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Geometrization of Perfect Fluids, Scalar Fields, and (2+1)-Dimensional Electromagnetic Fields

Dionisios Sotirios Krongos

Utah State University

GEOMETRIZATION OF PERFECT FLUIDS, SCALAR FIELDS, AND

(2+1)-DIMENSIONAL ELECTROMAGNETIC FIELDS

by

Dionisios Sotirios Krongos

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Approved:

Charles Torre, Ph.D. Major Professor	Ian Anderson, Ph.D. Committee Member
Maria Rodriguez, Ph.D. Committee Member	D. Richard Cutler, Ph.D. Interim Vice Provost of Graduate Studies

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ABSTRACT

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Dionisios Sotirios Krongos, Master of Science

Utah State University, 2021

Major Professor: Charles Torre, Ph.D.
Department: Physics

Rainich-type conditions giving a spacetime “geometrization” of matter fields in general relativity are reviewed and extended. Three types of matter are considered: perfect fluids, scalar fields, and electromagnetic fields. Necessary and sufficient conditions on a spacetime metric for it to be part of a perfect fluid solution of the Einstein equations are given. Formulas for constructing the fluid from the metric are obtained. All fluid results hold for any spacetime dimension. Geometric conditions on a metric which are necessary and sufficient for it to define a solution of the Einstein-scalar field equations and formulas for constructing the scalar field from the metric are unified and extended to arbitrary dimensions, to include a cosmological constant, and to include any self-interaction potential. Necessary and sufficient conditions on a $(2 + 1)$-dimensional spacetime metric for it to be an electrovacuum and formulas for constructing the electromagnetic field from the metric are obtained. Both null and non-null electromagnetic fields are treated. A number of examples and applications of these results are presented. Software implementations of these results are also included.

(52 pages)

PUBLIC ABSTRACT

Geometrization of Perfect Fluids, Scalar Fields, and (2+1)-dimensional Electromagnetic Fields

Dionisios Sotirios Krongos

The Rainich equations provide a purely geometrical interpretation of matter in terms of the gravitational field it generates. All this takes place within the geometrical formulation of gravity provided by Einstein's General Theory of Relativity. Rainich-type conditions giving spacetime ”goemetrizations” are reviewed and extended. Three types of matter are considered: perfect fluids, scalar fields, and electromagnetic fields.

I'd like to thank everyone who had to put up with me. You're all heroes.

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D. S. Krongos

CONTENTS

NOTATION

In this thesis we'll use the mostly positive metric signature $(- + \cdots +)$. Lowercase latin letters will run over the dimension of the spacetime 0 to $(n-1)$. We'll also work with geometrized units such that $\mathcal{G} = c = 1$.

Symbols

$\mathcal{M}$	Manifold.
x^a	Coordinates on manifold $\mathcal{M}$.
g_{ab}	Metric tensor on manifold $\mathcal{M}$.
$g = g_{ab}dx^a \otimes dx^b$	Metric in a coordinate basis.
${R_{abc}}^d$	Riemann curvature tensor.
$R_{ab} = {R_{adc}}^d$	Ricci tensor.
$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$	Einstein tensor.
$T = {T^a}_a$	Trace of tensor T.
$T_{ab} = T_{ba}$	Symmetric tensor.
$T_{ab} = -T_{ba}$	Antisymmetric tensor.
${T^{\mu_1 \cdots \mu_k}}_{\nu_1 \cdots \nu_l}$	Tensor of type (k, l).
$T_{(a_1 \cdots a_l)} = \dfrac{1}{l!} \sum\limits_{\pi} T_{a_{\pi(1)} \cdots a_{\pi(l)}}$	Symmetrization of tensor of type $(0, l)$.
δ_π	$+1$ for even permutations, -1 for odd permutations.
$T_{[a_1 \cdots a_l]} = \dfrac{1}{l!} \sum\limits_{\pi} \delta_\pi T_{a_{\pi(1)} \cdots a_{\pi(l)}}$	Antisymmetrization of tensor of type $(0, l)$.
$\psi_{,a} = \partial_a \psi$	Partial differentiation with respect to x^a.
∇	Metric compatible derivative operator.
${T^a}_{;b} = \nabla_b T^a$	Covariant differentiation.
ds^2	Line element.

CHAPTER 1

INTRODUCTION

In the general theory of relativity it is often possible to eliminate the matter fields from the Einstein-matter field equations and express the equations as local geometric conditions on the spacetime metric alone. This possibility was discovered by Rainich [1], who showed how to eliminate the Maxwell field from the Einstein-Maxwell equations, arriving at the "Rainich conditions", which give necessary and sufficient conditions on a spacetime metric for it to be a non-null electrovacuum. Rainich's work was made prominent by Misner and Wheeler [2], who advanced the "geometrization" program in which all matter was to be modeled as a manifestation of spacetime geometry. Over the subsequent years a variety of additional geometrization results have been found pertaining to electromagnetic fields, scalar fields, spinor fields, fluids, and so forth. See, *e.g.*, references [3–14]. Results such as these provide, at least in principle, a new way to analyze field equations and their solutions from a purely geometric point of view, just involving the metric.

The geometrization conditions which have been obtained over the years, while conceptually elegant, are generally more complicated than the original Einstein-matter field equations. For example, the Einstein-Maxwell equations are a system of variational second-order PDEs, while the Rainich conditions involve a system of non-variational fourth-order PDEs. For this reason one can understand why geometrization results have seen relatively little practical use in relativity and field theory. The current abilities of symbolic computational systems have, however, made the use of geometrization conditions viable for various applications. Indeed, the bulk of the non-null electrovacuum solutions presented in the treatise of reference [15] were verified using a symbolic computational implementation of the classical Rainich conditions. The purpose of this thesis is to compile a set of geometrization results for the Einstein field equations which involve the most commonly used matter fields and which are as comprehensive and as general as possible while at the same time in a form suitable for symbolic computational applications.

This last point requires some elaboration as it significantly constrains the type of geometric conditions which we shall deem suitable for our purposes. A suitable geometrization condition for our purposes will define an algorithm which takes as input a given spacetime metric and which determines, solely through algebraic and differentiation operations on the metric, whether the metric defines a solution of the Einstein equations with a given matter content. When the metric does define a solution, the matter fields shall be constructed directly from the metric via algebraic operations, differentiation, and integration.

We summarize our treatment for each of three types of matter fields and compare to existing results as follows.

Perfect Fluids

We give necessary and sufficient conditions on a spacetime metric for it to be part of a perfect fluid solution of the Einstein equations. Formulas for constructing the fluid from the metric are obtained. These results apply to spacetimes of any dimension greater than two and allow for a cosmological constant. No energy conditions or equations of state are imposed. Existing geometrization conditions for fluids can be found in [13], which generalize those found in four spacetime dimension in [12]. The conditions given in [13] and [12], while elegant, involve the existence of certain unspecified functions and so do not satisfy the computational criteria listed above. The results of [12] also include conditions which enforce equations of state, while the results of [13] enforce the dominant energy condition. Our conditions enforce neither of these since we are interested in geometrization conditions which characterize any type of fluid solutions. The geometrization conditions we obtain are built algebraically from the Einstein tensor and so involve up to two derivatives of the metric.

Scalar Fields

Geometric conditions on a metric which are necessary and sufficient for it to define a solution of the Einstein-scalar field equations and formulas for constructing the scalar field from the metric have been obtained by Kuchař for free fields in four spacetime dimensions without a cosmological constant [4]. These results apply to massless and massive fields. The results of [4] subsume related results in [3] and [6]. More recently, conditions for a symmetric tensor to be algebraically that of a free massless scalar field in any dimension, without cosmological constant, have been given in [13]. These conditions are necessary algebraically but are not sufficient for geometrization since additional differential conditions are required. Here we give necessary and sufficient conditions for a metric to define a solution of the Einstein-scalar field equations which generalize all these results. In particular, the results we obtain here hold in arbitrary dimensions, they allow for a cosmological constant, they allow for a mass, and they allow for a freely specifiable self-interaction potential. Null and non-null fields are treated. The geometrization conditions we have found for a scalar field necessarily involve both algebraic and differential conditions on the Einstein tensor; they involve up to three derivatives of the spacetime metric.

Electromagnetic Fields

Necessary and sufficient conditions on a four-dimensional spacetime metric for it to be a non-null electrovacuum were given by Rainich [1] and Misner, Wheeler [2]. The null case has been investigated by Misner, Wheeler [2], Peres [7], Geroch [8], Bartrum [9], and Ludwig [10]. Building upon Ludwig's results, a set of geometrization conditions for null electrovacua has been given by Torre [14] via the Newman-Penrose formalism.

