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Using the Reshetikhin-Turaev Link Invariant Approach with Non-Semisimple Categories

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USING THE RESHETIKHIN–TURAEV LINK INVARIANT APPROACH WITH NON-SEMISIMPLE CATEGORIES

by

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ABSTRACT

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Invariants of knots and links are useful because they give rise to invariants of 3-manifolds. In particular, combinatorial link invariants give rise to combinatorial invariants of 3-manifolds, which are hard to come by using traditional methods from classical topology. The Reshetikhin–Turaev approach, which is based in quantum topology, develops link invariants using semisimple ribbon categories. However, a large class of algebraically interesting ribbon categories are non-semisimple and so give trivial link invariants via the Reshetikhin–Turaev method. We modify the Reshetikhin–Turaev method to make it suitable for non-semisimple ribbon categories. We discuss explicitly the following three examples: semisimple modules for the abelian quantum group, non-semisimple modules for $U_q(\mathfrak{gl}(1|1))$, and non-semisimple modules for the unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$.

(114 pages)

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Adam Robertson

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LIST OF SYMBOLS

L	a (framed) link
D	a diagram of a link
N_L	tubular neighborhood of L
RI	the first Reidemeister move
FRI	the first Reidemeister move for framed links
RII	the second Reidemeister move
RIII	the third Reidemeister move
$\mathbb{C}[G]$	the group algebra of the group G
$[-,-]$	the Lie bracket
$\otimes$	the tensor product
$\cdot$	multiplication operation of an algebra
i	the unit
Δ	the coproduct
ε	the counit
S	the antipode
τ	the (graded) swap map
$T(V)$	the tensor algebra of V
Vect_k	category of vector spaces over k
vect_k	category of finite dimensional vector spaces over k
F	the Reshetikhin–Turaev functor
R	a universal R-matrix
R_{12}	$R \otimes 1$
R_{23}	$1 \otimes R$
R_{13}	$\alpha_i \otimes 1 \otimes \beta_i$, where $R = \sum_i \alpha_i \otimes \beta_i$
$V^{\vee}$	dual of V
ev_V	left evaluation map
coev_V	left coevaluation map
$\widehat{\mathrm{ev}}_V$	right evaluation map
$\widehat{\mathrm{coev}}_V$	right coevaluation map
A_q	the abelian quantum group
$\mathbb{C}^{1\|1}$	the two-dimensional graded $\mathbb{C}$-vector space
$\mathbb{C}[\![\hbar]\!]$	ring of formal power series in $\hbar$ over $\mathbb{C}$
$\mathfrak{gl}(1\|1)$	the general linear Lie superalgebra
$U_q(\mathfrak{gl}(1\|1))$	the quantum universal enveloping algebra of $\mathfrak{gl}(1\|1)$
$U_\hbar(\mathfrak{gl}(1\|1))$	the $\hbar$-adic completion of $U_q(\mathfrak{gl}(1\|1))$
$\mathrm{Ann}(h_1 + h_2)$	the set of annihilators of $h_1 + h_2$
$K(\lambda)$	the Kac module of weight λ
$\hat{\otimes}$	completed tensor product
$\mathfrak{sl}_2(\mathbb{C})$	the special linear Lie algebra of order 2
$U(\mathfrak{sl}_2(\mathbb{C}))$	the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$
$U_q(\mathfrak{sl}_2(\mathbb{C}))$	the quantum group of $\mathfrak{sl}_2(\mathbb{C})$
$U_q^H(\mathfrak{sl}_2(\mathbb{C}))$	the unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$
$\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$	the restricted unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$
$\mathcal{T}_r$	the rth Chebyshev polynomial

Chapter 1

INTRODUCTION

1.1

Motivation

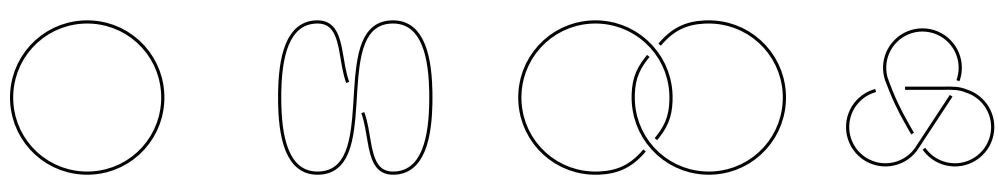
A mathematical *knot* resembles a physical knot in that it is a strand in $\mathbb{R}^3$ which can be looped, wrapped around itself and so forth after which the two ends of the strand must be joined to form a closed loop without boundary. A *link* is a generalization of a knot that is composed of one or more strands. A *framed link* is a link with a smooth choice of unit normal vectors at each point for each strand. Thus, we can think of each strand of a framed link as a ribbon with one edge the unframed link. When we draw links, we draw projections onto $\mathbb{R}^2$.

A few examples of unframed links are given in Figure 1.1. An example of a framed link is given in Figure 1.2.

Two links are equivalent (isotopic) if one can be deformed in $\mathbb{R}^3$ to match the other. For example, the knots in Figures 1.1a and 1.1b are equivalent because the twist in 1.1b can be undone. On the other hand, if we consider them as framed knots, they may not be isotopic. Figure 1.2 is one choice of framing of Figure 1.1b which is not isotopic to the framed unknot.

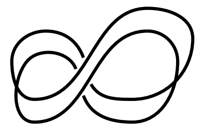
Equivalence of links partitions the set of links into equivalence classes. A primary goal of link theory is to construct link invariants, which are functions from the set of links onto a target set that are constant on equivalence classes of links. Ideally, comparison of objects in the target set should be easier than direct comparison of links.

Figure 1.1: Examples of Links



(a) The Unknot (b) Knot with a Twist (c) The Hopf Link (d) The Trefoil Knot

Figure 1.2: Framed Link



An important motivation for developing good link invariants is that link invariants are useful in the construction of invariants of 3-manifolds. The process is roughly as follows. Begin with a framed link L in the 3-sphere S^3 with $n \in \mathbb{Z}^+$ many components. Fatten each strand of L to its tubular neighborhood $N_L = \sqcup_n S^1 \times D^2$. Remove N_L from S^3. Now, along the boundary of $S^3 \setminus N_L$, glue in $\sqcup_n S^1 \times D^2$ by identifying the boundary of each component with the boundary of $S^3 \setminus N_L$ via the framing. The result is a closed, orientable 3-manifold which we label $S^3(L)$.

The Lickorish-Wallace theorem [27, Theorem 6] states that every compact, connected orientable 3-manifold can be obtained as $S^3(L)$ for some framed link L. Therefore, link invariants are useful in constructing invariants of 3-manifolds. This raises the following question:

Question. How can we construct interesting invariants of links?

In this thesis, we give a modern categorical approach for constructing link invariants.

1.2
Methods for Constructing Link Invariants

1.2.1
Classical Invariants

A rough pattern is that invariants which are easy to describe tend not to be very powerful or useful. For example, the simplest link invariant is the trivial invariant which maps every link to zero. This invariant is useless in practice. Another useless invariant is the identity map on the set of links.

A variety of link invariants exist which are either topological or combinatorial in nature. For example, one invariant simply counts the number of components in the link. This invariant, while extremely easy to describe, is quite limited. It can distinguish between the unknot and the Hopf link, but cannot distinguish between the Hopf link and a pair of disjoint unknots.

Another classical invariant is 3-colorability, see [11, Section 2.1]. A 3-coloring of a link diagram is an assignment of one of three colors, $\{1, 2, 3\}$ to each segment of the diagram in such a way that at each crossing either all three incident segments have the same color, or the segment crossing over is colored by a, while the two segments that form the strand crossing under are labeled by b and c, where $a, b, c \in \{1, 2, 3\}$ are mutually distinct. The 3-colorability invariant counts the total number of 3-colorings of the link. This invariant, while relatively simple to describe, is quite difficult to compute on an arbitrary knot.

An example of a classical invariant for framed links that uses the framing is the linking matrix. Choose an ordering of the components of the link, $\{1, \ldots, n\}$. Choose two distinct strands i and j. At each crossing of i and j, assign a positive sign if i crosses over j and a negative sign if j crosses over i. The linking number of i and j, denoted $L(i, j)$, is the sum of the signs of the crossings of i and j. The linking number of the ith strand with itself, labeled $L(i, i)$, is computed by first drawing an imaginary strand on the endpoints of each normal vector that gives the framing, then computing the linking number between the ith strand and this imaginary strand. We collect these linking numbers into a matrix M where the (i, j)th entry of M is $L(i, j)$. This invariant is combinatorial in nature, and gives rise to a combinatorial invariant of 3-manifolds.

1.2.2
Quantum Invariants

A more recent trend in link theory recasts the question of developing link invariants in terms of representations of algebraic objects. Consider a familiar object, such as a finite group or a Lie algebra. Given a diagram of a link, break it into elementary pieces consisting of cups, caps, crossings, and straight vertical lines. The strategy is to assign a morphism from the category of representations of the object to each elementary piece. Somewhat surprisingly, assuming the object satisfies the conditions of being a ribbon Hopf algebra, this strategy can be shown to produce a valid link invariant.

In 1990, Reshetikhin and Turaev formalized the algebraic approach of generating a link invariant by "categorifying" the set of knots and defining a functor from this category onto a ribbon category, see [23]. In their example, Reshetikhin and Turaev used the famous DeConcini-Kac quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$, which they proved to have a ribbon category of modules. In the particular example of the DeConcini-Kac quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$, the Reshetikhin–Turaev approach recovers the famous Jones polynomial, see [13] and [11, Section 6.4.1]. Reshetikhin and Turaev further showed that the link invariant gives rise to an invariant of 3-manifolds, called the Witten–Reshetikhin–Turaev invariant, see [28] and [22]. This invariant is the starting point of quantum topology and began a flurry of interest in finding other quantum invariants of links and 3-manifolds.

Question. Which ribbon categories produce interesting link invariants?

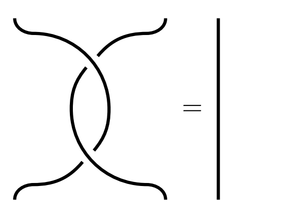
Some produce invariants which already have other presentations as classical invariants. For example, the representation theory of abelian quantum group, namely the cyclic group $\mathbb{Z}_\mathfrak{r}$ with a nonstandard quasi-triangular structure, recovers the linking matrix modulo $\mathfrak{r}$; this connection is explored in Chapter 5.

A limitation of the Reshetikhin–Turaev approach is that, as written, it yields a nonzero link invariant only if the ribbon category satisfies a very strong constraint called semisimplicity. In their work, Reshetikhin and Turaev *semisimplify* the category of $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2(\mathbb{C}))$-modules. Direct application of the Reshetikhin–Turaev approach to a non-semisimple category gives the zero map on every link. However, some very interesting and naturally occurring ribbon categories are non-semisimple. This leads to another interesting question:

Question. Can we modify Reshetikhin–Turaev to produce interesting link invariants from non-semisimple categories?

We show in this thesis how to modify the Reshetikhin–Turaev process using a simple cutting procedure, following Geer, Patureau-Mirand, and Turaev, to produce nontrivial link invariants from non-semisimple categories, see [9].

Classical representation theory produces symmetric monoidal categories, which are degenerate ribbon categories. Familiar examples include representations of finite groups, Lie algebras, Lie groups, etc. Symmetric categories are poor candidates for creating link invariants because they lose topological information. In a symmetric category, the following relation holds:



Thus, a link invariant constructed with a symmetric category amounts to an invariant under which all of the strands of the link can be separated into disjoint unknots. Representation theory of quantum groups produces non-symmetric ribbon categories. This allows us to import techniques from quantum algebra into topology.

1.3
Thesis Work

In this thesis, we construct three examples of link invariants from ribbon categories. The first example is built from a quantum deformed representation theory of the finite group $\mathbb{Z}_r$. This example was done previously in [19] and [10]; however, we provide complete detail and supply some of the required calculations that are omitted in the original explanation. We prove the following theorem:

Theorem 1.3.1 (Theorems 5.1.2 and 5.2.1)**.** *Let A_q be the group algebra of $\mathbb{Z}_r$ with Hopf algebra structure as given in Example 3.1.24 and R-matrix and twist as in Definition 5.1.1. Then the modules of A_q form a finite semisimple ribbon category. Moreover, the invariant that is produced from applying the Reshetikhin–Turaev method is the linking matrix modulo r.*

The second example is a quantization of the super Lie algebra $\mathfrak{gl}(1|1)$. This example has also been done previously, see [26], [24], and [15]. We prove the following theorem:

Theorem 1.3.2 (Theorems 6.4.14, 6.5.2, and 6.5.5)**.** *Let $U_q(\mathfrak{gl}(1|1))$ be the quantum enveloping superalgebra of $\mathfrak{gl}(1|1)$ at generic q as defined in Definition 6.2.4. The category of weight modules of $U_q(\mathfrak{gl}(1|1))$ is a non-semisimple ribbon category. Although the Reshetikhin–Turaev process applied to this category gives a trivial invariant, the cutting procedure described in Theorem 6.5.5 makes the invariant nontrivial.*

We compute this example in complete detail. We first prove that the category of modules is generically semisimple and use this to give a categorical construction of its ribbon structure. We supply the details for all of the equations that must be checked to verify the category is ribbon. As in the original, we give the details of the cutting procedure. Furthermore, we compute an example to show that the resulting invariant is nontrivial.

The third example is the unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$, which is a modification of the DeConcini-Kac quantum group. We prove the following theorem:

Theorem 1.3.3 (Corollaries 7.3.1 and 7.4.7)**.** *Let $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ be the restricted unrolled quantum group at q a root of unity from Definition 7.1.2. The category of weight modules of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ is a non-semisimple ribbon category. The cutting procedure described in Theorem 6.5.5, adapted to $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$-modules, makes the invariant nontrivial.*

In this example, we show that the category of modules is generically semisimple and classify all simple objects. Generic simplicity can again be used to show that there is a ribbon structure;

see the paper of Costantino, Geer, and Patureau-Mirand, [3]. We work out the details to check the cutting procedure.

The layout of the thesis is as follows. Chapters 2, 3, and 4 summarize prerequisite information from knot theory, representation theory, and category theory. In Chapters 5, 6, and 7 we study the three main examples: the abelian quantum group (Chapter 5), $\mathsf{U}_{\mathsf{q}}(\mathfrak{gl}(1|1))$ (Chapter 6), and the unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$ (Chapter 7).

Throughout this paper assume, unless otherwise indicated, that all vector spaces are finite dimensional and are over the field $\mathbb{C}$.

Chapter 2
KNOT THEORY BACKGROUND

In this chapter we record some elementary definitions from knot and link theory. In particular, we define link invariants and state Reidemeister's Theorem, which gives a set of moves on link diagrams that should be checked to verify that a candidate function is a link invariant. Those unfamiliar with knot theory should consult [11] for a more gentle introduction and thorough treatment of core ideas.

2.1
Definitions of a Knot and a Link

Intuitively, a knot can be thought of as a piece of string which is entangled with itself and with the two ends glued together to form a closed loop. A link is a more general version of a knot that is made up of n many strands.

Definition 2.1.1 (Link and Knot)**.** *A link is a disjoint piecewise linear embedding of* n *copies of the unit circle into* $\mathbb{R}^3$*. The image of each unit circle is called a component. A knot is a link with exactly one component.*

Definition 2.1.2 (Framed Link and Framed Knot)**.** *A framed link is a disjoint piecewise linear embedding of* n *annuli into* $\mathbb{R}^3$*. The image of each annulus is called a component. A framed knot is a framed link with exactly one component.*

When visualizing links, we typically draw them as regular projections onto $\mathbb{R}^2$ in such a way that when two strands are crossed we remember which strand crosses over the other.

Definition 2.1.3 (Regular Projection)**.** *A regular projection* $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ *of a link is a projection which satisfies the following conditions:*

1. *Each point in* $\mathbb{R}^2$ *is the image of at most two points in* $\mathbb{R}^3$ *(points in* $\mathbb{R}^2$ *which are the image of two points in* $\mathbb{R}^3$ *are called* ***double points****),*

Figure 2.1: Link Diagram

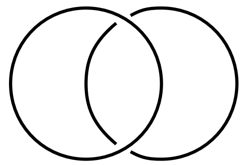
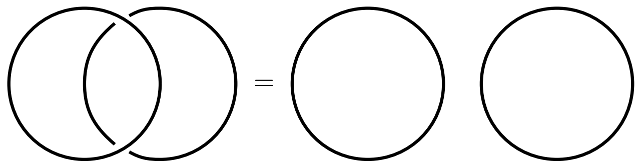


Figure 2.2: Isotopic Links



2. *There are only finitely many double points, and*

3. *In a small neighborhood around each double point, the left strand above the double point is the right strand below the double point.*

It can be shown that every link admits a regular projection, see [11, Theorem 1.13].

Definition 2.1.4 (Link Diagram)**.** *A link diagram of a link is a regular projection with an assignment of an over or under crossing to every double point.*

By convention, we represent crossings using breaks in the strands of our link diagrams. The broken strand is assumed to cross under the unbroken strand. For example, Figure 2.1, the strand on the left sits on top of the strand on the right.

Framed links can be represented using the same diagrams as standard links; we use this convention throughout the paper. The difference between framed links and standard links is that framed links admit twists. A *twist* of a link is a region in which the normal vector that gives the framing spirals around strand as it moves along it. Because the strand is a closed loop, each strand has some whole number of complete twists and no partial twists. By adding a loop into the strand, the vectors normal to each point of the strand can be made to all point in the same direction, see [11, section 3.1]. When the normal vectors all point out toward the reader, this is called the *blackboard framing.* Throughout this paper, we assume the blackboard framing whenever framed link diagrams are drawn.

By inspection, it is clear that the link with diagram in Figure 2.1 can be pulled apart to become two disjoint copies of the unknot, see Figure 2.2. This leads to a definition of link equivalence, or isotopy.

Definition 2.1.5 (Isotopy of (Framed) Links). *Two (framed) links* L *and* L' *are (framed) isotopic if there is a family of homeomorphisms* $h_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0, 1]$, *such that the following conditions are satisfied:*

1. $h_0 = \mathrm{id}$,
2. $h_1(L) = L'$, *and*
3. *The map* $\Pi : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3 \times [0, 1]$ *defined by* $(x, t) \mapsto (h_t(x), t)$ *is a homeomorphism.*

2.2
Link Invariants

Isotopy of links is an equivalence relation on the set of links. One of the central questions in the study of knot theory is to determine the equivalence classes. Link invariants are functions which are constructed to assist in accomplishing this goal.

Definition 2.2.1 (Link Invariant). *Given a set* s, *a (framed) link invariant* f *is a function*

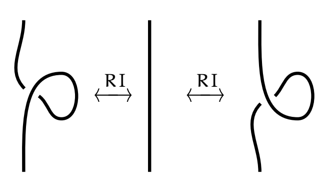
$$f : \frac{\text{(framed) links}}{\text{(framed) isotopy}} \to s.$$

Because we do not typically study links directly, we want to define link invariants, not on the set of links, but on the set of link diagrams. The Reidemeister theorem allows us to do this.

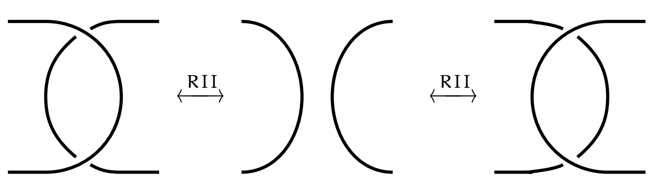
We first state the Reidemeister theorem for standard (not framed) links.

Theorem 2.2.2 ([11, Theorem 1.26]). *Two links are isotopic if and only if they are related by a finite sequence of the following moves (called the Reidemeister moves):*

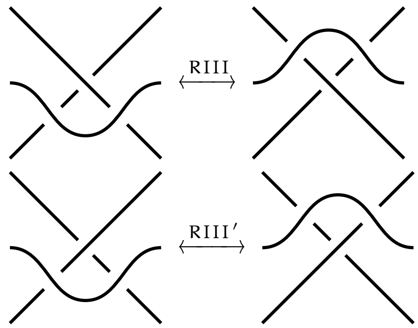
0. *(Planar isotopy) Any strand can be pulled, bent, and moved freely along the plane as long as the movement does not add or remove any crossings.*
1. *(*RI*)*



2. *(*RII*)*



3. (RIII)



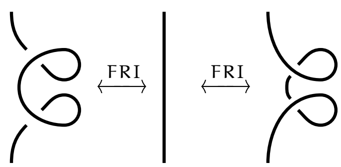
In other words the Reidemeister theorem implies that

$$\frac{\text{links}}{\text{isotopy}} \simeq \frac{\text{link diagrams}}{\text{Reidemeister moves}}.$$

The framed link version of Reidemeister's theorem is almost exactly the same as the standard version.

