$ for $\hat{\Delta}$ and $\check{\Delta}$, respectively, as

\[
\hat{\Delta} = \left\{ \hat{U}_1 = \hat{X}_1 + A_2 \hat{X}_2, \hat{U}_2 = \alpha \partial_r \right\},
\]

\[
\check{\Delta} = \left\{ \check{U}_1 = \check{X}_1 + B_2 \check{X}_2, \check{U}_2 = \beta \partial_t \right\}
\]

where

\[
A_2 = \frac{\epsilon p\alpha^4 + \epsilon ru\alpha^3 + \epsilon ru\alpha^2 - \alpha^2 u^2 - r^2 u^2}{\alpha^2 u^2},
\]

\[
B_2 = \frac{\epsilon q\beta^4 + \epsilon tu\beta^3 + \epsilon tu\beta^2 - \beta t^2 u^2 - t^2 u^2}{\beta^2 u^2}.
\]

We then compute the sequences of vector fields

\[
\hat{S}_1 = [\hat{U}_1, \hat{U}_2], \quad \hat{S}_2 = [\hat{U}_1, \hat{S}_1], \quad \hat{S}_3 = [\hat{U}_1, \hat{S}_2],
\]

\[
\check{S}_1 = [\check{U}_1, \check{U}_2], \quad \check{S}_2 = [\check{U}_1, \check{S}_1], \quad \check{S}_3 = [\check{U}_1, \check{S}_2].
\]
which, as usual, constitute local bases \( \mathfrak{V} = \{ \mathfrak{S}_i \}_{i=1}^3 \) and \( \mathfrak{V} = \{ \mathfrak{S}_j \}_{j=1}^3 \) for the Vessiot algebra of (8.17).

Upon restricting to the integral manifold \( \tilde{M} \) given by \( \tilde{I}_0 = \tilde{I}_2 = 0 \), we see that the vector fields \( \mathfrak{S}_i \) restrict to vector fields \( \tilde{T}_i \) on \( \tilde{M} \) which satisfy the structure equations,

\[
[\tilde{T}_1, \tilde{T}_2] = -\tilde{T}_3, \quad [\tilde{T}_1, \tilde{T}_3] = -c\tilde{T}_2 - c\tilde{T}_3,
\]

and similarly, after restricting to the integral manifold \( \tilde{M} \) given by \( \tilde{I}_0 = \tilde{I}_2 = 0 \), we see that the vector fields \( \mathfrak{S}_i \) restrict to vector fields \( \hat{T}_i \) on \( \tilde{M} \) which satisfy the structure equations

\[
[\hat{T}_1, \hat{T}_2] = -\hat{T}_3, \quad [\hat{T}_1, \hat{T}_3] = -c\hat{T}_2 - c\hat{T}_3.
\]

From this we see that the abstract Vessiot algebra has structure equations

\[
[e_1, e_2] = -e_3, \quad [e_1, e_3] = -e_2 - ce_3, \quad [e_2, e_3] = 0.
\]

By introducing the change of basis

\[
f_1 = e_3 - \left( \frac{c}{2} - \frac{\sqrt{c^2 + 4\epsilon}}{2} \right) e_2, \\
f_2 = e_2 + \left( \frac{c - \sqrt{c^2 + 4\epsilon}}{2\epsilon} \right) e_3, \\
f_3 = \left( \frac{c^2 - c\sqrt{c^2 + 4\epsilon} + 4\epsilon}{2\epsilon\sqrt{c^2 + 4\epsilon}} \right) e_1,
\]

the structure equations become

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = \lambda e_2,
\]

where

\[
\lambda = \frac{(c - \sqrt{c^2 + 4})(c^2 - c\sqrt{c^2 + 4} + 4)}{4\sqrt{c^2 + 4}}.
\]
These are the structure equations defining the abstract Lie algebra $\mathfrak{g}_{3,1}$ in [28], and therefore we see that the actions $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{H}}$ must be prolongations of $p_{3,5}$ in Table A.1 to $J^3(\mathbb{R}, \mathbb{R})$. As in previous examples, it remains to show how the parameter $\alpha$ relates to the equation (8.17).

This concludes the classification of Darboux integrable $f$-Gordon equations at order two in terms of their fundamental invariants. We list the results of this chapter in the table below.
Table 8.3: Darboux Integrable $f$-Gordon Equations at Order Two

<table>
<thead>
<tr>
<th>Class</th>
<th>Equation</th>
<th>( \mathcal{J} )</th>
<th>( \mathcal{J} )</th>
<th>( \mathcal{M} )</th>
<th>( \mathcal{M} )</th>
<th>Vessiot Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>( s = 0 )</td>
<td>( p_{1,1} )</td>
<td>( p_{1,1} )</td>
<td>( J^2(R, R) )</td>
<td>( J^2(R, R) )</td>
<td>( n_{1,1} )</td>
</tr>
<tr>
<td>(II)</td>
<td>( s = \frac{p}{x + y} )</td>
<td>( p_{2,1} )</td>
<td>( p_{2,1} )</td>
<td>( J^2(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( 2n_{1,1} )</td>
</tr>
<tr>
<td>(III)</td>
<td>( s = qe^u )</td>
<td>( p_{2,2} )</td>
<td>( p_{2,2} )</td>
<td>( J^3(R, R) )</td>
<td>( J^2(R, R) )</td>
<td>( s_{2,1} )</td>
</tr>
<tr>
<td>(IV)</td>
<td>( s = \frac{2u}{(x + y)^2} )</td>
<td>( p_{3,7} )</td>
<td>( p_{3,7} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( 3n_{1,1} )</td>
</tr>
<tr>
<td>(V)</td>
<td>( s = ap + bq - abu )</td>
<td>( p_{3,7} )</td>
<td>( p_{3,7} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( 3n_{1,1} )</td>
</tr>
<tr>
<td>(VI)</td>
<td>( s = e^u )</td>
<td>( p_{3,9} )</td>
<td>( p_{3,9} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( sl(2) )</td>
</tr>
<tr>
<td>(VII)</td>
<td>( s = \frac{2u + x + y}{(u + x)(u + y)}pq )</td>
<td>( p_{3,8} )</td>
<td>( p_{3,8} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( s_{3,1} )</td>
</tr>
<tr>
<td>(VIII.1)</td>
<td>( s = e^u \sqrt{p^2 + 1} )</td>
<td>( p_{3,1} )</td>
<td>( p_{3,9} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( sl(2) )</td>
</tr>
<tr>
<td>(VIII.2)</td>
<td>( s = e^u \sqrt{p^2 - 1} )</td>
<td>( p_{3,1} )</td>
<td>( p_{3,9} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( sl(2) )</td>
</tr>
<tr>
<td>(VIII.3)</td>
<td>( s = e^u \sqrt{1 - p^2} )</td>
<td>( p_{3,2} )</td>
<td>( p_{3,9} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( sl(2) )</td>
</tr>
<tr>
<td>(IX)</td>
<td>( s = \frac{2\sqrt{pq}}{x + y} )</td>
<td>( p_{3,3} )</td>
<td>( p_{3,3} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( n_{3,1} )</td>
</tr>
<tr>
<td>(X.1–X.3)</td>
<td>( s = \sqrt{p^2 + 1}\sqrt{q^2 + 1} )</td>
<td>( p_{3,5} )</td>
<td>( p_{3,5} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( sl(2) )</td>
</tr>
<tr>
<td>(X.4)</td>
<td>( s = \sqrt{1 - p^2}\sqrt{1 - q^2} )</td>
<td>( p_{3,10} )</td>
<td>( p_{3,10} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( sl(2) )</td>
</tr>
<tr>
<td>(XI)</td>
<td>( s = \frac{\sqrt{\pm p^2 + 1}\sqrt{\pm q^2 + 1}}{C(u)} )</td>
<td>—</td>
<td>—</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( sl(2) )</td>
</tr>
<tr>
<td>(XI.5)</td>
<td>( s = \frac{\sqrt{1 - p^2}\sqrt{1 - q^2}}{\sin u} )</td>
<td>( p_{3,12} )</td>
<td>( p_{3,12} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( sl(3) )</td>
</tr>
<tr>
<td>(XII)</td>
<td>( s = \frac{\alpha(p)\beta(q)}{x + y} )</td>
<td>( p_{3,3} )</td>
<td>( p_{3,3} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( n_{3,1} )</td>
</tr>
<tr>
<td>(XIII)</td>
<td>( s + \frac{\alpha(p)\beta(q)}{u} = 0 )</td>
<td>( p_{3,5} )</td>
<td>( p_{3,5} )</td>
<td>( J^3(R, R) )</td>
<td>( J^3(R, R) )</td>
<td>( s_{3,1} )</td>
</tr>
</tbody>
</table>
CHAPTER 9
THE GOURSAT EQUATION AND EQUATIONS OF ZHIBER AND SOKOLOV

In this chapter, we compute the fundamental invariants of all previously known non-linear $f$-Gordon equations which are Darboux integrable at order three. The first of these equations was analyzed by Goursat [19] and is given by

$$u_{xy} = \frac{4\sqrt{u_xu_y}}{x+y}.$$  \hspace{1cm} (9.1)

Later, Zhiber and Sokolov [38] exhibited two new examples of equations of this type. The first is

$$u_{xy} = \frac{P_1(u_x)Q_1(u_y)}{u}$$ \hspace{1cm} (9.2)

where $P_1 = P_1(u_x)$ and $Q_1 = Q_1(u_y)$ are defined implicitly by

$$P_1P_1' + P_1 = 2u_x, \quad Q_1Q_1' + Q_1 = 2u_y.$$

The second example of Zhiber and Sokolov is

$$u_{xy} = \frac{P_1^2(P_1 - 1)Q_1(Q_1 - 1)^2}{6u + y} + \frac{Q_1^2(Q_1 - 1)P_1(P_1 - 1)^2}{6u + x}$$ \hspace{1cm} (9.3)

where $P_1 = P_1(u_x)$ and $Q_1 = Q_1(u_y)$ are defined implicitly by

$$\frac{1}{3}P_1^3 - \frac{1}{2}P_1^2 = u_x, \quad \frac{1}{3}Q_1^3 - \frac{1}{2}Q_1^2 = u_y.$$

We summarize the fundamental invariants for these equations in the following theorem.
Theorem 9.0.1. Each of the above equations have Vessiot distributions given by the first prolongation of the Hilbert-Cartan distribution

\[ \{ \partial_x + \phi_2^2 \partial_z + \phi_1 \partial_{\phi} + \phi_2 \partial_{\phi_1}, \partial_{\phi_2} \}. \]

The Vessiot group of each equation is 5-dimensional, and the action of the Vessiot group has infinitesimal generators given by the first prolongation of the vector fields of

[i] [N, 23] in Table B.3 for equation (9.1),

[ii] [S, 14] in Table B.3 for equation (9.2),

[iii] and \( p_{5,3} \) with \( \alpha = 3 \) in Table A.3 for equation (9.3).

Remark.

[i] In [2], Anderson and Fels give the quotient representation for equations of the form

\[ u_{xy} = \frac{2n\sqrt{u_xu_y}}{x+y}, \]

in terms of prolongations of Hilbert-Cartan distributions and the action of their Vessiot group. These equations are Darboux integrable at order \( n+1 \), and generalize (9.1).

[ii] In [38], Zhiber and Sokolov note that the general solution to (9.2) contains four arbitrary constants occurring implicitly in the equations (9.6). Upon reinterpreting these equations as rank 2 distributions, we can see that the Vessiot distributions are indeed the first prolongation of Hilbert-Cartan distributions. However, the equations (9.6) below give little insight into the characterization of the Vessiot group.

[iii] As always, these equations are completely characterized by their fundamental invariants, meaning any other equation with the same Vessiot distributions and action of the Vessiot group as one of these equations will be contact-equivalent to that equation.

[iv] Finally, we emphasize that each of the actions generated by the vector fields of [N, 23], [S, 14], and \( p_{5,7} \) with \( \alpha = 3 \) are intransitive on their underlying 5-dimensional
manifold. We will see in Chapter 10 that this is an essential property of the Vessiot group for $f$-Gordon equations which are Darboux integrable at order three.

In the following sections, we provide a detailed calculation of the fundamental invariants for each of these equations. We note that since each equation is invariant upon interchanging $x$ with $y$ (and the respective derivatives), the fundamental invariants for each equation will be identical for each characteristic system, meaning that it will suffice to compute the Vessiot distribution and Vessiot algebra for only a single characteristic system. Along with this, we also give the reconstruction of equations (9.1) and (9.2) by realizing them as the quotient of the direct sum of their Vessiot distributions by the diagonal action of its Vessiot group. We give an alternative representation of equation (9.3) as a rank 4 distribution of linear vector fields.

9.1 The Goursat Equation

We first consider the Goursat equation

$$u_{xy} = \frac{4\sqrt{u_x u_y}}{x + y}, \quad (9.4)$$

In terms of the standard coordinates $x, y, u, p_1, q_1, p_2, q_2, p_3, q_3$ where $p_i = \frac{\partial^i u}{\partial x^i}$ and $q_j = \frac{\partial^j u}{\partial y^j}$, the prolonged total vector fields, restricted to the equation manifold, are

$$D_x = \partial_x + p_1 \partial_u + p_2 \partial_{p_1} + \frac{4\sqrt{p_1 q_1}}{x + y} \partial_{q_1} + p_3 \partial_{p_2} + A_1 \partial_{q_2}$$

$$D_y = \partial_y + q_1 \partial_u + \frac{4\sqrt{p_1 q_1}}{x + y} \partial_{p_1} + q_2 \partial_{q_1} + B_1 \partial_{p_2} + q_3 \partial_{q_2}$$

where

$$A_1 = \frac{2 p_1 q_2}{(x + y) \sqrt{p_1 q_1}} + \frac{4 q_1 (2\sqrt{p_1 q_1} - p_1)}{(x + y)^2 \sqrt{p_1 q_1}} \quad \text{and} \quad B_1 = \frac{2 q_1 p_2}{(x + y) \sqrt{p_1 q_1}} + \frac{4 p_1 (2\sqrt{p_1 q_1} - q_1)}{(x + y)^2 \sqrt{p_1 q_1}}.$$
The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ is then given by

$$\hat{\Delta} = \left\{ \hat{X}_1 = D_x + D_y^2(u_{xy})\partial_{q_3}, \; \hat{X}_2 = \partial_{p_3} \right\},$$

$$\tilde{\Delta} = \left\{ \tilde{X}_1 = D_y + D_x^2(u_{xy})\partial_{p_3}, \; \tilde{X}_2 = \partial_{q_3} \right\}.$$

The intermediate integrals are given by

$$\hat{I}_0 = y, \quad \hat{I}_3 = \frac{1}{\sqrt{q_1}} \left( q_3 + \frac{4q_2}{x + y} - \frac{q_2^2}{2q_1} + \frac{4q_1}{(x + y)^2} \right),$$

and

$$\tilde{I}_0 = x, \quad \tilde{I}_3 = \frac{1}{\sqrt{p_1}} \left( p_3 + \frac{4p_2}{x + y} - \frac{p_2^2}{2p_1} + \frac{4p_1}{(x + y)^2} \right).$$

In utilizing Theorem 4.2.2, these invariants allow us to construct commuting bases $\{\hat{U}_i\}$ and $\{\tilde{U}_j\}$ for $\hat{\Delta}$ and $\tilde{\Delta}$, respectively, as

$$\hat{\Delta} = \left\{ \hat{U}_1 = \hat{X}_1 - \sqrt{p_1}\hat{X}_1(\hat{I}_3)\hat{X}_2, \; \hat{U}_2 = \sqrt{p_1}\hat{X}_2 \right\},$$

$$\tilde{\Delta} = \left\{ \tilde{U}_1 = \tilde{X}_1 - \sqrt{q_1}\tilde{X}_1(\tilde{I}_3)\tilde{X}_2, \; \tilde{U}_2 = \sqrt{q_1}\tilde{X}_2 \right\}.$$

We then compute the sequences of vector fields

$$\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2], \quad \tilde{S}_4 = [\tilde{U}_1, \tilde{S}_3], \quad \tilde{S}_5 = [\tilde{U}_1, \tilde{S}_4],$$

$$\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2], \quad \tilde{S}_4 = [\tilde{U}_1, \tilde{S}_3], \quad \tilde{S}_5 = [\tilde{U}_1, \tilde{S}_4].$$

We also note that $[\tilde{U}_2, \tilde{S}_i] = 0$ for $1 \leq i \leq 4$, and $[\tilde{U}_2, \tilde{S}_5] = -(\tilde{I}_3)^{-1}\tilde{S}_5$.

These vector fields form bases $\hat{\mathfrak{G}} = \{\hat{S}_i\}_{i=1}^5$ and $\tilde{\mathfrak{G}} = \{\tilde{S}_j\}_{j=1}^5$ for the 5-dimensional Vessiot algebra of (9.1). In particular, the nonzero structure equations for $\hat{\mathfrak{G}}$ are

$$[\hat{S}_1, \hat{S}_4] = -\hat{S}_5, \quad [\hat{S}_2, \hat{S}_3] = \hat{S}_5,$$
and the nonzero structure equations for $\mathfrak{V}$ are

$$[\tilde{S}_1, \tilde{S}_4] = -\tilde{S}_5, \quad [\tilde{S}_2, \tilde{S}_3] = \tilde{S}_5.$$ 

So we see that, as abstract Lie algebras, both $\mathfrak{V}$ and $\tilde{\mathfrak{V}}$ are equivalent to $\mathfrak{n}_{5,3}$ of [28].

Furthermore, upon restricting $\tilde{\Delta}$ to the integral manifold $\hat{M}$ given by $\hat{I}_0 = \hat{I}_3 = 0$, we obtain the Vessiot distribution for the prolonged equation,

$$\hat{\mathcal{V}}^{(1)} = \tilde{\Delta}|_{\hat{M}} = \left\{ \partial_x + p_1 \partial_u + p_2 \partial_{p_1} + \frac{4\sqrt{p_1 q_1}}{x} \partial_{q_1} + p_3 \partial_{p_2} + \hat{A}_1 \partial_{q_2}, \partial_{p_3} \right\},$$

where

$$\hat{A}_1 = \frac{2p_1q_2}{x\sqrt{p_1 q_1}} + \frac{4q_1(2\sqrt{p_1 q_1} - p_1)}{x^2\sqrt{p_1 q_1}}.$$ 

After introducing the change of variables,

$$x = X, \quad y = Y, \quad u = U, \quad p_1 = P_1^2, \quad q_1 = Q_1^2, \quad p_2 = 2P_1P_2, \quad q_2 = 2Q_1Q_2, \quad p_3 = 2(P_2^2 + P_1P_3), \quad q_3 = 2(Q_2^2 + Q_1Q_3),$$

the prolonged Vessiot distribution $\hat{\mathcal{V}}^{(1)}$ becomes

$$\hat{\mathcal{V}}^{(1)} = \left\{ \partial_X + P_1^2 \partial_U + P_2 \partial_{P_1} + \frac{2\sqrt{P_1}}{X} \partial_{Q_1} + P_3 \partial_{P_2} - \frac{2P_1 - 4Q_1}{X^2} \partial_{Q_2}, \partial_{P_3} \right\}.$$ 

This is a rank 2 distribution defined on the 7-manifold $\hat{M}$ with growth vector $[2,3,4,5,7]$. In particular, we note that the second derived

$$(\hat{\mathcal{V}}^{(1)})'' = \left\{ \partial_X + P_1^2 \partial_U + \frac{2\sqrt{P_1}}{X} \partial_{Q_1} - \frac{2P_1 - 4Q_1}{X^2} \partial_{Q_2}, \partial_{P_1}, \partial_{P_2}, \partial_{P_3} \right\}.$$
has $\partial_{P_2}, \partial_{P_3}$ as Cauchy characteristics. In calculating the reduction of this distribution by its Cauchy characteristics, we obtain the rank 2 distribution

$$\tilde{\Delta}|_{\hat{M}_5} = \left\{ \partial_X + P_2^2 \partial_U + \frac{2\sqrt{P_1}}{X} \partial_{Q_1} - \frac{2P_1 - 4Q_1}{X^2} \partial_{Q_2}, \partial_{P_1} \right\}$$

defined on the 5-manifold $\hat{M}_5$ with coordinates $X, U, P_1, Q_1, Q_2$. One can then see by direct calculation that $\tilde{\mathcal{V}}^{(1)}$ is the second prolongation of $\tilde{\Delta}|_{\hat{M}_5}$.

By further introducing the change of variables,

$$X = x, \quad U = z - \frac{4}{x^3} (3\phi^2 - 3x\phi_1 + x^2 \phi_1^2), \quad P_1 = -\phi_2 - \frac{2}{x^2} (3\phi - 2x\phi_1),$$
$$Q_1 = \frac{2}{x^2} (3\phi - x\phi_1), \quad Q_2 = -\frac{2}{x^3} (6\phi - x\phi_1),$$

we see that $\tilde{\Delta}|_{\hat{M}_5}$ becomes

$$\tilde{\Delta}|_{\hat{M}_5} = \left\{ \partial_x + \phi_2^2 \partial_z + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1} + \left( \frac{12\phi - 10x\phi_1 + 4x^2 \phi_2}{x^3} \right) \partial_{\phi_2}, -\partial_{\phi_2} \right\}$$

$$\equiv \{ \partial_x + \phi_2^2 \partial_z + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1}, \partial_{\phi_2} \}.$$

But this is precisely the Hilbert-Cartan distribution. A similar argument shows that the Vessiot distribution $\tilde{\mathcal{V}}^{(1)}$ is the second prolongation of a distribution $\tilde{\Delta}|_{\hat{M}_5}$ which is equivalent to the Hilbert-Cartan distribution as well.

**Remark.** In general, finding this final change of variables can be quite difficult. We note that we could have alternatively shown $\tilde{\Delta}|_{\hat{M}_5}$ to be equivalent to the Hilbert-Cartan distribution by either

1. directly calculating the infinitesimal symmetries and noting that they form the real split-form of the 14-dimensional exceptional Lie algebra $\mathfrak{g}_2$, or

2. computing the Cartan quartic for this distribution and showing that it is identically zero.
The second alternative will be essential in showing that the Vessiot distributions are equivalent to the Hilbert-Cartan distribution in the following section.

Finally, since $\mathfrak{V}$ and $\mathfrak{W}$ are symmetry algebras of $\tilde{\Delta}$ and $\tilde{\Delta}$, respectively, their restrictions $\mathfrak{V}|_{\tilde{\mathcal{M}}_5}$ and $\mathfrak{W}|_{\tilde{\mathcal{M}}_5}$ will be symmetry algebras of $\tilde{\Delta}|_{\tilde{\mathcal{M}}_5}$ and $\tilde{\Delta}|_{\tilde{\mathcal{M}}_5}$. Moreover, since both $\tilde{\Delta}|_{\tilde{\mathcal{M}}_5}$ and $\tilde{\Delta}|_{\tilde{\mathcal{M}}_5}$ were identified to be Hilbert-Cartan distributions, we can realize both $\mathfrak{V}|_{\tilde{\mathcal{M}}_5}$ and $\mathfrak{W}|_{\tilde{\mathcal{M}}_5}$ as 5-dimensional subalgebras of $g_2$. In particular, there is only one 5-dimensional subalgebra of $g_2$ whose structure equations are equivalent to $n_{5,3}$, namely $[N, 23]$ of Table B.3. This proves part [i] of Theorem 9.0.1.

Before we leave this example, we wish to illustrate the second alternative method for showing $\tilde{\Delta}$ is the first prolongation of the Hilbert-Cartan distribution. First, we note that the vector fields

$$\hat{\mathcal{F}} = \{ \hat{S}_4, -\frac{1}{I_3} \hat{S}_5, -\hat{S}_3, \hat{S}_2, -\hat{S}_1, -\hat{U}_1, -\hat{U}_2, \hat{U}_1, \hat{U}_2 \}$$

form a frame on the 9-dimensional prolonged equation manifold whose dual coframe

$$\hat{\mathcal{F}}^* = \{ \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \pi^1, \pi^2, \chi^1, \chi^2 \}$$

has structure equations

$$d\theta^1 \equiv \theta^3 \land \pi^1 + \frac{(X + Y)^3}{6} \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2 \},$$
$$d\theta^2 \equiv \theta^3 \land \theta^4 + \frac{(X + Y)^3(P_1 + (X + Y)P_2)}{6} \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2 \},$$
$$d\theta^3 \equiv \theta^4 \land \pi^1 + \frac{(X + Y)^3}{6} \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2, \theta^3 \},$$
$$d\theta^4 \equiv \theta^5 \land \pi^1 + (X + Y) \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2, \theta^3, \theta^4 \},$$
$$d\theta^5 = \pi^1 \land \pi^2 + \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2, \theta^3, \theta^4, \theta^5 \}.$$
We note that $\chi^1 = dI_0$ and $\chi^2 = dI_3$. Restricting to the integral manifold where $\chi^1 = \chi^2 = 0$ further reduces these structure equations to

\begin{align*}
\theta_1^1 & \equiv \theta_3 \wedge \pi^1 \mod \{\theta^1, \theta^2\}, \\
\theta_2^2 & \equiv \theta_3 \wedge \theta^4 \mod \{\theta^1, \theta^2\}, \\
\theta_3^3 & \equiv \theta_4 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3\}, \\
\theta_4^4 & \equiv \theta_5 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3, \theta^4\}, \\
\theta_5^5 & = \pi^1 \wedge \pi^2 \mod \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\},
\end{align*}

which are precisely the congruences for the second prolongation of a rank 3 Monge system defined on a 5-manifold.

We can then further restrict the $\hat{\Delta}$ to the 5-manifold $\hat{M}_5$ with coordinates $X, U, P_1, Q_1,$ and $Q_2$, giving

$$\hat{\Delta}|_{\hat{M}_5} = \left\{ \partial_X + P_2^2 \partial_U + \frac{2\sqrt{P_1}}{X} \partial_{Q_1} - \frac{2P_1 - 4Q_1}{X^2} \partial_{Q_2}, \partial_{P_1} \right\}.$$ 

At this point, instead of introducing the change of variables taking the equation to the Hilbert-Cartan distribution, we can calculate the Cartan quartic using the FiveVariables package in Maple. In doing so, we see that the Cartan quartic vanishes and therefore $\hat{\Delta}|_{\hat{M}_5}$ must correspond to the Hilbert-Cartan distribution.

We now give the reconstruction of (9.1) by realizing it as a quotient of the direct sum of its Vessiot distributions by the diagonal action of its Vessiot group.

In terms of the coordinates $x, z, \phi, \phi_1, \phi_2, \phi_3$ and $y, w, \psi, \psi_1, \psi_2, \psi_3$, Vessiot distributions, defined on 6-dimensional manifolds $\hat{M}_6$ and $\tilde{M}_6$, are

\begin{align*}
\hat{\mathcal{V}} &= \{ \partial_x + \phi_2^2 \partial_z + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1} + \phi_3 \partial_{\phi_2}, \partial_{\phi_3} \}, \\
\tilde{\mathcal{V}} &= \{ \partial_y + \psi_2^2 \partial_w + \psi_1 \partial_\psi + \psi_2 \partial_{\psi_1} + \psi_3 \partial_{\psi_2}, \partial_{\psi_3} \},
\end{align*}
and the diagonal action, $G_{\text{diag}}$, of the Vessiot group is generated by the prolongation of the vector fields (see Appendix B)

\begin{align*}
Z_1 &= \tilde{X}_{10} - \tilde{X}_{10}, & Z_2 &= -\tilde{X}_{14} - \tilde{X}_{14}, & Z_3 &= -\tilde{X}_{11} + \tilde{X}_{11}, \\
Z_4 &= \tilde{X}_3 + \tilde{X}_3, & Z_5 &= -\tilde{X}_7 + \tilde{X}_7,
\end{align*}

where $\tilde{X}_i$ and $\tilde{X}_i$ are vector fields defined over $\tilde{M}_6$ or $\tilde{M}_6$, respectively.

The quotient of $\tilde{V} \oplus \tilde{V}$ by $G_{\text{diag}}$ is a rank 4 distribution $\Delta$ on the 7-manifold $M_7$ with coordinates $x, y, u, p_1, q_1, p_2, q_2$. The explicit formula for the quotient map $q : \tilde{M}_6 \times \tilde{M}_6 \to M_7$ is given by

\begin{align*}
x &= x, & y &= y, & u &= U, & p &= D_x(U), & q &= D_y(U), & r &= D_x^2(U), & t &= D_y^2(U),
\end{align*}

where $U$ is the lowest order joint differential invariant of $G_{\text{diag}},$

\begin{align*}
U &= -z - w + \frac{4(\phi_1^2 + \psi_1^2)}{x + y} + \frac{4\phi_1\psi_1}{x + y} - \frac{12(\phi + \psi)(\phi_1 + \psi_1)}{(x + y)^2} + \frac{12(\phi + \psi)^2}{(x + y)^3},
\end{align*}

and $D_x$ and $D_y$ are the total differential operators,

\begin{align*}
D_x &= \partial_x + \phi_2^2 \partial_z + \phi_1 \partial_\phi + \phi_2 \partial_\phi_1 + \phi_3 \partial_\phi_2, \\
D_y &= \partial_y + \psi_2^2 \partial_w + \psi_1 \partial_\psi + \psi_2 \partial_\psi_1 + \psi_3 \partial_\psi_2.
\end{align*}

Calculating the pushforward by $q$ gives

\begin{align*}
\Delta &= q_*(\tilde{V} \oplus \tilde{V}) \\
&= \left\{ \partial_x + p_1 \partial_u + p_2 \partial_{p_1} + \frac{4\sqrt{p_1 q_1}}{x + y} \partial_{q_1}, \partial_{p_2}, \partial_y + q_1 \partial_u + \frac{4\sqrt{p_1 q_1}}{x + y} \partial_{p_1} + q_2 \partial_{q_1}, \partial_{q_2} \right\}
\end{align*}

which is the rank 4 distribution corresponding to (9.1).
We can give an alternative representation of this distribution by instead taking the quotient map \( \tilde{q} : \tilde{M}_6 \times \tilde{M}_6 \to N_7 \) to be

\[
\begin{align*}
  z_1 &= x, \quad z_2 = y, \quad z_3 = U, \\
  z_4 &= \phi_2 - \frac{2(2\phi_1 + \psi_1)}{x + y} + \frac{6(\phi + \psi)}{(x + y)^2}, \\
  z_5 &= \psi_2 - \frac{2(\phi_1 + 2\psi_1)}{x + y} + \frac{6(\phi + \psi)}{(x + y)^2}, \\
  z_6 &= \phi_3 - \frac{6(\phi_1 + \psi_1)}{(x + y)^2} + \frac{12(\phi + \psi)}{(x + y)^3}, \\
  z_7 &= \psi_3 - \frac{6(\phi_1 + \psi_1)}{(x + y)^2} + \frac{12(\phi + \psi)}{(x + y)^3},
\end{align*}
\]

where the \( z_i \) are coordinates on \( N_7 \). Calculating the pushforward of \( \hat{V} \oplus \tilde{V} \) by \( \tilde{q} \) gives the hyperbolic distribution \( \nabla = \hat{V} \oplus \tilde{V} \) where \( \hat{V} = \{ \hat{Y}_1, \hat{Y}_2 \} \) and \( \tilde{V} = \{ \tilde{Y}_1, \tilde{Y}_2 \} \) are given by

\[
\begin{align*}
  \hat{Y}_1 &= \delta^2 \partial z_1 - \delta^2 z_4^2 \partial z_3 + \delta(\delta z_6 - 4z_4)\partial z_4 - 2\delta z_4 \partial z_5 - 6z_4 \partial z_7, \\
  \hat{Y}_2 &= \partial z_6, \\
  \tilde{Y}_1 &= \delta^2 \partial z_2 - \delta^2 z_5^2 \partial z_3 - 2\delta z_5 \partial z_4 + \delta(\delta z_7 - 4z_5)\partial z_5 - 6z_5 \partial z_6, \\
  \tilde{Y}_2 &= \partial z_7,
\end{align*}
\]

where \( \delta = z_1 + z_2 \).

The distribution \( \nabla \) satisfies all the hypotheses of Theorem 2.3.1 and therefore defines a PDE in the plane. Moreover, the characteristic distributions \( \hat{V} \) and \( \tilde{V} \) each admit a single first integral, \( z_2 \) and \( z_1 \), respectively while the first-derived of the \( \nabla \) has Cauchy characteristics \( A(\nabla') = \{ \partial z_6, \partial z_7 \} \). Since these invariants are annihilated by the Cauchy characteristics, they will survive on the 5-dimensional manifold \( \tilde{N} = N/A(\nabla') \), and we conclude that \( \nabla \) defines an f-Gordon equation.

Finally, we can see that \( \nabla \) is Darboux integrable at order three (or after one prolongation) by computing \( \nabla^{(1)} = \hat{V}^{(1)} \oplus \tilde{V}^{(1)} \) where

\[
\begin{align*}
  \hat{V}^{(1)} &= \{ \hat{Y}_1 = \hat{Y}_1 + (\delta^2 z_8 - 6z_4)\partial z_6, \hat{Y}_2 = \partial z_8 \}, \\
  \tilde{V}^{(1)} &= \{ \tilde{Y}_1 = \tilde{Y}_1 + (\delta^2 z_9 - 6z_4)\partial z_6, \tilde{Y}_2 = \partial z_9 \}.
\end{align*}
\]
These distributions have first integrals $\mathcal{J}_1 = z_2, \mathcal{J}_3 = z_9$ and $\mathcal{J}_1 = z_1, \mathcal{J}_3 = z_8$, respectively.

9.2 The First Equation of Zhiber and Sokolov

The first example of an $f$-Gordon equation which is Darboux integrable at order three presented by Zhiber and Sokolov [38] is given by

$$u_{xy} = \frac{\alpha(u_x)\beta(u_y)}{u}$$

(9.5)

where $\alpha = \alpha(u_x)$ and $\beta = \beta(u_y)$ are defined implicitly by

$$\alpha\alpha' + \alpha = 2u_x, \quad \beta\beta' + \beta = 2u_y.$$

Note that throughout our calculations, we will regularly substitute these relations when convenient.

As in the previous section, the prolonged total vector fields when restricted to the equation manifold are

$$D_x = \partial_x + p_1\partial_u + p_2\partial_{p_1} + \frac{\alpha\beta}{u}\partial_{q_1} + p_3\partial_{p_2} + A_1\partial_{q_2},$$

$$D_y = \partial_y + q_1\partial_u + \frac{\alpha\beta}{u}\partial_{p_1} + q_2\partial_{q_1} + B_1\partial_{p_2} + q_3\partial_{q_2}$$

where

$$A_1 = \frac{2\beta^2p_1}{u^2} - \frac{\alpha(\beta^2 - 2uq_2)q_1}{u^2\beta} - \frac{\alpha(\beta^2 + uq_2)}{u^2},$$

$$B_1 = \frac{2\alpha^2q_1}{u^2} - \frac{\beta(\alpha^2 - 2up_2)p_1}{u^2\alpha} - \frac{\beta(\alpha^2 + up_2)}{u^2}.$$
The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \check{\Delta}$ is then given by

$$\hat{\Delta} = \left\{ \hat{X}_1 = D_x + D_y^2(u_{xy})\partial_{q_3}, \hat{X}_2 = \partial_{p_3} \right\},$$

$$\check{\Delta} = \left\{ \check{X}_1 = D_y + D_x^2(u_{xy})\partial_{p_3}, \check{X}_2 = \partial_{q_3} \right\}.$$

The intermediate integrals are

$$\hat{I}_0 = y, \quad \hat{I}_3 = \frac{q_3}{\beta} + \frac{2(\beta - q_1)q_2^2}{\beta^3} + \frac{2(\beta + 2q_1)}{u\beta} + \frac{\beta(\beta + q_1)}{u^2},$$

and

$$\check{I}_0 = x, \quad \check{I}_3 = \frac{p_3}{\alpha} + \frac{2(\alpha - p_1)p_2^2}{\alpha^3} + \frac{2(\alpha + 2p_1)p_2}{u\alpha} + \frac{\alpha(\alpha + p_1)}{u^2}.$$

In utilizing Theorem 4.2.2, these invariants allow us to construct commuting bases $\{\hat{U}_i\}$ and $\{\check{U}_j\}$ for $\hat{\Delta}$ and $\check{\Delta}$, respectively, as

$$\hat{\Delta} = \left\{ \hat{U}_1 = \hat{X}_1 + \hat{X}_1(\check{I}_3)\partial_{p_3}, \hat{U}_2 = \alpha \hat{X}_2 \right\},$$

$$\check{\Delta} = \left\{ \check{U}_1 = \check{X}_1 + \check{X}_1(\check{I}_3)\partial_{q_3}, \check{U}_2 = \beta \check{X}_2 \right\}.$$

We then compute the sequences of vector fields

$$\hat{S}_1 = [\hat{U}_1, \hat{U}_2], \quad \hat{S}_2 = [\hat{U}_1, \hat{S}_1], \quad \hat{S}_3 = [\hat{U}_1, \hat{S}_2], \quad \hat{S}_4 = [\hat{U}_1, \hat{S}_3], \quad \hat{S}_5 = [\hat{U}_1, \hat{S}_4],$$

$$\check{S}_1 = [\check{U}_1, \check{U}_2], \quad \check{S}_2 = [\check{U}_1, \check{S}_1], \quad \check{S}_3 = [\check{U}_1, \check{S}_2], \quad \check{S}_4 = [\check{U}_1, \check{S}_3], \quad \check{S}_5 = [\check{U}_1, \check{S}_4].$$

Additionally, $[\hat{U}_2, \hat{S}_1] = [\hat{U}_2, \hat{S}_2] = 0,$ and

$$[\hat{U}_2, \hat{S}_3] = 2\hat{S}_1, \quad [\hat{U}_2, \hat{S}_4] = 4\hat{S}_4, \quad [\hat{U}_2, \hat{S}_5] = 8\check{I}_3 \hat{S}_1 + (\check{I}_3)^{-1}\hat{S}_5.$$
These vector fields form bases $\mathcal{H} = \{\hat{S}_i\}_{i=1}^5$ and $\mathcal{H} = \{\bar{S}_i\}_{i=1}^5$ for the 5-dimensional Vessiot algebra of (9.2). In particular, the structure equations for $\mathcal{H}$ are

\[
[\hat{S}_1, \hat{S}_2] = -2\hat{S}_1, \quad [\hat{S}_1, \hat{S}_3] = -2\hat{S}_2, \quad [\hat{S}_1, \hat{S}_4] = -8\bar{I}_3\hat{S}_1 + 4\bar{S}_3 - \frac{1}{I_3}\bar{S}_5, \quad [\hat{S}_1, \hat{S}_5] = -8\bar{I}_3\hat{S}_2,
\]

\[
[\hat{S}_2, \hat{S}_3] = 8\bar{I}_3\hat{S}_1 - 6\bar{S}_3 + \frac{1}{I_3}\bar{S}_5, \quad [\hat{S}_2, \hat{S}_4] = 4\bar{I}_3\hat{S}_2 - \bar{S}_4, \quad [\hat{S}_2, \hat{S}_5] = 32\bar{I}_3^2\hat{S}_1 - 28\bar{I}_3\bar{S}_3 + 5\bar{S}_5,
\]

\[
[\hat{S}_3, \hat{S}_4] = -32\bar{I}_3^2\hat{S}_1 + 32\bar{I}_3\bar{S}_3 - 6\bar{S}_5, \quad [\hat{S}_3, \hat{S}_5] = -8\bar{I}_3^2\hat{S}_2 + 2\bar{I}_3\bar{S}_4,
\]

\[
[\hat{S}_4, \hat{S}_5] = 128\bar{I}_3^3\hat{S}_1 - 144\bar{I}_3^2\bar{S}_3 + 28\bar{I}_3\bar{S}_5.
\]

This Lie algebra has nontrivial Levi decomposition $\mathcal{H} = \mathfrak{sl}(2) \oplus 2\mathfrak{n}_{1,1}$.

The vector fields $\mathcal{F} = \{\hat{S}_i, \hat{U}_1, \hat{U}_2, \bar{U}_1, \bar{U}_2\}_{i=1}^5$ form a frame on the 9-dimensional equation manifold, and after applying the change of frame

\[
\mathcal{F} = \left\{\hat{S}_4, -\frac{1}{I_3}\hat{S}_5, -\bar{S}_3, \bar{S}_2, -\hat{S}_1, -\hat{U}_1, -\hat{U}_2, \bar{U}_1, \bar{U}_2 \right\},
\]

we see that the dual coframe $\mathcal{F}^* = \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \pi^1, \pi^2, \chi^1, \chi^2\}$ has structure equations

\[
d\theta^1 \equiv \theta^3 \wedge \pi^1 + \frac{u^3}{3(\alpha - p_1)(\beta - q_1)(\alpha + 2p_1)} \chi^1 \wedge \chi^2 \mod \{\theta^1, \theta^2\},
\]

\[
d\theta^2 \equiv \theta^3 \wedge \theta^4 + \frac{(\alpha^2 + 2up_2)u^2}{6(\alpha - p_1)(\beta - q_1)(\alpha + 2p_1)} \chi^1 \wedge \chi^2 \mod \{\theta^1, \theta^2\},
\]

\[
d\theta^3 \equiv \theta^4 \wedge \pi^1 - \frac{(\alpha^2 - 6\alpha p_1 + 8p_2u)u^2}{6(\alpha - p_1)(\beta - q_1)(\alpha + 2p_1)} \chi^1 \wedge \chi^2 \mod \{\theta^1, \theta^2, \theta^3\},
\]

\[
d\theta^4 \equiv \theta^5 \wedge \pi^1 + T_4 \chi^1 \wedge \chi^2 \mod \{\theta^1, \theta^2, \theta^3, \theta^4\},
\]

\[
d\theta^5 \equiv \pi^1 \wedge \pi^2 + T_5 \chi^1 \wedge \chi^2 \mod \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\}
\]

where $\chi^1 = d\hat{I}_0$, $\chi^2 = d\hat{I}_3$,

\[
T_4 = -\frac{(\alpha^5 - 5p_1\alpha^4 + (5up_2 - 6p_2^2)\alpha^3 + 2u(5p_1p_2 + 2up_3)\alpha^2 + 8u^2p_2^2\alpha - 8u^2p_1p_2^2)u}{3\alpha^3(\alpha - p_1)(\beta - q_1)(\alpha + 2p_1)}.
\]
and
\[ T_5 = \frac{p_1 \alpha^4 + p_1^2 \alpha^3 - u(2p_1p_2 + up_3)\alpha^2 - u^2 p_2^2 \alpha + 2u^2 p_1p_2^2}{\alpha^3(\alpha - p_1)(\beta - q_1)}. \]

However, upon restricting to the integral manifold given by \( \chi^1 = \chi^2 = 0 \), we see that the structure equations become

\[
\begin{align*}
  d\theta^1 &\equiv \theta^3 \wedge \pi^1 \mod \{\theta^1, \theta^2\}, \\
  d\theta^2 &\equiv \theta^3 \wedge \theta^4 \mod \{\theta^1, \theta^2\}, \\
  d\theta^3 &\equiv \theta^4 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3\}, \\
  d\theta^4 &\equiv \theta^5 \wedge \pi^1 \mod \{\theta^1, \theta^2, \theta^3, \theta^4\}, \\
  d\theta^5 &\equiv \pi^1 \wedge \pi^2 \mod \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\}
\end{align*}
\]

which are precisely the congruences for the second prolongation of a rank 3 Monge system on a 5-manifold.

The restriction of \( \hat{\Delta} \) to the 5-manifold \( \hat{M}_5 \) with coordinates \( x, u, p_1, q_1, q_2 \) is

\[ \hat{\Delta}|_{\hat{M}_5} = \left\{ \partial_x + p_1 \partial_u + p_2 \partial_{p_1} + \frac{\alpha \beta}{u} \partial_{q_1} + A_1 \partial_{q_2}, \partial_{p_1} \right\}. \]

Using the \texttt{FiveVariables} package in Maple, we see that the Cartan quartic for \( \hat{\Delta}|_{\hat{M}_5} \) vanishes and conclude that \( \hat{\Delta}|_{\hat{M}_5} \) must be locally equivalent to the Hilbert-Cartan distribution

\[ \{ \partial_x + \phi_2 \partial_2 + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1}, \partial_{\phi_2} \}. \]

As before, this allows us to realize \( \hat{\mathfrak{G}}|_{\hat{M}_5} \) as a 5-dimensional subalgebra of \( \mathfrak{g}_2 \), and since there is only one 5-dimensional subalgebra of \( \mathfrak{g}_2 \) whose structure equations are equivalent to \( \mathfrak{sl}(2) \oplus 2\mathfrak{n}_{1,1} \), we conclude the Vessiot algebra must be \([S, 14]\) from Table B.3. This gives the following characterization of (9.2). This proves part [ii] of Theorem 9.0.1.
We now give the reconstruction of (9.2) by realizing it as a quotient of the direct sum of its Vessiot distributions by the diagonal action of its Vessiot group.

In [38], Zhiber and Sokolov note that the general solution to (9.2) contains four arbitrary constants occurring implicitly in the pairs of equations

\[
\begin{align*}
C_1'(x) &= \frac{z}{\sqrt{z'}} , \\
C_2'(x) &= \frac{1}{\sqrt{z'}}
\end{align*}
\] and

\[
\begin{align*}
\bar{C}_1'(y) &= \frac{w}{\sqrt{w'}} , \\
\bar{C}_2'(y) &= \frac{1}{\sqrt{w'}}
\end{align*}
\] . (9.6)

The vector field systems for each pair are

\[
\begin{align*}
\left\{ \partial_x + \frac{z}{\sqrt{z'}} \partial_{C_1} + \frac{1}{\sqrt{z'}} \partial_{C_2} + z' \partial_z, \partial_z' \right\} 
\quad \text{and} \quad
\left\{ \partial_y + \frac{w}{\sqrt{w'}} \partial_{\bar{C}_1} + \frac{1}{\sqrt{w'}} \partial_{\bar{C}_2} + w' \partial_w, \partial_w' \right\}
\end{align*}
\]

Each vector field system is a rank 2 distribution defined on a 5-manifold with growth vector (2,3,5). It can further be checked that each distribution has a 14-dimensional symmetry algebra and each has vanishing Cartan tensor. From this, we conclude that these distributions are equivalent to Hilbert-Cartan distributions. We can therefore use these distributions, in these coordinates, when we construct the quotient.

In terms of the coordinates \(x, z, z_1, z_2, C_1, C_2\) and \(y, w, w_1, w_2, \bar{C}_1, \bar{C}_2\), the Vessiot distributions are

\[
\begin{align*}
\breve{\mathcal{V}} &= \left\{ \partial_x + \frac{z}{\sqrt{z_1}} \partial_{C_1} + \frac{1}{\sqrt{z_1}} \partial_{C_2} + z_1 \partial_z + z_2 \partial_{z_1}, \partial_{z_2} \right\} , \\
\breve{\mathcal{V}} &= \left\{ \partial_y + \frac{w}{\sqrt{w_1}} \partial_{\bar{C}_1} + \frac{1}{\sqrt{w_1}} \partial_{\bar{C}_2} + w_1 \partial_w + w_2 \partial_{w_1}, \partial_{w_2} \right\} .
\end{align*}
\]
The diagonal action $G_{\text{diag}}$ of the Vessiot group is generated by the vector fields,

$$Z_1 = \partial C_2 - \partial C_1, \quad Z_2 = 2\partial C_1 + 2\partial C_1,$$

$$Z_3 = \frac{C_1}{2} \partial C_2 - \frac{z^2}{2} \partial z - z z_1 \partial z_1 - \left(z z_2 + z_1^2\right) \partial z_2$$

$$- \frac{\bar{C}_1}{2} \partial \bar{C}_2 + \frac{w^2}{2} \partial w + w w_1 \partial w_1 + \left(w w_2 + w^2\right) \partial w_2,$$

$$Z_4 = C_1 \partial C_1 - C_2 \partial C_2 + 2z \partial z + 2z_1 \partial z_1 + 2z_2 \partial z_2$$

$$+ \bar{C}_1 \partial C_1 - \bar{C}_2 \partial C_2 + 2w \partial w + 2w_1 \partial w_1 + 2w_2 \partial w_2,$$

$$Z_5 = 2C_2 \partial C_1 + 2\partial z - 2\bar{C}_2 \partial \bar{C}_1 - 2\partial w.$$

The quotient of $\hat{V} \oplus \tilde{V}$ by $G_{\text{diag}}$ is a rank 4 distribution $\Delta$ on the 7-manifold $M_7$ with coordinates $x, y, u, p_1, q_1, p_2, q_2$. The explicit formula for the quotient map $q : \hat{M}_6 \times \tilde{M}_6 \to M_7$ is given by

$$x = x, \quad y = y, \quad u = U, \quad p = D_x(U), \quad q = D_y(U), \quad r = D_x^2(U), \quad t = D_y^2(U),$$

where $U$ is the lowest order joint differential invariant of $G_{\text{diag}},$

$$U = \left(\frac{\bar{C}_1 - C_1 + z(C_2 + \bar{C}_2)(\bar{C}_1 - C_1 - w(C_2 + \bar{C}_2))}{z + w}\right)^{1/3},$$

and $D_x$ and $D_y$ are the total differential operators,

$$D_x = \partial_x + \frac{z}{\sqrt{z_1}} \partial C_1 + \frac{1}{\sqrt{z_1}} \partial C_2 + z_1 \partial z_1 + z_2 \partial z_2,$$

$$D_y = \partial_y + \frac{w}{\sqrt{w_1}} \partial C_1 + \frac{1}{\sqrt{w_1}} \partial C_2 + w_1 \partial w_1 + w_2 \partial w_1.$$

Calculating the pushforward of $\hat{V} \oplus \tilde{V}$ by $q$ gives the rank 4 distribution corresponding to the equation

$$u_{xy} = \frac{2^{4/3}}{u} \left(\frac{V - 2^{2/3}V^{4/3}p + 2^{4/3}V^{2/3}p^2}{V^{2/3}(2^{1/2}V^{2/3} - 2p)}\right) \left(\frac{W + 2^{2/3}W^{4/3}q - 2^{4/3}W^{2/3}q^2}{W^{2/3}(2q - 2^{1/2}W^{2/3})}\right), \quad (9.7)$$
where

\[ V = \sqrt{4p^3 + 1} + 1 \quad \text{and} \quad W = \sqrt{4q^3 + 1} - 1. \]

Finally, by setting

\[ \alpha(p) = \frac{2^{2/3}(V - 2^{2/3}V^{4/3}p + 2^{4/3}V^{2/3}p^2)}{V^{2/3}(2^{1/2}V^{2/3} - 2p)}, \quad \beta(q) = \frac{2^{2/3}(\sqrt{4}q^{4/3}q - 2^{4/3}W^{2/3}q^2)}{W^{2/3}(2q - 2^{1/2}W^{2/3})}, \]

we see that (9.7) becomes (9.2), and a direct computation shows that \( \alpha(p) \) and \( \beta(q) \) satisfy

\[(p - \alpha)(\alpha + 2p)^2 + 1 = 0 \quad \text{and} \quad (q - \beta)(\beta + 2q)^2 + 1 = 0,
\]

and therefore satisfy the differential relations \( \alpha' + \alpha = 2p \) and \( \beta' + \beta = 2q \).

### 9.3 The Second Equation of Zhiber and Sokolov

The second example of an \( f \)-Gordon equation which is Darboux integrable at order three presented by Zhiber and Sokolov [38] is given by

\[ u_{xy} = \frac{P_1^2(P_1 - 1)Q_1(Q_1 - 1)^2}{6u + y} + \frac{Q_1^2(Q_1 - 1)P_1(P_1 - 1)}{6u + x} \quad (9.8) \]

where \( P_1 = P_1(u_x) \) and \( Q_1 = Q_1(u_y) \) are defined implicitly by

\[ \frac{1}{3}P_1^3 - \frac{1}{2}P_1^2 = u_x, \quad \frac{1}{3}Q_1^3 - \frac{1}{2}Q_1^2 = u_y. \]

Due to the implicit definition of \( P_1 \) and \( Q_1 \), it will be beneficial to write the hyperbolic distribution associated to (9.3) in terms of \( P_1 \) and \( Q_1 \) rather than the standard coordinates \( u_x \) and \( u_y \).