All these results took advantage of special features of four dimensions, *e.g.,* the Hodge dual of a 2-form is again a 2-form. To our knowledge, for the electromagnetic field in dimensions other than four only partial geometrization results have been obtained, limited to the "algebraic" part of the Rainich conditions [13]. As we shall show, in (2+1) dimensions it is possible to complete the geometrization process for the electromagnetic field for non-null fields and for null fields [**?**]. The simplifying feature of three spacetime dimensions is that the problem of geometrization of an electromagnetic field can be reduced to that of a scalar field, whose solution is known [4, 16].

Besides proving various geometrization theorems, we provide a number of modest illustrations of the theorems which hopefully serve to clarify their structure and usage. All these illustrations were accomplished using the *DifferentialGeometry* package in *Maple*, amply demonstrating the amenability of our results to symbolic computation.

The thesis will conclude with a chapter of software implementations of the results presented. There will be a section for code on perfect fluids which consists of geometrization conditions and reconstruction of the fluid. There will be a section for code on scalar fields and (2 + 1)-dimensional electromagnetic fields which consists of geometrization conditions and reconstruction of the field. Since the case of (2 + 1)-dimensional electromagnetic fields is reduced to that of scalar fields, the code for scalar fields covers the (2 + 1)-dimensional electromagnetic field case.

This code has been used to find and verify solutions to various field equations. Due to the limited required input to find solutions to the field equations, it is quite simple to explore certain cases. As an example, this code has been used to systematically go through three dimensional metrics to uncover solutions in topologically massive gravity with an electromagnetic field. An additional feature of these geometrization procedures is that it facilitates the study of solutions with non-inheriting matter fields. As far as we know new solutions to the various field equations have been obtained.

The results from this thesis are from the two papers published in the Journal of Mathematical Physics by Krongos and Torre [16], [17], and this work has already been cited.

CHAPTER 2

PERFECT FLUIDS

2.1 Conditions on Perfect Fluids

Let (M, g) be an n-dimensional spacetime, $n > 2$, with signature $(-++\cdots+)$. Let $\mu\colon M \to \mathbf{R}$ and $p\colon M \to \mathbf{R}$ be functions on M. Let u be a unit timelike vector field on M, that is, $g_{ab}u^a u^b = -1$. The Einstein equations for a perfect fluid are

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = q\Big\{(\mu + p)u_a u_b + pg_{ab}\Big\}. \tag{2.1}$$

Here R_{ab} is the Ricci tensor of g_{ab}, $R = g^{ab}R_{ab}$ is the Ricci scalar, Λ is the cosmological constant, and $q = 8\pi\mathcal{G}/c^4$ with $\mathcal{G}$ being Newton's constant. We note that the cosmological and Newton constants can be absorbed into the definition of the fluid. With

$$\tilde{\mu} = q\mu + \Lambda, \quad \tilde{p} = qp - \Lambda, \tag{2.2}$$

the Einstein equations take the form

$$R_{ab} - \frac{1}{2}Rg_{ab} = (\tilde{\mu} + \tilde{p})u_a u_b + \tilde{p}g_{ab}. \tag{2.3}$$

If there exist functions $\tilde{\mu}$ and $\tilde{p}$ and a timelike unit vector field u on M such that (2.3) holds, we say that (M, g) is a perfect fluid spacetime. Note that if $\tilde{\mu} + \tilde{p} = 0$ in some open set $\mathcal{U} \subset M$ then the spacetime is actually an Einstein space on $\mathcal{U}$. In what follows, when we speak of perfect fluid spacetimes we assume that $\tilde{\mu} + \tilde{p} \neq 0$ at each point of the spacetime. Note also that we have not imposed any energy conditions, equations of state, or thermodynamic properties. These additional considerations are examined, for example, in reference [12].

In the following we will use the trace-free Ricci (or trace-free Einstein) tensor S_{ab}:

$$S_{ab} = R_{ab} - \frac{1}{n}Rg_{ab} = G_{ab} - \frac{1}{n}Gg_{ab}. \tag{2.4}$$

We will also need the following elementary result.

Proposition 1. *Let Q_{ab} be a covariant, symmetric, rank-2 tensor on an n-dimensional vector space V. Q_{ab} satisfies*

$$Q_{a[b}Q_{c]d} = 0 \tag{2.5}$$

if and only if there exists a covector $v_a \in V^*$ such that

$$Q_{ab} = \pm v_a v_b. \tag{2.6}$$

Proof. Eq. (2.6) clearly implies (2.5). We now show that (2.5) implies (2.6). From Sylvester's law of inertia there exists a basis for V^*, denoted by $\beta_i, i = 1, 2, \ldots, n$, in which Q_{ab} is diagonal with components given by ± 1, 0. In this basis, using index-free notation, Q takes the form:

$$Q = \sum_{i=1}^{n} a_i \beta_i \otimes \beta_i, \tag{2.7}$$

where $a_i \in \{-1, 0, 1\}$. In this basis, eq. (2.5) takes the form

$$\sum_{i,j=1}^{n} a_i a_j (\beta_i \otimes \beta_i \otimes \beta_j \otimes \beta_j - \beta_i \otimes \beta_j \otimes \beta_i \otimes \beta_j) = 0. \tag{2.8}$$

Consequently, $a_i a_j = 0$ for all $i \neq j$, from which it follows that all but one of the $\{a_i\}$ are zero (assuming $Q \neq 0$). If the basis is labeled so that a_1 is the non-zero component then $v = \sqrt{|a_1|}\beta_1$, and (2.6) follows. $\square$

The following theorem gives a simple set of Rainich-type conditions for a perfect fluid spacetime.

Theorem 1. *Let (M, g) be an n-dimensional spacetime, $n > 2$. Define*

$$\alpha = -\left[\frac{n^2}{(n-1)(n-2)} S_a^b S_b^c S_c^a\right]^{1/3}. \tag{2.9}$$

The metric g defines a perfect fluid spacetime if and only if

$$(1) \quad \alpha \neq 0, \tag{2.10}$$

$$(2) \quad K_{a[b}K_{c]d} = 0, \tag{2.11}$$

$$(3) \quad K_{ab}v^a v^b > 0, \text{ for some } v^a, \tag{2.12}$$

where

$$K_{ab} = \frac{1}{\alpha} S_{ab} - \frac{1}{n} g_{ab}. \tag{2.13}$$

Proof. We begin by showing the conditions are necessary. Suppose the Einstein equations (2.3) are satisfied for some $(g, \tilde{\mu}, \tilde{p}, u)$. The trace-free Einstein tensor takes the form

$$S_{ab} = (\tilde{\mu} + \tilde{p}) \left(u_a u_b + \frac{1}{n} g_{ab} \right). \tag{2.14}$$

Equations (2.14) and (2.9) yield

$$\alpha = \tilde{\mu} + \tilde{p}, \tag{2.15}$$

which implies condition (1) since we are always assuming that $\tilde{\mu} + \tilde{p} \neq 0$. From (2.15), (2.13), and (2.3) we have that

$$K_{ab} = u_a u_b, \tag{2.16}$$

which implies conditions (2) and (3).

We now check that conditions (1)–(3) are sufficient. Condition (1) permits K_{ab} to be defined. From Proposition 1, condition (2) implies there exists a covector field u_a such that $K_{ab} = \pm u_a u_b$, while condition (3) picks out the positive sign, $K_{ab} = u_a u_b$. From the definition (2.13) of K_{ab} it follows that $g_{ab}u^a u^b = -1$. Defining $\tilde{\mu}$ and $\tilde{p}$ via

$$\tilde{\mu} + \tilde{p} = \alpha, \quad \tilde{p} = \frac{1}{n}(G + \alpha), \tag{2.17}$$

it follows the perfect fluid Einstein equations (2.3) are satisfied. □

From the proof of this theorem we obtain a prescription for construction of the fluid variables from a metric satisfying the conditions (1)–(3).