Theorem 2.2.3 ([11, Theorem 3.5]). *Two framed links are isotopic if and only if they are related by a finite sequence of moves* 0), 2), *and* 3) *from Theorem 2.2.2 and the following move:*

1. *(FRI)*



The standard (RI) move allows us to pull a looped string straight, while (FRI) move allows us to "cancel out" twists.

We now give an example of a simple link invariant, see [11, Theorem 3.9], where the proof that this function is an invariant is given as an exercise for the reader.

Example 2.2.4. *Let* L *be a framed oriented link with* n *components, and let* D *be a diagram for* L. *Label the components* $\{1, \ldots, n\}$. *For* $i, j \in \{1, \ldots, n\}$, *the number* $M(i, j)$ *is the sum of the signs of the crossings between components* i *and* j *in* D. *Each* $M(i, j)$ *is called the linking number of* i *and* j. *If* $i = j$, *then* $M(i, i)$ *is called the self-linking number of* i. *The self-linking number of the* i*th strand is calculated by drawing an imaginary strand at the tip of each normal vector of the* i*th strand and summing the signs of the crossings of the* i*th strand with this imaginary strand. These linking numbers are independent of the choice of diagram, and are an invariant of framed oriented links. We collect the linking numbers of a link into an* $n \times n$ *matrix* M, *called the linking matrix, in which the* (i, j)*th entry is the linking number of* i *and* j.

Chapter 3

HOPF ALGEBRA AND LIE ALGEBRA BACKGROUND

In the following chapter, we give a very brief introduction to Hopf algebras and Lie algebras. Hopf algebras are bialgebras, meaning they have a compatible algebra and coalgebra structure, with an antipode. The three main examples of objects from which we derive ribbon categories, A_q, $U_q(\mathfrak{gl}(1|1))$, and the unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$ are all Hopf algebras. Refer to [4] or [21] for a more thorough treatment of Hopf algebras.

We briefly discuss modules over a ring and give as an example the modules of the group algebra of the cyclic group $\mathbb{Z}_r$. These modules are the objects in the ribbon category that we study in Chapter 5.

We also give a brief introduction to Lie algebras, paying particular attention to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ plays an essential role in the construction of the unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$.

Let k be a field. Throughout this section, all vector spaces and tensor products are assumed to over k, unless otherwise specified.

3.1

Algebras, Coalgebras, Bialgebras, and Hopf Algebras

Definition 3.1.1 (Algebra)**.** *An associative, unital algebra is a vector space* A *over* k *with linear maps* $\cdot : A \otimes A \to A$*, called multiplication, and* $i : k \to A$*, called the unit, such that the following diagrams commute:*

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\cdot\otimes \mathrm{id}} & A\otimes A \\ {\scriptstyle \mathrm{id}\otimes\cdot}\downarrow & & \downarrow{\scriptstyle \cdot} \\ A\otimes A & \xrightarrow[\cdot]{} & A \end{array} \qquad \begin{array}{ccc} k\otimes A & \xrightarrow{i\otimes \mathrm{id}} & A\otimes A \\ {\scriptstyle l}\downarrow & \swarrow{\scriptstyle \cdot} & \\ A & & \end{array} \qquad \begin{array}{ccc} A\otimes k & \xrightarrow{\mathrm{id}\otimes i} & A\otimes A \\ {\scriptstyle r}\downarrow & \swarrow{\scriptstyle \cdot} & \\ A & & \end{array} \tag{3.1}$$

where l *is the canonical isomorphism (left scalar multiplication)*

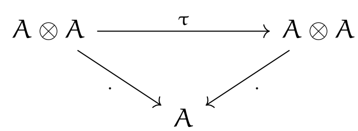
$$l : k \otimes A \to A$$
$$\lambda \otimes a \mapsto \lambda a$$

and r *is right scalar multiplication.*

The left diagram is the condition that multiplication is associative. The other two diagrams imply the existence of a vector $1 \in A$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$, so that $i(\lambda) = \lambda 1$ for $\lambda \in k$.

In this paper, we assume that all algebras are associative and unital.

Definition 3.1.2 (Commutative Algebra)**.** *A commutative algebra is an algebra such that the following diagram commutes:*



where $\tau : A \otimes B \to B \otimes A$ *is the "swap map" defined by* $\tau(a \otimes b) = b \otimes a$*.*

Example 3.1.3. *The field* k *is a commutative algebra over itself with multiplication given by the standard product and unit the identity element* $1 \in k$*.*

Example 3.1.4. *Let the ground field be* $\mathbb{C}$*. Let* G *be a finite group. Let* $\mathbb{C}[G]$ *be a vector space over* $\mathbb{C}$ *of dimension* $|G|$*. Choose a basis and label each basis vector with an element of* G*. An arbitrary element in* $\mathbb{C}[G]$ *is a sum of the form*

$$\sum_{g \in G} a_g e_g,$$

where $a_g \in \mathbb{C}$*. Define multiplication of basis vectors as*

$$e_g \cdot e_h = e_{gh}.$$

Extend this product linearly to define multiplication on $\mathbb{C}[G]$*.*

Define the unit map $i(\lambda) = \lambda e_0$ *for* $\lambda \in \mathbb{C}$*, where* $0 \in G$ *is the identity element. These choices of product and unit make* $\mathbb{C}[G]$ *into an algebra, called the group algebra of* G*. Note that this algebra is commutative exactly when the group is abelian.*

Example 3.1.5. *Let* V *be a vector space over* $\mathbb{C}$*. Let* $T^n(V) = V^{\otimes n}$ *for* $n \geq 1$*, and define* $T^0(V) = k$*.*

The tensor algebra $\mathsf{T}(V)$ *is the vector space*

$$\bigoplus_{n=0}^{\infty} \mathsf{T}^n(V)$$

with multiplication given by the canonical isomorphisms

$$\mathsf{T}^n(V) \otimes \mathsf{T}^m(V) \simeq \mathsf{T}^{n+m}(V)$$

extended linearly and unit $1 \in \mathsf{T}^0(V)$.

Lemma 3.1.6. *The tensor algebra* $\mathsf{T}(V)$ *is commutative if and only if the dimension of* V *is* 1.

Proof. Assume that V has dimension $n \geq 2$. Let v_1 and v_2 be linearly independent vectors in V and extend them to a basis $\{v_1, v_2, \ldots, v_n\}$ of V. A basis for $\mathsf{T}^2(V) = V \otimes V$ is $\{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$. Both $v_1 \cdot v_2 = v_1 \otimes v_2$ and $v_2 \cdot v_1 = v_2 \otimes v_1$ are basis vectors, so $v_1 \cdot v_2 \neq v_2 \cdot v_1$.

If the dimension of V is 1, then $V \simeq \mathbb{C}$. A vector in $\mathsf{T}^k(\mathbb{C})$ is of the form $c_1 \otimes \cdots \otimes c_k = c_1 \cdots c_k (1 \otimes \cdots \otimes 1)$. Since scalars commute, $\mathsf{T}(\mathbb{C})$ is commutative. In fact, $\mathsf{T}(\mathbb{C})$ is isomorphic to $\mathbb{C}[x]$ by the map $\phi : \mathsf{T}(\mathbb{C}) \to \mathbb{C}[x]$ which sends 1 to x. □

Example 3.1.7. *Let* A *be an algebra. Consider the vector space* $A \otimes A$. *Define a multiplication on* $A \otimes A$ *by*

$$\begin{aligned} \cdot : (A \otimes A) \otimes (A \otimes A) &\to A \otimes A \\ (a_1 \otimes a_2) \otimes (a_3 \otimes a_4) &\mapsto a_1 \cdot a_3 \otimes a_2 \cdot a_4, \end{aligned}$$

where $\cdot$ *on the right hand side is the product in* A. *Define a unit on* $A \otimes A$ *by*

$$\begin{aligned} i : k \otimes k &\to A \otimes A \\ \lambda \otimes \lambda' &\mapsto i(\lambda) \otimes i(\lambda') \end{aligned}$$

where i *on the right is the unit in* A *(recall that* $k \otimes k \simeq k$*). Then* $A \otimes A$ *is an algebra.*

Throughout this paper, this is the assumed algebra structure on $A \otimes A$.

Definition 3.1.8 (Ideal)**.** *An ideal of an algebra* A *is a vector subspace* $I \subset A$ *such that* $\cdot$ *restricts to a map*

$$\cdot : A \otimes I \oplus I \otimes A \to I.$$

Definition 3.1.9 (Algebra Homomorphism)**.** *An algebra homomorphism* $\phi : A \to B$ *is a homomorphism of the underlying vector spaces which also satisfies the following condition:*

$$\phi(a \cdot a') = \phi(a) \cdot \phi(a')$$

for all $a, a' \in A$.

The condition that ϕ is an algebra homomorphism is equivalent to the condition that the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\cdot_A} & A \\ {\scriptstyle \phi\otimes\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ B \otimes B & \xrightarrow[\cdot_B]{} & B \end{array}$$

where $\cdot_A$ and $\cdot_B$ represent multiplication in A and B, respectively.

Definition 3.1.10 (Coalgebra)**.** *A coassociative, counital coalgebra is a vector space* A *over* k *with linear maps* $\Delta : A \to A \otimes A$*, called the coproduct, and* $\varepsilon : A \to k$*, called the counit, such that the following diagrams commute:*

$$\begin{array}{ccc} A\otimes A\otimes A & \xleftarrow{\Delta\otimes \mathrm{id}} & A\otimes A \\ {\scriptstyle \mathrm{id}\otimes\Delta}\uparrow & & \uparrow{\scriptstyle\Delta} \\ A\otimes A & \xleftarrow[\Delta]{} & A \end{array} \qquad \begin{array}{ccc} k\otimes A & \xleftarrow{\varepsilon\otimes \mathrm{id}} & A\otimes A \\ {\scriptstyle \hat{l}}\uparrow & \nearrow_{\Delta} & \\ A & & \end{array} \qquad \begin{array}{ccc} A\otimes k & \xleftarrow{\mathrm{id}\otimes\varepsilon} & A\otimes A \\ {\scriptstyle \hat{r}}\uparrow & \nearrow_{\Delta} & \\ A & & \end{array} \tag{3.2}$$

where $\hat{l}$ *is the canonical isomorphism*

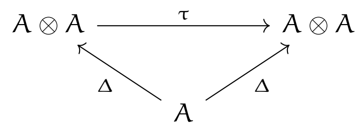
$$\begin{aligned} \hat{l} : A &\to k \otimes A \\ a &\mapsto 1 \otimes a \end{aligned}$$

and $\hat{r}$ *is the canonical isomorphism*

$$\begin{aligned} \hat{r} : A &\to A \otimes k \\ a &\mapsto a \otimes 1. \end{aligned}$$

Intuitively, the definition of a coalgebra is the same as the definition of an algebra but with all of the arrows in the diagrams reversed. In this paper, we assume that all coalgebras are coassociative and counital.

Definition 3.1.11 (Cocommutative Coalgebra)**.** *A cocommutative coalgebra is a coalgebra such that* *the following diagram commutes:*



Example 3.1.12. *The ground field* k *is a coalgebra with coproduct* $\Delta(\lambda) = \lambda \otimes 1$ *for* $\lambda \in k$ *and counit the identity map.*

Example 3.1.13. *Let* $\mathbb{C}[G]$ *be the group algebra of a finite group* G*. Define the coproduct on basis vectors in* $\mathbb{C}[G]$ *to be* $\Delta(e_g) = e_g \otimes e_g$*. Define the counit on basis vectors to be* $\varepsilon(e_g) = 1$*. This coalgebra is cocommutative.*

Example 3.1.14. *Let* $T(V)$ *be the tensor algebra. For the term* $T^0(V) = k$*, define*

$$\Delta(\lambda) = \lambda \otimes 1$$

for $\lambda \in k$*. Note that* $\lambda \otimes 1 = 1 \otimes \lambda$*. For* $n \in \mathbb{Z}^{>0}$*, define on* $v_1 \cdots v_n \in T^n(V)$*,*

$$\Delta(v_1 \cdots v_n) = \sum_{i=0}^{n} \sum_{\sigma \in sh_{i,n-i}} v_{\sigma(1)} \cdots v_{\sigma(i)} \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)} \tag{3.3}$$

where $sh_{i,n-i}$ *denotes the* $(i, n-i)$*-shuffles of* $\{1, \ldots, n\}$*. Define the counit to be the canonical projection onto* $T^0(V)$*. These choices of coproduct and counit make* $T(V)$ *into a coalgebra. Notice that*

$$\tau\left(\sum_{i=0}^{n} \sum_{\sigma \in sh_{i,n-i}} v_{\sigma(1)} \cdots v_{\sigma(i)} \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)}\right) = \sum_{i=0}^{n} \sum_{\sigma \in sh_{i,n-i}} v_{\sigma(i+1)} \cdots v_{\sigma(n)} \otimes v_{\sigma(1)} \cdots v_{\sigma(i)}.$$

Because σ *ranges over all shuffles, each term in Equation* (3.3) *appears once in the sum above. Therefore,* $T(V)$ *is cocommutative.*

Example 3.1.15. *Let* A *be a coalgebra. Consider the vector space* $A \otimes A$*. Define a coproduct on* $A \otimes A$ *by*

$$\begin{aligned} \Delta : A \otimes A &\to A \otimes A \otimes A \otimes A \\ a_1 \otimes a_2 &\mapsto (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)(a_1 \otimes a_2) \end{aligned}$$

where Δ *on the right is the coproduct in* A*. Define a counit by*

$$\varepsilon : A \otimes A \to k \otimes k \simeq k$$
$$a_1 \otimes a_2 \mapsto \varepsilon(a_1) \otimes \varepsilon(a_2)$$

where ε *on the right is the counit in* A*. Then* $A \otimes A$ *is a coalgebra.*

Throughout this paper, this is the assumed coalgebra structure on $A \otimes A$.

Definition 3.1.16 (Coideal)**.** *A coideal of a coalgebra A is a vector subspace* $I \subset A$ *such that*

$$\Delta : I \to A \otimes I \oplus I \otimes A.$$

Definition 3.1.17 (Coalgebra Homomorphism)**.** *A coalgebra homomorphism* $\phi : B \to A$ *is a homomorphism of the underlying vector spaces for which the following diagram commutes:*

$$\begin{array}{ccc} A \otimes A & \xleftarrow{\Delta_A} & A \\ \uparrow{\scriptstyle \phi\otimes\phi} & & \uparrow{\scriptstyle \phi} \\ B \otimes B & \xleftarrow[\Delta_B]{} & B \end{array}$$

where Δ_A *and* Δ_B *are the coproducts on* A *and* B*, respectively.*

Given a vector space with an algebra structure and a coalgebra structure, a priori there is no guarantee that these two structures are compatible. The following theorem is a compatibility condition.

Theorem 3.1.18. *Let* A *be a* k*-vector space with algebra structure* $(A, \cdot, i)$ *and coalgebra structure* (A, Δ, ε)*. Then* $\cdot$ *and* i *are homomorphisms of coalgebras if and only if* Δ *and* ε *are homomorphisms of algebras.*

Proof. The proof amounts to checking that the commuting diagrams implied by each condition are the same. The multiplication map $\cdot$ is a coalgebra homomorphism if and only if the following diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\quad \cdot \quad} & A \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & \\ A \otimes A \otimes A \otimes A & & \downarrow{\scriptstyle \Delta} \\ \downarrow{\scriptstyle id\otimes\tau\otimes id} & & \\ A \otimes A \otimes A \otimes A & \xrightarrow[\cdot\otimes\cdot]{} & A \otimes A \end{array} \tag{3.4}$$

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\cdot} & A \\ {\scriptstyle \varepsilon\otimes\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ k \otimes k & \xrightarrow[\rho]{} & k \end{array} \tag{3.5}$$

where $\rho : k \otimes k \to k$ is the canonical isomorphism. The unit map i is a coalgebra homomorphism if and only if the following diagrams commute:

$$\begin{array}{ccc} k & \xrightarrow{i} & A \\ {\scriptstyle \rho^{-1}}\downarrow & & \downarrow{\scriptstyle \Delta} \\ k \otimes k & \xrightarrow[i\otimes i]{} & A \otimes A \end{array} \tag{3.6}$$

$$\begin{array}{ccc} k & \xrightarrow{i} & A \\ & {\scriptstyle \mathrm{id}}\searrow \quad \swarrow{\scriptstyle \varepsilon} & \\ & k & \end{array} \tag{3.7}$$

Now, Δ is an algebra homomorphism if and only if Diagrams (3.4) and (3.6) commute, while ε is an algebra homomorphism if and only if Diagrams (3.5) and (3.7) commute. □

Definition 3.1.19 (Bialgebra)**.** *A bialgebra is a vector space* A *with product* $\cdot$ *and unit* i *such that* $(A, \cdot, i)$ *is an algebra, as well as a coproduct* Δ *and counit* ε *such that* (A, Δ, ε) *is a coalgebra, and with the condition that the coproduct and counit are algebra homomorphisms.*

Note that by Theorem 3.1.18, this definition implies that $\cdot$ and i are coalgebra homomorphisms. The definition could equivalently assume this condition.

Example 3.1.20. *Recall that the group algebra of a finite group is both an algebra, see example 3.1.4, and a coalgebra, see example 3.1.13. We show that these structures satisfy the compatibility condition that* Δ *and* ε *are algebra homomorphisms. Because* Δ *and* ε *are* $\mathbb{C}$*-linear, it is sufficient to check compatibility on a basis. Let* $g, h \in G$*, so that* e_g *and* e_h *are basis vectors in* $\mathbb{C}[G]$*. To check that* Δ *is an algebra homomorphism, we first verify Diagram* (3.4)*. In the counterclockwise direction, we have*

$$\begin{aligned} e_g \otimes e_h &\xmapsto{\Delta\otimes\Delta} e_g \otimes e_g \otimes e_h \otimes e_h \\ &\xmapsto{\mathrm{id}\otimes\tau\otimes\mathrm{id}} e_g \otimes e_h \otimes e_g \otimes e_h \\ &\xmapsto{\cdot\otimes\cdot} e_{gh} \otimes e_{gh}. \end{aligned}$$

In the clockwise direction,

$$\begin{aligned} e_g \otimes e_h &\stackrel{\cdot}{\mapsto} e_{gh} \\ &\stackrel{\Delta}{\mapsto} e_{gh} \otimes e_{gh}. \end{aligned}$$

Second, we verify Diagram (3.5)*. In the counterclockwise direction, we have, for* $\lambda \in \mathbb{C}$*,*

$$\begin{aligned} \lambda &\xmapsto{\rho^{-1}} \lambda \otimes 1 \\ &= \lambda e_0 \otimes e_0. \end{aligned}$$

In the clockwise direction, we have

$$\begin{aligned} \lambda &\stackrel{i}{\mapsto} \lambda e_0 \\ &\stackrel{\Delta}{\mapsto} \lambda(e_0 \otimes e_0). \end{aligned}$$

The proof that ε *is an algebra homomorphism is similar. Therefore,* $\mathbb{C}[G]$ *is a bialgebra.*

Example 3.1.21. *We again return to the example of the tensor algebra from example 3.1.14. Our choices of coproduct and counit were specifically chosen to be algebra homomorphisms. Thus,* $T(V)$ *is a bialgebra.*

Definition 3.1.22 (Antipode)**.** *An antipode of a bialgebra is a* k*-linear map* $S : A \to A$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} & A \otimes A & \xrightarrow{S \otimes \mathrm{id}} & A \otimes A & \\ {\scriptstyle \Delta} \nearrow & & & & \searrow {\scriptstyle \cdot} \\ A & \xrightarrow{\varepsilon} & k & \xrightarrow{i} & A \\ {\scriptstyle \Delta} \searrow & & & & \nearrow {\scriptstyle \cdot} \\ & A \otimes A & \xrightarrow[\mathrm{id} \otimes S]{} & A \otimes A & \end{array} \tag{3.8}$$

The commutativity of this diagram implies that $S(a \cdot a') = S(a') \cdot S(a)$ for all $a, a' \in A$, see [14, Theorem III.3.4]. Such a map is called an algebra antihomomorphism.

Definition 3.1.23 (Hopf Algebra)**.** *A Hopf algebra is a bialgebra with an antipode.*

We now introduce a slight change in notation. For a Hopf algebra, we denote the vector space by H instead of by A and the Hopf algebra by the tuple $(H, \cdot, i, \Delta, \varepsilon, S)$.