In terms of the standard coordinates, the total vector fields on the 2-jet are given by

\[ D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xy} \partial_{u_y} \quad \text{and} \quad D_y = \partial_y + u_y \partial_u + u_{xy} \partial_{u_x} + u_{yy} \partial_{u_y}. \]
Let \( P_2 = D_x(P_1) \) and \( Q_2 = D_y(Q_1) \). Then in terms of the adapted coordinates \( P_1, Q_1, P_2, Q_2 \), the total vector fields are

\[
D_x = \partial_x + \left( \frac{1}{3} P_1^3 - \frac{1}{2} P_1^2 \right) \partial_u + P_2 \partial_{P_1} + A_1 \partial_{Q_1},
\]

\[
D_y = \partial_y + \left( \frac{1}{3} Q_1^3 - \frac{1}{2} Q_2^2 \right) \partial_u + B_1 \partial_{P_1} + Q_2 \partial_{Q_1},
\]

where \( A_1 \) and \( B_1 \) are functions of \( x, y, u, P_1, Q_1, P_2 \) and \( Q_2 \). To explicitly determine \( A_1 \) and \( B_1 \), we calculate

\[
\begin{align*}
\frac{u_{xy}}{Q_1^2 - Q_1} &= \frac{D_x(u_y)}{D_x(u_y)} = D_x \left( \frac{1}{3} Q_1^3 - \frac{1}{2} Q_1^2 \right) = (Q_1^2 - Q_1) A_1, \\
\frac{u_{xy}}{P_1^2 - P_1} &= \frac{D_y(u_x)}{D_y(u_x)} = D_x \left( \frac{1}{3} P_1^3 - \frac{1}{2} P_1^2 \right) = (P_1^2 - P_1) B_1.
\end{align*}
\]

Solving and substituting (9.3) gives

\[
A_1 = \frac{u_{xy}}{Q_1^2 - Q_1} = \left( P_1^2 - P_1 \right) \left[ P_1 Q_1 \left( \frac{1}{6u + x} + \frac{1}{6u + y} \right) - \frac{P_1}{6u + y} - \frac{Q_1}{6u + x} \right],
\]

\[
B_1 = \frac{u_{xy}}{P_1^2 - P_1} = \left( Q_1^2 - Q_1 \right) \left[ P_1 Q_1 \left( \frac{1}{6u + x} + \frac{1}{6u + y} \right) - \frac{P_1}{6u + y} - \frac{Q_1}{6u + x} \right].
\]

Now, let \( D_x^{(3)} \) and \( D_y^{(3)} \) denote the total vector fields on the 3-jet,

\[
D_x^{(3)} = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xy} \partial_{u_y} + u_{xxx} \partial_{u_{xx}} + u_{xyy} \partial_{u_{yy}},
\]

\[
D_y^{(3)} = \partial_y + u_y \partial_u + u_{xy} \partial_{u_x} + u_{yy} \partial_{u_y} + u_{xxy} \partial_{u_{xx}} + u_{yyy} \partial_{u_{yy}},
\]

\[
D_x^{(3)} = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xy} \partial_{u_y} + u_{xxx} \partial_{u_{xx}} + u_{xyy} \partial_{u_{yy}},
\]

\[
D_y^{(3)} = \partial_y + u_y \partial_u + u_{xy} \partial_{u_x} + u_{yy} \partial_{u_y} + u_{xxy} \partial_{u_{xx}} + u_{yyy} \partial_{u_{yy}},
\]
and append $P_2 = D_x^{(3)}(P_1)$ and $Q_2 = D_y^{(3)}(Q_1)$ to our list of adapted coordinates. In terms of the adapted coordinates, $D_x^{(3)}$ and $D_y^{(3)}$ become

$$D_x^{(3)} = \partial_x + \left( \frac{1}{3} P_1^3 - \frac{1}{2} P_1^2 \right) \partial_u + P_2 \partial_{P_1} + A_1 \partial_{Q_1} + P_3 \partial_{P_2} + A_2 \partial_{Q_2},$$

$$D_y^{(3)} = \partial_y + \left( \frac{1}{3} Q_1^3 - \frac{1}{2} Q_2^2 \right) \partial_u + B_1 \partial_{P_1} + Q_2 \partial_{Q_1} + B_2 \partial_{P_2} + Q_3 \partial_{Q_2}$$

where $A_2$ and $B_2$ are functions of $x, y, u, P_1, Q_1, P_2, Q_2, P_3,$ and $Q_3$. To determine $A_2$ and $B_2$, we note that since

$$u_{yy} = D_y(u_y) = D_y \left( \frac{1}{3} Q_1^3 - \frac{1}{2} Q_2^2 \right) = (Q_1^2 - Q_1) \ D_y(Q_1) = (Q_1^2 - Q_1) \ Q_2,$$

we have

$$u_{xyy} = D_x^{(3)}(u_{yy}) = D_x^{(3)}((Q_1^2 - Q_1)Q_2) = (Q_1^2 - Q_1) D_x^{(3)}(Q_2) + D_x^{(3)}(Q_1^2 - Q_1) Q_2$$

$$= (Q_1^2 - Q_1) A_2 + D_x^{(3)}(Q_1^2 - Q_1) Q_2 = (Q_1^2 - Q_1) A_2 + (2Q_1 - 1) A_1 Q_2.$$ But then we see, by solving for $A_2$,

$$A_2 = \frac{u_{xyy}}{Q_1^2 - Q_1} - \frac{(2Q_1 - 1) A_1 Q_2}{Q_1^2 - Q_1} = \frac{u_{xyy}}{Q_1^2 - Q_1} - \frac{(2Q_1 - 1) Q_2}{(Q_1^2 - Q_1)^2} u_{xy} = D_y^{(3)}(A_1).$$

Similarly, since

$$u_{xx} = D_x(u_x) = D_x \left( \frac{1}{3} P_1^3 - \frac{1}{2} P_1^2 \right) = (P_1^2 - P_1) D_x(P_1) = (P_1^2 - P_1) P_2,$$

we have

$$u_{xxy} = D_y^{(3)}(u_{xx}) = D_y^{(3)}((P_1^2 - P_1)P_2) = (P_1^2 - P_1) D_y^{(3)}(P_2) + D_y^{(3)}(P_1^2 - P_1) P_2$$

$$= (P_1^2 - P_1) B_2 + D_y^{(3)}(P_1^2 - P_1) P_2 = (P_1^2 - P_1) B_2 + (2P_1 - 1) B_1 P_2,$$
and solving for $B_2$ gives

$$B_2 = \frac{u_{xxy}}{P_1^2 - P_1} - \frac{(2P_1 - 1)B_1 P_2}{P_1^2 - P_1} = \frac{u_{xxy}}{(P_1^2 - P_1)} - \frac{(2P_1 - 1)P_2}{(P_1^2 - P_1)^2} u_{xy} = D_x^{(3)}(B_1).$$

Finally, by once again prolonging the total vector fields in terms of the standard coordinates, and further appending adapted coordinates $P_3 = D_x^{(4)}(P_2)$ and $Q_3 = D_y^{(4)}(Q_2)$, we can construct the hyperbolic distribution associated to (9.3) in terms of the adapted coordinates $\{x, y, u, P_i, Q_i\}_{i=1}^3$ as $\Delta = \hat{\Delta} \oplus \check{\Delta}$, where

$$\hat{\Delta} = \left\{ \hat{X}_1 = D_x^{(3)} + \left( (D_y^{(3)})^2 u_{xy} \right) \partial_{Q_3}, \hat{X}_2 = \partial_{P_3} \right\},$$

$$\check{\Delta} = \left\{ \check{X}_1 = D_y^{(3)} + \left( (D_x^{(3)})^2 u_{xy} \right) \partial_{P_3}, \check{X}_2 = \partial_{Q_3} \right\}.$$

In terms of the adapted coordinates, the intermediate integrals are $\tilde{I}_0 = y$, $\tilde{I}_0 = x$,

$$\hat{I}_3 = \hat{X}_1 \left[ \ln \left( Q_2 - \frac{Q_1^3(Q_1 - 1)}{6u + x} - \frac{Q_1(Q_1 - 1)^3}{6u + y} \right) \right] - \left( \frac{Q_1^2(Q_1 - 1)}{6u + x} + \frac{Q_1(Q_1 - 1)^2}{6u + y} \right),$$

$$\check{I}_3 = \check{X}_1 \left[ \ln \left( P_2 - \frac{P_1(P_1 - 1)^3}{6u + x} - \frac{P_1^3(P_1 - 1)}{6u + y} \right) \right] - \left( \frac{P_1(P_1 - 1)^2}{6u + x} + \frac{P_1^2(P_1 - 1)}{6u + y} \right).$$

We then further adapt our coordinates to these invariants by introducing the transformation

$$\Phi = \begin{cases} \hat{j}_0 = y, & \hat{j}_0 = x, & \hat{j}_3 = \hat{i}_3, & \hat{j}_3 = \check{i}_3, & z = u, \\ p_1 = P_1, & q_1 = Q_1, & p_2 = P_2, & q_2 = Q_2. \end{cases}$$

To obtain the hyperbolic distribution in these coordinates, we calculate

$$\hat{\Delta} = \left\{ \hat{W}_1 = \Phi \cdot \hat{X}_1, \hat{W}_2 = \partial_{\hat{i}_3} \right\}, \quad \check{\Delta} = \left\{ \check{W}_1 = \Phi \cdot \check{X}_1, \check{W}_2 = \partial_{\check{i}_3} \right\}.$$
In order to construct bases such that $\hat{\Delta}$ and $\tilde{\Delta}$ commute, we introduce matrices

$$ A = \begin{pmatrix} 1 & \hat{W}_1(\hat{J}_3) \\ 0 & \hat{W}_2(\hat{J}_3) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & \tilde{W}_1(\tilde{J}_3) \\ 0 & \tilde{W}_2(\tilde{J}_3) \end{pmatrix}, $$

and form the vector fields $\hat{U}_i = A_j^i \hat{X}_i$ and $\tilde{U}_i = B_j^i \tilde{X}_i$. We then compute the sequence of vector fields

$$ \hat{S}_1 = [\hat{U}_1, \hat{U}_2], \quad \hat{S}_2 = [\hat{U}_1, \hat{S}_1], \quad \hat{S}_3 = [\hat{U}_1, \hat{S}_2], \quad \hat{S}_4 = [\hat{U}_1, \hat{S}_3], \quad \hat{S}_5 = [\hat{U}_1, \hat{S}_4], $$

$$ \tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_1, \tilde{S}_2], \quad \tilde{S}_4 = [\tilde{U}_1, \tilde{S}_3], \quad \tilde{S}_5 = [\tilde{U}_1, \tilde{S}_4]. $$

Additionally, $[\hat{U}_2, \hat{S}_1] = [\tilde{U}_2, \tilde{S}_2] = 0$, and

$$ [\hat{U}_2, \hat{S}_3] = \hat{S}_2, \quad [\hat{U}_2, \hat{S}_4] = 2\hat{S}_3, \quad [\hat{U}_2, \hat{S}_5] = \hat{P}_3^2 \hat{S}_3 - 2\tilde{\mathcal{I}}_3 \hat{S}_3 + 4\hat{S}_4. $$

These vector fields form bases $\hat{\mathfrak{H}} = \{\hat{S}_i\}_{i=1}^5$ and $\hat{\mathfrak{F}} = \{\tilde{S}_i\}_{i=1}^5$ for the 5-dimensional Vessiot algebra of (9.3). In particular, the structure equations for $\hat{\mathfrak{H}}$ are

$$ [\hat{S}_1, \hat{S}_2] = -\hat{S}_2, \quad [\hat{S}_1, \hat{S}_3] = -\hat{S}_3, \quad [\hat{S}_1, \hat{S}_4] = 2\hat{J}_3 \hat{S}_3 - 2\hat{S}_4 - \hat{J}_3^2 \hat{S}_2, $$

$$ [\hat{S}_1, \hat{S}_5] = 4\hat{J}_3 \hat{S}_4 - 3\hat{S}_5 - 2\hat{J}_3^2 \hat{S}_3, \quad [\hat{S}_2, \hat{S}_3] = \hat{S}_4 - 2\hat{J}_3 \hat{S}_3 + \hat{J}_3^2 \hat{S}_2, $$

$$ [\hat{S}_2, \hat{S}_4] = \hat{S}_5 - 2\hat{J}_3 \hat{S}_4 + \hat{J}_3^2 \hat{S}_3, \quad [\hat{S}_2, \hat{S}_5] = 4\hat{J}_3 \hat{S}_5 - 13\hat{J}_3^2 \hat{S}_4 + 14\hat{J}_3^3 \hat{S}_3 - 5\hat{J}_3^4 \hat{S}_2, $$

$$ [\hat{S}_3, \hat{S}_4] = \hat{J}_3 \hat{S}_5 - 3\hat{J}_3^2 \hat{S}_4 + 3\hat{J}_3^3 \hat{S}_3 - \hat{J}_3^4 \hat{S}_2, \quad [\hat{S}_3, \hat{S}_5] = 4\hat{J}_3^2 \hat{S}_5 - 14\hat{J}_3^3 \hat{S}_4 + 16\hat{J}_3^4 \hat{S}_3 - 6\hat{J}_3^5 \hat{S}_2, $$

$$ [\hat{S}_4, \hat{S}_5] = 3\hat{J}_3 \hat{S}_5 - 11\hat{J}_3^4 \hat{S}_4 + 13\hat{J}_3^5 \hat{S}_3 - 5\hat{J}_3^6 \hat{S}_2. $$

Upon making the change of basis,

$$ \hat{S}_1 \mapsto -e_1, \quad \hat{S}_2 \mapsto e_3, \quad \hat{S}_3 \mapsto e_3 - \hat{J}_3 e_2, \quad \hat{S}_4 \mapsto \hat{P}_3^2 e_2 - \hat{J}_3^2 e_3 + \hat{J}_3 e_4, \quad \hat{S}_5 \mapsto -2\hat{P}_3^3 e_2 + 5\hat{J}_3^4 e_3 - 4\hat{J}_3^5 e_4 + \hat{J}_3^6 e_5, $$

where $e_i$ are the basis vectors for the 5-dimensional space.
the structure equations become

\[
\begin{align*}
\end{align*}
\]

The structure equations for \( \tilde{\mathfrak{g}} \) are exactly the same upon replacing \( \hat{S} \) and \( \hat{J}_3 \) with \( \tilde{S} \) and \( \tilde{J}_3 \). As abstract Lie algebras, \( \hat{\mathfrak{g}} \) and \( \tilde{\mathfrak{g}} \) are equivalent to \( s_{5,35} \) with \( a = 1 \) [28].

The vector fields \( \hat{\mathscr{F}} = \{ \hat{S}_i, \hat{U}_1, \hat{U}_2, \hat{U}_1, \hat{U}_2 \}_{i=1}^5 \) form a frame on the 9-dimensional equation manifold with dual coframe \( \hat{\mathscr{F}}^* = \{ \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \pi^1, \pi^2, \chi^1, \chi^2 \} \).

After making the change of coframe,

\[
\hat{\mathscr{F}}^* = \{ \theta^5, \theta^4, \theta^3, \theta^2, \theta^1, \pi^1 + \tilde{J}_3 \theta^3, \pi^2, \chi^1, \chi^2 \},
\]

we see that the structure equations become

\[
\begin{align*}
d\theta^1 &\equiv \theta^2 \land \pi^1 + T_1 \chi^1 \land \chi^2 \mod \{ \theta^1 \}, \\
d\theta^2 &\equiv \theta^3 \land \pi^1 + T_2 \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2 \}, \\
d\theta^3 &\equiv \theta^4 \land \pi^1 + T_3 \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2, \theta^3 \}, \\
d\theta^4 &\equiv \theta^5 \land \pi^1 + T_4 \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2, \theta^3, \theta^4 \}, \\
d\theta^5 &\equiv -\pi^1 \land \pi^2 + \chi^1 \land \chi^2 \mod \{ \theta^1, \theta^2, \theta^3, \theta^4, \theta^5 \},
\end{align*}
\]

where \( \chi^1 = d\tilde{\theta}_0, \chi^2 = d\tilde{\theta}_3 \), and the functions \( T_i = T_i(z, p_1, q_1, p_2, q_2, \tilde{J}_0, \tilde{J}_0, \tilde{J}_3, \tilde{J}_3) \) have been omitted due to their length. Upon restricting to the integral manifolds where \( \chi^1 = \chi^2 = 0 \), we see that these congruences are precisely the Goursat congruences for the contact distribution on \( J^5(\mathbb{R}, \mathbb{R}) \). Moreover, a direct calculation shows that both the derived flag and the weak derived flag of \( \tilde{\Delta} \) and \( \tilde{\Delta} \) have growth \((2, 3, 4, 5, 6, 7)\), giving the same result.

We can then conclude that \( \hat{\mathfrak{g}} \) must be a 5-dimensional symmetry algebra of the contact distribution on \( J^5(\mathbb{R}, \mathbb{R}) \) whose abstract structure is equivalent to \( s_{5,35} \) with \( a = 1 \). The only such symmetry algebra is given by \( p_{5,3} \) with \( \alpha = 3 \) in Table A.3. This proves part [iii]
As in the previous two examples, we can theoretically reconstruct (9.3) by calculating the quotient of two contact distributions on $J^4(\mathbb{R}, \mathbb{R})$ by the prolonged diagonal action of $p_{5,3}$ with $\alpha = 3$. However, the formulas in this calculation quickly become unmanageable, and so we now use the quotient construction to give an alternative representation of (9.3) which is much simpler in that it is completely specified using linear vector fields.

In terms of the coordinates $x, u, u_1, u_2, u_3, u_4$ and $y, v, v_1, v_2, v_3, v_4$, the Vessiot distributions are

$$\hat{V} = \{ \partial_x + u_1 \partial_u + u_2 \partial_{u_1} + u_3 \partial_{u_2} + u_4 \partial_{u_3}, \partial_{u_4} \},$$

$$\check{V} = \{ \partial_y + v_1 \partial_v + v_2 \partial_{v_1} + v_3 \partial_{v_2} + v_4 \partial_{v_3}, \partial_{v_4} \}.$$ 

The quotient of $\hat{V} \oplus \check{V}$ by $G_{\text{diag}}$ is a rank 4 distribution $\Delta$ on the 7-manifold $M_7$ with coordinates $z_1, z_2, z_3, z_4, z_5, z_6, z_7$. The explicit formula for the quotient map $q : \hat{M}_6 \times \check{M}_6 \to M_7$ is given by

$$z_1 = \frac{u_1 + v_1}{(x - y)^2} - \frac{2(u - v)}{(x - y)^3}, \quad z_2 = \frac{u_2 - u_1 - v_1}{x - y} - \frac{(x - y)^2}{(x - y)^3}, \quad z_3 = \frac{v_2 - u_1 - v_1}{x - y} - \frac{(x - y)^2}{(x - y)^3}$$

$$z_4 = u_3, \quad z_5 = v_3, \quad z_6 = (x - y)u_4, \quad z_7 = (x - y)v_4.$$
Calculating the pushforward of $\mathcal{V} \oplus \mathcal{V}$ by $q$ gives the rank 4 hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ where

$$\hat{\Delta} = \{-(3z_1 - z_2)\partial_{z_1} - (2z_2 - z_4)\partial_{z_2} - (z_2 + z_3)\partial_{z_3} + z_6\partial_{z_4} + z_6\partial_{z_6} + z_7\partial_{z_7}, \partial_{z_8}\},$$

$$\tilde{\Delta} = \{(3z_1 + z_3)\partial_{z_1} + (z_2 + z_3)\partial_{z_2} + (2z_3 + z_5)\partial_{z_3} + z_7\partial_{z_5} - z_6\partial_{z_6} - z_7\partial_{z_7}, \partial_{z_7}\}.$$  

We can also see that since the growth vector of $\Delta$ is $(4,6,7)$, $\Delta$ has no Cauchy characteristics, and $\Delta^{(1)}$ has the two Cauchy characteristics $\partial_{z_6}, \partial_{z_7}$, $\Delta$ must define a PDE in the plane. Moreover, since, by construction, the equation defined by $\Delta$ has the same fundamental invariants as (9.3), the must be equivalent.
CHAPTER 10
DARBOUX INTEGRABLE $f$-GORDON EQUATIONS – ORDER THREE

In this chapter, we characterize the fundamental invariants of $f$-Gordon equations Darboux integrable at order three (with particular structure) and describe the quotient construction of these equations in terms of their fundamental invariants. In particular, we restrict our study to the most interesting case where the prolonged characteristic distributions each admit exactly two first integrals. We find that all equations of this type must come from the quotient of the prolongation of either $(2,3,4,5)$-distributions or $(2,3,5)$-distributions by 5-dimensional symmetry groups of these distributions acting with codimension 1 orbits on 5-dimensional manifolds. We then provide a list of all such 5-dimensional symmetry algebras of contact distributions on $J^5(\mathbb{R}, \mathbb{R})$ and of prolonged $(2,3,5)$-distributions of root types $[\infty]$ and $[4]$.

10.1 The Fundamental Invariants

The following theorem characterizes the fundamental invariants of $f$-Gordon equations which are Darboux integrable at order three and whose characteristic distributions each admit exactly two first integrals.

**Theorem 10.1.1.** Let $\Delta$ be a hyperbolic distribution defined by an $f$-Gordon equation which is Darboux integrable at order three and whose prolonged characteristic distributions each admit exactly two first integrals. Then the corresponding Vessiot distributions of $\Delta$ are either contact distributions on $J^4(\mathbb{R}, \mathbb{R})$ or the first prolongation of $(2,3,5)$-distributions. The Vessiot group of $\Delta$ is either the prolongation of a 5-dimensional contact symmetry group on $J^1(\mathbb{R}, \mathbb{R})$ to $J^4(\mathbb{R}, \mathbb{R})$ acting with codimension 1 orbits on $J^3(\mathbb{R}, \mathbb{R})$ or the prolongation of a 5-dimensional symmetry group of a $(2,3,5)$-distribution with codimension 1 orbits.

**Remark.** As noted in Chapter 9, equations (9.1), (9.2), and (9.3) all satisfy these properties, in particular that the action of their respective Vessiot groups have codimension 1 orbits.
on \(J^3(\mathbb{R}, \mathbb{R})\) or the underlying 5-dimensional manifold of a \((2,3,5)\)-distribution.

**Proof.** Let \(\Delta = \hat{\Delta} \oplus \tilde{\Delta}\) be a rank 4 distribution on 7-manifold \(M\) corresponding to an \(f\)-Gordon equation which is Darboux integrable at order three, and further suppose that the prolonged characteristic distributions \(\hat{\Delta}^{(1)}\) and \(\tilde{\Delta}^{(1)}\) each admit exactly two first integrals on the 9-dimensional manifold \(M^{(1)}\). This immediately implies that the Vessiot group is 5-dimensional.

Since \(\Delta\) corresponds to an \(f\)-Gordon equation, the characteristic distributions must satisfy

\[
\text{rank}(\hat{\Delta}') = \text{rank}(\tilde{\Delta}') = 3, \quad \text{and} \quad \text{rank}(\hat{\Delta}'') = \text{rank}(\tilde{\Delta}'') = 4,
\]

Moreover, since \(\hat{\Delta}\) and \(\tilde{\Delta}\) each must admit a first integral of order less than or equal to one, we further see that

\[
\text{rank}(\hat{\Delta}^{(\infty)}) = \text{rank}(\tilde{\Delta}^{(\infty)}) = 6
\]

This implies that the derived dimensions for \(\hat{\Delta}\) and \(\tilde{\Delta}\) must be either \((2,3,4,5,6)\) or \((2,3,4,6)\).

A simple calculation shows that weak derived dimensions of \(\hat{\Delta}\) and \(\tilde{\Delta}\) are \((2,3,4,5,6)\). The restriction of each characteristic distribution to their respective integral manifolds \(\hat{M}\) and \(\tilde{M}\) given by the level set of the first integral gives the Vessiot distributions \(\hat{V}\) and \(\tilde{V}\) which must have derived distributions \((2,3,4,5,6)\) or \((2,3,4,6)\) and weak derived dimensions are \((2,3,4,5,6)\).

In either case, the derived distributions \(\hat{V}'\) and \(\tilde{V}'\) must each admit a single Cauchy characteristic by which we can deprolong each distribution giving

\[
\hat{W} = \hat{V}' / A(\hat{V}') \quad \text{and} \quad \tilde{W} = \tilde{V}' / A(\tilde{V}')
\]

on 5-manifolds \(\hat{N} = \hat{M} / A(\hat{V}')\) and \(\tilde{N} = \tilde{M} / A(\tilde{V}')\). The derived dimensions for \(\hat{W}\) and \(\tilde{W}\) are then either \((2,3,4,5)\) or \((2,3,5)\). If the derived dimensions are \((2,3,4,5)\), then the weak
derived dimensions will be as well, and we conclude that the deprolonged Vessiot distribution is locally equivalent to the standard contact distribution on $J^3(\mathbb{R}, \mathbb{R})$ by Theorem 2.5.2, meaning the Vessiot distribution will be locally equivalent to the standard contact distribution on $J^4(\mathbb{R}, \mathbb{R})$. If the derived dimensions of the deprolonged Vessiot distribution are $(2,3,5)$, then we can immediately conclude that the Vessiot distribution itself is the prolongation of a $(2,3,5)$ distribution.

Finally, since the Vessiot group is a symmetry group of the Vessiot distributions, we see that it must be a 5-dimensional contact symmetry group acting on $J^4(\mathbb{R}, \mathbb{R})$ or the first prolongation of a 5-dimensional symmetry group of a $(2,3,5)$-distribution. Since there is a 1-1 correspondence between the two first-order first integrals on $M/A(\Delta')$ and differential invariants of the Vessiot group acting on $\hat{N}$ and $\bar{N}$, the Vessiot group must act with codimension 1 orbits on the 5-dimensional manifolds $\hat{N}$ and $\bar{N}$ described above.

\[ \Box \]

10.2 The Quotient Construction

In this section, we give a converse to the previous theorem. This gives the general quotient construction of $f$-Gordon equations which are Darboux integrable at order three.

**Theorem 10.2.1.** Let $\hat{V}$ and $\bar{V}$ be two rank 2 distributions on 5-manifolds $\hat{M}$ and $\bar{M}$ with nonintegrable first-derived systems and common 5-dimensional symmetry group $G$. Let $\hat{V}^{(1)}, \bar{V}^{(1)}$ be the prolongations of $\hat{V}, \bar{V}$ to the 6-manifolds, $\hat{M}^{(1)}, \bar{M}^{(1)}$, and denote the infinitesimal generators of the diagonal action of $G$ by $\Gamma_{\text{diag}}$ and the prolongation of these infinitesimal generators by $\Gamma_{\text{diag}}^{(1)}$. Suppose that

[i] $G$ acts freely on $\hat{M}^{(1)}$ and $\bar{M}^{(1)}$,

[ii] $\Gamma_{\text{diag}} \cap (\hat{V} \oplus \bar{V})' = \{0\}$,

[iii] $\Gamma_{\text{diag}}^{(1)} \cap (\hat{V}^{(1)} \oplus \bar{V}^{(1)})$ is 1-dimensional, and

(iv) $G$ acts on $\hat{M}$ and $\bar{M}$ with codimension 1 orbits.
Then the quotient distribution

\[ \Delta = \left( \tilde{\mathcal{V}}^{(1)} \oplus \tilde{\mathcal{V}}^{(1)} \right)/G^{(1)}_{\text{diag}} \]

is a rank 4 distribution defining an \( f \)-Gordon equation which is Darboux integrable at order three and not at order two.

Remark. Conditions [i], [ii], and [iii] are sufficient for \( \Delta \) to define a Monge-Ampère equation which is Darboux integrable at order three. Condition [iv] is further required if \( \Delta \) is to define an \( f \)-Gordon equation. As we previously remarked in Theorem 8.2.1, this condition is no longer automatic since \( G \) is 5-dimensional.

Proof. Let \( \tilde{\mathcal{V}} \) and \( \tilde{\mathcal{V}} \) be two rank 2 distributions on 5-manifolds \( \tilde{M} \) and \( \tilde{M} \) with nonintegrable first-derived systems and common 5-dimensional symmetry group \( G \) acting on \( \tilde{M} \) and \( \tilde{M} \) with codimension 1 orbits and differential invariants \( \tilde{I} \) and \( \tilde{I} \). We begin our proof by showing that we may always choose a basis of \( G^{(1)} \)-invariant vector fields for the first and second derived systems of the prolongations of \( \tilde{\mathcal{V}}^{(1)} \) and \( \tilde{\mathcal{V}}^{(1)} \). This will be essential in our construction of the quotient distribution \( \Delta \).

The prolongation of \( \tilde{\mathcal{V}} \) on \( \tilde{M}^{(1)} = \tilde{M} \times \mathbb{R} \) is

\[ \tilde{\mathcal{V}}^{(1)} = \{ \tilde{\gamma}_1 = \tilde{X}_1 + \lambda \tilde{X}_2, \tilde{\gamma}_6 = \partial_\lambda \} \]

where \( \lambda \) is the coordinate on \( \mathbb{R} \). The first-derived of \( \tilde{\mathcal{V}}^{(1)} \) is then \( (\tilde{\mathcal{V}}^{(1)})' = \{ \tilde{\gamma}_1, \tilde{\gamma}_6, \tilde{\gamma}_2 \} \) where

\[ \tilde{\gamma}_2 = [\tilde{\gamma}_1, \tilde{\gamma}_6] = [\tilde{X}_1 + \lambda \tilde{X}_2, \partial_\lambda] = -\lambda \tilde{X}_2. \]

It then follows that since

\[ [\tilde{\gamma}_6, \tilde{\gamma}_2] = [\partial_\lambda, -\lambda \tilde{X}_2] = -\tilde{X}_2 = \frac{1}{\lambda} \tilde{\gamma}_2 \in (\tilde{\Delta}^{(1)})', \]

\( \tilde{\gamma}_6 \) is a Cauchy characteristic of \( (\tilde{\mathcal{V}}^{(1)})' \). By Proposition 3.2.3, if \( \hat{\Gamma} \) is a symmetry of \( \tilde{\mathcal{V}}^{(1)} \), then it will be a symmetry of \( (\tilde{\mathcal{V}}^{(1)})' \). Since the set of Cauchy characteristics of a distribution
form an ideal in the set of infinitesimal symmetries, we see that any multiple of $\tilde{Y}_6$ will remain a Cauchy characteristic. But then by the argument made in the proof of Lemma 3.3.3, we see that there exists a nonzero function $f$ such $\tilde{Y}_6^G = f\tilde{Y}_6$ is a $G^{(1)}$-invariant Cauchy characteristic of $(\tilde{V}^{(1)})'$.

Utilizing our argument in the proof of Theorem 3.3.2, we may then adjust $\tilde{Y}_1$ to obtain $G^{(1)}$-invariant vector fields $\tilde{Y}_1^G$ and $\tilde{Y}_2^G = [\tilde{Y}_1^G, \tilde{Y}_6^G]$. We therefore have a $G^{(1)}$-invariant basis for $(\tilde{\Delta}^{(1)})'$ given by $(\tilde{\Delta}^{(1)})' = \{\tilde{Y}_1^G, \tilde{Y}_2^G, \tilde{Y}_6^G\}$ where $\tilde{Y}_6^G$ is a Cauchy characteristic. Finally, since the bracket of two invariant vector fields, remains invariant, we can construct a $G^{(1)}$-invariant basis for $(\tilde{V}^{(1)})''$ given by $(\tilde{V}^{(1)})'' = \{\tilde{Y}_1^G, \tilde{Y}_2^G, \tilde{Y}_3^G, \tilde{Y}_6^G\}$ where $\tilde{Y}_3^G = [\tilde{Y}_1^G, \tilde{Y}_2^G]$. The same argument yields $G^{(1)}$-invariant bases for $(\tilde{V}^{(1)})' = \{\tilde{Y}_1^G, \tilde{Y}_2^G, \tilde{Y}_6^G\}$ where $\tilde{Y}_6^G$ is a Cauchy characteristic, and $(\tilde{V}^{(1)})'' = \{\tilde{Y}_1^G, \tilde{Y}_2^G, \tilde{Y}_3^G, \tilde{Y}_6^G\}$.

On the 12-dimensional product manifold, $M_{12} = \tilde{M}^{(1)} \times \tilde{M}^{(1)}$ we construct the frame $\mathcal{F} = \{\tilde{Y}_1^G, \tilde{Y}_2^G, \tilde{Y}_6^G, \tilde{Y}_1^G, \tilde{Y}_2^G, \tilde{Y}_6^G, Z, \Gamma_i\}_{i=1}^5$ where $\Gamma_i$ are infinitesimal generators of the diagonal action of $G^{(1)}$ on $M_{12}$ and

$$Z = \mu[\tilde{Y}_1^G, \tilde{Y}_2^G] + \nu[\tilde{Y}_1^G, \tilde{Y}_2^G]$$

for some $G^{(1)}$-invariant functions $\mu, \nu$ not both zero.

The structure equations on $M_{12}$ satisfy

$$[\tilde{Y}_1^G, \tilde{Y}_6^G] = \tilde{Y}_2^G, \quad [\tilde{Y}_1^G, \tilde{Y}_2^G] = \tilde{Y}_3^G, \quad [\tilde{Y}_2^G, \tilde{Y}_3^G] = \tilde{Y}_4^G, \quad [\tilde{Y}_4^G, \tilde{Y}_5^G] = \tilde{Y}_6^G,$$

$$[\tilde{Y}_1^G, \tilde{Y}_6^G] = \tilde{Y}_2^G, \quad [\tilde{Y}_1^G, \tilde{Y}_2^G] = \tilde{Y}_3^G, \quad [\tilde{Y}_2^G, \tilde{Y}_3^G] = \tilde{Y}_4^G, \quad [\tilde{Y}_3^G, \tilde{Y}_4^G] = \tilde{Y}_5^G,$$

$$[\tilde{Y}_1^G, \tilde{Y}_6^G] = \tilde{Y}_2^G, \quad [\tilde{Y}_2^G, \tilde{Y}_3^G] = \tilde{Y}_4^G, \quad [\tilde{Y}_3^G, \tilde{Y}_4^G] = \tilde{Y}_5^G,$$

$$[\tilde{Y}_a^G, \tilde{Y}_b^G] = 0, \quad [\Gamma_i, \tilde{Y}_a^G] = 0, \quad [\Gamma_i, \tilde{Y}_b^G] = 0, \quad [\Gamma_i, Z] = 0.$$

The vector fields $\tilde{Y}_1^G, \tilde{Y}_2^G, \tilde{Y}_6^G, \tilde{Y}_1^G, \tilde{Y}_2^G, \tilde{Y}_6^G, Z$ are all $G^{(1)}$-invariant and descend to vector fields $\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_6, \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_6, Z$ on the 7-manifold $M = (\tilde{M}^{(1)} \times \tilde{M}^{(1)})/\Gamma_{\text{diag}}$. From the structure equations on $M_{12}$, we can read off all the properties of $\Delta = \{\tilde{Y}_1, \tilde{Y}_6, \tilde{Y}_1, \tilde{Y}_6\}$. In particular,
we immediately see that $\Delta$ is hyperbolic with characteristic distributions

$$\hat{\Delta} = \{\bar{Y}_1, \bar{Y}_6\} \quad \text{and} \quad \tilde{\Delta} = \{\bar{Y}_1, \bar{Y}_6\},$$

and the first derived system is

$$\Delta' = \{\bar{Y}_1, \bar{Y}_2, \bar{Y}_6, \bar{Y}_1, \bar{Y}_2, \bar{Y}_6\},$$

and the second derived system $\Delta''$ must have rank 7 due to dimensional constraints. Moreover, we see that $\bar{Y}_6, \bar{Y}_6$ are Cauchy characteristics of $\Delta'$. Therefore, $\Delta$ defines a hyperbolic PDE in the plane by Theorem 2.3.1.

Since

$$\hat{\Delta}' = \{\bar{Y}_1, \bar{Y}_2, \bar{Y}_6\}, \quad \hat{\Delta}'' = \{\bar{Y}_1, \bar{Y}_2, \bar{Y}_3, \bar{Y}_6\},$$

$$\tilde{\Delta}' = \{\bar{Y}_1, \bar{Y}_2, \bar{Y}_6\}, \quad \tilde{\Delta}'' = \{\bar{Y}_1, \bar{Y}_2, \bar{Y}_3, \bar{Y}_6\},$$

we conclude by Theorem 2.3.5 that $\Delta$ defines a hyperbolic Monge-Ampère equation.

A hyperbolic distribution will define an $f$-Gordon equation if each of its characteristic systems admit at least one first integral of order less than or equal to one. Since both $\hat{I}$ and $\tilde{I}$ are $\Gamma_{\text{diag}}$-invariant, we see that $\hat{I}$ and $\tilde{I}$ will descend to the quotient.

Clearly, $\bar{Y}_1(\hat{I}) = \bar{Y}_6(\hat{I}) = 0$ and $\bar{Y}_1(\tilde{I}) = \bar{Y}_6(\tilde{I}) = 0$, so $\hat{\Delta}$ and $\tilde{\Delta}$ each have an invariant of at most second order. However, since $\bar{Y}_6, \bar{Y}_6$ are Cauchy characteristics of $\hat{\Delta}'$ and $\tilde{\Delta}'$, respectively, if we let $\rho : M \to M/\{\bar{Y}_6, \bar{Y}_6\}$ be the standard reduction map, we can calculate the reduced distribution

$$(\hat{\Delta} \oplus \tilde{\Delta})' / \{\bar{Y}_6, \bar{Y}_6\} = \{\bar{X}_1, \bar{X}_2, \bar{X}_1, \bar{X}_2\}$$

where $\bar{X}_i = \rho_* \bar{Y}_i$ and $\bar{X}_i = \rho_* \bar{Y}_i$.

This reduced distribution is again hyperbolic with characteristic distributions $\{\bar{X}_1, \bar{X}_2\}$ and $\{\bar{X}_1, \bar{X}_2\}$. By construction, the prolongation of the reduced distribution is precisely
\[ \Delta, \text{ and since } \tilde{I} \text{ and } \hat{I} \text{ were respectively } \tilde{Y}_6 \text{ and } \hat{Y}_6 \text{-invariant, we again see that will be invariants on the reduced distribution, and therefore must be of order less than or equal to one. Therefore, } \Delta \text{ defines an } f \text{-Gordon equation.} \]

Finally, since the prolonged orbit dimensions of } G \text{ on } \tilde{M} \text{ and } \hat{M} \text{ are 4, 5, 5 while the dimension of the prolonged manifolds are 5, 6, 7, we see that the prolonged characteristic distributions } \tilde{\Delta}^{(1)} \text{ and } \hat{\Delta}^{(1)} \text{ will each have an additional invariant. We therefore conclude that } \Delta \text{ will be Darboux integrable after one prolongation; in other words, } \Delta \text{ defines and } f \text{-Gordon equation which is Darboux integrable at order three.} \]

10.3 Symmetry Algebras of \((2,3,5)\)-Distributions of Root Type \([4]\)

Theorem 10.2.1 allows for the construction of } f \text{-Gordon equations which are Darboux integrable at order three given two rank 2 distributions on 5-manifolds with nonintegrable derived systems and an appropriate 5-dimensional symmetry group. When the distributions are equivalent to the standard contact distribution or the Hilbert-Cartan distribution, all of the 5-dimensional symmetry algebras are known via the classification of Doubrov \([14]\) and can readily be analyzed to find which will give rise to } f \text{-Gordon equations Darboux integrable at order three. It remains, however, to classify the 5-dimensional symmetry algebras of } (2,3,5)\text{-distributions of the other root types. As a first step, consider } (2,3,5)\text{-distributions of root type } [4].

In Cartan’s original analysis of distributions of this type \([12]\), he states that every such distribution can be encoded by the equation

\[ z' = -\frac{1}{2} \left( y'' + \frac{10}{3} I(y')^2 + (1 + I^2 - I'')y'' \right), \quad (10.1) \]

where } I = I(x) \text{ is an invariant. Cartan then splits this classification based upon whether the invariant } I \text{ is constant or non-constant. In the case when } I \text{ is constant, the full symmetry algebra of } (10.1) \text{ is 7-dimensional, and when } I \text{ is non-constant, the symmetry algebra of } (10.1) \text{ is 6-dimensional. Doubrov and Govorov later showed in } [15] \text{ that Cartan’s analysis of these distributions was incomplete and missed the single } (2,3,5)\text{-distribution encoded by }
the equation

\[ z' = y + (y'')^{1/3}, \]  

(10.2)

which has 6-dimensional symmetry algebra.

In the following three sections, we give a classification of all inequivalent 5-dimensional subalgebras of these classes of (2,3,5)-distributions. We note that when \( I \) is constant, the class of distributions encoded by (10.1) can instead be encoded by the simpler equation

\[ z' = (y'')^m \quad m \neq -1, 0, 1, \frac{1}{3}, \frac{2}{3}, 2, \]

when \( I \neq \pm 3/4 \). In this case, the relationship between the invariant \( I \) and the parameter \( m \) is given by

\[
I^2 = -\frac{9}{4} \frac{(2m^2 - 2m + 1)^2}{(m - 2)(3m - 1)(3m - 2)(m + 1)}. 
\]

When \( I = \pm 3/4 \), the class of distributions encoded by (10.1) can instead be encoded by the equation \( z' = \ln(y'') \).

## 10.4 Root Type [4] with \( I(x) \) Constant

In the case where the invariant \( I(x) \) is constant, Cartan [12] states the symmetry algebra \( g \) of such a system is generically 7-dimensional and is given by the structure equations

\[
\begin{align*}
\{ &d\omega^1 = 2\omega^1 \wedge \pi^1 + \omega^2 \wedge \pi^2 + \omega^3 \wedge \omega^4, \\
&d\omega^2 = \omega^2 \wedge \pi^1 + \omega^3 \wedge \omega^5, \\
&d\omega^3 = I\omega^2 \wedge \omega^5 + \omega^3 \wedge \pi^1 + \omega^4 \wedge \omega^5, \\
&d\omega^4 = \frac{4}{3}I\omega^3 \wedge \omega^5 + \omega^4 \wedge \pi^1 + \omega^5 \wedge \pi^2, \\
&d\omega^5 = d\pi^1 = 0, \\
&d\pi^2 = \pi^2 \wedge \pi^1 - I\omega^4 \wedge \omega^5 + \omega^2 \wedge \omega^5. \\
\}
\]

(10.3)
If we denote the dual basis to \( \{ \omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \pi^1, \pi^2 \} \) by \( \{ \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6, \bar{e}_7 \} \), then we may make the change of basis,

\[
e_1 = -\bar{e}_1, \quad e_5 = \bar{e}_7, \quad e_6 = \bar{e}_5, \quad e_7 = -\bar{e}_6,
\]

which puts the nilradical in standard form. The multiplication table for \( g \) in this basis is given by

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We note that, with the exception of \( \text{ad}(e_6) \), the structure equations are in simple form. We therefore focus on transforming the matrix

\[
\text{ad}(e_6) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{4}{3}I & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & -I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

to standard normal forms while preserving the nilradical. This restricts the allowable transformations to those given by the Lie algebra of derivations of the nilradical. In particular, we focus on the semisimple part, which fixes \( e_1 \) and allows for transformations among the vectors \( e_2, e_3, e_4, e_5 \). Upon considering the matrix representation for these transformations,
we see that after restricting to the subspace $\Sigma = \text{span}\{e_2, e_3, e_4, e_5\}$, the resulting set of ten $4 \times 4$ matrices, which we will denote as $\mathcal{R}$, preserves the skew-symmetric bilinear form

$$K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

such that for any given matrix $L \in \mathcal{R}$, the condition $LK + KL^T = 0$ is satisfied. Along with this, we also see that $\mathcal{R}$ is closed under the matrix commutator and satisfies the Jacobi identity. We therefore conclude the representation is the standard matrix representation for the real symplectic algebra $\mathfrak{sp}(2, \mathbb{R})$.

Furthermore, the restriction of $\text{ad}(e_6)$ to $\Sigma$ gives the $4 \times 4$ matrix

$$X = \text{ad}(e_6)|_\Sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ I & 0 & 1 & 0 \\ 0 & \frac{4}{3}I & 0 & -1 \\ 1 & 0 & -I & 0 \end{pmatrix},$$

which also satisfies $XK + KX^T = 0$, so we see that $X \in \mathfrak{sp}(2, \mathbb{R})$. This leads us to the observation that the problem of finding normal forms for $X$ is equivalent to the problem of classifying the orbits of $\mathfrak{sp}(2, \mathbb{R})$ under the adjoint action of its Lie group.

To make this analysis, we first consider a general element in $\mathfrak{sp}(2, \mathbb{R})$,

$$Y = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_3 \\ y_8 & y_9 & -y_6 & -y_2 \\ y_{10} & y_8 & -y_5 & y_1 \end{pmatrix}.$$
The characteristic polynomial of $Y$ is of the form

$$
\lambda^4 + a_1(y_1, \ldots, y_{10})\lambda^2 + a_2(y_1, \ldots, y_{10}),
$$

and so we see that if $\lambda_i$ is an eigenvalue of $Y$, then so is $-\lambda_i$. The full characterization of the orbits is then given by the discriminant,

$$
\delta = \sqrt{a_1^2 - 4a_2}.
$$

If $\delta > 0$, then the normal form for $Y$ is

$$
\tilde{Y}_1 = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & -\lambda_1 & 0 \\
0 & 0 & 0 & -\lambda_2
\end{pmatrix}
$$

with $\lambda_1 \neq \lambda_2$.

if $\delta = 0$, then the normal form for $Y$ is either

$$
\tilde{Y}_2 = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & -\lambda_1 & 0 \\
0 & 0 & 0 & -\lambda_1
\end{pmatrix}
$$

or

$$
\tilde{Y}_3 = \begin{pmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_3 & 0 & 0 \\
0 & 0 & -\lambda_1 & 1 \\
0 & 0 & 0 & -\lambda_1
\end{pmatrix},
$$

and if $\delta < 0$, then

$$
\tilde{Y}_4 = \begin{pmatrix}
a_1 & -b_1 & 0 & 0 \\
b_1 & a_1 & 0 & 0 \\
0 & 0 & a_2 & -b_2 \\
0 & 0 & b_2 & a_2
\end{pmatrix}
$$

where $\lambda_1 = a_1 + ib_1$, $\lambda_2 = a_2 + ib_2$. 
Now, the characteristic polynomial for $X$ is given by

$$\lambda^4 - \frac{10}{3} I \lambda^2 + I^2 + 1$$

and has discriminant

$$\delta = \frac{64}{9} I^2 - 4.$$ 

This leads us to the following three cases:

(i) $I > 3/4$ or $I < -3/4$,

(ii) $I = \pm 3/4$, and


We note that in Case (i), the values for $I$ correspond to values $m \neq -1, 0, 1, \frac{1}{3}, \frac{2}{3}, 2$ while Cases (ii) and (iii) do not correspond to distributions associated to equations of the form $z' = (y'')^m$. We also note that in all cases, we may set the eigenvalue $\lambda_1 = 1$, provided $\lambda_1 \neq 0$, by scaling $\text{ad}(e_6)$. We see that the second eigenvalue $\lambda_2 = \mu$ is related to the invariant $I$ via

$$\mu = \frac{5I - \sqrt{16I^2 - 9}}{9(I^2 + 1)}.$$

In doing so, we can obtain the following standard forms for the symmetry algebra $\mathfrak{g}$ for each of the above cases.

**Case (i).** In the case where $I > 3/4$ or $I < -3/4$, the standard form for the multiplication table, after rescaling $\text{ad}(e_6)$, is given by
Case (ii). In the case where $I = \pm 3/4$, the standard form for the multiplication table, after rescaling $\text{ad}(e_6)$, is given by

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Case (iii). In the case where $-3/4 < I < 3/4$, the standard form for the multiplication table, after rescaling $\text{ad}(e_6)$, is given by

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10.4.1 Subalgebras of the Symmetry Algebra of \( z' = (y'')^m \)

In this section, we consider the 5-dimensional subalgebras of the symmetry algebra of the Monge equation

\[
z' = (y'')^m, \quad m \neq -1, 0, 1, \frac{1}{3}, \frac{2}{3}, 2.
\]  

(10.4)

The symmetry algebra \( g \) of (10.4) is 7-dimensional, and in terms of the jet coordinates \( x, z, y, y_1, y_2 \), is generated by the vector fields

\[
\begin{align*}
\Gamma_1 &= \frac{1}{m - 1} \partial_y, \\
\Gamma_2 &= \frac{my_1^{m-1}}{m - 1} \partial_x + \frac{my_2^{m-1}}{2m - 1} \partial_z + \frac{m y y_2^{m-1} - z}{m - 1} \partial_y + y_2^m, \\
\Gamma_3 &= \frac{x}{m - 1} \partial_x + \frac{1}{m - 1} \partial_{y_1}, \\
\Gamma_4 &= \partial_x, \\
\Gamma_5 &= \partial_z, \\
\Gamma_6 &= -x \partial_x + (2m - 1)z \partial_z + y_1 \partial_{y_1} + 2y_2 \partial_{y_2}, \\
\Gamma_7 &= x \partial_x + z \partial_z + 2y \partial_y + y_1 \partial_{y_1},
\end{align*}
\]

provided \( m \neq 1/2 \). We consider the special case where \( m = 1/2 \) at the end of this section.

If we let \( e_i = \Gamma_i \) for \( 1 \leq i \leq 7 \), and let \( \mu = 2m - 1 \), then the multiplication table is equivalent to that of case (i),

<table>
<thead>
<tr>
<th>( g )</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
<th>( e_3 )</th>
<th>( e_4 )</th>
<th>( e_5 )</th>
<th>( e_6 )</th>
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<td>( \cdot )</td>
<td>( e_1 )</td>
<td>( \cdot )</td>
<td>( -\mu e_3 )</td>
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<td>( e_4 )</td>
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<td>( \mu e_4 )</td>
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<tr>
<td>( e_5 )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( -e_5 )</td>
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<td>( e_7 )</td>
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</table>

We note that \( \mu \neq 0 \), as \( m \neq 1/2 \). We can then calculate the full automorphism group \( \mathcal{G} \) of \( g \) and split \( \mathcal{G} \) into two subgroups \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) corresponding to the nondiagonal and diagonal
matrices

\[
G_1 = \begin{pmatrix}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
0 & 1 & 0 & 0 & 0 & x_4 \\
0 & 0 & 1 & 0 & 0 & -\mu x_3 \\
0 & 0 & 0 & 1 & 0 & -\mu x_2 \\
0 & 0 & 0 & 0 & 1 & x_1 \\
0 & 0 & 0 & 0 & 0 & -x_1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
G_2 = \begin{pmatrix}
z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_1/z_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z_1/z_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

We denote the actions of the matrix groups on \( \mathfrak{g} \) by \( \varphi_1 \) and \( \varphi_2 \) such that if \( v = v^i e_i \in \mathfrak{g} \), then

\[
\varphi_1 = G_1[v^1, v^2, v^3, v^4, v^5, v^6, v^7]^T \quad \text{and} \quad \varphi_2 = G_2[v^1, v^2, v^3, v^4, v^5, v^6, v^7]^T.
\]
As before, we seek all 5-dimensional subalgebras of $\mathfrak{g}$, and do so by writing five arbitrary vectors $X_i = A_i^j e_j$ such that

$$[X_1, X_2, X_3, X_4, X_5] = [e_1, e_2, e_3, e_4, e_5, e_6, e_7],$$

and requiring that the matrix $A = [A_i^j]$ is full rank along with the additional requirement that the resulting vectors $X_i$ satisfy the Jacobi identity.

We begin by letting $\varphi_1$ act on the vectors $X_i$. In doing so, we see that the row vectors $A^6$ and $A^7$ are invariant, and so, we consider cases depending upon the dimension of the possible subspaces formed by these two vectors in $\mathbb{R}^5$. We note that in order to preserve the structure of our subalgebras, we only allow column operations in our analysis. This leads to the following five normal forms for the $2 \times 5$ block $S$ generated by $A^6$ and $A^7$,

- $\text{rank}(S) = 0 : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,
- $\text{rank}(S) = 1 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \lambda \neq 0$,
Case I

When the subspace generated by $A^6$ and $A^7$ is 0-dimensional, we see that both $A^6$ and $A^7$ must be identically zero, and immediately obtain the 5-dimensional subspace

$$\Sigma_1 = \text{span}\{e_1, e_2, e_3, e_4, e_5\}.$$ 

This subspace additionally satisfies the Jacobi identity, and therefore defines the subalgebra

$$a_1 = \text{span}\{e_1, e_2, e_3, e_4, e_5\}.$$ 

Case II

In the case where $S$ is of the form

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we see that the coefficient matrix $A$ has the initial form

$$A = \begin{pmatrix} A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\ A_1^2 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\ A_1^3 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\ A_1^4 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\ A_1^5 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

We can transform the vector $X_1$ to

$$\tilde{X}_1 = \varphi_1(X_1) = e_6 + A_1^1 e_1,$$
by setting
\[ x_1 = -A_1^5, \quad x_2 = \frac{A_1^4}{\mu}, \quad x_3 = \frac{A_1^3}{\mu}, \quad x_4 = -A_1^2. \]

After renaming \( A = G_1 A \), we see that \( A \) is of the form
\[
A = \begin{pmatrix}
A_1^1 & A_1^2 & A_1^3 & A_1^4 & A_1^5 \\
0 & A_2^1 & A_2^2 & A_2^3 & A_2^4 \\
0 & A_3^1 & A_3^2 & A_3^3 & A_3^4 \\
0 & A_4^1 & A_4^2 & A_4^3 & A_4^4 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Now, the vector \( A^5 \) becomes \( \varphi_1 \)-invariant, and we consider cases where either \( A^5 = 0 \) or not.

If \( A^5 = 0 \), then we may take as our 5-dimensional subspace,
\[
\Sigma_2 = \text{span}\{e_1, e_2, e_3, e_4, e_6\}
\]
which additionally satisfies the Jacobi identity, and so we obtain the subalgebra
\[
a_2 = \text{span}\{e_1, e_2, e_3, e_4, e_6\}.
\]

If \( A^5 \neq 0 \), then by renormalizing and taking linear combinations of the vectors \( X_i \), we can take it to the form \( A^5 = [0, 1, 0, 0, 0] \). At this point the five vectors are of the form
\[
X_1 = e_6 + A_1^1 e_1, \quad X_2 = e_5 + A_2^1 e_1 + A_2^2 e_2 + A_2^3 e_3 + A_2^4 e_4, \\
X_i = A_i^1 e_1 + A_i^2 e_2 + A_i^3 e_3 + A_i^4 e_4, \quad 3 \leq i \leq 5.
\]
From here, we see that there are only three possible subspaces of this form which satisfy the Jacobi identity, namely

\[ a_3 = \text{span}\{e_1, e_2, e_3, e_5, e_6\}, \]
\[ a_4 = \text{span}\{e_1, e_2, e_4, e_5, e_6\}, \]
\[ a_5 = \text{span}\{e_1, e_3, e_4, e_5, e_6\}. \]

Case III

In the case where \( S \) is of the form

\[
S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

we see that the coefficient matrix \( A \) has the initial form

\[
A = \begin{pmatrix}
A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
A_1^2 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
A_1^3 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
A_1^4 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
A_1^5 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

By letting \( \varphi_1 \) act, we see that we can transform the vector \( X_1 \) to the vector

\[ \tilde{X}_1 = \varphi_1(X_1) = e_7, \]

by setting

\[ x_1 = A_1^5, \quad x_2 = A_1^4, \quad x_3 = -A_1^3, \quad x_4 = -A_1^2, \quad x_5 = -A_1^1. \]
After renaming $A = G_1 A$, we see

$$
A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
0 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Now, the vector $A^5$ becomes $\varphi_1$-invariant, and so we consider cases where either $A^5 = 0$ or not.

When $A^5 = 0$, we immediately obtain the subspace

$$
\Sigma_6 = \text{span}\{e_1, e_2, e_3, e_4, e_7\}
$$

which additionally satisfies the Jacobi identity, and therefore gives the subalgebra

$$
a_6 = \text{span}\{e_1, e_2, e_3, e_4, e_7\}.
$$

When $A^5 \neq 0$, by remormalizing an taking linear combinations of the vectors $X_i$, we can take it to the form $A^5 = [0, 1, 0, 0, 0]$. The five vectors are then of the form

$$
X_1 = e_7, \quad X_2 = e_5 + A_2^1 e_1 + A_2^2 e_2 + A_2^3 e_3 + A_2^4 e_4, \\
X_j = A_j^1 e_1 + A_j^2 e_2 + A_j^3 e_3 + A_j^4 e_4, \quad 3 \leq j \leq 5.
$$
From here, we see that in order for the subspaces to satisfy the Jacobi identity, they must be of one of the following forms:

\[
\Sigma^{(\alpha, \beta, \gamma)}_{7, 1} = \operatorname{span}\{e_1, e_2 + \alpha e_4, e_3 + \beta e_4, e_5 + \gamma e_4, e_7\},
\]
\[
\Sigma^{(\alpha, \beta, \gamma)}_{7, 2} = \operatorname{span}\{e_1, e_2 + \alpha e_3, e_4 + \beta e_3, e_5 + \gamma e_3, e_7\},
\]
\[
\Sigma^{(\alpha, \beta, \gamma)}_{7, 3} = \operatorname{span}\{e_1, e_3 + \alpha e_2, e_4 + \beta e_2, e_5 + \gamma e_2, e_7\}.
\]

These subspaces can be seen to be identical for several parameter values, however. To give a list of subalgebras generated from these subspaces which are inequivalent for all values of \(\alpha, \beta, \gamma\), we begin by considering cases where either 0, 1, 2, or all 3 of the parameter values are zero, and see which are equivalent under the action of \(\varphi_2\).

When \(\alpha = \beta = \gamma = 0\), we immediately obtain the following inequivalent subspaces:

\[
\Sigma^{(0, 0, 0)}_{7, 1} = \operatorname{span}\{e_1, e_3, e_4, e_5, e_7\},
\]
\[
\Sigma^{(0, 0, 0)}_{7, 2} = \operatorname{span}\{e_1, e_2, e_4, e_5, e_7\},
\]
\[
\Sigma^{(0, 0, 0)}_{7, 3} = \operatorname{span}\{e_1, e_2, e_3, e_5, e_7\}.
\]

When two of \(\alpha, \beta, \gamma\) are zero and the other is nonzero, after normalizing the remaining parameter using the action of \(\varphi_2\), we obtain the following inequivalent subspaces:

\[
\Sigma^{(\alpha, 0, 0)}_{7, 1} = \operatorname{span}\{e_1, e_2 + e_3, e_4, e_5, e_7\} = \Sigma^{(\alpha, 0, 0)}_{7, 2},
\]
\[
\Sigma^{(0, \beta, 0)}_{7, 1} = \operatorname{span}\{e_1, e_2 + e_4, e_3, e_5, e_7\} = \Sigma^{(0, \beta, 0)}_{7, 3},
\]
\[
\Sigma^{(0, 0, \gamma)}_{7, 1} = \operatorname{span}\{e_1, e_2 + e_5, e_3, e_4, e_7\},
\]
\[
\Sigma^{(0, \beta, 0)}_{7, 2} = \operatorname{span}\{e_1, e_2, e_3 + e_4, e_5, e_7\} = \Sigma^{(0, \beta, 0)}_{7, 3},
\]
\[
\Sigma^{(0, 0, \gamma)}_{7, 2} = \operatorname{span}\{e_1, e_2, e_3 + e_5, e_4, e_7\},
\]
\[
\Sigma^{(0, 0, \gamma)}_{7, 3} = \operatorname{span}\{e_1, e_2, e_3, e_4 + e_5, e_7\}.
\]
Similarly, when only one of the parameter values is zero, we obtain after normalizing using the action of $\varphi_2$, the following inequivalent subspaces:

$$\Sigma_{7,1}^{(\alpha,\beta,\gamma)} = \text{span}\{e_1, e_2 + e_4, e_3 + e_5, e_7\} = \Sigma_{7,2}^{(\alpha,\beta,\gamma)} = \Sigma_{7,3}^{(\alpha,\beta,\gamma)},$$

$$\Sigma_{7,1}^{(\alpha,\gamma)} = \text{span}\{e_1, e_2 + e_5, e_3 + e_4, e_7\} = \Sigma_{7,2}^{(\alpha,\gamma)},$$

$$\Sigma_{7,1}^{(0,\beta,\gamma)} = \text{span}\{e_1, e_2 + e_5, e_3 + e_4, e_5\} = \Sigma_{7,3}^{(\alpha,\beta,\gamma)},$$

$$\Sigma_{7,2}^{(0,\beta,\gamma)} = \text{span}\{e_1, e_2, e_3 + e_5, e_4 + e_5, e_7\} = \Sigma_{7,3}^{(0,\beta,\gamma)}. $$

Finally, when none of the parameter values are zero, we obtain after normalization by $\varphi_2$, the single subspace

$$\Sigma_{7,1}^{(\alpha,\beta,\gamma)} = \text{span}\{e_1, e_2 + e_5, e_3 + e_5, e_4 + \lambda e_5, e_7\} = \Sigma_{7,2}^{(\alpha,\beta,\gamma)} = \Sigma_{7,3}^{(\alpha,\beta,\gamma)}, \quad \lambda \neq 0.$$
This gives the following complete list of inequivalent subalgebras of this form as

\[
\begin{align*}
\mathfrak{a}_7 &= \text{span}\{e_1, e_2, e_3, e_5, e_7\}, \\
\mathfrak{a}_8 &= \text{span}\{e_1, e_2, e_4, e_5, e_7\}, \\
\mathfrak{a}_9 &= \text{span}\{e_1, e_3, e_4, e_5, e_7\}, \\
\mathfrak{a}_{10} &= \text{span}\{e_1, e_2, e_3, e_4 + e_5, e_7\}, \\
\mathfrak{a}_{11} &= \text{span}\{e_1, e_2, e_3 + e_5, e_4, e_7\}, \\
\mathfrak{a}_{12} &= \text{span}\{e_1, e_2 + e_4, e_3, e_5, e_7\}, \\
\mathfrak{a}_{13} &= \text{span}\{e_1, e_2 + e_5, e_3, e_4, e_7\}, \\
\mathfrak{a}_{14} &= \text{span}\{e_1, e_2 + e_3, e_4, e_5, e_7\}, \\
\mathfrak{a}_{15} &= \text{span}\{e_1, e_2, e_3 + e_5, e_4 + e_7\}, \\
\mathfrak{a}_{16} &= \text{span}\{e_1, e_2 + e_5, e_3, e_4 + e_5, e_7\}, \\
\mathfrak{a}_{17} &= \text{span}\{e_1, e_2 + e_5, e_3 + e_5, e_4, e_7\}, \\
\mathfrak{a}_{18} &= \text{span}\{e_1, e_2 + e_4, e_3 + e_4, e_5, e_7\}, \\
\mathfrak{a}_{19} &= \text{span}\{e_1, e_2 + e_5, e_3 + e_5, e_4 + \lambda e_5, e_7\}, \quad \lambda \neq 0.
\end{align*}
\]

Case IV

In the case where \(S\) is of the form

\[
S = \begin{pmatrix}
\lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \lambda \neq 0,
\]

\[\lambda \neq 0,\]
we see that the coefficient matrix $A$ has the initial form

$$A = \begin{pmatrix}
A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
A_1^2 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
A_1^3 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
A_1^4 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
A_1^5 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
\lambda & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix} \cdot$$

Upon applying $\varphi_1$, we see that we can transform the vector $X_1$ to the vector

$$\tilde{X}_1 = \varphi_1(X_1) = e_7 + \lambda e_6,$$

provided $\lambda \neq \pm 1, \pm \mu^{-1}$, by setting

\begin{align*}
x_1 &= -\frac{A_1^1}{\lambda - 1}, \quad x_2 = \frac{A_1^2}{\lambda \mu + 1}, \quad x_3 = \frac{A_1^3}{\lambda \mu - 1}, \quad x_4 = -\frac{A_1^4}{\lambda + 1}, \\
x_5 &= \frac{-A_1^1 \lambda^4 \mu^2 + A_1^2 A_5^1 \lambda^3 \mu^2 - A_1^3 A_4^1 \lambda^3 \mu + A_1^4 A_3^1 \lambda \mu + A_1^5 A_2^1 \lambda - A_1^0}{(\lambda^2 \mu^2 - 1)(\lambda^2 - 1)}.
\end{align*}

After renaming $A = G_1 A$, we see

$$A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
0 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
\lambda & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},$$
so that the vectors $X_i$ are of the form

\[
X_1 = e_7 + \lambda e_6, \\
X_j = A_j^1e_1 + A_j^2e_2 + A_j^3e_3 + A_j^4e_4 + A_j^5e_5, \quad 2 \leq j \leq 5.
\]

We then see that by requiring these vectors satisfy the Jacobi identity, the only possible subalgebras generated by the $X_i$ are

\[
\mathfrak{a}_{20} = \text{span}\{e_1, e_2, e_4, e_5, e_7 + \lambda e_6\}, \\
\mathfrak{a}_{21} = \text{span}\{e_1, e_2, e_3, e_4, e_7 + \lambda e_6\}, \\
\mathfrak{a}_{22} = \text{span}\{e_1, e_3, e_4, e_5, e_7 + \lambda e_6\}, \\
\mathfrak{a}_{23} = \text{span}\{e_1, e_2, e_3, e_5, e_7 + \lambda e_6\}.
\]

**Special Case: $\lambda = 1$**

In the special case where $\lambda = 1$, we can transform the vector $X_1$ to

\[
\tilde{X}_1 = \varphi_1(X_1) = e_7 + e_6 + \tau e_5
\]

by setting

\[
x_2 = \frac{A_1^4}{\mu + 1}, \quad x_3 = \frac{A_1^3}{\mu - 1}, \quad x_4 = -\frac{A_1^2}{2}, \\
x_5 = \frac{A_1^2\tau \mu^2 - A_1^2\mu^2 x_1 - 2A_1^4 \mu^2 - 2A_1^3 A_1^4 \mu - A_1^2 \tau + A_1^2 x_1 + 2A_1^1}{2 (\mu + 1)(\mu - 1)}
\]
After renaming $A = G_1 A$, we see

$$A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
\tau & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
\lambda & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

The row vector $A^4$ is then seen to be $\varphi_1$-invariant, and so we consider cases depending on whether or not $A^4 = 0$. 