Corollary 1. *If (M, g) is an n-dimensional spacetime satisfying the conditions of Theorem 1 then it is a perfect fluid spacetime with energy density $\tilde{\mu}$ and pressure $\tilde{p}$ given by*

$$\tilde{\mu} = \frac{1}{n}[(n-1)\alpha - G], \quad \tilde{p} = \frac{1}{n}(\alpha + G), \tag{2.18}$$

and fluid velocity u^a determined (up to an overall sign) from the quadratic condition

$$u_a u_b = K_{ab}. \tag{2.19}$$

2.2 Examples

2.2.1 Example: A static, spherically symmetric perfect fluid

Here we use Theorem 1 to find fluid solutions. Consider the following simple ansatz for a class of static, spherically symmetric spacetimes:

$$g = -r^2 dt \otimes dt + f(r) dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi). \tag{2.20}$$

Here f is a function to be determined. Computation of the tensor field K and imposition the quadratic condition $K_{a[b}K_{c]d} = 0$ leads to a system of non-linear ordinary differential equations for $f(r)$ which can be reduced to:

$$rf' + 2f - f^2 = 0. \tag{2.21}$$

This has the 1-parameter family of solutions:

$$f(r) = \frac{2}{1 + \lambda r^2}, \tag{2.22}$$

where λ and r are restricted by $1 + \lambda r^2 > 0$ to give the metric Lorentz signature. With $f(r)$ so determined, the scalar α and the tensor K are computed to be

$$\alpha = \frac{1}{r^2}, \quad K = r^2 dt \otimes dt, \tag{2.23}$$

from which it immediately follows that all 3 conditions of Theorem 1 are satisfied.

From Corollary 1 the energy density, $\tilde{\mu}$, pressure $\tilde{p}$ and 4-velocity u are given by

$$\tilde{\mu} = \frac{1}{2}[\frac{1}{r^2} - 3\lambda], \quad \tilde{p} = \frac{1}{2}[\frac{1}{r^2} + 3\lambda], \quad u = \frac{1}{r}\partial_t. \tag{2.24}$$

If desired, one can interpret this solution as admitting a cosmological constant $\Lambda = -\frac{3}{2}\lambda$ and a stiff equation of state, $\mu = p = 1/(2qr^2)$.

2.2.2 Example: A class of 5-dimensional cosmological fluid solutions

In this example we use Theorem 1 to construct a class of cosmological perfect fluid solutions on a 5-dimensional spacetime (M, g) where $M = \mathbf{R} \times \Sigma^4$ with $\Sigma^4 = \mathbf{R}^3 \times \mathbf{R}$ being homogeneous and anisotropic. We start from a 4-parameter family of metrics of the form

$$g = -dt \otimes dt + r_0^2 t^{2b}(dx \otimes dx + dy \otimes dy + dz \otimes dz) + R_0^2 t^{2\beta} dw \otimes dw, \tag{2.25}$$

where r_0, R_0, b, and β are parameters to be determined. This metric defines a family of spatially flat 3+1 dimensional FRW-type universes with (x, y, z) coordinates, each with an extra dimension w described with its own scale factor. Using Theorem 1 we select metrics from this set which solve the perfect fluid Einstein equations.

Condition (2) of Theorem 1 leads to a system of algebraic equations for b and β from which we have found 3 solutions:

$$\text{(i) } b = \frac{\beta(\beta-1)}{\beta+2}, \quad \text{(ii) } b = -\frac{1}{3}(\beta-1), \quad \text{(iii) } b = \beta. \tag{2.26}$$

Case (i) can be eliminated from consideration since

$$b = \frac{\beta(\beta-1)}{\beta+2} \quad \Longrightarrow \quad K = -R_0^2 t^{2\beta} dw \otimes dw, \tag{2.27}$$

which violates condition (3) of Theorem 1. Cases (ii) and (iii) satisfy all the conditions of Theorem 1. Using Corollary 1 we obtain solutions of the Einstein equations as follows:

$$\text{(ii) } b = -\frac{1}{3}(\beta-1) \quad \Longrightarrow \quad K = dt \otimes dt, \quad u = \partial_t, \quad \tilde{\mu} = \tilde{p} = -\frac{1}{3}\frac{2\beta^2-\beta-1}{t^2} \tag{2.28}$$

$$\text{(iii) } b = \beta \quad \Longrightarrow \quad K = dt \otimes dt, \quad u = \partial_t \quad \tilde{\mu} = \frac{6\beta^2}{t^2}, \tilde{p} = -\frac{3\beta(2\beta-1)}{t^2}. \tag{2.29}$$

Case (ii) is anisotropic (except when $b = \beta = \frac{1}{4}$) and allows for any combination of expansion and contraction for the (x, y, z) and w dimensions. Case (iii) is isotropic in all four spatial dimensions. Both cases are singular as $t \to 0$.

CHAPTER 3
SCALAR FIELDS

3.1 Conditions on Scalar Fields

Kuchař has given Rainich-type geometrization conditions for a minimally-coupled, free scalar field in four spacetime dimensions without a cosmological constant [4]. Here we generalize his treatment to a real scalar field with any self-interaction, in any dimension, and including the possibility of a cosmological constant.

The Einstein-scalar field equations for a spacetime (M, g) with a minimally coupled real scalar field ψ, with self interaction potential $V(\psi)$, and with cosmological constant Λ are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = q\left\{\psi_{;a}\psi_{;b} - \frac{1}{2}g^{lm}\psi_{;l}\psi_{;m}g_{ab} - V(\psi)g_{ab}\right\}, \tag{3.1}$$

$$g^{lm}\psi_{;lm} - V'(\psi) = 0, \tag{3.2}$$

where $q = 8\pi\mathcal{G}/c^4$ and we use a semicolon to denote the usual torsion-free, metric-compatible covariant derivative.

We distinguish two classes of solutions to the Einstein-scalar field equations. If a solution has $\psi_{;a}\psi_{;}{}^{a} \neq 0$ everywhere we say that the solution is *non-null*. If the solution has $\psi_{;a}\psi_{;}{}^{a} = 0$ everywhere we say that the solution is *null*.

The Rainich-type conditions we shall obtain require the following extension of Proposition 1 (*c.f.* [4]).

Proposition 2. *Let Q_{ab} be a symmetric tensor field on a manifold M. Then (locally on M) there exists a function ϕ such that*

$$Q_{ab} = \pm\phi_{;a}\phi_{;b} \tag{3.3}$$

if and only if Q_{ab} satisfies

$$(1)\ Q_{a[b}Q_{c]d} = 0, \tag{3.4}$$

$$(2)\ Q_{ab}Q_{c[d;e]} + Q_{ac}Q_{b[d;e]} + Q_{bc;[d}Q_{e]a} = 0. \tag{3.5}$$

Proof. Using Proposition 1, condition (1) is necessary and sufficient for the existence of a 1-form v_a such that $Q_{ab} = \pm v_a v_b$. We then have

$$Q_{bc;[d}Q_{e]a} = v_a v_b v_{c;[d}v_{e]} + v_a v_c v_{b;[d}v_{e]}, \tag{3.6}$$

and

$$Q_{c[d;e]} = \pm(v_{c;[e}v_{d]} + v_c v_{[d;e]}), \tag{3.7}$$

and hence

$$Q_{ab}Q_{c[d;e]} + Q_{ac}Q_{b[d;e]} + Q_{bc;[d}Q_{e]a} = 2v_a v_b v_c v_{[d;e]}. \tag{3.8}$$

Consequently, if (3.3) holds then condition (2) holds and, conversely, condition (2) implies the 1-form v_a is closed, hence locally exact. □

We shall use the following notation:

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}, \quad G = G^a_a, \quad {}_2G_{ab} = G^c_a G_{cb}, \quad {}_2G = G_{ab}G^{ab}, \tag{3.9}$$

$${}_3G_{ab} = G^c_a G^d_c G_{db}, \quad {}_3G = G^c_a G^d_c G^a_d. \tag{3.10}$$

3.2 Free Massless Scalar Fields

3.2.1 Non-Null Scalar Fields

The geometrization theorems we shall obtain will depend upon whether the self-interaction potential $V(\psi)$ is present. We begin with a free, massless, non-null scalar field.