Example 3.1.24. *The group algebra of a finite group, which is a bialgebra by example 3.1.20, is a Hopf algebra with the antipode defined on basis vectors as* $S(e_g) = e_{g^{-1}}$*. To verify the antipode diagram, we need only check Diagram* (3.8) *on basis vectors. We first check the center path of the figure. Let* e_g *be a basis vector. We have*

$$e_g \overset{\varepsilon}{\mapsto} 1 \overset{i}{\mapsto} e_0$$

Along the top branch of the diagram, we have

$$e_g \overset{\Delta}{\longmapsto} e_g \otimes e_g \overset{S\otimes \mathrm{id}}{\longmapsto} e_{g^{-1}} \otimes e_g \overset{\cdot}{\mapsto} e_0.$$

Along the bottom branch, we have

$$e_g \overset{\Delta}{\longmapsto} e_g \otimes e_g \overset{\mathrm{id}\otimes S}{\longmapsto} e_g \otimes e_{g^{-1}} \overset{\cdot}{\mapsto} e_0.$$

Example 3.1.25. *We define an antipode in stages on homogeneous vectors in the tensor algebra* $T(V)$*, which is a bialgebra by example 3.1.21. On* $T^0(V) = k$*, define*

$$S(\lambda) = \lambda.$$

For $v \in V$*, define*

$$S(v) = -v.$$

For $v_1 \cdots v_n \in V^{\otimes n}$*, define*

$$S(v_1 \cdot \cdots \cdot v_n) = (-1)^n v_n \cdot \cdots \cdot v_1.$$

We verify that this candidate for an antipode satisfies Diagram (3.8)*. It suffices to check a homogeneous element of degree* n*; commutativity on* $T(V)$ *follows by linearity.*

We consider two cases. First, assume $n = 0$*. Then the middle path of the diagram is the identity map, while the upper path of the diagram,*

$$c \overset{\Delta}{\mapsto} c \otimes 1 \overset{S\otimes \mathrm{id}}{\longmapsto} c \otimes 1 \overset{\cdot}{\mapsto} c,$$

is also the identity map. One can easily verify that the bottom path of the diagram is also the identity map.

Now, assume that $n \geq 1$*. Let* $v_1 \otimes \cdots \otimes v_n \in T^n(V)$*. Following the middle path of the diagram, we have*

$$c \overset{\epsilon}{\mapsto} 0 \overset{i}{\mapsto} 0.$$

We examine the top path of the diagram. The coproduct yields

$$v_1 \cdots v_n \overset{\Delta}{\mapsto} \sum_{i=0}^{n} \sum_{\sigma \in sh_{i,n-1}} (-1)^i v_{\sigma(1)} \cdots v_{\sigma(i)} \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)}.$$

Observe that, in each term of this sum, $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ *and* $\sigma(i+1) < \cdots < \sigma(n)$*. Thus, there are* 2^{n-1} *terms with a leading* v_1 *(i.e.* $\sigma(1) = 1$*) and* 2^{n-1} *terms without.*

Consider the following two summands:

$$v_1 v_{\sigma(2)} \cdots v_{\sigma(i)} \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)} \tag{3.9}$$

$$v_{\sigma(2)} \cdots v_{\sigma(i)} \otimes v_1 v_{\sigma(i+1)} \cdots v_{\sigma(n)} \tag{3.10}$$

Note that these two summands differ only in the placement of v_1*:* (3.9) *has a leading* v_1 *while* (3.10) *does not. We compute*

$$\cdot \circ (S \otimes \mathrm{id})(v_1 v_{\sigma(2)} \cdots v_{\sigma(i)} \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)}) = (-1)^i v_{\sigma(i)} v_{\sigma(i-1)} \cdots v_1 v_{\sigma(i+1)} \cdots v_{\sigma(n)}.$$

$$\cdot \circ (S \otimes \mathrm{id})(v_{\sigma(2)} \cdots v_{\sigma(i)} \otimes v_1 v_{\sigma(i+1)} \cdots v_{\sigma(n)}) = (-1)^{i-1} v_{\sigma(i)} v_{\sigma(i-1)} \cdots v_1 v_{\sigma(i+1)} \cdots v_{\sigma(n)}.$$

Because these two terms differ only by a sign, they add to zero. Because the sum

$$\cdot \circ (S \otimes \mathrm{id}) \circ \Delta(v_1 \cdots v_n)$$

ranges over all shuffles, every term with a leading v_1 *cancels out exactly one term without a leading* v_1*. Thus,*

$$\cdot \circ (S \otimes \mathrm{id}) \circ \Delta(v_1 \cdots v_n) = 0.$$

Commutativity of the bottom part of the diagram is analogous.

Definition 3.1.26 (Hopf Ideal)**.** *A Hopf ideal of a Hopf algebra* H *is a vector subspace* $I \subset H$ *which* *is an ideal of the underlying algebra, for which the coproduct factors as*

$$\Delta : I \to H \otimes I \oplus I \otimes H,$$

for which $\varepsilon(I) = 0$, *and for which* $S(I) \subset I$.

The condition on the coproduct and counit mean that I is an ideal of the underlying coalgebra.

Theorem 3.1.27. *The quotient of a Hopf algebra* H *by a Hopf ideal* I *is a Hopf algebra.*

Proof. We use the maps defined for H to build maps which satisfy the properties of a Hopf algebra for H/I. Let m^* and Δ^* be the multiplication and coproduct maps for H. We define $m : H/I \otimes H/I \to H/I$ by

$$m(h + I, h' + I) = m^*(h, h') + I$$

for all $h \in H$. This map is well-defined because, for example, for $i \in I$,

$$\begin{aligned} m(h + i + I, h' + I) &= m^*(h + i, h') + I \\ &= m^*(h, h') + m^*(i, h') + I \\ &= m^*(h, h') + I. \end{aligned}$$

Because m^* is bilinear and associative, m is also bilinear and associative.

The map

$$\begin{aligned} \psi : H/I \otimes H/I &\to (H \otimes H)/(I \otimes H + H \otimes I) \\ a + I \otimes b + I &\mapsto a \otimes b + I \otimes H + H \otimes I \end{aligned}$$

is a vector space isomorphism. We define the coproduct $\Delta : H/I \to H/I \otimes H/I$ by

$$\Delta(h + I) = \Delta^*(h) + I \otimes H + H \otimes I.$$

By the linearity and coassociativity of Δ^*, Δ is linear and coassociative. Because

$$\begin{aligned}\Delta(m(h+I, h+I)) &= \Delta(m^*(h,h')+I)\\ &= \Delta^*(m^*(h,h')) + I\otimes H + H\otimes I\\ &= m^*(\Delta^*(h),\Delta^*(h')) + I\otimes H + H\otimes I \quad \text{(because } \Delta^* \text{ is an algebra homomorphism)}\\ &= m(\Delta(h),\Delta(h')),\end{aligned}$$

Δ is an algebra homomorphism.

Lastly, define the antipode $S : H/I$ by

$$S(h+I) = S^*(h) + I,$$

where S^* is the antipode for H. This map is well-defined because, for example, for $i \in I$,

$$\begin{aligned}S(h+i+I) &= S^*(h+i)+I\\ &= S^*(h)+S^*(i)+I\\ &= S^*(h)+I.\end{aligned}$$

We check that this condition satisfies the diagram in Definition 3.1.22. The top path of the diagram applied to $h + I \in H/I$ is

$$\begin{aligned}m\circ(S\otimes \text{id})\circ\Delta(h+I) &= m\circ(S\otimes\text{id})(\Delta^*(h)+I\otimes H+H\otimes I)\\ &= m(S^*\otimes \text{id}(\Delta^*(h))+I\otimes H+H\otimes I)\\ &= m^*(S^*\otimes\text{id}(\Delta^*(h)))+I.\end{aligned}$$

The bottom part of the diagram is the composition

$$\begin{aligned}m\circ(\text{id}\otimes S)\circ\Delta(h+I) &= m\circ(\text{id}\otimes S)(\Delta^*(h)+I\otimes H+H\otimes I)\\ &= m(\text{id}\otimes S^*(\Delta^*(h))+I\otimes H+H\otimes I)\\ &= m^*(\text{id}\otimes S^*(\Delta^*(h)))+I.\end{aligned}$$

The middle path of the diagram is

$$i \circ \varepsilon(h + I) = i(\varepsilon(h)).$$

The antipode diagram for H/I commutes because we reduce it to the antipode diagram for H, which commutes by assumption.

We conclude that $(H/I, m, i, \Delta, \varepsilon, S)$ is a Hopf algebra. □

3.2
Modules

Definition 3.2.1. *Let A be an algebra. A left A-module is a vector space M and an action of R on M; that is, a map $R \times M \to M$ which satisfies the following properties:*

1. $(r + s)m = rm + sm$ *for all* $r, s \in R, m \in M$.
2. $(rs)m = r(sm)$ *for all* $r, s \in R, m \in M$.
3. $r(m + n) = rm + rn$ *for all* $r \in R, m, n \in M$.

A right R-module is defined analogously with multiplication by elements of the ring on the right instead of on the left.

Definition 3.2.2. *Let M be an R-module. An R-submodule of M is a subspace $N \subset M$ which is closed under the action of R.*

Theorem 3.2.3. *Given a finite group G, modules of $k[G]$ are in bijection with representations of G.*

Proof. Let V be a $k[G]$-module. Let $g, h \in G$ and $v, w \in V$, and $c \in k$. We denote the action of e_g on V by $\phi(e_g)$. Then

$$\begin{aligned} \phi(e_g)(v + cw) &= e_g \cdot v + ce_g \cdot w \\ &= \phi(e_g)(v) + c\phi(e_g)(w). \end{aligned}$$

Therefore, $\phi(e_g)$ is k-linear. Now, since $\phi(e_0)$ is the identity map and $\phi(e_{g^{-1}})(\phi(e_g)(v)) = v$,

$\phi(e_g)$ is invertible. Furthermore,

$$\begin{aligned} \phi(e_g e_h)(\nu) &= e_g e_h \cdot \nu \\ &= e_g \cdot (e_h \cdot \nu) \\ &= \phi(e_g)(\phi(e_h)(\nu)), \end{aligned}$$

so ϕ is a homomorphism. Then ϕ is a representation.

Given a representation of G, running the above procedure in reverse yields a $k[G]$-module. □

Example 3.2.4. *Consider the additive cyclic group $\mathbb{Z}_r$ over $\mathbb{C}$. By Maschke's Lemma, all of its representations are semisimple. The simple representations are all one-dimensional. Moreover, the action of $\mathbb{Z}_r$ on each simple representation is uniquely determined by how $1 \in \mathbb{Z}_r$ acts. Because $1^r = 1$, $\mathbb{Z}_r$ acts on each simple representation by an rth root of unity.*

Definition 3.2.5 (Projective Module)**.** *Let M, N, and P be modules of an algebra A such that the sequence $M \to N \to 0$ is exact. The module P is projective if every A-module homomorphism from P into N lifts to an R-module homomorphism into M.*

In other words, P is projective if, given $f \in \mathrm{Hom}_R(P, N)$, there is a lift $F \in \mathrm{Hom}_R(P, M)$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & P & & \\ & \overset{F}{\swarrow} & \downarrow f & & \\ M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

3.3
Lie Algebras

In this section, all vector spaces and tensor products are over $\mathbb{C}$, unless otherwise specified. A good reference for introductory Lie algebra theory is [6].

Definition 3.3.1. *A Lie algebra is a vector space $\mathfrak{g}$ together with a $\mathbb{C}$-bilinear operation, called the Lie bracket,*

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

which satisfies the following properties:

1. $[-,-]$ *is skew-symmetric (i.e. $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$), and*

2. *the Jacobi identity,* $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0$*, holds for all* $x,y,z\in\mathfrak{g}$.

Note that the first property of the Lie bracket is equivalent to the condition that the Lie bracket is alternating; i.e. $[x,x]=0$ for all $x\in\mathfrak{g}$. The Lie bracket is not assumed to be an associative operation. Instead, the Jacobi identity "measures" the nonassociativity as follows: by property 1), $[[x,y],z]=-[z,[x,y]]$. We rewrite the Jacobi Identity as

$$[x,[y,z]]-[[x,y],z]=-[y,[z,x]]$$

and see that the term $-[y,[z,x]]$ measures the difference $x\cdot(y\cdot z)-(x\cdot y)\cdot z$, where $\cdot$ denotes the bracket.

We examine a few examples of Lie algebras.

Example 3.3.2. *Let* $\mathfrak{g}$ *be a complex vector space and define the Lie bracket by* $[x,y]=0$ *for all* $x,y\in\mathfrak{g}$ *(this bracket is called the zero bracket). Then* $\mathfrak{g}$ *is a Lie algebra. Because the zero bracket is the only commutative Lie bracket, any Lie algebra with the zero bracket is called an abelian Lie algebra.*

Example 3.3.3. *The vector space* $\mathbb{R}^3$ *with the cross product (denoted* $\times$*) as the Lie bracket is a Lie algebra over* $\mathbb{R}$*. We show that the cross product satisfies the Jacobi identity by making use of the triple product expansion,* $x\times(y\times z)=y(x\cdot z)-z(x\cdot y)$ *for all* $x,y,z\in\mathbb{R}^3$*, where* $\cdot$ *denotes the dot product. The Jacobi identity is*

$$\begin{aligned}[x,[y,z]]+[y,[z,x]]+[z,[x,y]] &= x\times(y\times z)+y\times(z\times x)+z\times(x\times y)\\ &= y(x\cdot z)-z(x\cdot y)+z(y\cdot x)-x(y\cdot z)+x(z\cdot y)-y(z\cdot x)\\ &= 0.\end{aligned}$$

Example 3.3.4. *Let* A *be an algebra and take the Lie bracket to be the commutator bracket, defined on* $A\times A$ *by*

$$[x,y]:=xy-yx\quad \textit{for all } x,y\in A,$$

where xy *denotes standard matrix multiplication. Then* A *is a Lie algebra.* A *is an abelian Lie algebra if and only if* A *is a commutative algebra.*

Example 3.3.5. *Let* $\mathfrak{g}$ *be a Lie algebra and* $T(\mathfrak{g})$ *the tensor algebra as defined in Example 3.1.5. Consider the ideal generated by* $xy-yx-[x,y]$ *for all* $x,y\in\mathfrak{g}$*. By Theorem 3.1.27,* $T(\mathfrak{g})/I$ *is a Hopf algebra.*

Example 3.3.6. *Let* V *be an* n*-dimensional vector space over* $\mathbb{C}$*. The set of all linear maps from* V *to* V*, denoted by* $\mathfrak{gl}(V)$*, is again a vector space over* $\mathbb{C}$*. Define the Lie bracket by*

$$[x, y] := x \circ y - y \circ x \quad \textit{for all } x, y \in \mathfrak{gl}(V).$$

Then $\mathfrak{gl}(V)$ *is a Lie algebra. We justify this claim by showing that the Jacobi identity holds:*

$$\begin{aligned}
&[x,[y,z]] + [y,[z,x]] + [z,[x,y]] \\
&= x \circ (y \circ z - z \circ y) - (y \circ z - z \circ y) \circ x + y \circ (z \circ x - x \circ z) - (z \circ x - x \circ z) \circ y \\
&+ z \circ (x \circ y - y \circ x) - (x \circ y - y \circ x) \circ z \\
&= x \circ y \circ z - x \circ z \circ y - y \circ z \circ x + z \circ y \circ x + y \circ z \circ x - y \circ x \circ z - z \circ x \circ y + x \circ z \circ y \\
&+ z \circ x \circ y - z \circ y \circ x - x \circ y \circ z + y \circ x \circ z \\
&= 0.
\end{aligned}$$

If we fix a basis for V*, then the linear maps in* $\mathfrak{gl}(V)$ *can be expressed as* $n \times n$ *matrices. Denote by* $\mathfrak{gl}_n(\mathbb{C})$ *the vector space of these matrices. With the unit map* $i : \mathbb{C} \to \mathfrak{gl}_n(\mathbb{C})$ *defined on generators by* $i(1) = I_n$*, where* I_n *is the* $n \times n$ *identity matrix,* $\mathfrak{gl}_n(\mathbb{C})$ *is an algebra. Therefore, with the same choice of Lie bracket as in the previous example,*

$$[x, y] := xy - yx \quad \textit{for all } x, y \in \mathfrak{gl}_n(\mathbb{C}),$$

$\mathfrak{gl}_n(\mathbb{C})$ *is a Lie algebra.*

Example 3.3.7. *The trace map* $tr : \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}$ *acts on a matrix* $x \in \mathfrak{gl}_n(\mathbb{C})$ *by summing the diagonal entries in* x*. Because the trace is linear, the subset* $\mathfrak{sl}_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \,|\, tr(x) = 0\}$ *is a vector subspace. The same choice of bracket as for* $\mathfrak{gl}_n(\mathbb{C})$ *makes* $\mathfrak{sl}_n(\mathbb{C})$ *a Lie algebra.*

Definition 3.3.8. *Given two Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$*, a Lie algebra homomorphism* $\phi : \mathfrak{g} \to \mathfrak{h}$ *is a vector space homomorphism which satisfies the the following property:*

$$\phi([x, y]) = [\phi(x), \phi(y)] \quad \textit{for all } x, y \in \mathfrak{g}.$$

Example 3.3.9. *The inclusion homomorphism* $i : \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ *defined by* $i(x) = x$ *is a Lie algebra homomorphism.*

Example 3.3.10. *Consider the trace map from* $\mathfrak{gl}_n(\mathbb{C})$ *to the Lie algebra* $\mathbb{C}$ *bracket given by standard multiplication (note that* $\mathbb{C}$ *is an abelian Lie algebra). The trace map is a Lie algebra homomorphism because*

$$tr([x, y]) = tr(xy - yx) = 0.$$

Note that the kernel of the trace map is $\mathfrak{sl}_n(\mathbb{C})$*.*

Definition 3.3.11. *Let* $\mathfrak{g}$ *be a Lie algebra. A Lie subalgebra* $\mathfrak{h}$ *is a vector subspace of* $\mathfrak{g}$ *such that for all* $x, y \in \mathfrak{h}$*,*

$$[x, y] \in \mathfrak{h}.$$

Definition 3.3.12. *Let* $\mathfrak{g}$ *be a Lie algebra. A Lie ideal is a Lie subalgebra* $I \subseteq \mathfrak{g}$ *such that for all* $x \in I$ *and all* $y \in \mathfrak{g}$*,*

$$[x, y] \in I.$$

Note that if $[x, y]$ is in I, then $[y, x] = -[x, y]$ is also in I. Therefore, we do not distinguish between left and right ideals.

Example 3.3.13. *The trivial Lie ideals for any Lie algebra* $\mathfrak{g}$ *are* $\mathfrak{g}$ *itself and* $\{0\}$*.*

Example 3.3.14. *The Lie algebra* $\mathfrak{sl}_n(\mathbb{C})$ *is a Lie ideal of* $\mathfrak{gl}_n(\mathbb{C})$ *because, if* $x \in \mathfrak{sl}_n(\mathbb{C})$ *and* $y \in \mathfrak{gl}_n(\mathbb{C})$*, then*

$$tr([x, y]) = tr(xy - yx) = tr(xy) - tr(yx) = tr(xy) - tr(xy) = 0.$$

This example motivates the first part of the following theorem:

Theorem 3.3.15. *The kernel of a Lie algebra homomorphism is a Lie ideal. The image of a Lie algebra homomorphism is a Lie subalgebra.*

Proof. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. We first show that the kernel of ϕ is a Lie ideal. Let $x \in \ker \phi$ and let $y \in \mathfrak{g}$. We have

$$\begin{aligned} \phi([x, y]) &= [\phi(x), \phi(y)] \\ &= [0, \phi(y)] \\ &= 0. \end{aligned}$$

Therefore, $[x, y] \in \ker(\phi)$, and $\ker(\phi)$ is a Lie ideal.

We now show that the image of ϕ is a Lie subalgebra. Let $u, v \in \mathrm{im}(\phi)$. Then $u = \phi(a)$ and $v = \phi(b)$ for some $a, b \in \mathfrak{g}$. Now,

$$\begin{aligned}[u, v] &= [\phi(a), \phi(b)] \\ &= \phi([a, b])\end{aligned}$$

Therefore, $[u, v] \in \mathrm{im}(\phi)$, so $\mathrm{im}(\phi)$ is a Lie subalgebra. $\square$

Having defined Lie ideals, we can now state and prove the First Isomorphism Theorem for Lie algebras.