If $A^4 = 0$, then we immediately obtain the subspace

$$\Sigma_{24} = \text{span}\{e_1, e_2, e_3, e_5, e_6 + e_7\}$$

which additionally satisfies the Jacobi identity giving the subalgebra

$$\mathfrak{a}_{24} = \text{span}\{e_1, e_2, e_3, e_5, e_6 + e_7\}.$$  

If $A^4 \neq 0$, then by renormalizing and taking linear combinations, we can take it to $A^4 = [0, 1, 0, 0, 0]$. From here, we see that the vectors are of the form

$$X_1 = e_7 + \lambda e_6 + \tau e_5, \quad X_2 = e_4 + A_2^1 e_1 + A_2^2 e_2 + A_2^3 e_3 + A_2^5 e_5,$$

$$X_j = A_1^j e_1 + A_2^j e_2 + A_3^j e_3 + A_5^j e_5, \quad 3 \leq j \leq 5,$$
and requiring that they satisfy the Jacobi identity gives the subalgebras

\[ a_{25} = \text{span}\{e_1, e_2, e_4, e_5, e_6 + e_7\}, \]
\[ a_{26} = \text{span}\{e_1, e_3, e_4, e_5, e_6 + e_7\}, \]
\[ \text{span}\{e_1, e_2, e_3, e_4, e_7 + e_6 + \tau e_5\}, \]

the third of which can be normalized to one of two subalgebras depending on whether or not \( \tau = 0 \). If \( \tau = 0 \), then we obtain the subalgebra

\[ a_{27} = \text{span}\{e_1, e_2, e_4, e_6 + e_7\}. \]

If \( \tau \neq 0 \), then by let \( \varphi_2 \) act and taking \( z_2 = \tau z_1 \), we obtain the subalgebra

\[ a_{28} = \text{span}\{e_1, e_2, e_3, e_4, e_5 + e_6 + e_7\}. \]

**Special Case: \( \lambda = -1 \)**

In the special case where \( \lambda = -1 \), an almost identical analysis can be preformed yielding the following subalgebras:

\[ a_{29} = \text{span}\{e_1, e_2, e_3, e_4, e_6 - e_7\}, \]
\[ a_{30} = \text{span}\{e_1, e_2, e_3, e_4, e_5 - e_7\}, \]
\[ a_{31} = \text{span}\{e_1, e_2, e_3, e_5, e_6 - e_7\}, \]
\[ a_{32} = \text{span}\{e_1, e_3, e_4, e_5, e_6 - e_7\}, \]
\[ a_{33} = \text{span}\{e_1, e_3, e_4, e_5, e_6 - e_7 - e_2\}. \]

**Special Case: \( \lambda = \mu^{-1} \)**

In the special case where \( \lambda = \mu^{-1} \), we can transform the vector \( X_1 \) to

\[ \tilde{X}_1 = \varphi_1(X_1) = e_7 + \mu^{-1}e_6 + \tau e_3, \]
by setting

\[ x_1 = \frac{\tau \mu}{\mu - 1}, \quad x_2 = \frac{A_4^1}{2}, \quad x_4 = -\frac{A_3^1 \mu}{\mu + 1}, \]

\[ x_5 = -\frac{1}{2} \frac{A_3^3 A_4^1 \mu^2 + A_4^1 \mu^2 x_3 + 2 A_4^1 \mu^2 + 2 A_4^2 \mu - A_4^1 A_3^1 - x_3 A_4^1 - 2 A_4^1}{\mu^2 - 1}. \]

After renaming \( A = G_1 A \), we see

\[
A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
\tau & A_3^2 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_4^2 & A_3^4 & A_4^4 & A_5^4 \\
0 & A_5^2 & A_5^3 & A_4^5 & A_5^5 \\
\mu^{-1} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

and the vectors \( X_i \) are of the form

\[ X_1 = e_7 + \mu^{-1} e_6 + \tau e_3, \quad X_j = A_1^j e_1 + A_2^2 e_2 + A_4^3 e_3 + A_4^4 e_4 + A_5^5 e_5, \quad 2 \leq j \leq 5. \]

Requiring that these vectors satisfy the Jacobi identity gives the following subalgebras:

\[ a_{34} = \text{span}\{e_1, e_2, e_3, e_4, e_6 + \mu e_7\}, \]

\[ a_{35} = \text{span}\{e_1, e_3, e_4, e_5, e_6 + \mu e_7\}, \]

\[ a_{36} = \text{span}\{e_1, e_2, e_3, e_5, e_6 + \mu e_7\}, \]

\[ \text{span}\{e_1, e_2, e_4, e_5, e_6 + \mu e_7 + \tau e_3\}. \]
The last subalgebra can be normalized to one of two subalgebras depending on whether or not \( \tau = 0 \). If \( \tau = 0 \), then we obtain the subalgebra

\[
a_{37} = \text{span}\{e_1, e_2, e_4, e_5, e_6 + \mu e_7\}.
\]

If \( \tau \neq 0 \), then by letting \( \varphi_2 \) act and taking \( z_3 = 1/\mu \tau \), we obtain the subalgebra

\[
a_{38} = \text{span}\{e_1, e_2, e_4, e_5, e_6 + \mu e_7 + e_3\}.
\]

**Special Case: \( \lambda = -\mu^{-1} \)**

In the special case where \( \lambda = -\mu^{-1} \), an almost identical analysis can be performed yielding the subalgebras

\[
\begin{align*}
  a_{39} &= \text{span}\{e_1, e_2, e_3, e_4, e_6 - \mu e_7\}, \\
  a_{40} &= \text{span}\{e_1, e_2, e_4, e_5, e_6 - \mu e_7\}, \\
  a_{41} &= \text{span}\{e_1, e_3, e_4, e_5, e_6 - \mu e_7\}, \\
  a_{42} &= \text{span}\{e_1, e_2, e_3, e_5, e_6 - \mu e_7\}, \\
  a_{43} &= \text{span}\{e_1, e_2, e_3, e_5, e_6 - \mu e_7 + e_4\}.
\end{align*}
\]

**Case V**

Finally, in the case where \( S \) is of the form

\[
S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
\]
we see that the coefficient matrix $A$ has the initial form

$$A = \begin{pmatrix}
A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
A_1^2 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
A_1^3 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
A_1^4 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
A_1^5 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}. $$

Upon applying $\varphi_1$, we see that we can transform the vector $X_1$ to the vector $\tilde{X}_1 = \varphi_1(X_1) = e_7,$ by setting

$$x_1 = A_1^5, \quad x_2 = A_1^4, \quad x_3 = -A_1^3, \quad x_4 = -A_1^2, \quad x_5 = -A_1^1. $$

After renaming $A = G_1 A$, we see that $A$ is of the form

$$A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
0 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}. $$
and the vectors $X_i$ are

$$X_1 = e_7, \quad X_2 = e_6 + A^1_2 e_1 + A^2_2 e_2 + A^3_2 e_3 + A^4_2 e_4 + A^5_2 e_5,$$

$$X_j = A^j_1 e_1 + A^j_2 e_2 + A^j_3 e_3 + A^j_4 e_4 + A^j_5 e_5, \quad 3 \leq j \leq 5.$$

Requiring that these vectors satisfy the Jacobi identity immediately yields the following subalgebras

$$\mathfrak{a}_{44} = \text{span}\{e_1, e_3, e_5, e_6, e_7\},$$

$$\mathfrak{a}_{45} = \text{span}\{e_1, e_3, e_4, e_6, e_7\},$$

$$\mathfrak{a}_{46} = \text{span}\{e_1, e_2, e_3, e_6, e_7\},$$

$$\mathfrak{a}_{47} = \text{span}\{e_1, e_2, e_4, e_6, e_7\},$$

$$\mathfrak{a}_{48} = \text{span}\{e_1, e_2, e_5, e_6, e_7\}.$$

This concludes our classification of all inequivalent 5-dimensional subalgebras of the symmetry algebra of $z' = (y'')^m$ where $m \neq -1, 0, 1, \frac{1}{3}, \frac{2}{3}, 2, \text{ or } \frac{1}{2}$. We tabulate these subalgebras in Table 10.1 below.
Table 10.1: 5-Dimensional Subalgebras of the Symmetry Algebra of $z' = (y'')^m$

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Vanishing Lie Determinant</th>
</tr>
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<tbody>
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<td>$a_1$</td>
<td>$e_1, e_2, e_3, e_4, e_5$</td>
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</tr>
<tr>
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<td>$e_1, e_2, e_3, e_4, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$e_1, e_2, e_3, e_5, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$e_1, e_2, e_4, e_5, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$e_1, e_3, e_4, e_5, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_6$</td>
<td>$e_1, e_2, e_3, e_4, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_7$</td>
<td>$e_1, e_2, e_3, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_8$</td>
<td>$e_1, e_2, e_4, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_9$</td>
<td>$e_1, e_3, e_4, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{10}$</td>
<td>$e_1, e_2, e_3, e_4 + e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>$e_1, e_2, e_3 + e_5, e_4, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>$e_1, e_2 + e_4, e_3, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{13}$</td>
<td>$e_1, e_2 + e_5, e_3, e_4, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{14}$</td>
<td>$e_1, e_2 + e_3, e_4, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{15}$</td>
<td>$e_1, e_2 + e_3 + e_5, e_4 + e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{16}$</td>
<td>$e_1, e_2 + e_5, e_3, e_4 + e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{17}$</td>
<td>$e_1, e_2 + e_5, e_3 + e_5, e_4, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{18}$</td>
<td>$e_1, e_2 + e_4, e_3 + e_4, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{19}$</td>
<td>$e_1, e_2 + e_5, e_3 + e_5, e_4 + \lambda e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{20}$</td>
<td>$e_1, e_2, e_3, e_5, e_7 + \lambda e_6$</td>
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</tr>
<tr>
<td>$a_{21}$</td>
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<td>✓</td>
</tr>
<tr>
<td>$a_{22}$</td>
<td>$e_1, e_3, e_4, e_5, e_7 + \lambda e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{23}$</td>
<td>$e_1, e_2, e_3, e_5, e_7 + \lambda e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{24}$</td>
<td>$e_1, e_2, e_3, e_5, e_6 + e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{25}$</td>
<td>$e_1, e_2, e_4, e_5, e_6 + e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{26}$</td>
<td>$e_1, e_3, e_4, e_5, e_6 + e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{27}$</td>
<td>$e_1, e_2, e_5, e_4, e_6 + e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{28}$</td>
<td>$e_1, e_2, e_3, e_4, e_5 + e_6 + e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{29}$</td>
<td>$e_1, e_2, e_3, e_4, e_6 - e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{30}$</td>
<td>$e_1, e_2, e_4, e_5, e_6 - e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{31}$</td>
<td>$e_1, e_2, e_3, e_5, e_6 - e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{32}$</td>
<td>$e_1, e_3, e_4, e_5, e_6 - e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{33}$</td>
<td>$e_1, e_3, e_4, e_5, e_6 - e_7 - e_2$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{34}$</td>
<td>$e_1, e_2, e_3, e_4, e_6 + \mu e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{35}$</td>
<td>$e_1, e_3, e_4, e_5, e_6 + \mu e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{36}$</td>
<td>$e_1, e_2, e_3, e_5, e_6 + \mu e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{37}$</td>
<td>$e_1, e_2, e_4, e_5, e_6 + \mu e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{38}$</td>
<td>$e_1, e_2, e_4, e_5, e_6 + \mu e_7 + e_3$</td>
<td>✓</td>
</tr>
<tr>
<td>$a_{39}$</td>
<td>$e_1, e_2, e_3, e_4, e_6 - \mu e_7$</td>
<td>✓</td>
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</table>
5-Dimensional Subalgebras of the Symmetry Algebra of $z' = (y'')^m$ (cont.)

<table>
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<th>Generators</th>
<th>Vanishing Lie Determinant</th>
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<tbody>
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<td>$a_{40}$</td>
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<tr>
<td>$a_{41}$</td>
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<tr>
<td>$a_{42}$</td>
<td>$e_1, e_2, e_3, e_5, e_6 - \mu e_7$</td>
<td></td>
</tr>
<tr>
<td>$a_{43}$</td>
<td>$e_1, e_2, e_3, e_5, e_6 - \mu e_7 + e_4$</td>
<td></td>
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<tr>
<td>$a_{44}$</td>
<td>$e_1, e_3, e_5, e_6, e_7$</td>
<td></td>
</tr>
<tr>
<td>$a_{45}$</td>
<td>$e_1, e_3, e_4, e_6, e_7$</td>
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</tr>
<tr>
<td>$a_{46}$</td>
<td>$e_1, e_2, e_3, e_6, e_7$</td>
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</tr>
<tr>
<td>$a_{47}$</td>
<td>$e_1, e_2, e_4, e_6, e_7$</td>
<td></td>
</tr>
<tr>
<td>$a_{48}$</td>
<td>$e_1, e_2, e_5, e_6, e_7$</td>
<td></td>
</tr>
</tbody>
</table>

Note: All parameter values $\lambda$ are nonzero and $\mu = 2m - 1$ where $m \neq -1, 0, 1, \frac{1}{3}, \frac{2}{3}, 2, \text{ or } \frac{1}{2}$. The $e_i = \Gamma_i$ where

$$
\begin{align*}
\Gamma_1 &= \frac{1}{m-1} \partial_y, \\
\Gamma_2 &= \frac{m y_2^{m-1}}{m-1} \partial_x + \frac{m y_1 y_2^{m-1}}{2m-1} \partial_z + \frac{m y_1 y_2^{m-1} - z}{m-1} \partial_y + y_2^m \partial_z, \\
\Gamma_3 &= \frac{x}{m-1} \partial_y + \frac{1}{m-1} \partial_y, \quad \Gamma_4 = \partial_x, \quad \Gamma_5 = \partial_z, \\
\Gamma_6 &= -x \partial_x + (2m - 1) z \partial_z + y_1 \partial y_1 + 2y_2 \partial y_2, \\
\Gamma_7 &= x \partial_x + z \partial_z + 2y \partial y + y_1 \partial y_1.
\end{align*}
$$
We now repeat this classification in the special case where \( m = \frac{1}{2} \). Due to the similarities in the classification, we only provide the generators for the symmetry algebra, the subgroups of the automorphism group used in our normalizations, and the list of inequivalent subalgebras of the symmetry algebra.

The symmetry algebra \( \mathfrak{g} \) of (10.4) is again 7-dimensional, and in terms of the coordinates \( x, z, y, y_1, y_2 \), is generated by the vector fields

\[
\begin{align*}
\Gamma_1 &= 2\partial_y, \\
\Gamma_2 &= 2\partial_x, \\
\Gamma_3 &= \partial_z, \\
\Gamma_4 &= x\partial_y + \partial_{y_1}, \\
\Gamma_5 &= -\frac{1}{\sqrt{y_2}}\partial_x + \frac{\ln y_2}{2}\partial_z + \left(2z - \frac{y_1}{\sqrt{y_2}}\right)\partial_y + \sqrt{y_2}\partial_{y_1}, \\
\Gamma_6 &= -x\partial_x + y_1\partial_{y_1} + 2y_2\partial_{y_2}, \\
\Gamma_7 &= x\partial_x + z\partial_z + 2y\partial_y + y_1\partial_{y_1}.
\end{align*}
\]

If we let \( e_i = \Gamma_i \) for \( 1 \leq i \leq 7 \), then the multiplication table becomes

<table>
<thead>
<tr>
<th></th>
<th>( e_1 )</th>
<th>( e_2 )</th>
<th>( e_3 )</th>
<th>( e_4 )</th>
<th>( e_5 )</th>
<th>( e_6 )</th>
<th>( e_7 )</th>
</tr>
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<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>( 2e_1 )</td>
</tr>
<tr>
<td>( e_2 )</td>
<td>\cdot</td>
<td>\cdot</td>
<td>( e_1 )</td>
<td>\cdot</td>
<td>( -e_2 )</td>
<td>( e_2 )</td>
<td>\cdot</td>
</tr>
<tr>
<td>( e_3 )</td>
<td>\cdot</td>
<td>\cdot</td>
<td>( e_1 )</td>
<td>\cdot</td>
<td>( e_3 )</td>
<td>\cdot</td>
<td>\cdot</td>
</tr>
<tr>
<td>( e_4 )</td>
<td>\cdot</td>
<td>\cdot</td>
<td>( e_4 )</td>
<td>\cdot</td>
<td>( e_4 )</td>
<td>\cdot</td>
<td>\cdot</td>
</tr>
<tr>
<td>( e_5 )</td>
<td>\cdot</td>
<td>\cdot</td>
<td>( -e_3 )</td>
<td>( e_5 )</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
</tr>
<tr>
<td>( e_6 )</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
</tr>
<tr>
<td>( e_7 )</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
</tr>
</tbody>
</table>
with

\[
\text{ad}(e_6) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

so that we again correspond to Case (i) with \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \).

We split the full automorphism group \( \mathfrak{g} \) of \( \mathfrak{g} \) into two subgroups \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) corresponding to the nondiagonal and diagonal matrices

\[
G_1 = \begin{pmatrix}
1 & x_1 & x_2 & x_3 & x_4 & x_6 \\
0 & 1 & 0 & 0 & 0 & -x_4 \\
0 & 0 & 1 & 0 & 0 & x_2 \\
0 & 0 & 0 & 1 & 0 & x_1 \\
0 & 0 & 0 & 0 & 1 & -x_1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
G_2 = \begin{pmatrix}
z_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & z_1 & 0 & 0 & 0 & 0 \\
0 & 0 & z_2 & 0 & 0 & 0 \\
0 & 0 & 0 & z_2/z_1 & 0 & 0 \\
0 & 0 & 0 & 0 & z_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

We then proceed as before with our analysis of the 5-dimensional subalgebras of \( \mathfrak{g} \). The results are given in Table 10.2 below.
<table>
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<th>Index</th>
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<tr>
<td>b₃</td>
<td>$e_1, e_3, e_4, e_5, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>b₄</td>
<td>$e_1, e_2, e_3, e_4, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>b₅</td>
<td>$e_1, e_2, e_3, e_4, e_5 + e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>b₆</td>
<td>$e_1, e_2, e_3, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>b₇</td>
<td>$e_1, e_2, e_3, e_4, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>b₈</td>
<td>$e_1, e_2, e_3, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>b₉</td>
<td>$e_1, e_3, e_4, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>b₁₀</td>
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<td>✓</td>
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<tr>
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<td>✓</td>
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<tr>
<td>b₁₃</td>
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<td>b₁₄</td>
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<tr>
<td>b₁₅</td>
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<td>✓</td>
</tr>
<tr>
<td>b₁₆</td>
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<tr>
<td>b₁₇</td>
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<tr>
<td>b₁₈</td>
<td>$e_1, e_3, e_4, e_5, e_7 + \alpha e_6$</td>
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<tr>
<td>b₁₉</td>
<td>$e_1, e_2, e_3, e_4, e_6 + e_7$</td>
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<td>b₂₄</td>
<td>$e_1, e_3, e_4, e_5, e_6 - e_7$</td>
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<tr>
<td>b₂₅</td>
<td>$e_1, e_2, e_3, e_5, e_6 - e_7$</td>
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<tr>
<td>b₂₆</td>
<td>$e_1, e_2, e_3, e_5, e_6 - e_7 - e_4$</td>
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<tr>
<td>b₂₇</td>
<td>$e_1, e_2, e_4, e_6, e_7$</td>
<td></td>
</tr>
<tr>
<td>b₂₈</td>
<td>$e_1, e_2, e_3, e_6, e_7$</td>
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<tr>
<td>b₂₉</td>
<td>$e_1, e_3, e_5, e_6, e_7$</td>
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</tr>
<tr>
<td>b₃₀</td>
<td>$e_1, e_3, e_4, e_6, e_7$</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** All parameter values $\alpha, \lambda, \tau$ are nonzero. Additionally, $\alpha \neq \pm 1$. The $e_i = \Gamma_i$ where

\[
\begin{align*}
\Gamma_1 &= 2\partial_y, \quad \Gamma_2 = 2\partial_x, \quad \Gamma_3 = \partial_z, \quad \Gamma_4 = x\partial_y + \partial_{y_1}, \\
\Gamma_5 &= -\frac{1}{\sqrt{y_2}}\partial_x + \frac{\ln y_2}{2}\partial_z + \left(2z - \frac{y_1}{\sqrt{y_2}}\right)\partial_y + \sqrt{y_2}\partial_{y_1}, \\
\Gamma_6 &= -x\partial_x + y_1\partial_{y_1} + 2y_2\partial_{y_2}, \quad \Gamma_7 = x\partial_x + z\partial_z + 2y\partial_y + y_1\partial_{y_1}.
\end{align*}
\]
10.4.2 Subalgebras of the Symmetry Algebra of $z' = \ln(y'')$ 

In this section, we consider the 5-dimensional subalgebras of the symmetry algebra of the Monge equation

\[ z' = \ln(y''). \]  

(10.5)

The symmetry algebra $g$ of (10.5) is 7-dimensional, and in terms of the jet coordinates $x, z, y, y_1, y_2$, is generated by the following vector fields

\[
\begin{align*}
\Gamma_1 &= \partial_y, \quad \Gamma_2 = -\partial_x + \partial_z, \quad \Gamma_3 = 2\partial_z, \\
\Gamma_4 &= -\frac{1}{2y_2} \partial_x - \frac{\ln(y_2)}{2y_2} + \frac{1}{2} \partial_z + \frac{x y_2 + z y_2 - y_1}{2y_2} \partial_y + \frac{\ln(y_2)}{2} \partial_{y_1}, \\
\Gamma_5 &= -x \partial_y - \partial_{y_1}, \quad \Gamma_6 = x \partial_x - (2x - z) \partial_z - y_1 \partial_{y_1} - y_2 \partial_{y_2}, \\
\Gamma_7 &= x \partial_x + z \partial_z + 2y \partial_y + y_1 \partial_{y_1}.
\end{align*}
\]

If we let $e_i = \Gamma_i$ for $1 \leq i \leq 7$, then the multiplication table for $g$ is precisely that of case (ii),

\[
\begin{array}{cccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2e_1 \\
e_2 & \cdot & \cdot & \cdot & e_1 & e_2 + e_3 & \cdot & \cdot \\
e_3 & \cdot & e_1 & \cdot & \cdot & \cdot & e_3 & \cdot \\
e_4 & \cdot & \cdot & \cdot & -e_4 - e_5 & e_4 & \cdot & \cdot \\
e_5 & \cdot & \cdot & \cdot & \cdot & \cdot & e_5 & \cdot \\
e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]
We can then calculate the full automorphism group $\mathfrak{G}$ of $\mathfrak{g}$ and split $\mathfrak{G}$ into the two subgroups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ corresponding to the nondiagonal and diagonal matrices

$$G_1 = \begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_1 x_4 + x_2 x_3 + x_2 x_4 & x_5 \\ 0 & 1 & 0 & 0 & 0 & x_4 & x_4 \\ 0 & x_6 & 1 & 0 & 0 & x_3 + x_4 + x_4 x_6 & x_3 + x_4 x_6 \\ 0 & 0 & 0 & 1 & 0 & x_2 & -x_2 \\ 0 & 0 & 0 & -x_6 & 1 & x_1 + x_2 - x_2 x_6 & x_2 x_6 - x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$G_2 = \begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1/z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1/z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

We denote the actions of the matrix groups on $\mathfrak{g}$ by $\varphi_1$ and $\varphi_2$, respectively, such that if $v = v_i e_i \in \mathfrak{g}$, then

$$\varphi_1 = G_1[v^1, v^2, v^3, v^4, v^5, v^6, v^7]^T \quad \text{and} \quad \varphi_2 = G_2[v^1, v^2, v^3, v^4, v^5, v^6, v^7]^T.$$
As before, we seek all 5-dimensional subalgebras of \( g \), and do so by writing five arbitrary vectors \( X_i = A^j_i e_j \) such that

\[
[X_1, X_2, X_3, X_4, X_5] = [e_1, e_2, e_3, e_4, e_5, e_6, e_7]
\]

\[
\begin{pmatrix}
A_1^1 & A_1^2 & A_1^3 & A_1^4 & A_1^5 \\
A_2^1 & A_2^2 & A_2^3 & A_2^4 & A_2^5 \\
A_3^1 & A_3^2 & A_3^3 & A_3^4 & A_3^5 \\
A_4^1 & A_4^2 & A_4^3 & A_4^4 & A_4^5 \\
A_5^1 & A_5^2 & A_5^3 & A_5^4 & A_5^5 \\
A_6^1 & A_6^2 & A_6^3 & A_6^4 & A_6^5 \\
A_7^1 & A_7^2 & A_7^3 & A_7^4 & A_7^5 \\
\end{pmatrix},
\]

and requiring that the matrix \( A = [A^j_i] \) is full rank along with the additional requirement that the resulting vectors \( X_i \) satisfy the Jacobi identity.

We begin by letting \( \varphi_1 \) act on the vectors \( X_i \). In doing so, we see that the row vectors \( A^6 \) and \( A^7 \) are invariant, and so, we consider cases depending upon the dimension of the possible subspaces formed by these two vectors in \( \mathbb{R}^5 \). We note that in order to preserve the structure of our subalgebras, we only allow column operations in our analysis. This leads to the following five normal forms for the \( 2 \times 5 \) block \( S \) generated by \( A^6 \) and \( A^7 \),

\[
\text{rank}(S) = 0 : \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
\text{rank}(S) = 1 : \quad \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}
\lambda & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \lambda \neq 0,
\]

each of which we consider separately.
Case I

When the subspace generated by $A^6$ and $A^7$ is 0-dimensional, we see that both $A^6$ and $A^7$ must be identically zero, and immediately obtain the 5-dimensional subspace

$$\Sigma_1 = \text{span}\{e_1, e_2, e_3, e_4, e_5\}.$$ 

This subspace additionally satisfies the Jacobi identity, and therefore defines the subalgebra

$$c_1 = \text{span}\{e_1, e_2, e_3, e_4, e_5\}.$$ 

Case II

In the case where $S$ is of the form

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we see that the coefficient matrix $A$ has the initial form

$$A = \begin{pmatrix} A^1_1 & A^1_2 & A^1_3 & A^1_4 & A^1_5 \\ A^2_1 & A^2_2 & A^2_3 & A^2_4 & A^2_5 \\ A^3_1 & A^3_2 & A^3_3 & A^3_4 & A^3_5 \\ A^4_1 & A^4_2 & A^4_3 & A^4_4 & A^4_5 \\ A^5_1 & A^5_2 & A^5_3 & A^5_4 & A^5_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

We can transform the vector $X_1$ to

$$\tilde{X}_1 = \varphi_1(X_1) = e_6 + A^1_1 e_1,$$
by setting

\[ x_1 = A_4^1 - A_5^1, \quad x_2 = -A_4^1, \quad x_3 = A_1^2 - A_5^3, \quad x_4 = -A_1^2. \]

After renaming \( A = G_1 A \), we see that \( A \) is of the form

\[
A = \begin{pmatrix}
A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Now, the vector \( A^4 \) becomes \( \varphi_1 \)-invariant, and so we consider cases where either \( A^4 = 0 \) or not.

If \( A^4 = 0 \), then we may take as our 5-dimensional subspace,

\[ \Sigma_2 = \text{span}\{e_1, e_2, e_3, e_5, e_6\} \]

which additionally satisfies the Jacobi identity, and so we obtain the subalgebra

\[ c_2 = \text{span}\{e_1, e_2, e_3, e_5, e_6\}. \]

If \( A^4 \neq 0 \), then by renomalizing and taking linear combinations of the vectors \( X_1 \), we can take it to the form \( A^4 = [0, 1, 0, 0, 0] \). By again letting \( \varphi_1 \) act, we can transform the vector \( X_2 \) to

\[
\tilde{X}_2 = \varphi_1(X_2) = e_4 + A_2^3 e_3 + A_2^2 e_2 + A_2^1 e_1
\]
by setting $x_6 = A_2^5$. After again remaning $A = G_1A$, we see

$$A = \begin{pmatrix}
A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & 0 & A_2^3 & A_3^3 & A_5^3 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

so that the five vectors are of the form

$$X_1 = e_6 + A_1^1e_1, \quad X_2 = e_4 + A_2^3e_3 + A_2^2e_2 + A_2^1e_1,$$
$$X_3 = A_1^3e_1 + A_3^2e_2 + A_3^3e_3 + A_3^5e_5,$$
$$X_4 = A_4^1e_1 + A_4^2e_2 + A_4^3e_3 + A_4^5e_5,$$
$$X_5 = A_5^1e_1 + A_5^2e_2 + A_5^3e_3 + A_5^5e_5.$$

We then see that there is only one subspace of this form which satisfies the Jacobi identity, namely

$$c_3 = \text{span}\{e_1, e_3, e_4, e_5, e_6\}.$$  

**Case III**

In the case where $S$ is of the form

$$S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},$$
we see that the coefficient matrix $A$ has the initial form

$$A = \begin{pmatrix}
A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
A_3^3 & A_3^3 & A_4^3 & A_5^3 \\
A_4^4 & A_4^4 & A_4^4 & A_5^4 \\
A_5^5 & A_5^5 & A_5^5 & A_5^5 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

By letting $\varphi_1$ act, we see that we can transform the vector $X_1$ to the vector

$$\tilde{X}_1 = \varphi_1(X_1) = e_7,$$

by setting

$$x_1 = A_1^5, \quad x_2 = A_1^4, \quad x_3 = -A_1^3, \quad x_4 = -A_1^2, \quad x_5 = -A_1^1.$$

After renaming $A = G_1 A$, we see

$$A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
0 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

Now, the vector $A^4$ becomes $\varphi_1$-invariant, and so we consider cases where either $A^4 = 0$ or not.
If $A^4 = 0$, then we may take as our 5-dimensional subspace

$$ \Sigma_4 = \text{span}\{e_1, e_2, e_3, e_5, e_7\}. $$

These vectors additionally satisfy the Jacobi identity, and so we obtain the subalgebra

$$ e_4 = \text{span}\{e_1, e_2, e_3, e_5, e_7\}. $$

If $A^4 \neq 0$, then by renomalizing and taking linear combinations of the vectors $X_i$, we can take it to the form $A^4 = [0, 1, 0, 0, 0]$. We can then transform the vector $X_2$ to

$$ \tilde{X}_2 = \varphi_1(X_2) = e_4 + A^3_2 e_3 + A^2_2 e_2 + A^1_2 e_1 $$

by setting $x_6 = A^5_2$. After again renaming $A = G_1 A$, we see

$$ A = \begin{pmatrix}
0 & A^1_2 & A^1_3 & A^1_4 & A^1_5 \\
0 & A^2_2 & A^2_3 & A^2_4 & A^2_5 \\
0 & A^3_2 & A^3_3 & A^3_4 & A^3_5 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & A^5_3 & A^5_4 & A^5_5 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}, $$

so that the five vectors are of the form

$$ X_1 = e_7, \quad X_2 = e_4 + A^3_2 e_3 + A^2_2 e_2 + A^1_2 e_1, $$
$$ X_3 = A^1_3 e_1 + A^3_3 e_2 + A^3_3 e_3 + A^5_3 e_5, $$
$$ X_4 = A^1_4 e_1 + A^3_4 e_2 + A^3_4 e_3 + A^5_4 e_5, $$
$$ X_5 = A^1_5 e_1 + A^3_5 e_2 + A^3_5 e_3 + A^5_5 e_5. $$
In requiring that these vectors satisfy the Jacobi identity, we see three possible subspaces can occur:

\[ \Sigma_{5,1}^{(\alpha,\beta,\gamma)} = \text{span}\{e_1, e_2 + \alpha e_5, e_3 + \beta e_5, e_4 + \gamma e_5, e_7\}, \]
\[ \Sigma_{5,2}^{(\alpha,\beta,\gamma)} = \text{span}\{e_1, e_2 + \alpha e_3, e_4 + \beta e_3, e_5 + \gamma e_3, e_7\}, \]
\[ \Sigma_{5,3}^{(\alpha,\beta,\gamma)} = \text{span}\{e_1, e_3 + \alpha e_2, e_4 + \beta e_2, e_5 + \gamma e_2, e_7\}. \]

These subspaces can be seen to be identical for several parameter values. To give a list of subalgebras generated from these subspaces which are inequivalent for all values of \(\alpha, \beta, \gamma\), we begin by considering cases where either 0, 1, 2, or all 3 of the parameter values are zero and see which are equivalent under the action of \(\varphi_2\).

When \(\alpha = \beta = \gamma = 0\), we immediately obtain the following inequivalent subspaces:

\[ \Sigma_{5,1}^{(0,0,0)} = \text{span}\{e_1, e_2, e_3, e_4, e_7\}, \]
\[ \Sigma_{5,2}^{(0,0,0)} = \text{span}\{e_1, e_2, e_4, e_5, e_7\}, \]
\[ \Sigma_{5,3}^{(0,0,0)} = \text{span}\{e_1, e_3, e_4, e_5, e_7\}. \]

When two of \(\alpha, \beta, \gamma\) are zero and the other is nonzero, after normalizing the remaining parameter value (when possible) using the action of \(\varphi_2\), we obtain the following inequivalent subspaces (\(\lambda \neq 0\)):

\[ \Sigma_{5,1}^{(\alpha,0,0)} = \text{span}\{e_1, e_2 + e_5, e_3, e_4, e_7\} = \Sigma_{5,3}^{(0,0,\gamma)}, \]
\[ \Sigma_{5,1}^{(0,\beta,0)} = \text{span}\{e_1, e_2, e_3 + e_5, e_4, e_7\} = \Sigma_{5,2}^{(0,0,\gamma)}, \]
\[ \Sigma_{5,1}^{(0,0,\gamma)} = \text{span}\{e_1, e_2, e_3, e_4 + \lambda e_5, e_7\}, \]
\[ \Sigma_{5,2}^{(\alpha,0,0)} = \text{span}\{e_1, e_2 + \lambda e_3, e_4, e_5, e_7\} = \Sigma_{5,3}^{(\alpha,0,0)}, \]
\[ \Sigma_{5,2}^{(0,\beta,0)} = \text{span}\{e_1, e_2, e_3 + e_4, e_5, e_7\}, \]
\[ \Sigma_{5,3}^{(0,\beta,0)} = \text{span}\{e_1, e_2 + e_4, e_3, e_5, e_7\}. \]
Similarly, when only one of the parameter values is zero, after normalizing using the action of $\varphi_2$, we obtain the following inequivalent subspaces ($\lambda \neq 0$):

$$\Sigma^{(a,\beta,0)}_{5,1} = \text{span}\{e_1, e_2 + \lambda e_5, e_3, e_4, e_7\} = \Sigma^{(a,0,\gamma)}_{5,2} = \Sigma^{(a,0,\gamma)}_{5,3},$$

$$\Sigma^{(\alpha,0,\gamma)}_{5,1} = \text{span}\{e_1, e_2 + e_5, e_3, e_4 + \lambda e_5, e_7\} = \Sigma^{(0,\beta,\gamma)}_{5,3},$$

$$\Sigma^{(0,\beta,\gamma)}_{5,1} = \text{span}\{e_1, e_2, e_3 + e_5, e_4 + \lambda e_5, e_7\} = \Sigma^{(0,\beta,\gamma)}_{5,2},$$

$$\Sigma^{(a,\beta,0)}_{5,2} = \text{span}\{e_1, e_2 + e_3, e_4, e_5, e_7\}.$$

Finally, when none of the parameter values are zero, after normalization by $\varphi_2$, we obtain the single subspace

$$\Sigma^{(a,\beta,\gamma)}_{5,1} = \text{span}\{e_1, e_2 + \lambda e_5, e_3 + e_5, e_4 + \tau e_5, e_7\} = \Sigma^{(a,\beta,\gamma)}_{5,2} = \Sigma^{(a,\beta,\gamma)}_{5,3}.$$

This gives the following complete list of inequivalent subalgebras of this form. Note that for each, $\lambda, \tau \neq 0$.

$$c_5 = \text{span}\{e_1, e_2, e_3, e_4, e_7\},$$

$$c_6 = \text{span}\{e_1, e_2, e_4, e_5, e_7\},$$

$$c_7 = \text{span}\{e_1, e_3, e_4, e_5, e_7\},$$

$$c_8 = \text{span}\{e_1, e_2 + e_4, e_3, e_5, e_7\},$$

$$c_9 = \text{span}\{e_1, e_2, e_3 + e_4, e_5, e_7\},$$

$$c_{10} = \text{span}\{e_1, e_2 + e_5, e_3, e_4, e_7\},$$

$$c_{11} = \text{span}\{e_1, e_2, e_3 + e_5, e_4, e_7\},$$

$$c_{12} = \text{span}\{e_1, e_2, e_3, e_4 + \lambda e_5, e_7\},$$

$$c_{13} = \text{span}\{e_1, e_2 + \lambda e_3, e_4, e_5, e_7\},$$

$$c_{14} = \text{span}\{e_1, e_2 + \lambda e_4, e_3 + e_4, e_5, e_7\},$$

$$c_{15} = \text{span}\{e_1, e_2, e_3 + e_5, e_4 + \lambda e_5, e_7\}.$$
\( \epsilon_{16} = \text{span}\{e_1, e_2 + e_5, e_3, e_4 + \lambda e_5, e_7\} \),
\( \epsilon_{17} = \text{span}\{e_1, e_2 + \lambda e_5, e_3, e_4, e_7\} \),
\( \epsilon_{18} = \text{span}\{e_1, e_2 + \lambda e_5, e_3 + e_5, e_4 + \tau e_5, e_7\} \).

**Case IV**

In the case where \( S \) is of the form

\[
S = \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \lambda \neq 0,
\]

we see that the coefficient matrix \( A \) has the initial form

\[
A = \begin{pmatrix}
A_1^1 & A_1^2 & A_1^3 & A_1^4 & A_1^5 \\
A_2^2 & A_2^3 & A_2^4 & A_2^5 \\
A_3^3 & A_3^4 & A_3^5 \\
A_4^4 & A_4^5 \\
A_5^5 \\
\lambda & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Upon applying \( \varphi_1 \), we see that we can transform the vector \( X_1 \) to the vector

\[
\tilde{X}_1 = \varphi_1(X_1) = e_7 + \lambda e_6,
\]

provided \( \lambda \neq \pm 1 \), by setting

\[
x_1 = \frac{\lambda A_1^4 - \lambda A_1^5 + A_1^7}{(\lambda - 1)^2}, \quad x_2 = \frac{A_1^4}{1 - \lambda}, \quad x_3 = \frac{\lambda A_1^7 - \lambda A_1^5 - A_1^3}{(\lambda + 1)^2}, \quad x_4 = -\frac{A_1^2}{\lambda + 1},
\]

\[
x_5 = -\frac{A_1^4 \lambda^4 + (A_1^2 A_1^4 - A_1^2 A_1^5 - A_1^3 A_1^4) \lambda^3 - 2 A_1^4 \lambda^2 + (A_1^2 A_1^4 + A_1^3 A_1^5 + A_1^3 A_1^4) \lambda + A_1^1}{(\lambda - 1)^2(\lambda + 1)^2}.
\]
Case IV.a: $\lambda \neq 0, \pm 1$ After renaming $A = G_1 A$, we see

$$A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
0 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
\lambda & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

Now, the vector $A^4$ is $\varphi_1$-invariant, so we consider cases where either $A^4 = 0$ or not.

If $A^4 = 0$, then we can immediately take as our subspace

$$\Sigma_{19} = \text{span}\{e_1, e_2, e_3, e_5, e_7 + \lambda e_6\}.$$  

These vectors additionally satisfy the Jacobi identity, and so we obtain the subalgebra

$$c_{19}^\lambda = \text{span}\{e_1, e_2, e_3, e_5, e_7 + \lambda e_6\}.$$  

If $A^4 \neq 0$, then by renormalizing and taking linear combinations of the vectors $X_i$, we can take it to the form $A^4 = [0, 1, 0, 0, 0]$. We can then transform the vector $X_2$ to

$$\tilde{X}_2 = \varphi_1(X_2) = e_4 + A_2^3 e_3 + A_2^2 e_2 + A_2^1 e_1$$
by setting \( x_6 = A_2^5 \). After renaming \( A = G_1 A \), we see

\[
A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & A_3^5 & A_4^5 & A_5^5 \\
\lambda & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

so that the five vectors are of the form

\[
X_1 = e_7 + \lambda e_6, \quad X_2 = e_4 + A_3^3 e_3 + A_2^2 e_2 + A_1^1 e_1, \\
X_3 = A_3^1 e_1 + A_2^2 e_2 + A_1^3 e_3 + A_5^5 e_5, \\
X_4 = A_4^1 e_1 + A_3^2 e_2 + A_2^3 e_3 + A_4^5 e_5, \\
X_5 = A_5^1 e_1 + A_5^2 e_2 + A_3^3 e_3 + A_5^5 e_5.
\]

Requiring that these vectors satisfy the Jacobi identity results in a single Lie algebra

\[
c_{20}^\lambda = \text{span}\{e_1, e_3, e_4, e_5, e_7 + \lambda e_6\}.
\]

**Case IV.b: \( \lambda = 1 \)** In the case where \( \lambda = 1 \), the coefficient matrix \( A \) has the form

\[
A = \begin{pmatrix}
A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
A_1^2 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
A_1^3 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
A_1^4 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
A_1^5 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
We can then transform the vector $X_1$ to the vector

$$\tilde{X}_1 = \varphi_1(X_1) = e_7 + e_6 + A_1^5 e_5 + A_1^4 e_4 + A_1^1 e_1$$

by setting

$$x_3 = \frac{A_1^2 - 2A_1^3}{4} \quad \text{and} \quad x_4 = -\frac{A_1^2}{2}.$$

We can then further transform $\tilde{X}_1$ to

$$\tilde{\tilde{X}}_1 = \varphi_1(\tilde{X}_1) = e_7 + e_6 + A_1^4 e_4$$

by setting $x_5 = -A_1^1$ and $x_2 = A_1^4 x_6 - A_1^5$.

At this point, $A$ is of the form

$$A = \begin{pmatrix} 0 & A_1^1 & A_1^2 & A_1^3 & A_1^4 \\ 0 & A_2^1 & A_2^2 & A_2^3 & A_2^4 \\ 0 & A_3^1 & A_3^2 & A_3^3 & A_3^4 \\ A_4^1 & A_4^2 & A_4^3 & A_4^4 & A_4^5 \\ 0 & A_5^1 & A_5^2 & A_5^3 & A_5^4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the vector $A^4$ is $\varphi_1$-invariant. We consider cases where the subspace generated by $\{A^4, A^6, A^7\}$ is either 2 or 3-dimensional.

This subspace is 2-dimensional when $A^4$ is proportional to $A^7$; that is when $A^4 = \tau A^7$. If so, we may immediately take as our 5-dimensional subspace

$$\text{span}\{e_1, e_2, e_3, e_5, e_7 + e_6 + \tau e_4\}$$
which can then be transformed to the subspace

\[ \Sigma_{21} = \text{span}\{e_1, e_2, e_3, e_5, e_7 + e_6\}, \]

if \( \tau = 0 \), and to

\[ \Sigma_{22} = \text{span}\{e_1, e_2, e_3, e_5, e_7 + e_6 + e_4\}, \]

if \( \tau \neq 0 \) by letting \( \varphi_2 \) act and setting \( z_2 = \tau z_1 \). Each of these subspaces additionally satisfy the Jacobi identity and give the subalgebras

\[ c_{21} = \text{span}\{e_1, e_2, e_3, e_5, e_7 + e_6\} \quad \text{and} \quad c_{22} = \text{span}\{e_1, e_2, e_3, e_5, e_7 + e_6 + e_4\}. \]

If the subspace generated by \( \{A^4, A^6, A^7\} \) is 3-dimensional, then by rescaling and taking linear combinations, we may take \( A^4 = [0, 1, 0, 0, 0] \), so that \( A \) is of the form

\[
A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & 1 & 0 & 0 & 0 \\
0 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

and the five vectors are given by

\[
X_1 = e_7 + e_6, \quad X_2 = e_4 + A_2^5 e_5 + A_3^3 e_3 + A_2^2 e_2 + A_2^1 e_1, \\
X_3 = A_3^1 e_1 + A_3^2 e_2 + A_3^3 e_3 + A_2^5 e_5, \\
X_4 = A_4^1 e_1 + A_4^2 e_2 + A_4^3 e_3 + A_2^5 e_5, \\
X_5 = A_5^1 e_1 + A_5^2 e_2 + A_5^3 e_3 + A_5^5 e_5.
\]
Requiring that these vectors satisfy the Jacobi identity results in a single Lie algebra

\[ \mathfrak{c}_{23} = \text{span}\{e_1, e_3, e_4, e_5, e_6 + e_7\}. \]

Case IV.c: \( \lambda = -1 \) In the case where \( \lambda = -1 \), the coefficient matrix \( A \) has the form

\[
A = \begin{pmatrix}
A_1^1 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
A_1^2 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
A_1^3 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
A_1^4 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
A_1^5 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
-1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

We can then transform the vector \( X_1 \) to the vector

\[
\tilde{X}_1 = \varphi_1(X_1) = e_7 - e_6 + A_1^3e_3 + A_1^2e_2 + A_1^1e_1
\]

by setting

\[
x_1 = \frac{2A_1^5 - A_1^4}{4} \quad \text{and} \quad x_2 = \frac{A_1^4}{2}.
\]

We can then further transform \( \tilde{X}_1 \) to

\[
\ddot{X}_1 = \varphi_1(\tilde{X}_1) = e_7 - e_6 + A_1^2e_2
\]

by setting \( x_5 = -A_1^1 \) and \( x_4 = A_1^2x_6 + A_1^3 \).
At this point, \( A \) is of the form

\[
A = \begin{pmatrix}
0 & A^1_2 & A^1_3 & A^1_4 & A^1_5 \\
A^2_1 & A^2_2 & A^2_3 & A^2_4 & A^2_5 \\
0 & A^3_2 & A^3_3 & A^3_4 & A^3_5 \\
0 & A^4_2 & A^4_3 & A^4_4 & A^4_5 \\
0 & A^5_2 & A^5_3 & A^5_4 & A^5_5 \\
-1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 
\end{pmatrix},
\]

and the vector \( A^4 \) is \( \varphi_1 \)-invariant. Here, we consider cases where either \( A^4 = 0 \) or it is not.

If \( A^4 = 0 \), then we immediately obtain the subspace

\[
\Sigma_{24} = \text{span}\{e_1, e_2, e_3, e_5, e_7 - e_6\}.
\]

These vectors additionally satisfy the Jacobi identity, and therefore give the subalgebra

\[
\mathfrak{e}_{24} = \text{span}\{e_1, e_2, e_3, e_5, e_7 - e_6\}.
\]

If \( A^4 \neq 0 \), then by renormalizing and taking linear combinations, we may take \( A^4 = [0, 1, 0, 0, 0] \), so that \( A \) becomes

\[
A = \begin{pmatrix}
0 & A^1_2 & A^1_3 & A^1_4 & A^1_5 \\
A^2_1 & A^2_2 & A^2_3 & A^2_4 & A^2_5 \\
0 & A^3_2 & A^3_3 & A^3_4 & A^3_5 \\
0 & 1 & 0 & 0 & 0 \\
0 & A^5_2 & A^5_3 & A^5_4 & A^5_5 \\
-1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 
\end{pmatrix}.
\]
The vectors $X_i$ are then

\[ X_1 = e_7 - e_6, \quad X_2 = e_4 + A_5^1 e_5 + A_3^3 e_3 + A_2^2 e_2 + A_1^1 e_1, \]
\[ X_3 = A_3^1 e_1 + A_3^2 e_2 + A_3^3 e_3 + A_3^5 e_5, \]
\[ X_4 = A_4^1 e_1 + A_4^2 e_2 + A_4^3 e_3 + A_4^5 e_5, \]
\[ X_5 = A_5^1 e_1 + A_5^2 e_2 + A_5^3 e_3 + A_5^5 e_5. \]

Requiring that these vectors satisfy the Jacobi identity results in a single Lie algebra

\[ \text{span}\{e_1, e_3, e_4, e_5, e_7 - e_6 + \tau e_2\}. \]

However, if $\tau = 0$, we obtain the subalgebra

\[ c_{25} = \text{span}\{e_1, e_3, e_4, e_5, e_7 - e_6\}, \]

and if $\tau \neq 0$, then we obtain the subalgebra

\[ c_{26} = \text{span}\{e_1, e_3, e_4, e_5, e_7 - e_6 + e_2\} \]

by letting $\varphi_2$ act and setting $z_2 = 1/\tau$.

**Case V**

Finally, in the case where $S$ is of the form

\[ S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \]
we see that the coefficient matrix $A$ has the initial form

$$
A = \begin{pmatrix}
A_1 & A_2 & A_3 & A_4 & A_5 \\
A_2 & A_1 & A_3 & A_4 & A_5 \\
A_3 & A_2 & A_3 & A_4 & A_5 \\
A_4 & A_2 & A_3 & A_4 & A_5 \\
A_5 & A_2 & A_3 & A_4 & A_5 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Upon applying $\varphi_1$, we see that we can transform the vector $X_1$ to the vector

$$
\tilde{X}_1 = \varphi_1(X_1) = e_7,
$$

by setting

$$
x_1 = A_1^5, \quad x_2 = A_1^4, \quad x_3 = -A_1^3, \quad x_4 = -A_1^2, \quad x_5 = -A_1^1.
$$

After renaming $A = G_1 A$, we see that $A$ is of the form

$$
A = \begin{pmatrix}
0 & A_2^1 & A_3^1 & A_4^1 & A_5^1 \\
0 & A_2^2 & A_3^2 & A_4^2 & A_5^2 \\
0 & A_2^3 & A_3^3 & A_4^3 & A_5^3 \\
0 & A_2^4 & A_3^4 & A_4^4 & A_5^4 \\
0 & A_2^5 & A_3^5 & A_4^5 & A_5^5 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Now, the vector $A^4$ becomes $\varphi_1$-invariant, and so we consider cases where either $A^4 = 0$ or not.
If $A^4 = 0$, then the vectors $X_i$ are of the form

$$
X_1 = e_7, \quad X_2 = e_6 + A_2^5 e_5 + A_2^3 e_3 + A_2^2 e_2 + A_2^1 e_1, \\
X_3 = A_3^1 e_1 + A_3^2 e_2 + A_3^3 e_3 + A_3^5 e_5, \\
X_4 = A_4^1 e_1 + A_4^2 e_2 + A_4^3 e_3 + A_4^5 e_5, \\
X_5 = A_5^1 e_1 + A_5^2 e_2 + A_5^3 e_3 + A_5^5 e_5.
$$

Requiring that these vectors satisfy the Jacobi identity gives two subalgebras,

$$
c_{27} = \text{span}\{e_1, e_2, e_3, e_6, e_7\} \quad \text{and} \quad c_{28} = \text{span}\{e_1, e_3, e_5, e_6, e_7\}.
$$

If $A^4 \neq 0$, then by renormalizing and taking linear combinations, we may take $A^4 = [0, 0, 1, 0, 0]$. In doing so, the vectors $X_i$ are of the form,

$$
X_1 = e_7, \quad X_2 = e_6 + A_2^5 e_5 + A_2^3 e_3 + A_2^2 e_2 + A_2^1 e_1, \\
X_3 = e_4 + A_3^1 e_1 + A_3^2 e_2 + A_3^3 e_3 + A_3^5 e_5, \\
X_4 = A_4^1 e_1 + A_4^2 e_2 + A_4^3 e_3 + A_4^5 e_5, \\
X_5 = A_5^1 e_1 + A_5^2 e_2 + A_5^3 e_3 + A_5^5 e_5.
$$

Requiring that these vectors satisfy the Jacobi identity gives a single subalgebra

$$
c_{29} = \text{span}\{e_1, e_4, e_5, e_6, e_7\}.
$$

We tabulate these results in Table 10.3 below.
Table 10.3: 5-Dimensional Subalgebras of the Symmetry Algebra of $z' = \ln(y'')$

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Vanishing Lie Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$e_1, e_2, e_3, e_4, e_5$</td>
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</tr>
<tr>
<td>$c_2$</td>
<td>$e_1, e_2, e_3, e_5, e_6$</td>
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</tr>
<tr>
<td>$c_3$</td>
<td>$e_1, e_3, e_4, e_5, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$e_1, e_2, e_3, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$e_1, e_2, e_3, e_4, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_6$</td>
<td>$e_1, e_2, e_4, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_7$</td>
<td>$e_1, e_3, e_4, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_8$</td>
<td>$e_1, e_2 + e_4, e_3, e_5, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_9$</td>
<td>$e_1, e_2, e_4 + e_5, e_6, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{10}$</td>
<td>$e_1, e_2 + e_5, e_4, e_4, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>$e_1, e_2, e_3 + e_5, e_4, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{12}$</td>
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<td>✓</td>
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<tr>
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<td>✓</td>
</tr>
<tr>
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</tr>
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<td>$c_{17}$</td>
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<td>✓</td>
</tr>
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<td>$c_{18}$</td>
<td>$e_1, e_2 + \lambda e_5, e_3 + e_5, e_4 + \tau e_5, e_7$</td>
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<tr>
<td>$c_{19}$</td>
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</tr>
<tr>
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<td>$c_{21}$</td>
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<td>✓</td>
</tr>
<tr>
<td>$c_{22}$</td>
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<td>✓</td>
</tr>
<tr>
<td>$c_{23}$</td>
<td>$e_1, e_3, e_4, e_5, e_6 + e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{24}$</td>
<td>$e_1, e_2, e_3, e_5, e_7 - e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{25}$</td>
<td>$e_1, e_3, e_4, e_5, e_7 - e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{26}$</td>
<td>$e_1, e_3, e_4, e_5, e_7 - e_6 + e_2$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{27}$</td>
<td>$e_1, e_2, e_5, e_6, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{28}$</td>
<td>$e_1, e_3, e_5, e_6, e_7$</td>
<td>✓</td>
</tr>
<tr>
<td>$c_{29}$</td>
<td>$e_1, e_4, e_5, e_6, e_7$</td>
<td>✓</td>
</tr>
</tbody>
</table>

Note: The $e_i = \Gamma_i$ where

\[
\Gamma_1 = \partial_y, \quad \Gamma_2 = -\partial_x + \partial_z, \quad \Gamma_3 = 2\partial_z,
\]
\[
\Gamma_4 = -\frac{1}{2y_2} \partial_x - \frac{\ln(y_2)}{2y_2} \partial_z + \frac{xy_2 + zy_2 - y_1}{2y_2} \partial_y + \frac{\ln(y_2)}{2} \partial_{y_1},
\]
\[
\Gamma_5 = -x\partial_y - \partial_{y_1}, \quad \Gamma_6 = x\partial_x - (2x - z)\partial_z - y_1\partial_{y_1} - y_2\partial_{y_2},
\]
\[
\Gamma_7 = x\partial_x + z\partial_z + 2y\partial_y + y_1\partial_{y_1}.
\]
10.5 Root Type [4] with $I(x)$ Non-Constant

In this section, we consider the case where the invariant $I$ associated to the (2,3,5)-distribution corresponding to the equation

$$z' = -\frac{1}{2} \left( y'' + \frac{10}{3} I(y')^2 + (1 + I^2 - I'')y'' \right)$$

is non-constant. Cartan [12] proves that the symmetry algebra $g$ of such a system is 6-dimensional and is given by the following structure equations

$$\begin{align*}
  d\omega^1 &= 2\omega^1 \wedge \pi^1 + \omega^2 \wedge \pi^2 + \omega^3 \wedge \omega^4, \\
  d\omega^2 &= \omega^2 \wedge \pi^1, \\
  d\omega^3 &= \omega^3 \wedge \pi^1, \\
  d\omega^4 &= \omega^4 \wedge \pi^1, \\
  d\pi^1 &= 0, \\
  d\pi^2 &= \pi^2 \wedge \pi^1. 
\end{align*}$$

If we denote the dual basis to $\{\omega^1, \omega^2, \omega^3, \omega^4, \pi^1, \pi^2\}$ by $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6\}$, then we may introduce the change of basis

$$e_1 = -\bar{e}_1, \quad e_5 = \bar{e}_6, \quad e_6 = -\bar{e}_5,$$

which puts the nilradical in standard form. The multiplication table for $g$ in this basis is given by
The generators for the symmetry algebra $\mathfrak{g}$ are given by the vector fields

\[
\begin{align*}
\Gamma_1 &= \partial_z, \quad \Gamma_6 = 2z\partial_z + y\partial_y + y_1\partial_{y_1} + y_2\partial_{y_2}, \quad 2 \leq i \leq 5, \\
\Gamma_i &= -\left(\frac{10}{3}yI B_i'(x) + y_1 B_i''(x) - y B''_i(x)\right) \partial_z + B_i(x)\partial_y + B_i'(x)\partial_{y_1} + B''_i(x)\partial_{y_2},
\end{align*}
\]

where $B_i(x)$ are linearly independent solutions to the fourth-order ODE

\[
B'''' - \frac{10}{3} (IB'' + I'B') + (1 + I^2 - I'')B = 0.
\]

Immediately, we see that $\mathfrak{g}$ is intransitive on the 5-dimensional manifold with coordinates $x, z, y, y_1, y_2$.