Theorem 2. *Let (M, g) be an n-dimensional spacetime, $n > 2$. The following are necessary and sufficient conditions on g such that there exists a scalar field ψ with (g, ψ) defining a local, non-null solution to the Einstein-scalar equations (3.1), (3.2) with $V(\psi) = 0$:*

$$ {}_2G - \frac{1}{n}G^2 \neq 0, \tag{3.11}$$

$$ A_{;i} = 0, \tag{3.12}$$

$$ H_{a[b}H_{c]d} = 0, \tag{3.13}$$

$$ H_{ab}H_{c[d;e]} + H_{ac}H_{b[d;e]} + H_{bc;[d}H_{e]a} = 0, \tag{3.14}$$

$$ H_{ab}w^a w^b > 0, \quad \text{for some } w^a, \tag{3.15}$$

where we define

$$ A = \frac{1}{2}\frac{\frac{1}{n}G\,{}_2G - {}_3G}{{}_2G - \frac{1}{n}G^2}, \tag{3.16}$$

and

$$ H_{ab} = G_{ab} + \frac{1}{2-n}\left(G + 2A\right)g_{ab}. \tag{3.17}$$

Proof. We begin by showing the conditions are necessary. Suppose (g, ψ) define a non-null solution to the Einstein-scalar field equations with $V(\psi) = 0$. From the Einstein equations

(3.1) we have

$$G_{ab} - \frac{1}{n} G g_{ab} = q \left(\psi_{;a} \psi_{;b} - \frac{1}{n} g^{lm} \psi_{;l} \psi_{;m} g_{ab} \right), \tag{3.18}$$

and

$${}_2G_{ab} - \frac{1}{n}\, {}_2G g_{ab} = -2q\Lambda \left(\psi_{;a} \psi_{;b} - \frac{1}{n} g^{lm} \psi_{;l} \psi_{;m} g_{ab} \right). \tag{3.19}$$

These two equations imply

$${}_2G_{ab} - \frac{1}{n}\, {}_2G g_{ab} = -2\Lambda (G_{ab} - \frac{1}{n} G g_{ab}). \tag{3.20}$$

Now multiplying (3.20) by G^{ab} we get

$${}_3G - \frac{1}{n} G\, {}_2G = -2\Lambda ({}_2G - \frac{1}{n} G^2), \tag{3.21}$$

while contracting (3.18) with G^{ab} gives

$${}_2G - \frac{1}{n} G^2 = \frac{n-1}{n} q^2 \left(g^{ab} \psi_{;a} \psi_{;b} \right)^2 \neq 0. \tag{3.22}$$

Equation (3.21) then implies

$$A \equiv \frac{1}{2} \frac{\frac{1}{n} G\, {}_2G - {}_3G}{{}_2G - \frac{1}{n} G^2} = \Lambda. \tag{3.23}$$

It follows that $A_{;i} = 0$ and then, from the Einstein equations, we have that

$$H_{ab} = q \psi_{;a} \psi_{;b}, \tag{3.24}$$

from which follow the rest of the conditions, (3.13), (3.14), (3.15).

Conversely, suppose the conditions (3.11)–(3.15) are satisfied. From Proposition 2, equations (3.13), (3.14), (3.15) imply that there exists a function ψ such that $H_{ab} = q\psi_{;a}\psi_{;b}$, while (3.12) implies that $A = const. \equiv \Lambda$. Together, these results imply that:

$$G_{ab} = q\left(\psi_{;a}\psi_{;b} - \frac{1}{2}g_{ab}g^{mn}\psi_{;m}\psi_{;n}\right) - \Lambda g_{ab} \tag{3.25}$$

so that the Einstein equations are satisfied by (g, ψ). The contracted Bianchi identity now implies

$$\psi_{;c}g^{ab}\psi_{;ab} = 0, \tag{3.26}$$

and the non-null condition (3.11) yields $g^{ab}\psi_{;a}\psi_{;b} \neq 0$, which enforces the scalar field equation (3.2). □

From the proof just given it is clear the scalar field is determined from the metric by solving a system of quadratic equations followed by a simple integration.

Corollary 2. *Let (M, g) satisfy the conditions of Theorem 2. Then (g, ψ) satisfy the Einstein-scalar field equations, with $V = 0$, with $\Lambda = A$, and with ψ determined up to an additive constant and up to a sign by*

$$\psi_{;a}\psi_{;b} = \frac{1}{q}H_{ab}. \tag{3.27}$$

3.2.2 Null Scalar Fields

We turn to the special case of free, massless, null solutions, that is, solutions in which $g^{ab}\psi_{;a}\psi_{;b} = 0$.

Theorem 3. *Let (M, g) be an n-dimensional spacetime, $n > 2$. The following are necessary and sufficient conditions on g such that there exists a scalar field ψ with (g, ψ) defining a local, null solution of the Einstein-scalar field equations with $V(\psi) = 0$:*

$$R = \frac{2n}{n-2}\Lambda, \tag{3.28}$$

$$S_{a[b}S_{c]d} = 0, \tag{3.29}$$

$$S_{ab}S_{c[d;e]} + S_{ac}S_{b[d;e]} + S_{bc;[d}S_{e]a} = 0, \tag{3.30}$$

$$S_{ab}w^a w^b > 0 \quad \text{for some } w^a, \tag{3.31}$$

where S_{ab} is the trace-free Ricci tensor.

Proof. Suppose the Einstein equations (3.1) with $V(\psi) = 0$ are satisfied for some (g, ψ) where $g^{lm}\psi_{;l}\psi_{;m} = 0$. Taking the trace of (3.1) leads to (3.28) and the trace-free part yields

$$S_{ab} = q\psi_{;a}\psi_{;b}, \tag{3.32}$$

from which (3.29), (3.30), and (3.31) follow.

Conversely, using Proposition 2, conditions (3.29), (3.30), and (3.31) imply that locally there exists a function ψ such that

$$S_{ab} = q\psi_{;a}\psi_{;b}. \tag{3.33}$$

Note that this implies $g^{lm}\psi_{;l}\psi_{;m} = 0$. Using (3.28) and S_{ab} to construct the Einstein tensor leads to the free, massless, null field Einstein equations:

$$G_{ab} = q\psi_{;a}\psi_{;b} - \Lambda g_{ab}. \tag{3.34}$$

The contracted Bianchi identity, along with $g^{lm}\psi_{;l}\psi_{;m} = 0$, then implies the field equation

$$g^{ab}\psi_{;ab} = 0. \tag{3.35}$$

□

Corollary 3. *Let (M, g) satisfy the conditions of Theorem 3. Then (g, ψ) satisfy the Einstein-scalar field equations with $V = 0$, and ψ is determined up to an additive constant and up to a sign by*

$$\psi_{;a}\psi_{;b} = \frac{1}{q} S_{ab}. \tag{3.36}$$

3.3 Real Massless Scalar Fields with Potential

We now turn to geometrization conditions which describe the case with $V(\psi) \neq 0$. We define $\tilde{V} = qV(\psi) + \Lambda$. We always assume that $V(\psi)$ has been specified such that there exists an inverse function $W : \mathbf{R} \to \mathbf{R}$ with $W(\tilde{V}(\psi)) = \psi$, and $\tilde{V}(W(x)) = x$.

3.3.1 Non-Null Scalar Field with Potential

Theorem 4. *Let (M, g) be an n-dimensional spacetime, $n > 2$. The following are necessary and sufficient conditions on g such that there exists a scalar field ψ with (g, ψ) defining a non-null solution to the Einstein-scalar equations (3.1), (3.2), where $\tilde{V} = qV + \Lambda$ has inverse W:*

$${}_2G - \frac{1}{n} G^2 \neq 0, \tag{3.37}$$

$$H_{ab} = qW'^2(A) A_{;a} A_{;b}, \tag{3.38}$$

where we define

$$A = \frac{1}{2} \frac{\frac{1}{n} G \, {}_2G - {}_3G}{{}_2G - \frac{1}{n} G^2}. \tag{3.39}$$

and

$$H_{ab} = G_{ab} + \frac{1}{2-n} \left(G + 2A\right) g_{ab}. \tag{3.40}$$

Proof. The proof is along the same lines as the proof of Theorem 2. To see that the conditions (3.37), (3.38) are necessary, we start from the Einstein equations (3.1), from which it follows that the scalar field is non-null only if

$$ {}_2G - \frac{1}{n}G^2 \neq 0. \tag{3.41} $$

It also follows from (3.1) that

$$ A \equiv \frac{1}{2}\frac{\frac{1}{n}G\,{}_2G - {}_3G}{{}_2G - \frac{1}{n}G^2} = \tilde{V}(\psi), \tag{3.42} $$

so that

$$ \psi = W(A), \tag{3.43} $$

and

$$ H_{ab} = q\psi_{;a}\psi_{;b} = qW'^2(A)A_{;a}A_{;b}. \tag{3.44} $$

Conversely, assuming the metric satisfies conditions (3.37) and (3.38), if we set

$$ \psi \equiv W(A), \tag{3.45} $$

so that $A = \tilde{V}(\psi)$, then from (3.37), (3.38) the scalar field is not null and the Einstein equations are satisfied. The contracted Bianchi identity then implies the scalar field equation (3.2) is satisfied as before. □

Corollary 4. *If a metric g satisfies the conditions of Theorem 4, then there exists a non-null solution (g, ψ) to the Einstein-scalar field equations (3.1), (3.2) where*

$$ \psi = \tilde{V}^{-1}(A). \tag{3.46} $$

3.3.2 Null Scalar Field with Potential

Finally we consider the null case with a given self-interaction potential, invertible as before.