Theorem 3.3.16 (The First Isomorphism Theorem for Lie Algebras)**.** *Let* $\phi : \mathfrak{g} \to \mathfrak{h}$ *be a Lie algebra homomorphism. Then there exists an isomorphism*

$$\mathrm{im}(\phi) \simeq \mathfrak{g} \big/ \ker(\phi).$$

In particular, if ϕ *is surjective, then*

$$\mathfrak{h} \simeq \mathfrak{g} \big/ \ker(\phi).$$

Proof. As vector spaces, $\mathfrak{g} \big/ \ker(\phi)$ and $\mathrm{im}(\phi)$ are isomorphic by the First Isomorphism Theorem for vector spaces. Call that isomorphism $\phi' : \mathfrak{g} \big/ \ker(\phi) \to \mathrm{im}(\phi)$. Since $\ker(\phi)$ is an ideal of $\mathfrak{g}$, $\phi' : \mathfrak{g} \big/ \ker(\phi) \to \mathrm{im}(\phi)$ is a Lie algebra homomorphism by construction. $\square$

3.3.1

Representations of Lie Algebras

Recall from example 3.3.6 that for a vector space V, the vector space of $\mathbb{C}$-linear endomorphisms on V, denoted $\mathfrak{gl}(V)$, is a Lie algebra with Lie bracket defined by

$$[x, y] = x \circ y - y \circ x \quad \text{for all } x, y \in \mathfrak{gl}(V).$$

We use $\mathfrak{gl}(V)$ to define a representation of a Lie algebra.

Definition 3.3.17. *A representation of a Lie algebra* $\mathfrak{g}$ *is a vector space* V *and a Lie algebra homomorphism*

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

In particular, ρ *must satisfy the compatibility condition that for all* $g, h \in \mathfrak{g}$,

$$\begin{aligned}\rho([g,h]) &= [\rho(g), \rho(h)] \\ &= \rho(g) \circ \rho(h) - \rho(h) \circ \rho(g).\end{aligned}$$

Because the objects $\mathfrak{gl}(V)$ are linear transformations, a representation of a Lie algebra allows us to introduce tools from linear algebra into our study of Lie algebras.

Example 3.3.18. *The trivial representation of a Lie algebra* $\mathfrak{g}$ *is the zero map.*

Example 3.3.19. *Let* A *be an algebra. With the commutator bracket* A *becomes a Lie algebra. The map* $\rho : A \to \mathfrak{gl}(A)$ *defined by multiplication in* A *is a representation of* A.

Example 3.3.20. *The fundamental representation of* $\mathfrak{gl}_n(\mathbb{C})$ *is*

$$\rho : \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C}),$$

where ρ *is the identity map.*

Given a Lie algebra homomorphism $\phi : \mathfrak{h} \to \mathfrak{g}$ and a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, the restricted representation of ρ by $\mathfrak{h}$ is the map

$$\mathfrak{h} \to \mathfrak{g} \to \mathfrak{gl}(V).$$

Example 3.3.21. *The inclusion map* $\phi : \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ *defined by* $\phi(x) = x$ *is a Lie algebra homomorphism. We can restrict the representation in Example 3.3.20 along* $\mathfrak{sl}_n(\mathbb{C})$.

Example 3.3.22. *The adjoint representation of a Lie algebra* $\mathfrak{g}$ *is the map* $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *defined by*

$$ad[g](x) = [g, x] \quad \textit{for all } g, x \in \mathfrak{g}.$$

Note that $ad(g)$ *is a map from* $\mathfrak{g}$ *to* $\mathfrak{g}$*; in the map above,* x *is the argument of the map* $ad(g)$*. We* *show that this map satisfies the compatibility condition for a representation. Let* $x, y, z \in \mathfrak{g}$.

$$\begin{aligned} ad([x,y])(z) &= [[x,y],z] \\ &= -[z,[x,y]] && \textit{(by skew-symmetry)} \\ &= [x,[y,z]] + [y,[z,x]] && \textit{(by the Jacobi identity)} \\ &= [x,[y,z]] - [y,[x,z]] && \textit{(by skew-symmetry and linearity in the second component)} \\ &= ad[x](ad[y](z)) - ad[y](ad[x](z)). \end{aligned}$$

Definition 3.3.23. *A subrepresentation of a representation* (V, ρ) *of a Lie algebra* $\mathfrak{g}$ *is a subspace* $W \subseteq V$ *with the property that*

$$\rho(x)w \in W \quad \textit{for all } w \in W \textit{ and for all } x \in \mathfrak{g}.$$

Example 3.3.24. *Every representation* V *has two trivial subrepresentations:* 0 *and* V *itself.*

Definition 3.3.25. *A subrepresentation is called irreducible or simple if it has no nontrivial subrepresentations.*

In the following theorem we argue that the direct sum of representations is naturally a representation.

Theorem 3.3.26. *Let* V *and* W *be vector spaces and let* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *and* $\lambda : \mathfrak{g} \to \mathfrak{gl}(W)$ *be representations of a Lie algebra* $\mathfrak{g}$*. Then the map* $\eta : \mathfrak{g} \to \mathfrak{gl}(V \oplus W)$ *defined by*

$$\eta[g](v \oplus w) = \rho(g)v \oplus w + v \oplus \lambda(g)w \quad \textit{for all } v \in V, w \in W, \textit{and } g \in \mathfrak{g}$$

is a representation.

Proof. We have

$$\begin{aligned} \eta([g,h])(v \oplus w) &= \rho([g,h])v \oplus w + v \oplus \lambda([g,h])w \\ &= (\rho(g)\rho(h) - \rho(h)\rho(g))v \oplus w + v \oplus (\lambda(g)\lambda(h) - \lambda(h)\lambda(g))w \\ &= \rho(g)\rho(h)v \oplus w + v \oplus \lambda(g)\lambda(h)w - (\rho(h)\rho(g)v \oplus w + v \oplus \lambda(h)\lambda(g)w) \\ &= \eta[g](\eta[h](v \oplus w)) - \eta[h](\eta[g](v \oplus w)). \end{aligned}$$

□

In the following theorem we argue that the tensor product of representations is a representation.

Theorem 3.3.27. *Let* V *and* W *be vector spaces and let* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *and* $\lambda : \mathfrak{g} \to \mathfrak{gl}(W)$ *be representations of a Lie algebra* $\mathfrak{g}$*. Then* $\delta : \mathfrak{g} \to \mathfrak{gl}(V \otimes W)$ *is a representation of* $\mathfrak{g}$*, where*

$$\delta[g](v \otimes w) = \rho(g)v \otimes w + v \otimes \lambda(g)w \quad \textit{for all } v \in V, w \in W, \textit{and } g \in \mathfrak{g}.$$

Proof. We have

$$\begin{aligned} \delta([g,h])(v \otimes w) &= \rho([g,h])v \otimes w + v \otimes \lambda([g,h])w \\ &= \rho(g)\rho(h)v \otimes w - \rho(h)\rho(g)v \otimes w + v \otimes \lambda(g)\lambda(h)w - v \otimes \lambda(h)\lambda(g)w \\ &= \rho(g)\rho(h)v \otimes w + v \otimes \lambda(g)\lambda(h)w - (\rho(h)\rho(g)v \otimes w + v \otimes \lambda(h)\lambda(g)w) \\ &= \delta[g](\delta[h](v \otimes w)) - \delta[h](\delta[g](v \otimes w)). \end{aligned}$$

□

3.4
The Special Linear Lie Algebra $\mathfrak{sl}_2(\mathbb{C})$

We now turn our full attention to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Recall that $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of 2×2 matrices with zero trace and Lie bracket given by

$$[x, y] = xy - yx \quad \text{for all } x, y \in \mathfrak{sl}_2(\mathbb{C}).$$

We will use the following standard basis throughout our discussion of $\mathfrak{sl}_2(\mathbb{C})$:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The commutation relations are

$$\begin{array}{lll} [h,h] = 0 & [h,e] = 2e & [h,f] = -2f \\ [e,h] = -2e & [e,e] = 0 & [e,f] = h \\ [f,h] = 2f & [f,e] = -h & [f,f] = 0. \end{array}$$

3.4.1

Irreducible Representations

Every finite dimensional irreducible representation can be decomposed into a direct sum of weight spaces with respect to the action of h.

Theorem 3.4.1. *Let* (V, ρ) *be a finite dimensional irreducible representation of* $\mathfrak{sl}_2(\mathbb{C})$*. Let* $v \in V$ *an eigenvector of* $\rho(h)$ *with eigenvalue* $\lambda \in \mathbb{C}$*. The following statements are implied:*

1. $\rho(e)v$ *is zero or an eigenvector of* $\rho(h)$ *with eigenvalue* $\lambda + 2$*.*

2. $\rho(f)v$ *is zero or an eigenvector of* $\rho(h)$ *with eigenvector* $\lambda - 2$*.*

Proof. We prove statement 1 and note that the proof of statement 2 is similar. By assumption,

$$\begin{aligned}\rho([h, e])v &= \rho(h)\rho(e)v - \rho(e)\rho(h)v \\ &= \rho(h)\rho(e)v - \lambda\rho(e)v.\end{aligned}$$

Recall that $[h, e] = 2e$, so

$$\rho([h, e])v = 2\rho(e)v.$$

Setting these equal, we conclude that

$$\rho(h)\rho(e)v = (\lambda + 2)\rho(e)v.$$

□

The proof of the preceding claim allows for the possibility of infinitely many eigenvectors of $\rho(h)$. However, because eigenvectors which correspond to distinct eigenvalues are linearly independent, and because the representation is finite dimensional, there must be finitely many linearly independent eigenvectors.

Theorem 3.4.2. *Let* (V, ρ) *be a finite dimensional irreducible representation of* $\mathfrak{sl}_2(\mathbb{C})$*. There exists an eigenvector* $w \in V$ *of* $\rho(h)$ *such that* $\rho(e)w = 0$ *and which generates* V*.*

Proof. Let $v \in V$ be an eigenvector of the linear map $\rho(h) : V \to V$. By statement 1) in Theorem 3.4.1, if j and k are both positive integers with $j \neq k$ and $\rho^j(e)v$ and $\rho^k(e)v$ are both nonzero, then $\rho^j(e)v$ and $\rho^k(e)v$ are linearly independent. Because V is finite dimensional, there must be some $k > 0$ such that $\rho^j(e)v = 0$ for all $j \geq k$.

Let $k \geq 0$ be the minimal value such that $\rho^k(e)v \neq 0$ and $\rho^{k+1}(e)v = 0$. Fix $w = \rho^k(e)v$.

Now, by Theorem 3.4.1, repeated action of f on w gives a set of linearly independent weight vectors of V. Because V is finite dimensional, there exists some l such that $\rho^l(f)w \neq 0$ and $\rho^{l+1}(f)w = 0$. Then

$$W = \{w, \rho(f)w, \ldots, \rho^l(f)w\}$$

is a linearly independent set. We claim that the span of W is a subrepresentation of V. Obviously, W is invariant under the actions of f and h, so we need only consider how e acts on W.

Let i be a nonnegative integer less than k. We prove by induction that

$$\rho(e)\rho(f)^i w = i(\lambda - i + 1)\rho(f)^{i-1}w.$$

In the case when $i = 0$, $\rho(e)w = 0$ by assumption.

Assume that $\rho(e)\rho(f)^i w = i(\lambda - i + 1)\rho(f)^{i-1}w$. We will show that the statement holds for $i + 1$. We have

$$\begin{aligned}
\rho(e)\rho(f)^{i+1}w &= \rho(e)\rho(f)\rho(f)^i w \\
&= [\rho([e,f]) + \rho(f)\rho(e)]\,\rho(f)^i w \\
&= \rho(h)\rho(f)^i w + \rho(f)\rho(e)\rho(f)^i w \\
&= (\lambda - 2i)\rho(f)^i w + i(\lambda - i + 1)\rho(f)^i w \\
&= (i + 1)(\lambda - i)\rho(f)^i w.
\end{aligned}$$

Therefore, the span of W is a subrepresentation of V. Since V is irreducible by assumption, W spans V. □

Corollary 3.4.3. *Any finite dimensional irreducible representation* V *of* $\mathfrak{sl}_2(\mathbb{C})$ *is isomorphic to an irreducible representation of highest weight* $\dim(V) - 1$*. Moreover, two finite dimensional irreducible representations are isomorphic if and only if they have equal dimension.*

Definition 3.4.4 (Highest Weight Vector)**.** *Any vector satisfying the conditions in Theorem 3.4.2 is called a highest weight vector.*

Let $T(\mathfrak{sl}_2(\mathbb{C}))$ be the Tensor algebra of $\mathfrak{sl}_2(\mathbb{C})$ from Example 3.1.5. In particular, it is the algebra generated by e, f, and h and with coproduct

$$\Delta(e) = e \otimes 1 + 1 \otimes e, \quad \Delta(h) = h \otimes 1 + 1 \otimes h, \quad \Delta(f) = f \otimes 1 + 1 \otimes f$$

and antipode

$$S(e) = -e, \quad S(h) = -h, \quad S(f) = -f.$$

Consider the ideal generated by the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

By Theorem 3.1.27, the quotient is a Hopf algebra, which we denote $U(\mathfrak{sl}_2(\mathbb{C}))$.

Chapter 4

RIBBON CATEGORY AND RIBBON HOPF ALGEBRA BACKGROUND

In this chapter, we motivate a definition for a particular type of category, called a ribbon category. A good introductory reference to category theory is [16]. We also discuss a particular type of Hopf algebra, called a ribbon Hopf algebra, which has the property that its category of modules is ribbon. We conclude this chapter by defining the Reshetikhin–Turaev functor, which we use to construct link invariants using ribbon categories.

Throughout this section, we write $V \in \mathcal{C}$ to denote that V is an object of the category $\mathcal{C}$.

4.1

Semisimple and Ribbon Categories

Definition 4.1.1 (Semisimple Category)**.** *Let $\mathcal{C}$ be an abelian category. A nonzero object $U \in \mathcal{C}$ is simple if its only subobjects are 0 and U. An object V is called semisimple if it is a direct sum of simple objects. A category $\mathcal{C}$ is semisimple if every object in $\mathcal{C}$ is semisimple.*

For an introduction to semisimple categories, see [7, Sections 1.2-1.5].

Example 4.1.2. *The category of finite dimensional vector spaces over $\mathbb{C}$ is a semisimple category. The simple objects are one-dimensional vector spaces. Every vector space can be written as a direct sum of one-dimensional subspaces.*

Definition 4.1.3 (Tensor Product in a Category)**.** *Given a category $\mathcal{C}$, a tensor product on $\mathcal{C}$ is a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.*

In particular, the statement that $\otimes$ is a functor means that it assigns an object $V \otimes W \in \mathcal{C}$ to each pair $(V, W) \in \mathcal{C} \times \mathcal{C}$ and a morphism $f \otimes g : V \otimes W \to V' \otimes W'$ to each pair of morphisms $f : V \to V'$ and $g : W \to W'$. Furthermore, this assignment satisfies the following conditions:

1. $(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g')$, and
2. $\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes W}$.

Definition 4.1.4 (Monoidal Category)**.** *A monoidal category* $\mathcal{C}$ *is a category with a tensor product* $\otimes$*, an object* $\mathbb{1}$*, called the unit, and the following data:*

1. *A natural isomorphism*

$$a : ((-) \otimes (-)) \otimes (-) \overset{\cong}{\Longrightarrow} (-) \otimes ((-) \otimes (-))$$

called the associator, with components

$$a_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W);$$

2. *A natural isomorphism*

$$\lambda : (\mathbb{1} \otimes (-)) \overset{\cong}{\Longrightarrow} (-)$$

with components

$$\lambda_U : \mathbb{1} \otimes U \to U;$$

3. *A natural isomorphism*

$$\rho : (-) \otimes \mathbb{1} \overset{\cong}{\Longrightarrow} (-)$$

with components

$$\rho_U : U \otimes \mathbb{1} \to U.$$

These isomorphisms satisfy the condition that the following diagrams commute:

1. *The triangle identity:*

$$\begin{array}{ccc} (U \otimes \mathbb{1}) \otimes V & \xrightarrow{a_{U\mathbb{1}V}} & U \otimes (\mathbb{1} \otimes V) \\ & {\scriptstyle \rho_U \otimes \mathrm{id}_V} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_U \otimes \lambda_V} & \\ & U \otimes V & \end{array} \tag{4.1}$$

where id_V *is the identity map on* V*.*

2. The pentagon identity:

$$\begin{array}{ccc} & ((U \otimes V) \otimes W) \otimes X & \\ & {\scriptstyle a_{UVW} \otimes \mathrm{id}_X} \swarrow \qquad \searrow {\scriptstyle a_{U \otimes V W X}} & \\ (U \otimes (V \otimes W)) \otimes X & & (U \otimes V) \otimes (W \otimes X) \\ {\scriptstyle a_{U V \otimes W X}} \big\downarrow & & \big\downarrow {\scriptstyle a_{U V W \otimes X}} \\ U \otimes ((V \otimes W) \otimes X) & \xrightarrow[\mathrm{id}_U \otimes a_{VWX}]{} & U \otimes (V \otimes (W \otimes X)) \end{array} \tag{4.2}$$

For additional detail on monoidal categories, see [17] or [20, Chapter 1].

Definition 4.1.5 (Braided Monoidal Category)**.** *A braided monoidal category $\mathcal{C}$ is a monoidal category with a natural isomorphism*

$$c : (-) \otimes (-) \overset{\cong}{\Longrightarrow} \tau \circ (-) \otimes (-)$$

with components

$$c_{U,V} : U \otimes V \to V \otimes U$$

satisfying the following diagrams (frequently referred to as the "hexagon diagrams")

$$\begin{array}{ccccc} (U \otimes V) \otimes W & \xrightarrow{a_{UVW}} & U \otimes (V \otimes W) & \xrightarrow{c_{U,V \otimes W}} & (V \otimes W) \otimes U \\ {\scriptstyle c_{U,V} \otimes \mathrm{id}_W} \big\downarrow & & & & \big\downarrow {\scriptstyle a_{VWU}} \\ (V \otimes U) \otimes W & \xrightarrow[a_{VUW}]{} & V \otimes (U \otimes W) & \xrightarrow[\mathrm{id}_V \otimes c_{U,W}]{} & V \otimes (W \otimes U) \end{array} \tag{4.3}$$

$$\begin{array}{ccccc} U \otimes (V \otimes W) & \xrightarrow{a^{-1}_{UVW}} & (U \otimes V) \otimes W & \xrightarrow{c_{U \otimes V, W}} & W \otimes (U \otimes V) \\ {\scriptstyle \mathrm{id}_U \otimes c_{V,W}} \big\downarrow & & & & \big\downarrow {\scriptstyle a^{-1}_{WUV}} \\ U \otimes (W \otimes V) & \xrightarrow[a^{-1}_{UWV}]{} & (U \otimes W) \otimes V & \xrightarrow[c_{U,W} \otimes \mathrm{id}_V]{} & (W \otimes U) \otimes V \end{array} \tag{4.4}$$

Note that the naturality of c means that for any morphisms $f : V \to V'$ and $g : W \to W'$ we have

$$(g \otimes f) c_{V,W} = c_{V',W'}(f \otimes g).$$

Definition 4.1.6 (Symmetric Monoidal Category)**.** *A braided monoidal category is called symmetric if $c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W}$ for all $V, W \in \mathcal{C}$.*

Definition 4.1.7 (Vect_k)**.** *The category of vector spaces over a field k is $Vect_k$. The subcategory of finite dimensional vector spaces is $vect_k$.*

Example 4.1.8. *The category* $\mathsf{vect}_{\mathsf{k}}$ *is a symmetric monoidal category with the standard definition of the tensor product of vector spaces. The unit is the ground field as a vector space over itself.*

Definition 4.1.9 (Twist)**.** *A twist* θ *in a braided monoidal category* $\mathcal{C}$ *is a natural isomorphism*

$$\theta = (-) \overset{\cong}{\Longrightarrow} (-)$$

with components

$$\theta_V : V \to V$$

such that for all $V, W \in \mathcal{C}$*,*

$$\theta_{V \otimes W} = c_{W,V} c_{V,W} (\theta_V \otimes \theta_W).$$

Note that naturality of these isomorphisms means that for any morphism $f : V \to W$, we have $\theta_W f = f \theta_V$.

Definition 4.1.10 (Left Duality)**.** *Let* $\mathcal{C}$ *be a monoidal category and let* $V \in \mathcal{C}$*. A left dual of* V *is a triple* $(V^\vee, \mathrm{ev}_V, \mathrm{coev}_V)$ *where*

1. $V^\vee \in \mathcal{C}$*,*
2. $\mathrm{ev}_V : V \otimes V^\vee \to \mathbb{1}$*, and*
3. $\mathrm{coev}_V : \mathbb{1} \to V^\vee \otimes V$*,*

subject to the following conditions:

1. $(\mathrm{id}_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}_V) = \mathrm{id}_V$*, and*
2. $(\mathrm{coev}_V \otimes \mathrm{id}_{V^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes \mathrm{ev}_V) = \mathrm{id}_{V^\vee}$*.*

The morphisms ev_V and coev_V are called the evaluation and coevaluation maps, respectively.