We then calculate the full automorphism group $\mathfrak{G}$ of $\mathfrak{g}$ and split $\mathfrak{G}$ into the three subgroups $\mathfrak{G}_1, \mathfrak{G}_2,$ and $\mathfrak{G}_3$ given by the upper triangular, lower triangular, and diagonal matrix groups

\[
G_1 = \begin{pmatrix}
1 & x_1 & x_4 & x_6 & x_8 & x_9 \\
0 & 1 & x_2 & x_5 & x_7 & -x_1 x_7 + x_2 x_6 - x_4 x_5 + x_8 \\
0 & 0 & 1 & x_3 & -x_2 x_3 + x_5 & x_1 x_2 x_3 - x_1 x_5 - x_3 x_4 + x_6 \\
0 & 0 & 0 & 1 & -x_2 & x_1 x_2 - x_4 \\
0 & 0 & 0 & 0 & 1 & -x_1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
\[
G_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & y_1 & 1 & 0 & 0 \\
0 & y_3 & y_2 & 1 & 0 \\
0 & y_4 & -y_1y_2 + y_3 & -y_1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} ,
G_3 = \begin{pmatrix}
z_1 & 0 & 0 & 0 & 0 \\
0 & z_2 & 0 & 0 & 0 \\
0 & 0 & z_3 & 0 & 0 \\
0 & 0 & 0 & z_1/z_3 & 0 \\
0 & 0 & 0 & z_2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} .
\]

We denote the actions of these matrix groups on \( \mathfrak{g} \) by \( \varphi_1, \varphi_2, \) and \( \varphi_3, \) respectively, such that if \( v = v^i e_i \in \mathfrak{g}, \) then

\[
\varphi_j(v) = G_j^\mathbb{T} \begin{pmatrix} v^1, v^2, v^3, v^4, v^5 \end{pmatrix} , \quad 1 \leq j \leq 3.
\]

In order to classify all 5-dimensional subalgebras of \( \mathfrak{g} \), we begin by constructing five arbitrary vectors in \( \mathfrak{g} \), requiring that they

(i) be linearly independent, and

(ii) satisfy the Jacobi identity.

If we write the five vectors as \( X_i = A_i^j e_j \) such that

\[
[X_1 \ X_2 \ X_3 \ X_4 \ X_5] = [e_1 \ e_2 \ e_3 \ e_4 \ e_5 \ e_6]
\]

\[
\begin{pmatrix}
A_1^1 & A_1^2 & A_1^3 & A_1^4 & A_1^5 \\
A_2^1 & A_2^2 & A_2^3 & A_2^4 & A_2^5 \\
A_3^1 & A_3^2 & A_3^3 & A_3^4 & A_3^5 \\
A_4^1 & A_4^2 & A_4^3 & A_4^4 & A_4^5 \\
A_5^1 & A_5^2 & A_5^3 & A_5^4 & A_5^5 \\
A_6^1 & A_6^2 & A_6^3 & A_6^4 & A_6^5
\end{pmatrix},
\]

then (i) is equivalent to requiring that the matrix \( A = [A_i^j] \) be full rank. As we make our analysis, we will use these actions to find standard forms for the possible 5-dimensional subalgebras.
We begin by letting $\varphi_1$ act on the vectors $X_i$. In doing so, we see that the row vector $A^6$ is invariant, and so, we consider two cases depending on whether or not $A^6 = 0$. If it is, then the $e_6$-component of each $X_i$ vanishes, and we may take as a 5-dimensional subspace

$$\Sigma_1 = \text{span}\{e_1, e_2, e_3, e_4, e_5\}.$$ 

This subspace additionally satisfies (ii), and therefore defines the subalgebra

$$\mathfrak{h}_1 = \text{span}\{e_1, e_2, e_3, e_4, e_5\}.$$ 

If $A^6 \neq 0$, then we may without loss of generality take $A^6_1 = 1$ and transform the vector $X_1$ to

$$\tilde{X}_1 = \varphi_1(X_1) = e_6,$$

by setting

$$x_1 = A^5_1, \quad x_4 = A^4_1, \quad x_6 = -A^3_1, \quad x_8 = -A^2_1, \quad x_9 = -A^1_1.$$ 

We can then eliminate the $e_6$-component from each of the remaining vectors, so that $A^6_j = 0$ for $2 \leq j \leq 5$, and after renaming $A = [\varphi_1]A$, we see that $A$ is of the form

$$A = \begin{pmatrix} 0 & A^1_2 & A^1_3 & A^1_4 & A^1_5 \\ 0 & A^2_2 & A^2_3 & A^2_4 & A^2_5 \\ 0 & A^3_2 & A^3_3 & A^3_4 & A^3_5 \\ 0 & A^4_2 & A^4_3 & A^4_4 & A^4_5 \\ 0 & A^5_2 & A^5_3 & A^5_4 & A^5_5 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. $$
Now, $A^5$ becomes $\varphi_1$-invariant, and as before, we consider cases depending on whether or not $A^5 = 0$. If $A^5 = 0$, then we may take as our subspace

$$\Sigma_2 = \text{span}\{e_1, e_2, e_3, e_4, e_6\}.$$  

This subspace additionally satisfies $(ii)$, and therefore defines the subalgebra

$$\mathfrak{h}_2 = \text{span}\{e_1, e_2, e_3, e_4, e_6\}.$$  

If $A^5 \neq 0$, we may set $A^5_2 = 1$ and transform the vector $X_2$ to

$$\bar{X}_2 = \varphi_1(X_2) = e_5 + A^1_2 e_1$$

by setting $x_2 = A^4_2$, $x_5 = -A^3_2$, and $x_7 = -A^2_2$. We can then further eliminate all $e_5$-components from the remaining vectors such that $A^5_j = 0$ for $3 \leq j \leq 5$. After again renaming $A = [\varphi_1]A$, we see that $A$ is now takes the form

$$A = \begin{pmatrix}
0 & A^1_2 & A^1_3 & A^1_4 & A^1_5 \\
0 & 0 & A^2_3 & A^2_4 & A^2_5 \\
0 & 0 & A^3_3 & A^3_4 & A^3_5 \\
0 & 0 & A^4_3 & A^4_4 & A^4_5 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

The vector $A^4$ then becomes $\varphi_1$-invariant, and if $A^4 = 0$, then we may take as our subspace

$$\Sigma_3 = \text{span}\{e_1, e_2, e_3, e_5 + \alpha e_1, e_6\} = \text{span}\{e_1, e_2, e_3, e_5, e_6\}.$$
This subspace additionally satisfies (\(ii\)), and therefore defines the subalgebra

\[ h_3 = \text{span}\{e_1, e_2, e_3, e_5, e_6\}. \]

If \(A^4 \neq 0\), then we set \(A^4_3 = 1\) and transform the vector \(X_3\) to

\[ \tilde{X}_3 = \varphi_1(X_3) = e_4 + A^1_3 e_1 + A^2_3 e_2, \]

by setting \(x_3 = -A^3_3\). After once more renaming \(A = [\varphi_1]A\), we see

\[
A = \begin{pmatrix}
0 & A^1_2 & A^1_3 & A^1_4 & A^1_5 \\
0 & 0 & A^2_3 & A^2_4 & A^2_5 \\
0 & 0 & 0 & A^3_4 & A^3_5 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

so that the five vectors are of the form

\[
X_1 = e_6, \quad X_2 = e_5 + A^1_2 e_1, \quad X_3 = e_4 + A^1_3 e_1 + A^2_3 e_2, \\
X_4 = A^1_4 e_1 + A^2_4 e_2 + A^3_4 e_3 + A^4_4 e_4 + A^5_4 e_5, \\
X_5 = A^1_5 e_1 + A^2_5 e_2 + A^3_5 e_3 + A^4_5 e_4 + A^5_5 e_5.
\]

In requiring that these vectors satisfy (\(ii\)), we see three possible cases can occur:

\[
\text{span}\{e_1, \alpha e_2 + e_3, e_4 + \beta e_2, e_5, e_6\}, \\
\text{span}\{e_1, e_2 + \alpha e_3, \beta e_3 + e_4, e_5, e_6\}, \\
\text{span}\{e_1, e_2 + \alpha e_4, e_3 + \beta e_4, e_5, e_6\},
\]
however upon further inspection, we see that each is isomorphic to the Lie algebra

\[ \mathfrak{h}_4^{\alpha,\beta} = \text{span}\{e_1, \alpha e_2 + e_3, \beta e_2 + e_4, e_5, e_6\}. \]

Upon applying the automorphism group of scalings \( G_3 \), we see that \( \mathfrak{h}_4^{\alpha,\beta} \) can be transformed to

\[ \varphi_3(\mathfrak{h}_4^{\alpha,\beta}) = \text{span}\left\{e_1, e_3 + \frac{z_2 \alpha}{z_3} e_2, e_4 + \frac{z_2 z_3 \beta}{z_1} e_2, e_5, e_6\right\}. \]

We can then consider four possible cases depending on whether or not \( \alpha = 0 \) or \( \beta = 0 \). If \( \alpha = \beta = 0 \), then we immediately obtain the subalgebra

\[ \mathfrak{d}_{4,1} = \text{span}\{e_1, e_3, e_4, e_5, e_6\}. \]

If \( \alpha \neq 0 \) and \( \beta = 0 \), then we can obtain the algebra

\[ \mathfrak{d}_{4,2} = \text{span}\{e_1, e_3 + e_2, e_4, e_5, e_6\} \]

by setting \( z_3 = z_2 \alpha \). If \( \alpha = 0 \) and \( \beta \neq 0 \), then we can obtain the algebra

\[ \mathfrak{d}_{4,3} = \text{span}\{e_1, e_3, e_4 + e_2, e_5, e_6\} \]

by setting \( z_1 = z_2 z_3 \beta \). Finally, if both \( \alpha \) and \( \beta \) are nonzero, then we can obtain the algebra

\[ \mathfrak{d}_{4,4} = \text{span}\{e_1, e_3 + e_2, e_4 + e_2, e_5, e_6\} \]

by setting \( z_3 = z_2 \alpha \) and \( z_1 = z_2 z_3 \beta = z_2^2 \alpha \beta \).

We now let \( G_2 \) act on each of these subalgebras. Applying \( \varphi_2 \), we first see that \( \mathfrak{d}_{4,1} \) is completely invariant. We can then take

\[ \mathfrak{d}_{4,2} = \varphi_2(\mathfrak{d}_{4,2}) = \text{span}\{e_1, e_2, e_4, e_5, e_6\} \]
by setting $y_1 = -1$,

$$\mathfrak{d}_{4,3} = \varphi_2(\mathfrak{d}_{4,3}) = \text{span}\{e_1, e_2, e_3, e_5, e_6\}$$

by setting $y_2 = 0$ and $y_3 = -1$, and

$$\mathfrak{d}_{4,4} = \varphi_2(\mathfrak{d}_{4,4}) = \text{span}\{e_1, e_2, e_3, e_5, e_6\}$$

by setting $y_2 = 1$ and $y_3 = -1$.

The subalgebras $\mathfrak{d}_1, \mathfrak{d}_2,$ and $\mathfrak{d}_3$ are all invariant under the actions of $\varphi_2$ and $\varphi_3$, and we therefore conclude that there are five inequivalent 5-dimensional symmetry algebras of Monge systems of root type $[4]$ with non-constant invariant $I(x)$. They are

$$\mathfrak{d}_1 = \text{span}\{e_1, e_2, e_3, e_4, e_5\},$$
$$\mathfrak{d}_2 = \text{span}\{e_1, e_2, e_3, e_4, e_6\},$$
$$\mathfrak{d}_3 = \text{span}\{e_1, e_2, e_3, e_5, e_6\},$$
$$\mathfrak{d}_4 = \text{span}\{e_1, e_3, e_4, e_5, e_6\},$$
$$\mathfrak{d}_5 = \text{span}\{e_1, e_2, e_4, e_5, e_6\}.$$ 

Furthermore, as abstract Lie algebras $\mathfrak{d}_1 \cong \mathfrak{n}_{5,3}$ and $\mathfrak{d}_2 \cong \mathfrak{d}_3 \cong \mathfrak{d}_4 \cong \mathfrak{d}_5 \cong \mathfrak{s}_{5,22}$. These symmetry algebras are given in Table 10.5 below.
Table 10.4: 5-Dimensional Subalgebras of the Symmetry Algebra of Root Type [4] Distributions with $I(x)$ Non-constant

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Vanishing Lie Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{g}_1$</td>
<td>$e_1, e_2, e_3, e_4, e_5$</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathfrak{g}_2$</td>
<td>$e_1, e_2, e_3, e_4, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathfrak{g}_3$</td>
<td>$e_1, e_2, e_3, e_5, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathfrak{g}_4$</td>
<td>$e_1, e_3, e_4, e_5, e_6$</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathfrak{g}_5$</td>
<td>$e_1, e_2, e_4, e_5, e_6$</td>
<td>✓</td>
</tr>
</tbody>
</table>

Note: The $e_i = \Gamma_i$ where

$$
\Gamma_1 = \partial_z, \quad \Gamma_6 = 2z\partial_z + y\partial_y + y_1\partial_{y_1} + y_2\partial_{y_2}, \quad 2 \leq i \leq 5,
$$

$$
\Gamma_i = -\left(\frac{10}{3} yIB_i'(x) + y_1B_i''(x) - yB_i'''(x)\right)\partial_z + B_i(x)\partial_y + B_i'(x)\partial_{y_1} + B_i''(x)\partial_{y_2},
$$

and $B_i(x)$ are linearly independent solutions to the fourth-order ODE

$$
B'''' - \frac{10}{3} (IB'' + IB'') + (1 + I^2 - I'')B = 0.
$$

10.6 The Exceptional Case of Doubrov and Govorov

In [15], Doubrov and Govorov show that Cartan’s original analysis of root type [4] systems missed the rank 2 distributions encoded by the Monge equation

$$
z' = y + (y'')^{1/3}.
$$

(10.7)

The symmetry algebra $\mathfrak{g}$ of (10.7) is 6-dimensional, solvable, and is isomorphic to $\mathfrak{sl}(2) \rtimes \mathfrak{n}_{3,1}$. In terms of coordinates $x, z, y, y_1, y_2$, $\mathfrak{g}$ is generated by vector fields

$$
\Gamma_1 = -x\partial_x + y\partial_y + 2y_1\partial_{y_1} + 3y_2\partial_{y_2}, \quad \Gamma_2 = y\partial_x + \frac{y^2}{2}\partial_z - y_1^2\partial_{y_1} - 3y_1y_2\partial_{y_2},
$$

$$
\Gamma_3 = \frac{x^2}{2}\partial_z + x\partial_y + \partial_{y_1}, \quad \Gamma_4 = x\partial_z + \partial_y, \quad \Gamma_5 = \partial_x, \quad \Gamma_6 = -\partial_z.
$$
If we let \( e_i = \Gamma_i \) for \( 1 \leq i \leq 6 \), then the multiplication table for \( g \) is

\[
\begin{array}{c|cccccc}
\mathfrak{g} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
e_1 & \cdot & 2e_2 & -2e_3 & -e_4 & e_5 & \cdot \\
e_2 & \cdot & e_1 & -e_5 & \cdot & \cdot & \\
e_3 & \cdot & \cdot & -e_4 & \cdot & \cdot & \\
e_4 & \cdot & \cdot & e_6 & \cdot & \cdot & \\
e_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\end{array}
\]

From this table we clearly see that \( \mathfrak{sl}(2) = \{e_1, e_2, e_3\} \) is an ideal in \( \mathfrak{n}_{3,1} = \{e_4, e_5, e_6\} \). This leads us to utilize the following lemma.

**Lemma 10.6.1.** Let \( \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \) be a Lie algebra where \( \mathfrak{a} \) is an ideal in \( \mathfrak{b} \). If the vectors \( X_i = A^\mu_i a_\mu + B^\nu_i b_\nu \) form a basis for a subalgebra of \( \mathfrak{g} \) with \( a_\mu \in \mathfrak{a} \) and \( b_\nu \in \mathfrak{b} \), then the vectors \( A^\mu_i a_\mu \) form a subalgebra of \( \mathfrak{a} \).

In order to construct all possible subalgebras of \( \mathfrak{g} \), we can begin by constructing all 2-dimensional subalgebras of \( \mathfrak{sl}(2) \). They are \( \mathfrak{h}^0_1 = \{e_1, e_2\} \) and \( \mathfrak{h}^0_2 = \{e_1, e_3\} \). This means the only 5-dimensional subalgebras of \( \mathfrak{g} \) are

\[
\mathfrak{h}_1 = \{e_1, e_2, e_4, e_5, e_6\} \quad \text{and} \quad \mathfrak{h}_2 = \{e_1, e_3, e_4, e_5, e_6\}.
\]

These results are summarized in Table 10.5 below.

**Table 10.5: 5-Dimensional Subalgebras of the Symmetry Algebra of \( z' = y + (y')^{1/3} \)**

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Vanishing Lie Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathfrak{h}_1 )</td>
<td>( e_1, e_2, e_4, e_5 )</td>
<td></td>
</tr>
<tr>
<td>( \mathfrak{h}_2 )</td>
<td>( e_1, e_2, e_5, e_6 )</td>
<td></td>
</tr>
</tbody>
</table>
Note: The $e_i = \Gamma_i$, where

$$
\Gamma_1 = -x \partial_x + y \partial_y + 2y_1 \partial_{y_1} + 3y_2 \partial_{y_2}, \quad \Gamma_2 = y \partial_x + \frac{y^2}{2} \partial_z - y_1 \partial_{y_1} - 3y_1 y_2 \partial_{y_2},
$$

$$
\Gamma_3 = \frac{x^2}{2} \partial_z + x \partial_y + \partial_{y_1}, \quad \Gamma_4 = x \partial_z + \partial_y, \quad \Gamma_5 = \partial_x, \quad \Gamma_6 = -\partial_z.
$$

10.7 Construction of $f$-Gordon Equations Darboux Integrable at Order Three

In this section, we use Theorem 10.2.1 and the classification of 5-dimensional symmetry algebras of contact distributions, Hilbert-Cartan distributions, and $(2,3,5)$-distributions of root type [4] to construct several hyperbolic distributions for $f$-Gordon equations which are Darboux integrable at order three. In particular, we find those 5-dimensional symmetry algebras which satisfy the codimension 1 orbit condition given by [iv] in Theorem 10.2.1. The complete list of these equations is given by Theorem 10.7.1. This construction greatly expands the number of $f$-Gordon equations which are Darboux integrable at order three.

**Theorem 10.7.1.** Let $\mathcal{V}, \tilde{\mathcal{V}}$ be two copies of the same rank 2-distribution $\mathcal{V}$ defined on 5-dimensional manifolds $\tilde{\mathcal{M}}, \tilde{\mathcal{M}}$ with common symmetry group $G$, and let $G^{(1)}_{\text{diag}}$ denote the first prolongation of the diagonal action of $G$. Then the quotient distribution

$$
\Delta = (\mathcal{V}^{(1)} \oplus \tilde{\mathcal{V}}^{(1)})/G^{(1)}_{\text{diag}}
$$

defines a nonlinear $f$-Gordon equation which is Darboux integrable after one prolongation in the following cases:

[I] $\mathcal{V}$ has derived dimensions $(2,3,4,5)$ and the action of $G$ has infinitesimal generators

$$
\Gamma = \{ \partial_x, x \partial_x + 3y \partial_y + 2y_1 \partial_{y_1} + y_2 \partial_{y_2}, \partial_y, x \partial_y + \partial_{y_1}, x^2 \partial_y + 2x \partial_{y_1} + 2\partial_{y_2} \}.
$$
[II] $\mathcal{V}$ has derived dimensions $(2,3,5)$, the Cartan tensor vanishes identically, and the action of $G$ has infinitesimal generators given by

$[N, 23], [S, 14], [S, 46]_{a=2}, [S, 49]_{a=2}, \text{ or } [S, 51]$ in Table B.3.

[III] $\mathcal{V}$ has derived dimensions $(2,3,5)$, the Cartan tensor has a single root of multiplicity four with constant fundamental invariant $I^2 \neq -1, \frac{9}{16}$, and the action of $G$ has infinitesimal generators given by $a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, \text{ or } a_{18}$ in Table 10.1.

[IV] $\mathcal{V}$ has derived dimensions $(2,3,5)$, the Cartan tensor has a single root of multiplicity four with constant fundamental invariant $I^2 = -1$, and the action of $G$ has infinitesimal generators given by $b_1, b_6, b_7, b_{10}, b_{11}, b_{13}, b_{14}, \text{ or } b_{15}$ in Table 10.2.

[V] $\mathcal{V}$ has derived dimensions $(2,3,5)$, the Cartan tensor has a single root of multiplicity four with constant fundamental invariant $I^2 = \frac{9}{16}$, and the action of $G$ has infinitesimal generators given by $c_1, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, \text{ or } c_{18}$ in Table 10.3.

[VI] $\mathcal{V}$ has derived dimensions $(2,3,5)$, the Cartan tensor has a single root of multiplicity four with non-constant fundamental invariant $I$, and the action of $G$ has infinitesimal generators given by $d_1, d_2, d_3, d_4, d_5$ in Table 10.5.

**Remark.** In the above theorem,

1. the action $\Gamma$ of Case [I] is the unique action on $J^3(\mathbb{R}, \mathbb{R})$ which pseudo-stabilizes;
2. the distribution $\mathcal{V}$ of Case [II] corresponds to the Hilbert-Cartan equation, $z' = (y'')^2$;
3. the distribution $\mathcal{V}$ of Case [III] corresponds to $z' = (y'')^m$ with $m \neq -1, 0, 1, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 2$;
4. the distribution $\mathcal{V}$ of Case [IV] corresponds to the equation $z' = (y'')^{1/2}$;
5. the distribution $\mathcal{V}$ of Case $[V]$ corresponds to the equation $z' = \ln(y'')$.

We also note that for each choice of action, the resulting $f$-Gordon equation obtained from the quotient is unique up to contact equivalence. Consequently, this theorem greatly expands the list of inequivalent $f$-Gordon equations which are Darboux integrable after one prolongation. That said, we do not claim that our list constitutes a complete classification of such equations.

**Remark.** The list given in Theorem 10.7.1 includes all previously known nonlinear $f$-Gordon equations which are Darboux integrable at order three. In particular,

1. the quotient given by the action $\Gamma$ in Case $[I]$ corresponds to the second equation of Zhiber and Sokolov,

$$u_{xy} = \frac{P_1^2(P_1 - 1)Q_1(Q_1 - 1)^2}{6u + y} + \frac{Q_1^2(Q_1 - 1)P_1(P_1 - 1)^2}{6u + x}$$

where $P_1 = P_1(u_x)$ and $Q_1 = Q_1(u_y)$ are defined implicitly by

$$\frac{1}{3}P_1^3 - \frac{1}{2}P_1^2 = u_x, \quad \frac{1}{3}Q_1^3 - \frac{1}{2}Q_1^2 = u_y;$$

2. the quotient given by the action $[N,23]$ in Case $[II]$ corresponds to the equation

$$u_{xy} = \frac{4\sqrt{u_x u_y}}{x + y};$$

3. and the quotient given by the action $[S,14]$ in Case $[II]$ corresponds to the equation

$$u_{xy} = \frac{P_1(u_x)Q_1(u_y)}{u}$$

where $P_1 = P_1(u_x)$ and $Q_1 = Q_1(u_y)$ are defined implicitly by

$$P_1P_1' + P_1 = 2u_x, \quad Q_1Q_1' + Q_1 = 2u_y.$$
We now present several examples of this quotient construction. In particular, we give the explicit $f$-Gordon equation defined by quotient of the first prolongation of two Hilbert-Cartan distributions by the action $[S, 46]_{a=2}$ in Case II. For the subsequent examples, we represent the equation with a hyperbolic distribution. Future work will focus on finding the explicit equations corresponding to each of the actions listed in Theorem 10.7.1 above.

Example 10.7.2. For our first example, we let $\mathcal{V}$ and $\mathcal{V}'$ be two copies of the Hilbert-Cartan distribution on 5-dimensional manifolds $\mathcal{M} = \{x, z, u, u_1, u_2\}$ and $\mathcal{M}' = \{y, w, v, v_1, v_2\}$, respectively, so that

$\mathcal{V} = \{\partial_x + u_2^2 \partial_z + u_1 \partial_u + u_2 \partial_{u_1}, \partial_{u_2}\}$ and $\mathcal{V}' = \{\partial_y + v_2^2 \partial_w + v_1 \partial_v + v_2 \partial_{v_1}, \partial_{v_2}\}$.

The first prolongation of these distributions to 6-dimensional manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}'^{(1)}$ are given by

$\mathcal{V}^{(1)} = \{\partial_x + u_2^2 \partial_z + u_1 \partial_u + u_2 \partial_{u_1} + u_3 \partial_{u_2}, \partial_{u_3}\}$,  
$\mathcal{V}'^{(1)} = \{\partial_y + v_2^2 \partial_w + v_1 \partial_v + v_2 \partial_{v_1} + v_3 \partial_{v_2}, \partial_{v_3}\}$,

and the diagonal action of the 5-dimensional symmetry algebra $[S,46]_{a=2}$, written in a more convenient basis, on the product manifold $\mathcal{M}^{(1)} \times \mathcal{M}'^{(1)}$ is generated by the vector fields

$Z_1 = \partial_z + \partial_w, \quad Z_2 = \partial_u + \partial_v, \quad Z_3 = x \partial_u + \partial_{u_1} + y \partial_v + \partial_{v_1}, \quad Z_4 = u_1 \partial_z + \frac{x^2}{4} \partial_u + \frac{x}{2} \partial_{u_1} + \frac{1}{2} \partial_{u_2} + v_1 \partial_w + \frac{y^2}{4} \partial_v + \frac{y}{2} \partial_{v_1} + \frac{1}{2} \partial_{v_2}, \quad Z_5 = 2z \partial_z + u \partial_u + u_1 \partial_{u_1} + u_2 \partial_{u_2} + u_3 \partial_{u_3} + 2w \partial_w + v \partial_v + v_1 \partial_{v_1} + v_2 \partial_{v_2} + v_3 \partial_{v_3}$.

The quotient of $\mathcal{V}^{(1)} \oplus \mathcal{V}'^{(1)}$ by $\Gamma_{\text{diag}}^{(1)}$ is a rank 4 distribution $\Delta$ on the 7-manifold $\mathcal{M}$ with coordinates $z_1, z_2, z_3, z_4, z_5, z_6, z_7$. The explicit formula for the quotient map $q : \mathcal{M}^{(1)} \times \mathcal{M}'^{(1)} \rightarrow \mathcal{M}$ is
$\tilde{M}^{(1)} \times \tilde{M}^{(1)} \to M$ is given by

\[
\begin{align*}
  z_1 &= x, \quad z_2 = y, \quad z_3 = \frac{(z-w)\delta^3 + 4u_1v_1\delta^2 - 4(u-v)(u_1+v_1)\delta + 4(u-v)^2}{(2u-2v-(u_1+v_1)\delta)\delta^3}, \\
  z_4 &= \frac{u_1-v_1-\delta u_2}{(2u-2v-(u_1+v_1)\delta)\delta}, \quad z_5 = \frac{u_1-v_2-\delta v_2}{(2u-2v-(u_1+v_1)\delta)\delta}, \\
  z_6 &= -\frac{u_3}{2u-2v-(u_1+v_1)\delta}, \quad z_7 = -\frac{v_3}{2u-2v-(u_1+v_1)\delta},
\end{align*}
\]

where $\delta = x - y$.

Calculating the pushforward of $\tilde{\mathcal{V}} \oplus \tilde{\mathcal{V}}$ by $q$ give the rank 4 hyperbolic distribution

$\Delta = \tilde{\Delta} \oplus \tilde{\Delta}$ where $\tilde{\Delta} = \{\tilde{X}_1, \tilde{X}_2\}$ and $\tilde{\Delta} = \{\tilde{X}_1, \tilde{X}_2\}$ are given by

\[
\begin{align*}
  \tilde{X}_1 &= \partial_{z_1} - \frac{3 - 2\delta^2 z_4 - \delta^4 z_4}{\delta^4} \partial_{z_3} - \frac{z_4 - \delta z_6 + \delta^2 z_4}{\delta} \partial_{z_4} - \frac{z_4(1 + \delta^2 z_5)}{\delta} \partial_{z_5} \\
  &\quad - \delta z_4 z_7 \partial_{z_7}, \\
  \tilde{X}_2 &= \partial_{z_6}, \\
  \tilde{X}_1 &= \partial_{z_2} + \frac{3 + 2\delta^2 z_5 - \delta^4 z_5}{\delta^4} \partial_{z_3} + \frac{z_5(1 - \delta^2 z_4)}{\delta} \partial_{z_4} + \frac{z_5 + \delta z_7 - \delta^2 z_5}{\delta} \partial_{z_5} \\
  &\quad - \delta z_5 z_6 \partial_{z_6}, \\
  \tilde{X}_2 &= \partial_{z_7},
\end{align*}
\]

where $\delta = z_1 - z_2$. It is easily verified that the derived dimensions for $\Delta$ are (4,6,7), that $\Delta$ has no Cauchy characteristics, and that $\Delta'$ has Cauchy characteristics

$$\mathcal{A}(\Delta') = \{\partial_{z_6}, \partial_{z_7}\},$$

so $\Delta$ defines a hyperbolic PDE in the plane. Moreover, $\tilde{\Delta}$ has a single first integral $z_2$, and $\tilde{\Delta}$ has a single first integral $z_1$, each of which survive on the 5-manifold $\tilde{M} = M/\mathcal{A}(\Delta')$. This implies that the PDE defined by $\Delta$ must be an $f$-Gordon equation. Finally, the first
prolongation of $\Delta$ is given by $\Delta^{(1)} = \tilde{\Delta}^{(1)} \oplus \bar{\Delta}^{(1)}$ where

$$\tilde{\Delta}^{(1)} = \left\{ \tilde{X}_1 + z_8 \partial_{z_8} + (z_4 z_7 - z_5 z_7 + \delta^2 z_4 z_5 z_7 - \delta z_4 z_9) \partial_{z_9}, \partial_{z_8} \right\},$$

$$\bar{\Delta}^{(1)} = \left\{ \bar{X}_1 + z_9 \partial_{z_9} + (z_4 z_6 - z_5 z_6 + \delta^2 z_4 z_5 z_6 - \delta z_5 z_8) \partial_{z_8}, \partial_{z_9} \right\}.$$ 

The first integrals of $\tilde{\Delta}^{(1)}$ and $\bar{\Delta}^{(1)}$ are

$$\tilde{I}_0 = z_2, \quad \tilde{I}_3 = \frac{z_9 + (z_1 - z_2) z_5 z_7}{z_7} \quad \text{and} \quad \bar{I}_0 = z_1, \quad \bar{I}_3 = \frac{z_8 + (z_1 - z_2) z_4 z_6}{z_6},$$

and we conclude that $\Delta$ is Darboux integrable after one prolongation.

In order to find the explicit $f$-Gordon equation corresponding to $\Delta$, we calculate the second prolongation of the distributions $\hat{V}$ and $\hat{W}$ to 7-dimensional manifolds $\hat{M}^{(2)}$ and $\hat{M}^{(2)}$ giving

$$\hat{V}^{(2)} = \{ \partial_x + u_2^2 \partial_z + u_1 \partial_u + u_2 \partial_{u_1} + u_3 \partial_{u_2} + u_4 \partial_{u_3}, \partial_{u_4} \},$$

$$\hat{W}^{(2)} = \{ \partial_y + v_2^2 \partial_w + v_1 \partial_v + v_2 \partial_{v_1} + v_3 \partial_{v_2} + v_4 \partial_{v_3}, \partial_{v_4} \},$$

and the diagonal action $\Gamma^{(2)}_{\text{diag}}$ on the product manifold $\hat{M}^{(2)} \times \hat{M}^{(2)}$ generated by vector fields

$$Z_1^{(1)} = Z_1, \quad Z_2^{(1)} = Z_2, \quad Z_3^{(1)} = Z_3, \quad Z_4^{(1)} = Z_4, \quad Z_5^{(1)} = Z_5 + u_4 \partial_{u_4} + v_4 \partial_{v_4}.$$ 

The lowest order joint-invariant for this action is given by

$$u = \frac{4(z - w)}{(2u - 2v + \delta(u_1 + v_1))^2} - \frac{4(u_1 - v_1)}{(2u - 2v + \delta(u_1 + v_1))^2 \delta}.$$

If we define total differential operators,

$$D_x = \partial_x + u_2^2 \partial_z + u_1 \partial_u + u_2 \partial_{u_1} + u_3 \partial_{u_2} + u_4 \partial_{u_3},$$

$$D_y = \partial_y + v_2^2 \partial_w + v_1 \partial_v + v_2 \partial_{v_1} + v_3 \partial_{v_2} + v_4 \partial_{v_3},$$

$$D_z = \partial_z + u_2^2 \partial_{z_2} + u_1 \partial_{z_1}.$$
then we can define first order invariants \( p = D_x(u) \) and \( q = D_y(u) \) which satisfy

\[
p = \alpha u + \frac{\alpha^2}{(x-y)^2} \quad \text{and} \quad q = \beta u - \frac{\beta^2}{(x-y)^2}.
\]  

(10.8)

where the functions \( \alpha \) and \( \beta \) are explicitly given by

\[
\alpha = \frac{2((u_1 - v_1) - \delta u_2)}{\delta(u_1 + v_1) - 2(u - v)} \quad \text{and} \quad \beta = \frac{2((u_1 - v_1) - \delta v_2)}{\delta(u_1 + v_1) - 2(u - v)}.
\]

In addition, \( \alpha \) and \( \beta \) satisfy the differential relations

\[
D_y(\alpha) = D_x(\beta) = \frac{\alpha \beta}{2} - \frac{\alpha - \beta}{x-y}.
\]  

(10.9)

Calculating \( s = D_y(p) = D_x(q) \), and substituting (10.9), gives the \( f \)-Gordon equation

\[
s = \left( \frac{3\alpha \beta \delta - 2\alpha + 2\beta}{2\delta} \right) u + \frac{\alpha \beta((\alpha - \beta) \delta + 2)}{\delta^3}
\]  

(10.10)

where \( \delta = x - y \) and \( \alpha, \beta \) satisfy the system (10.9). This equation is defined strictly on the 7-manifold, and by construction, is an integrable extension of (10.8).

**Example 10.7.3.** We now take \( \hat{V} \) and \( \tilde{V} \) to be two copies of the \((2,3,5)\)-distribution generated by the equation \( z' = (y'')^{m} \) with \( m \neq -1, 0, 1, \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \), or 2 on 5-dimensional manifolds \( \hat{M} = \{ x, z, u, u_1, u_2 \} \) and \( \tilde{M} = \{ y, w, v, v_1, v_2 \} \), respectively, so that

\[
\hat{V} = \{ \partial_x + u_2^m \partial_z + u_1 \partial_u + u_2 \partial_{u_1}, \partial_{u_2} \} \quad \text{and} \quad \tilde{V} = \{ \partial_y + v_2^m \partial_z + v_1 \partial_v + v_2 \partial_{v_1}, \partial_{v_2} \}.
\]

The first prolongation of these distributions to the 6-dimensional manifolds \( \hat{M}^{(1)} \) and \( \tilde{M}^{(1)} \) are given by

\[
\hat{V}^{(1)} = \{ \partial_x + u_2^m \partial_z + u_1 \partial_u + u_2 \partial_{u_1} + u_3 \partial_{u_2}, \partial_{u_3} \},
\]

\[
\tilde{V}^{(1)} = \{ \partial_y + v_2^m \partial_z + v_1 \partial_v + v_2 \partial_{v_1} + v_3 \partial_{v_2}, \partial_{v_3} \}.
\]
The diagonal action of the 5-dimensional symmetry algebra $a_9$ of Table 10.1 on the product manifold $\hat{M}^{(1)} \times \bar{M}^{(1)}$ is generated by the vector fields

\[
Z_1 = \partial_u + \partial_v, \quad Z_2 = x\partial_u + \partial_{u_1} + y\partial_v + \partial_{v_1}, \quad Z_3 = \partial_x + \partial_y, \quad Z_4 = \partial_z + \partial_w,
\]
\[
Z_5 = x\partial_x + z\partial_z + 2u\partial_u + u_1\partial_{u_1} - u_3\partial_{u_3} + y\partial_y + w\partial_w + 2v\partial_v + v_1\partial_{v_1} - v_3\partial_{v_3}.
\]

The quotient of $V^{(1)} \oplus V^{(1)}$ by $\Gamma^{(1)}_{\text{diag}}$ is a rank 4 distribution $\Delta$ on the 7-manifold $M$ with coordinates $z_1, z_2, z_3, z_4, z_5, z_6, z_7$. The explicit formula for the quotient map $q : \hat{M}^{(1)} \times \bar{M}^{(1)} \rightarrow M$ is given by

\[
\begin{align*}
 z_1 &= u_2, \quad z_2 = v_2, \quad z_3 = \frac{z - w}{x - y}, \quad z_4 = \frac{u - v - (x - y)u_1}{(x - y)^2}, \quad z_5 = \frac{u - v - (x - y)v_1}{(x - y)^2}, \\
 z_6 &= -u_3(x - y), \quad z_7 = -v_3(x - y)
\end{align*}
\]

Calculating the pushforward of $\hat{V} \oplus \bar{V}$ by $q$ gives the rank 4 hyperbolic distribution $\Delta = \hat{\Delta} \oplus \bar{\Delta}$ where

\[
\begin{align*}
 \hat{\Delta} &= \left\{ \hat{X}_1 = z_6\partial_{z_1} + (z_3 - z_1^m)\partial_{z_3} + (z_1 + 2z_4)\partial_{z_4} + (z_4 + z_5)\partial_{z_5} - z_7\partial_{z_7}, \hat{X}_2 = \partial_{z_6} \right\}, \\
 \bar{\Delta} &= \left\{ \bar{X}_1 = z_7\partial_{z_2} - (z_3 - z_2^m)\partial_{z_3} - (z_4 + z_5)\partial_{z_4} + (z_2 - 2z_5)\partial_{z_5} + z_6\partial_{z_6}, \bar{X}_2 = \partial_{z_7} \right\}.
\end{align*}
\]

It is easily verified that the derived dimensions for $\Delta$ are $(4,6,7)$, that $\Delta$ has no Cauchy characteristics, and that $\Delta'$ has Cauchy characteristics

\[\mathcal{A}(\Delta') = \{\partial_{z_6}, \partial_{z_7}\}.\]

Theorem 2.3.1 them implies that $\Delta$ defines a hyperbolic PDE in the plane. Moreover, $\hat{\Delta}$ has a single first integral $z_2$, and $\bar{\Delta}$ has a single first integral $z_1$, each of which survive on the 5-manifold $\bar{M} = M/\mathcal{A}(\Delta')$. This implies that the PDE defined by $\Delta$ must be an $f$-Gordon
equation. Finally, the first prolongation of $\Delta$ is given by $\Delta^{(1)} = \hat{\Delta}^{(1)} \oplus \tilde{\Delta}^{(1)}$ where

$$\hat{\Delta}^{(1)} = \left\{ \hat{X}_1 - (z_6 - z_8)\partial_{z_6} - 2z_9\partial_{z_9}, \partial_{z_8} \right\} \quad \text{and} \quad \tilde{\Delta}^{(1)} = \left\{ \tilde{X}_1 + (z_7 + z_9)\partial_{z_7} + 2z_8\partial_{z_8}, \partial_{z_9} \right\}.$$ 

The first integrals of $\hat{\Delta}^{(1)}$ and $\tilde{\Delta}^{(1)}$ are

$$\hat{I}_0 = z_2, \quad \hat{I}_3 = \frac{z_9}{z_7} \quad \text{and} \quad \tilde{I}_0 = z_1, \quad \tilde{I}_3 = \frac{z_8}{z_6},$$

and we conclude that $\Delta$ is Darboux integrable after one prolongation.

**Example 10.7.4.** Though this is not always the case, the previous example has a simple analog where $m = 1/2$. The distributions on the 5-dimensional manifolds $\hat{M}$ and $\tilde{M}$ are given by

$$\hat{V} = \{ \partial_x + \sqrt{w_2}\partial_z + u_1\partial_u + u_2\partial_{u_1}, \partial_{u_2} \} \quad \text{and} \quad \tilde{V} = \{ \partial_y + \sqrt{v_2}\partial_w + v_1\partial_v + v_2\partial_{v_1}, \partial_{v_2} \},$$

and the action of $a_9$ is equivalent to that of $b_7$ in Table 10.2. The prolonged diagonal action $\Gamma^{(1)}_{\text{diag}}$ remains exactly the same, as does the quotient map to the 7-dimensional manifold $M$.

Calculating the pushforward of $\hat{V} \oplus \tilde{V}$ by the quotient map $q : \hat{M}^{(1)} \times \tilde{M}^{(1)} \to M$ gives the rank 4 hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ where

$$\hat{\Delta} = \left\{ \hat{X}_1 = z_6\partial_{z_1} + (z_3 - \sqrt{z_7})\partial_{z_3} + (z_1 + 2z_4)\partial_{z_4} + (z_4 + z_5)\partial_{z_5} - z_7\partial_{z_7}, \hat{X}_2 = \partial_{z_6} \right\},$$

$$\tilde{\Delta} = \left\{ \tilde{X}_1 = z_7\partial_{z_2} - (z_3 - \sqrt{z_2})\partial_{z_3} - (z_4 + z_5)\partial_{z_4} + (z_2 - 2z_5)\partial_{z_5} + z_6\partial_{z_6}, \tilde{X}_2 = \partial_{z_7} \right\},$$

and again it can be verified that $\Delta$ defines an $f$-Gordon equation which is Darboux integrable after one prolongation.

**Example 10.7.5.** Let $\hat{V}$ and $\tilde{V}$ be two copies of the $(2,3,5)$-distribution generated by the equation $z' = \ln(y'')$ on 5-dimensional manifolds $\hat{M} = \{ x, z, u, u_1, u_2 \}$ and
\( \tilde{M} = \{ y, w, v, v_1, v_2 \} \), respectively, so that

\[
\hat{\mathcal{V}} = \{ \partial_x + \ln(u_2)\partial_z + u_1\partial_u + u_2\partial_{u_1}, \partial_{u_2} \} \quad \text{and} \quad \tilde{\mathcal{V}} = \{ \partial_y + \ln(v_2)\partial_z + v_1\partial_v + v_2\partial_{v_1}, \partial_{v_2} \}.
\]

The first prolongation of these distributions to the 6-dimensional manifolds \( \tilde{M}^{(1)} \) and \( \tilde{M}^{(1)} \) are given by

\[
\hat{\mathcal{V}}^{(1)} = \{ \partial_x + \ln(u_2)\partial_z + u_1\partial_u + u_2\partial_{u_1} + u_3\partial_{u_2}, \partial_{u_3} \},
\]

\[
\tilde{\mathcal{V}}^{(1)} = \{ \partial_y + \ln(v_2)\partial_z + v_1\partial_v + v_2\partial_{v_1} + v_3\partial_{v_2}, \partial_{v_3} \}.
\]

If we quotient by the prolonged diagonal action of \( c_4 \) in Table 10.3, we can again obtain an analog to the distribution in Example 10.7.3. The diagonal action of \( c_4 \) on the product manifold \( \tilde{M}^{(1)} \times \tilde{M}^{(1)} \) is generated by the vector fields

\[
Z_1 = \partial_u + \partial_v, \quad Z_2 = -\partial_x + \partial_z - \partial_y + \partial_w, \quad Z_3 = 2\partial_x + 2\partial_w,
\]

\[
Z_4 = -x\partial_u - u_1\partial_{u_1} - y\partial_v - v_1\partial_{v_1},
\]

\[
Z_5 = x\partial_x + z\partial_z + 2u\partial_u + u_1\partial_{u_1} - u_3\partial_{u_3} + y\partial_y + w\partial_w + 2v\partial_v + v_1\partial_{v_1} - v_3\partial_{v_3}.
\]

The quotient of \( \hat{\mathcal{V}}^{(1)} \oplus \tilde{\mathcal{V}}^{(1)} \) by \( \Gamma^{(1)}_{\text{diag}} \) is a rank 4 distribution \( \Delta \) on the 7-manifold \( M \) with coordinates \( z_1, z_2, z_3, z_4, z_5, z_6, z_7 \). The explicit formula for the quotient map \( q : \hat{M}^{(1)} \times \tilde{M}^{(1)} \to M \) is given by

\[
z_1 = u_2, \quad z_2 = v_2, \quad z_3 = \frac{z - w}{x - y}, \quad z_4 = \frac{u - v - (x - y)u_1}{(x - y)^2}, \quad z_5 = \frac{u - v - (x - y)v_1}{(x - y)^2},
\]

\[
z_6 = -u_3(x - y), \quad z_7 = -v_3(x - y)
\]
Calculating the pushforward of $\tilde{V} \oplus \tilde{V}$ by $q$ gives the rank 4 hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) where

\[
\hat{\Delta} = \left\{ \hat{X}_1 = z_6 \partial_{z_1} + (z_3 - \ln(z_1)) \partial_{z_3} + (z_1 + 2z_4) \partial_{z_4} + (z_4 + z_5) \partial_{z_5} - z_7 \partial_{z_7}, \hat{X}_2 = \partial_{z_6} \right\},
\]

\[
\tilde{\Delta} = \left\{ \tilde{X}_1 = z_7 \partial_{z_2} - (z_3 - \ln(z_2)) \partial_{z_3} - (z_4 + z_5) \partial_{z_4} + (z_2 - 2z_5) \partial_{z_5} + z_6 \partial_{z_6}, \tilde{X}_2 = \partial_{z_7} \right\}.
\]

It is easily verified that the derived dimensions for \( \Delta \) are (4,6,7), that \( \Delta \) has no Cauchy characteristics, and that \( \Delta' \) has Cauchy characteristics

\[
\mathcal{A}(\Delta') = \{ \partial_{z_6}, \partial_{z_7} \}.
\]

Theorem 2.3.1 then implies that \( \Delta \) defines a hyperbolic PDE in the plane. Moreover, \( \hat{\Delta} \)
has a single first integral \( z_2 \), and \( \tilde{\Delta} \) has a single first integral \( z_1 \), each of which survive on the 5-manifold \( \tilde{M} = M/A(\Delta') \). This implies that the PDE defined by \( \Delta \) must be an \( f \)-Gordon equation. Finally, the first prolongation of \( \Delta \) is given by \( \Delta^{(1)} = \hat{\Delta}^{(1)} \oplus \tilde{\Delta}^{(1)} \) where

\[
\hat{\Delta}^{(1)} = \left\{ \hat{X}_1 - (z_6 - z_8) \partial_{z_6} - 2z_9 \partial_{z_9}, \partial_{z_8} \right\} \quad \text{and} \quad \tilde{\Delta}^{(1)} = \left\{ \tilde{X}_1 + (z_7 + z_9) \partial_{z_7} + 2z_8 \partial_{z_8}, \partial_{z_9} \right\}.
\]

The first integrals of \( \hat{\Delta}^{(1)} \) and \( \tilde{\Delta}^{(1)} \) are

\[
\hat{I}_0 = z_2, \quad \hat{I}_3 = \frac{z_9}{z_7} \quad \text{and} \quad \tilde{I}_0 = z_1, \quad \tilde{I}_3 = \frac{z_8}{z_6},
\]

and we conclude that \( \Delta \) is Darboux integrable after one prolongation.
Part IV

Miscellanea
CHAPTER 11
EQUATIONS OF STRICT MONGE-AMPIÈRE TYPE

As previously stated in the last remark of Section 8.2, every classical Darboux integrable equation of Monge-Ampère type presented in the Goursat’s treatise [19] is equivalent to an \( f \)-Gordon equation. In this chapter, we give two examples of hyperbolic distributions which define Darboux integrable equations of Monge-Ampère type that are not equivalent to \( f \)-Gordon equations. The first equation is Darboux integrable at order four, and the second equation is Darboux integrable at order five. The construction of these equations gives rise to a family of equations of Monge-Ampère type which are Darboux integrable order \( k \geq 2 \).

11.1 An Example Darboux Integrable at Order Four

In this section, we use the quotient theory of Darboux integrable systems to construct a hyperbolic distribution for a nonlinear equation of Monge-Ampère type which is Darboux integrable at order four and is not equivalent to an \( f \)-Gordon equation.

Let \( J^5(\mathbb{R}, \mathbb{R}) \times J^5(\mathbb{R}, \mathbb{R}) \) be the product of jet spaces with coordinates

\[
(x, u, u_1, u_2, u_3, u_4, y, v, v_1, v_2, v_3, v_4, v_5),
\]

and let \( \hat{\mathcal{V}} \) and \( \tilde{\mathcal{V}} \) be the canonical contact distributions on \( J^5(\mathbb{R}, \mathbb{R}) \) given by

\[
\hat{\mathcal{V}} = \{ \partial_x + u_1 \partial_u + u_2 \partial_u_1 + u_3 \partial_u_2 + u_4 \partial_u_3 + u_5 \partial_u_4, \partial_u_5 \},
\]

\[
\tilde{\mathcal{V}} = \{ \partial_y + v_1 \partial_v + v_2 \partial_v_1 + v_3 \partial_v_2 + v_4 \partial_v_3 + v_5 \partial_v_4, \partial_v_5 \}.
\]

Let \( G \) be the 7-dimensional symmetry group of both \( \hat{\mathcal{V}} \) and \( \tilde{\mathcal{V}} \) acting diagonally on \( J^5(\mathbb{R}, \mathbb{R}) \times J^5(\mathbb{R}, \mathbb{R}) \) whose infinitesimal generators are given by the 5-fold prolongation of the vector

\[
\{ \partial_x + u_1 \partial_u + u_2 \partial_u_1 + u_3 \partial_u_2 + u_4 \partial_u_3 + u_5 \partial_u_4, \partial_u_5 \},
\]

and

\[
\{ \partial_y + v_1 \partial_v + v_2 \partial_v_1 + v_3 \partial_v_2 + v_4 \partial_v_3 + v_5 \partial_v_4, \partial_v_5 \}.
\]
fields

\[ Z_1 = \partial_x + \partial_y, \quad Z_2 = x \partial_x + 5u \partial_u + y \partial_y + 5v \partial_v, \quad Z_3 = \partial_u + \partial_v, \]

\[ Z_4 = x \partial_u + y \partial_v, \quad Z_5 = x^2 \partial_u + y^2 \partial_v, \quad Z_6 = x^3 \partial_u + x^3 \partial_v, \quad Z_7 = x^4 \partial_u + y^4 \partial_v. \]

The quotient of \( \hat{V} \oplus \tilde{V} \) by the diagonal action of \( G \) is a rank 4 distribution \( \Delta \) defined on the 7-dimensional manifold \( M \) with coordinates \( z_1, z_2, z_3, z_4, z_5, z_6, z_7 \). The explicit formula for the quotient map \( q : J^5(\mathbb{R}, \mathbb{R}) \times J^5(\mathbb{R}, \mathbb{R}) \to M \) is given by by

\[
\begin{align*}
z_1 &= \frac{u_2 - v_2}{(x - y)^3} - \frac{6(u_1 - v_1)}{(x - y)^4} + \frac{12(u - v)}{(x - y)^5}, \\
z_2 &= \frac{u_3}{(x - y)^2} - \frac{2(2u_1 + v_1)}{(x - y)^3} + \frac{6(u_1 - v_1)}{(x - y)^4}, \\
z_3 &= \frac{v_3}{(x - y)^2} - \frac{2(u_1 + 2v_1)}{(x - y)^3} + \frac{6(u_1 - v_1)}{(x - y)^4}, \\
z_4 &= \frac{u_4}{x - y} - \frac{u_3 - v_3}{(x - y)^2}, \\
z_5 &= \frac{v_4}{x - y} - \frac{u_3 - v_3}{(x - y)^2}, \\
z_6 &= u_5, \quad z_7 = v_5.
\end{align*}
\]

Calculating the pushforward of \( \hat{V} \oplus \tilde{V} \) by \( q \) gives the rank 4 hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) where

\[
\hat{\Delta} = \begin{cases}
\hat{X}_1 = (5z_1 - z_2) \partial_{z_1} + (5z_2 + z_3 - z_4) \partial_{z_2} - 2(z_2 - z_3) \partial_{z_3} + (2z_4 - z_5) \partial_{z_4} + (z_4 + z_5) \partial_{z_5}, \\
\hat{X}_2 = \partial_{z_6},
\end{cases}
\]

\[
\tilde{\Delta} = \begin{cases}
\tilde{X}_1 = (5z_1 - z_3) \partial_{z_1} + 2(z_2 - z_3) \partial_{z_2} + (z_2 + 5z_3 + z_5) \partial_{z_3} + (z_4 + z_5) \partial_{z_4} + (2z_5 + z_7) \partial_{z_5}, \\
\tilde{X}_2 = \partial_{z_7}.
\end{cases}
\]

The distribution \( \Delta \) has derived flag dimensions \((4, 6, 7)\), no Cauchy characteristics, and it’s derived system \( \Delta' \) has two Cauchy characteristics \( \{ \partial_{z_6}, \partial_{z_7} \} \). By Theorem 2.3.1
Δ therefore defines a PDE in the plane. Moreover since, \( \Delta \) and \( \bar{\Delta} \) each have derived dimensions \((2, 3, 4, 5, 6)\), we conclude that \( \Delta \) is of Monge-Ampère type by Theorem 2.3.5. The distribution \( \Delta \) does not define an \( f \)-Gordon equation however, since \( \Delta \) admits only \( z_7 \) as a first integral and \( \bar{\Delta} \) admits only \( z_6 \) as a first integral, neither of which exist on the 5-dimensional manifold \( M/\{\partial z_6, \partial z_7\} \).

Finally after calculating the second prolongation of \( \Delta \) to the 11-dimensional manifold \( M^{(2)} \) with coordinates \( \{z_i\}_{i=1}^{11} \) we obtain \( \Delta^{(2)} = \hat{\Delta}^{(2)} \oplus \tilde{\Delta}^{(2)} \) where

\[
\begin{align*}
\hat{\Delta}^{(2)} &= \begin{cases} 
\hat{Y}_1 = \hat{X}_1 + z_8 \partial z_6 + z_{10} \partial z_8 - z_9 \partial z_9 - (z_9 + 2z_{11}) \partial z_{11}, \\
\hat{Y}_2 = \partial z_{10},
\end{cases} \\
\tilde{\Delta}^{(2)} &= \begin{cases} 
\tilde{Y}_1 = \tilde{X}_1 + z_9 \partial z_7 - z_8 \partial z_8 + z_{11} \partial z_9 - (2z_{10} + z_8) \partial z_{10}, \\
\tilde{Y}_2 = \partial z_{11},
\end{cases}
\end{align*}
\]

define the hyperbolic distribution \( \Delta^{(2)} \). We see that \( \hat{\Delta}^{(2)} \) has first integrals

\[
\hat{I}_2 = z_6, \quad \hat{I}_4 = \frac{z_{10} - z_8}{z_8^2},
\]

and \( \tilde{\Delta}^{(2)} \) has first integrals

\[
\tilde{I}_2 = z_7, \quad \tilde{I}_4 = \frac{z_{11} - z_9}{z_9^2}.
\]

We therefore conclude that \( \Delta \) corresponds to a Monge-Ampère equation which is Darboux integrable at order four and is not equivalent to an \( f \)-Gordon equation.

### 11.2 An Example Darboux Integrable at Order Five

We now use the quotient theory of Darboux integrable systems to construct a hyperbolic distribution for a nonlinear equation of Monge-Ampère type which is Darboux integrable at order five and is not equivalent to an \( f \)-Gordon equation.

Let \( J^6(\mathbb{R}, \mathbb{R}) \times J^6(\mathbb{R}, \mathbb{R}) \) be the product of jet spaces with coordinates \((x, u, u_i, y, v, v_i)\)
where $1 \leq i \leq 6$. Let $G$ be the 9-dimensional symmetry group of both $\hat{\nabla}$ and $\tilde{\nabla}$ acting diagonally on $J^6(\mathbb{R}, \mathbb{R}) \times J^6(\mathbb{R}, \mathbb{R})$ whose infinitesimal generators are given by the 6-fold prolongation of the vector fields

$$Z_1 = \partial_x + \partial_y, \quad Z_2 = x\partial_x + 7u\partial_u + y\partial_y + 7v\partial_v, \quad Z_3 = \partial_u + \partial_v,$$

$$Z_4 = x\partial_u + y\partial_v, \quad Z_5 = x^2\partial_u + y^2\partial_v, \quad Z_6 = x^3\partial_u + x^3\partial_v, \quad Z_7 = x^4\partial_u + y^4\partial_v,$$

$$Z_8 = x^5\partial_u + y^5\partial_v, \quad Z_9 = x^6\partial_u + y^6\partial_v.$$

The quotient of $\hat{\nabla} \oplus \tilde{\nabla}$ by the diagonal action of $G$ is a rank 4 distribution $\Delta$ defined on the 7-dimensional manifold $M$ with coordinates $z_1, z_2, z_3, z_4, z_5, z_6, z_7$. The explicit formula for the quotient map $q : J^6(\mathbb{R}, \mathbb{R}) \times J^6(\mathbb{R}, \mathbb{R}) \to M$ is given by by

$$z_1 = \frac{u_3 + v_3}{(x-y)^3} + \frac{12(u_2 + v_2)}{(x-y)^5} + \frac{60(u_1 + v_1)}{(x-y)^6} - \frac{120(u - v)}{(x-y)^7},$$

$$z_2 = \frac{u_4}{(x-y)^4} - \frac{3(3u_3 - v_3)}{(x-y)^4} + \frac{12(3u_2 + 2v_2)}{(x-y)^5} - \frac{60(u_1 - v_1)}{(x-y)^6},$$

$$z_3 = \frac{v_4}{(x-y)^4} - \frac{3(u_3 - 3v_3)}{(x-y)^4} + \frac{12(2u_2 + 3v_2)}{(x-y)^5} - \frac{60(u_1 - v_1)}{(x-y)^6},$$

$$z_4 = \frac{u_5}{(x-y)^3} - \frac{2(2u_4 + v_4)}{(x-y)^3} + \frac{6(u_3 - v_3)}{(x-y)^4},$$

$$z_5 = \frac{v_5}{(x-y)^3} + \frac{2(u_4 + 2v_4)}{(x-y)^3} - \frac{6(u_3 - v_3)}{(x-y)^4},$$

$$z_6 = u_6 \frac{1}{x-y} - \frac{u_5 - v_5}{(x-y)^2}, \quad z_7 = \frac{v_6}{x-y} - \frac{u_5 - v_5}{(x-y)^2}.$$

Calculating the pushforward of $\hat{\nabla} \oplus \tilde{\nabla}$ by $q$ gives the rank 4 hyperbolic distribution $\Delta = \tilde{\Delta} \oplus \tilde{\Delta}$ where the characteristic distributions $\tilde{\Delta}$ and $\tilde{\Delta}$ are generated by

$$\hat{\chi}_1 = (7z_1 - z_2)\partial_{z_1} + (8z_2 - 2z_3 - z_4)\partial_{z_2} + 3(z_2 + z_3)\partial_{z_3} + (5z_4 + z_5 - z_6)\partial_{z_4} - (2z_4 - 2z_5)\partial_{z_5} + 2z_6\partial_{z_6} + (z_6 + z_7)\partial_{z_7},$$

$$\hat{\chi}_2 = \partial_{z_6},$$
and

\[ \tilde{X}_1 = (7z_1 + z_3)\partial_{z_1} + 3(z_2 + z_3)\partial_{z_2} - (2z_2 - 8z_3 - z_5)\partial_{z_3} + 2(z_4 - z_5)\partial_{z_4} \\
+ (z_4 + 5z_5 + z_7)\partial_{z_5} + (z_6 + z_7)\partial_{z_6} + 2z_7\partial_{z_7}, \]

\[ \tilde{X}_2 = \partial_{z_7}. \]

Again, one can show that \( \Delta \) satisfies the hypotheses of Theorem 2.3.1 and therefore corresponds to a PDE in the plane. The characteristic distributions have derived dimensions \((2,3,4,5,6,7)\), and consequently, have no first integrals. This implies that the corresponding PDE cannot be an \( f \)-Gordon equation, however, it will be Monge-Ampère type by Theorem 2.3.5.