Theorem 5. *Let (M, g) be an n-dimensional spacetime, $n > 2$. There exists a scalar field ψ with (g, ψ) defining a local, null solution to the Einstein-scalar equations (3.1), (3.2) (with potential described by $\tilde{V} = qV + \Lambda$ and $W = \tilde{V}^{-1}$) if and only if either*

$$S_{ab} = qW'^2(B)B_{;a}B_{;b} \neq 0, \tag{3.47}$$

or

$$S_{ab} = 0, \quad B_{;a} = 0, \quad V'(W(B)) = 0, \tag{3.48}$$

where

$$B = \frac{n-2}{2n}R. \tag{3.49}$$

Proof. To see that this condition is necessary, assume the Einstein-scalar field equations hold for a metric g and a null scalar field ψ. It follows that

$$\tilde{V} = \frac{n-2}{2n}R, \quad S_{ab} = q\psi_{;a}\psi_{;b}, \tag{3.50}$$

so that $\psi = W(B)$ and condition (3.47) or condition (3.48) follows, depending upon whether $\psi_{;a}$ vanishes or not. Conversely, defining $\psi = W(B)$, it follows that $R = \frac{2n}{n-2}\tilde{V}(\psi)$ and, using (3.47) if $S_{ab} \neq 0$, the Einstein equations (3.1) are satisfied and the scalar field is null. The contracted Bianchi identity implies

$$\psi_{;b}\left[\psi_{;a}{}^{a} - V'(\psi)\right] = 0, \tag{3.51}$$

which implies (3.2) if $S_{ab} \neq 0$ since $\psi_{;b} \neq 0$ by (3.47). If $S_{ab} = 0$, and $B_{;a} = 0$, the scalar field equation follows from $V'(W(B)) = 0$. $\square$

Corollary 5. *If a metric g satisfies the conditions of Theorem 5, then there exists a null solution (g, ψ) to the Einstein-scalar field equations (3.1), (3.2) where*

$$\psi = \tilde{V}^{-1}(B). \tag{3.52}$$

3.4 Examples

3.4.1 Example: A non-inheriting scalar field solution

The static, spherically symmetric fluid spacetime (2.20), (2.22) also satisfies the geometrization conditions for a massless free scalar field contained in Theorem 2. Starting with the metric

$$g = -r^2 dt \otimes dt + \frac{2}{1+\lambda r^2} dr \otimes dr + r^2 (d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi). \tag{3.53}$$

and calculating A and H from (3.16) and (3.17) gives

$$A = -\frac{3}{2}\lambda, \quad H_{ab} dx^a \otimes dx^b = dt \otimes dt, \tag{3.54}$$

so that, according to Corollary 2, the scalar field is given by

$$\psi = \pm \frac{1}{\sqrt{q}} t + constant, \tag{3.55}$$

and the cosmological constant is given by $\Lambda = -\frac{3}{2}\lambda$. This solution (with $\lambda = 0$) was exhibited in ref. [18]. We remark that while the spacetime is static the scalar field is clearly not static and so represents an example of a "non-inheriting" solution to the Einstein-scalar field equations. Non-inheriting solutions of the Einstein-Maxwell equations are well-known [15]. Geometrization conditions, which depend solely upon the metric, treat inheriting and non-inheriting matter fields on the same footing.

We have been able to find an analogous family of non-inheriting solutions in 2+1 dimensions from an analysis of the geometrization conditions in Theorem 2. In coordinates (t, r, θ) the spacetime metric takes the form:

$$g = -\frac{1}{2\Lambda} dt \otimes dt + \frac{1}{b - 2\Lambda r^2} dr \otimes dr + r^2 d\theta, \tag{3.56}$$

where b is a constant. This metric yields $A = \Lambda$ and

$$H = dt \otimes dt, \tag{3.57}$$

so that from Corollary 2 the scalar field is given by

$$\psi = \pm \frac{1}{\sqrt{q}} t + constant. \tag{3.58}$$

It is straightforward to verify that the metric and scalar field so-defined satisfy the Einstein-scalar field equations (3.1) and (3.2) with $V = 0$.

3.4.2 Example: No-go results for spherically symmetric null scalar field solutions

We use Theorem 3 to show that there are no null solutions to the free, massless Einstein-scalar field equations if the spacetime is static and spherically symmetric, provided the spherical symmetry orbits are not null. We also show that there are no spherically symmetric null solutions with null spherical symmetry orbits. Both results hold with or without a cosmological constant. Since these results follow directly from the geometrization conditions they apply whether or not the scalar field inherits the spacetime symmetries.

We first consider a static, spherically symmetric spacetime in which the spherical symmetry orbits are not null. We use coordinates chosen such that the metric takes the form:

$$g = -f(r)dt \otimes dt + h(r)dr \otimes dr + R^2(r)(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi), \tag{3.59}$$

for some non-zero functions f, h, R. The condition (3.29) applied to (3.59) yields:

$$R'' = \frac{1}{2}\left(\frac{1}{h}h' + \frac{1}{f}f'\right)R', \tag{3.60}$$

$$f'' = 2f\left(\frac{R'}{R}\right)^2 + \frac{1}{2h}h'f' + \frac{1}{2f}f'^2 - 2\frac{fh}{R^2}. \tag{3.61}$$

These conditions force the trace-free Ricci tensor to vanish, whence the scalar field vanishes and we have an Einstein space. Consequently there are no non-trivial null solutions to the Einstein-scalar field equations in which the spacetime is static and spherically symmetric with non-null spherical symmetry orbits.

Next we consider a spherically symmetric spacetime in which the spherical symmetry orbits are null. In this case there exist coordinates (v, r, θ, ϕ) such that the metric takes the form:

$$g = w(v,r)(dv \otimes dr + dr \otimes dv) + u(v,r)dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi), \tag{3.62}$$

for some functions $w \neq 0$ and u. Calculation of conditions (3.28), (3.29) for metrics (3.62) reveals they are are incompatible. Consequently there are no null solutions to the Einstein-scalar field equations in this case. Since (3.62) is not actually static, but merely spherically symmetric, this proves that there are no Einstein-free-scalar field null solutions for spacetimes which are spherically symmetric with null symmetry orbits.

3.4.3 Example: Self-interacting scalar fields

Fonarev [19] has found a 1-parameter family of non-null spherically symmetric solutions to the Einstein-scalar field equations with a potential energy function which is an exponential function of the scalar field. Here we verify these solutions directly from the metric using Theorem 4.

In coordinates (t, r, θ, ϕ) and with $q = 1$ the metric in reference [19] takes the form:

$$\begin{aligned} g = &-e^{8\alpha^2\beta t}(1 - \frac{2m}{r})^{\delta} dt \otimes dt + e^{2\beta t}(1 - \frac{2m}{r})^{-\delta} dr \otimes dr \\ &+ e^{2\beta t}(1 - \frac{2m}{r})^{1-\delta} r^2 (d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi), \end{aligned} \tag{3.63}$$

where $m > 0$ is a free parameter,

$$\delta = \frac{2\alpha}{\sqrt{4\alpha^2 + 1}}, \tag{3.64}$$

and α and β parametrize the scalar field potential and cosmological constant via

$$\tilde{V}(\psi) = \beta^2(3-4\alpha^2)\exp\left(-\sqrt{8}\alpha\psi\right). \tag{3.65}$$

The inverse of the potential function is given by

$$W(x) = -\frac{\sqrt{2}}{4\alpha}\ln\left(\frac{x}{\beta^2(3-4\alpha^2)}\right). \tag{3.66}$$

Using the metric (3.63) to calculate A in (3.42) gives

$$A = (3-4\alpha^2)\beta^2 e^{-8\alpha^2\beta t}\left(1-\frac{2m}{r}\right)^{\frac{2\alpha}{\sqrt{4\alpha^2+1}}}. \tag{3.67}$$

Calculating the tensor H in (3.40) yields

$$\begin{aligned} H =& 8\beta^2\alpha^2 dt\otimes dt + \frac{4\alpha m\beta}{\sqrt{4\alpha^2+1}r(r-2m)}(dt\otimes dr + dr\otimes dt) \\ &+ \frac{2m^2}{(4\alpha^2+1)r^2(r-2m)^2}dr\otimes dr, \end{aligned} \tag{3.68}$$

and it follows that (3.38) is satisfied. Therefore, from Theorem 4, the metric (3.63) does indeed define a scalar field solution with the potential (3.65). Using Corollary 4, the scalar field is calculated to be

$$\psi = \sqrt{2}\left\{2\alpha\beta t + \frac{1}{2\sqrt{4\alpha^2+1}}\ln\left(1-\frac{2m}{r}\right)\right\}, \tag{3.69}$$

in agreement with Fonarev [19].