Left duality is compatible with the twist if for all $V \in \mathcal{C}$,

$$(\theta_V \otimes \mathrm{id}_{V^\vee}) \circ \mathrm{coev}_V = (\mathrm{id}_V \otimes \theta_{V^\vee}) \circ \mathrm{coev}_V.$$

Right duality is defined similarly. The right evaluation and coevaluation maps are denoted $\widehat{\mathrm{ev}}_V$ and $\widehat{\mathrm{coev}}_V$.

We now give the definition of a rigid monoidal category. See also [7, Definition 2.10.2].

Definition 4.1.11 (Rigid Monoidal Category)**.** *A rigid monoidal category is a monoidal category in which every object has a left and a right dual such that the following compositions are identity morphisms (these relations are often called the snake relations):*

$$V \xrightarrow{\mathrm{coev}_V \otimes \mathrm{id}_V} (V \otimes V^\vee) \otimes V \xrightarrow{a_{V,V^\vee,V}} V \otimes (V^\vee \otimes V) \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} V, \tag{4.5}$$

$$V^\vee \xrightarrow{\mathrm{id}_{V^\vee} \otimes \mathrm{coev}_V} V^\vee \otimes (V \otimes V^\vee) \xrightarrow{a^{-1}_{V^\vee,V,V^\vee}} (V^\vee \otimes V) \otimes V^\vee \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{V^\vee}} V^\vee, \tag{4.6}$$

$$V \xrightarrow{\mathrm{id}_V \otimes \widehat{\mathrm{coev}}_V} V \otimes (V^\vee \otimes V) \xrightarrow{a^{-1}_{V,V^\vee,V}} (V \otimes V^\vee) \otimes V \xrightarrow{\widehat{\mathrm{ev}}_V \otimes \mathrm{id}_V} V, \tag{4.7}$$

$$V^\vee \xrightarrow{\widehat{\mathrm{coev}}_V \otimes \mathrm{id}_{V^\vee}} (V^\vee \otimes V) \otimes V^\vee \xrightarrow{a_{V^\vee,V,V^\vee}} V^\vee \otimes (V \otimes V^\vee) \xrightarrow{\mathrm{id}_{V^\vee} \otimes \widehat{\mathrm{ev}}_V} V^\vee. \tag{4.8}$$

Definition 4.1.12 (Ribbon Category)**.** *A ribbon category is a braided rigid monoidal category $\mathcal{C}$ with a compatible twist.*

Definition 4.1.13 ($\Bbbk$-linear Category)**.** *Let $\Bbbk$ be a field. A $\Bbbk$-linear category which is enriched over $Vect_{\Bbbk}$.*

The definition implies that in a $\Bbbk$-linear category, given any two objects V and W, the set of morphisms from V to W forms a vector space and the composition of such morphisms is $\Bbbk$-linear.

Theorem 4.1.14 ([23, Theorem 4.6])**.** *Let H be a Hopf algebra. The category of homogeneous H-colored ribbon directed graphs forms a ribbon category.*

In this category, the strands of the graphs are morphisms between objects, which are sets of discrete points. Considering the set of links as a subset of the set of graphs, this gives a categorification of links.

A major goal of this paper is to develop examples of ribbon categories that can be used to construct invariants by generalizing the approach in [23]. In the next section we discuss how to construct a link invariant using a ribbon category.

4.2
Ribbon Hopf Algebra

In this section, we give a very basic overview of some of the fundamental structure of a ribbon Hopf algebra. Interested readers can refer to [20, Chapter 4] for a thorough treatment of ribbon Hopf algebras.

Definition 4.2.1 (Universal R-matrix)**.** *Let* H *be a Hopf algebra. A universal* R*-matrix is an invertible element* $R \in H \otimes H$ *which satisfies the following equations:*

1. $\tau \circ \Delta(h) = R\Delta(h)R^{-1}$ *for all* $h \in H$,
2. $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$, *and*
3. $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$,

where $\tau : H \otimes H \to H \otimes H$ *is the swap map* $\tau(h \otimes j) = j \otimes h$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, *and* $R_{13} = \alpha_i \otimes 1 \otimes \beta_i$ *where we write* $R = \sum_i \alpha_i \otimes \beta_i$.

Definition 4.2.2 (Quasitriangular Hopf Algebra)**.** *A quasitriangular Hopf is a pair* (H, R) *where* H *is a Hopf algebra and* R *is a universal* R*-matrix.*

Example 4.2.3. *If* H *is a cocommutative Hopf algebra, then* $R = 1$ *is a universal* R*-matrix.*

Speaking loosely, for an arbitrary quasitriangular Hopf algebra the R-matrix measures how far the Hopf algebra is from being cocommutative, see the first property in Definition 4.2.1.

Definition 4.2.4 (Pivot)**.** *Given a Hopf algebra* H *and a universal* R*-matrix, write* $R = \sum_i \alpha_i \otimes \beta_i$. *Define* $u \in H$ *as* $u = \sum_i S(\beta_i)\alpha_i$, *where* S *is the antipode.*

Theorem 4.2.5 ([20, Propositions 4.1 and 4.2])**.** *Let* (H, R) *be a quasitriangular Hopf algebra. The elements* R *and* u *satisfy the following properties:*

1. u *is invertible with inverse* $u^{-1} = \sum_i S^{-1}(\beta_i')\alpha_i'$, *where we write* $R^{-1} = \sum_i \alpha_i' \otimes \beta_i'$;
2. $S^2(h) = uhu^{-1}$ *for all* $h \in H$;
3. *The quantized Yang-Baxter equation holds, which is* $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$;
4. $(\varepsilon \otimes \mathrm{id})R = 1 = (\mathrm{id} \otimes \varepsilon)R$ *where* ε *is the counit;*
5. $(S \otimes \mathrm{id})R = R^{-1} = (\mathrm{id} \otimes S^{-1})R$;
6. $(S \otimes S)R = R$;
7. $u \otimes u \cdot R = R \cdot u \otimes u$.

Definition 4.2.6 (Twist)**.** *Given a quasitriangular Hopf algebra* (H, R), *a twist* $\nu \in H$ *is an element with the following properties:*

1. ν *is central in* H,

2. $\nu^2 = S(u)u$*,*

3. $\Delta(\nu) = (\tau(R)R)^{-1} \cdot (\nu \otimes \nu)$*,*

4. $S(\nu) = \nu$*, and*

5. $\varepsilon(\nu) = 1$*.*

A quasitriangular Hopf algebra with a twist element is called a ribbon Hopf algebra.

We denote a ribbon Hopf algebra by (H, R, ν). The following theorem motivates our study of ribbon Hopf algebras in the setting of category theory.

Theorem 4.2.7 ([14, Proposition XIV.6.2])**.** *The category of modules of a ribbon Hopf algebra is ribbon.*

4.3
Reshetikhin–Turaev Link Invariant Approach

Our goal is to develop a link invariant using a ribbon category. In order to do this, we use a major theorem from [23, section 5.1].

Fix a field k. Let $\mathcal{R}$ be a k-linear ribbon category with unit $\mathbb{1}$. Before stating the main theorem, we note that every framed, $\mathcal{R}$-colored (link and) tangle determines a homogeneous $\mathcal{R}$-colored directed ribbon graph. Thus, in the sense of the theorem, we consider our links as being in the category from Example 4.1.14. We label that category $\mathcal{H}_{\mathcal{R}}$.

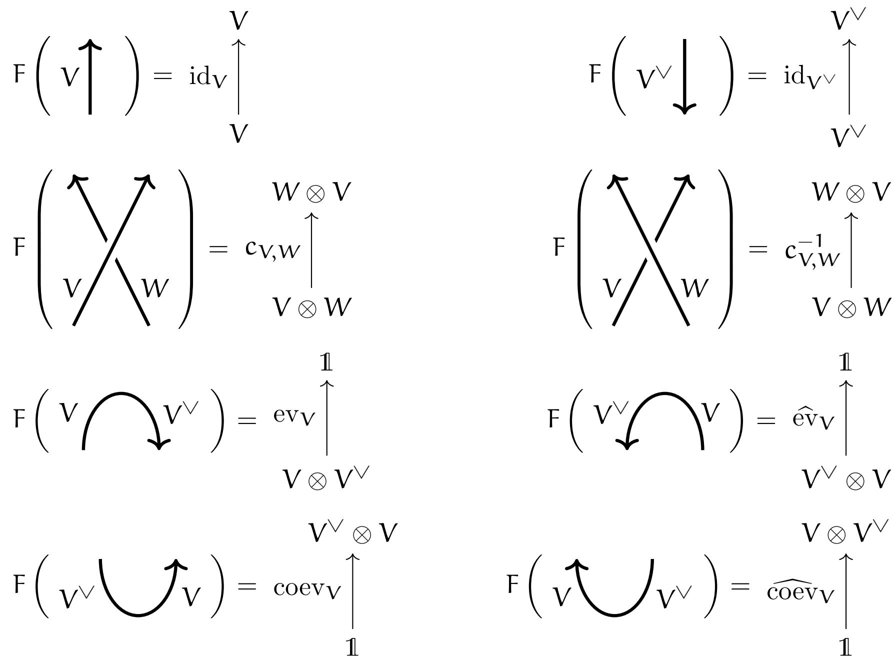
Reshetikhin and Turaev prove the following theorem:

Theorem 4.3.1 ([23, section 5.4])**.** *There exists a unique covariant functor* $F : \mathcal{H}_{\mathcal{R}} \to \mathcal{R}$ *satisfying the following properties:*

1. F *preserves the tensor product.*

2. For any $V \in \mathcal{R}$*,* F *transforms the object* (V, ϵ) *into* $V_i^{\epsilon_i}$*, where* $V^1 = V$ *and* $V^{-1} = V^{\vee}$*.*

3. F *transforms the following tangles into morphisms in* $\mathcal{R}$*:*



In practice, given a ribbon category, the following approach gives a link invariant:

1. Let L be a framed oriented link. Fix an ordering of the components of L. Color each component by an object in $\mathcal{R}$.

2. Choose a diagram D of L.

3. Break D into elementary pieces, which are the tangles in property 3) of Theorem 4.3.1.

4. Transform each elementary piece into a morphism in $\mathcal{R}$ by F.

5. Compose these morphisms. By convention, diagrams are read, and morphisms composed, from bottom to top.

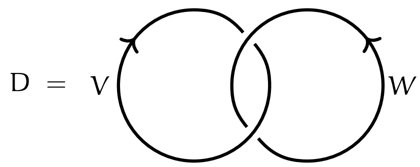
The ribbon category is assumed to be k-linear. Therefore, given a link diagram D, $\mathsf{F}(\mathsf{D})$ is simply an endomorphism of the unit. The vector space of endomorphsims of $\mathbb{1}$ is isomorphic to $\mathsf{k} \cdot \mathrm{id}_{\mathbb{1}}$. Thus, F effectively gives a map from the set of links into the ground field. We use the following notation to denote the image of F: given a link diagram D,

$$\mathsf{F}(\mathsf{D}) = \langle \mathsf{D} \rangle \cdot \mathrm{id}_{\mathbb{1}}.$$

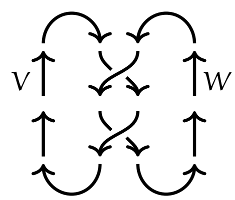
Corollary 4.3.2. *Let* L *be a link. The assignment* $\mathsf{L} \mapsto \langle \mathsf{D}_\mathsf{L} \rangle$ *where* D_L *is any diagram of* L*, is an isotopy invariant of L.*

See [23, Section 5] for a precise explanation of how Corollary 4.3.2 follows from Theorem 4.3.1.

Example 4.3.3. *Let* H *be a ribbon Hopf algebra and* $\mathcal{R}$ *the category of* H*-modules. Let* L *be the Hopf link colored by modules* $\mathsf{V}, \mathsf{W} \in \mathcal{R}$*. We make the following choice of diagram* D *for* L*:*



We break D *into the following elementary pieces:*



Then $\mathsf{F}(\mathsf{D}) = (\widehat{\mathrm{coev}}_\mathsf{V} \otimes \mathrm{coev}_W) \circ (\mathrm{id}_{V^\vee} \otimes \mathsf{c}_{W,V} \otimes \mathrm{id}_{W^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes \mathsf{c}_{V,W} \otimes \mathrm{id}_{W^\vee}) \circ (\mathrm{ev}_\mathsf{V} \otimes \widehat{\mathrm{ev}}_W)$.

Throughout the rest of this paper, we discuss the link invariant obtained using each of three different ribbon categories: the modules of the abelian quantum group, the modules of $\mathfrak{gl}(1|1)$, and a subcategory of modules of the unrolled quantum group.

Chapter 5
ABELIAN QUANTUM GROUP

The first example of a ribbon category that we consider is the category of modules of the abelian quantum group. We prove that the category is ribbon by showing that the abelian quantum group is a ribbon Hopf algebra and invoking Theorem 4.2.7. This example was originally given, albeit with less detail, in [19]. This example is also briefly treated in [18, Examples 2.1.6 and 2.1.11]. The case when r is odd is treated in great detail in [10].

Throughout this section, fix $r \geq 2$ an integer and fix $q \in \mathbb{C}^*$ a primitive rth root of unity.

5.1
Ribbon Category of Modules

Definition 5.1.1 (Abelian Quantum Group)**.** *Let $\mathbb{Z}_r$ be the additive cyclic group of order r. Define $A_q = \mathbb{C}[\mathbb{Z}_r]$ the group algebra of $\mathbb{Z}_r$ with the Hopf algebra structure as in example 3.1.24. Define $R \in A_q \otimes A_q$ by*

$$R = \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-ij} e_i \otimes e_j.$$

Define a twist on A_q by

$$\nu = \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-i(i+j)} e_j.$$

Theorem 5.1.2. (A_q, R, ν) *is a ribbon Hopf algebra.*

A proof of this theorem is discussed in [19, Section 7]. Our proof repeatedly uses a helpful trick, which we prove here as a lemma.

Lemma 5.1.3. *For any function $f : \mathbb{Z}_r \to \mathbb{Z}_r$,*

$$\frac{1}{r} \sum_{a,b \in \mathbb{Z}_r} q^{bf(a)} = \sum_{\substack{a \in \mathbb{Z}_r \\ f(a)=0}} 1.$$

Proof. By the geometric formula, if $a \neq 1$ is an rth root of unity, then

$$\sum_{b\in\mathbb{Z}_r} a^b = \frac{1-a^r}{1-a} = 0, \tag{5.1}$$

We examine the sum

$$\frac{1}{r}\sum_{a\in\mathbb{Z}_r}\sum_{b\in\mathbb{Z}_r} q^{bf(a)}.$$

on a fixed index a. For each a, there are two cases:

1. If $f(a) = 0$, then $\sum_{b\in\mathbb{Z}_r} q^{bf(a)} = \sum_{b\in\mathbb{Z}_r} q^0 = r$.

2. Assume $f(a) \neq 0$. Then $q^{f(a)}$ is an rth root of unity not equal to 1, so

$$\sum_{b\in\mathbb{Z}_r} q^{bf(a)} = \sum_{b\in\mathbb{Z}_r} \left(q^{f(a)}\right)^b,$$

   which is zero by formula (5.1).

The result follows. □

Proof of Theorem 5.1.2. We first show that R is a universal R-matrix by showing that it satisfies the conditions in Definition 4.2.1.

We first show that R is a unit. Define

$$R^{-1} = \frac{1}{r}\sum_{k,l\in\mathbb{Z}_r} q^{kl} e_{-k}\otimes e_{-l}.$$

Now, the product

$$RR^{-1} = \frac{1}{r^2}\sum_{i,j,k,l\in\mathbb{Z}_r} q^{-ij+kl} e_{i-k}\otimes e_{j-l}.$$

Set $m = i-k$ and $n = j-l$. Then the sum above can be written as

$$\frac{1}{r^2}\sum_{k,l,m,n\in\mathbb{Z}_r} q^{-(m+k)(n+l)+kl} e_m\otimes e_n = \frac{1}{r^2}\sum_{k,l,m,n\in\mathbb{Z}_r} q^{-mn-kn-ml} e_m\otimes e_n.$$

We use Lemma 5.1.3, with k as b. We have

$$\frac{1}{r^2}\sum_{k,l,m,n\in\mathbb{Z}_r} q^{-mn-kn-ml} e_m\otimes e_n = \frac{1}{r}\sum_{l,m,n\in\mathbb{Z}_r} q^{-mn-ml}\frac{1}{r}\sum_{k\in\mathbb{Z}_r} q^{-kn} e_m\otimes e_n$$

By Lemma 5.1.3, if $n\neq 0$, then $\sum_{k\in\mathbb{Z}_r} q^{-kn} = 0$. We can therefore assume that $n = 0$. The sum reduces to

$$\frac{1}{r}\sum_{l,m\in\mathbb{Z}_r} q^{-ml} e_m\otimes e_0.$$

By Lemma 5.1.3 again, we can assume that $m = 0$, so the sum simplifies to $e_0\otimes e_0 = 1$.

Because A_q is commutative, $R^{-1}R = 1$ as well.

We now show that $\tau\circ\Delta(x) = R\Delta(x)R^{-1}$ for all $x\in A_q$. Because A_q is cocommutative,

$$\tau\circ\Delta(x) = \Delta(x).$$

Because A_q is commutative, so too is $A_q\otimes A_q$. Therefore,

$$\begin{aligned} R\Delta(x)R^{-1} &= RR^{-1}\Delta(x) \\ &= \Delta(x). \end{aligned}$$

Thus, $\tau\circ\Delta(x) = R\Delta(x)R^{-1}$.

Now, we show that $(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}$. The proof that $(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23}$ is very similar and is left to the reader. We have

$$R_{13}R_{12} = \frac{1}{r^2}\sum_{i,j,k,l\in\mathbb{Z}_r} q^{-ij-kl} e_{i+k}\otimes e_j\otimes e_l$$

Let $m = i + k$. Then we can rewrite the sum above as

$$\frac{1}{r^2}\sum_{i,j,l,m\in\mathbb{Z}_r} q^{-ij-(m-i)l} e_m\otimes e_j\otimes e_l = \frac{1}{r^2}\sum_{j,l,m\in\mathbb{Z}_r} q^{-ml}\sum_{i\in\mathbb{Z}_r} q^{-i(j-l)} e_m\otimes e_j\otimes e_l.$$

By Lemma 5.1.3, the terms of the sum are zero unless $l = j$. The sum simplifies to

$$\frac{1}{r}\sum_{j,m\in\mathbb{Z}_r} q^{-jm} e_m\otimes e_j\otimes e_j.$$

On the other hand,

$$(\mathrm{id} \otimes \Delta)(R) = \frac{1}{r} \sum_{a,b \in \mathbb{Z}_r} q^{-ab} e_a \otimes e_b \otimes e_b.$$

Therefore, the desired relation holds.

Thus, (A_q, R) is a quasitriangular Hopf algebra.

Using Definition 4.2.4, we write

$$u = \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-ij} S(e_j) e_i = \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-ij} e_{i-j}.$$

We now show that the twist satisfies the conditions in Definition 4.2.6.

The first condition, that ν is central in A_q, follows because A_q is commutative.

We now show that $\nu^2 = S(u)u$. On the left, we have

$$\begin{aligned}
\nu^2 &= \frac{1}{r^2} \left(\sum_{i,j \in \mathbb{Z}_r} q^{-i(i+j)} e_j \right) \left(\sum_{k,l \in \mathbb{Z}_r} q^{-k(k+l)} e_l \right) \\
&= \frac{1}{r^2} \left(\sum_{i,j,k,l \in \mathbb{Z}_r} q^{-i(i+j)-k(k+l)} e_{j+l} \right).
\end{aligned}$$

We examine this sum term-by-term. The coefficient for e_m is

$$\begin{aligned}
\frac{1}{r^2} \sum_{\substack{i,j,k,l \in \mathbb{Z}_r \\ j+l=m}} q^{-i(i+j)-k(k+l)} &= \frac{1}{r^2} \sum_{i,j,k \in \mathbb{Z}_r} q^{-i(i+j)-k(k+m-j)} \\
&= \frac{1}{r^2} \sum_{i,j,k \in \mathbb{Z}_r} q^{-i^2-ij-k^2-km+kj} \\
&= \frac{1}{r} \sum_{i,k \in \mathbb{Z}_r} q^{-i^2-k^2-km} \frac{1}{r} \sum_{j} q^{j(k-i)} \\
&= \frac{1}{r} \sum_{i \in \mathbb{Z}_r} q^{-2i^2-im}
\end{aligned}$$

where the last line follows by using Lemma 5.1.3 to conclude that the right sum is zero unless $k = i$.