Finally after calculating the third prolongation of \( \Delta \) to the 13-dimensional manifold \( M^{(3)} \) with coordinates \( \{z_i\}_{i=1}^{13} \) we obtain the hyperbolic distribution \( \Delta^{(3)} = \tilde{\Delta}^{(3)} \oplus \bar{\Delta}^{(3)} \) where \( \tilde{\Delta}^{(3)} \) and \( \bar{\Delta}^{(3)} \) are generated by

\[ \tilde{Y}_1 = \tilde{X}_1 + (2z_6 - z_8)\partial_{z_6} + (z_6 + z_7)\partial_{z_7} - z_{10}\partial_{z_8} - z_{12}\partial_{z_{10}} - z_{11}\partial_{z_{11}} + (z_{11} - 2z_{13})\partial_{z_{13}}, \]

\[ \tilde{Y}_2 = \partial_{z_{12}}, \]

\[ \tilde{Y}_1 = \tilde{X}_1 + (z_6 + z_7)\partial_{z_6} + (2z_7 - z_9)\partial_{z_7} - z_{11}\partial_{z_9} - z_{10}\partial_{z_{10}} - z_{13}\partial_{z_{11}} + (z_{10} - 2z_{12})\partial_{z_{12}}, \]

\[ \tilde{Y}_2 = \partial_{z_{13}}. \]

We see that \( \tilde{\Delta}^{(3)} \) has first integrals

\[ \hat{I}_3 = z_8, \quad \hat{I}_5 = \frac{z_{12} - z_{10}}{z_{10}^2}, \]

and \( \bar{\Delta}^{(3)} \) has first integrals

\[ \bar{I}_3 = z_9, \quad \bar{I}_5 = \frac{z_{13} - z_{11}}{z_{11}^2}. \]
We therefore conclude that $\Delta$ corresponds to a Monge-Ampère equation which is Darboux integrable at order five and is not equivalent to an $f$-Gordon equation.

**Remark.** In each of the above examples, the infinitesimal generators for the action of the group $G$ were given by prolongations of actions of type 1.7 in [27]. In general, these actions have generators

$$\left\{ \partial_x, x \partial_x + \alpha u \partial_u, x \partial_u, \ldots, x^{k-1} \partial_u \right\},$$

where the corresponding group is of dimension $k + 2$. In the case where $\alpha = k$, Olver states that this is the unique action whose prolonged orbit dimensions *pseudo-stabilize* at order $k + 1$, meaning that the group will admit differential invariants at orders $k$ and $k + 2$, but not at order $k + 1$.

Our construction of the previous examples can easily be generalized to generate a family of hyperbolic distributions which correspond to non-$f$-Gordon Monge-Ampère equations which are Darboux integrable at arbitrarily high orders.
CHAPTER 12
EQUATIONS OF GOURSAT TYPE

In this chapter, we calculate the fundamental invariants for two Darboux integrable equations of Goursat type.

12.1 Equation 1: \( u_{xx} = f(u_{xy}) \)

In [19], Goursat shows that every equation of the form

\[ r = f(s) \quad \text{with} \quad f_s \neq 0 \]

is Darboux integrable at order two. The hyperbolic distribution \( \Delta = \hat{\Delta} \oplus \tilde{\Delta} \) for (12.1) is given by

\[
\hat{\Delta} = \left\{ \hat{X}_1 = \partial_x + p \partial_u + f(s) \partial_p + s \partial_q, \hat{X}_2 = f'(s) \partial_x + \partial_t \right\}, \\
\tilde{\Delta} = \left\{ \tilde{X}_1 = \partial_x - f'(s) \partial_y + (p - f'(s)q) \partial_u + (f(s) - f'(s)s) \partial_p + (s - f'(s)t) \partial_q, \tilde{X}_2 = \partial_t \right\}.
\]

Immediately, we can check that the derived flag dimensions for \( \hat{\Delta} \) are \((2,3,5)\), and the derived flag dimensions for \( \tilde{\Delta} \) are \((2,3,4,5)\). Therefore, by Theorem 2.3.5, we see that (12.1) is indeed of Goursat type.

The characteristic distributions \( \hat{\Delta}, \tilde{\Delta} \) have first integrals

\[
\hat{I}_0 = y, \quad \hat{I}_2 = t - \int \frac{1}{f'} ds, \quad \text{and} \quad \tilde{I}_0 = y + xf', \quad \tilde{I}_1 = p - xf - xsf', \quad \tilde{I}_2 = s.
\]

Using Theorem 4.2.2, we construct the commuting bases of vector fields,

\[
\hat{\Delta} = \left\{ \hat{U}_1 = \frac{1}{f'} \hat{X}_1, \hat{U}_2 = \frac{1}{f'} \hat{X}_2 - \frac{x f''}{f'} \hat{X}_1 \right\} \quad \text{and} \quad \tilde{\Delta} = \left\{ \tilde{U}_1 = -\frac{1}{f'} \tilde{X}_1, \tilde{U}_2 = \tilde{X}_2 \right\}.
\]
We then compute the sequences of vector fields

\[ \hat{S}_1 = [\hat{U}_2, \hat{U}_1] = \partial_p + \frac{1}{f'} \partial_q, \quad \hat{S}_2 = [\hat{S}_1, \hat{U}_1] = \frac{1}{f'} \partial_u, \]

\[ \tilde{S}_1 = [\tilde{U}_2, \tilde{U}_1] = \partial_q, \quad \tilde{S}_2 = [\tilde{S}_1, \tilde{U}_1] = \partial_u. \]

These vector fields constitute local bases for $\hat{\mathfrak{H}} = \{\hat{S}_1, \hat{S}_2\}$ and $\tilde{\mathfrak{H}} = \{\tilde{S}_1, \tilde{S}_2\}$ for the 2-dimensional abelian Vessiot algebra of (12.1).

Upon restricting $\Delta$ to the 5-dimensional integral manifold $\hat{M}$ given by $\hat{I}_0 = \hat{I}_2 = 0$, we obtain the Vessiot distribution

\[ \hat{\nu} = \{\partial_x + p\partial_u + f\partial_p + s\partial_q, \partial_s\}, \]

and similarly, the restriction of $\Delta$ to the 4-dimensional integral manifold $\tilde{M}$ given by $\tilde{I}_0 = \tilde{I}_1 = \tilde{I}_2 = 0$, we obtain the Vessiot distribution

\[ \tilde{\nu} = \{\partial_x + (f(0)x + f'(0)q) \partial_u - f'(0)t\partial_q, \partial_t\}. \]

The derived flag dimensions of the Vessiot distribution $\hat{\nu}$ are (2,3,4), and so by Engel’s theorem, we immediately see that $\hat{\nu}$ is locally equivalent to the standard contact distribution on $J^2(\mathbb{R}, \mathbb{R})$. Moreover, since the restriction of $\hat{\mathfrak{H}}$ to $\hat{M}$ is a symmetry algebra of $\hat{\nu}$, we conclude that $\hat{\mathfrak{H}}$ corresponds to $\mathfrak{p}_{2,1}$.

The derived flag dimensions of the Vessiot distribution $\tilde{\nu}$ are (2,3,5). Using the FiveVariables package in Maple, we find that the Cartan quartic is given by

\[
F(x, y) = \left(10(f')^4(f'')^3 f^{(6)} - 80(f')^4(f''')^2(f''') f^{(5)} - 51(f')^4(f'')^2(f^{(4)})^2 + 336(f')^4(f''')^2 f^{(4)} - 224(f')^4(f''')^4 + 30(f')^3(f'')^4 f^{(5)} - 146(f')^3(f''')^2 f^{(4)} + 128(f')^3(f''')^2(f''')^3 - 6(f')^2(f''')^3 f^{(4)} + 4(f')^2(f''')^4 f^{(2)} + 12f'(f''')^6 f''' - 9(f''')^8 \right) x^4.
\]
We therefore conclude that $\hat{V}$ is either of root type $[\infty]$, in which case the action of the Vessiot algebra $\hat{\mathcal{V}}$ corresponds to either $[N, 11]$, $[N, 12]$, or $[N, 18]$ with $a = \frac{1}{3}$ in Table B.1, or $\hat{V}$ is of root type $[4]$.

12.2 Equation 2: $u_{xx}u_{xy} = u_x$

As we showed in Example 2.3.7, the equation

$$rs = p$$

(12.2)

is of Goursat type. In [19], Goursat shows this equation to be Darboux integrable at order three. The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ for the first prolongation of (12.2) is given by $\hat{\Delta} = \{\hat{X}_1, \hat{X}_2\}$ and $\tilde{\Delta} = \{\tilde{X}_1, \tilde{X}_2\}$ where

$$\hat{X}_1 = \partial_x + p\partial_u + \frac{p}{s}\partial_p + s\partial_q - \frac{p\sigma - s^2}{s^2}\partial_s + \sigma\partial_s + \frac{\sigma(2p\sigma - s^2)}{ps}\partial_r,$$

$$\hat{X}_2 = \partial_\sigma - \frac{s^2}{p}\partial_r,$$

$$\tilde{X}_1 = \partial_x + p\partial_u + \frac{p(s^2 + q)}{s^2}\partial_u + \frac{2p}{s}\partial_p + \frac{s^3 + pt}{s^2}\partial_q + \partial_s + \frac{s^2\sigma + pt}{s^2}\partial_t + \frac{\sigma(2p\sigma - s^2)}{s^3}\partial_\sigma,$$

$$\tilde{X}_2 = \partial_\tau.$$

The first integrals for $\hat{\Delta}$ are

$$\hat{I}_0 = y, \quad \hat{I}_3 = \tau + \frac{\sigma s^2}{p},$$

and the first integrals for $\tilde{\Delta}$ are

$$\tilde{I}_0 = s + x, \quad \tilde{I}_1 = y - \frac{xp}{s^2}, \quad \tilde{I}_2 = \frac{p}{s^2}, \quad \tilde{I}_3 = \frac{1}{\sigma s} - \frac{2p}{s^3}.$$
Using Theorem 4.2.2, we construct commuting bases \( \{\bar{U}_i\} \) and \( \{\bar{U}_j\} \) for \( \bar{\Delta} \) and \( \bar{\Delta} \), respectively, as

\[
\bar{\Delta} = \left\{ \bar{U}_1 = -\frac{s^2}{sp} \hat{X}_1 + 6\sigma^2 p^2 - 5\sigma s^2 p + s^4 \frac{X_2}{s^3 p}, \bar{U}_2 = -\sigma^2 s \hat{X}_2 \right\},
\]

\[
\bar{\Delta} = \left\{ \bar{U}_1 = \frac{s^2 X_1}{p} - \frac{2\sigma^2 sp - \sigma s^3}{p^2} \bar{X}_2, \bar{U}_2 = \bar{X}_2 \right\}.
\]

We can then calculate the sequence of vector fields,

\[
\bar{S}_1 = \left[ \bar{U}_1, \bar{U}_2 \right], \quad \bar{S}_2 = \left[ \bar{U}_1, \bar{S}_1 \right], \quad \bar{S}_3 = \left[ \bar{U}_1, \bar{S}_2 \right], \quad \bar{S}_4 = \left[ \bar{U}_1, \bar{S}_3 \right], \quad \bar{S}_5 = \left[ \bar{U}_1, \bar{S}_4 \right],
\]

\[
\tilde{S}_1 = \left[ \tilde{U}_1, \tilde{U}_2 \right] = -\partial_t, \quad \tilde{S}_2 = \left[ \tilde{U}_1, \tilde{S}_1 \right] = \partial_q, \quad \tilde{S}_3 = \left[ \tilde{U}_1, \tilde{S}_2 \right] = -\partial_u,
\]

and since

\[
\hat{\Delta}^{(\infty)} \cap \bar{\Delta}^{(\infty)} = \text{span}\{\partial_u, \partial_q, \partial_t\} = \text{span}\{\tilde{S}_3, \tilde{S}_4, \tilde{S}_5\} = \text{span}\{\bar{S}_1, \bar{S}_2, \bar{S}_3\},
\]

we take as bases for the Vessiot algebra of (12.2) \( \hat{\mathfrak{H}} = \{\hat{S}_3, \hat{S}_4, \hat{S}_5\} \) and \( \bar{\mathfrak{H}} = \{\bar{S}_1, \bar{S}_2, \bar{S}_3\} \).

The structure equations for \( \hat{\mathfrak{H}} \) are clearly

\[
[\hat{S}_1, \hat{S}_2] = [\hat{S}_1, \hat{S}_3] = [\hat{S}_2, \hat{S}_3] = 0,
\]

and though the generators for \( \hat{\mathfrak{H}} \) are more complicated, a simple calculation shows that

\[
[\hat{S}_1, \hat{S}_2] = [\hat{S}_1, \hat{S}_2] = [\hat{S}_2, \hat{S}_3] = 0,
\]

as well. As an abstract Lie algebra, the Vessiot algebra of (12.2) is the 3-dimensional abelian Lie algebra \( 3n_{1,1} \) in [28].

Upon restricting \( \bar{\Delta} \) to the 5-dimensional integral manifold \( \bar{M} \) given by \( \bar{I}_0 = \bar{I}_1 = \bar{I}_3 = 0 \) and \( \bar{I}_2 = 1 \), we obtain the Vessiot distribution for the prolongation of (12.2)

\[
\check{\mathfrak{V}} = \left\{ \partial_x + (x^2 + q)\partial_u + (x + t)\partial_q + \left( \tau + \frac{1}{2} \right) \partial_t, \partial_r \right\}
\]
which has both derived and weak derived dimensions \((2,3,4,5)\). By Theorem 2.5.2, we conclude that \(\hat{\mathcal{V}}\) is locally equivalent to the standard contact distribution on \(J^3(\mathbb{R}, \mathbb{R})\), and therefore the corresponding Vessiot distribution \(\hat{\mathcal{W}}\) for \((12.2)\) is the standard contact distribution on \(J^2(\mathbb{R}, \mathbb{R})\). Knowing this, we further conclude that the action \(\hat{\mathcal{V}}\) must correspond to \(p_{3,1}\) in Table A.1.

Upon restricting \(\hat{\Delta}\) to the 7-dimensional integral manifold \(\hat{M}\) given by \(\hat{I}_0 = \hat{I}_3 = 0\), we obtain the Vessiot distribution for the prolongation of \((12.2)\)

\[
\hat{\mathcal{V}} = \left\{ \partial_x + p \partial_u + \frac{p}{s} \partial_p + s \partial_q - \frac{sp - s^2}{s^2} \partial_s + \sigma \partial_t, \partial_\sigma \right\}.
\]

The derived flag dimensions for \(\hat{\mathcal{V}}\) are \((2,3,4,6,7)\), and the derived distribution \(\hat{\mathcal{V}}'\) has Cauchy characteristic \(\partial_\sigma\). Calculating the reduction of \(\hat{\mathcal{V}}'\) by \(\partial_\sigma\) gives the Vessiot distribution for \((12.2)\)

\[
\hat{\mathcal{W}} = \hat{\mathcal{V}}' / \{ \partial_\sigma \} = \left\{ \partial_x + p \partial_u + \frac{p}{s} \partial_p + s \partial_q + \frac{s^2}{p} \partial_t, \partial_s - \frac{s^2}{p} \partial_t \right\}
\]

The derived flag dimensions for \(\hat{\mathcal{W}}\) are \((2,3,5,6)\).

In [7], Anderson and Kruglikov classify \((2,3,5,6)\)-distributions into three types based upon their symbol algebras. The symbol algebra for \(\hat{\mathcal{W}}\) is

<table>
<thead>
<tr>
<th>symbol((\hat{\mathcal{W}}))</th>
<th>(e_1)</th>
<th>(e_2)</th>
<th>(e_3)</th>
<th>(e_4)</th>
<th>(e_5)</th>
<th>(e_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_1)</td>
<td>\cdot</td>
<td>(-e_3)</td>
<td>(e_4)</td>
<td>(e_6)</td>
<td>(-e_6)</td>
<td>\cdot</td>
</tr>
<tr>
<td>(e_2)</td>
<td>(e_3)</td>
<td>\cdot</td>
<td>(e_5)</td>
<td>(-e_6)</td>
<td>(e_6)</td>
<td>\cdot</td>
</tr>
</tbody>
</table>

According to their classification, this implies \(\hat{\mathcal{W}}\) must correspond to an equation of the form

\[
z'' = F(x, z, z', y, y', y'').
\] (12.3)

The maximal symmetry algebra for equations of the form \((12.3)\) is 8-dimensional, and any equation of the form \((12.3)\) which realizes this symmetry algebra is internally equivalent
to the equation

\[ z'' = (y'')^2. \] (12.4)

Upon calculating the symmetry algebra for \( \mathcal{W} \), we see that it is 8-dimensional and generated by the vector fields

\[ \begin{align*}
\Gamma_1 &= \partial_x, \quad \Gamma_2 = \partial_u, \quad \Gamma_3 = \partial_q, \quad \Gamma_4 = \partial_t, \\
\Gamma_5 &= -u \partial_u - p \partial_p + \partial_t, \\
\Gamma_6 &= x \partial_x + 4u \partial_u + 3p \partial_p + 2q \partial_q + s \partial_s, \\
\Gamma_7 &= \frac{p}{2} \partial_x + \frac{p^2}{4} \partial_u + u \partial_q + \frac{p}{2} \partial_s + \left( q - \frac{s^2}{2} \right) \partial_t, \\
\Gamma_8 &= \frac{s^2}{p} \partial_x + (q + s^2) \partial_u + 2s \partial_p + \frac{s^3 + pt}{p} \partial_q + \frac{s^2}{p} \partial_s.
\end{align*} \]

We therefore conclude that \( \mathcal{W} \) is equivalent to the rank 2 distribution generated by the equation (12.4). The action \( \mathcal{H} \) consequently must correspond to a 3-dimensional subalgebra of this symmetry algebra.

**Remark.** This example motivates a study of the equivalence problem for (2,3,5,6)-distributions on 6-dimensional manifolds. Indeed, by classifying these distributions, one can construct several new examples of Goursat and generic type equations which are Darboux integrable at order three or higher.
CHAPTER 13
EQUATIONS OF GENERIC TYPE

In this chapter, we calculate the fundamental invariants of two second order hyperbolic PDE of generic type which are Darboux integrable at order two. The first is the classical equation

$$3u_{xx}u_{yy}^3 + 1 = 0$$

which was originally studied by Goursat [19] and was shown by The [32] to have maximal 9-dimensional symmetry algebra.

The second equation is

$$3u_{xx}u_{yy}^3 = u_{xy}^3$$

which we believe to be a new example of a generic equation with 7-dimensional symmetry algebra which is Darboux integrable at order two.

13.1 Equation 1: $3u_{xx}u_{yy}^3 + 1 = 0$

We first consider the equation

$$3u_{xx}u_{yy}^3 + 1 = 0. \quad (13.1)$$

The hyperbolic distribution $\Delta = \hat{\Delta} \oplus \tilde{\Delta}$ is given by

$$\hat{\Delta} = \{ \hat{X}_1 = \partial_x + \frac{1}{t^2} \partial_y + \frac{pt^2 + q}{t^2} \partial_u + \frac{3st - 1}{3t^3} \partial_p + \frac{st + 1}{t} \partial_q, \hat{X}_2 = \partial_s + t^2 \partial_t \},$$

$$\tilde{\Delta} = \{ \tilde{X}_1 = \partial_x - \frac{1}{t^2} \partial_y + \frac{pt^2 - q}{t^2} \partial_u - \frac{3st + 1}{3t^3} \partial_p + \frac{st - 1}{t} \partial_q, \tilde{X}_2 = \partial_s - t^2 \partial_t \},$$
and the invariants are given by
\[
\tilde{I}_1 = \frac{st}{t} + 1, \quad \tilde{I}_2 = \frac{(q - xs)t - x}{t}, \quad \tilde{I}_1 = \frac{1 - st}{t}, \quad \tilde{I}_2 = \frac{(q - xs)t + x}{t}.
\]

Theorem 4.2.2 allows us to construct commuting bases \(\{\tilde{U}_i\}\) and \(\{\tilde{U}_j\}\) for \(\tilde{\Delta}\) and \(\tilde{\Delta}\), respectively, as
\[
\tilde{\Delta} = \left\{ \tilde{U}_1 = -\frac{xt}{2} \tilde{X}_1 - \frac{t^2}{2} \tilde{X}_2, \tilde{U}_2 = \frac{t}{2} \tilde{X}_1 \right\},
\]
\[
\tilde{\Delta} = \left\{ \tilde{U}_1 = -\frac{xt}{2} \tilde{X}_1 - \frac{t^2}{2} \tilde{X}_2, \tilde{U}_2 = -\frac{t}{2} \tilde{X}_1 \right\}.
\]

We then compute the sequence of vector fields
\[
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_2, \tilde{S}_1],
\]
\[
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1], \quad \tilde{S}_3 = [\tilde{U}_2, \tilde{S}_1].
\]

These vector fields form bases \(\tilde{\mathfrak{S}} = \{\tilde{S}_i\}_{i=1}^3\) and \(\tilde{\mathfrak{S}} = \{\tilde{S}_i\}_{i=1}^3\) for the 3-dimensional Vessiot algebra of (13.1). In particular, the structure equations for \(\tilde{\mathfrak{S}}\) are
\[
[\tilde{S}_1, \tilde{S}_2] = [\tilde{S}_1, \tilde{S}_3] = [\tilde{S}_2, \tilde{S}_3] = 0,
\]
and the structure equations for \(\tilde{\mathfrak{S}}\) are
\[
[\tilde{S}_1, \tilde{S}_2] = [\tilde{S}_1, \tilde{S}_3] = [\tilde{S}_2, \tilde{S}_3] = 0.
\]

As abstract Lie algebras, both \(\tilde{\mathfrak{S}}\) and \(\tilde{\mathfrak{S}}\) can be identified as \(3\mathfrak{n}_{1,1}\) in [28].

Furthermore, upon restricting \(\tilde{\Delta}\) to the integral manifold \(\tilde{M}\) given by \(\tilde{I}_1 = \tilde{I}_2 = 0\), we obtain the Vessiot distribution,
\[
\hat{\mathcal{V}} = \tilde{\mathcal{V}}|_{\tilde{M}} = \left\{ \frac{1}{t^2} \partial_x + \frac{1}{t} \partial_y + p \partial_u - \frac{4}{3t^3} \partial_p, \partial_t \right\}.
\]
This is a rank 2 distribution on the 5-dimensional manifold with coordinates \(x, y, u, p, t\) whose growth vector is \([2, 3, 5]\) and whose symmetry algebra is 14-dimensional. Additionally, using the FiveVariables package, we see that the Cartan quartic vanishes identically. We therefore, conclude that \(\mathcal{V}\) is equivalent to the Hilbert-Cartan distribution

\[
\{\partial_x + \phi_2^2 \partial_z + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1}, \partial_{\phi_2}\}.
\]

A similar calculation shows that the Vessiot distribution \(\mathcal{V}\) obtained from the restriction of \(\Delta\) to the integral manifold \(\tilde{M}\) given by \(\tilde{I}_1 = \tilde{I}_2 = 0\) is equivalent to the Hilbert-Cartan distribution as well.

This allows us to realize \(\mathcal{W}|_{\tilde{M}}\) and \(\mathcal{W}|_{\tilde{M}}\) as 3-dimensional subalgebras of \(\mathfrak{g}_2\), however, there are three 3-dimensional abelian subalgebras of \(\mathfrak{g}_2\) which are equivalent to \(3n_{1,1}\) in Table B.1. To identify which subalgebra corresponds to \(\mathcal{W}|_{\tilde{M}}\) and \(\mathcal{W}|_{\tilde{M}}\), we calculate their normalizer in the full 14-dimensional symmetry algebra of \(\mathcal{V}\) and \(\mathcal{V}\), respectively, and see that both are 9-dimensional. From this, we see that the Vessiot algebra for (13.1) is equivalent to \([N, 11]\) of Table B.1. This gives the following characterization of (13.1).

**Theorem 13.1.1.** The equation

\[
3u_{xx}u_{yy}^3 + 1 = 0
\]

is completely characterized by having Hilbert-Cartan distributions as its Vessiot distributions and having Vessiot group generated by the diagonal action of \([N, 11]\) in Table B.1.

We now give the reconstruction of (13.1) by realizing it as a quotient of the direct sum of its Vessiot distributions by the diagonal action of its Vessiot group.

In terms of the coordinates \(x, z, \phi, \phi_1, \phi_2\) and \(y, w, \psi, \psi_1, \psi_2\), the Vessiot distributions, defined on 5-dimensional manifolds \(\tilde{M}_5\) and \(\tilde{M}_5\), are

\[
\mathcal{V} = \{\partial_x + \phi_2^2 \partial_z + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1}, \partial_{\phi_2}\}
\]

\[
\tilde{\mathcal{V}} = \{\partial_y + \psi_2^2 \partial_w + \psi_1 \partial_\psi + \psi_2 \partial_{\psi_1}, \partial_{\psi_2}\}.
\]
and the diagonal action, $G_{\text{diag}}$, is generated by the vector fields

$$Z_1 = \partial_{\phi} + \partial_{\psi}, \quad Z_2 = x\partial_{\phi} + y\partial_{\psi} + \partial_{\phi_1}, \quad Z_3 = \partial_z + \partial_w.$$  

The quotient of $\hat{V} \oplus \tilde{V}$ by $G_{\text{diag}}$ is a rank 4 distribution defined on the 7-manifold $M_7$ with coordinates $z_1, z_2, z_3, z_4, z_5, z_6, z_7$. The explicit formula for the quotient map $q: \hat{M}_5 \times \tilde{M}_5 \to M_7$ is given by

$$z_1 = -\phi + \psi + x\phi_1 + y\psi_1 - \frac{(x-y)(2x\phi_2 + y\phi_2 + x\psi_2 + 2y\psi_2)}{6},$$
$$z_2 = -\phi_1 + \psi_1 + \frac{(x+y)(\phi_2 + \psi_2)}{2},$$
$$z_3 = -z + w + \frac{(x-y)(\phi_2^2 + \phi_2 \psi_2 + \psi_2^2)}{3}, \quad z_4 = \frac{2(\psi_2 - \phi_2)}{x-y}, \quad z_5 = \frac{2(x\psi_2 - y\phi_2)}{x-y},$$
$$z_6 = \phi_2, \quad z_7 = \psi_2.$$

Calculating the pushforward of $\hat{V} \oplus \tilde{V}$ by $q$ gives the rank 4 distribution

$$\Delta = \{ \partial_{z_1} + z_4 \partial_{z_3} + R \partial_{z_4}, \partial_{z_2} + z_5 \partial_{z_3} + S \partial_{z_4} + T \partial_{z_5}, \partial_{z_6}, \partial_{z_7} \},$$

where

$$R = \frac{3z_4^3}{(z_6 - z_7)^3},$$
$$S = \frac{3z_2^2(z_5 - z_6 - z_7)}{(z_6 - z_7)^3},$$
$$T = \frac{(3z_2^2 - 6z_4 z_6 - 6z_4 z_7 + 4z_6^2 + 4z_6 z_7 + 4z_7^2) z_4}{(z_6 - z_7)^3}.$$

We can then see that $R, S, T$ satisfy the relationship

$$3(RT - S^2)^3 - R^4 = 0.$$
This leads us to make the partial Legendre transformation

\[ z_1 \mapsto z_4, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3 - z_1 z_4, \quad z_4 \mapsto -z_1, \quad z_5 \mapsto z_5, \quad z_6 \mapsto z_6, \quad z_7 \mapsto z_7, \]

and in doing so, we see that \( \Delta \) becomes

\[ \Delta = \{ \partial z_1 + z_4 \partial z_3 + \bar{R} \partial z_4 + \bar{S} \partial z_5, \partial z_2 + z_5 \partial z_3 + \bar{S} \partial z_4 + \bar{T} \partial z_5, \partial z_6, \partial z_7 \}, \]

where

\[ \bar{R} = \left( \frac{z_6 - z_7}{z_1} \right)^3, \quad \bar{S} = \frac{z_5 - z_6 - z_7}{z_1}, \quad \bar{T} = -\frac{z_1}{z_6 - z_7}. \]

We then see that \( \bar{R}, \bar{S}, \bar{T} \) satisfy the relationship

\[ 3\bar{R}\bar{T}^3 + 1 = 0, \]

so that upon making the final transformation

\[ x = z_1, \quad y = z_2, \quad u = z_3, \quad u_x = z_4, \quad u_y = z_5, \quad u_{xy} = \frac{z_5 - z_6 - z_7}{z_1}, \quad u_{yy} = -\frac{z_1}{z_6 - z_7}, \]

\( \Delta \) becomes the rank 4 distribution corresponding to (13.1).

13.2 Equation 2: \( 3u_{xx}u_{yy}^3 = u_{xy}^3 \)

We now consider the equation

\[ 3u_{xx}u_{yy}^3 = u_{xy}^3, \quad (13.2) \]
To obtain the hyperbolic distribution $\Delta = \Delta \oplus \bar{\Delta}$, we first let

\[
X_1 = \partial_x + p\partial_u + \frac{s^3}{3t^3}\partial_p + s\partial_q, \quad X_2 = \partial_y + q\partial_u + s\partial_p + t\partial_q, \quad X_3 = \partial_s, \quad X_4 = \partial_t,
\]

\[
\hat{\lambda} = \frac{-s^2 + s\sqrt{s^2 - 4st^2}}{2t^3}, \quad \bar{\lambda} = \frac{-s^2 - s\sqrt{s^2 - 4st^2}}{2t^3},
\]

and then by following the theory of Section 2.3, we see that the hyperbolic distribution is given by

\[
\hat{\Delta} = \{ \hat{X}_1 = X_1 + \hat{\lambda} X_2 + D_y \left( \frac{s^3}{3t^3} \right) X_3, \quad \hat{X}_2 = X_4 - \hat{\lambda} X_3 \},
\]

\[
\bar{\Delta} = \{ \bar{X}_1 = X_1 + \bar{\lambda} X_2 + D_y \left( \frac{s^3}{3t^3} \right) X_3, \quad \bar{X}_2 = X_4 - \bar{\lambda} X_3 \}.
\]

The invariants are

\[
\hat{I}_1 = q - \frac{(2st^2 + s\sqrt{s^2 - 4st^2} - s^2)x}{2t^2}, \quad \hat{I}_2 = \frac{s - 2t^2 + \sqrt{s^2 - 4st^2}}{st^2},
\]

\[
\bar{I}_1 = q - \frac{(2st^2 - s\sqrt{s^2 + 4st^2} + s^2)x}{2t^2}, \quad \bar{I}_2 = \frac{s(s - 2t^2 + \sqrt{s^2 - 4st^2})}{t^2}.
\]

We can then introduce the transformation to invariant coordinates

\[
\Phi = \begin{cases} 
  y = y, & u = u, & p = p, & \hat{J}_1 = \hat{I}_1, & \hat{J}_2 = \sqrt{2\hat{I}_2}, \\
  \bar{J}_1 = \bar{I}_1, & \bar{J}_2 = \sqrt{2\bar{I}_2},
\end{cases}
\]

and calculate

\[
\hat{\Delta} = \{ \hat{W}_1 = \Phi_* \hat{X}_1, \quad \hat{W}_2 = \Phi_* \hat{X}_2 \}; \quad \bar{\Delta} = \{ \bar{W}_1 = \Phi_* \bar{X}_1, \quad \bar{W}_2 = \Phi_* \bar{X}_2 \}. 
\]
Theorem 4.2.2 allows us to construct commuting bases \( \{ \hat{U}_i \} \) and \( \{ \tilde{U}_j \} \) for \( \hat{\Delta} \) and \( \tilde{\Delta} \), respectively, as

\[
\hat{\Delta} = \{ \hat{U}_1 = \hat{W}_1(\hat{J}_1)\hat{W}_1, \hat{U}_2 = \hat{W}_2(\hat{J}_1)\hat{W}_1 + \hat{W}_2(\hat{J}_2)\hat{W}_2 \},
\]

\[
\tilde{\Delta} = \{ \tilde{U}_1 = \tilde{W}_1(\tilde{J}_1)\tilde{W}_1, \tilde{U}_2 = \tilde{W}_2(\tilde{J}_1)\tilde{W}_1 + \tilde{W}_2(\tilde{J}_2)\tilde{W}_2 \}.
\]

We then compute the sequence of vector fields

\[
\hat{S}_1 = [\hat{U}_1, \hat{U}_2] = \frac{2}{\tilde{f}_2^2} \partial_y + \frac{2(\tilde{J}_1\tilde{J}_2^2\tilde{f}_2 - 16\tilde{J}_1)}{\tilde{f}_2^2(\tilde{f}_2^2 - 16)} \partial_u - \frac{1}{2} \partial_p,
\]

\[
\hat{S}_2 = [\hat{U}_1, \hat{S}_1] = \frac{2}{\tilde{f}_2^2} \partial_u,
\]

\[
\hat{S}_3 = [\hat{U}_2, \hat{S}_1] = -\frac{4}{\tilde{f}_2^2} \partial_y - \frac{4\tilde{J}_1}{\tilde{f}_3} \partial_u,
\]

\[
\tilde{S}_1 = [\tilde{U}_1, \tilde{U}_2] = \frac{1}{2} \partial_y + \frac{\tilde{J}_1\tilde{f}_2^2\tilde{J}_2 - 16\tilde{J}_1}{2(\tilde{f}_2^2 - 16)} \partial_u - \frac{2}{\tilde{f}_2^2} \partial_p,
\]

\[
\tilde{S}_2 = [\tilde{U}_1, \tilde{S}_1] = \frac{1}{2} \partial_u,
\]

\[
\tilde{S}_3 = [\tilde{U}_2, \tilde{S}_1] = \frac{16(\tilde{J}_1 - \tilde{J}_1^2)}{2(16 - \tilde{f}_2^2 \tilde{f}_2^2)} \partial_u + \frac{4}{\tilde{f}_2^2} \partial_p.
\]

These vector fields form bases \( \hat{\mathfrak{g}} = \{ \hat{S}_i \}_{i=1}^3 \) and \( \tilde{\mathfrak{g}} = \{ \tilde{S}_i \}_{i=1}^3 \) for the 3-dimensional Vessiot algebra of (13.1). In particular, the structure equations for \( \hat{\mathfrak{g}} \) are

\[
[\hat{S}_1, \hat{S}_2] = [\hat{S}_1, \hat{S}_3] = [\hat{S}_2, \hat{S}_3] = 0,
\]

and the structure equations for \( \tilde{\mathfrak{g}} \) are

\[
[\tilde{S}_1, \tilde{S}_2] = [\tilde{S}_1, \tilde{S}_3] = [\tilde{S}_2, \tilde{S}_3] = 0.
\]

As abstract Lie algebras, both \( \hat{\mathfrak{g}} \) and \( \tilde{\mathfrak{g}} \) can be identified as \( 3n_{1,1} \) in [28].
Furthermore, upon restricting $\tilde{\Delta}$ to the integral manifold $\tilde{M}$ given by $\tilde{J}_1 = 0, \tilde{J}_2 = 4$, we obtain the Vessiot distribution,

$$\tilde{\mathcal{V}} = \Delta|_{\tilde{M}}$$

$$= \left\{ \partial_y - \frac{\tilde{J}_1 + \tilde{J}_1 \tilde{J}_2 + 2p\tilde{J}_2 - 2p\tilde{J}_2^2}{(\tilde{J}_2 + 1)(\tilde{J}_2^2 - 1)} \partial_u - \frac{\tilde{J}_2(\tilde{J}_2^2 - 2)}{12} \partial_p - \frac{(\tilde{J}_2 - 1)^2}{4\tilde{J}_1} \partial_{\tilde{J}_2}, \partial_{\tilde{J}_1} + \frac{\tilde{J}_2^2 - 1}{2\tilde{J}_1 \tilde{J}_2} \partial_{\tilde{J}_2} \right\}.$$  

This is a rank 2 distribution on the 5-dimensional manifold with coordinates $y, u, p, \tilde{J}_1, \tilde{J}_2$ whose growth vector is $(2,3,5)$ and whose symmetry algebra is 14-dimensional. Additionally, using the FiveVariables package, we see that the Cartan quartic vanishes identically. We therefore, conclude that $\tilde{\mathcal{V}}$ is equivalent to the Hilbert-Cartan distribution

$$\{\partial_x + \phi_2^2 \partial_z + \phi_1 \partial_\phi + \phi_2 \partial_{\phi_1}, \partial_{\phi_2}\}.$$  

A similar calculation shows that the Vessiot distribution $\mathcal{V}$ obtained by restricting $\Delta$ to the integral manifold $\tilde{M}$ given by $\tilde{J}_1 = 0, \tilde{J}_2 = 4$ is equivalent to the Hilbert-Cartan distribution as well.

This allows us to realize $\mathcal{H}|_{\tilde{M}}$ and $\mathcal{H}|_{\tilde{M}}$ as 3-dimesional subalgebras of $\mathfrak{g}_2$, however, there are three 3-dimensional abelian subalgebras of $\mathfrak{g}_2$ which are equivalent to $3\mathfrak{n}_{1,1}$ in Table B.1. To identify which subalgebra corresponds to $\mathcal{H}|_{\tilde{M}}$ and $\mathcal{H}|_{\tilde{M}}$, we calculate their normalizer in the full 14-dimensional symmetry algebra of $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}$, respectively, and see that both are 7-dimensional. From this, we see that the Vessiot algebra for (13.2) is equivalent to $[N, 12]$ of Table B.1. This gives the following characterization of (13.2).

**Theorem 13.2.1.** The equation

$$3u_{xx}u_{yy}^3 = u_{xy}^3$$

is completely characterized by having Hilbert-Cartan distributions as its Vessiot distributions and having Vessiot group generated by the diagonal action of $[N, 12]$ in Table B.1.
We now give the reconstruction of (13.2) by realizing it as a quotient of the direct sum of its Vessiot distributions by the diagonal action of its Vessiot group.

In terms of the coordinates $x, z, \phi, \phi_1, \phi_2$ and $y, \psi, \psi_1, \psi_2$, the Vessiot distributions, defined on 5-dimensional manifolds $M_5$ and $\tilde{M}_5$, are

$$
\mathcal{V} = \{ \partial_x + \phi_2^2 \partial_z + \phi_1 \partial_{\phi} + \phi_2 \partial_{\phi_1}, \partial_{\phi_2} \}
$$

$$
\tilde{\mathcal{V}} = \{ \partial_y + \psi_2^2 \partial_w + \psi_1 \partial_{\psi} + \psi_2 \partial_{\psi_1}, \partial_{\psi_2} \},
$$

and the diagonal action, $G_{\text{diag}}$, is generated by the vector fields

$$
Z_1 = \partial_{\phi} + \partial_{\psi}, \quad Z_2 = \partial_x + \partial_y, \quad Z_3 = \partial_z + \partial_w.
$$

The quotient of $\mathcal{V} \oplus \tilde{\mathcal{V}}$ by $G_{\text{diag}}$ is a rank 4 distribution defined on the 7-manifold $M_7$ with coordinates $z_1, z_2, z_3, z_4, z_5, z_6, z_7$. The explicit formula for the quotient map $q : \tilde{M}_5 \times \tilde{M}_5 \to M_7$ is given by

$$
z_1 = \frac{\phi_2^2 - \psi_2^2}{\phi_1 - \psi_1}, \quad z_2 = (x - y) - \frac{2(\phi_1 - \psi_1)}{\phi_2 + \psi_2},
$$

$$
z_3 = w - z - \frac{(\phi - \psi)(\phi_2^2 - \psi_2^2)}{\phi_1 - \psi_1} + \frac{4(\phi_1 + 2\psi_1)\phi_2^2 + 4(\phi_1 - \psi_1)\phi_2\psi_2 - (2\phi_1 + 4\psi_2)\psi_2^2}{3(\phi_2 + \psi_2)^2},
$$

$$
z_4 = \phi - \psi - \frac{2(\phi_1 - \psi_1)(\phi_1\phi_2 + 2\phi_1\psi_2 + 2\phi_2\psi_1 + \psi_1\psi_2)}{3(\phi_2 + \psi_2)^2},
$$

$$
z_5 = \frac{\phi_1\psi_2^2 - \phi_2^2\psi_1}{\phi_1 - \psi_1}, \quad z_6 = \phi_2, \quad z_7 = \psi_2.
$$

Calculating the pushforward of $\mathcal{V} \oplus \tilde{\mathcal{V}}$ by $q$ gives the rank 4 distribution

$$
\Delta = \{ \partial_{z_1} + z_4 \partial_{z_2} + R \partial_{z_4} + S \partial_{z_5}, \partial_{z_6} + z_5 \partial_{z_3} + S \partial_{z_4} + T \partial_{z_5}, \partial_{z_6}, \partial_{z_7} \},
$$

where

$$
R = \frac{(z_6 - z_7)^3}{3z_1^3}, \quad S = \frac{z_5 - z_6z_7}{z_1}, \quad T = -\frac{z_1z_6z_7}{z_6 - z_7}.$$
We can then see that \( R, S, T \) satisfy the relationship

\[
3RT^3 - (z_1S - z_5)^3 = 0.
\]

However, by further introducing the transformation

\[
\begin{align*}
  z_1 &\mapsto \frac{1}{z_1}, & z_2 &\mapsto z_2, & z_3 &\mapsto \frac{z_3}{z_1}, & z_4 &\mapsto z_1z_4 - z_3, & z_5 &\mapsto \frac{z_5}{z_1}, \\
  z_6 &\mapsto z_1z_6 - z_5, & z_7 &\mapsto \frac{z_7}{z_1},
\end{align*}
\]

we see that \( \Delta \) becomes

\[
\Delta = \{ \partial_{z_1} + z_4 \partial_{z_3} + \tilde{R} \partial_{z_4} + \tilde{S} \partial_{z_5}, \partial_{z_2} + z_5 \partial_{z_3} + \tilde{S} \partial_{z_4} + \tilde{T} \partial_{z_5}, \partial_{z_6}, \partial_{z_7} \},
\]

where

\[
\tilde{R} = \frac{(z_7 + z_1z_5 - z_1^2z_6)^3}{3z_1^3}, \quad \tilde{S} = \frac{(z_5 - z_1z_6)z_7}{z_1}, \quad \tilde{T} = \frac{(z_1z_6 - z_5)z_7}{z_1z_6 - z_1z_5 - z_7}.
\]

We then see that \( \tilde{R}, \tilde{S}, \tilde{T} \) satisfy the relationship

\[
3\tilde{R}^3 = \tilde{S}^3,
\]

so that upon making the final transformation

\[
\begin{align*}
  x &= z_1, & y &= z_2, & u &= z_3, & u_x &= z_4, & u_y &= z_5, \\
  u_{xy} &= \frac{(z_5 - z_1z_6)z_7}{z_1}, & u_{yy} &= \frac{(z_1z_6 - z_5)z_7}{z_1z_6 - z_1z_5 - z_7},
\end{align*}
\]

\( \Delta \) becomes the rank 4 distribution corresponding to (13.2).
CHAPTER 14
FUTURE WORK

The results of this dissertation demonstrate the utility of the quotient theory of Darboux integrable systems developed by Anderson, Fels, and Vassiliou to the general study of Darboux integrable systems. Absent from our presentation are several projects which we have partially completed and hope to finish in the near future. We briefly list them here.

**Project 1: (Linear Equations)**

We noted in Chapter 7 that coefficients $a(x, y), b(x, y)$ of Darboux integrable linear equations of the form

$$u_{xy} = a(x, y)u_x + b(x, y)u_y - a(x, y)b(x, y)u$$

are closely related to solutions of the $A_2$-Toda lattice

$$\ln(-a_x)_{xy} = -2a_x + b_y \quad \text{and} \quad \ln(-b_y)_{xy} = a_x - 2b_y.$$

As part of our future work, we plan to give explicit formulas for these coefficients in terms of the arbitrary functions appearing in the solutions to the $A_2$-Toda lattice system as well as show how these functions correspond with those appearing in the action of the equation’s Vessiot group.

**Project 2: (Invariant Classification of Moutard Equations)**

In Chapter 6, we gave an invariant classification of Darboux integrable linear equations. Motivated by [2], we hope to prove the following conjecture.
Conjecture. A second order hyperbolic partial differential equation of Moutard type

\[ u_{xy} + \frac{\partial}{\partial x}(a(x,y)e^u) - \frac{\partial}{\partial y}(b(x,y)e^{-u}) + c(x,y) = 0, \]

is Darboux integrable at some order if and only if its Vessiot distributions are each locally equivalent to the standard contact systems on \( J^r(\mathbb{R}, \mathbb{R}) \) and \( J^s(\mathbb{R}, \mathbb{R}) \), and the diagonal action of its Vessiot algebra is generated by the (prolonged) vector fields

\[ Z_i = \zeta_i(x) \partial_u + \zeta_i(y) \partial_v \quad \text{and} \quad W = u \partial_u + v \partial_v, \quad \text{with} \quad 1 \leq i \leq r + s - 4. \]

Project 3: (Invariant Classification of Goursat Equations)

In [29], Sokolov and Zhiber state that equations of the form

\[ u_{xy} = A_n(x,y)\sqrt{pq} \]

are Darboux integrable at order \( n \) if the coefficient \( A_n(x,y) \) is a solution to the \( C_n \)-Toda lattice. Anderson and Fels [2] show that when \( A_n = 2n/(x+y) \), the equation always comes from the reduction of systems corresponding to the equation \( z' = (y^{(n)})^2 \). We believe that the Vessiot group must be either abelian or 1-step solvable, however it remains unclear how the action of this group relates to the coefficient function \( A_n(x,y) \). Our goal here is to understand the structure of this class of equations in general using the quotient theory of Darboux integrable systems.

Project 4: (Cartan’s Analysis of Algebraically Special (2,3,5)-Distributions)

Though Cartan’s original solution to the equivalence problem for (2,3,5)-distributions [12] is valid over \( \mathbb{R} \), his subsequent analysis of (2,3,5)-distributions by their root types was incomplete. In particular, for distributions of root type [2,2], he only considers the case
where the Cartan quartic factors as

\[ F(x, y) = 6x^2y^2. \]

This misses the case where the quartic factors as \( F(x, y) = (x^2 + y^2)^2 \). We note that Zelenko [37] gave examples of distributions of each of these types. As part of this project, we intend to give a complete analysis of distributions of root type \([2,2]\) over \( \mathbb{R} \).

Additionally, Doubrov and Govorov [15] gave the exceptional distribution of root type \([4]\), but somewhat surprisingly did not explain how Cartan’s analysis failed to account for this example. We wish to understand this.

Finally, in the root type \([3,1]\) case, the symmetry group is at most 5-dimensional, but it is unclear if the symmetry group can act intransitively. Research into this matter could give rise to new classes of Darboux integrable \( f \)-Gordon equations.

**Project 5:** *(Alternative Forms of Root Type \([4]\) Systems with \( I(x) \) Nonconstant)*

Consistent with Cartan’s root type \([4]\) distributions given by the equation

\[ z' = -\frac{1}{2} \left( y'' + \frac{10}{3} I(y')^2 + (1 + I^2 - I'')y'' \right), \]

with \( I(x) \) nonconstant, the distribution generated by the equation

\[ z' = f(y''), \quad f_{y''y''} \neq 0, \]

is of root type \([4]\) and has generically has 6-dimensional symmetry algebra. We are interested to see how, and in what cases, the invariant \( I(x) \) is related to the function \( f(y'') \). In particular, we wish to know at what extent every \((2,3,5)\)-distribution root type \([4]\) can be encoded this way.

**Project 6:** *(Integrable Extensions and Bäcklund Transformations)*

In [3] Anderson and Fels showed that if two Darboux integrable equations have the
same Vessiot group, they can be related with a Bäcklund transformation. We plan to show how this can be done for several of the new examples brought to light by this dissertation.

Along with this, Zhiber and Sokolov [38] shows that their second example

$$u_{xy} = \frac{A(u_x)B(u_y)}{u}$$

is the 2-fold integrable extension of the Liouville equation $v_{xy} = e^v$. One of our goals is to understand this and similar relationships for the other $f$-Gordon equations which are Darboux integrable at order three.

**Project 7: (f-Gordon Equations Darboux Integrable at Order Four)**

The equation $u_{xy} = \frac{6\sqrt{pq}}{x + y}$ is Darboux integrable at order four and is known to be a quotient of distributions corresponding to the equation $z' = (y'')^2$ which has 7-dimensional symmetry group with 5-dimensional orbits on the 6-dimensional manifold with coordinates $x, z, y, y', y'', y'''$. If we consider quotients of distributions corresponding to the equation $z'' = (y'')^2$, which has 8-dimensional symmetry algebra, we can ask if there exist 7-dimensional subalgebras of its symmetry algebra with 5-dimensional orbits on the 6-dimensional manifold with coordinates $x, z', y, y', y''$. In doing so, we can determine if there exist other $f$-Gordon equation which are Darboux integrable at order four.

**Project 8: (Explicit Formulas for New Examples of f-Gordon Equations Darboux Integrable at Order Three)**

As previously stated, one of our future goals is to find explicit equations corresponding to the distributions presented in Chapter 10 which are Darboux integrable at order three.

**Project 9: (Symmetry Analysis of Darboux Integrable Equations)**

Let $\Delta = (\hat{\mathcal{V}} \oplus \check{\mathcal{V}})/G_{\text{diag}}$. The normalizer of $G_{\text{diag}}$ in the symmetry algebra of $\hat{\mathcal{V}} \oplus \check{\mathcal{V}}$ quotients to symmetries of $\Delta$. We intend to use several of the distributions coming from
the quotient construction, particularly those with simple representations, to check the conjecture that the normalizer fully determines the symmetries of $\Delta$. This phenomenon was first noted for the equation $3rt^3 + 1 = 0$ by Anderson.

**Project 10: (First-Order Systems Darboux Integrable After One Prolongation)**

In our analysis of $f$-Gordon equations which are Darboux integrable at order three (or after one prolongation), we found in multiple cases that the resulting equation is an integrable extension of a first order system which is Darboux integrable after one prolongation. From the quotient perspective, these systems arise as the quotient of two rank 2 distributions on 6-dimensional manifolds by a 4-dimensional symmetry group. By further studying these systems, we hope to gain deeper insights into the structure of Darboux integrable systems.

**Project 11: (Elliptic Darboux Integrable Equations)**

Provisional work has been done on using the quotient theory of Darboux integrable systems to classify nonlinear elliptic Darboux integrable equations at order two. We intend to continue this work to provide an analogous list to Biesecker’s list of hyperbolic $f$-Gordon equations which are Darboux integrable at order two. As a prototypical example, we consider the elliptic Liouville equation, $u_{xx} + u_{yy} = e^u$. 
REFERENCES


APPENDICES
Appendix A

LIE ALGEBRAS OF VECTOR FIELDS IN THE PLANE

In [27], Olver gives the complete classification of all finite-dimensional Lie algebras of vector fields in $\mathbb{R}^2$ originally studied by Lie in [23]. Here, we provide Olver’s list of Lie algebras of vector fields in $\mathbb{R}^2$ up to dimension five. For each table, the functions $\eta_i(x)$ satisfy a $k$th order constant coefficient homogeneous linear ordinary differential equation. The functions $\zeta_i(x)$ are arbitrary smooth functions. For each Lie algebra, we provide the corresponding algebraic structure given in [28].