CHAPTER 4
ELECTROMAGNETIC FIELDS

4.1 Conditions on Electromagnetic Fields in (2+1) dimensions

Let (M, g) be a $(2+1)$-dimensional spacetime with signature $(-++)$. The Einstein-Maxwell equations with electromagnetic 2-form F and cosmological constant Λ are given by Einstein's equations

$$G_{ab} + \Lambda g_{ab} = q\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{de}F^{de}\right) \tag{4.1}$$

along with the source-free Maxwell equations

$$F_{ab\,;}{}^a = 0, \quad F_{[ab\,;\,c]} = 0. \tag{4.2}$$

Here a semi-colon denotes covariant differentiation with respect to the Christoffel connection, G_{ab} is the Einstein tensor, and $q > 0$ represents Newton's constant. All fields on M will be assumed to be smooth.

We note that the Einstein-Maxwell equations admit a discrete symmetry: if (g, F) is a solution to these equations then so is $(g, -F)$. For this reason the electromagnetic field can be recovered from the geometry only up to a sign.

We say an electromagnetic field is *null* in a given region $U \subset M$ if $F_{ab}F^{ab} = 0$ on U, and the field is *non-null* in a region $U \subset M$ if $F_{ab}F^{ab} \neq 0$ on U. As in (3+1) dimensions, the null and non-null cases must be treated separately.

Define

$$G = G^a_a, \quad {}_2G = G^a_bG^b_a, \quad {}_3G = G^a_bG^b_cG^c_a. \tag{4.3}$$

The (2+1)-dimensional version of the Rainich geometrization of non-null electromagnetic fields is as follows.

4.1.1 Non-Null Electromagnetic Fields

Theorem 6. *Let (M, g) be a (2+1)-dimensional spacetime. The following are necessary and sufficient conditions on g such that on $U \subset M$ there exists a non-null electromagnetic field F with (g, F) being a solution of the Einstein-Maxwell equations (4.1)–(4.2):*

$$ {}_2G - \frac{1}{3}G^2 \neq 0, \tag{4.4} $$

$$ H_{ab}w^a w^b > 0, \qquad \textit{for some } w^a, \tag{4.5} $$

$$ B = \Lambda, \tag{4.6} $$

$$ H_{a[b}H_{c]d} = 0, \tag{4.7} $$

$$ H_{ab}H_{c[d;e]} + H_{ac}H_{b[d;e]} + H_{bc;[d}H_{e]a} = 0, \tag{4.8} $$

where

$$ B = \frac{1}{2}\frac{\frac{1}{3}G\,{}_2G - {}_3G}{{}_2G - \frac{1}{3}G^2}, \tag{4.9} $$

and

$$ H_{ab} = G_{ab} - (G + 2B)g_{ab}. \tag{4.10} $$

These conditions hold everywhere on U.

Corollary 6. *Let a metric g satisfy the conditions of Theorem 6. Then (g, F) satisfy the Einstein-Maxwell equations on U with the non-null electromagnetic field F determined up* *to a sign from*

$$F_{ab} = \epsilon_{abc}v^c, \quad v_a v_b = \frac{1}{q}H_{ab}. \tag{4.11}$$

4.1.2 Null Electromagnetic Fields

The geometrization in the null case is as follows.

Theorem 7. *Let (M, g) be a (2+1)-dimensional spacetime. The following are necessary and sufficient conditions on g such that on $U \subset M$ there exists a null electromagnetic field F with (g, F) being a solution of the Einstein-Maxwell equations (4.1)–(4.2):*

$$G = -3\Lambda, \tag{4.12}$$

$$S_{ab}w^a w^b > 0 \quad \text{for some } w^a, \tag{4.13}$$

$$S_{a[b}S_{c]d} = 0, \tag{4.14}$$

$$S_{ab}S_{c[d;e]} + S_{ac}S_{b[d;e]} + S_{bc;[d}S_{e]a} = 0, \tag{4.15}$$

where $S_{ab} = G_{ab} - \frac{1}{3}G^c_c\, g_{ab}$ is the trace-free Einstein (or Ricci) tensor. These conditions hold everywhere on U.

Corollary 7. *Let a metric g satisfy the conditions of Theorem 7. Then (g, F) satisfy the Einstein-Maxwell equations on U with a null electromagnetic field F determined up to a sign from*

$$F_{ab} = \epsilon_{abc}v^c, \quad v_a v_b = \frac{1}{q}S_{ab}. \tag{4.16}$$

As also happens in (3+1) dimensions, for both the null and non-null cases the Rainich conditions split into conditions which are algebraic in the Einstein (or Ricci) tensor and conditions which involve derivatives of the Einstein tensor. In (3+1) dimensions the non-null Rainich conditions involve up to 4 derivatives of the metric, while the null Rainich conditions can involve as many as 5 derivatives [14]. From the above theorems, in (2+1) dimensions both the null and non-null conditions involve up to 3 derivatives of the metric.

4.1.3 Proofs

We now prove the results stated in the previous section. The electromagnetic field F is a two-form in three spacetime dimensions, so (at least locally) we can express it as the Hodge dual of a one-form v,

$$F_{ab} = \epsilon_{ab}{}^{c} v_c, \quad v_a = -\frac{1}{2}\epsilon_{abc} F^{bc}, \tag{4.17}$$

where ϵ_{abc} is the volume form defined by the Lorentz signature metric, and which satisfies

$$\epsilon^{abc}\epsilon_{def} = -3!\delta^{[a}_{d}\delta^{b}_{e}\delta^{c]}_{f}. \tag{4.18}$$

The Einstein-Maxwell equations (4.1)–(4.2) can then be rewritten as

$$G_{ab} + \Lambda g_{ab} = q\left(v_a v_b - \frac{1}{2} g_{ab} v_c v^c\right), \tag{4.19}$$

$$v_{[a;b]} = 0 = v^{a}_{;a}. \tag{4.20}$$

These equations are locally equivalent to gravity coupled to a scalar field, where the scalar field ϕ is massless and minimally-coupled. The correspondence is via $v_a = \nabla_a \phi$. Consequently, the geometrization runs along the same lines as the scalar field case, found in the previous chapter.

We begin with Theorem 6. To see that the conditions are necessary, we consider a metric g and non-null electromagnetic field F satisfying the Einstein-Maxwell equations. From (4.19) it follows that

$$_2G - \frac{1}{3}G^2 = \frac{2}{3}q^2\left(v_c v^c\right)^2 \neq 0, \tag{4.21}$$

$$B = \Lambda, \quad H_{ab} = q v_a v_b, \tag{4.22}$$

and

$$H_{ab}H_{c[d;e]} + H_{ac}H_{b[d;e]} + H_{bc;[d}H_{e]a} = 2q^2 v_a v_b v_c v_{[d;e]} = 0, \tag{4.23}$$

from which it follows that the conditions (4.4)–(4.8) in Theorem 6 are necessary.

Conversely, suppose equations (4.4)–(4.8) are satisfied. From Eqs. (4.5) and (4.7) there exists a one-form v_a such that

$$H_{ab} = q v_a v_b. \tag{4.24}$$

(See the previous chapter for a proof.) Equation (4.4) implies $v_a v^a \neq 0$. Equation (4.8) becomes $2v_a v_b v_c v_{[d;e]} = 0$, so that $v_{[a;b]} = 0$. Taking account of condition (4.6), we now have

$$G_{ab} = q\left(v_a v_b - \frac{1}{2}g_{ab}v_c v^c\right) - \Lambda g_{ab}, \tag{4.25}$$

$$v_{[a;b]} = 0, \ \ v_a v^a \neq 0. \tag{4.26}$$

From the contracted Bianchi identity, $\nabla^b G_{ab} = 0$, we get

$$v_b v^a_{;a} = 0, \tag{4.27}$$

so that the Einstein-Maxwell equations are satisfied. The construction of the electromagnetic field from the metric described in Corollary 6 follows from solving the algebraic relations (4.24) for v_a and then using (4.17).

The null case, described in Theorem 7 and Corollary 7, is established as follows. As before, begin by assuming the Einstein-Maxwell equations are satisfied in the null case, that is, with $v_a v^a = 0$. The trace and trace-free parts of the Einstein equations yield, respectively,

$$G = -3\Lambda, \quad S_{ab} = q v_a v_b, \tag{4.28}$$

These equations and the Maxwell equations $v_{[a;b]} = 0$ imply the necessity of the conditions listed in Theorem 7. Conversely, granted the conditions of Theorem 7, it follows in a similar fashion as in the proof of Theorem 6 that equations (4.28) hold with $v_a v^a = 0$, and that the Maxwell equations $v_{[a;b]} = 0$ are satisfied. The contracted Bianchi identity again implies $v^a_{;a} = 0$. The construction of the electromagnetic field from the metric described in Corollary 7 follows from solving the algebraic relations (4.28) for v_a and then using (4.17).