On the right hand side, we have

$$\begin{aligned}
S(u)u &= \frac{1}{r^2} \left(\sum_{i,j \in \mathbb{Z}_r} q^{-ij} e_{j-i} \right) \left(\sum_{k,l \in \mathbb{Z}_r} q^{-kl} e_{k-l} \right) \\
&= \frac{1}{r^2} \sum_{i,j,k,l \in \mathbb{Z}_r} q^{-ij-kl} e_{j-i+k-l}.
\end{aligned}$$

The coefficient for e_m is

$$\begin{aligned}\frac{1}{r^2}\sum_{\substack{i,j,k,l\in\mathbb{Z}_r\\ j-i+k-l=z}} q^{-ij-kl} &= \frac{1}{r^2}\sum_{i,j,k\in\mathbb{Z}_r} q^{-ij-k(j-i+k-m)}\\ &= \frac{1}{r^2}\sum_{i,j,k\in\mathbb{Z}_r} q^{-ij-kj+ik-k^2+km}\\ &= \frac{1}{r}\sum_{i,k\in\mathbb{Z}_r} q^{ik-k^2+km}\frac{1}{r}\sum_{j\in\mathbb{Z}_r} q^{j(-i-k)}\\ &= \frac{1}{r}\sum_{i\in\mathbb{Z}_r} q^{-2i^2-im}\end{aligned}$$

where the last equality again follows by using Lemma 5.1.3 to conclude that the rightmost sum is zero unless $k=-i$. Since the coefficients agree for each term, the two sums are equal.

Next, we show that $\Delta(\nu)=(\nu\otimes\nu)\cdot(\tau(R)R)^{-1}$. On the left, we have

$$\Delta(\nu)=\frac{1}{r}\sum_{i,j\in\mathbb{Z}_r} q^{-i(i+j)}e_j\otimes e_j.$$

On the right, we have

$$\begin{aligned}(\tau(R)R)^{-1}\cdot(\nu\otimes\nu) &= R^{-2}\cdot(\nu\otimes\nu)\\ &= \frac{1}{r^4}\sum_{\substack{i,j,k,l,\\ s,t,u,w\in\mathbb{Z}_r}} q^{ij+kl-s(s+t)-u(u+w)}e_{-i-k+t}\otimes e_{-j-l+w}.\end{aligned}$$

We set $m=-i-k+t$ and $n=-j-l+w$. Then the sum above is equal to

$$\begin{aligned}&\frac{1}{r^4}\sum_{\substack{i,j,k,l,\\ m,n,s,u\in\mathbb{Z}_r}} q^{ij+kl-s(s+m+i+k)-u(u+n+j+l)}e_m\otimes e_n\\ &= \frac{1}{r^4}\sum_{\substack{i,j,k,l,\\ m,n,s,u\in\mathbb{Z}_r}} q^{i(j-s)+l(k-u)-s(s+m+k)-u(u+n+j)}e_m\otimes e_n\\ &= \frac{1}{r^3}\sum_{\substack{j,k,l,\\ m,n,s,u\in\mathbb{Z}_r}} q^{l(k-u)-s(s+m+k)-u(u+n+j)}\frac{1}{r}\sum_{i\in\mathbb{Z}_r} q^{i(j-s)}e_m\otimes e_n.\end{aligned}$$

By Lemma 5.1.3, the right sum is nonzero only when $j=s$. The sum simplifies to

$$\frac{1}{r^2}\sum_{\substack{j,k,m,\\ n,u\in\mathbb{Z}_r}} q^{-j(j+m+k)-u(u+n+j)}\frac{1}{r}\sum_{l\in\mathbb{Z}_r} q^{l(k-u)}e_m\otimes e_n.$$

Again, the right sum is nonzero only when $k = u$. Simplifying further,

$$\frac{1}{r^2} \sum_{j,k,m,n \in \mathbb{Z}_r} q^{-j(j+m+k)-k(k+n+j)} e_m \otimes e_n.$$

Set $z = j + k$ The sum equals

$$\begin{aligned}
\frac{1}{r^2} \sum_{j,z,m,n \in \mathbb{Z}_r} q^{-j(m+z)-(z-j)(z+n)} e_m \otimes e_n &= \frac{1}{r^2} \sum_{j,z,m,n \in \mathbb{Z}_r} q^{-j(m-n)-z(z+n)} e_m \otimes e_n \\
&= \frac{1}{r} \sum_{z,m,n \in \mathbb{Z}_r} q^{-z(z+n)} \frac{1}{r} \sum_{j \in \mathbb{Z}_r} q^{-j(m-n)} e_m \otimes e_n \\
&= \frac{1}{r} \sum_{z,m \in \mathbb{Z}_r} q^{-z(z+m)} e_m \otimes e_m \\
&= \Delta(\nu).
\end{aligned}$$

Next, we show that $S(\nu) = \nu$. We have

$$\begin{aligned}
S(\nu) &= \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-i(i+j)} e_{-j} \\
&= \frac{1}{r} \sum_{k,l \in \mathbb{Z}_r} q^{k(-k-l)} e_l \\
&= \nu
\end{aligned}$$

where the second equality follows from setting $k = -i$ and $l = -j$.

Finally, we show that $\varepsilon(\nu) = 1$. We have

$$\begin{aligned}
\varepsilon(\nu) &= \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-i(i+j)} \\
&= \frac{1}{r} \sum_{i \in \mathbb{Z}_r} \sum_{j \in \mathbb{Z}_r} q^{-i^2-ij} \\
&= 1
\end{aligned}$$

by Lemma 5.1.3. □

5.2
Using the Reshetikhin–Turaev Link Invariant Approach

We now explore the invariant constructed using the Reshetikhin–Turaev approach with the ribbon Hopf algebra A_q. In this section, we work out the details of calculations which are stated in [19, section 7].

Recall from Example 3.2.4 that the modules of A_q are all semisimple and that there are exactly r many simple modules, all of which are one-dimensional. We label these representations V_k, $k \in \{0, \cdots, r-1\}$, where the generator $e_1 \in A_q$ acts on V_k by q^k. Let $\mathsf{Mod}(A_q)$ be the ribbon category of modules of A_q.

We compute on $v_k \in V_k$,

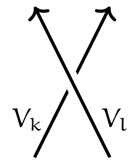
$$\begin{aligned} u \cdot v_k &= \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-ij} e_{i-j} \cdot v_k \\ &= \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-ij+(i-j)k} v_k \\ &= \frac{1}{r} \sum_{j \in \mathbb{Z}_r} \sum_{i \in \mathbb{Z}_r} q^{-i(j-k)-jk} v_k \\ &= q^{-k^2} v_k \end{aligned}$$

where the last equality holds by Lemma 5.1.3. Also,

$$\begin{aligned} v \cdot v_k &= \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-i(i+j)} e_j \cdot v_k \\ &= \frac{1}{r} \sum_{i,j \in \mathbb{Z}_r} q^{-i(i+j)+jk} v_k \\ &= \frac{1}{r} \sum_{i \in \mathbb{Z}_r} \sum_{j \in \mathbb{Z}_r} q^{j(-i+k} - i^2 v_k \\ &= q^{-k^2} e_j \end{aligned}$$

where the last line holds by Lemma 5.1.3. Therefore, $v^{-1}u \cdot v_k = v_k$, which implies that $u^{-1}v \cdot v_k = v_k$. Thus, when evaluating the composition after acting on a diagram of a framed link by F, there are no constants which need to be considered by the $\widehat{\text{coev}}$ or $\widehat{\text{ev}}$ maps, just as there are no constants introduced by coev and ev. Recall that in Theorem 4.2.7 we defined $\widehat{\text{ev}}$ to act by $v^{-1}u$ and $\widehat{\text{coev}}$ to act by $u^{-1}v$.

Now, we consider the braidings of irreducible representations V_k and V_l. In the case

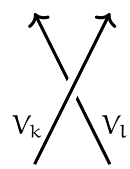


(5.2)

we have

$$\begin{aligned} v_k \otimes v_l &\to \frac{1}{r}\sum_{i,j\in\mathbb{Z}_r} q^{-ij} e_j \cdot v_l \otimes e_i \cdot v_k \\ &= \frac{1}{r}\sum_{i,j\in\mathbb{Z}_r} q^{-ij+jl+ik} v_l \otimes v_k \\ &= \frac{1}{r}\sum_{i\in\mathbb{Z}_r}\sum_{j\in\mathbb{Z}_r} q^{j(-i+l)+ik} v_l \otimes v_k \\ &= q^{kl} v_l \otimes v_k. \end{aligned}$$

A similar computation, in the case



gives

$$v_k \otimes v_l \to q^{-kl} v_l \otimes v_k.$$

Let L be a framed oriented link with an ordering of components from 1 to n and a coloring given by $c = (i_1, \cdots, i_n)$. By Corollary 4.3.2, the functor $F : \mathcal{H} \to \mathsf{Mod}(A_q)$ gives a link invariant. We can identify it with a familiar classical link invariant.

Theorem 5.2.1. *Given a framed link* L *with diagram* D*,*

$$\langle D \rangle = q^{\frac{1}{2}c^T M(D) c} \tag{5.3}$$

where $M(D)$ *is the linking matrix of* D *from Example 2.2.4.*

Proof. We first consider the evaluation and coevaluation maps. By the compactness and closedness of links, each link diagram has an equal number of coevaluation and evaluation maps. Because the evaluation and coevaluation maps contribute a factor of $q^{-i_k^2}$ and $q^{i_k^2}$, respectively, the scalars from these maps cancel each other out in the invariant.

Now we examine the effect of a single crossing on both sides of Equation (5.3). Assume that the lth component crosses over the kth component. Then we are in the case of Equation (5.2). As we computed, the scalar that comes from this crossing is $q^{i_k i_l}$.

We now work out the effect of this crossing on the right hand side. First, we write the linking matrix as the following sum

$$M(D) = \sum_{i,j} M_{i,j} E_{i,j}$$

where $M_{i,j}$ is the number of crossings of the ith and jth components with sign and $E_{i,j}$ is the matrix with a 1 in the (i,j) position and zeros elsewhere. Written this way, the product $c^T M(D) c$ can be easily rewritten as

$$\begin{aligned} c^T M(D) c &= \sum_{i,j} M_{i,j} c^T E_{i,j} c \\ &= \sum_{i,j} M_{i,j} c_i c_j. \end{aligned}$$

In the linking matrix, the crossing contributes a value of $+1$ to $M_{l,k}$ and -1 to $M_{k,l}$. Therefore, on the right hand side of the equation, this crossing contributes a value of

$$q^{\frac{1}{2}\cdot(-1\cdot(-i_k i_l)+1\cdot(i_l i_k))} = q^{i_k i_l}.$$

Since each crossing in the link contributes the same scalar to both sides of Equation (5.3), the two sides must be equal. □

We see from this example that the Reshetikhin–Turaev approach applied to a semisimple category yields a nontrivial link invariant. Morevoer, we recognize the invariant as new presentation of an already familiar classical invariant. In the next section we see an example of the Reshetikhin–Turaev method applied to a non-semisimple ribbon category.

Chapter 6
THE QUANTUM LIE SUPERALGEBRA OF $\mathfrak{gl}(1|1)$

Our goal in this section is to construct a nontrivial link invariant using representations of the quantum Lie superalgebra of $\mathfrak{gl}(1|1)$. This example serves two purposes. Firstly, the calculations needed to prove that the quantum Lie superalgebra has a braided category of modules are simpler than in the case of the unrolled quantum group. Therefore, in this example, we can identify the necessary parts of the theory without going into the weeds on lengthy computations, giving a satisfactory preview with which to compare the unrolled quantum group. Secondly, this example is interesting in its own right because it is a relatively easy link invariant to construct using a non-semisimple category.

This example is explained in [26, Sections 2 and 7] and [24].

6.1
Superalgebras, Supercoalgebras, Superbialgebras, and Hopf Superalgebras

In the following discussion we deal with *super* objects. The prefix *super* denotes a $\mathbb{Z}_2$-grading on the object. The development of this section roughly follows Section 3.1, although with fewer examples and less commentary. A good reference for the material in this section is [25, Section 3.1].

Fix a ground field k.

Definition 6.1.1 (Super Vector Space)**.** *A super vector space* A *is a* $\mathbb{Z}_2$*-graded vector space over* k *with a decomposition into two subspaces*

$$A = A_0 \oplus A_1, \quad 0, 1 \in \mathbb{Z}_2.$$

Vectors in A_0 are homogeneous of degree zero and vectors in A_1 are homogeneous of degree one. Homogeneous vectors of degree zero are called *even*, while homogeneous vectors of degree one are called *odd*. For a homogeneous vector $a \in A$, we denote by $|a|$ the degree of a. Anytime the degree of a vector is taken, it is assumed that the vector is homogeneous.

Example 6.1.2. *Given two super vector spaces* A *and* B*, the tensor product* $A \otimes B$ *is a super vector space with grading given by the rule* $|a \otimes b| = |a| + |b|$.

Define a graded swap map τ on super vector spaces as

$$\tau : A \otimes A \to A \otimes A$$
$$a_1 \otimes a_2 = (-1)^{|a_1||a_2|} a_2 \otimes a_1.$$

A map from a super vector space to a super vector space is said to be of degree zero if it preserves degree.

Definition 6.1.3 (Superalgebra)**.** *A superalgebra* A *is a super vector space over* k *with degree zero maps* $\cdot : A \otimes A \to A$*, called multiplication, and* $i : k \to A$*, called the unit, which satisfy the associativity and unital conditions in Equation* (3.1) *in Definition 3.1.1.*

The statement that $\cdot$ is a degree zero map means that for homogeneous elements a_i and a_j of degrees i and j respectively,

$$a_i \cdot a_j \in A_{i+j},$$

where A_{i+j} is the subspace of A of degree $i + j$.

Compare the following with Example 3.1.7.

Example 6.1.4. *Let* A *be a superalgebra. The super vector space* $A \otimes A$ *is a superalgebra with multiplication*

$$\cdot : (A \otimes A) \otimes (A \otimes A) \to A \otimes A$$
$$a_1 \otimes a_2 \cdot a_3 \otimes a_4 = (-1)^{|a_2||a_3|} a_1 a_3 \otimes a_2 a_4.$$

The sign in the definition of the product comes from τ*.*

Definition 6.1.5 (Supercoalgebra)**.** *A supercoalgebra* A *is a super vector space over* k *with a degree zero maps* $\Delta : A \to A \otimes A$*, called the coproduct, and* $\varepsilon : A \to k$*, called the counit, which satisfy the coassociativity and counital conditions in Equation* (3.2) *of Definition 3.1.10.*

Theorem 3.1.18 gives a compatibility condition for algebras and coalgebras. A modification of the proof of that theorem for super vector spaces gives a proof of the following theorem:

Theorem 6.1.6. *Let* A *be a vector space over* k *with superalgebra structure* $(A, \cdot, i)$ *and supercoalgebra structure* (A, Δ, ε)*. Then* $\cdot$ *and* i *are supercoalgebra homomorphisms if and only if* Δ *and* ε *are superalgebra homomorphisms.*

Definition 6.1.7 (Superbialgebra)**.** *A superbialgebra* A *is an algebra which is also a supercoalgebra which satisfies the condition that the coproduct and counit are superalgebra homomorphisms.*

Definition 6.1.8 (Hopf Superalgebra)**.** *A Hopf superalgebra* H *is a superbialgebra with a degree zero map* $S : A \to A$*, called the antipode, which satisfies the commutativity of the diagram in Equation* (3.8) *in Definition 3.1.22.*

A Hopf superideal of a Hopf superalgebra is a Hopf ideal that respects the grading. The proof of Theorem 3.1.27 can be easily modified to prove the following theorem:

Theorem 6.1.9. *The quotient of a Hopf superalgebra by a Hopf superideal is a Hopf superalgebra.*

6.2
Lie Superalgebras

In this section, we define Lie superalgebras and give our main example, the Lie superalgebra $\mathfrak{gl}(1|1)$.

Fix a ground field k. Compare the following definition with Definition 3.3.1.

Definition 6.2.1 (Lie Superalgebra)**.** *A Lie superalgebra is a super vector space* $\mathfrak{g}$ *over* k *together with a bilinear map*

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
$$(v, w) \mapsto [v, w]$$

which satisfies the following conditions: for all $x, y, z \in \mathfrak{g}$*,*

1. $[x, y] = -(-1)^{|x||y|}[y, x]$ *and*
2. *The super Jacobi identity holds:* $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$

Notice that if $\mathfrak{g} = \mathfrak{g}_0$ is homogeneous of degree 0 then the conditions above reduce to the conditions from the standard definition of a Lie algebra.

While there are many interesting examples of Lie superalgebras, for the purposes of this paper we will consider a single example, which we will use to construct a link invariant.

Let $\mathbb{C}$ be the ground field. Let $\mathbb{C}^{1|1}$ be the two-dimensional super vector space with decomposition

$$\mathbb{C}^{1|1} = \mathbb{C}_0 \oplus \mathbb{C}_1,$$

where $\mathbb{C}_0$ is the one-dimensional even subspace and $\mathbb{C}_1$ is the one-dimensional odd subspace.

Let $\mathfrak{gl}(1|1)$ be the super vector space of endomorphisms of $\mathbb{C}^{1|1}$ with the following naturally defined grading: even elements are those which preserve the degree of homogeneous vectors in $\mathbb{C}^{1|1}$, while odd elements are those which swap degree. Define a bracket, called the supercommutator, on homogeneous endomorphisms $a, b \in \mathfrak{gl}(1|1)$ by

$$[a, b] = ab - (-1)^{|a||b|} ba.$$

With this choice of bracket, $\mathfrak{gl}(1|1)$ is a Lie superalgebra. We think of the endomorphisms in $\mathfrak{gl}(1|1)$ as 2×2 matrices and use the following convenient choice of basis:

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

All four basis vectors are homogeneous. The vectors h_1 and h_2 have degree 0 while e and f have degree 1. In this basis, the following relations can be easily calculated:

$$[h_1, e] = e, \quad [h_1, f] = -f, \quad [h_2, e] = -e, \quad [h_2, f] = f,$$

$$[h_1, h_2] = 0, \quad [e, f] = h_1 + h_2, \quad [e, e] = 0, \quad [f, f] = 0.$$

Note that the final two relations do not follow from the definition of a Lie superalgebra since e and f are odd. They must be checked by hand.

Let $\mathfrak{h} \subset \mathfrak{gl}(1|1)$ be the Cartan subalgebra, which is the span of h_1 and h_2. Let $\mathfrak{h}^\vee$ be the dual of $\mathfrak{h}$. Let $\{\varepsilon_1, \varepsilon_2\}$ be the basis dual to h_1, h_2. We use the following shorthand for evaluation using dual vectors: for $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^\vee$, we write $\langle h, \lambda \rangle = \lambda(h)$. Let $\alpha = \varepsilon_1 - \varepsilon_2$. We can write any

$h \in \mathfrak{h}$ as $h = ah_1 + bh_2$ for scalars $a, b \in \mathbb{C}$. Then

$$\begin{aligned}[h, e] &= a[h_1, e] + b[h_2, e] \\ &= ae - be \\ &= \langle h, \alpha \rangle e.\end{aligned}$$

A similar calculation shows that $[h, f] = -\langle h, \alpha \rangle f$. From these calculations, we see that e and f are roots with weights α and $-\alpha$.

Let $q \in \mathbb{C}^* \setminus \{\pm 1\}$ not be a root of unity. Define $Q(\mathfrak{gl}(1|1))$ to be the unital $\mathbb{C}$-superalgebra with generators given by

$$\begin{aligned}&q^h, \quad h \in \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \quad \text{(degree } 0\text{)} \\ &E, F \quad \text{(degree } 1\text{)}\end{aligned}$$

and relations $q^h q^{h'} = q^{h+h'}$ and $q^0 = 1$. Define a coproduct Δ, counit ϵ, and antipode S on generators

$$\begin{aligned}\Delta(q^h) &= q^h \otimes q^h & \epsilon(q^h) &= 1 & S(q^h) &= q^{-h} \\ \Delta(E) &= E \otimes K^{-1} + 1 \otimes E & \epsilon(E) &= 0 & S(E) &= -EK \\ \Delta(F) &= F \otimes 1 + K \otimes F & \epsilon(F) &= 0 & S(F) &= -FK^{-1}\end{aligned}$$

where $K = q^{h_1+h_2}$. Extend Δ and ϵ to superalgebra homomorphisms and S to a superalgebra antihomomorphism. In particular, for homogeneous elements $a, b \in Q(\mathfrak{gl}(1|1))$, $S(ab) = (-1)^{|a||b|} S(b)S(a)$.