Table A.1: One, Two, and Three-Dimensional Lie Algebras of Vector Fields in the Plane

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Winternitz</th>
<th>Olver Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{1.1}$</td>
<td>$\eta_1(x)\partial_u$</td>
<td>$n_{1.1}$</td>
<td>3.1</td>
</tr>
<tr>
<td>$p_{2.1}$</td>
<td>$\partial_x, \eta_1(x)\partial_u$</td>
<td>$2n_{1.1}$</td>
<td>1.5</td>
</tr>
<tr>
<td>$p_{2.2}$</td>
<td>$\zeta_1(x)\partial_u, u\partial_u$</td>
<td>$s_{2.1}$</td>
<td>3.2</td>
</tr>
<tr>
<td>$p_{3.1}$</td>
<td>$\partial_x, x\partial_x - u\partial_u, x^2\partial_x - 2xu\partial_u$</td>
<td>$sl(2)$</td>
<td>1.1</td>
</tr>
<tr>
<td>$p_{3.2}$</td>
<td>$\partial_x, x\partial_x - u\partial_u, x^2\partial_x - (2xu + 1)\partial_u$</td>
<td>$sl(2)$</td>
<td>1.2</td>
</tr>
<tr>
<td>$p_{3.3}$</td>
<td>$\partial_x, \eta_1(x)\partial_u, \eta_2(x)\partial_u$</td>
<td>$n_{3.1}$</td>
<td>1.5</td>
</tr>
<tr>
<td>$p_{3.4}$</td>
<td>$\partial_x, u\partial_u, \eta_1(x)\partial_u$</td>
<td>$s_{2.1} \oplus n_{1.1}$</td>
<td>1.6</td>
</tr>
<tr>
<td>$p_{3.5}$</td>
<td>$\partial_x, x\partial_x + \alpha u\partial_u, \partial_u$</td>
<td>$s_{3.1}$</td>
<td>1.7</td>
</tr>
<tr>
<td>$p_{3.6}$</td>
<td>$\partial_x, x\partial_x + (u + x)\partial_u, \partial_u$</td>
<td>$s_{3.2}$</td>
<td>1.8</td>
</tr>
<tr>
<td>$p_{3.7}$</td>
<td>$\zeta_1(x)\partial_u, \zeta_2(x)\partial_u, \zeta_3(x)\partial_u$</td>
<td>$3n_{1.1}$</td>
<td>3.1</td>
</tr>
<tr>
<td>$p_{3.8}$</td>
<td>$\zeta_1(x)\partial_u, \zeta_2(x)\partial_u, u\partial_u$</td>
<td>$s_{3.1}$</td>
<td>3.2</td>
</tr>
<tr>
<td>$p_{3.9}$</td>
<td>$\partial_u, u\partial_u, u^2\partial_u$</td>
<td>$sl(2)$</td>
<td>3.3</td>
</tr>
<tr>
<td>$p_{3.10}$</td>
<td>$\partial_x, \partial_u, \alpha(x\partial_x + u\partial_u) + u\partial_x - x\partial_u$</td>
<td>$s_{3.3}$</td>
<td>6.1</td>
</tr>
<tr>
<td>$p_{3.11}$</td>
<td>$\partial_x, x\partial_x + u\partial_u, (x^2 - u^2)\partial_x + 2xu\partial_u$</td>
<td>$sl(2)$</td>
<td>6.2</td>
</tr>
<tr>
<td>$p_{3.12}$</td>
<td>$u\partial_x - x\partial_u, (1 + x^2 - u^2)\partial_x + 2xu\partial_u,$</td>
<td>$so(3)$</td>
<td>6.3</td>
</tr>
<tr>
<td></td>
<td>$2xu\partial_x + (1 - x^2 + u^2)\partial_u$</td>
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</tbody>
</table>
Table A.2: Four-Dimensional Lie Algebras of Vector Fields in the Plane

<table>
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<th>Generators</th>
<th>Winternitz</th>
<th>Olver Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>p4.1</td>
<td>( \partial_x, x \partial_x, u \partial_u, x^2 \partial_x - xu \partial_u )</td>
<td>( \mathfrak{sl}(2) \oplus \mathfrak{n}_{1,1} )</td>
<td>1.3</td>
</tr>
<tr>
<td>p4.2</td>
<td>( \partial_x, \eta_1(x) \partial_u, \ldots, \eta_3(x) \partial_u )</td>
<td>( \mathfrak{n}_{4,1} )</td>
<td>1.5</td>
</tr>
<tr>
<td>p4.3</td>
<td>( \partial_x, u \partial_u, \eta_1(x) \partial_u, \eta_2(x) \partial_u )</td>
<td>( \mathfrak{s}_{4,11} )</td>
<td>1.6</td>
</tr>
<tr>
<td>p4.4</td>
<td>( \partial_x, x \partial_x + au \partial_u, \partial_u, xu \partial_u )</td>
<td>( \mathfrak{s}_{4,8} )</td>
<td>1.7</td>
</tr>
<tr>
<td>p4.5</td>
<td>( \partial_x, x \partial_x + (2u + x^2) \partial_u, \partial_u, xu \partial_u )</td>
<td>( \mathfrak{s}_{4,10} )</td>
<td>1.8</td>
</tr>
<tr>
<td>p4.6</td>
<td>( \partial_x, 2x \partial_x, \partial_u )</td>
<td>( \mathfrak{s}<em>{2,1} \oplus \mathfrak{s}</em>{2,1} )</td>
<td>1.9</td>
</tr>
<tr>
<td>p4.7</td>
<td>( \partial_x, 2x \partial_x, x^2 \partial_x, \partial_u )</td>
<td>( \mathfrak{sl}(2) \oplus \mathfrak{n}_{1,1} )</td>
<td>1.10</td>
</tr>
<tr>
<td>p4.8</td>
<td>( \zeta_1(x) \partial_u, \ldots, \zeta_4(x) \partial_u )</td>
<td>( 4 \mathfrak{n}_{1,1} )</td>
<td>3.1</td>
</tr>
<tr>
<td>p4.9</td>
<td>( \zeta_1(x) \partial_u, \ldots, \zeta_3(x) \partial_u, u \partial_u )</td>
<td>( \mathfrak{s}_{4,3} )</td>
<td>3.2</td>
</tr>
<tr>
<td>p4.10</td>
<td>( \partial_x, \partial_u, x \partial_x + u \partial_u, u \partial_x - x \partial_u )</td>
<td>( \mathfrak{s}_{4,12} )</td>
<td>6.4</td>
</tr>
</tbody>
</table>

Table A.3: Five-Dimensional Lie Algebras of Vector Fields in the Plane

<table>
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<th>Generators</th>
<th>Winternitz</th>
<th>Olver Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>p5.1</td>
<td>( \partial_x, \eta_1(x) \partial_u, \ldots, \eta_4(x) \partial_u )</td>
<td>( \mathfrak{n}_{5,5} )</td>
<td>1.5</td>
</tr>
<tr>
<td>p5.2</td>
<td>( \partial_x, u \partial_u, \eta_1(x) \partial_u, \ldots, \eta_3(x) \partial_u )</td>
<td>( \mathfrak{s}_{5,37} )</td>
<td>1.6</td>
</tr>
<tr>
<td>p5.3</td>
<td>( \partial_x, x \partial_x + au \partial_u, \partial_u, xu \partial_u, x^2 \partial_u )</td>
<td>( \mathfrak{s}_{5,35} )</td>
<td>1.7</td>
</tr>
<tr>
<td>p5.4</td>
<td>( \partial_x, x \partial_x + (3u + x^2) \partial_u, \partial_u, xu \partial_u, x^2 \partial_u )</td>
<td>( \mathfrak{s}_{5,34} )</td>
<td>1.8</td>
</tr>
<tr>
<td>p5.5</td>
<td>( \partial_x, x \partial_x, u \partial_u, \partial_u, x^2 \partial_u )</td>
<td>( \mathfrak{s}_{5,44} )</td>
<td>1.9</td>
</tr>
<tr>
<td>p5.6</td>
<td>( \partial_x, 2x \partial_x + u \partial_u, x^2 \partial_x + xu \partial_u, \partial_u, x \partial_u )</td>
<td>( \mathfrak{sl}(2) \times 2 \mathfrak{n}_{1,1} )</td>
<td>1.10</td>
</tr>
<tr>
<td>p5.7</td>
<td>( \partial_x, x^2 \partial_x, x \partial_u, \partial_u )</td>
<td>( \mathfrak{gl}(2) \times 2 \mathfrak{n}_{1,1} )</td>
<td>1.11</td>
</tr>
<tr>
<td>p5.8</td>
<td>( \zeta_1(x) \partial_u, \ldots, \zeta_5(x) \partial_u )</td>
<td>( 5 \mathfrak{n}_{1,1} )</td>
<td>3.1</td>
</tr>
<tr>
<td>p5.9</td>
<td>( \zeta_1(x) \partial_u, \ldots, \zeta_4(x) \partial_u, u \partial_u )</td>
<td>( \mathfrak{s}_{5,9} )</td>
<td>3.2</td>
</tr>
<tr>
<td>p5.10</td>
<td>( \partial_x, \partial_u, x \partial_x - u \partial_u, u \partial_x, x \partial_u )</td>
<td>( \mathfrak{sl}(2) \times 2 \mathfrak{n}_{1,1} )</td>
<td>6.5</td>
</tr>
</tbody>
</table>
Appendix B

SUBALGEBRAS OF $\mathfrak{g}_2$

In [14], Doubrov gives a classification for all subalgebras of the real split form of the exceptional Lie algebra $\mathfrak{g}_2$. Here we list the subalgebras of dimensions three and five from his classification, but written in a Chevalley basis. In terms of this Chevalley basis, the infinitesimal generators for $\mathfrak{g}_2$ acting on the five-dimensional manifold with local coordinates \{x, z, y, y_1, y_2\} are as follows:

\[
X_1 = -x \partial_x - 3y \partial_y - 2y_1 \partial_{y_1} - y_2 \partial_{y_2} - 3z \partial_z,
\]
\[
X_2 = y \partial_y + y_1 \partial_{y_1} + y_2 \partial_{y_2} + 2z \partial_z,
\]
\[
X_3 = 3x \partial_y + 3 \partial_{y_1},
\]
\[
X_4 = \left(\frac{4y_1^2}{3} - 2yy_2\right) \partial_x + \left(zy - 2yy_1y_2 + \frac{8y_1^3}{9}\right) \partial_y + \left(zy_1 - yy_2^2\right) \partial_{y_1}
\]
\[+ \left(zy_2 - 2y_1y_2^2\right) \partial_{y_2} + \left(z^2 - \frac{2yy_2^3}{3}\right) \partial_z,
\]
\[
X_5 = (8y_1 - 6xy_2) \partial_x + (3xz - 6xy_1y_2 + 4y_1^2) \partial_y + (3z - 3xy_2^2) \partial_{y_1} - 2y_2^2 \partial_{y_2} - 2xy_2^3 \partial_z,
\]
\[
X_6 = 12 \partial_x,
\]
\[
X_7 = -12 \partial_y,
\]
\[
X_8 = 24y_2 \partial_x - (12z - 24y_1y_2) \partial_y + 12y_2^2 \partial_{y_1} + 8y_2^3 \partial_z,
\]
\[
X_9 = \left(\frac{4xy_1}{3} - \frac{x^2y_2}{2} - y\right) \partial_x + \left(\frac{x^2z}{4} + \frac{2xy_1^2}{3} - \frac{x^2y_1y_2}{2}\right) \partial_y + \left(\frac{xz}{2} - \frac{x^2y_2}{4} - \frac{y_1^3}{3}\right) \partial_{y_1}
\]
\[+ \left(\frac{z}{2} - \frac{xy_2^2}{3} + \frac{y_1y_2}{3}\right) \partial_{y_2} + \left(zy_1 - \frac{x^2y_2^3}{6}\right) \partial_z,
\]
\[
X_{10} = \partial_z,
\]
The subalgebras of $\mathfrak{g}_2$ in this basis are given in the table below. We index these subalgebras as in [Doubrov].

### Table B.1: Three-Dimensional Subalgebras of $\mathfrak{g}_2$

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Winternitz</th>
<th>Change of Basis</th>
<th>Normalizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>[N, 11]</td>
<td>$X_{10}$, $-X_{14}$, $-X_{11}$</td>
<td>$3n_1, 1$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>9</td>
</tr>
<tr>
<td>[N, 12]</td>
<td>$X_{10}$, $-X_{14}$, $X_3$</td>
<td>$3n_1, 1$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>7</td>
</tr>
<tr>
<td>[N, 13]</td>
<td>$X_{10}$, $-X_{14}$, $-X_7$</td>
<td>$n_3, 1$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>7</td>
</tr>
<tr>
<td>[N, 14]</td>
<td>$X_{10}$, $-X_{11}$, $X_3$</td>
<td>$n_3, 1$</td>
<td>$[3e_1, e_2, -e_3]$</td>
<td>7</td>
</tr>
<tr>
<td>[N, 15]</td>
<td>$X_{10}$, $-X_{14}$, $-X_7 - X_{12}$</td>
<td>$n_3, 1$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>5</td>
</tr>
<tr>
<td>[N, 16]</td>
<td>$X_{10}$, $-X_{11}$, $-X_{14}$, $X_3$</td>
<td>$n_3, 1$</td>
<td>$[3e_1, e_2, -e_3]$</td>
<td>6</td>
</tr>
<tr>
<td>[N, 17]</td>
<td>$X_{10}$, $-X_{11}$, $-X_{14}$, $X_3$</td>
<td>$n_3, 1$</td>
<td>$[3e_1, e_2, -e_3]$</td>
<td>6</td>
</tr>
<tr>
<td>[N, 18]</td>
<td>$X_{10}$, $X_3 - X_7 - aX_{11} - X_{14}$</td>
<td>$n_3, 1$</td>
<td>$[(3a - 1)e_1, e_2, e_3]$</td>
<td>6</td>
</tr>
<tr>
<td>[N, 18]</td>
<td>$X_{10}$, $X_3 - X_7 - \frac{1}{2}X_{11} - X_{14}$</td>
<td>$3n_1, 1$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>6</td>
</tr>
<tr>
<td>[NS, 9]</td>
<td>$-X_{14} + X_{11}, -2X_1 - 3X_2 + X_{10}$</td>
<td>$g_{3, 1}$</td>
<td>$[e_1, e_2, \frac{1}{2}e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[NS, 10]</td>
<td>$X_{10}, -X_{11}, X_1 - X_{14}$</td>
<td>$g_{3, 1}$</td>
<td>$[e_1, e_2, \frac{1}{2}e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[NS, 11]</td>
<td>$X_{10}, -X_{14}, X_1 + X_2 - X_{11}$</td>
<td>$g_{3, 1}$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[NS, 12]</td>
<td>$-X_{14}, X_3, -2X_1 - 3X_2 + X_{10}$</td>
<td>$g_{3, 1}$</td>
<td>$[e_1, e_2, \frac{1}{2}e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[NS, 13]</td>
<td>$X_{10}, -X_{14}, -X_1 - 2X_2 + X_3$</td>
<td>$g_{3, 1}$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>4</td>
</tr>
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</table>
Three-Dimensional Subalgebras of $\mathfrak{g}_2$ (cont.)

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Winternitz</th>
<th>Change of Basis</th>
<th>Normalizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>[NS, 14]</td>
<td>$X_{10}, X_3, X_1 - X_{14}$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, \frac{1}{2}e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[NS, 15]</td>
<td>$X_{10}, -X_{14}, -\frac{1}{2}X_1 - \frac{3}{2}X_2 - X_7$</td>
<td>$\mathfrak{s}_{3,2}$</td>
<td>$[2e_1, -3e_2, \frac{2}{3}e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[NS, 16]</td>
<td>$X_{10}, -X_7, X_1 - X_{14}$</td>
<td>$\mathfrak{s}_{3,2}$</td>
<td>$[e_1, 3e_2, \frac{1}{3}e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[NS, 17]</td>
<td>$X_{10}, -X_{11}, -X_1 - 2X_2 + X_3$</td>
<td>$\mathfrak{s}_{3,2}$</td>
<td>$[e_1, \frac{1}{3}e_2, e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[NS, 18]</td>
<td>$X_{10}, X_3, X_1 + X_2 - X_{11}$</td>
<td>$\mathfrak{s}_{3,2}$</td>
<td>$[e_1, -\frac{1}{3}e_2, e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[R, 4]</td>
<td>$-X_7, X_1 + X_2, X_{13}$</td>
<td>$\mathfrak{sl}(2)$</td>
<td>$[e_1, -e_2, e_3]$</td>
<td>6</td>
</tr>
<tr>
<td>[R, 6]</td>
<td>$-X_{12}, -2X_1 - 3X_2, X_6$</td>
<td>$\mathfrak{sl}(2)$</td>
<td>$[e_1, -e_2, e_3]$</td>
<td>6</td>
</tr>
<tr>
<td>[R, 8]</td>
<td>$X_3 - X_{14}, -X_2, X_8 - X_9$</td>
<td>$\mathfrak{sl}(2)$</td>
<td>$[e_1, -2e_2, e_3]$</td>
<td>3</td>
</tr>
<tr>
<td>[R, 9]</td>
<td>$-X_7 - X_{12}, 3X_6 + 5X_{13}, -X_1 - 4X_2$</td>
<td>$\mathfrak{sl}(2)$</td>
<td>$[2e_1, -e_3, e_2]$</td>
<td>3</td>
</tr>
<tr>
<td>[S, 3]</td>
<td>$X_{10}, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>$\mathfrak{s}<em>{2,1} \oplus \mathfrak{n}</em>{1,1}$</td>
<td>$\text{---}$</td>
<td>3</td>
</tr>
<tr>
<td>[S, 8]</td>
<td>$-X_{11}, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>$\mathfrak{s}<em>{2,1} \oplus \mathfrak{n}</em>{1,1}$</td>
<td>$\text{---}$</td>
<td>3</td>
</tr>
<tr>
<td>[S, 16]</td>
<td>$X_{10}, -X_{14}, (1 - 2a)X_1 + (1 - 3a)X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, e_3], a = \frac{1}{3}(\alpha + 1)$</td>
<td>4</td>
</tr>
<tr>
<td>[S, 16]</td>
<td>$X_{10}, -X_{14}, -\frac{1}{3}X_1 - X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>6</td>
</tr>
<tr>
<td>[S, 16]</td>
<td>$X_{10}, -X_{14}, (1 - 2a)X_1 + (1 - 3a)X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, e_3], a = \frac{1}{3}(\alpha + 1)$</td>
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<tr>
<td>[S, 16]</td>
<td>$X_{10}, -X_{14}, -\frac{1}{3}X_1$</td>
<td>$\mathfrak{s}<em>{2,1} \oplus \mathfrak{n}</em>{1,1}$</td>
<td>$\text{---}$</td>
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<tr>
<td>[S, 17]</td>
<td>$X_{10}, -X_{14}, -2X_1 - 3X_2$</td>
<td>$\mathfrak{s}<em>{2,1} \oplus \mathfrak{n}</em>{1,1}$</td>
<td>$\text{---}$</td>
<td>4</td>
</tr>
<tr>
<td>[S, 19]</td>
<td>$X_{10}, -X_{11}, (a - 2)X_1 + (a - 3)X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, ae_3], a = \frac{1}{\alpha}$</td>
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<tr>
<td>[S, 20]</td>
<td>$X_{10}, -X_{11}, X_1 + X_2$</td>
<td>$\mathfrak{s}<em>{2,1} \oplus \mathfrak{n}</em>{1,1}$</td>
<td>$\text{---}$</td>
<td>4</td>
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<tr>
<td>[S, 21]</td>
<td>$X_{10}, X_3 - X_{11}, -X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, \frac{1}{2}e_3]$</td>
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<tr>
<td>[S, 22]</td>
<td>$X_{10}, -X_7 - X_{12}, -X_1 - 4X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, \frac{1}{3}e_3]$</td>
<td>3</td>
</tr>
<tr>
<td>[S, 23]</td>
<td>$-X_{11} - X_{14}, X_3 - 3X_7, -X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>3</td>
</tr>
<tr>
<td>[S, 25]</td>
<td>$-X_{14}, X_3, (a - 2)X_1 + (a - 3)X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, -\frac{1}{a-3}e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[S, 25]*</td>
<td>$-X_{14}, X_3, -X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, e_3]$</td>
<td>4</td>
</tr>
<tr>
<td>[S, 25]*</td>
<td>$-X_{14}, X_3, X_1$</td>
<td>$\mathfrak{s}<em>{2,1} \oplus \mathfrak{n}</em>{1,1}$</td>
<td>$\text{---}$</td>
<td>4</td>
</tr>
<tr>
<td>[S, 25]*</td>
<td>$-X_{14}, X_3, 2X_1 - 3X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, \frac{1}{3}e_3]$</td>
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</tr>
<tr>
<td>[S, 26]</td>
<td>$-X_{14}, X_3, X_1 + X_2$</td>
<td>$\mathfrak{s}_{3,1}$</td>
<td>$[e_1, e_2, -e_3]$</td>
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### Table B.2: Four-Dimensional Subalgebras of $\mathfrak{g}_2$

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<th>Generators</th>
<th>Normalizer</th>
<th>Vanishing Lie Det.</th>
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<tr>
<td>[N, 19]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3$</td>
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<tr>
<td>[N, 20]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7$</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>[N, 21]</td>
<td>$X_{10}, -X_{14}, X_3 - X_{11}, -X_7$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[N, 22]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7 - X_{12}$</td>
<td>6</td>
<td></td>
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<tr>
<td>[NS, 19]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_1 - 2X_2 + X_3$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[NS, 20]</td>
<td>$X_{10}, -X_{11}, X_3, X_1 - X_{14}$</td>
<td>5</td>
<td></td>
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<tr>
<td>[NS, 21]</td>
<td>$X_{10}, -X_{14}, X_3, X_1 + X_2 - X_{11}$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[NS, 22]</td>
<td>$X_{10}, -X_{11}, -X_7, X_1 - X_{14}$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[NS, 23]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -\frac{1}{2}X_1 - \frac{3}{2}X_2 - X_7$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[NS, 24]</td>
<td>$X_{10}, -X_{14}, -X_7, X_1 + X_2 - X_{11}$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[R, 5]</td>
<td>$-X_7, X_1 + X_2, -2X_1 - 3X_2, X_{13}$</td>
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<tr>
<td>[R, 7]</td>
<td>$-X_{12}, X_1 + X_2, -2X_1 - 3X_2, X_6$</td>
<td>4</td>
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<tr>
<td>[S, 2]</td>
<td>$X_{10}, -X_{12}, -2X_1 - 3X_2, X_6$</td>
<td>5</td>
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<tr>
<td>[S, 7]</td>
<td>$-X_{11}, -X_7, X_1 + X_2, X_{13}$</td>
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<td>[S, 15]</td>
<td>$X_{10}, -X_{14}, X_1 + X_2, -2X_1 - 3X_2$</td>
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<tr>
<td>[S, 18]</td>
<td>$X_{10}, -X_{11}, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>4</td>
<td></td>
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<tr>
<td>[S, 24]</td>
<td>$-X_{14}, X_3, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>4</td>
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</table>
### Four-Dimensional Subalgebras of $\mathfrak{g}_2$ (cont.)

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<th>Vanishing Lie Det.</th>
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<tbody>
<tr>
<td>[S, 30]</td>
<td>$X_{10}, -X_{14}, -X_{11}, (1 - 2a)X_1 + (1 - 3a)X_2$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[S, 30]</td>
<td>$X_{10}, -X_{14}, -X_{11}, (1 - 2a)X_1 + (1 - 3a)X_2$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[S, 31]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -2X_1 - 3X_2$</td>
<td>5</td>
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</tr>
<tr>
<td>[S, 33]</td>
<td>$X_{10}, -X_{14}, X_3, (a - 2)X_1 + (a - 3)X_2$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[S, 34]</td>
<td>$X_{10}, -X_{14}, X_3, X_1 + X_2$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[S, 36]</td>
<td>$X_{10}, -X_{14}, -X_7, (a - 2)X_1 + (a - 3)X_2$</td>
<td>5</td>
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<tr>
<td>[S, 37]</td>
<td>$X_{10}, -X_{14}, -X_7, X_1 + X_2$</td>
<td>6</td>
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</tr>
<tr>
<td>[S, 39]</td>
<td>$X_{10}, -X_{11}, X_3, (a - 2)X_1 + (a - 3)X_2$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[S, 40]</td>
<td>$X_{10}, -X_{11}, X_3, X_1 + X_2$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>[S, 41]</td>
<td>$X_{10}, -X_{14}, -X_7 - X_{12}, -X_1 - 4X_2$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>[S, 42]</td>
<td>$X_{10}, -X_{11} - X_{14}, X_3, -X_2$</td>
<td>4</td>
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</tr>
<tr>
<td>[S, 43]</td>
<td>$X_{10}, -X_{11} - X_{14}, X_3, -X_2$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>[S, 44]</td>
<td>$X_{10}, X_3 - X_7, -aX_{11} - X_{14}, -X_2$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>[S, 44]</td>
<td>$X_{10}, X_3 - X_7, -X_{14}, -X_2$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>[S, 44]</td>
<td>$X_{10}, X_3 - X_7, -X_{14}, -X_2$</td>
<td>4</td>
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</tbody>
</table>
Table B.3: Five-Dimensional Subalgebras of $\mathfrak{g}_2$

<table>
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<tr>
<th>Index</th>
<th>Generators</th>
<th>Winternitz</th>
<th>Change of Basis</th>
<th>Normalizer</th>
<th>Vanishing Lie Det.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[N, 23]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7$</td>
<td>5,3</td>
<td>$[e_1, e_2, -\frac{1}{2} e_3, e_4, e_5]$</td>
<td>9</td>
<td>✓</td>
</tr>
<tr>
<td>[N, 24]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}$</td>
<td>5,2</td>
<td>$[6e_2, 6e_1, -2e_3, e_4, e_5]$</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>[N, 25]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7 - X_{12}$</td>
<td>5,6</td>
<td>$[6e_1, 6e_2, -2e_3, e_4, e_5]$</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>[NS, 25]</td>
<td>$X_{10}, -X_{11}, X_3, -X_7, X_1 - X_{14}$</td>
<td>5,26</td>
<td>$[-3e_1, e_2, e_3, e_4, -e_1 - 3e_4, e_5]$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[NS, 26]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, -X_1 - 2X_2 + X_3$</td>
<td>5,26</td>
<td>$[-\frac{3}{2} e_1, -3e_2, e_4, e_1 - e_3, \frac{1}{2} e_5]$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[NS, 27]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -\frac{1}{2} X_1 - \frac{3}{2} X_2 - X_7$</td>
<td>5,26</td>
<td>$[\frac{3}{2} e_1, e_3, -\frac{1}{2} e_4, e_1 - \frac{3}{2} e_2, e_5]$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[NS, 28]</td>
<td>$X_{10}, -X_{14}, X_3, -X_7, X_1 + X_2 - X_{11}$</td>
<td>5,26</td>
<td>$[3e_1, 3e_2, e_4, 3e_1 + e_3, -e_5]$</td>
<td>6</td>
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</tr>
<tr>
<td>[NS, 29]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_{12}, -X_1 - 2X_2 + X_3$</td>
<td>5,21</td>
<td>$[-\frac{3}{2} e_2, -\frac{1}{2} e_4, e_3, 3e_1, e_5]$</td>
<td>7</td>
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</tr>
<tr>
<td>[NS, 30]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_2 - X_{12}$</td>
<td>5,21</td>
<td>$[\frac{3}{2} e_1, -\frac{1}{2} (2e_3 + e_4), -3e_2 - e_3, -3e_2, e_5]$</td>
<td>6</td>
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</tr>
<tr>
<td>[S, 1]</td>
<td>$X_{10}, -X_{12}, X_1 + X_2, -2X_1 - 3X_2, X_6$</td>
<td>$s\mathfrak{l}(2) \oplus s_2, 1$</td>
<td>—</td>
<td>5</td>
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<tr>
<td>[S, 6]</td>
<td>$-X_{11}, -X_7, X_1 + X_2, -2X_1 - 3X_2, X_{13}$</td>
<td>$s\mathfrak{l}(2) \oplus s_2, 1$</td>
<td>—</td>
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<tr>
<td>[S, 14]</td>
<td>$X_{10}, -X_{14}, -X_7, X_1 + X_2, X_{13}$</td>
<td>$s\mathfrak{l}(2) \oplus 2n_{1, 1}$</td>
<td>$[e_4, -e_4, e_5, e_2, -e_1]$</td>
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<tr>
<td>[S, 29]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>5,41</td>
<td>$[e_3, e_1, e_2, e_5, e_4]$</td>
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<td>[S, 32]</td>
<td>$X_{10}, -X_{14}, X_3, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>5,41</td>
<td>$[e_1, e_2, e_3, -e_4 - \frac{1}{2} e_5, -\frac{1}{2} e_5]$</td>
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<td>[S, 35]</td>
<td>$X_{10}, -X_{14}, -X_7, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>5,44</td>
<td>$[-2e_1, e_2, -2e_3, e_4 + \frac{2}{3} e_5, \frac{1}{3} e_5]$</td>
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<td>[S, 38]</td>
<td>$X_{10}, -X_{11}, X_3, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>5,44</td>
<td>$[6e_1, e_2, -2e_3, e_4 + e_5, e_5]$</td>
<td>5</td>
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<tr>
<td>[S, 46]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, (a - 2)X_1 + (a - 3)X_2$</td>
<td>5,22</td>
<td>$[3e_1, e_4, e_5, e_2, a e_5]$, $a = 1 + \frac{1}{3}$</td>
<td>6</td>
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<tr>
<td>Index</td>
<td>Generators</td>
<td>Winternitz</td>
<td>Change of Basis</td>
<td>Normalizer</td>
<td>Vanishing Lie Det.</td>
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<tr>
<td>[S, 46]_{a=2}</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_2$</td>
<td>$\mathfrak{g}_{5,22}$</td>
<td>$[e_1, e_2, e_3, e_4, e_5]$</td>
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<td>[S, 46]_{a=3/2}</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -\frac{1}{2}X_1 - \frac{3}{2}X_2$</td>
<td>$\mathfrak{g}_{5,22}$</td>
<td>$[e_1, e_2, e_3, e_4, e_5]$</td>
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<td>$\mathfrak{g}_{5,22}$</td>
<td>$[e_1, e_2, e_3, e_4, e_5]$</td>
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<td>$\mathfrak{g}_{5,22}$</td>
<td>$[e_1, e_2, e_3, e_4, e_5]$</td>
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<td>[S, 47]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, X_1 + X_2$</td>
<td>$\mathfrak{g}_{5,30}$</td>
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</tr>
<tr>
<td>[S, 49]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -(a - 2)X_1 + (a - 3)X_2$</td>
<td>$\mathfrak{g}_{5,22}$</td>
<td>$[e_1, e_2, e_3, e_4, e_5, e_5]$; $a = \frac{3\beta + 3}{\beta + 2}$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[S, 49]_{a=1}</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, -X_1 - 2X_2$</td>
<td>$\mathfrak{g}_{5,22}$</td>
<td>—</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>[S, 49]_{a=3/2}</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, -\frac{1}{2}X_1 - \frac{3}{2}X_2$</td>
<td>$\mathfrak{g}_{5,22}$</td>
<td>—</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[S, 49]_{a=2}</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, -X_2$</td>
<td>$\mathfrak{g}_{5,22}$</td>
<td>—</td>
<td>6</td>
<td>✓</td>
</tr>
<tr>
<td>[S, 49]_{a=3}</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, X_1$</td>
<td>$\mathfrak{g}_{5,22}$</td>
<td>—</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[S, 50]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, X_1 + X_2$</td>
<td>$\mathfrak{g}<em>{4,8} \oplus \mathfrak{n}</em>{1,1}$</td>
<td>—</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>[S, 51]</td>
<td>$X_{10}, -X_{14}, X_3 - X_{11}, -X_7, -X_2$</td>
<td>$\mathfrak{g}_{5,22}$</td>
<td>$[e_1, e_2, e_3, e_5, e_4]$</td>
<td>5</td>
<td>✓</td>
</tr>
<tr>
<td>[S, 52]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7 - X_{12}, -X_1 - 4X_2$</td>
<td>$\mathfrak{g}_{5,35}$</td>
<td>$[e_1, e_2, e_3, e_4, e_5]$</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>
Table B.4: Six-Dimensional Subalgebras of $g_2$

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Normalizer</th>
<th>Vanishing Lie Det.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[N, 26]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}$</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>[NS, 31]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_2 - X_{12}$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[NS, 32]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, -\frac{1}{2}X_1 - \frac{3}{2}X_2 - X_7$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[R, 10]</td>
<td>$-X_{14}, X_3, X_1 + X_2, -2X_1 - 3X_2, -X_9, X_8$</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>[S, 13]</td>
<td>$X_{10}, -X_{14}, -X_7, X_1 + X_2, -2X_1 - 3X_2, X_{13}$</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>[S, 28]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, X_1 + X_2, X_{13}$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 48]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, X_1 + X_2, -2X_1 - 3X_2$</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>[S, 56]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, (a - 2)X_1 + (a - 3)X_2$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 56]$_{a=1/2}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -\frac{3}{2}X_1 - \frac{5}{2}X_2$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 56]$_{a=1,3}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, (a - 2)X_1 + (a - 3)X_2$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 56]$_{a=2}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_2$</td>
<td></td>
<td>9 ✓</td>
</tr>
<tr>
<td>[S, 57]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, X_1 + X_2$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 61]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, (1 - 2a)X_1 + (1 - 3a)X_2$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 61]$_{a=0}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, X_1 + X_2$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 61]$_{a=1/2,1}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, (1 - 2a)X_1 + (1 - 3a)X_2$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 61]$_{a=2/3}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, -\frac{1}{2}X_1 - X_2$</td>
<td></td>
<td>9 ✓</td>
</tr>
<tr>
<td>[S, 62]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, -2X_1 - 3X_2$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>[S, 63]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7 - X_{12}, -X_1 - 4X_2$</td>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>
Table B.5: Seven-Dimensional Subalgebras of $\mathfrak{g}_2$

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Normalizer</th>
<th>Vanishing Lie Det.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[S, 27]</td>
<td>$X_{10}, -X_{14}, -X_{11}, -X_7, X_1 + X_2, -2X_1 - 3X_2, X_{13}$</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>[S, 55]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>[S, 60]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>[S, 65]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}, (a - 2)X_1 + (a - 3)X_2$</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>[S, 65]$^a_{3/2}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}, -\frac{1}{2}X_1 - \frac{3}{2}X_2$</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>[S, 65]$^a_{5/3}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}, -\frac{1}{3}X_1 - \frac{2}{3}X_2$</td>
<td>8</td>
<td>✓</td>
</tr>
<tr>
<td>[S, 65]$^a_{0.2}$</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}, -X_2$</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>[S, 66]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}, X_1 + X_2$</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Table B.6: Eight-Dimensional Subalgebras of $\mathfrak{g}_2$

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Normalizer</th>
<th>Vanishing Lie Det.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[R, 11]</td>
<td>$X_{10}, -X_{14}, -X_7, X_1 + X_2, -2X_1 - 3X_2, X_{13}, X_8, -X_4$</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>[S, 54]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}, -2X_1 - 3X_2, X_6$</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>[S, 59]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, -X_7, X_1 + X_2, X_{13}$</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>[S, 64]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}, X_1 + X_2, -2X_1 - 3X_2$</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Table B.7: Nine-Dimensional Subalgebras of $\mathfrak{g}_2$

<table>
<thead>
<tr>
<th>Index</th>
<th>Generators</th>
<th>Normalizer</th>
<th>Vanishing Lie Det.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[S, 53]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_7, -X_{12}, X_1 + X_2, -2X_1 - 3X_2, X_6$</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>[S, 58]</td>
<td>$X_{10}, -X_{14}, -X_{11}, X_3, -X_{12}, -X_7, X_1 + X_2, -2X_1 - 3X_2, X_{13}$</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>
C.1 Fundamental Invariants of the Goursat Equation

The following Maple worksheet shows the calculation of the fundamental invariants of the Goursat equation (9.1).

Fundamental Invariants for the Goursat Equation

\[ u_{xy} = \frac{4 \sqrt{u_u u_v}}{x+y}. \]

Commuting Characteristics and Vessiot Algebras

Initialize manifold (using coordinates that eliminate square roots)

\[ \text{DGEnvironment}[\text{Coordinate}][\{x, y, u, p_1, q_1, p_2, q_2, p_3, q_3\}, \text{M9}]; \]

The UHat and UChk vector fields for the commuting characteristic distributions are given by

\[ \text{UHatList := subsets}\{(X = x, Y = y, U = u, P_1 = p_1, Q_1 = q_1, P_2 = p_2, Q_2 = q_2, P_3 = p_3, Q_3 = q_3)\}; \]

(1.1)

\[ \text{UChkList := subsets}\{(X = x, Y = y, U = u, P_1 = p_1, Q_1 = q_1, P_2 = p_2, Q_2 = q_2, P_3 = p_3, Q_3 = q_3)\}; \]

(1.2)

Appendix C
Maple Worksheets
Calculate ChkS vector fields:

\[
UChk := \left[ \frac{\partial_y + q_1^2 \partial_x + 2 q_y}{x + y} \partial_{p_1} + q_2 \partial_{q_1} + \frac{4 p_1 - 2 q_1}{(x + y)^2} \partial_{p_2} + q_3 \partial_{q_3} \\
+ \frac{4 (x + y) y^2 - 12 p_1 + 4 q_1}{(x + y)^3} \partial_{p_3} + \frac{4 (x + y) q_3 + 2 q_2 - 2 q_2 y + 2 q_1}{(x + y)^3} \partial_{q_3} \partial_{q_3} \right] \tag{1.3}
\]

The invariants are

\[
\text{HatI} := \text{FirstIntegrals}(UHat) \Rightarrow \text{HatI} := \left[ y, \frac{x + y)^2 q_3 + 4 q_2 x + 4 q_2 y + 2 q_1}{(x + y)^2} \right] \tag{1.4}
\]

\[
\text{ChkI} := \text{FirstIntegrals}(UChk) \Rightarrow \text{ChkI} := \left[ x, \frac{x + y)^2 p_3 + 4 p_2 x + 4 p_2 y + 2 p_1}{(x + y)^2} \right] \tag{1.5}
\]

Calculate HatS vector fields:

\[
S1 := \text{LieDerivative}(UHat[1], UHat[2]) ;
S2 := \text{LieDerivative}(UHat[1], S1) ;
S3 := \text{LieDerivative}(UHat[1], S2) ;
S4 := \text{LieDerivative}(UHat[1], S3) ;
S5 := \text{LieDerivative}(UHat[1], S4) ;
\]

\[
S1 := -\partial_{p_2} + \frac{4}{x + y} \partial_{p_3} \\
S2 := \partial_{p_1} - \frac{4}{x + y} \partial_{p_2} + \frac{14}{(x + y)^2} \partial_{p_3} \\
S3 := -2 p_1 \partial_u - \frac{4}{x + y} \partial_{p_1} - \frac{2}{x + y} \partial_{q_1} - \frac{10}{(x + y)^2} \partial_{p_2} + \frac{2}{(x + y)^2} \partial_{q_1} + \frac{32}{(x + y)^3} \partial_{p_3} \\
S4 := \frac{2 (p_2 x + p_2 y + 4 p_1)}{x + y} \partial_u + \frac{6}{(x + y)^2} \partial_{p_1} - \frac{6}{(x + y)^2} \partial_{q_1} - \frac{12}{(x + y)^3} \partial_{p_2} \\
+ \frac{12}{(x + y)^3} \partial_{q_2} + \frac{36}{(x + y)^4} \partial_{p_3} - \frac{36}{(x + y)^4} \partial_{q_3} \\
S5 := -2 \left( p_3 x^2 + 2 p_3 x y + p_3 y^2 + 4 p_2 x + 4 p_2 y + 2 p_1 \right) \partial_u \tag{1.6}
\]

Calculate ChkS vector fields:

\[
T1 := \text{LieDerivative}(UChk[1], UChk[2]) ;
T2 := \text{LieDerivative}(UChk[1], T1) ;
T3 := \text{LieDerivative}(UChk[1], T2) ;
T4 := \text{LieDerivative}(UChk[1], T3) ;
T5 := \text{LieDerivative}(UChk[1], T4) ;
\]

\[
T1 := -\partial_{q_2} + \frac{4}{x + y} \partial_{q_3} \\
T2 := \partial_{q_1} - \frac{4}{x + y} \partial_{q_2} + \frac{14}{(x + y)^2} \partial_{q_3} \\
T3 := -2 q_1 \partial_u - \frac{2}{x + y} \partial_{p_1} + \frac{4}{x + y} \partial_{q_1} + \frac{2}{(x + y)^2} \partial_{p_2} - \frac{10}{(x + y)^2} \partial_{q_2} - \frac{4}{(x + y)^3} \partial_{p_3} \\
+ \frac{32}{(x + y)^3} \partial_{q_3} \\
T4 := -2 \left( q_2 x + q_2 y + 4 q_1 \right) \partial_u - \frac{6}{(x + y)^2} \partial_{p_1} + \frac{6}{(x + y)^2} \partial_{q_1} + \frac{12}{(x + y)^3} \partial_{p_2} \\
- \frac{12}{(x + y)^3} \partial_{q_2} - \frac{36}{(x + y)^4} \partial_{p_3} + \frac{36}{(x + y)^4} \partial_{q_3} \tag{1.7}
\]
Hat Side

We break this section into three parts.

- In the first section, we calculate the restriction of \( \text{HatDelta} \) and \( \text{HatS} \) to the 7-dimensional integral manifold given by \( \text{HatI} = 0 \), that is \( \text{I1} = \text{I2} = 0 \).
- In the second section, we write \( \text{HatDelta} \) and \( \text{HatS} \) on a 6-dimensional manifold.
- In the third section, we write \( \text{HatDelta} \) and \( \text{HatS} \) on a 5-dimensional manifold.

Restriction to 7-manifold

\[
T_5 := -\frac{2(q_3 x^2 + 2 q_3 x y + q_3 y^2 + 4 q_2 x + 4 q_2 y + 2 q_1)}{(x+y)^2}
\tag{1.7}
\]

Hat Delta := DGsimplify([UHat[1], UHat[2]]);

\[
\text{HatDelta} := \begin{bmatrix}
\partial_x + p_1 \partial_u + p_2 \partial_{p_1} + \frac{2 p_1}{x+y} \partial_{q_1} + p_3 \partial_{p_2} - \frac{2(p_1 - 2 q_1)}{(x+y)^2} \partial_{q_2} \\
+ 2 \left( -2 p_3 x^2 - 4 p_3 x y - 2 p_3 y^2 + p_2 x + p_2 y + 2 p_1 \right) \partial_{p_3} \\
+ 4 \left( q_2 x + q_2 y + p_1 - 3 q_1 \right) \frac{\partial_{q_2} \partial_{p_3}}{(x+y)^3}
\end{bmatrix}
\tag{2.1.1}
\]

The ChkS vector fields are symmetries of \( \text{HatDelta} \)

\[
\text{LieDerivative}([T1, T2, T3, T4, T5], \text{HatDelta});
\]

\[
\begin{bmatrix}
0 \partial_x, 0 \partial_u, 0 \partial_{p_1}, 0 \partial_{p_2}, 0 \partial_{q_1}, 0 \partial_{q_2}, 0 \partial_{I1}, 0 \partial_{I2}
\end{bmatrix}
\tag{2.1.2}
\]

We now write \( \text{HatDelta} \) and ChkS in terms of the invariant coordinates \( \text{HatI} \). Begin by initializing the manifold.

\[
\text{DGEnvironment}[\text{Coordinate}][\{x, u, p_1, p_2, p_3, q_1, q_2, I_1, I_2\}, \text{NineH}];
\]

\[
\text{Manifold: NineH}
\tag{2.1.3}
\]

Initialize the map \( \chi : \text{M9} \rightarrow \text{Nine} \) and its inverse.

\[
\text{chi} := \text{Transformation}(\text{M9}, \text{NineH}, [x = x, u = u, p_1 = p_1, p_2 = p_2, p_3 = p_3, q_1 = q_1, q_2 = q_2, I_1 = \text{HatI}[1], I_2 = \text{HatI}[2] ]);
\]

\[
\chi := x = x, u = u, p_1 = p_1, p_2 = p_2, p_3 = p_3, q_1 = q_1, q_2 = q_2, I_1 = y, I_2
\tag{2.1.4}
\]

\[
\text{invchi} := \text{InverseTransformation}(\text{chi});
\]

\[
\text{invchi} := x = x, y = I_1, u = u, p_1 = p_1, q_1 = q_1, p_2 = p_2, q_2 = q_2, p_3 = p_3, q_3
\] \tag{2.1.5}

Calculate the pushforward \( \text{HatDelta} \) and ChkS

\[
\text{HatDelta1} := \text{simplify}(\text{Pushforward}(\text{chi}, \text{invchi}, \text{HatDelta}));
\]

\[
\text{HatDelta1} := \begin{bmatrix}
\partial_x + p_1 \partial_u + p_2 \partial_{p_1} + p_3 \partial_{p_2} \\
+ 4 I_1^2 p_3 + \left( -8 p_3 x + 2 p_2 \right) I_1 - 4 q_2 x^2 + 2 p_2 x + 4 p_1 \frac{\partial_{p_3}}{x + I_1} + 2 \frac{p_1}{x + I_1} \partial_{q_1}
\end{bmatrix}
\tag{2.1.6}
\]
Now we want to calculate the restriction of \( \hat{\Delta}_1 \) and \( \text{ChkS}_1 \) to the integral manifold given by \( I_1 = I_2 = 0 \). Begin by defining the restricted 7-manifold.

Define the inclusion map

\[
\iota := \text{Transformation}(\text{SevenH}, \text{NineH}, [x = x, u = u, p_1 = p_1, p_2 = p_2, p_3 = p_3, q_1 = q_1, q_2 = q_2, I_1 = 0, I_2 = 0]);
\]

Calculate the restriction of \( \hat{\Delta}_1 \) to Seven. Note that this means \( \iota^* X = Y \).

\[
\hat{\Delta}_2 := \text{PullbackVector}(\iota, \hat{\Delta}_1);
\]

Check the definition.

\[
\text{simplify}(\text{evalDG}(\text{Pushforward}(\iota, \hat{\Delta}_2[1]) - \text{eval}(\hat{\Delta}_1[1], I_1 = 0)));
\]

\[
\text{evalDG}(\text{Pushforward}(\iota, \hat{\Delta}_2[2]) - \hat{\Delta}_1[2]);
\]

Calculate the restriction of \( \text{ChkS}_1 \) to Seven, and again, check the definition. Before we do this, we note that \( \text{ChkS}_1[5] \) has a common factor of \( I_2 \) that can be divided out.

\[
\text{ChkS}_2 := [\text{seq}(\text{PullbackVector}(\iota, \text{ChkS}_1[i]), i = 1..4), \text{PullbackVector}(\iota, \text{ChkS}_1[5]/I_2)];
\]

Check that these vector fields are still symmetries of \( \hat{\Delta}_2 \).

\[
\text{LieDerivative}(\text{ChkS}_2, \hat{\Delta}_2);
\]

\[
\text{DFHat} := \text{CanonicalBasis}(\text{DerivedFlag}(\hat{\Delta}_2)[2]);
\]

**Projection to 6-manifold**

In order to write \( \hat{\Delta}_2 \) on the 6-manifold, we calculate its derived and note that it has a single Cauchy characteristic.
Define the six manifold given by \( p_3 = 0 \)

\[
DGEnvironment[Coordinate](\{x, u, p_1, p_2, q_1, q_2\}, SixH); \\
Manifold: SixH
\]

Define the projection map from Seven to Six

\[
p_1 := Transformation(SevenH, SixH, \{x = x, u = u, p_1 = p_1, p_2 = p_2, q_1 = q_1, q_2 = q_2\}); \\
\pi_1 := x = x, u = u, p_1 = p_1, p_2 = p_2, q_1 = q_1, q_2 = q_2
\]

Note that the derived of HatDelta2 is \( p_1 \)-projectable

\[
LieDerivative(D_p_3, DFHat); \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
HatDelta3 := CanonicalBasis(DGsimplify(Pushforward(pi_1, DFHat))) \\
HatDelta3 := \begin{bmatrix}
0 + p_1^2 \partial_x + p_2 \partial_{p_1} + \frac{2 p_1 l}{x} \partial_{q_1} - \frac{2 (p_1 - 2 q_1)}{x^2} \partial_{q_2} \partial_{p_1} \partial_{p_2}
\end{bmatrix}
\]

As are the vector fields of ChkS2

\[
LieDerivative(D_p_3, ChkS2); \\
[0, 0, 0, 0, 0, 0]
\]

\[
ChkS3 := Pushforward(pi_1, ChkS2);
\]

Again check that ChkS3 are symmetries of HatDelta3

\[
GetComponents(simplify(LieDerivative(ChkS3, HatDelta3)), HatDelta3); \\
\begin{bmatrix}
[0, 0], [0, 0], [0, 0], \frac{4}{x}, [0, 0], [0, 0], [0, 0]
\end{bmatrix}
\]

\[
LieAlgebraData(ChkS3); \\
\begin{bmatrix}
\end{bmatrix}
\]

**Projection to 5-manifold**

In order to write HatDelta3 on a 5-manifold, we again calculate it's derived and note that it has a single Cauchy characteristic.

\[
DFHat1 := CanonicalBasis(DerivedFlag(HatDelta3)[2]); \\
DFHat1 := \begin{bmatrix}
0 + p_1^2 \partial_x + p_2 \partial_{p_1} + \frac{2 p_1 l}{x} \partial_{q_1} - \frac{2 (p_1 - 2 q_1)}{x^2} \partial_{q_2} \partial_{p_1} \partial_{p_2}
\end{bmatrix}
\]

\[
CauchyCharacteristics(Annihilator(DFHat1)); \\
\begin{bmatrix}
0
\end{bmatrix}
\]

Define the 5-manifold given by \( p_2 = 0 \)

\[
DGEnvironment[Coordinate](\{x, u, p_1, q_1, q_2\}, FiveH); \\
Manifold: FiveH
\]

Define the projection map from Six to Five
\begin{verbatim}
\textbf{pi2 := Transformation(SixH, FiveH, \{x = x, u = u, p1 = p1, q1 = q1, q2 = q2\});}
\textbf{x2 := x=x, u=u, p1=p1, q1=q1, q2=q2} \quad (2.3.4)

Note that the derived of HatDelta3 is pi2-projectable

\textbf{LieDerivative(D_p2, DFHat1);}
\textbf{HatDelta4 := CanonicalBasis(Pushforward(pi2, DFHat1))[1..2];}
\textbf{HatDelta4 := \left[ \frac{\partial + pl^2 \partial_u + 2 pl}{x} \partial_{ql} - \frac{2 (pl - 2 ql)}{x^2} \partial_{q2} \partial_{pl} \right]} \quad (2.3.6)

As are the vector fields of ChkS3

\textbf{LieDerivative(D_p2, ChkS3);}
\textbf{ChkS4 := Pushforward(pi2, ChkS3);
ChkS4 := \left[ -\frac{6}{x^2} \partial_{pl} + \frac{6}{x^2} \partial_{ql} - \frac{12}{x^2} \partial_{q2} - 2 \frac{\partial}{u} \right]} \quad (2.3.7)

Again, these vector fields are symmetries of HatDelta4

\textbf{GetComponents(LieDerivative(ChkS4, HatDelta4), HatDelta4);
\left[ 0, 0, 0, 0, 0, 0, 0, 0, 0 \right]} \quad (2.3.8)

We can further check that the Lie determinant vanishes for ChkS4

0 \wedge \partial_u \wedge \partial_{pl} \wedge \partial_{ql} \wedge \partial_{q2}} \quad (2.3.9)

Conclusion: The following is our HatDelta and ChkS on the 5-manifold.

\textbf{HatDelta4;}
\textbf{HatDelta4 := \left[ \frac{\partial + pl^2 \partial_u + 2 pl}{x} \partial_{ql} - \frac{2 (pl - 2 ql)}{x^2} \partial_{q2} \partial_{pl} \right]} \quad (2.1)

\textbf{ChkS4;}
\textbf{ChkS4 := \left[ -\frac{6}{x^2} \partial_{pl} + \frac{6}{x^2} \partial_{ql} - \frac{12}{x^2} \partial_{q2} - 2 \frac{\partial}{u} \right]} \quad (2.2)

\subsection*{Chk Side}

We break this section into three parts.

- In the first section, we calculate the restriction of HatDelta and HatS to the 7-dimensional integral manifold given by HatI = 0, that is I1 = I2 = 0.
- In the second section, we write HatDelta and HatS on a 6-dimensional manifold.
- In the third section, we write HatDelta and HatS on a 5-dimensional manifold.
\end{verbatim}
Restriction to 7-manifold

\[ \begin{align*}
\text{Restriction to 7-manifold} & \quad &: \quad \text{DGsimplify}([\text{UChk}[1], \text{UChk}[2]]); \\
\text{ChkDelta} & := \begin{bmatrix}
\frac{\partial}{\partial y} + q1^2 \frac{\partial}{\partial q1} + \frac{q2}{x+y} \frac{\partial}{\partial p1} + q2 \frac{\partial}{\partial q2} + \frac{2 (2 p1 - q1)}{(x+y)^2} \frac{\partial}{\partial q1} + q3 \frac{\partial}{\partial q2} \\
- \frac{4 (-p2 x - p2 y + 3 p1 - q1)}{(x+y)^3} \frac{\partial}{\partial q3}
\end{bmatrix} \\
& \quad \text{LieDerivative}([S1, S2, S3, S4, S5], \text{ChkDelta}); \\
& \quad \begin{bmatrix}
0 \frac{\partial}{\partial y} + 4 p2 x + 4 p2 y + 2 p1 \\
0 \frac{\partial}{\partial x} + 4 p2 x + 4 p2 y + 2 p1 \\
0 \frac{\partial}{\partial u} + 4 p2 x + 4 p2 y + 2 p1 \\
0 \frac{\partial}{\partial p1} + 4 p2 x + 4 p2 y + 2 p1 \\
0 \frac{\partial}{\partial p2} + 4 p2 x + 4 p2 y + 2 p1 \\
0 \frac{\partial}{\partial q1} + 4 p2 x + 4 p2 y + 2 p1 \\
0 \frac{\partial}{\partial q2} + 4 p2 x + 4 p2 y + 2 p1 \\
0 \frac{\partial}{\partial q3} + 4 p2 x + 4 p2 y + 2 p1
\end{bmatrix}
\end{align*} \]
ChkDelta2 := \[ \frac{q}{u} y \left[ \frac{q}{u} y + \frac{2 q}{u} \frac{q}{u} y + \frac{2 (2 p - q l)}{y^2} \frac{p}{p} y + q \frac{q}{q} y + q \frac{2}{q} y \right] \]

(3.1.9)

Check the definition.

> evalDG(Pushforward(iota, ChkDelta2[1]) - DGsimplify(subs(J1 = 0, ChkDelta1[1])));
0

> evalDG(Pushforward(iota, ChkDelta2[2]) - ChkDelta1[2]);
0

Calculate the restriction of ChkS1 to Seven, and again, check the definition. Before we do this, we note that HatS1[5] has a common factor of J2 that can be divided out.

> evalDG(HatS1[5]/J2);

Check that these vector fields are still symmetries of HatDelta2.

> LieDerivative(HatS2, ChkDelta2);

Projection to 6-manifold

In order to write ChkDelta2 on the 6-manifold, we calculate it's derived and note that it has a single Cauchy characteristic.

> DFChk := CanonicalBasis(DerivedFlag(ChkDelta2)[2]);

(3.2.1)

> CauchyCharacteristics(Annihilator(DFChk));

Define the six manifold given by \( q_3 = 0 \)

> DGEnvironment[Coordinate](\[y,u,p1,p2,q1,q2\], SixC);

Manifold: SixC

Define the projection map from Seven to Six

> pi1 := Transformation(SevenC, SixC, \[y = y, u = u, p1 = p1, p2 = p2, q1 = q1, q2 = q2\]);

(3.2.4)

Note that the derived of ChkDelta2 is pi1-projectable

> LieDerivative(D_q3, DFChk);

> ChkDelta3 := CanonicalBasis(DGsimplify(Pushforward(pi1, DFChk)))[1..2];
\[
\text{ChkDelta3} := \left[ \frac{\partial}{\partial y} + q l^2 \frac{\partial}{\partial u} + \frac{2 q l}{y} \frac{\partial}{\partial p_1} + \frac{2 (2 p l - q l)}{y^2} \frac{\partial}{\partial p_2} + q 2 \frac{\partial}{\partial q_1} \frac{\partial}{\partial q_2} \right] \quad (3.2.6)
\]

As are the vector fields of HatS2

\[
\text{LieDerivative}(D_{q3}, \text{HatS2});
\]

\[
\begin{bmatrix}
0 \frac{\partial}{\partial y}, 0 \frac{\partial}{\partial y}, 0 \frac{\partial}{\partial y}, 0 \frac{\partial}{\partial y}
\end{bmatrix}
\]  \quad (3.2.7)

\[
\text{HatS3} := \text{Pushforward}(\pi_1, \text{HatS2});
\]

Again check that HatS3 are symmetries of ChkDelta3

\[
\text{GetComponents}(\text{simplify}(\text{LieDerivative}(\text{HatS3}, \text{ChkDelta3})), \text{ChkDelta3});
\]

\[
\left[ \begin{bmatrix} 0, 0 \end{bmatrix}, \begin{bmatrix} 0, 0 \end{bmatrix}, \begin{bmatrix} 0, 0 \end{bmatrix}, \begin{bmatrix} 0, 0 \end{bmatrix}, \begin{bmatrix} 0, 0 \end{bmatrix} \right]
\]  \quad (3.2.8)

\section*{Projection to 5-manifold}

In order to write ChkDelta3 on a 5-manifold, we again calculate its derived and note that it has a single Cauchy characteristic.

\[
\text{DFChk1} := \text{CanonicalBasis}(\text{DerivedFlag}(\text{ChkDelta3})[2]);
\]

\[
\text{DFChk1} := \left[ \frac{\partial}{\partial y} + q l^2 \frac{\partial}{\partial u} + \frac{2 q l}{y} \frac{\partial}{\partial p_1} + \frac{2 (2 p l - q l)}{y^2} \frac{\partial}{\partial p_2} \frac{\partial}{\partial q_1} \frac{\partial}{\partial q_2} \right] \quad (3.3.1)
\]

\[
\text{CauchyCharacteristics}(\text{Annihilator}(\text{DFChk1}));
\]

\[
\left[ 0, q_2 \right]
\]  \quad (3.3.2)

Define the 5-manifold given by \(q_2 = 0\)

\[
\text{DGEnvironment}[\text{Coordinate}](\left[ y, u, p_1, p_2, q_1 \right], \text{FiveC});
\]

\[
\text{Manifold: FiveC}
\]  \quad (3.3.3)

Define the projection map from Six to Five

\[
\pi_2 := \text{Transformation}(\text{SixC}, \text{FiveC}, [y = y, u = u, p_1 = p_1, p_2 = p_2, q_1 = q_1]);
\]

\[
\pi_2 := y = y, u = u, p_1 = p_1, p_2 = p_2, q_1 = q_1 \]  \quad (3.3.4)

Note that the derived of ChkDelta3 is \(\pi_2\)-projectable

\[
\text{LieDerivative}(D_{q2}, \text{DFChk1});
\]

\[
\begin{bmatrix}
0 \frac{\partial}{\partial y}, 0 \frac{\partial}{\partial y}, 0 \frac{\partial}{\partial y}
\end{bmatrix}
\]  \quad (3.3.5)

\[
\text{ChkDelta4} := \text{CanonicalBasis}(\text{Pushforward}(\pi_2, \text{DFChk1}))[1..2];
\]

\[
\text{ChkDelta4} := \left[ \frac{\partial}{\partial y} + q l^2 \frac{\partial}{\partial u} + \frac{2 q l}{y} \frac{\partial}{\partial p_1} + \frac{2 (2 p l - q l)}{y^2} \frac{\partial}{\partial p_2} \frac{\partial}{\partial q_1} \frac{\partial}{\partial q_2} \right] \quad (3.3.6)
\]

As are the vector fields of HatS3

\[
\text{LieDerivative}(D_{q2}, \text{HatS3});
\]

\[
\begin{bmatrix}
0 \frac{\partial}{\partial y}, 0 \frac{\partial}{\partial y}, 0 \frac{\partial}{\partial y}, 0 \frac{\partial}{\partial y}
\end{bmatrix}
\]  \quad (3.3.7)

\[
\text{HatS4} := \text{Pushforward}(\pi_2, \text{HatS3});
\]

\[
\text{HatS4} := \left[ -\frac{\partial}{\partial p^2}, \frac{\partial}{\partial p^1} - \frac{4}{y} \frac{\partial}{\partial q_2}, -2 p l \frac{\partial}{\partial p_1} + \frac{4}{y} \frac{\partial}{\partial p_1} - \frac{10}{y^2} \frac{\partial}{\partial p_1} - \frac{2}{y} \frac{\partial}{\partial q_1} - \frac{2 (p l^2 + 4 p l)}{y} \frac{\partial}{\partial u} \right] + \frac{6}{y^2} \frac{\partial}{\partial p_1} - \frac{12}{y^2} \frac{\partial}{\partial q_1} - 2 \frac{\partial}{\partial u} \]  \quad (3.3.8)

Again, these vector fields are symmetries of ChkDelta4
> GetComponents(LieDerivative(HatS4, ChkDelta4), ChkDelta4);
\[
\begin{bmatrix}
[[0, 0], [0, 0]], [[0, 0], [0, 0]], \\
[0, -\frac{2}{y^2}], [0, 0]
\end{bmatrix}
\begin{bmatrix}
[[0, 0], [0, 0]]
\end{bmatrix}
\begin{bmatrix}
0, 0, \\
0, \frac{12}{y^2}, [0, 0]
\end{bmatrix}
\begin{bmatrix}
[[0, 0], [0, 0]]
\end{bmatrix}
\]

We can further check that the Lie determinant vanishes for ChkS4

\[
0, 0, 0, 0, 0
\]

Conclusion: The following is our ChkDelta and HatS on the 5-manifold.

> ChkDelta4;
\[
\left[\partial_y + q l^2 \partial_u + \frac{2 q l}{y} \partial_{p l} + \frac{2 (2 p l - q l)}{y^2} \partial_{p l} \partial_{q l}\right]
\]

> HatS4;
\[
\left[\begin{array}{c}
-\frac{1}{y^2} \partial_{p l} - \frac{6}{y^2} \partial_{q l} - 2 \partial_u \\
- \frac{4}{y} \partial_{p l} - 2 p l \partial_u + \frac{4}{y} \partial_{p l} - \frac{10}{y^2} \partial_{q l} - \frac{2}{y^2} \partial_{q l} - \frac{2 (p l + q l)}{y} \partial_u + \frac{6}{y^2} \partial_{p l} \\
- \frac{12}{y^2} \partial_{p l} - \frac{6}{y^2} \partial_{q l} - 2 \partial_u
\end{array}\right]
\]
C.2 Reconstruction of the Goursat Equation

The following Maple worksheet shows the calculations for the reconstruction of the Goursat equation (9.1).