4.2 Example: BTZ Black Hole

As an illustration of these geometrization conditions we investigate static, rotationally symmetric solutions to the Einstein-Maxwell equations. Begin with the following ansatz for the metric:

$$g = -f(r)\, dt \otimes dt + \frac{1}{f(r)}\, dr \otimes dr + r^2 d\theta \otimes d\theta, \tag{4.29}$$

where $f(r)$ is to be determined by the geometrization conditions. The algebraic condition (4.14) from Theorem 7 would imply the metric (4.29) is Einstein, so there can be no electromagnetic field in the null case. In the non-null case the conditions of Theorem 6 reduce to a remarkably simple linear third-order differential equation

$$f'''(r) + \frac{1}{r} f''(r) - \frac{1}{r^2} f'(r) = 0, \tag{4.30}$$

which has the solution

$$f(r) = c_1 + c_2 \ln r + c_3 r^2, \tag{4.31}$$

where c_1, c_2, and c_3 are constants of integration. Eq. (4.5) requires $c_2 < 0$. From equation (4.6) the form of $f(r)$ given in (4.31) corresponds to a cosmological constant

$$\Lambda = -c_3, \tag{4.32}$$

and, from Corollary 6, to an electromagnetic field

$$F = \pm \frac{\sqrt{-c_2}}{r} dt \wedge dr. \tag{4.33}$$

With the identifications

$$c_1 = -M, \quad c_2 = -\frac{1}{2}Q^2, \quad c_3 = \frac{1}{\ell^2} \tag{4.34}$$

we obtain the static charged BTZ solution [20].

4.3 Extension to other metric theories of gravity

The geometrization conditions obtained here can be extended to other metric theories of (2 + 1)-dimensional gravity coupled to electromagnetism provided the action functional S for the system takes the form

$$S = S_1[g] - \frac{2}{q}\Lambda V[g] + S_2[g, F], \tag{4.35}$$

where S_1 is diffeomorphism invariant, S_2 is the usual action for the electromagnetic field on a three-dimensional spacetime with metric g, and V is the volume functional. In the field equations, theorems, and corollaries given above one simply makes the replacement

$$G^{ab} \longrightarrow E^{ab} = -\frac{1}{\sqrt{|g|}} \frac{\delta S_1}{\delta g_{ab}}. \tag{4.36}$$

The identity $E^{ab}{}_{;b} = 0$ still holds because of the diffeomorphism invariance of S_1; all the proofs remain unchanged.

4.3.1 Example: Topologically Massive Gravity

As a simple application of this result, we suppose the action S_1 is a linear combination of the Einstein-Hilbert action and the Chern-Simons action constructed from the metric-compatible connection. The field equations are the Maxwell equations (4.2) along with

$$\alpha G_{ab} + \beta Y_{ab} + \Lambda g_{ab} = q\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{de}F^{de}\right), \tag{4.37}$$

where Y_{ab} is the Cotton-York tensor [**?**] and α, β are constants. These are the equations of topologically massive gravity [21] coupled to the electromagnetic field. We ask whether there are any solutions of the pp-wave type, admitting a covariantly constant null vector field. Using the usual metric ansatz

$$g = -2du \odot dv + dx \otimes dx + f(u,x)\, du \otimes du, \tag{4.38}$$

it follows that condition (4.4) in Theorem 6 is not satisfied, so only null solutions are possible. For this metric $E^a_a = 0$; (4.12) then implies we can only get a solution for $\Lambda = 0$. The conditions (4.13) – (4.15) of Theorem 7 reduce to

$$\alpha\frac{\partial^3 f}{\partial x^3} + \beta\frac{\partial^4 f}{\partial x^4} = 0, \tag{4.39}$$

with solution (assuming $\beta \neq 0$)

$$f(u,x) = a_0(u) + a_1(u)x - a_2(u)x^2 + b(u)e^{-\frac{\alpha}{\beta}x}, \tag{4.40}$$

where $\alpha\, a_2(u) > 0$, and $a_0(u), a_1(u), a_2(u), b(u)$ are otherwise arbitrary. From Corollary 7 the electromagnetic field is given by

$$F = \sqrt{\alpha a_2(u)/q}\, du \wedge dx. \tag{4.41}$$

Evidently, the term in $f(u,x)$ quadratic in x determines (or is determined by) the electromagnetic field. The York tensor vanishes, *i.e.*, the metric is conformally flat, if and only if $b(u) = 0$.

CHAPTER 5
SOFTWARE IMPLEMENTATIONS OF RESULTS

One of the major goals of our project was to create geometrization conditions on some of the most common matter fields in such a way that the conditions could be implemented on the computer. For this to work we needed to write the geometrization conditions where the corresponding computational algorithms could accept the minimal input for the problem and make minimal decisions. The code is split into two parts. First are the geometrization conditions for the various matter fields. Second are the functions which reconstruct the field given a metric which satisfies the conditions. The input for these algorithms is a metric tensor g. The output of the geometrization condition functions is verification whether the given metric is a solution to Einstein's equations with the corresponding matter field. In the case that the metric fails to satisfy the geometrization conditions, a set of equations which the metric must satisfy for it to be a solution to Einstein's equations can be requested. These algorithms neglect various physical properties (such as energy conditions) and only examine the problem mathematically. For the functions which reconstruct the field associated with the solution the input is a metric tensor which satisfies the geometrization conditions, and the output is the desired field. This code was used to test the theorems and calculate the examples throughout the text.

The code included below is for perfect fluids, real scalar fields, and $(2 + 1)$-dimensional electromagnetic fields. In the case of $(2 + 1)$-dimensional electromagnetic fields, the problem was reduced to that of scalar fields, so the same code is used.

5.1 Perfect Fluids

5.1.1 Perfect Fluid Conditions

The PerfectFluidCondition function corresponds to Theorem 1 and verifies whether a metric corresponds to a perfect fluid solution of Einstein's equations. Optionally, it returns a set of equations which a metric must satisfy to be a perfect fluid solution.

```
PerfectFluidCondition := proc(g, {output := "TF"})
   local dim, S, alpha, H, condition1, Z;

   dim := nops( DGinfo("FrameBaseVectors")):
   S := TraceFreeRicciTensor(g):

  # alpha is defined in Eq. (2.9)
   S2 := TensorInnerProduct(g, S, S, tensorindices = [2]):
   S3 := TensorInnerProduct(g, S, S2):
   alpha := -(dim^2 / ((dim - 1)*(dim - 2))*S3)^(1/3):

  # H is defined as K in Eq. (2.13)
   H := evalDG( 1/alpha*S - 1/dim*g):

  # condition1 is defined in Eq. (2.11)
   condition1 := SymmetrizeIndices( H &t H, [2, 3], "SkewSymmetric"):

   if (output = "TF") then
      Z := DGinfo( condition1, "CoefficientSet"):
      if (Z <> {0}) then
         return false;
      end if;
   end if;

   if (output = "TF") then
      return true:
   else
      condition1;
   end if;
```

```
end proc:
```

5.1.2 Perfect Fluid Reconstruction

The PerfectFluidData function is given a perfect fluid spacetime metric and returns the four velocity u, the energy-density μ, and the pressure p corresponding to the metric.

```
PerfectFluidData := proc(g)
   local dim, S, R, alpha, beta, u, m, a, H, frameVectors, manifoldName,
      frameForms;

   dim := nops( DGinfo("FrameBaseVectors")):
   manifoldName := DGinfo( "CurrentFrame"):
   frameForms := DGinfo(manifoldName, "FrameBaseForms"):
   frameVectors := DGinfo( manifoldName, "FrameBaseVectors"):
   S := TraceFreeRicciTensor(g):
   R := RicciScalar(g):

  # alpha is defined in Eq. (2.9)
   S2 := TensorInnerProduct(g, S, S, tensorindices = [2]):
   S3 := TensorInnerProduct(g, S, S2):
   alpha := -(dim^2 / ((dim - 1)*(dim - 2))*S3)^(1/3):

   # beta corresponds to the pressure as defined in Eq. (2.18)
   beta := 1/dim*(R*(1 - dim/2) + alpha);

   # H is defined as K in Eq. (2.13)
   H := evalDG( 1/alpha*S - 1/dim*g):

   for m from 1 by 1 to dim do
      if (Hook( [frameVectors[m], frameVectors[m]], H) <> 0) then
         a[m] := sqrt( Hook( [frameVectors[m], frameVectors[m]], H)):
```

```
        else
            a[m] := 0:
        end if;
    end do;

    u := RaiseLowerIndices(InverseMetric(g), DGzip( a, frameForms, "plus"), [1]);

   # four velocity, energy-density, and pressure are returned in this order
    u, simplify(alpha - beta), simplify(beta);

end proc:
```