Lemma 6.2.2. *The structure above makes* $Q(\mathfrak{gl}(1|1))$ *into a Hopf superalgebra.*

Proof. By construction, Δ and ϵ are algebra homomorphisms. We need to show that these maps satisfy the diagram in Definition 3.1.22 on all generators. We calculate the diagram for E. The middle path of the diagram is

$$E \overset{\epsilon}{\longmapsto} 0 \overset{i}{\longmapsto} 0.$$

The top path of the diagram is

$$E \overset{\Delta}{\longmapsto} E \otimes K^{-1} + 1 \otimes E \overset{S \otimes \mathrm{id}}{\longmapsto} -EK \otimes K^{-1} + 1 \otimes E \overset{\cdot}{\longmapsto} -E + E = 0.$$

The bottom path of the diagram is

$$\mathsf{E} \overset{\Delta}{\longmapsto} \mathsf{E} \otimes \mathsf{K}^{-1} + 1 \otimes \mathsf{E} \overset{\mathrm{id} \otimes S}{\longmapsto} \mathsf{E} \otimes \mathsf{K} + 1 \otimes (-\mathsf{E}\mathsf{K}) \overset{\cdot}{\longmapsto} \mathsf{E}\mathsf{K} - \mathsf{E}\mathsf{K} = 0.$$

The calculations to check the other generators are similar. □

Let I be the ideal of $Q(\mathfrak{gl}(1|1))$ generated by

$$q^h E - q^{\langle h, \alpha \rangle} E q^h, \quad EF + FE - \frac{K - K^{-1}}{q - q^{-1}},$$
$$q^h F - q^{-\langle h, \alpha \rangle} F q^h, \quad E^2, \quad F^2.$$

Lemma 6.2.3. *The ideal* I *is a Hopf superideal.*

Proof. We check the conditions that $S(I) \subset I$ on the relation $EF + FE = \frac{K - K^{-1}}{q - q^{-1}}$ and leave the rest of the proof to the reader. We have

$$\begin{aligned} S\left(EF + FE - \frac{K - K^{-1}}{q - q^{-1}}\right) &= -S(F)S(E) - S(E)S(F) - \frac{S(K) - S(K^{-1})}{q - q^{-1}} \\ &= -FK^{-1}EK - EKFK^{-1} + \frac{K - K^{-1}}{q - q^{-1}} \\ &= -q^{-\langle h_1 + h_2, \alpha \rangle}(FE + EF) + \frac{K - K^{-1}}{q - q^{-1}} \\ &= -\left(EF + FE - \frac{K - K^{-1}}{q - q^{-1}}\right) \in I. \square \end{aligned}$$

Definition 6.2.4. *The quantum enveloping superalgebra of* $\mathfrak{gl}(1|1)$ *is* $U_q(\mathfrak{gl}(1|1)) = Q(\mathfrak{gl}(1|1))/I$.

By Theorem 6.1.9, $U_q(\mathfrak{gl}(1|1))$ is a Hopf superalgebra.

Note that in $U_q(\mathfrak{gl}(1|1))$, K is a central element, which implies that $EF + FE$ is also central. This makes the calculations in the following sections, which verify important properties about the modules of $U_q(\mathfrak{gl}(1|1))$, easier than the corresponding calculations for the DeConcini-Kac quantum group or the unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$.

6.3
Simple Modules

We now consider simple modules of $U_q(\mathfrak{gl}(1|1))$. Let $P = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2 \subset \mathfrak{h}^\vee$. We define a $\mathbb{Z}_2$-grading on P by

$$|\varepsilon_1| = 0, \quad |\varepsilon_2| = 1.$$

We consider only finite dimensional weight $U_q(\mathfrak{gl}(1|1))$-modules. A weight $U_q(\mathfrak{gl}(1|1))$-modules is a supermodule M such that

$$M = \bigoplus_{\lambda \in P} M_\lambda, \quad q^h|_{M_\lambda} = q^{\langle h,\lambda\rangle} \mathrm{id}_{M_\lambda}, \quad |M_\lambda| = |\lambda|.$$

In other words, M decomposes into weight spaces M_λ with integral weights $\lambda \in P$ such that q^h acts by $q^{\langle h,\lambda\rangle}$ on M_λ.

Since h_1 and $h_1 + h_2$ span $\mathbb{Z}h_1 \oplus \mathbb{Z}h_2$, every q^h, $h \in \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$, can be written as a product of integer powers of q^{h_1} and K. Thus, when we define modules of $U_q(\mathfrak{gl}(1|1))$, we need only define the actions of q^{h_1}, K, E, and F.

Let $\mathrm{Ann}(h_1 + h_2) \subset \mathfrak{h}^\vee$ be the subgroup of annihilators of the vector $h_1 + h_2$. For $\lambda \in \mathfrak{h}^\vee$, if $\lambda \in \mathrm{Ann}(h_1 + h_2)$, we say that λ is *atypical*. If $\lambda \notin \mathrm{Ann}(h_1 + h_2)$, we say that λ is *typical*.

Assume that λ is atypical. Define a one-dimensional representation, denoted $L(\lambda)$, as $\mathbb{C} \cdot v_0$ for a vector v_0 with the following properties:

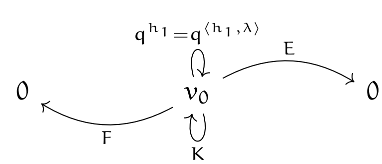
$$\begin{aligned} Kv_0 &= q^{\langle h_1+h_2,\lambda\rangle} v_0 \\ &= q^0 v_0 \\ &= v_0, \end{aligned}$$

$$q^{h_1} v_0 = q^{\langle h_1,\lambda\rangle} v_0,$$

$$Ev_0 = 0, \text{ and } Fv_0 = 0.$$

We verify that the relations in I hold. We check the relation $EF + FE = \frac{K-K^{-1}}{q-q^{-1}}$; the rest can be easily verified. We have $(EF + FE)v_0 = 0$ and $(K - K^{-1})v_0 = v_0 - v_0 = 0$.

The data of $L(\lambda)$ is recorded in the figure below:



We now define a two-dimensional representation, called the Kac module and denoted $K(\lambda)$.

Let v_0 and v_1 be basis vectors for $K(\lambda)$. Define the action of $U_q(\mathfrak{gl}(1|1))$ on $K(\lambda)$ by the following set of equations:

$$\begin{array}{cccc} Ev_1 = v_0, & Ev_0 = 0, & Fv_1 = 0, & Fv_0 = [\lambda]v_1, \\ Kv_1 = q^{\langle h_1+h_2,\lambda\rangle}v_1, & Kv_0 = q^{\langle h_1+h_2,\lambda\rangle}v_0, & q^{h_1}v_1 = q^{-\langle h_1,\lambda-\alpha\rangle}v_1, & q^{h_1}v_0 = q^{\langle h_1,\lambda\rangle}v_0, \end{array} \tag{6.1}$$

where $[\lambda] = \frac{q^{\langle h_1+h_2,\lambda\rangle}-q^{-\langle h_1+h_2,\lambda\rangle}}{q-q^{-1}}$. Again, we check that the relations in I hold for this representation. Most are easy to verify. We check the relation $EF + FE = \frac{K-K^{-1}}{q-q^{-1}}$. For v_0, we have

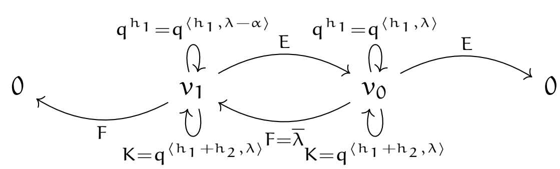
$$\begin{aligned} (EF + FE)v_0 &= [\lambda]Ev_1 \\ &= [\lambda]v_0 \end{aligned}$$

and

$$\begin{aligned} \frac{K-K^{-1}}{q-q^{-1}}v_0 &= \frac{q^{\langle h_1+h_2,\lambda\rangle} - q^{-\langle h_1+h_2,\lambda\rangle}}{q-q^{-1}}v_0 \\ &= [\lambda]v_0. \end{aligned}$$

A similar computation shows that the equation holds for v_1.

The following diagram captures the data of this representation:



(6.2)

If λ is typical, then $\langle h_1 + h_2, \lambda\rangle \neq 0$. Since q is not a root of unity, $q^{\langle h_1+h_2,\lambda\rangle} \neq q^{-\langle h_1+h_2,\lambda\rangle}$, so $[\lambda]$ is nonzero. This means that $Fv_0 \neq 0$. The module $K(\lambda)$ has no nontrivial submodule because E sends v_1 onto the span of v_0 and F sends v_0 onto the span of v_1. Therefore, $K(\lambda)$ is simple. This is the second class of irreducible representations. We label all such irreducible representations $L(\lambda)$. On the other hand, if λ is atypical, then $[\lambda] = 0$, so $Fv_0 = 0$. It follows that the span of v_0 is a submodule of $K(\lambda)$, so $K(\lambda)$ is reducible. However, because E maps the span of v_1 onto the span of v_0, $K(\lambda)$ is indecomposable. In this case, there is a short exact sequence

$$0 \to L(\lambda - \alpha) \to K(\lambda) \to L(\lambda) \to 0$$

where, because λ and $\lambda-\alpha$ are both atypical, $L(\lambda)$ and $L(\lambda-\alpha)$ are one-dimensional irreducible representations. We can conclude from our analysis of $K(\lambda)$ that the category of modules of $U_q(\mathfrak{gl}(1|1))$ is not semisimple.

We now explore tensor products of the representations listed above.

Assume that λ and μ are both atypical. Then $L(\lambda)$ and $L(\mu)$ are both one-dimensional modules. Let v_0^λ and v_0^μ be weight vectors of these two modules. The tensor product

$$L(\lambda)\otimes L(\mu)$$

is also a one-dimensional module, so it must be simple. The action of q^{h_1} on $L(\lambda)\otimes L(\mu)$ is

$$\begin{aligned} q^{h_1}\cdot(v_0^\lambda\otimes v_0^\mu) &= (q^{h_1}\otimes q^{h_1})(v_0^\lambda\otimes v_0^\mu)\\ &= q^{\langle h_1,\lambda\rangle}v_0^\lambda\otimes q^{\langle h_1,\mu\rangle}v_0^\mu\\ &= q^{\langle h_1,\lambda+\mu\rangle}v_0^\lambda\otimes v_0^\mu. \end{aligned}$$

It is easy to check that E and F both kill $v_0^\lambda\otimes v_0^\mu$ and that K acts by 1. Thus,

$$L(\lambda)\otimes L(\mu)\simeq L(\lambda+\mu).$$

If λ and μ are both typical, then $L(\lambda)\otimes L(\mu)$ is four-dimensional. Let v_0^λ and v_1^λ be weight vectors in $L(\lambda)$ with degrees $|v_0^\lambda|=|\lambda|$ and $|v_1^\lambda|=|\lambda|+1$, with the action of $U_q(\mathfrak{gl}(1|1))$ is as in (6.1). Similarly, let v_0^μ, and v_1^μ be weight vectors in $L(\mu)$ of degrees $|\mu|$ and $|\mu|+1$, respectively, with the same $U_q(\mathfrak{gl}(1|1))$ action. The following is a weight basis of $L(\lambda)\otimes L(\mu)$: $v_0^\lambda\otimes v_0^\mu$ (weight $\lambda+\mu$), $v_1^\lambda\otimes v_0^\mu$ (weight $\lambda+\mu-\alpha$), $v_0^\lambda\otimes v_1^\mu$ (weight $\lambda+\mu-\alpha$), and $v_1^\lambda\otimes v_1^\mu$ (weight $\lambda+\mu-2\alpha$).

Theorem 6.3.1. *Assume that* $\lambda,\mu\notin \mathrm{Ann}(h_1+h_2)$ *and* $\lambda+\mu\notin\mathrm{Ann}(h_1+h_2)$. *Then*

$$L(\lambda)\otimes L(\mu)\simeq L(\lambda+\mu)\oplus L(\lambda+\mu-\alpha).$$

Proof. We compute

$$E(v_1^\lambda\otimes v_1^\mu) = v_0^\lambda\otimes q^{-\langle h_1+h_2,\mu\rangle}v_1^\mu+(-1)^{|\lambda|+1}v_1^\lambda\otimes v_0^\mu. \tag{6.3}$$

$$F(v_0^\lambda\otimes v_0^\mu) = [\lambda]v_1^\lambda\otimes v_0^\mu+(-1)^{|\lambda|}q^{\langle h_1+h_2,\lambda\rangle}v_0^\lambda\otimes[\mu]v_1^\mu. \tag{6.4}$$

From these equations, we see that $E(v_1^\lambda \otimes v_1^\mu)$ and $F(v_0^\lambda \otimes v_0^\mu)$ are linear combinations of the basis vectors $v_0^\lambda \otimes v_1^\mu$ and $v_1^\lambda \otimes v_0^\mu$. We compute the determinant of the matrix of coefficients:

$$\det \begin{pmatrix} q^{-\langle h_1+h_2,\mu\rangle} & (-1)^{|\lambda|+1} \\ (-1)^\lambda q^{\langle h_1+h_2,\lambda\rangle}[\mu] & [\lambda] \end{pmatrix} = q^{-\langle h_1+h_2,\mu\rangle}[\lambda] + q^{\langle h_1+h_2,\lambda\rangle}[\mu]$$
$$= [\lambda+\mu].$$

Since the determinant is nonzero, $E(v_1^\lambda \otimes v_1^\mu)$ and $F(v_0^\lambda \otimes v_0^\mu)$ are linearly independent. Thus, a basis for $L(\lambda) \otimes L(\mu)$ is given by

$$\{v_0^\lambda \otimes v_0^\mu, v_1^\lambda \otimes v_1^\mu, E(v_1^\lambda \otimes v_1^\mu), F(v_0^\lambda \otimes v_0^\mu)\}.$$

Now consider the subspace spanned by $v_1 = v_1^\lambda \otimes v_1^\mu$ and $v_0 = E(v_1^\lambda \otimes v_1^\mu)$. Then $Ev_0 = 0$, $Fv_0 \in \text{span}(v_1)$ is nonzero, $Fv_1 = 0$, and $q^{h_1}v_0 = q^{\langle \lambda+\mu-\alpha\rangle}v_0$. Thus, this subspace is a submodule isomorphic to $L(\lambda+\mu-\alpha)$.

Similarly, we can show that $v_0^\lambda \otimes v_0^\mu$ and $F(v_0^\lambda \otimes v_0^\mu)$ span a subrepresentation isomorphic to $L(\lambda+\mu)$. □

Theorem 6.3.2. *Assume that* $\lambda, \mu \notin Ann(h_1+h_2)$ *and* $\lambda+\mu \in Ann(h_1+h_2)$. *Then*

$$L(\lambda) \otimes L(\mu)$$

is indecomposable and has a Jordan-Hölder filtration

$$0 \subset M \subset Q \subset L(\lambda) \otimes L(\mu)$$

where $M \simeq L(\lambda+\mu-\alpha)$, $Q/M \simeq L(\lambda+\mu-2\alpha) \oplus L(\lambda+\mu)$ *and* $L(\lambda) \otimes L(\mu)/Q \simeq L(\lambda+\mu-\alpha)$.

Proof. Because $\lambda+\mu$ is atypical, $q^{\langle h_1+h_2,\lambda\rangle} = q^{-\langle h_1+h_2,\mu\rangle}$ and $[\lambda] = -[\mu]$.

Combining equations (6.3) and (6.4), we get

$$F(v_0^\lambda \otimes v_0^\mu) = [\lambda]v_1^\lambda \otimes v_0^\mu + (-1)^{|\lambda|+1}[\lambda]q^{-\langle h_1+h_2,\mu\rangle}v_0^\lambda \otimes v_1^\mu$$
$$= (-1)^{|\lambda|+1}[\lambda](q^{-\langle h_1+h_2,\mu\rangle}v_0^\lambda \otimes v_1^\mu + (-1)^{|\lambda|+1}v_1^\lambda + v_0^\mu)$$
$$= (-1)^{|\lambda|+1}[\lambda]E(v_1^\lambda \otimes v_1^\mu).$$

Acting by F on $F(v_0^\lambda \otimes v_0^\mu)$ gives 0, as does acting by E, so this vector spans a one-dimensional subrepresentation, isomorphic to $L(\lambda+\mu-\alpha)$. We label this submodule M.

Now, consider the quotient $L(\lambda)\otimes L(\mu)/M$. In this quotient, $F(v_0^\lambda \otimes v_0^\mu) = 0$. In $L(\lambda)\otimes L(\mu)$, $E(v_0^\lambda \otimes v_0^\mu) = 0$, so in the quotient, $v_0^\lambda \otimes v_0^\mu$ spans a one-dimensional submodule isomorphic to $L(\lambda+\mu)$. Similarly, $v_1^\lambda \otimes v_1^\mu$ spans a one-dimensional submodule isomorphic to $L(\lambda+\mu-2\alpha)$ in the quotient. Notice that, in $L(\lambda)\otimes L(\mu)$, both of the vectors above are sent, by E and F, to M. Thus, $\operatorname{span}(v_0^\lambda \otimes v_0^\mu, v_1^\lambda \otimes v_1^\mu) \oplus M$ is a submodule of $L(\lambda)\otimes L(\mu)$, which we label Q.

Now, $L(\lambda)\otimes L(\mu)/Q$ is a one-dimensional submodule spanned by $v_0^\lambda \otimes v_1^\mu$. Therefore, this representation is isomorphic to $L(\lambda+\mu-\alpha)$. □

Theorem 6.3.3. *The dual of the two-dimensional representation* $L(\lambda)$ *is isomorphic to* $L(\alpha-\lambda)$.

Proof. Assume that $\{v_0^\lambda, v_1^\lambda\}$ is a weight basis of $L(\lambda)$ as in (6.1). Then $\{(v_0^\lambda)^\vee, (v_1^\lambda)^\vee\}$ is the basis dual and $|v_0^\lambda| = |(v_0^\lambda)^\vee| = |\lambda|$ while $|v_1^\lambda| = |(v_1^\lambda)^\vee| = |\lambda|+1$. We calculate

$$\begin{aligned} E(v_0^\lambda)^\vee &= -(-1)^{|\lambda|} q^\lambda (v_1^\lambda)^\vee, & E(v_1^\lambda)^\vee &= 0,\\ F(v_0^\lambda)^\vee &= 0, & F(v_1^\lambda)^\vee &= (-1)^{|\lambda|}[\lambda] q^{-\lambda}(v_0^\lambda)^\vee,\\ q^{h_1}(v_0^\lambda)^\vee &= q^{-\langle h_1,\lambda\rangle}(v_0^\lambda)^\vee, & q^{h_1}(v_1^\lambda)^\vee &= q^{-\langle h_1,\lambda-\alpha\rangle}(v_1^\lambda)^\vee. \end{aligned}$$

It is straightforward to check that the map

$$\begin{aligned} L(\alpha-\lambda) &\to L(\lambda)^\vee,\\ v_0^{\alpha-\lambda} &\mapsto -(-1)^{|\lambda|} q^\lambda (v_1^\lambda)^\vee,\\ v_1^{\alpha-\lambda} &\mapsto (v_0^\lambda)^\vee \end{aligned}$$

is an isomorphism of $U_q(\mathfrak{gl}(1|1))$-modules by verifying that $-(-1)^{|\lambda|} q^\lambda (v_1^\lambda)^\vee$ and $(v_0^\lambda)^\vee$ satisfy the conditions in (6.1). □

6.4
Braided and Ribbon Structure

In order to define link invariants, we need a ribbon category, so we need a braiding and a twist on the category of $U_q(\mathfrak{gl}(1|1))$-modules. Theorem 4.2.7 states that if $U_q(\mathfrak{gl}(1|1))$ is a ribbon Hopf algebra, then it has a ribbon category of modules. Unfortunately, $U_q(\mathfrak{gl}(1|1))$ is not a ribbon Hopf algebra because it is not possible to construct a universal R-matrix that lies in $U_q(\mathfrak{gl}(1|1))$. Such an element would be a power series and $U_q(\mathfrak{gl}(1|1))$ contains only polynomial elements. Nevertheless, we can construct a braiding on the category of $U_q(\mathfrak{gl}(1|1))$-modules by examining the R-matrix of the $\hbar$-adic completion of $U_q(\mathfrak{gl}(1|1))$ which is a ribbon Hopf algebra and to which Theorem 4.2.7 applies. We will use the R-matrix from the $\hbar$-adic completion to define a braiding on $U_q(\mathfrak{gl}(1|1))$-modules.