Reconstruction of the Goursat Equation

restart;
Preferences("ShowFramePrompt", false):
_EnvExplicit := true:

Load in database of subalgebras of \( g_2 \) and custom programs for calculating the prolongation of a vector fields and product manifolds.

read ("C:\Users\brand\OneDrive\Dissertation\BrandonsPrograms\DoubrovData[5D-v2].mm");with(DoubrovDatabase):
read ("C:\Users\brand\OneDrive\Dissertation\BrandonsPrograms\Prolong.mm");
read ("C:\Users\brand\OneDrive\Dissertation\BrandonsPrograms\ProductM.mm");

Here we realize the Goursat equation as the quotient

\[
\Delta = \left( \Delta_1^{(1)} \oplus \Delta_2^{(1)} \right) / G_{\text{diag}}
\]

where \( \Delta_1 \) and \( \Delta_2 \) are Hilbert-Cartan distributions

\[
\Delta = \left\{ \partial_x + x \partial_z + \partial_y + y \partial_1, \partial_y, \phi \partial_1, \phi \partial_2 \right\}
\]

and \( G_{\text{diag}} \) denotes the diagonal action of the group \( G \) generated by the first prolongation of the vector fields \([N, 23]\).

Initialize the 5-manifold.

DGEEnvironment[Coordinate]([x,z,y,y1,y2], M5);

Initialize the Hilbert-Cartan distribution.

Delta5 := evalDG([D_x + y2*D_z + y1*D_y + y2*D_y1, D_y2]);

\[
\Delta^5 := \left[ \partial_x + y \partial_z + y1 \partial_y + y2 \partial_1, \partial_y, \phi \partial_1, \phi \partial_2 \right]
\]

Initialize the infinitesimal generators for \( g_2 \).

Sym5 := DGzip(DGTable["g2"], DGinformation(M5, "FrameBaseVectors"));
Initialize the 5-dimensional subalgebra given by [N, 23].

\[ \Gamma_5 := \left[ \text{DGzip(DGTable["N", 23], Sym5)} \right] \]
\[ I_3 := \left[ \frac{y_1}{2} \frac{x_1}{12} + \frac{y}{12} \partial_z + \frac{x^2}{144} \partial_y + \frac{x^2}{48} \partial_{y_1} + \frac{x}{24} \partial_{y_2} + \frac{x^2}{2} \partial_y + \frac{1}{2} \right] \partial_{y_1} 3 \partial_y \partial_y \]
\[ + 3 \partial_{y_1} \partial_{y_2} \partial_y \]
\[ \text{LieAlgebraData}(\Gamma_5) ; \]
\[ [e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_2, e_3] = 0, [e_2, e_4] = 0, [e_2, e_5] = e_1, [e_3, e_4] = -3 e_1, [e_3, e_5] = 0, [e_4, e_5] = 0 \]

Initialize the 6-dimensional manifold for the prolongation.

\[ \Delta_{M12} := \left[ \text{DGzip(DGTable[Annihilator(\Delta_5), \{dy, dy_1, dy_2, dz, dx\})]} \right] \]
\[ \text{LieAlgebraData}(\Gamma_5); \]
\[ \text{ChangeFrame}(M_6); \]

Calculate the prolonged infinitesimal generators.

\[ \Delta_6 := \text{evalDG}([D_x + y_2^2 D_z + y_1 D_y + y_2 D_y_1 + y_3 D_y_2, D_y_3]) ; \]
\[ \Delta_6 := \left[ \frac{1}{4} 3 y_1 \partial_x + 3 \partial_{y_1} 12 \partial_y \right] \]
\[ \text{LieAlgebraData}(\Gamma_6) ; \]
\[ [e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_2, e_3] = 0, [e_2, e_4] = 0, [e_2, e_5] = e_1, [e_3, e_4] = -3 e_1, [e_3, e_5] = 0, [e_4, e_5] = 0 \]

The prolonged Hilbert-Cartan distribution is

\[ \Delta_{M12} := \text{evalDG}([D_x + y_2^2 D_z + y_1 D_y + y_2 D_y_1 + y_3 D_y_2, D_y_3]) ; \]
\[ \Delta_{M12} := \left[ \frac{1}{4} 3 y_1 \partial_x + 3 \partial_{y_1} 12 \partial_y \right] \]
\[ \text{LieAlgebraData}(\Gamma_6) ; \]
\[ [e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_2, e_3] = 0, [e_2, e_4] = 0, [e_2, e_5] = e_1, [e_3, e_4] = -3 e_1, [e_3, e_5] = 0, [e_4, e_5] = 0 \]

Calculate the dual Pfaffian system to the Hilbert-Cartan distribution.

\[ \text{Delta}_{M12} := \left[ \text{DGzip(DGTable[Annihilator(\Delta_5), \{dy, dy_1, dy_2, dz, dx\})]} \right] \]
\[ \text{LieAlgebraData}(\Gamma_5); \]
\[ \text{ChangeFrame}(M_6); \]

Calculate the dual Pfaffian system to the Hilbert-Cartan distribution.

\[ \text{Delta}_{M12} := \left[ \text{DGzip(DGTable[Annihilator(\Delta_5), \{dy, dy_1, dy_2, dz, dx\})]} \right] \]
\[ \text{LieAlgebraData}(\Gamma_5); \]
\[ \text{ChangeFrame}(M_6); \]

Calculate the direct sum of the prolonged Hilbert-Cartan distributions on M_12.

\[ \text{Delta}_{M12} := \left[ \text{DGzip(DGTable[Annihilator(\Delta_5), \{dy, dy_1, dy_2, dz, dx\})]} \right] \]
\[ \text{LieAlgebraData}(\Gamma_5); \]
\[ \text{ChangeFrame}(M_6); \]
The total differential operators are

\[ \frac{\partial}{\partial z} + u^2 \frac{\partial}{\partial x} + u l \frac{\partial}{\partial y} + u^2 \frac{\partial}{\partial x} + u^3 \frac{\partial}{\partial y} + v l \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y} + v l \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y} \]

Adjust the diagonal action to obtain the vector fields given in Section 8.1.

\[ Z_1 := \text{evalDG} \left( \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi1} \right), \text{pi1}, \text{Gamma6}[1] \right) - \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi2} \right), \text{pi2}, \text{Gamma6}[1] \right) \right) ; \]

\[ Z_2 := \text{evalDG} \left( \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi1} \right), \text{pi1}, \text{Gamma6}[2] \right) + \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi2} \right), \text{pi2}, \text{Gamma6}[2] \right) \right) ; \]

\[ Z_3 := \text{evalDG} \left( \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi1} \right), \text{pi1}, \text{Gamma6}[3] \right) - \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi2} \right), \text{pi2}, \text{Gamma6}[3] \right) \right) ; \]

\[ Z_4 := \text{evalDG} \left( \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi1} \right), \text{pi1}, \text{Gamma6}[4] \right) + \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi2} \right), \text{pi2}, \text{Gamma6}[4] \right) \right) ; \]

\[ Z_5 := \text{evalDG} \left( \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi1} \right), \text{pi1}, \text{Gamma6}[5] \right) - \text{Pushforward} \left( \text{InverseTransformation} \left( \text{pi2} \right), \text{pi2}, \text{Gamma6}[5] \right) \right) ; \]

\[ \text{DeltaM12} := \text{collect} \left( \left( -w z \right) x^3 + \left( -3 w^3 z^3 \right) y + 4 u l^2 + 4 u l v + 4 \right) \]

\[ \text{Xi} := \text{collect} \left( \left( \left( -w z \right) x^3 + \left( -3 w^3 z^3 \right) y + 4 u l^2 + 4 u l v + 4 \right) \right) \]

\[ \text{Yi} := \text{collect} \left( \left( \left( -w z \right) x^3 + \left( -3 w^3 z^3 \right) y + 4 u l^2 + 4 u l v + 4 \right) \right) \]

\[ \text{UX} := \text{factor} \left( \text{LieDerivative} \left( \text{Xtot}, \text{U} \right) \right) ; \]

\[ \text{UX} := -u^2 x^2 + 2 u l x y + u^2 y^2 - 4 u l x - 4 u l y - 2 v l x - 2 v l y + 6 u + 6 v \]

\[ \left( x + y \right)^2 \]
Initialize the quotient map to the equation manifold.

\[ U_{XX} := \text{factor}(\text{LieDerivative}(\text{Ytot}, \text{U})); \]
\[ U_{XX} := \frac{1}{(x+y)^5} \left( 2 \left( 2 u^2 x^2 + 2 u^2 x y + u^2 y^2 - 4 u l y - 2 v l x - 2 v l y + 6 u + 6 v \right) \right) \]
\[ \text{UXX} := \text{factor}(\text{LieDerivative}(\text{Xtot}, \text{UX})); \]
\[ \text{UXX} := \frac{1}{(x+y)^5} \left( 2 \left( 2 u^2 x^2 + 2 u^2 x y + u^2 y^2 - 4 u l y - 2 v l x - 2 v l y + 6 u + 6 v \right) \right) \]

This is the rank 4 distribution defining the Goursat equation.

\[ \text{DGEnvironment}[\text{Coordinate}]\{[x, y, u, p, q, r, t], \text{M7}\}; \]
Manifold: M7

Initialize the 7-dimensional equation manifold.

\[ \text{quot} := \text{Transformation}(\text{M12}, \text{M7}, \{x = x, y = y, u = \text{U}, p = \text{UX}, q = \text{UY}, r = \text{UXX}, t = \text{UYY}\}); \]
\[ \text{quot} := x = x, y = y, u = \frac{4 u l^2}{x+y} + \frac{4 u l v l}{x+y} - \frac{12 (u+v) u l}{(x+y)^2} + \frac{4 v l^2}{x+y} - \frac{12 (u+v) v l}{(x+y)^2} - w - z \]
\[ q = \left( \frac{u^2 x^2 + 2 u^2 x y + u^2 y^2 - 4 u l x - 4 u l y - 2 v l x - 2 v l y + 6 u + 6 v}{(x+y)^4} \right)^2, \]
\[ r = \frac{1}{(x+y)^5} \left( 2 \left( 2 u^2 x^2 + 2 u^2 x y + u^2 y^2 - 4 u l x - 4 u l y - 2 v l x - 2 v l y + 6 u + 6 v \right) \right) \}

Calculate the pushforward of DeltaM12

\[ \text{DeltaM7} := \text{simplify}(\text{CanonicalBasis}(\text{Pushforward}(\text{quot}, \text{invquot}, \text{DeltaM12}), \text{symbolic})); \]
\[ \text{DeltaM7} := \left[ \partial_s + p \partial_u + r \partial_r + \frac{4 \sqrt{q} \sqrt{p}}{x+y} \partial_q, q \partial_u + q \partial_r + \frac{4 \sqrt{q} \sqrt{p}}{x+y} \partial_q + r \partial_r, \partial_r \right] \]

This is the rank 4 distribution defining the Goursat equation.
C.3 Fundamental Invariants of the First Equation of Zhiber and Sokolov

The following Maple worksheet shows the calculations of the fundamental invariants of the first equation of Zhiber and Sokolov (9.2).

```
restart;
libname := "C:\FiveVariablesMLA-2", "C:\Program Files\Maple 2016\lib":
Preferences("ShowFramePrompt", false):
Here, we calculate the fundamental invariants for the first equation of Zhiber and Sokolov

\[ u_{xy} = \frac{\alpha(u_{y})\beta(u_{y})}{u}. \]

\[ (1.1) \]

\[ (1.2) \]

Fundamental Invariants for the First Equation of Zhiber and Sokolov

Here, we calculate the fundamental invariants for the first equation of Zhiber and Sokolov

```
Commuting Characteristics and Vessiot Algebras

Here are the side conditions for the equation.

```
DGSuppress([A(p1), B(q1)]);
EQ := {diff(A(p1), p1) = -(-2*p1+A(p1))/A(p1), diff(B(q1), q1) = -(-2*q1+B(q1))/B(q1)};
EQ
```

```
\[
\begin{align*}
\frac{dA}{dp1} &= -\frac{2 p1 + A(p1)}{A}, \\
\frac{dB}{dq1} &= -\frac{2 q1 + B(q1)}{B} \\
\end{align*}
\]

(1.1)
```

```
Initialize manifold (with side conditions)
```
```
DGEnvironment[Manifold]([x, y, u, p1, q1, p2, q2, p3, q3], M9, sideconditions = EQ);
```
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```
The invariants are

\[ U_{hat} := D\text{Gzip}(U_{hat}\text{List, Dginformation(M9, "FrameBaseVectors")}); \]

\[ U_{hat} := \left[ \partial_y + p_1 \partial_u + p_2 \partial_{q_1} + A B q_1 u^2 - \frac{(2 p_1 A + A) B}{u^2} \right] \partial_{\theta_5} + \frac{1}{A^3} \left( 2 A^7 p_1 + 2 A^6 p_2 u - A^5 p_1 p_2 u \right) \]

\[ - 2 A^3 p_3 u^2 + 2 A^3 p_1^2 p_2 u - 4 A^3 p_1 p_3 u^2 - 4 A^3 p_3^2 u^2 - 4 A^3 p_1 p_2 u^2 \]

\[ - 5 A^3 p_3 u^3 + 8 A^4 p_1^2 p_3 u^2 + 6 A^2 p_1 p_2 u p_3 u^3 - 2 A^2 p_3 u^3 + 14 A p_1 p_2 u^3 \]

\[ - 12 p_1^2 p_3^2 u^2 \left[ \partial_{\theta_3} - \frac{1}{A^3} \left( 2 \left( \frac{dA}{d\theta_1} \right) A B^5 p_1 + 2 \left( \frac{dB}{d\theta_1} \right) A^7 q_2 u^2 \right) \right] \]

\[ - 3 A^3 B^3 - 3 B^3 q_1 A^2 - 2 A^2 B^3 q_2 u + 4 A^2 q_1 q_2 u B^2 + q_3 A^2 u^2 B^2 \]

\[ + 4 A^2 q_2 u B - 2 q_3 A^2 u^2 B q_1 - 2 A^3 B q_2 u^2 + 4 A^3 B p_1 + 6 B q_1 A p_1 \]

\[ + 6 A^3 p_1 q_2 u - 12 A B p_3 q_1 q_2 u - 4 B^4 p_1^2 u^2 \right] \partial_{\theta_4} A \partial_{\theta_3} \]

\[ U_{chk} := D\text{Gzip}(U_{chk}\text{List, Dginformation(M9, "FrameBaseVectors")}); \]

\[ U_{chk} := \left[ \partial_y + q_1 \partial_u + A B \partial_{p_1} + q_2 \partial_{q_1} + \right] \left[ - 2 p_1 A B \right] \partial_{\theta_5} + \frac{1}{A^3} \left( 2 A^7 p_1 + 2 A^6 p_2 u - A^5 p_1 p_2 u \right) \]

\[ - 2 A^3 B^3 p_1^2 u^2 + 2 A^3 B^3 p_2 u + 6 A^3 B p_2 q_1 u + 4 B^4 p_1 p_2 u A^3 + p_3 B^2 u^2 A^2 \]

\[ - 12 A^3 B p_1 q_2 u + 4 B^4 p_1^2 p_2 u A - 2 p_3 B^2 u^2 A p_1 - 2 A B^2 p_2 u^2 \right] \partial_{\theta_3} \]

\[ + \frac{1}{B^2 u^2} \left( 2 B^2 q_1 + 2 B^2 q_1^2 + B^2 q_2 u - B^2 q_1 u - 2 B^2 q_3 u^2 + 2 B^2 q_1^2 q_2 u \right) \partial_{\theta_3} \]

\[ - 4 B^2 q_1 q_3 u^2 - 4 B^2 q_2 u^2 - 4 B^3 q_1 q_2 u^2 - 5 B^2 q_2 q_3 u^3 + 8 B^2 q_1^2 q_2 u^2 \]

\[ + 6 B^2 q_1 q_2 q_3 u^3 - 2 B^2 q_1^2 u^2 + 14 B q_1 q_3 u^3 - 12 q_1^2 q_2 u^2 \right] \partial_{\theta_4}, B \partial_{\theta_3} \]

The invariants are

\[ H_{12} := \left[ q_1, q_3 B q_1 + 2 B(q_1) - q_1/B(q_1) \right]; \]
\[
\text{HatId} := \left[ y, \frac{q_3}{B} + \frac{2}{B^2}(B-q_1)q_2^2 + \frac{2}{uB}(2q_1+B)q_2 + \frac{B(q_1+B)}{u^2} \right] \quad (1.5)
\]

\[
\text{ChkI} := \left[ x, \frac{p_3}{A(p_1)} + 2*(A(p_1) - p_1)/A(p_1)^3p_2^2 + 2*(2*p_1 + A(p_1))/(u*A(p_1))*p_2 + A(p_1)*(p_1 + A(p_1))/u^2 \right]; \quad (1.6)
\]

Calculate HatS vector fields:
\[
\begin{align*}
S_1 & := \text{LieDerivative}(\text{UHat}[1], \text{UHat}[2]) : \\
S_2 & := \text{LieDerivative}(\text{UHat}[1], S_1) : \\
S_3 & := \text{LieDerivative}(\text{UHat}[1], S_2) : \\
S_4 & := \text{LieDerivative}(\text{UHat}[1], S_3) : \\
S_5 & := \text{LieDerivative}(\text{UHat}[1], S_4) :
\end{align*}
\]

The HatS vector fields form a basis for the Vessiot algebra with structure equations:
\[
\text{factor} (\text{VessiotAlgebra}([S_1, S_2, S_3, S_4, S_5], \{j2= \text{ChkI}[2], \{p_3\}], \text{HatAlg}));
\]

\[
\begin{align*}
[e_1, e_2] & = -2 \, e_1, \quad [e_1, e_3] = -2 \, e_2, \quad [e_1, e_4] = -8 \, j2 \, e_1 + 4 \, e_3 - \frac{1}{j2} \, e_5, \quad [e_1, e_5] = -8 \, j2 e_2 + 32 \, j2^2 e_1 - 28 j2 e_3 + 5 e_5, \quad [e_1, e_4] = -8 \, j2 e_2 - 32 \, j2^2 e_1 + 32 \, j2 e_3 - 6 e_5, \quad [e_1, e_5] = -32 j2^2 e_1 - 32 \, j2 e_3 - 6 e_5, \quad [e_2, e_3] = 8 \, j2 e_1 - 6 e_3 + \frac{1}{j2} \, e_5, \quad [e_2, e_4] = 4 \, j2 e_2 - 4 \, e_4, \quad [e_2, e_5] = 32 \, j2^2 e_1 \\
[e_3, e_4] & = -32 \, j2^2 e_1 + 32 \, j2 e_3 - 6 e_5, \quad [e_3, e_5] = -8 \, j2^2 e_2 + 128 j2^3 e_1 - 144 j2^2 e_3 + 28 j2 e_5
\end{align*}
\]

Calculate ChkS vector fields:
\[
\begin{align*}
T_1 & := \text{LieDerivative}(\text{UChk}[1], \text{UChk}[2]) : \\
T_2 & := \text{LieDerivative}(\text{UChk}[1], T_1) : \\
T_3 & := \text{LieDerivative}(\text{UChk}[1], T_2) : \\
T_4 & := \text{LieDerivative}(\text{UChk}[1], T_3) : \\
T_5 & := \text{LieDerivative}(\text{UChk}[1], T_4) :
\end{align*}
\]

The ChkS vector fields form a basis for the Vessiot algebra with structure equations:
\[
\text{LD} := \text{factor} (\text{VessiotAlgebra}([T_1, T_2, T_3, T_4, T_5], \{i2= \text{HatI}[2], \{p_3\}], \text{ChkAlg}));
\]

\[
\begin{align*}
[e_1, e_2] & = -2 \, e_1, \quad [e_1, e_3] = -2 \, e_2, \quad [e_1, e_4] = -8 \, i2 \, e_1 + 4 \, e_3 - \frac{1}{i2} \, e_5, \quad [e_1, e_5] = -8 \, i2 e_2 + 32 \, i2^2 e_1 - 28 i2 e_3 + 5 e_5, \quad [e_1, e_4] = -8 \, i2 e_2 - 32 \, i2^2 e_1 + 32 \, i2 e_3 - 6 e_5, \quad [e_1, e_5] = -32 i2^2 e_1 - 32 \, i2 e_3 - 6 e_5, \quad [e_2, e_3] = 8 \, i2 e_1 - 6 e_3 + \frac{1}{i2} \, e_5, \quad [e_2, e_4] = 4 \, i2 e_2 - 4 \, e_4, \quad [e_2, e_5] = 32 i2^2 e_1 \\
[e_3, e_4] & = -32 i2^2 e_1 + 32 i2 e_3 - 6 e_5, \quad [e_3, e_5] = -8 i2^2 e_2 + 128 i2^3 e_1 - 144 i2^2 e_3 + 28 i2 e_5
\end{align*}
\]

Initialize the Vessiot algebra
\[
\text{DGEnvironment}[\text{LieAlgebra}](\text{LD});
\]

Check Levi decomposition:
\[
\text{LeviDecomposition}(\text{ChkAlg});
\]

This is the only 5D algebra with nontrivial Levi decomposition.
We now show that these systems satisfy the congruences for the second prolongation of a rank 3 Monge system defined on a 5 manifold.

We begin by constructing the following frame on the 9-dimensional equation manifold:

\[
\text{HatFr} := [S1, S2, S3, S4, S5, \text{op(UHat)}, \text{op(UChk)}];
\]

Apply change of frame and calculate structure equations:

\[
\text{DGEnvironment}[\text{AnholonomicFrame}] \left( \text{evalDG}([\text{HatFr}[4], -1/\text{ChkI}[2] \times \text{HatFr}[5], -\text{HatFr}[3], \text{HatFr}[2], -\text{HatFr}[1], -\text{HatFr}[6], -\text{HatFr}[7], \text{HatFr}[8], \text{HatFr}[9]]), \text{HF1}, \text{formlabels} = ' [o1, o2, o3, o4, o5, pi1, pi2, chi1, chi2]' \right);
\]

Define DGMod command to calculate congruences:

\[
\text{DGMod} := \text{proc(}\theta, J) \text{DGsubs([seq(J[i] = 0 &mult DGinformation(DGinformation("CurrentFrame"), "FrameBaseForms")[1], i = 1..nops(J))], \theta); end:}
\]

The congruences on the 9-manifold are

\[
\text{strH1 := factor(DGMod(ExteriorDerivative(o1), [o1, o2]));}
\]

\[
\text{strH2 := factor(DGMod(ExteriorDerivative(o2), [o1, o2]));}
\]

\[
\text{strH3 := factor(DGMod(ExteriorDerivative(o3), [o1, o2, o3]));}
\]

\[
\text{strH4 := factor(DGMod(ExteriorDerivative(o4), [o1, o2, o3, o4]));}
\]

\[
\text{strH5 := factor(DGsimplify(DGsubs(convert(EQ, list), DGMod(ExteriorDerivative(o5), [o1, o2, o3, o4, o5]))))};
\]

Note that \( \chi^1, \chi^2 \) are equal to the exterior derivative of the HatI invariants.

\[
\text{DGsimplify(DGsubs(convert(EQ, list), ExteriorDerivative(HatI)))}
\]

\[
[\chi^1, \chi^2]
\]

Upon restricting to the 7-dimensional integral manifold given by \( \chi^1 = \chi^2 = 0 \), the congruences equations become

\[
\text{DGsubs([chi1 = 0 &w chi1, chi2 = 0 &w chi1], strH1);}
\]

\[
\text{DGsubs([chi1 = 0 &w chi1, chi2 = 0 &w chi1], strH2);}
\]
We break this section into three parts.

These are precisely the congruences for the second prolongation of a rank 3 Monge system defined on a 5-manifold.

In the first section, we calculate the restriction of \( \hat{\Delta} \) and \( \hat{S} \) to the 7-dimensional integral manifold given by \( \hat{I}_1 = 0 \), that is \( I_1 = I_2 = 0 \).

In the second section, we write \( \hat{\Delta} \) and \( \hat{S} \) on a 5-dimensional manifold.

In the third section, we write \( \hat{\Delta} \) and \( \hat{S} \) on a 5-dimensional manifold.

---

**Hat Side**

We break this section into three parts.

- In the first section, we calculate the restriction of \( \hat{\Delta} \) and \( \hat{S} \) to the 7-dimensional integral manifold given by \( \hat{I}_1 = 0 \), that is \( I_1 = I_2 = 0 \).
- In the second section, we write \( \hat{\Delta} \) and \( \hat{S} \) on a 6-dimensional manifold.
- In the third section, we write \( \hat{\Delta} \) and \( \hat{S} \) on a 5-dimensional manifold.

**Restriction to 7-manifold**

\[
\hat{\Delta} := \text{DGsimplify}([\text{UH}[1], \text{UH}[2]])
\]

(3.1.1)
The ChkS vector fields are symmetries of HatDelta

\[ \text{LieDerivative} \{ [T1, T2, T3, T4, T5], \text{HatDelta} \}; \]
\[
\begin{bmatrix}
[0 \partial_x 0 \partial_y 0 \partial_z]
[0 \partial_x 0 \partial_y 1 \partial_z]
[0 \partial_x 0 \partial_y 0 \partial_z]
[0 \partial_x 0 \partial_y 0 \partial_z]
[0 \partial_x 0 \partial_y 0 \partial_z]
\end{bmatrix}
\]  \hspace{1cm} (3.1.2)

We now write HatDelta and ChkS in terms of the invariant coordinates HatI. Begin by initializing the manifold.

\[ \text{DGEnvironment[Manifold]} \{ \{x, u, p1, p2, p3, q1, q2, I1, I2 \}, \text{NineH}, \text{sideconditions} = \text{EQ} \}; \]
\[
\text{Manifold: NineH}
\]  \hspace{1cm} (3.1.3)

Initialize the map chi: M9 \rightarrow Nine and it's inverse.

\[ \text{chi} := \text{Transformation} (M9, \text{NineH}, \{x = x, u = u, p1 = p1, p2 = p2, p3 = p3, q1 = q1, q2 = q2, I1 = \text{HatI}[1], I2 = \text{HatI}[2] \}); \]
\[
\chi := x = x, \ u = u, \ p1 = p1, \ p2 = p2, \ p3 = p3, \ q1 = q1, \ q2 = q2, \ I1 = y, \ I2 = \frac{q3}{B} \]
\hspace{1cm} + \frac{2 (B - q1) q2^2}{u B} + \frac{2 (2 q1 + B) q2}{u B} + \frac{B (q1 + B)}{u^2} \]
\[ \text{invchi} := \text{InverseTransformation} (\text{chi}); \]
\[
\text{invchi} := x = x, \ y = I1, \ u = u, \ p1 = p1, \ q1 = q1, \ p2 = p2, \ q2 = q2, \ p3 = p3, \ q3 = \]
\[-12 B^2 u^3 + B^2 + 2 u q2 B^2 + 4 u q2 q1 B^2 + 2 u^2 q2^2 B - 2 u^2 q2^2 q1 \]
\hspace{1cm} u^2 B^2
\]  \hspace{1cm} (3.1.4)

Pushforward HatDelta and ChkS

\[ \text{HatDelta1} := \text{Pushforward} (\text{chi}, \text{invchi}, \text{HatDelta}); \]
\[
\text{HatDelta1} := \left[ \begin{array}{c}
\partial_x + p1 \partial_u + p2 \partial_{p1} + p3 \partial_{p2} + \frac{1}{A^2} (2 A^4 p1 + 2 A^6 p1^2 + A^8 p2 u)
- A^5 p1 p2 u - 2 A^5 p3 u^2 + 2 A^4 p1^2 p2 u - 4 A^4 p1 p3 u^2 - 4 A^4 p2^2 u
- 4 A^3 p1 p2^2 u^2 - 5 A^3 p2 p3 u^3 + 8 A^3 p1^2 p2^2 u^2 + 6 A^2 p1^2 p2 p3 u^3 - 2 A^2 p3 u^4
+ 14 A p1 p2^3 u^3 - 12 p1^2 p2^3 u^3 \partial_{p3} + \frac{AB}{u^2} \partial_{q1} + \left\{ - \frac{AB q1}{u^2} + \frac{(2 p1 - A) B^2}{u^2} \right\} \partial_{p3} + \frac{AB}{u^2} \partial_{q1} + \left\{ - \frac{AB q1}{u^2} + \frac{(2 p1 - A) B^2}{u^2} \right\} \partial_{p3} + \frac{AB}{u^2} \partial_{q1}
- A (-2 q1 + B) q2 \partial_{q2} + \frac{AB}{u^2} \partial_{q1} + \left\{ - \frac{AB q1}{u^2} + \frac{(2 p1 - A) B^2}{u^2} \right\} \partial_{p3} + \frac{AB}{u^2} \partial_{q1} + \left\{ - \frac{AB q1}{u^2} + \frac{(2 p1 - A) B^2}{u^2} \right\} \partial_{p3} + \frac{AB}{u^2} \partial_{q1}
\end{array} \right]
\]  \hspace{1cm} (3.1.5)

Now we want to calculate the restriction of HatDelta1 and ChkS1 to the integral manifold given by I1 = I2 = 0. Begin by defining the restricted 7-manifold.

\[ \text{DGEnvironment[Manifold]} \{ \{x, u, p1, p2, p3, q1, q2 \}, \text{SevenH}, \text{sideconditions} = \text{EQ} \}; \]
\[
\text{Manifold: SevenH}
\]  \hspace{1cm} (3.1.6)

Define the inclusion map

\[ \text{iota} := \text{Transformation} (\text{SevenH}, \text{NineH}, \{x = x, \ u = u, \ p1 = p1, \ p2 = p2, \ p3 = p3, \ q1 = q1, \ q2 = q2, \ I1 = 0, \ I2 = 0 \}); \]
\[
\iota := x = x, \ u = u, \ p1 = p1, \ p2 = p2, \ p3 = p3, \ q1 = q1, \ q2 = q2, \ I1 = 0, \ I2 = 0
\]  \hspace{1cm} (3.1.7)

Calculate the restriction of HatDelta1 to Seven. Note that this means \( \iota X_p = \iota X_p \).

\[ \text{HatDelta2} := \text{PullbackVector} (\text{iota}, \text{HatDelta}); \]
\[
\text{HatDelta2} := \left[ \begin{array}{c}
\partial_x + p1 \partial_u + p2 \partial_{p1} + p3 \partial_{p2} + \frac{1}{A^2} (2 A^4 p1 + 2 A^6 p1^2 + A^8 p2 u)
\end{array} \right]
\]  \hspace{1cm} (3.1.8)
In order to write HatDelta2 on the 6-manifold, we calculate it's derived and note that it has a single Cauchy characteristic.

\[ DFHat := \text{CanonicalBasis}(\text{DerivedFlag}(\text{HatDelta2})[2]); \]
\[
\begin{align*}
DFHat &= \begin{bmatrix}
\frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial u} + p_2 \frac{\partial}{\partial q_1} + \frac{A B}{u} \frac{\partial}{\partial q_1} \\
- B^3 A - 2 B^3 p_1 + B^2 q_1 A + A B q_2 u - 2 A q_1 q_2 u \frac{\partial}{\partial q_2} \frac{\partial}{\partial p_3} \frac{\partial}{\partial p_1}
\end{bmatrix}
\end{align*}
\]
\[ \begin{align*}
\text{CauchyCharacteristics}(\text{Annihilator}(DFHat));
\end{align*}
\]
Define the six manifold given by \( p_3 = 0 \)

```plaintext
DGEnvironment[Manifold]([x,u,p1,p2,q1,q2], SixH, sideconditions = EQ);
Manifold: SixH
```

Define the projection map from Seven to Six

```plaintext
pi1 := Transformation(SevenH, SixH, [x = x, u = u, p1 = p1, p2 = p2, q1 = q1, q2 = q2]);
```

Note that the derived of HatDelta2 is \( \pi_1 \)-projectable

```plaintext
LieDerivative(D_p3, DFHat);
```

As are the vector fields of ChkS2

```plaintext
simplify(LieDerivative(D_p3, ChkS2));
```

Again check that ChkS3 are symmetries of HatDelta3

```plaintext
GetComponents(simplify(LieDerivative(ChkS3, HatDelta3)), HatDelta3);
```
\[-6 B A^2 p_2^2 q_1 u^2 + 12 B p_1^2 p_2 q_1 u^2 - 8 u^2 p_1 p_2 A^3 q_2 + 4 A^2 p_1^2 p_2 q_2 u^2\]
\[-2 A p_1 p_2^2 q_2 u^3 + 9 B^2 A^3 p_1 + 14 A^4 B^2 p_1^2 - 3 A^6 q_1 B - A^6 q_2 u\]
\[-12 u p_1 p_2 A^3 q_1 B + 24 B A^3 p_1 p_2 q_1 u - 6 B A p_1 p_2^2 q_1 u^2 + 4 p_1^2 p_2^2 q_2 u^3\]
\[+ 2 A^4 B^2 p_2 u + 2 B^2 A^2 p_2^2 u^2 - 4 B^2 p_1^2 p_2^2 u^2 - 3 B A^5 p_1 q_1 - 18 A^5 B p_1^2 q_1\]
\[+ 3 A^5 p_1 q_2 u - 2 A^3 q_2 u p_1^2 + 4 A^2 q_2 u^2 p_2 - 2 A^2 p_2^2 q_2 u^3\]
\[(2 q_1 + B) \left( -2 p_1 + A \right) \left( B^2 - 3 B q_1 - q_2 u \right) \right]\left[ 0, - \frac{1}{A^2 u} \right] (19 B A^6)
\[+ 39 B A^6 p_1 + 6 B A^4 p_1^2 + 36 B A^3 p_2 u - 72 B A^3 p_1 p_2 u - 12 B A^5 p_1^2 p_2 u\]
\[+ 6 B A^2 p_2^2 u^2 + 6 B A p_1 p_2^2 u^2 - 12 B p_1^2 p_2^2 u^2 - 26 A^6 q_1 - 18 A^5 p_1 q_1\]
\[+ 20 A^4 p_1^2 q_1 u - 8 A^4 p_2 q_1 u + 16 A^3 p_1 p_2 q_1 u + 40 A^2 p_2^2 q_1 u u\]
\[+ 20 A^2 p_2^2 q_1 u^2 - 20 A p_1 p_2^2 q_1 u^2 + 40 p_1^2 p_2^2 q_1 u^2\]\[(3.3.2)\]

**Projection to 5-manifold**

In order to write HatDelta3 on a 5-manifold, we again calculate it's derived and note that it has a single Cauchy characteristic.

\[\text{DFHat1} := \text{CanonicalBasis}(\text{DerivedFlag}(\text{HatDelta3})[2]);\] (3.3.1)
\[\text{DFHat1} := \begin{bmatrix}
\frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial u} + \frac{A B}{u} \frac{\partial}{\partial q_1} \\
\frac{B}{u^2} A - 2 B^2 p_1 + B^2 q_1 A + A B q_2 u - 2 A q_1 q_2 u \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_2}
\end{bmatrix}\]

\[\text{CauchyCharacteristics}(\text{Annihilator}(\text{DFHat1}));\]
\[\left[ \frac{\partial}{\partial q_2} \right]\]

Define the 5-manifold given by p_2 = 0

\[\text{DGEnvironment}[\text{Manifold}][\{x, u, p_1, q_1, q_2\}, \text{FiveH}, \text{sideconditions} = \text{EQ}];\]
\[\text{Manifold: FiveH} \]

Define the projection map from Six to Five

\[\text{pi2} := \text{Transformation}(\text{SixH}, \text{FiveH}, [x = x, \ u = u, \ p_1 = p_1, \ q_1 = q_1, \ q_2 = q_2]);\]
\[\mathfrak{z}_2 := x = x, \ u = u, \ p_1 = p_1, \ q_1 = q_1, \ q_2 = q_2 \]

(3.3.4)

Note that the derived of HatDelta3 is pi2-projectable

\[\text{LieDerivative}(\text{D}_{p_2}, \text{DFHat1});\]
\[\left[ 0 \frac{\partial}{\partial x}, 0 \frac{\partial}{\partial x}, 0 \frac{\partial}{\partial x} \right]\]

(3.3.5)

\[\text{HatDelta4} := \text{CanonicalBasis}(\text{Pushforward}(\text{pi2}, \text{DFHat1}))[1..2];\]
\[\text{HatDelta4} := \begin{bmatrix}
\frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial u} + \frac{A B}{u} \frac{\partial}{\partial q_1}
\end{bmatrix}\]

(3.3.6)
Again, these are the vector fields of ChkS3

\[ \text{ChkS4} := \text{Pushforward}(\pi_2, \text{ChkS3}); \]

\[ \begin{align*}
0 \, \partial_x \, 0 \, \partial_x &\quad - \left( \frac{2 \, q_1 - B}{A \, u} \right) \partial_{q_2} \\
- 4 \left( q_1 + \frac{B}{2} \right) \left( q_2 \, u + 3 \, B \, q_1 - \frac{B^2}{2} \right) \left( \frac{p_1 - A}{2} \right) &\quad \partial_{q_2} \\
- \left( \frac{10 \, q_1 - 3 \, B}{A \, u} \right) \left( \frac{2 \, p_1 - A}{p_2} \right) &\quad \partial_{q_2} \\
\end{align*} \] (3.3.7)

\[ \begin{align*}
- \frac{B^3}{4} \, A \, p_1 + B^2 \, q_1 \, A + A \, B \, q_2 \, u - 2 \, A \, q_1 \, q_2 \, u \, \partial_{q_2} \, 0 \, p_1 &\quad - \frac{2 \, B^3}{4} \, A \, p_1 + B^2 \, q_1 \, A + A \, B \, q_2 \, u - 2 \, A \, q_1 \, q_2 \, u \, \partial_{q_2} \, 0 \, p_1 \\
\end{align*} \] (3.3.8)

Again, these vector fields are symmetries of \( \text{HatDelta4} \)

\[ \text{GetComponents}(\text{LieDerivative}(\text{ChkS4}, \text{HatDelta4}), \text{HatDelta4}); \]

\[ \begin{bmatrix}
[0, 0, \{0, 0\}, \{0, 0, 0\}] \begin{bmatrix}
0, \ \left(3 \, A \, B - 10 \, q_1 \right) u^2 \left(2 \, B^3 + 12 \, B \, p_1 \right)
0, \ \left(-2 \, p_1 + A \right) \left(2 \, B^3 + 12 \, B \, p_1 \right)
0, \ \left(-2 \, p_1 + A \right) \left(2 \, B^3 + 12 \, B \, p_1 \right)
0, \ \left(-2 \, p_1 + A \right) \left(2 \, B^3 + 12 \, B \, p_1 \right)
\end{bmatrix}
\end{bmatrix} \] (3.3.9)
We can further check that the Lie determinant vanishes for $\text{ChkS}4$

\begin{verbatim}
0 \partial_t \wedge \partial_u \wedge \partial_{p1} \wedge \partial_{q1} \wedge \partial_{q2}
\end{verbatim}

(3.3.10)

Conclusion: The following is our $\text{HatDelta}$ and $\text{ChkS}$ on the 5-manifold.

\begin{verbatim}
> HatDelta4:
\frac{\partial_x + p1 \partial_u + \frac{4 B}{u} \partial_{q1} - \frac{B^3 A - 2 B^3 p1 + B^2 q1 A + A B q2 u - 2 A q1 q2 u}{B u^2} \partial_{q2} \partial_{p1}}{\partial_{p1}}
\end{verbatim}

(3.1)

\begin{verbatim}
> ChkS4:
- B \partial_{q2} B \partial_{q1} - \frac{2 B^3}{B u} + 4 q1 B^2 + 3 u q2 B - 2 u q2 q1 \partial_u - B \partial_u \\
+ A (\frac{-2 q1 + B}{u}) \partial_{p1} + 2 \left(\frac{B^3}{B u} + \frac{2 B \partial q1 + q2 u}{u} \partial_{q1} - \frac{1}{B u^2} (3 B^5 + 9 q1 B^4 + 10 q1^2 B^3 + 6 u q2 B^3 + 8 u q2 q1 B^2 + 4 u^2 q2^2 B - 4 u^2 q2 q1 - 8 u q2 q1^2 B) \right) \partial_{q2},

- 2 B^3 + 4 q1 B^2 + u q2 B + 2 u q2 q1 \partial_{u} \\
+ A \left(\frac{B^3 - q1 B^2 - 6 B q1^2 - u q2 B - 2 u q2 q1}{B u^2} \partial_{q2} \\
+ \frac{B^3}{B^2 u} + 5 q1 B^2 + 6 B q1 q2 + 2 u q2 B + 4 u q2 q1 \partial_{q1} - \frac{1}{B^2 u^2} \left(2 \left(\frac{B^6 + 4 q1 B^4}{B u^2} + 7 q1^2 B^4 + u q2 B^4 + 6 B q1^3 + 5 u q2 q1 B^3 + 3 u q2 q1^2 B^2 + u^2 q2^2 B^2ight.ight.ight.

- 6 B q1^2 q2 u - 4 u^2 q2^2 q1^2) \partial_{q2} - \left.5 B + 2 q1 \right) \partial_{p1} + \frac{A (3 B - 10 q1)}{u} \partial_{p1} + 2 \left(\frac{5 B^2 + 10 B q1 + 4 q2 u}{u} \partial_{q1} - \frac{1}{B^2 u^2} (13 B^5 + 43 q1 B^4 + 50 q1^2 B^3 + 26 u q2 B^3 + 32 u q2 q1 B^2 - 40 u q2 q1^2 B + 16 u^2 q2^2 B - 16 u^2 q2 q1) \right) \partial_{q2}
\end{verbatim}

(3.2)

Calculate Cartan quartic.

\begin{verbatim}
> Omega := CanonicalBasis(Annihilator(HatDelta4), [du, dq1, dq2, dp1, dx]):
\Omega := \left[-p1 dx + du, -\frac{A B}{u} dx + dq1, \frac{B^3 A - 2 B^3 p1 + B^2 q1 A + A B q2 u - 2 A q1 q2 u}{B u^2} dx + dq2\right]
\end{verbatim}

(3.3)

\begin{verbatim}
> LC := LiftedCoframe(Omega, FR, sideconditions = EQ, coframelevel = 4):
> CQ := simplify(CartanQuartic(LC)):
\end{verbatim}

(3.4)

Since the Cartan quartic is identically zero, we conclude that the distribution $\text{HatDelta}4$ is locally equivalent to the Hilbert-Cartan distribution.
C.4 Reconstruction of the First Equation of Zhiber and Sokolov

The following Maple worksheet shows the calculations for the reconstruction of the first equation of Zhiber and Sokolov (9.2).

---

**Reconstruction of the First Equation of Zhiber and Sokolov**

```
> restart;
> Preferences("ShowFramePrompt", false):
> read ("C:\Users\brand\OneDrive\Dissertation\BrandonsPrograms\Prod uctM.mm");
```

Here, we realize the first equation of Zhiber and Sokolov as the quotient

\[ \Delta = \frac{\Delta_1^{(1)} \oplus \Delta_2^{(1)}}{G_{\text{diag}}}, \]

where \( \Delta_1^{(1)} \) and \( \Delta_2^{(1)} \) are the first prolongation of Hilbert-Cartan distributions using the coordinates given by Zhiber and Sokolov, namely

\[
\Delta_1^{(1)} = \left\{ \frac{\partial}{\partial x} + \frac{z}{\sqrt{z_1}} \frac{\partial}{\partial z_1} + \frac{1}{\sqrt{z_1}} \frac{\partial}{\partial z_2} + z_i \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_1} \right\}
\]

and

\[
\Delta_2^{(1)} = \left\{ \frac{\partial}{\partial y} + \frac{w}{\sqrt{w_1}} \frac{\partial}{\partial w_1} + \frac{1}{\sqrt{w_1}} \frac{\partial}{\partial w_2} + w_i \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_1} \right\},
\]

and \( G_{\text{diag}} \) denotes the diagonal action of the group \( G \) generated by the first prolongation of the vector fields [S, 14] from Table B.3 (written in these coordinates).

---

Initialize the 6-manifold.

```
> DGEnvironment[Coordinate]([x, y, u, z, z1, z2], M6);
Manifold: M6
```

Initialize the Hilbert-Cartan distribution.

```
> OmegaM6 := evalDG([dy - z/sqrt(z1)*dx, du - 1/sqrt(z1)*dx, dz - z1*dx, dz1 - z2*dx]);
```

```
\[
OmegaM6 := \left[ -\frac{z}{\sqrt{z_1}} dx + dy, -\frac{1}{\sqrt{z_1}} dx + du, -z_1 dx + dz, -z_2 dx + dz_1 \right]
\]

> DeltaM6 := Annihilator(OmegaM6);
```

```
\[
\DeltaM6 := \left[ \frac{\partial}{\partial x} + \frac{z}{\sqrt{z_1}} \frac{\partial}{\partial y} + \frac{1}{\sqrt{z_1}} \frac{\partial}{\partial u} + z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right]
\]

---

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Initialize the 5-dimensional subalgebra given by \([S, 14]\).

\[
\text{GammaM6} := \text{evalDG}\{(D_u, 2*D_y, y/2*D_u-(z^2)/2*D_z - z*z1*D_z1 -(z^2+z1^2)*D_z2, y*D_y -u*D_u+2*z*D_z +2*z1*D_z1 +2*z2*D_z2, 2*u*D_y + 2*D_z)\};
\]

\[
\text{GammaM6} := \left[ \begin{array}{c}
\frac{\partial}{\partial u}, 2 \frac{\partial}{\partial y}, \frac{\partial}{\partial u} - \frac{z^2}{z} \frac{\partial}{\partial z} - z z1 \frac{\partial}{\partial z1} - (z^2 + z1^2) \frac{\partial}{\partial z2}, y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u} + 2 z \frac{\partial}{\partial z} + 2 z1 \frac{\partial}{\partial z1} + 2 z2 \frac{\partial}{\partial z2}, 2 u \frac{\partial}{\partial y} + 2 \frac{\partial}{\partial z} \end{array} \right]
\]

(4)

Initialize the 12-dimensional product manifold.

\[
\text{ProductM}(M6, M6, M12, \text{coordinates} = [x, u1, u2, z, z1, z2, y, v1, v2, w, w1, w2], \text{projections} = [\pi1, \pi2], \text{inclusions} = [\iota1, \iota2]);
\]

Initialize \(D11, D12\).

\[
\DeltaM12 := [\text{seq}(\text{Pushforward}(\text{InverseTransformation}(\pi1), \pi1, i), i = \text{DeltaM6}), \text{seq}(\text{Pushforward}(\text{InverseTransformation}(\pi2), \pi2, i), i = \text{DeltaM6})];
\]

\[
\DeltaM12 := \left[ \begin{array}{c}
\frac{\partial}{\partial u}, 2 \frac{\partial}{\partial v}, \frac{\partial}{\partial u} + \frac{1}{\sqrt{z^2}} \frac{\partial}{\partial v} + z z1 \frac{\partial}{\partial z1} + 2 z2 \frac{\partial}{\partial z2}, \frac{\partial}{\partial y} + \frac{w}{\sqrt{w1}}, \frac{\partial}{\partial y} + \frac{1}{\sqrt{w1}}, \frac{\partial}{\partial y}
\end{array} \right]
\]

(5)

The dual Pfaffian system is

\[
\text{OmegaM12} := \text{CanonicalBasis}(\text{Annihilator}(\DeltaM12), [du1, dv1, du2, dv2, dz2, dw2, dz1, dw1, dz, dw, dx, dy]);
\]

\[
\text{OmegaM12} := \left[ \begin{array}{c}
- \frac{z}{\sqrt{z^2}} dx + du1, - \frac{w}{\sqrt{w1}} dy + dv1, - \frac{1}{\sqrt{z^2}} dx + du2, - \frac{1}{\sqrt{w1}} dy + dv2, - z z1 dx + dz1, - w2 dy + dw1, - z z2 dx + dz2, - w1 dy + dw
\end{array} \right]
\]

(6)

Initialize the diagonal action of \([S, 14]\) on M12.

\[
\text{Z1} := \text{evalDG}(\text{Pushforward}(\text{InverseTransformation}(\pi1), \pi1, \text{GammaM6}[1]) + a1*\text{Pushforward}(\text{InverseTransformation}(\pi2), \pi2, \text{GammaM6}[1]));
\]

\[
\text{Z1} := \frac{\partial}{\partial u} + a1 \frac{\partial}{\partial v2}
\]

(7)

\[
\text{Z2} := \text{evalDG}(\text{Pushforward}(\text{InverseTransformation}(\pi1), \pi1, \text{GammaM6}[2]) + a2*\text{Pushforward}(\text{InverseTransformation}(\pi2), \pi2, \text{GammaM6}[2]));
\]

\[
\text{Z2} := 2 \frac{\partial}{\partial u} + 2 a2 \frac{\partial}{\partial v1}
\]

(8)

\[
\text{Z3} := \text{evalDG}(\text{Pushforward}(\text{InverseTransformation}(\pi1), \pi1, \text{GammaM6}[3]) + a3*\text{Pushforward}(\text{InverseTransformation}(\pi2), \pi2, \text{GammaM6}[3]));
\]

\[
\text{Z3} := \frac{u1}{2} \frac{\partial}{\partial u} - \frac{z^2}{2} \frac{\partial}{\partial z} - z z1 \frac{\partial}{\partial z1} - (z z2 + z^2) \frac{\partial}{\partial z2} + \frac{a3 v1}{2} \frac{\partial}{\partial v2} - \frac{a3 w2}{2} \frac{\partial}{\partial w} - a3 w1 \frac{\partial}{\partial w1}
\]

(9)

\[
\text{Z4} := \text{evalDG}(\text{Pushforward}(\text{InverseTransformation}(\pi1), \pi1, \text{GammaM6}[4]) + a4*\text{Pushforward}(\text{InverseTransformation}(\pi2), \pi2, \text{GammaM6}[4]));
\]

\[
\text{Z4} := \frac{u1}{2} \frac{\partial}{\partial u} - u2 \frac{\partial}{\partial u} + 2 z \frac{\partial}{\partial z} + z z1 \frac{\partial}{\partial z1} + 2 z z2 \frac{\partial}{\partial z2} + a4 v1 \frac{\partial}{\partial v2} - a4 v2 \frac{\partial}{\partial v2} + 2 a4 w \frac{\partial}{\partial w}
\]

(10)

\[
\text{Z5} := \text{evalDG}(\text{Pushforward}(\text{InverseTransformation}(\pi1), \pi1, \text{GammaM6}[5]) + a5*\text{Pushforward}(\text{InverseTransformation}(\pi2), \pi2, \text{GammaM6}[5]));
\]

(11)
We now show that the above equation decomposes as in (9.7).

The higher-order differential invariants are

\[
\Gamma_{M12} = \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right] u + \frac{1}{\sqrt{z}} \frac{\partial}{\partial z} u + \frac{1}{\sqrt{z}} \frac{\partial}{\partial z} \left( z \frac{\partial u}{\partial z} - u \frac{\partial z}{\partial z} \right)
\]

The total differential operators are

\[
D_x := \Delta_{M12}[1];
\]
\[
D_y := \Delta_{M12}[3];
\]

The higher-order differential invariants are

\[
UX := \text{simplify(LieDerivative(Dx, U), symbolic)};
\]
\[
UY := \text{simplify(LieDerivative(Dy, U), symbolic)};
\]
\[
UXY := \text{simplify(LieDerivative(Dx, UY), symbolic)};
\]

We now show that the above equation decomposes as in (9.7).

\[
CEQ := \text{simplify(solve(%, {C}), symbolic)};
\]
\[ CEQ := \left\{ C = \frac{2}{9} \left( (-2 \left( z (u_2 + v_2) - u_l + v_l \right)^2 w_l^{3/2} + (w + z)^2 (u_2 + v_2) w + u_l \\
- v_l) \right) \left( (u_2 + v_2) w + u_l - v_l \right)^2 z_l^{3/2} - \frac{1}{2} (w + z)^3 (z (u_2 + v_2) - u_l + v_l) \right\} \]

\[
\frac{((u_2 + v_2) w + u_l - v_l)^{4/3} \sqrt{z_l} (w + z)^{8/3} \sqrt{w_l} ((u_2 + v_2) w + u_l - v_l)^{4/3}}{((u_2 - v_2) z + u_l - v_l)^{4/3} \sqrt{z_l} (w + z)^{8/3} \sqrt{w_l} ((u_2 + v_2) w + u_l - v_l)^{4/3}}
\]

\[ \text{C1} := \text{rhs}(\text{CEQ}[1]); \]

\[ \text{C1} := \frac{2}{9} \left( (-2 \left( z (u_2 + v_2) - u_l + v_l \right)^2 w_l^{3/2} + (w + z)^2 (u_2 + v_2) w + u_l \\
- v_l) \right) \left( (u_2 + v_2) w + u_l - v_l \right)^2 z_l^{3/2} - \frac{1}{2} (w + z)^3 (z (u_2 + v_2) - u_l + v_l) \right\} \]

\[
\frac{((u_2 - v_2) z + u_l - v_l)^{4/3} \sqrt{z_l} (w + z)^{8/3} \sqrt{w_l} ((u_2 + v_2) w + u_l - v_l)^{4/3}}{((u_2 - v_2) z + u_l - v_l)^{4/3} \sqrt{z_l} (w + z)^{8/3} \sqrt{w_l} ((u_2 + v_2) w + u_l - v_l)^{4/3}}
\]

\[ \text{u5ans} := \text{solve}(\text{UX} = \text{ux}, \{z_l\}[1], \text{symbolic}); \]

\[ \text{v5ans} := \text{simplify}(\text{solve}(\text{UY} = \text{uy}, \{w_1\}[1], \text{symbolic}); \]

\[ \text{ans} := \text{u5ans } \text{union } \text{v5ans}; \]

\[ \text{rationalize}(\text{simplify}(\text{subs} \text{(ans, C1, symbolic)}), \text{symbolic}); \]

\[ \text{1/64} \text{ux}_6^3 \text{uy}_6^3 \left( \left( 2/3 \text{uy} \left( \sqrt{4 \text{w}_y^3 + 1} - 1 \right)^{4/3} - 2 \left( \sqrt{4 \text{w}_y^3 + 1} - 1 \right)^{2/3} \text{w}_y^2 + \sqrt{4 \text{w}_y^3 + 1} \\
1/3 \text{w}_x^3 \right) - 1 \right) \left( \left( \sqrt{4 \text{w}_x^3 + 1} + 1 \right)^{2/3} \text{w}_x^2 - \frac{1}{2} \left( \sqrt{4 \text{w}_x^3 + 1} + 1 \right) \left( \sqrt{4 \text{w}_x^3 + 1} + 1 \right)^{1/3} \right) \]

\[ + 4 \text{w}_y^2 \left( \sqrt{4 \text{w}_y^3 + 1} - 1 \right)^{1/3} + 2 \text{w}_y 2^{1/3} \sqrt{4 \text{w}_y^3 + 1} - 2 \text{w}_y 2^{1/3} \left( \sqrt{4 \text{w}_x^3 + 1} + 1 \right)^{1/3} + 2 \text{w}_x 2^{1/3} \sqrt{4 \text{w}_x^3 + 1} \]

\[ + 2 \text{w}_x 2^{1/3} \right) 2^{1/3} \left( \text{uy}^3 + \frac{1}{2} \sqrt{4 \text{w}_y^3 + 1} + \frac{1}{2} \right) \]
We see that $C_2 = C$ decomposes into the product of $TM_1$ and $TM_2$.  

Furthermore, $TM_1$ and $TM_2$ have the form

\[
TM_1 := \frac{1}{64} \left( \frac{1}{w^6 y^6} \right) \frac{1}{2} \left( 2^{2/3} w^{3/2} \left( \sqrt{4 w y^3 + 1} - 1 \right)^{2/3} - 2 \left( \sqrt{4 w y^3 + 1} - 1 \right)^{2/3} y^2 w^{5/3} \right) + \sqrt{4 w y^3 + 1} - 1 \left( \frac{1}{w^2 y^2} + 1 \right)^{2/3} 2^{1/3} w^2 - \frac{1}{2} \left( \sqrt{4 w y^3 + 1} + 1 \right) \left( \sqrt{4 w y^3 + 1} + 1 \right)^{1/3} 2^{2/3} w x - 1 \left( \sqrt{4 w y^3 + 1} + 1 \right)^{1/3} 2^{2/3} w x - \frac{1}{2} \left( \sqrt{4 w y^3 + 1} + 1 \right) \left( \sqrt{4 w y^3 + 1} + 1 \right)^{1/3} 2^{2/3} w x - 1 \left( \sqrt{4 w y^3 + 1} + 1 \right)^{1/3} 2^{2/3} w x \right)
\]

\[
TM_2 := \frac{1}{64} \left( \frac{1}{w^6 y^6} \right) \frac{1}{2} \left( 2^{2/3} w^{3/2} \left( \sqrt{4 w y^3 + 1} - 1 \right)^{2/3} - 2 \left( \sqrt{4 w y^3 + 1} - 1 \right)^{2/3} y^2 w^{5/3} \right) + \sqrt{4 w y^3 + 1} - 1 \left( \frac{1}{w^2 y^2} + 1 \right)^{2/3} 2^{1/3} w^2 + \sqrt{4 w y^3 + 1} + 1 \left( \frac{1}{w^2 y^2} + 1 \right)^{2/3} 2^{1/3} w x - \frac{1}{2} \left( \sqrt{4 w y^3 + 1} + 1 \right) \left( \sqrt{4 w y^3 + 1} + 1 \right)^{1/3} 2^{2/3} w x - \frac{1}{2} \left( \sqrt{4 w y^3 + 1} + 1 \right) \left( \sqrt{4 w y^3 + 1} + 1 \right)^{1/3} 2^{2/3} w x \right)
\]
C.5  Fundamental Invariants of the Second Equation of Zhiber and Sokolov

The following Maple worksheet shows the calculations of the fundamental invariants of the second equation of Zhiber and Sokolov (9.3).