5.2 Scalar Fields and Electromagnetic Fields

5.2.1 Scalar Field Conditions

The SFC function corresponds to Theorem 2 and Theorem 3, and verifies whether a metric corresponds to a non-null or null scalar field solution, respectively. Optionally, it returns a set of equations which a metric must satisfy to be a scalar field solution.

```
SFC := proc(g, {output := "TF"})
    local G, Gtrace, Gtwo, Gthree, Gdown, Lambda, H, test, C, dim, HHS, covH, p1,
        p2, p3, condition1, condition2, condition3, Z;

    dim := nops( DGinfo("FrameBaseVectors")):
    G := DGsimplify( EinsteinTensor(g)):
    Gdown := RaiseLowerIndices(g, G, [1, 2]):
    Gtrace := DGsimplify( ContractIndices( RaiseLowerIndices(g, G, [1]), [[1,
        2]])):
    # Gtwo is defined in Eq. (3.9)
    Gtwo := DGsimplify( TensorInnerProduct(g, G, G)):
    # Gthree is defined in Eq. (3.10)
    Gthree := DGsimplify( ContractIndices( RaiseLowerIndices(g, G, [1]) &t
        RaiseLowerIndices(g, G, [1]) &t RaiseLowerIndices(g, G, [1]), [[2, 3], [4,
        5], [1, 6]])):

    # test is defined in Eq. (3.11)
    test := simplify( (Gtwo - 1/dim*Gtrace*Gtrace)):

    # We test the condition in Eq. (3.11) to see if the scalar field is non-null
        or null
    if (test <> 0) then
        # Lambda corresponds to Λ as defined in Eq. (3.16)
        Lambda := simplify( 1/2*(1/dim* Gtrace*Gtwo - Gthree)*(Gtwo -
            1/dim*Gtrace*Gtrace)^(-1)):
```

```
    # H is defined in Eq. (3.17)
    H := DGsimplify( evalDG( Gdown + Lambda*g + 1/2*( Gtrace + dim*Lambda)*(1
        - dim/2)^(-1)*g));
else
    # Lambda is defined in Eq. (3.28)
    Lambda := -Gtrace/dim;
    # H is the trace-free Ricci tensor
    H := evalDG( Gdown - 1/dim*Gtrace*g);
end if;

C := Christoffel(g):
HHS := DGsimplify( SymmetrizeIndices( H &t H, [2, 3], "SkewSymmetric")):
covH := CovariantDerivative(H, C):

p1 := SymmetrizeIndices( H &t covH, [4, 5], "SkewSymmetric"):
p2 := RearrangeIndices( p1, [1, 3, 2, 4, 5]):
p3 := SymmetrizeIndices( RearrangeIndices( H &t covH, [1, 4, 2, 3, 5]), [4,
    5], "SkewSymmetric"):


# condition1 is Eq. (3.13)
condition1 := HHS:
# condition2 is Eq. (3.14)
condition2 := evalDG( p1 + p2 - p3):
# condition3 is Eq. (3.12)
condition3 := CovariantDerivative( Lambda, C):

if output = "TF" then
    Z := DGinfo(condition1 , "CoefficientSet"):
    if (Z <> {0}) then
        return false
    end if;
```

```
    end if;

    if (output = "TF") then
        Z := DGinfo(condition2 , "CoefficientSet"):
        if (Z <> {0}) then
            return false
        end if;
    end if;

    if (output = "TF") then
        Z := DGinfo(condition3 , "CoefficientSet"):
        if (Z <> {0}) then
            return false
        end if;
    end if;

    if (output = "TF") then
        true
    else
        condition1, condition2, condition3
    end if;

end proc:
```

5.2.2 Scalar Field Reconstruction

The SF function is given a scalar field spacetime metric and returns the associated scalar field corresponding to the metric.

```
SF := proc(g0)
```

```
local g, manifoldName, coordinates, frameForms, frameVectors, numVars, C, m,
    a, A, b, B, eq, phiSol, aa, dim, G, Gdown, Gtrace, Gtwo, Gthree, Lambda,
    H, test;

g := DifferentialGeometry:-evalDG(g0):
manifoldName := DGinfo( "CurrentFrame"):
coordinates := DGinfo(manifoldName, "FrameIndependentVariables"):
frameForms := DGinfo(manifoldName, "FrameBaseForms"):
frameVectors := DGinfo( manifoldName, "FrameBaseVectors"):
numVars := nops(frameForms):
C := Christoffel(g):

dim := nops( DGinfo("FrameBaseVectors")):
G := DGsimplify( EinsteinTensor(g)):
Gdown := RaiseLowerIndices(g, G, [1, 2]):
Gtrace := DGsimplify( ContractIndices( RaiseLowerIndices(g, G, [1]), [[1,
    2]])):
# Gtwo is defined in Eq. (3.9)
Gtwo := DGsimplify( TensorInnerProduct(g, G, G)):
# Gthree is defined in Eq. (3.10)
Gthree := DGsimplify( ContractIndices( RaiseLowerIndices(g, G, [1]) &t
    RaiseLowerIndices(g, G, [1]) &t RaiseLowerIndices(g, G, [1]), [[2, 3], [4,
    5], [1, 6]])):

# test is the requirement defined in Eq. (3.11)
test := (Gtwo - 1/dim*Gtrace*Gtrace):

# test is defined in Eq. (3.11)
if (test <> 0) then
    # Lambda corresponds to A as defined in Eq. (3.16)
    Lambda := simplify( 1/2*(1/dim* Gtrace*Gtwo - Gthree)*(Gtwo -
        1/dim*Gtrace*Gtrace)^(-1)):
```

```
    # H is defined in Eq. (3.17)
    H := DGsimplify( evalDG( Gdown + Lambda*g + 1/2*( Gtrace + dim*Lambda)*(1
        - dim/2)^(-1)*g));
else
    # Lambda is defined in Eq. (3.28)
    Lambda := -Gtrace/dim;
    # H is the trace-free Ricci tensor
    H := evalDG( Gdown - 1/dim*Gtrace*g);
end if;

for m from 1 by 1 to numVars do
    if (Hook( [frameVectors[m], frameVectors[m]], H) <> 0) then
        a[m] := sqrt( Hook( [frameVectors[m], frameVectors[m]], H)):
    else
        a[m] := 0:
    end if;
end do;

# solve for the scalar field
A := DGzip( a, frameForms, "plus");
eq := DGinfo( evalDG( convert(A, DGtensor) - CovariantDerivative(
    b(op(coordinates)), C)), "CoefficientSet"):
phiSol := pdsolve( eq):

# return the solution or solutions for the scalar field
if (nops([phiSol]) = 1) then
    phiSol := rhs( op(simplify( pdsolve( eq), symbolic)));
else
    phiSol := pdsolve( eq);
    aa := {}:
    for m from 1 to nops([phiSol]) do
        aa := aa union {rhs( phiSol[m][1])}
```

```
        end do;
    end if;
end proc:
```

5.2.3 Utility: Cosmological Constant

The function Lcheck is a utility function to compute the cosmological constant given a scalar field spacetime metric.

```
Lcheck := proc(g)
local G, Gtrace, Gtwo, Gthree, Gdown, Lambda, dim;

    dim := nops( DGinfo("FrameBaseVectors")):
    G := DGsimplify( EinsteinTensor(g)):
    Gdown := RaiseLowerIndices(g, G, [1, 2]):
    Gtrace := DGsimplify( ContractIndices( RaiseLowerIndices(g, G, [1]), [[1,
        2]])):
    Gtwo := DGsimplify( TensorInnerProduct(g, G, G)):
    Gthree := DGsimplify( ContractIndices( RaiseLowerIndices(g, G, [1]) &t
        RaiseLowerIndices(g, G, [1]) &t RaiseLowerIndices(g, G, [1]), [[2, 3], [4,
        5], [1, 6]])):
    # Lambda is defined in Eq. (3.16)
    Lambda := simplify( 1/2*(1/dim* Gtrace*Gtwo - Gthree)*(Gtwo -
        1/dim*Gtrace*Gtrace)^(-1)):
end proc:
```

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