```
restart;
Preferences("ShowFramePrompt", false):

Here, we calculate the fundamental invariants for the second equation of Zhiber and Sokolov

\[u_{xy} = \frac{P_1(P_1 - 1)Q_1(Q_1 - 1)^2}{6u + y} + \frac{Q_1^2(P_1 - 1)^2}{6u + x}\]

Commuting Characteristics and Vessiot Algebras

Initialize the 9-dimensional equation manifold (in terms of invariant coordinates described in Section 9.3).

```

The UHat and UChk vector fields are given by

```
DGEnvironment[Coordinate]([x,y,u,p1,q1,p2,q2,i2,j2], M9);
```

```
[321]
```
UchkList := \begin{bmatrix} 0, 1, (1/3) q_1^3 - (1/2) q_1^2, q_1 (q_1 - 1) \end{bmatrix};

Uhat := DGzip(UhatList, Dsignormalize(M9, "FrameBaseVectors"));

\[ Uhat = \begin{pmatrix}
\frac{1}{9} u_1 - \frac{4}{9} u_2 + \frac{1}{9} u_3 + \frac{1}{9} u_4 - \frac{1}{9} u_5 - \frac{1}{9} u_6 - \frac{1}{9} u_7 - \frac{1}{9} u_8 - \frac{1}{9} u_9
\end{pmatrix};

+ \frac{1}{3} (p + 1) (12 p q_1 u_1 + p q_1 q_2 + p q_2 q_3 - p q_3 q_4 + p q_4 q_5 - p q_5 q_6 + p q_6 q_7 - p q_7 q_8 + p q_8 q_9 + p q_9 q_{10}) - \frac{1}{3} (288 p^3 q_1^4 u_1 + 32 p^3 q_1^4 u_2 - 96 p^3 q_1^4 u_3 + 144 p^3 q_1^4 u_4 - 144 p^3 q_1^4 u_5 + 72 p^3 q_1^4 u_6 - 36 p^3 q_1^4 u_7 + 18 p^3 q_1^4 u_8 - 6 p^3 q_1^4 u_9 + p^3 q_1^4 u_{10})

+ 48 p^3 q_1^4 u_2 - 48 p^3 q_1^4 u_3 + 32 p^3 q_1^4 u_4 - 32 p^3 q_1^4 u_5 - 32 p^3 q_1^4 u_6 + 32 p^3 q_1^4 u_7 - 32 p^3 q_1^4 u_8 + 32 p^3 q_1^4 u_9 - 32 p^3 q_1^4 u_{10}.

\]
Calculate HatS vector fields:

\[ S_1 := \text{LieDerivative}(\text{UHat}[1], \text{UHat}[2]) ; \]
\[ S_2 := \text{LieDerivative}(\text{UChk}[1], \text{UChk}[2]) ; \]

The invariants are

\[ \text{HatI} := \text{FirstIntegrals} (\text{UHat}) ; \]
\[ \text{HatI} := [y, z] \]

\[ \text{ChkI} := \text{FirstIntegrals} (\text{UChk}) ; \]
\[ \text{ChkI} := [x, y] \]
Initialize the Vessiot algebra

\[ \text{The ChkS vector fields form a basis for the Vessiot algebra with structure equations:} \]

\[ \text{Calculate ChkS vector fields:} \]

\[ \text{The ChkS vector fields form a basis for the Vessiot algebra with structure equations} \]

\[ \text{Initialize the Vessiot algebra} \]

\[ \text{Perform the following change of basis.} \]

\[ \text{DGEnvironment[LieAlgebra}(LD)\}; \]

\[ \text{Lie algebra: HatAlg} \]

\[ \text{MultiplicationTable(HatAlg1)}; \]
This is the abstract Lie algebra $\mathfrak{g}_{3,35}$ with $a = 1$.

### Congruences for Prolonged Monge System

We now show that these systems satisfy the Goursat congruences for the contact distribution on $J^* (R, R)$.

We begin by constructing the following frame on the 9-dimensional equation manifold:

\[ \text{HatFr} := \{ S1, S2, S3, S4, S5, \text{op(UHat)}, \text{op(UCHk)} \} \]

Calculate the structure equations for the frame.

\[ \text{LD} := \text{simplify(VessiotAlgebra(HatFr, [i2, j2], alg))} \]

Initialize the frame:

\[ \text{DGEnvironment}[\text{LieAlgebra}](\text{LD}, \text{formlabels} = ' [o1, o2, o3, o4, o5, pi1, pi2, chi1, chi2]') \]

Apply change of coframe

\[ \text{DGEnvironment}[\text{AnholonomicCoframe}](\text{evalDG([o5, o4, o3, o2, o1, pi1 + 1*o2 + j2*o3, pi2, chi1, chi2])), \text{HF1}, \text{formlabels} = ' [o1, o2, o3, o4, o5, pi1, pi2, chi1, chi2]') \]

Define DGMod command to calculate congruences:

\[ \text{DGMod} := \text{proc(theta, J)} \]
\[ \quad \text{DGsubs}([\text{seq}(J[i] = 0 \mult \text{DGinformation(DGinformation("CurrentFrame"), "FrameBaseForms")}[1], i = 1..\text{nops}(J))], \theta) \];

The congruences on the 9-manifold are:

\[ \text{strH1} := \text{DGMod(ExteriorDerivative(o1), [o1])}; \]
\[ \text{strH1} := o2 \wedge \pi \wedge \left\{ (pt^2 q1 x - pt^2 q1 y - 6 pt^2 u - pt^2 x + 12 pt q1 u + 2 pt q1 y} \right. \]
\[ \left. - 6 q1 u - q1 y \right) (6 u + y)^3 (6 u + y)^3 \left/ (2 \left( 12 pt^4 u + pt^4 x + pt^4 y - 24 pt^3 u \right. \right. \]
\[ \left. - pt^3 x - 3 pt^3 y + 18 pt^2 u + 3 pt^3 y - 36 pt^2 u^2 - 6 pt^2 u - 6 pt^2 u - 2 p1 x y \right. \]
\[ \left. - 6 p1 u - p1 y ) \right\} \chi1 \wedge \chi2 \]
\[ \text{strH2} := \text{factor(DGMod(ExteriorDerivative(o2), [o1, o2])}); \]
\[ \text{strH2} := o3 \wedge \pi + \left\{ (-12 pt^5 q1 u x + 12 pt^5 q1 u y - pt^5 q1 x^2 + pt^5 q1 y^2 } \right. \]
\[ \left. + 72 j2 pt^5 q1 u^2 x - 72 j2 pt^5 q1 u^2 y + 12 j2 pt^5 q1 u x^2 - 12 j2 pt^5 q1 u y^2 \right. \]
\[ \left. + 2 j2 pt^5 q1 x^2 y - 2 j2 pt^5 q1 x y^2 + 72 pt^5 u^2 + 18 pt^5 u x + 6 pt^5 u y + pt^5 x^2 \right. \]
\[ \left. + pt^5 x y - 72 pt^5 q1 u^2 + 18 pt^5 q1 u x - 42 pt^5 q1 u y + pt^5 q1 x^2 + pt^5 q1 x y \right. \]
\[
\begin{align*}
-4p^4 & q_1 y^2 - 432 j_2 p^2 l^2 a^3 - 144 j_2 p^2 l^2 a^2 x - 72 j_2 p^2 l^2 a^2 y - 12 j_2 p^2 l^2 u x^2 \\
-24 j_2 p^2 l^2 u x y - 2 j_2 p^2 l^2 x^2 y + 864 j_2 p^2 l^2 u y + 144 j_2 p^2 l^2 u x^2 \\
+ 288 j_2 p^2 q_1 u^2 y + 48 j_2 p^4 q_1 u x y + 24 j_2 p^2 q_1 u y^2 + 4 j_2 p^2 q_1 x y^2 - 144 p^4 l^2 u^2 \\
- 30 p^4 u x - 18 p^4 u y - p^4 l^2 x^2 - 3 p^4 l^2 y + 144 p^4 q_1 u^2 - 12 p^4 l^2 q_1 u x \\
+ 60 p^3 l^2 q_1 u y - 2 p^3 l^2 q_1 x y + 6 l^3 q_1 q_1 y^2 + 36 p_2 l^2 q_1 u^2 x - 36 p_2 l^2 q_1 u^2 y \\
+ 6 j_2 p_2 q^2 u x^2 - 6 p_2 q_2 q_1 u^2 y + p_2 q_2 q_1 x y - p_2 q_2 q_1 y^2 - 432 j_2 q_1 a^2 \\
- 72 j_2 q_1 u^2 x - 144 j_2 q_1 u^2 y - 24 j_2 q_1 u y^2 - 12 j_2 q_1 l^2 y - 2 j_2 q_1 l^2 x \\
+ 108 p^3 l^2 u^2 + 18 p^3 l^2 u x + 18 p^3 l^2 y + 3 p^3 l^2 x y - 108 p^3 l^2 q_1 u^2 + 6 p^3 l^2 q_1 u x \\
- 42 p^3 l^2 q_1 u y + p^3 l^2 q_1 x y - 4 p^3 l^2 q_1 y^2 - 216 j_2 p^2 u a^2 - 72 j_2 p^2 a^2 x \\
- 36 p_2 u a^2 y - 6 j_2 p^2 u x^2 - 12 j_2 p^2 u x y - p_2 p^2 x y - 216 j_2 p^2 u^3 \\
+ 36 p_2 a^2 x^2 + 72 p_2 a^2 y^2 + 12 p_2 q_1 u x y + 6 p_2 q_1 u y^2 + p_2 q_1 x y^2 \\
- 36 p_2 l^2 x^2 - 6 p_2 l^2 u x y - 6 p_2 l^2 y^2 - 36 p_2 q_1 u x^2 + 12 j_2 q_1 u y + p_2 q_1 l^2 y^2 \\
+ 6 j_2 q_1 l^2 x^2 + 12 p_2 q_1 l^2 x y - 6 p_2 q_1 l^2 y^2 - 6 p_2 q_1 l^2 u x y - 6 p_2 q_1 l^2 u y^2 \\
- 6 j_2 q_1 l^2 u x^2 - 6 j_2 q_1 l^2 u y - 6 j_2 q_1 l^2 x y - 6 j_2 q_1 l^2 u x y - 6 j_2 q_1 l^2 u y^2 \\
(6 + u x)(6 + u y)^2 (12 p^4 u + p^4 l^4 + p^4 q_1 y^2 - 24 p^4 l^2 u - p^3 l^4 x - 3 p^3 l^4 y \\
+ 18 p^2 l^4 u + 3 p^2 l^4 y - 36 p_2 u^2 - 6 p_2 u x - 6 p_2 u y - 6 p_2 u - p_l y^3 \chi l \\
\wedge x^2
\end{align*}
\]
\[ \text{strH4 := } \text{factor}(\text{DGMod(ExteriorDerivative(o4), [o1, o2, o3, o4])}) \]

\[ \text{(2.6)} \]

\[ \begin{align*}
+ 4 j^2 p^1 q^1 x y^3 & - 7344 p^1 q^1 x^3 - 1368 p^1 q^1 u^2 x - 2304 p^1 q^1 u^2 y - 24 p^1 q^1 u x^2 \\
- 408 p^1 q^1 u x y & - 180 p^1 q^1 u^2 y^3 - 4 p^1 q^1 x^2 y - 30 p^1 q^1 x^2 y^3 - 432 p^1 q^1 u^2 x \\
+ 432 p^1 q^1 u^2 y & - 144 p^1 q^1 u x y + 144 p^1 q^1 u y^3 - 12 p^1 q^1 x y^3 + 12 p^1 q^1 y^3 \\
+ 15552 p^1 q^1 x y^3 & + 5184 p^1 q^1 u x u x + 5184 p^1 q^1 u x u y + 432 p^1 q^1 x u y^3 \\
+ 1728 p^1 q^1 u x y & + 432 p^1 q^1 x u x - 24 p^1 q^1 x u y - 24 p^1 q^1 u x y - 1728 p^1 q^1 u x y \\
+ 144 p^1 q^1 u x y & - 288 p^1 q^1 u x y - 48 p^1 q^1 u x y - 24 p^1 q^1 u x y - 1728 p^1 q^1 u x y \\
- 24 p^1 q^1 u x y & + 4 p^1 q^1 x y + 4 p^1 q^1 x u y - 4 p^1 q^1 u x y - 1728 p^1 q^1 u x y \\
- 5184 p^1 q^1 u x y & - 1296 p^1 q^1 x u x - 1296 p^1 q^1 x u y - 1296 p^1 q^1 x u y \\
+ 432 p^1 q^1 u x y & - 144 p^1 q^1 u x y - 48 p^1 q^1 u x y - 48 p^1 q^1 u x y - 1296 p^1 q^1 u x y \\
- 432 p^1 q^1 u x y & - 12 p^1 q^1 x y - 12 p^1 q^1 x u y - 12 p^1 q^1 u x y - 432 p^1 q^1 u x y \\
+ 2592 p^1 q^1 u x y & + 432 p^1 q^1 u x y + 256 p^1 q^1 u x y + 256 p^1 q^1 u x y - 12 p^1 q^1 u x y - 2 p^1 q^1 u x y \\
+ 72 p^1 q^1 u x y & - 72 p^1 q^1 u x y - 72 p^1 q^1 u x y - 72 p^1 q^1 u x y + 2 p^1 q^1 u x y \\
+ 2 p^1 q^1 u x y & - 2 p^1 q^1 u x y - 2 p^1 q^1 u x y - 2 p^1 q^1 u x y \\
+ (6 u + x) (6 u + y) & \{ (2 (12 p^4 u + p^4 x + p^4 y - 24 p^3 u - p^3 x - 3 p^3 y \\
+ 18 p^2 x + 3 p^2 y - 36 p^2 u - 6 p^2 u - 6 p^2 u - 6 p^2 u - 6 p^2 u - 6 p^2 u) \}\} \]

\]
K2592 j23 p12 u4 y K648 j23 p12 u3 x2 K1296 j23 p12 u3 x y K216 j23 p12 u3 y2

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K36 j23 p12 u2 x3 K216 j23 p12 u2 x2 y K108 j23 p12 u2 x y2 K12 j23 p12 u x3 y
K18 j23 p12 u x2 y2 Kj23 p12 x3 y2 C15552 j23 p1 q1 u5 C5184 j23 p1 q1 u4 x
C7776 j23 p1 q1 u4 y C432 j23 p1 q1 u3 x2 C2592 j23 p1 q1 u3 x y
C1296 j23 p1 q1 u3 y2 C216 j23 p1 q1 u2 x2 y C432 j23 p1 q1 u2 x y2
C72 j23 p1 q1 u2 y3 C36 j23 p1 q1 u x2 y2 C24 j23 p1 q1 u x y3 C2 j23 p1 q1 x2 y3
K5184 j22 p14 u4 K1944 j22 p14 u3 x K1512 j22 p14 u3 y K216 j22 p14 u2 x2
K540 j22 p14 u2 x y K108 j22 p14 u2 y2 K6 j22 p14 u x3 K54 j22 p14 u x2 y
K36 j22 p14 u x y2 Kj22 p14 x3 y K3 j22 p14 x2 y2 C5184 j22 p13 q1 u4
C432 j22 p13 q1 u3 x C3024 j22 p13 q1 u3 y K72 j22 p13 q1 u2 x2 C360 j22 p13 q1 u2 x y
C576 j22 p13 q1 u2 y2 K24 j22 p13 q1 u x2 y C84 j22 p13 q1 u x y2 C36 j22 p13 q1 u y3
K2 j22 p13 q1 x2 y2 C6 j22 p13 q1 x y3 C1296 j22 p1 p2 q1 u4 x K1296 j22 p1 p2 q1 u4 y
C432 j22 p1 p2 q1 u3 x2 K432 j22 p1 p2 q1 u3 y2 C36 j22 p1 p2 q1 u2 x3
C108 j22 p1 p2 q1 u2 x2 y K108 j22 p1 p2 q1 u2 x y2 K36 j22 p1 p2 q1 u2 y3
C12 j22 p1 p2 q1 u x3 y K12 j22 p1 p2 q1 u x y3 Cj22 p1 p2 q1 x3 y2 Kj22 p1 p2 q1 x2 y3
K6048 j2 p16 u3 K1512 j2 p16 u2 x K1512 j2 p16 u2 y K90 j2 p16 u x2
K324 j2 p16 u x y K90 j2 p16 u y2 Kj2 p16 x3 K12 j2 p16 x2 y K15 j2 p16 x y2
K1008 j2 p15 q1 u2 x C1008 j2 p15 q1 u2 y K48 j2 p15 q1 u x2 K240 j2 p15 q1 u x y
C288 j2 p15 q1 u y2 K8 j2 p15 q1 x2 y K12 j2 p15 q1 x y2 C20 j2 p15 q1 y3
C5184 j2 p14 p2 u4 C2160 j2 p14 p2 u3 x C1296 j2 p14 p2 u3 y C288 j2 p14 p2 u2 x2
C504 j2 p14 p2 u2 x y C72 j2 p14 p2 u2 y2 C12 j2 p14 p2 u x3 C60 j2 p14 p2 u x2 y
C24 j2 p14 p2 u x y2 C2 j2 p14 p2 x3 y C2 j2 p14 p2 x2 y2 C1728 j2 p13 p2 q1 u3 x
K1728 j2 p13 p2 q1 u3 y C360 j2 p13 p2 q1 u2 x2 C144 j2 p13 p2 q1 u2 x y
K504 j2 p13 p2 q1 u2 y2 C12 j2 p13 p2 q1 u x3 C84 j2 p13 p2 q1 u x2 y
K60 j2 p13 p2 q1 u x y2 K36 j2 p13 p2 q1 u y3 C2 j2 p13 p2 q1 x3 y
C4 j2 p13 p2 q1 x2 y2 K6 j2 p13 p2 q1 x y3 K720 p18 u2 K96 p18 u x K144 p18 u y
K3 p18 x2 K10 p18 x y K7 p18 y2 K576 p17 q1 u2 K72 p17 q1 u x K120 p17 q1 u y
K2 p17 q1 x2 K8 p17 q1 x y K6 p17 q1 y2 K7776 j23 q1 u5 K2592 j23 q1 u4 x
K3888 j23 q1 u4 y K216 j23 q1 u3 x2 K1296 j23 q1 u3 x y K648 j23 q1 u3 y2
K108 j23 q1 u2 x2 y K216 j23 q1 u2 x y2 K36 j23 q1 u2 y3 K18 j23 q1 u x2 y2
K12 j23 q1 u x y3 Kj23 q1 x2 y3 C3888 j22 p13 u4 C1296 j22 p13 u3 x
C1296 j22 p13 u3 y C108 j22 p13 u2 x2 C432 j22 p13 u2 x y C108 j22 p13 u2 y2
C36 j22 p13 u x2 y C36 j22 p13 u x y2 C3 j22 p13 x2 y2 K3888 j22 p12 q1 u4
K432 j22 p12 q1 u3 x K2160 j22 p12 q1 u3 y C36 j22 p12 q1 u2 x2 K288 j22 p12 q1 u2 x y
K396 j22 p12 q1 u2 y2 C12 j22 p12 q1 u x2 y K60 j22 p12 q1 u x y2 K24 j22 p12 q1 u y3
Cj22 p12 q1 x2 y2 K4 j22 p12 q1 x y3 K7776 j22 p1 p2 u5 K3888 j22 p1 p2 u4 x
K2592 j22 p1 p2 u4 y K648 j22 p1 p2 u3 x2 K1296 j22 p1 p2 u3 x y K216 j22 p1 p2 u3 y2
K36 j22 p1 p2 u2 x3 K216 j22 p1 p2 u2 x2 y K108 j22 p1 p2 u2 x y2 K12 j22 p1 p2 u x3 y
K18 j22 p1 p2 u x2 y2 Kj22 p1 p2 x3 y2 C7776 j22 p2 q1 u5 C2592 j22 p2 q1 u4 x
C3888 j22 p2 q1 u4 y C216 j22 p2 q1 u3 x2 C1296 j22 p2 q1 u3 x y C648 j22 p2 q1 u3 y2
C108 j22 p2 q1 u2 x2 y C216 j22 p2 q1 u2 x y2 C36 j22 p2 q1 u2 y3 C18 j22 p2 q1 u x2 y2
C12 j22 p2 q1 u x y3 Cj22 p2 q1 x2 y3 C6048 j2 p15 u3 C1296 j2 p15 u2 x
C1728 j2 p15 u2 y C48 j2 p15 u x2 C336 j2 p15 u x y C120 j2 p15 u y2 C8 j2 p15 x2 y
C20 j2 p15 x y2 C612 j2 p14 q1 u2 x K612 j2 p14 q1 u2 y C12 j2 p14 q1 u x2
C180 j2 p14 q1 u x y K192 j2 p14 q1 u y2 C2 j2 p14 q1 x2 y C13 j2 p14 q1 x y2
K15 j2 p14 q1 y3 K10368 j2 p13 p2 u4 K3888 j2 p13 p2 u3 x K3024 j2 p13 p2 u3 y
K432 j2 p13 p2 u2 x2 K1080 j2 p13 p2 u2 x y K216 j2 p13 p2 u2 y2 K12 j2 p13 p2 u x3

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\[-108 j^2 p_1^3 p_2 u x^2 y - 72 j^2 p_1^3 p_2 u x y^2 - 2 j^2 p_1^3 p_2 x^3 y - 6 j^2 p_1^3 p_2 x^2 y^2 \]
\[-1296 j^2 p_1^3 p_2 q_1 u^3 x + 1296 j^2 p_1^3 p_2 q_1 u^4 y - 216 j^2 p_1^3 p_2 q_1 u^5 x \]
\[-216 j^2 p_1^3 p_2 q_1 u^3 y + 432 j^2 p_1^3 p_2 q_1 u^2 y^2 - 72 j^2 p_1^3 p_2 q_1 u x y^2 \]
\[+ 36 j^2 p_1^3 p_2 q_1 u x y^2 + 36 j^2 p_1^3 p_2 q_1 u y^3 - 6 j^2 p_1^3 p_2 q_1 x y^2 \]
\[+ 6 j^2 p_1^3 p_2 q_1 x y^3 + 1296 j^2 p_2^3 q_1 u^4 x - 1296 j^2 p_2^3 q_1 u^4 y + 432 j^2 p_2^3 q_1 u^3 y \]
\[-36 j^2 p_2^3 q_1 u^3 x + 12 j^2 p_2^3 q_1 u^2 x y - 12 j^2 p_2^3 q_1 u x y^2 + j^2 p_2^3 q_1 x y^2 \]
\[-j^2 p_2^3 q_1 x y^2 + 1584 j^2 p_1^3 u^2 + 156 j^2 p_1^3 u x y + 372 j^2 p_1^3 u y + 20 j^2 p_1^3 x y \]
\[+ 21 j^2 p_1^3 y^2 + 1008 j^2 p_1^3 q_1 u^2 + 84 j^2 p_1^3 q_1 u x + 252 j^2 p_1^3 q_1 u y + 64 j^2 p_1^3 q_1 u^2 \]
\[+ 12 p_1^3 q_1 s x y + 15 p_1^3 q_1 y^2 - 864 j^2 p_1^3 q_2 u^2 x - 216 j^2 p_1^3 q_2 u^2 y - 216 j^2 p_1^3 q_2 u x^2 \]
\[+ 12 j^2 p_1^3 q_2 u x y - 48 j^2 p_1^3 q_2 u y^2 - 2 j^2 p_1^3 q_2 x^2 y - 2 j^2 p_1^3 q_2 y^2 x \]
\[-864 j^2 p_1^3 q_2 u x y - 216 j^2 p_1^3 q_2 u y^2 - 216 j^2 p_1^3 q_2 x^2 y - 216 j^2 p_1^3 q_2 y^2 x \]
\[-8 j^2 p_1^3 q_2 u x y + 72 j^2 q_2 p_1^3 q_1 u x y + 72 j^2 q_2 p_1^3 q_1 u y^2 - 6 j^2 q_2 p_1^3 q_1 x y \]
\[+ 6 j^2 q_2 p_1^3 q_1 y^2 + 7776 j^2 p_2^3 q_2 u^4 + 2592 j^2 p_2^3 q_2 u^2 x + 2592 j^2 p_2^3 q_2 u^2 y \]
\[+ 216 j^2 p_2^3 q_2 u y^2 + 864 j^2 p_2^3 q_2 u x^2 + 216 j^2 p_2^3 q_2 u x y + 72 j^2 p_2^3 q_2 u^2 x y + 6 j^2 p_2^3 q_2 u^2 y x + 432 j^2 q_1 p_2 q_1 u x y - 432 j^2 p_1 q_2 u x y \]
\[+ 72 j^2 p_1 q_2 u x^2 + 12 j^2 p_1 q_2 u y^2 + 12 j^2 p_1 q_2 u x^2 y + 12 j^2 p_1 q_2 u x y^2 \]
\[+ 24 j^2 p_1 q_2 u y^2 x + 12 j^2 q_2 p_1 q_1 u x y - 12 j^2 q_2 p_1 q_1 u y^2 - 12 j^2 q_1 p_2 q_1 u x y \]
\[+ 2 j^2 p_1 q_2 u x^2 y - 2 j^2 p_1 q_2 u x y^2 - 3888 j^2 p_2 q_2 u^2 x \]
\[+ 2592 j^2 p_2 q_2 u^2 y - 648 j^2 p_2 q_2 u x^2 y - 1296 j^2 p_2 q_2 u x y^2 - 36 j^2 p_2 q_2 u x^2 y \]
\[+ 18 j^2 p_2 q_2 u x y^2 - 12 j^2 p_2 q_2 u^2 y^2 - 12 j^2 p_2 q_2 u^2 x y \]
\[+ 20 j^2 q_2 p_2 q_2 u x y - 20 j^2 q_2 p_2 q_2 x y + 2592 j^2 p_2 q_2 u x y + 576 j^2 p_2 q_2 u x y - 720 j^2 p_2 u x y \]
\[+ 24 j^2 p_2 q_2 u x^2 + 144 j^2 q_2 p_2 q_2 u x y + 4 j^2 p_2 q_2 u y^2 + 4 j^2 p_2 q_2 u x^2 y + 4 j^2 p_2 q_2 u x y^2 \]
\[+ 1728 j^2 p_3 q_1 q_1 u^3 + 360 j^2 p_3 q_2 q_1 u^2 x + 504 j^2 p_3 q_2 q_1 u^2 y + 12 j^2 p_3 q_2 q_1 u x \]
\[+ 96 j^2 p_3 q_2 q_1 u x y + 36 j^2 q_2 p_3 q_1 u x y + 2 j^2 q_3 p_2 q_1 u x y + 6 j^2 p_3 q_1 q_1 u x y \]
\[+ 1296 j^2 p_3 q_1 u^3 + 216 j^2 p_3 q_1 u^2 x + 432 j^2 p_3 q_1 u^2 y + 72 j^2 p_3 q_1 u x y + 36 j^2 p_3 q_1 u x y \]
\[+ 6 j^2 p_3 q_1 q_1 u x y + 36 j^2 p_3 q_1 q_1 u x y + 12 j^2 p_3 q_1 q_1 u x y \]
\[+ 12 j^2 q_2 u x^2 + 12 j^2 q_2 u x y^2 + 12 j^2 q_2 u x y^2 + 12 j^2 q_2 u x y^2 \]
\[+ 24 j^2 p_2 q_1 q_2 u x y - 12 j^2 p_2 q_1 q_2 u x y - 12 j^2 p_1 q_2 q_1 u x y \]
\[+ 2 j^2 p_1 q_2 q_1 u x y - 2 j^2 p_1 q_2 q_1 x y^2 - 3888 j^2 p_2 q_2 u^2 x \]
\[-2592 j^2 p_2 q_2 u^2 y - 648 j^2 p_2 q_2 u x^2 y - 1296 j^2 p_2 q_2 u x y^2 - 36 j^2 p_2 q_2 u x^2 y \]
\[+ 18 j^2 p_2 q_2 u x y^2 - 12 j^2 p_2 q_2 u^2 y^2 - 12 j^2 p_2 q_2 u^2 x y \]
\[+ 20 j^2 q_2 p_2 q_2 u x y - 20 j^2 q_2 p_2 q_2 x y + 2592 j^2 p_2 q_2 u x y + 576 j^2 p_2 q_2 u x y - 720 j^2 p_2 u x y \]
\[+ 24 j^2 p_2 q_2 u x^2 + 144 j^2 q_2 p_2 q_2 u x y + 4 j^2 p_2 q_2 u y^2 + 4 j^2 p_2 q_2 u x^2 y + 4 j^2 p_2 q_2 u x y^2 \]
\[+ 1728 j^2 p_3 q_1 q_1 u^3 + 360 j^2 p_3 q_2 q_1 u^2 x + 504 j^2 p_3 q_2 q_1 u^2 y + 12 j^2 p_3 q_2 q_1 u x \]
\[+ 96 j^2 p_3 q_2 q_1 u x y + 36 j^2 q_2 p_3 q_1 u x y + 2 j^2 q_3 p_2 q_1 u x y + 6 j^2 p_3 q_1 q_1 u x y \]
\[+ 1296 j^2 p_3 q_1 u^3 + 216 j^2 p_3 q_1 u^2 x + 432 j^2 p_3 q_1 u^2 y + 72 j^2 p_3 q_1 u x y + 36 j^2 p_3 q_1 u x y \]
\[+ 6 j^2 p_3 q_1 q_1 u x y + 36 j^2 p_3 q_1 q_1 u x y + 12 j^2 p_3 q_1 q_1 u x y \]
\[+ 12 j^2 q_2 u x^2 + 12 j^2 q_2 u x y^2 + 12 j^2 q_2 u x y^2 + 12 j^2 q_2 u x y^2 \]
Upon restricting to the 7-dimensional integral manifold given by

\[ +36 p_2^2 q_1 u^2 x^2 + 144 p_2^2 q_1 u x y + 36 p_2^2 q_1 u^2 y^2 + 12 p_2^2 q_1 u x y^2 + 12 p_2^2 q_1 u^2 x^2 - 6 j_2 p_1^2 u^3 - 36 j_2 p_1^2 u^2 x - 72 j_2 p_1^2 u^2 y - 12 j_2 p_1^2 u x y - 6 j_2 p_1^2 u x y^2 - 12 p_1^4 u^2 y - 6 p_1^4 u x y - 6 p_1^4 u x y^2 + 2 p_1^4 u^2 y^2 - 21 p_1^4 u x y^2 - 72 p_1^3 q_1 u x y - 6 p_1^3 q_1 u x y^2 + 1728 p_1^2 p_2 u x^3 - 288 p_1^2 p_2 u x^2 y + 576 p_1^2 p_2 u x y^2 + 96 p_1^2 p_2 u x y^2 + 48 p_1^2 p_2 u y^2 + 8 p_1^2 p_2 x y^2 + 432 p_1 p_2 q_1 u^7 + 72 p_1 p_2 q_1 u^2 x + 144 p_1 p_2 q_1 u^2 y + 24 p_1 p_2 q_1 u x y + 12 p_1 p_2 q_1 u^2 + 2 p_1 p_2 q_1 x y - 1296 p_2^2 u^3 - 432 p_2^2 u^2 x - 36 p_2^2 u^2 y^2 - 12 p_2^2 u^2 x y - 12 p_2^2 u x^2 y^2 + 252 p_1^2 u^2 + 84 p_1^2 u^2 y + 7 p_1^2 y^2 + 36 p_1^2 q_1 u^2 + 12 p_1^2 q_1 u y + p_1^2 q_1 y^2 - 432 p_1 p_2 u^3 - 72 p_1 p_2 u^2 x - 144 p_1 p_2 u^2 y - 24 p_1 p_2 u x y - 12 p_1 p_2 u y^2 - 2 p_1 p_2 x y^2 - 36 p_1^2 u^2 y - 12 p_1^2 u x y - 12 p_1^2 u x y^2 + 12 p_1^2 u x + p_1^2 x + p_1^2 y - 12 p^3 x - 6 p_1^2 u^3 + 3 p_1^2 y - 36 p_2 u^3 - 6 p_2 u x y - 6 p_2 y - 2 p_2 x y - 6 p_1 u - p_1 y \] \times \chi_1 \wedge \chi_2

\[ \text{strH5} := \text{factor} (\text{DGMod} (\text{ExteriorDerivative} (\text{o5}), \{\text{o1}, \text{o2}, \text{o3}, \text{o4}, \text{o5}\})) ; \]

\[ \text{strH5} := - \pi l \wedge \pi 2 + \chi 2 \wedge \chi 2 \]

(2.7)

Note that \( \chi_1, \chi_2 \) are equal to the exterior derivative of the \( \hat{\chi} \) invariants.

Upon restricting to the 7-dimensional integral manifold given by \( \chi_1^2 = \chi_2^2 = 0 \), the congruences equations become

\[ \text{DGSubs} ([\text{chil} = 0 \&\& \text{chil}], \text{strH1}) ; \]

\[ o2 \wedge \pi l \]

(2.8)

\[ \text{DGSubs} ([\text{chil} = 0 \&\& \text{chil}], \text{strH2}) ; \]

\[ o3 \wedge \pi l \]

(2.9)

\[ \text{DGSubs} ([\text{chil} = 0 \&\& \text{chil}], \text{strH3}) ; \]

\[ o4 \wedge \pi l \]

(2.10)

\[ \text{DGSubs} ([\text{chil} = 0 \&\& \text{chil}], \text{strH4}) ; \]

\[ o5 \wedge \pi l \]

(2.11)

\[ \text{DGSubs} ([\text{chil} = 0 \&\& \text{chil}], \text{strH5}) ; \]

\[ - \pi l \wedge \pi 2 \]

(2.12)

These are precisely the Goursat congruences for the standard contact system on \( J^3 (R, R) \).

\[ \text{unprotect('pi1'), 'pi2'}, 'chil', 'chi2'); \]

\[ \text{unassign('pi1'), 'pi2'}, 'chil', 'chi2'); \]

\[ \textcolor{red}{\text{Hat Side}} \]

We break this section into three parts.

- In the first section, we calculate the restriction of HatDelta and HatS to the 7-dimensional integral manifold given by Hatl = 0, that is \( 11 = 12 = 0 \).
- In the second section, we write HatDelta and HatS on a 6-dimensional manifold.
- In the third section, we write HatDelta and HatS on a 5-dimensional manifold.

\[ \textcolor{red}{\text{Restriction to 7-manifold}} \]

\[ \text{M9 > HatDelta} := \text{DGsimplify} ([\text{UHat}[1], \text{UHat}[2]]) ; \]

The ChkS vector fields are symmetries of HatDelta.
We now write HatDelta and Chk S in terms of the invariant coordinates HatI. Begin by initializing the manifold.

\[ \text{M9} \to \text{DGEnvironment[Coordinate]}([x,u,p1,p2,j2,q1,q2,I1,I2], \text{NineH}); \]

**Manifold: NineH**

Initialize the map \( \chi: M9 \to \text{NineH} \) and its inverse.

\[ \chi := \text{Transformation}(\text{M9}, \text{NineH}, [x = x, u = u, p1 = p1, p2 = p2, j2 = j2, q1 = q1, q2 = q2, I1 = \text{HatI}[1], I2 = \text{HatI}[2]]); \]

\[ \text{invchi := InverseTransformation}(\chi); \]

Pushforward HatDelta and Chk S

\[ \text{Nine} \to \text{HatDelta1 := simplify(Pushforward(\chi, invchi, HatDelta));} \]
\[ + \frac{11 p_2^2}{108} x \left( H + \frac{x^2 p_2^2}{27} \right) p_1^2 + 216 \left( (j_2 + 7 p_2) u + \frac{1}{6} + \frac{j_2^2}{6} \right) \]

\[ + \frac{7 p_2^2}{6} x \left( u + \frac{H}{6} \right)^2 p_1 + 1296 \left( u + \frac{x}{6} \right) \left( u + \frac{H}{6} \right)^2 p_2 \left( u j_2 \right) + \frac{1}{6} \left( j_2 x - \frac{1}{6} \right) \frac{\partial}{\partial p_2} \]

\[ + \frac{1}{(6 u + H) (6 u + x)^2} \left( 432 (p_1 - 1) \left( p_1 - \frac{1}{2} \right) q_2 p_1 u^3 + 108 (p_1 - 1) \right) q_2 p_1 x + 108 (p_1 - 1) q_2 p_1 \left( p_1 - \frac{2}{3} \right) I_{1} + 288 (p_1 - 1) \left( p_1^3 - 720 p_1^2 - 288 p_1 + 36 \right) q_1^3 + 432 \left( p_1 - 1 \right) \left( p_1^2 - \frac{9}{8} p_1 + \frac{3}{4} \right) q_1^2 + \left( - \frac{1}{2} p_1^2 + \frac{1}{4} p_1^3 \right) q_1 + \frac{p_1^2}{4} \]

\[ -(p_1 - 1) \left( p_1^2 - \frac{9}{8} p_1 + \frac{3}{4} \right) q_1^2 + \left( - \frac{1}{2} p_1^2 + \frac{1}{4} p_1^3 \right) q_1 + \frac{p_1^2}{4} \]

\[ - \frac{p_1^2}{4} \right) p_1 x + 6 (p_1 - 1) \left( p_1 q_2 (p_1 - 1) I_{1} + 8 q_1^2 \left( p_1^2 - \frac{7}{8} p_1 \right) q_1 + \frac{3 p_1^2}{4} - \frac{3 p_1}{8} \right) I_{1} u \]

\[ + (q_2 (p_1 - 1) I_{1} + (q_1 - 1)^2 (p_1 q_1 - p_1 + 1) p_1^2 x + (p_1 - 1) I_{1} \left( q_2 (p_1 - 1) I_{1} + 6 \left( p_1 - \frac{1}{2} \right) q_1^2 (q_1 - 1)^2 \right) p_1 x + (1 + q_1 (p_1 - 1)) (p_1 - 1)^2 q_1^2 x^2 + (p_1 - 1) \left( q_2 (p_1 - 1) I_{1} + 6 \left( p_1 - \frac{1}{2} \right) q_1^2 (q_1 - 1)^2 \right) p_1 x + (1 + q_1 (p_1 - 1)) (p_1 - 1)^2 \right) \frac{\partial}{\partial q_1} \frac{\partial}{\partial j_2} \]

\[ \text{Now we want to calculate the restriction of } \hat{\Delta}_1 \text{ and } S_1 \text{ to the integral manifold given by } I_1 = I_2 = 0. \text{ Begin by defining the restricted 7-manifold.} \]

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\[ \text{Now we want to calculate the restriction of } \hat{\Delta}_1 \text{ and } S_1 \text{ to the integral manifold given by } I_1 = I_2 = 0. \text{ Begin by defining the restricted 7-manifold.} \]
Calculate the restriction of $\text{HatDelta}_1$ to Seven. Note that this means $t_\lambda \lambda_p = \eta_{qj} p_j$.

$$\text{HatDelta}_2 := \partial + \left( \frac{1}{3} p l^4 - \frac{1}{2} p l^3 \right) \partial_u + p_2 \partial_{p_j} \quad (2.1.1.8)$$

Seven > $\text{HatDelta}_2 := \text{PullbackVector}(\text{iota}, \text{HatDelta}_1)$;

$$\text{HatDelta}_2 := \partial + \left( \frac{1}{3} p l^4 - \frac{1}{2} p l^3 \right) \partial_u + p_2 \partial_{p_j} \quad (2.1.1.8)$$

Check the definition.

Seven > simplify(evalDG(Pushforward(\text{iota}, \text{HatDelta}_2[1]) - DGsimplify(subs(11 = 0, HatDelta[1])))));

$$0 \frac{\partial}{\partial x} \quad (2.1.1.9)$$

Nine > evalDG(Pushforward(\text{iota}, \text{HatDelta}_2[2]) - \text{HatDelta}[2]));

$$0 \quad (2.1.1.10)$$

Calculate the restriction of $\text{ChkS1}$ to Seven, and again, check the definition. Before we do this, we note that $\text{ChkS1}[5]$ has a common factor of $I_2$ that can be divided out.
Projection to 6-manifold

In order to write HatDelta2 on the 6-manifold, we calculate it's derived and note that it has a single Cauchy characteristic.

\[
\text{Define the projection map from Seven to Six}
\]

Note that the derived of HatDelta2 is pi1-projectable
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(2.1.2.6)

\[
\text{Seven} \rightarrow \text{HatDelta3} := \text{CanonicalBasis} (\text{DGsimplify}
\text{(Pushforward(p1, DHat))}) [1..2];
\]

\[
\text{HatDelta3} := \begin{bmatrix}
\frac{1}{3} p l^3 - \frac{1}{2} p l^2 \partial_u + p l^2 \partial_{p l}
\end{bmatrix}
\]

(2.1.2.7)

Again check that \(\text{ChkS3}\) are symmetries of \(\text{HatDelta3}\)

\[
\text{Six} \rightarrow \text{LieDerivative}(D_{i2}, \text{ChkS2});
\]

(2.1.3.1)

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(2.1.2.8)

\[
\text{Six} \rightarrow \text{ChkS3} := \text{Pushforward(p1, ChkS2)};
\]

(2.1.2.9)

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

In order to write \(\text{HatDelta3}\) on a 5-manifold, we again calculate it's derived and note that it has a single Cauchy characteristic.

\[
\text{Six} \rightarrow \text{DFHat1} := \text{CanonicalBasis} (\text{DerivedFlag} (\text{HatDelta3}) [2]);
\]

(2.1.3.2)
\[ +6 p_1^3 q_1^2 x^2 + 432 p_1^3 q_2^3 + 108 p_1^3 q_2 u^2 x + 6 p_1^3 q_2 u x^2 \\
+ 720 p_1^2 q_1^3 u^2 + 120 p_1^2 q_1^3 u x + p_1^2 q_1^3 x^2 + 216 p_1 q_1^3 u^2 \\
+ 18 p_1 q_1^3 u x - 144 p_1 q_1 u^2 - 48 p_1 q_1 u x - 4 p_1 q_1^3 u^3 - 432 p_1 q_1^2 u x^2 \\
- 90 p_1^2 q_1 u x - 3 p_1^2 q_1^3 x^2 - 648 p_1^2 q_2 u^3 - 144 p_1^2 q_2 u^2 x \\
- 6 p_1^2 q_2 x^2 - 288 p_1 q_1^3 u^2 - 36 p_1 q_1^3 u x - 36 q_1^3 u^2 + 36 p_1^3 u^2 \\
+ 12 p_1^3 u x + 12 p_1^3 x^2 + 108 p_1^2 q_1 u^2 + 36 p_1^2 q_1 u x + 3 p_1^2 q_1 x^2 \\
+ 108 p_1 q_1^3 u^2 + 18 p_1 q_1^3 u x + 216 p_1 q_2 u^3 + 36 p_1 q_2 u^2 x + 36 q_1^3 u^2 \\
- 36 p_1^2 u^2 - 12 p_1^2 u x - p_1^2 x^2 \right) \frac{\partial}{\partial q_1} \frac{\partial}{\partial p_1} \]

Six > CauchyCharacteristics(Annihilator(DFHat1))

\[
\begin{bmatrix}
0 \\
p_2
\end{bmatrix}
\] (2.1.3.2)

Define the 5-manifold given by \( p_2 = 0 \)

Six > DGEnvironment[Coordinate]({x, u, p_1, q_1, q_2}, FiveH);

\[ \text{Manifold: FiveH} \] (2.1.3.3)

Define the projection map from Six to Five

Five > pi2 := Transformation(SixH, FiveH, {x = u, p_1 = p_1, q_1 = q_1, q_2 = q_2});

\[ \pi_2 := x = u, p_1 = p_1, q_1 = q_1, q_2 = q_2 \] (2.1.3.4)

Note that the derived of HatDelta3 is \( \pi_2 \)-projectable

Six > LieDerivative(D_p2, D_Hat1);

\[ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \] (2.1.3.5)

Six > HatDelta4 := CanonicalBasis(Pushforward(pi2, D_Hat1))[1..2];

\[ \text{HatDelta4} := \begin{bmatrix}
\frac{\partial}{\partial q_2} + \frac{1}{3} p_1^3 - \frac{1}{2} p_1^2 \\
\frac{1}{6} u \left( \frac{1}{6} u + x \right)
\end{bmatrix} \frac{1}{12 p_1^2 q_1 u} \]

(2.1.3.6)

As are the vector fields of ChkS3
Five > simplify(LieDerivative(D_p2, ChkS3));

\[
0 \frac{\partial_x^2 \partial_z - \frac{1}{u^2 (6u + x)^2}}{36} \left( q^2 u^2 + \frac{q^2 x}{6} \right) \left( q^2 - q + \frac{1}{2} \right) (q^2 - 1) q_1 (q_1 - 1) \left( \frac{\partial_{p^2}}{u^3 (6u + x)^3} \right) \left( q^2 - q + \frac{1}{2} \right) (q^2 - 1) q_1 (q_1 - 1) \left( \frac{\partial_{p^2}}{u^3 (6u + x)^3} \right) \left( q^2 - q + \frac{1}{2} \right) (q^2 - 1) q_1 (q_1 - 1) \left( \frac{\partial_{p^2}}{u^3 (6u + x)^3} \right)
\]

(2.1.3.7)

Six > ChkS4 := simplify(Pushforward(pi2, ChkS3));

Again, these vector fields are symmetries of HatDelta4

Five > simplify(expand(GetComponents(LieDerivative(ChkS4, HatDelta4), HatDelta4)));

We can further check that the Lie determinant vanishes for ChkS4


HatDelta4 has the appropriate growth vectors.

Five > map(nops, DerivedFlag(HatDelta4));

map(nops, DerivedFlag(HatDelta4, flagtype = "WeakDerivedFlag"));
C.6 Reconstruction of the Second Equation of Zhiber and Sokolov

The following Maple worksheet shows the calculations for the reconstruction of the second equation of Zhiber and Sokolov (9.3).

Reconstruction of the

Second Equation of Zhiber and Sokolov

```
> restart;
> Preferences("ShowFramePrompt", false):

Here, we realize the second equation of Zhiber and Sokolov as the quotient

\[ D = \frac{\Delta_1^{(1)} \oplus \Delta_2^{(1)}}{G_{\text{diag}}}, \]

where \( \Delta_1^{(1)} \) and \( \Delta_2^{(1)} \) are the standard contact distributions on \( J^4_{(R, R)} \) and \( G_{\text{diag}} \) denotes the diagonal action of the group \( G \) generated by the first prolongation of the vector fields \( p_{5,3} \) with \( \alpha = 3 \) from Table A.1. As stated in Section 9.3, we give an alternative representation of second equation of Zhiber and Sokolov.

Initialize \( J^4_{(R, R)} \) and the direct sum of canonical contact systems.

```
> DGEnvironment[Coordinate]([x,u,u1,u2,u3,u4,y,v,v1,v2,v3,v4], M12);
```

```
> OmegaM12 := evalDG([du - u1*dx, du1 - u2*dx, du2 - u3*dx, du3 - u4*dx, dv - v1*dy, dv1 - v2*dy, dv2 - v3*dy, dv3 - v4*dy]);
```

```
> DeltaM12 := Annihilator(OmegaM12);
```

```
> GammaM12 := evalDG([D_x + D_y, x*D_x + 3*u*D_u + 2*u1*D_u1 + u2*D_u2 - u4*D_u4 + y*D_y + 3*v*D_v + 2*v1*D_v1 + v2*D_v2 - v4*D_v4, D_u + D_v, x*D_u + D_u1 + y*D_v + D_v1, x^2*D_u + 2*x*D_u1 + 2*D_u2 + y^2*D_v + 2*D_v2]);
```

The diagonal action of the 5-dimensional symmetry algebra given by Olver 1.7 with \( \alpha = 3 \) on \( J^4 \times J^4 \) is

```
> GammaM12 := evalDG([\partial_x - v_1 \partial_v, x*\partial_x + 3*u \partial_u + 2*u1 \partial_u1 + u2 \partial_u2 - u4 \partial_u4 + y*\partial_y + 3*v \partial_v + 2*v1 \partial_v1 - v_2 \partial_v2 + v_3 \partial_v3 + v_4 \partial_v4,])
```

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Calculate invariants of this action. These invariants serve as coordinates on the quotient manifold.

\[
\text{invquot} := \text{InverseTransformation}(\text{quot});
\]

\[
\text{quot} := z_1 = \frac{u_1}{(x-y)^2} + \frac{v_1}{(x-y)^2} - \frac{2(u-v)}{(x-y)^3}, \quad z_2 = -\frac{u_1}{(x-y)^2} + \frac{u_2}{x-y} + \frac{v_1}{(x-y)^2}, \quad z_3 = \frac{z_4}{(x-y)^2 + u_2/(x-y) + v_1/(x-y)^2}, \quad z_4 = v_1/(x-y)^2 + v_2/(x-y) - u_1/(x-y)^2, \quad z_5 = z_6 = (x-y) u_4, \quad z_7 = (x-y) v_4.
\]

\[
\text{simplex}(\text{LieDerivative}(\text{GammaM12}, \text{map}(\text{rhs}, \text{Inv})));
\]

\[
\text{DGEnvironment}[\text{Coordinate}][\{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}, \text{M7}]; \quad \text{Manifold: M7}
\]

Initialize the quotient manifold M7.

\[
\text{quot} := \text{Transformation}(\text{M12}, \text{M7}, \text{Inv});
\]

\[
\text{invquot} := \text{InverseTransformation}(\text{quot});
\]

\[
\text{quot} := z_1 = -\frac{2 C_1^2 C_2 C_1}{2 C_1^3} - C_1^2 C_3 z_6 - C_1 C_4 z_6^2 + z_1 z_6^3 + z_3 z_6^3, \quad u_1 = -\frac{2 C_1^2 C_2}{2 C_1^3} - C_1^2 C_3 z_6 + z_3 z_6^3, \quad u_2 = \frac{C_1 C_4 + z_2 z_6 - z_3 z_6}{-C_1}, \quad u_3 = z_4, \quad u_4 = z_5 z_6
\]

Calculate the pushforward of DeltaM12

\[
\text{X1} := \text{evalDG}(\text{simplify}(\text{Pushforward}(\text{quot}, \text{invquot}, \text{DeltaM12}))[1]*z_6/_{C_1});
\]

\[
\text{X2} := \text{evalDG}(\text{simplify}(\text{Pushforward}(\text{quot}, \text{invquot}, \text{DeltaM12}))[2]*_{C_1}/z_6);
\]

\[
\text{Y1} := \text{evalDG}(\text{simplify}(\text{Pushforward}(\text{quot}, \text{invquot}, \text{DeltaM12}))[3]*z_6/_{C_1});
\]

\[
\text{Y2} := \text{evalDG}(\text{simplify}(\text{Pushforward}(\text{quot}, \text{invquot}, \text{DeltaM12}))[4]*_{C_1}/z_6);
\]

\[
\text{HatDelta} := \{\text{X1}, \text{X2}\};
\]

\[
\text{HatDelta} = \{-3 z_1 - z_2\} \delta_{z_1} + (2 z_2 - z_3) \delta_{z_2} - (z_2 + z_3) \delta_{z_3} + z_6 \delta_{z_4} + z_6 \delta_{z_5} + z_7 \delta_{z_6} - z_7 \delta_{z_7}
\]

\[
\text{ChkDelta} := \{\text{Y1}, \text{Y2}\};
\]

\[
\text{ChkDelta} = \{3 z_1 + z_3\} \delta_{z_1} + (z_2 + z_3) \delta_{z_2} + (2 z_3 + z_5) \delta_{z_3} + z_7 \delta_{z_4} - z_6 \delta_{z_5} - z_7 \delta_{z_6} - z_7 \delta_{z_7}
\]

This is the rank 4 distribution giving the alternative representation of the second equation of Zhiber and Sokolov.
\[
\Delta := \left[ \text{op}(\text{HatDelta}), \text{op}(\text{ChkDelta}) \right];
\]

\[
\begin{align*}
\Delta & := \left[ -(3z_1 - z_2) \frac{\partial}{\partial z_1} - (2z_2 - z_4) \frac{\partial}{\partial z_2} - (z_2 + z_3) \frac{\partial}{\partial z_3} + z_6 \frac{\partial}{\partial z_4} + z_6 \frac{\partial}{\partial z_6} + z_7 \frac{\partial}{\partial z_7} \frac{\partial}{\partial z_7} \right] \\
& \quad \quad \quad + z_3 \frac{\partial}{\partial z_3} + (2z_3 + z_5) \frac{\partial}{\partial z_5} + z_7 \frac{\partial}{\partial z_7} - z_6 \frac{\partial}{\partial z_6} - z_7 \frac{\partial}{\partial z_7} \frac{\partial}{\partial z_7}
\end{align*}
\]  

We can verify that this distribution does indeed define a hyperbolic PDE in the plane.

\[
> \text{GetComponents}\left( \text{LieDerivative}(\text{HatDelta}, \text{ChkDelta}), \left[ \text{op}(\text{HatDelta}), \text{op}(\text{ChkDelta}) \right] \right);
\]

\[
\begin{align*}
& \left[ \left[ 1, 0, 1, 0 \right], \left[ 0, 0, 0, -1 \right], \left[ 0, -1, 0, 0 \right], \left[ 0, 0, 0, 0 \right] \right] \\
& \left[ 4, 6, 7 \right]
\end{align*}
\]  

\[
> \text{map}\left( \text{nops}, \text{DerivedFlag}(\Delta) \right);
\]

\[
\begin{align*}
& \left[ \left[ 0, 0, 0 \right], \left[ 0, 0, 0 \right] \right] \\
& \left[ 4, 6, 7 \right]
\end{align*}
\]  

\[
> \text{CauchyCharacteristics}\left( \text{Annihilator}\left( \text{DerivedFlag}(\Delta)[2] \right) \right);
\]

\[
\begin{align*}
& \left[ \frac{\partial}{\partial z_3} \frac{\partial}{\partial z_3} \right] \\
& \left[ 0, 0 \right]
\end{align*}
\]
Assistant Professor of Mathematics
Department of Mathematics
Southern Oregon University
Ashland, OR 97520; USA

Email: ashleyb@sou.edu
Website: brandonpashley.com
Citizenship: United States

Education

**UtaH State University**, PhD in Mathematical Sciences. Aug. 2016 – Present
Advisor: Professor Ian M. Anderson

Thesis: *Asymptotic Tracking and Disturbance Rejection of the Blood Glucose Regulation System.* Advisor: Professor Weijiu Liu


Teaching Experience

Assistant Professor: In my first term as an assistant professor at Southern Oregon University, I have taught additional sections of College Algebra and Calculus. Sept. 2021 – Present

Graduate School: I have taught over twenty courses during my academic career as a graduate student at Utah State University and the University of Central Arkansas. In particular, I have been the lead instructor for the following classes:

<table>
<thead>
<tr>
<th>Course</th>
<th>Description</th>
<th>Term(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Differential Equations</td>
<td>Standard course on techniques for solving ordinary differential equations. Encouraged student projects and presentations.</td>
<td>Summer 2019 – Fall 2019</td>
</tr>
<tr>
<td>Calculus (all levels)</td>
<td>Taught all levels of calculus multiple times. Responsibilities included writing lesson plans, giving lectures, supervising graders, creating exams, and creating supplementary videos.</td>
<td>Fall 2017 – Present</td>
</tr>
<tr>
<td>Trigonometry</td>
<td>Course serves as an introduction to trigonometric functions, polar coordinates, and complex numbers.</td>
<td>Summer 2016</td>
</tr>
<tr>
<td>College Algebra</td>
<td>Course covers standard precalculus material. Taught this course several times at the University of Central Arkansas.</td>
<td>Fall 2014 – Spring 2016</td>
</tr>
</tbody>
</table>

† The following courses have been taught via online broadcast:

Multivariable Calculus – Spring 2020, Summer 2020
Calculus II – Fall 2020

Research Interests

Geometry of Differential Equations, Lie Theory, Mathematical Physics
Research Publications


Papers in Preparation


Recent Conferences and Presentations

1. AMS/MAA Joint Mathematics Meeting, Denver, Colorado (2020)
   Invited Talk: Transformation Groups and The Method of Darboux.

2. Together We Teach Conference, Logan, Utah (2020)
   Invited Talk: Promoting Progressive Problems in the Classroom.

3. Together We Teach Conference, Logan, Utah (2019)
   Invited Talk: Design and Implementation of Proficiency-Based Homeworks.

   Poster: Extensions of AFV Theory of Darboux Integrable Equations.

5. DGCAMP Student Daze Conference, Logan, Utah (2018)
   Talk: Finding Zero Curvature Representations of Hyperbolic PDEs.

6. MAA Intermountain Section Conference (2018)

7. Together We Teach Conference, Logan, Utah (2018)
   Talk: A Consideration of Pre-Lecture Calculus Video Podcasts with Kaitlin Murphy.

8. DGCAMP Student Workshop, Logan, Utah (2017)
   Talk: Making New Differential Equations from Old.

Other Conferences and Presentations

1. UCA Student Research Symposium, Conway, Arkansas (2016)
   Poster: Nonclassical Symmetries of a Nonlinear Diffusion-Convection/Wave Equation and Equivalent Systems
   Poster: Asymptotic Tracking and Disturbance Rejection of the Blood Glucose Regulation System.

2. 78th Annual Meeting of the OK-AR Section of the MAA, Conway Arkansas (2016)
   Talk: Asymptotic Tracking and Disturbance Rejection of the Blood Glucose Regulation System.
   Talk: Asymptotic Tracking and Disturbance Rejection of the Blood Glucose Regulation System.

**Undergraduate Research Activities**
1. Compatibility & Bäcklund Transformations of the Loiuville Equation  
   June – Aug. 2014
   Analyzed mathematical structure and solutions of the Liouville equation. Connected solutions obtained from Bäcklund transformations to those obtained using compatibility methods. Solved systems of several nonlinear equations using Maple. Supervised by Professor Daniel Arrigo.

2. Hopf-Cole Type Transformations of a Viscous Burgers’ Equation  
   June 2012 – May 2014
   Classified the existence of transformations linking solutions of a viscous Burgers’ equation to various target PDEs using Maple. Collaborated with group members and presented poster at the MAA/AMS Joint Mathematics Meeting. Supervised by Professor Daniel Arrigo.

**Pedagogical Activities**
1. Together We Teach (Committee Member)  
   Developed material and scheduled weekly speakers for the Together We Teach graduate professional development program. Implemented mentoring groups into the program in which first year graduate students were paired with senior graduate students who would then regularly meet to discuss techniques to improve teaching.

2. Together We Teach (Participant)  
   Jan. 2018 – Present
   Worked with other graduate students in the USU Math/Stat department to improve graduate teaching. Attended several seminars aimed at improving various aspects of teaching. Presentations included topics such as writing weighted objectives based on appropriate learning levels, developing formative/summative assessments, designing rubrics, and implementing technology in the classroom.

**Honors and Awards**
1. Excellence in Research Award | 2021 (USU)
2. Excellence in Teaching Award | 2021, 2020, 2019, 2018 (USU)
3. Utah State University Graduate Student Teacher of the Year (Robins Award) | 2019 (USU)
4. College of Science Graduate Student Teacher of the Year Award | 2019 (USU)
5. Math/Stat Department Graduate Student Teacher of the Year Award | 2019 (USU)
6. Summer Research Funding | 2018 (USU)
7. Outstanding Graduate Teaching Assistant Award | 2015 (UCA)
8. O.L. Hughes Award for Outstanding Undergraduate in Mathematics | 2014 (UCA)
9. Dorothy Long Scholarship for Students of the CNSM | 2013 (UCA)
10. JMM Undergraduate Research Funds Grant Recipient | 2013 (UCA)
11. Student Undergraduate Research Funds Grant Recipient | 2012 (UCA)
12. Dean Scholarship Award | 2010 (UCA)