6.4.1

Motivating the Braiding

Let $\hbar$ be an indeterminate. Define $U_\hbar(\mathfrak{gl}(1|1))$ to be the topological $\mathbb{C}[[\hbar]]$-superalgebra generated by two elements in degree one, E and F, and two elements in degree zero, H_1 and H_2, subject to the following relations:

$$\begin{aligned} &H_1H_2 = H_2H_1, && H_iE - EH_i = \langle H_i, \alpha\rangle E, \quad H_iF - FH_i = -\langle H_i, \alpha\rangle F, \\ &EF + FE = \tfrac{e^{\hbar(H_1+H_2)} - e^{-\hbar(H_1+H_2)}}{e^{\hbar} - e^{-\hbar}}, && E^2 = F^2 = 0, \end{aligned}$$

Set $q = e^\hbar$ and $K = e^{\hbar(H_1+H_2)}$ and let $\hat{\otimes}$ be the completed tensor product. Define a coproduct $\Delta : U_\hbar(\mathfrak{gl}(1|1)) \to U_\hbar(\mathfrak{gl}(1|1))\hat{\otimes}U_\hbar(\mathfrak{gl}(1|1))$, counit $\epsilon : U_\hbar(\mathfrak{gl}(1|1)) \to \mathbb{C}[[\hbar]]$, and antipode $S : U_\hbar(\mathfrak{gl}(1|1)) \to U_\hbar(\mathfrak{gl}(1|1))$ as follows:

$$\begin{aligned} \Delta(E) &= E \otimes K^{-1} + 1 \otimes E & \epsilon(E) &= 0 & S(E) &= -EK \\ \Delta(F) &= F \otimes 1 + K \otimes F & \epsilon(F) &= 0 & S(F) &= -K^{-1}F \\ \Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i & \epsilon(H_i) &= 0 & S(H_i) &= -H_i \end{aligned}$$

and extend Δ and ϵ to superalgebra homomorphisms and S to a superalgebra antihomomorphism.

The reader can verify that the same calculations used in the proof of Lemma 6.2.2 show that $U_\hbar(\mathfrak{gl}(1|1))$ is also a Hopf superalgebra.

Define $R = \Theta\Upsilon \in U_\hbar(\mathfrak{gl}(1|1))\hat{\otimes}U_\hbar(\mathfrak{gl}(1|1))$, where

$$\begin{aligned} \Theta &= 1 + (q - q^{-1})F \otimes E, \text{ and} \\ \Upsilon &= e^{\hbar(H_1\otimes H_1 - H_2 \otimes H_2)}. \end{aligned}$$

We verify that R satisfies the conditions of the universal R-matrix. We first verify that R is invertible. The inverse of Υ is $e^{-\hbar(H_1\otimes H_1 - H_2\otimes H_2)}$. Define $\Theta^{-1} = 1 - (q - q^{-1})F \otimes E$. We check that Θ^{-1} is an inverse of Θ. We have

$$\begin{aligned} \Theta\Theta^{-1} &= (1 + (q - q^{-1})F \otimes E)(1 - (q - q^{-1})F \otimes E) \\ &= 1 + (q - q^{-1})^2 F^2 \otimes E^2 \\ &= 1, \end{aligned}$$

where the last equality holds because $F^2 = E^2 = 0$. Similarly,

$$\begin{aligned}\Theta^{-1}\Theta &= (1-(q-q^{-1})F\otimes E)(1+(q-q^{-1})F\otimes E)\\ &= 1+(q-q^{-1})^2F^2\otimes E^2\\ &= 1.\end{aligned}$$

We write the inverse of R as $R^{-1} = \Upsilon^{-1}\Theta^{-1}$.

Theorem 6.4.1 ([24, Proposition 2.4]). *The Hopf superalgebra* $(U_\hbar(\mathfrak{gl}(1|1)), R)$ *is a quasitriangular Hopf superalgebra.*

Proof. We need to check that R satisfies the conditions in Definition 4.2.1. First, to simplify calculations, we define the bar involution

$$\begin{aligned}{}^{-} : \mathbb{C}[\![\hbar]\!] &\to \mathbb{C}[\![\hbar]\!]\\ h &\mapsto h^{-1}.\end{aligned}$$

On $U_\hbar(\mathfrak{gl}(1|1))$, we set $\bar{E} = E$, $\bar{F} = F$, and $\bar{H_i} = H_i$. Define $\bar{\Delta} = (\bar{\ }\otimes\bar{\ }) \circ \Delta \circ \bar{\ }$. We show that the following equations hold for all $x \in U_\hbar(\mathfrak{gl}(1|1))$:

$$\Theta(\tau\circ\bar{\Delta})(x) = (\tau\circ\Delta)(x)\Theta, \tag{6.5}$$

$$\Upsilon\Delta(x) = (\tau\circ\bar{\Delta})(x)\Upsilon. \tag{6.6}$$

We prove equation (6.5) on generators. For $x = E$, we have

$$\begin{aligned}
\Theta(\tau\circ\bar{\Delta})(E) &= \Theta(\tau(E\otimes K + 1\otimes E))\\
&= \Theta(K\otimes E + E\otimes 1)\\
&= K\otimes E + E\otimes 1 + (q-q^{-1})FK\otimes E^2 - (q-q^{-1})FE\otimes E\\
&= K\otimes E + E\otimes 1 - (q-q^{-1})\left(-EF\otimes E + \frac{K-K^{-1}}{q-q^{-1}}\otimes E\right) \quad \text{(recall that } E^2 = 0)\\
&= K\otimes E + E\otimes 1 + (q-q^{-1})EF\otimes E - (K-K^{-1})\otimes E\\
&= K^{-1}\otimes E + E\otimes 1 + (q-q^{-1})EF\otimes E\\
&= K^{-1}\otimes E + E\otimes 1 + (q-q^{-1})(-K^{-1}F\otimes E^2 + EF\otimes E)\\
&= K^{-1}\otimes E + E\otimes 1 + (q-q^{-1})(K^{-1}\otimes E + E\otimes 1)(F\otimes E)\\
&= (\tau\circ\Delta)(E)\Theta.
\end{aligned}$$

The calculations for F, H_1, and H_2 are similar. Refer to [24], Lemma A.1 for the details of the calculations.

We also prove equation (6.6) on generators. Before doing so, we show how to pull $\Delta(E)$ past each factor of Υ. Note the following relations:

$$\begin{aligned}
(H_1\otimes H_1 - H_2\otimes H_2)(E\otimes K^{-1}) &= H_1E\otimes H_1K^{-1} - H_2E\otimes H_2K^{-1}\\
&= (EH_1 + \langle H_1,\alpha\rangle E)\otimes K^{-1}H_1 - (EH_2 + \langle H_2,\alpha\rangle E)\otimes K^{-1}H_2\\
&= E\otimes K^{-1}((H_1+1)\otimes H_1 - (H_2-1)\otimes H_2)\\
(H_1\otimes H_1 - H_2\otimes H_2)(1\otimes E) &= H_1\otimes H_1E - H_2\otimes H_2E\\
&= H_1\otimes(EH_1 + \langle H_1,\alpha\rangle E) - H_2\otimes(EH_2 + \langle H_2,\alpha\rangle E)\\
&= (1\otimes E)(H_1\otimes(H_1+1) - H_2\otimes(H_2-1)).
\end{aligned}$$

Because Υ is a power series, each term is of the form $\hbar^n(H_1\otimes H_1 - H_2\otimes H_2)^n$ for $n\in\mathbb{Z}^{\geq 0}$. Pulling $\Delta(E)$ past each factor of an arbitrary term yields

$$\begin{aligned}
\hbar^n(H_1\otimes H_1 - H_2\otimes H_2)^n(E\otimes K^{-1} + 1\otimes E) =& (E\otimes K^{-1})\hbar^n((H_1+1)\otimes H_1 - (H_2-1)\otimes H_2)^n\\
&+ (1\otimes E)\hbar^n(H_1\otimes(H_1+1) - H_2\otimes(H_2-1))^n.
\end{aligned}$$

This relation justifies the second equality of the following calculation:

$$\begin{aligned}\Upsilon\Delta(E) &= e^{\hbar(H_1\otimes H_1 - H_2\otimes H_2)}(E\otimes K^{-1} + 1\otimes E)\\ &= (E\otimes K^{-1})e^{\hbar((H_1+1)\otimes H_1-(H_2-1)\otimes H_2)} + (1\otimes E)e^{\hbar(H_1\otimes(H_1+1)-H_2\otimes(H_2-1))}\\ &= (E\otimes 1 + K\otimes E)e^{\hbar(H_1\otimes H_1-H_2\otimes H_2)}\\ &= (\tau\circ\bar{\Delta})(E)\Upsilon\end{aligned}$$

The calculation for F is similar, see [24, Lemma A.1]. The elements H_1 and H_2 commute with each other, so $\Upsilon\Delta(H_i) = \Delta(H_i)\Upsilon$. Furthermore, $(\tau\circ\bar{\Delta})(H_i) = H_i$, so $\Upsilon\Delta(H_i) = (\tau\circ\bar{\Delta})(H_i)\Upsilon$.

Recall that the first relation in Definition 4.2.1 is

$$(\tau\circ\Delta)(x) = R\Delta(x)R^{-1}$$

or, equivalently,

$$(\tau\circ\Delta)(x)R = R\Delta(x).$$

We prove this property holds for our choice of R as follows: for all $x\in U_\hbar(\mathfrak{gl}(1|1))$, we have

$$\begin{aligned}R\Delta(x) &= \Theta\Upsilon\Delta(x)\\ &= \Theta(\tau\circ\bar{\Delta})(x)\Upsilon \quad \text{(by Equation (6.6))}\\ &= (\tau\circ\Delta)(x)\Theta\Upsilon \quad \text{(by Equation (6.5))}\\ &= (\tau\circ\Delta)(x)R.\end{aligned}$$

In order to show that the second and third conditions in Definition 4.2.1 hold, we first prove the following equations:

$$(\Delta\otimes \mathrm{id})(\Theta) = \Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1}, \tag{6.7}$$

$$(\mathrm{id}\otimes\Delta)(\Theta) = \Theta_{13}\Upsilon_{13}\Theta_{12}\Upsilon_{13}^{-1}. \tag{6.8}$$

We prove Equation 6.7. The proof of Equation 6.8 is similar.

First, we compute

$$\begin{aligned}\Upsilon_{13}(1\otimes F\otimes E)\Upsilon_{13}^{-1} &= \Upsilon_{13}(1\otimes F\otimes E)e^{-\hbar(H_1\otimes 1\otimes H_1)}e^{-\hbar(H_2\otimes 1\otimes H_2)}\\ &= \Upsilon_{13}e^{-\hbar(H_1\otimes 1\otimes(H_1-1))}(1\otimes F\otimes E)e^{\hbar(H_2\otimes -\otimes H_2)}\\ &= \Upsilon_{13}e^{-\hbar(H_1\otimes 1\otimes(H_1-1))}e^{\hbar(H_2\otimes 1\otimes(H_2+1))}(1\otimes F\otimes E)\\ &= K\otimes F\otimes E.\end{aligned}$$

Now,

$$\begin{aligned}\Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1} &= (1+(q-q^{-1})F\otimes 1\otimes E)(1+(q-q^{-1})K\otimes F\otimes E)\\ &= 1+(q-q^{-1})F\otimes 1\otimes E+(q-q^{-1})K\otimes F\otimes E \quad \text{(again, } E^2=0).\end{aligned}$$

Now, since

$$\begin{aligned}(\Delta\otimes \mathrm{id})(\Upsilon) &= e^{\hbar(H_1\otimes 1\otimes H_1+1\otimes H_1\otimes H_1-H_2\otimes 1\otimes H_2-1\otimes H_2\otimes H_2)}\\ &= \Upsilon_{13}\Upsilon_{23},\end{aligned}$$

we have

$$\begin{aligned}(\Delta\otimes \mathrm{id})(R) &= (\Delta\otimes \mathrm{id})(\Theta)(\Delta\otimes \mathrm{id})(\Upsilon)\\ &= \Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1}\Upsilon_{13}\Upsilon_{23}\\ &= R_{13}R_{23}.\end{aligned}$$

The relation $(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}$ can be proved similarly. □

Recall that in section 4.2 we defined, for an arbitrary quasitriangular Hopf algebra $(H, \mathcal{R})$,

$$u = \sum_i S(\beta_i)\alpha_i$$

where we write $\mathcal{R}$ as $\sum_i \alpha_i \otimes \beta_i$. Because $U_\hbar(\mathfrak{gl}(1|1))$ is graded, we modify the definition of u accordingly:

$$u = \sum_i (-1)^{|\alpha_i||\beta_i|} S(\beta_i)\alpha_i$$

where we require that R decompose into homogeneous α_i and β_i. In Theorem 6.4.4 below, we give an explicit formula for u. Before doing this, we write the power series expansion of R and prove two useful lemmas. Let $\sum_i A_i \otimes B_i$ be the power series expansion of Υ, and note that $|A_i| = |B_i| = 0$. Then

$$\begin{aligned} R &= \Theta \sum_i A_i \otimes B_i \\ &= \sum_i A_i \otimes B_i + \sum_i (q - q^{-1}) F A_i \otimes E B_i. \end{aligned}$$

Lemma 6.4.2. $\sum_i S(B_i)A_i = e^{\hbar(H_2^2 - H_1^2)}$.

Proof. We have

$$\begin{aligned} \Upsilon = \sum_i A_i \otimes B_i &= \sum_n \frac{\hbar^n}{n!} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} H_1^{n-k} \otimes H_1^{n-k} \cdot H_2^k \otimes H_2^k \\ &= \sum_n \frac{\hbar^n}{n!} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} H_1^{n-k} H_2^k \otimes H_1^{n-k} H_2^k. \end{aligned}$$

This implies that

$$\begin{aligned} \sum_i S(B_i)A_i &= \sum_n \frac{\hbar^n}{n!} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} S(H_1^{n-k} H_2^k) H_1^{n-k} H_2^k \\ &= \sum_n \frac{\hbar^n}{n!} \sum_{k=0}^{n} (-1)^k (-1)^n \begin{pmatrix} n \\ k \end{pmatrix} H_1^{2(n-k)} H_2^{2k} \\ &= \sum_n \frac{\hbar^n}{n!} (-1)^n \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} H_1^{2(n-k)} H_2^{2k} \\ &= e^{\hbar(H_2^2 - H_1^2)}. \end{aligned}$$

□

Lemma 6.4.3. $S(B_i)EKF = EKFS(B_i)$.

Proof. We first show that H_1 and H_2 commute with the element EFK. We have

$$\begin{aligned} H_2EKF &= EKH_2F + \langle H_2, \alpha \rangle EKF \\ &= EK(FH_2 - \langle H_2, \alpha \rangle F) + \langle H_2, \alpha \rangle EKF \\ &= EKFH_2. \end{aligned}$$

The proof that H_1 commutes with EKF is a similar calculation. Since $S(B_i)$ is a polynomial in H_1 and H_2, the result follows. □

Theorem 6.4.4. $u = (1 + (q - q^{-1})EKF)e^{\hbar(H_2^2 - H_1^2)}$.

Proof. We have

$$\begin{aligned} u &= \sum_i S(B_i)A_i - \sum_i (q - q^{-1})S(EB_i)FA_i \\ &= \sum_i S(B_i)A_i - \sum_i (q - q^{-1})S(B_i)S(E)FA_i \\ &= \sum_i S(B_i)A_i + \sum_i (q - q^{-1})S(B_i)EKFA_i \\ &= \sum_i S(B_i)A_i + \sum_i (q - q^{-1})EKFS(B_i)A_i \quad \text{(by Lemma 6.4.3)} \\ &= (1 + (q - q^{-1})EKF) \sum_i S(B_i)A_i \\ &= (1 + (q - q^{-1})EKF)e^{\hbar(H_2^2 - H_1^2)} \quad \text{(by Lemma 6.4.2)}. \end{aligned}$$

□

By [20, Proposition 4.1], $S^2(x) = uxu^{-1}$. The equation $S^2 = \mathrm{id}$ can be checked on the generators of $U_q(\mathfrak{gl}(1|1))$. These two relations imply that u is central. We calculate $S(u)$:

$$\begin{aligned} S(u) &= S((1 + (q - q^{-1})EKF)e^{\hbar(H_2^2 - H_1^2)}) \\ &= S(e^{\hbar(H_2^2 - H_1^2)})S(1 + (q - q^{-1})EKF) \\ &= e^{\hbar(H_2^2 - H_1^2)}(1 - (q - q^{-1})K^{-1}FK^{-1}EK) \\ &= e^{\hbar(H_2^2 - H_1^2)}(1 - (q - q^{-1})FK^{-1}E). \end{aligned}$$

Define the twist $v = uK^{-1} = K^{-1}u$.

Theorem 6.4.5 ([24, Proposition 2.5])**.** $(U_\hbar(\mathfrak{gl}(1|1)), R, v)$ *is a ribbon Hopf algebra.*

Proof. We have already proved that $(U_\hbar(\mathfrak{gl}(1|1)), R)$ is quasitriangular, so we show that ν satisfies the conditions in Definition 4.2.6. Since u and K^{-1} are both central, ν is central. This is the first condition in the definition.

We now show that $u = S(u)K^2$. We have

$$\begin{aligned} u &= e^{\hbar(H_2^2 - H_1^2)}(1 + (q - q^{-1})EFK) \\ &= e^{\hbar(H_2^2 - H_1^2)}(1 + (K - K^{-1})K - (q - q^{-1})FEK) \\ &= e^{\hbar(H_2^2 - H_1^2)}(K^2 - (q - q^{-1})FEK) \\ &= S(u)K^2. \end{aligned}$$

It follows that the second and fourth conditions in the definition hold: $\nu^2 = u^2K^{-2} = uS(u)$, and

$$\begin{aligned} S(\nu) &= S(K^{-1}u) \\ &= S(u)K \\ &= uK^{-1} \\ &= \nu. \end{aligned}$$

The third condition holds because of the analogous condition on u in Theorem 4.2.5.

Finally, we have

$$\begin{aligned} \epsilon(u) &= \epsilon((1 + (q - q^{-1})EKF)e^{\hbar(H_2^2 - H_1^2)}) \\ &= \epsilon(1 + (q - q^{-1})EKF)\epsilon(e^{\hbar(H_2^2 - H_1^2)}) \\ &= 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \epsilon(\nu) &= \epsilon(K^{-1}u) \\ &= 1. \end{aligned}$$

So the fifth condition holds. □

6.4.2

Braided Category Structure

Our goal is to construct a ribbon structure on the category of finite dimensional $U_q(\mathfrak{gl}(1|1))$ weight modules. We cannot do this directly using the ribbon superalgebra structure on $U_\hbar(\mathfrak{gl}(1|1))$ because, as defined, R, u, and v are infinite sums. Instead, we use the R-matrix to motivate a reasonable choice of braiding on the category of $U_q(\mathfrak{gl}(1|1))$-modules. We then prove that the category is generically semisimple and pivotal, then appeal to a theorem of Geer and Patureau-Mirand to show that it is ribbon.

Let V and W be modules, and let $v \in V$ and $w \in W$ be weight vectors with weights λ and μ, respectively. Define $\mathbb{C}$-linear operators $\Theta_{V,W} : V \otimes W \to V \otimes W$ and $\Upsilon_{V,W} : V \otimes W \to V \otimes W$ on weight vectors as

$$\Theta_{V,W}(v \otimes w) = (1 + (q - q^{-1})F \otimes E)(v \otimes w) = v \otimes w + (q - q^{-1})(-1)^{|v|} Fv \otimes Ew$$

$$\Upsilon_{V,W}(v \otimes w) = q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(v \otimes w).$$

Define $c_{V,W} : V \otimes W \to W \otimes V$ as $c_{V,W}(v \otimes w) = \tau \circ \Theta_{V,W} \circ \Upsilon_{V,W}(v \otimes w)$.

Lemma 6.4.6. *The operator* $c_{V,W}$ *is* $U_q(\mathfrak{gl}(1|1))$*-linear.*

Proof. See Appendix. □

Lemma 6.4.7. *The operator* $c_{V,W}$ *satisfies the hexagon axioms.*

Proof. Let U, V, and W be weight $U_q(\mathfrak{gl}(1|1))$-modules. Let $u \in U$, $v \in V$, and $w \in W$ be weight vectors with weights λ, μ, and ν, respectively. The hexagon axioms are Equations (4.3) and (4.4) in Definition 4.1.5. We work over a strict category, so we suppress the associators. Thus, we verify the equations

$$c_{U,V\otimes W}(u \otimes v \otimes w) = \mathrm{id}_V \otimes c_{U,W} \circ c_{U,V} \otimes \mathrm{id}_W(u \otimes v \otimes w), \text{ and} \tag{6.9}$$

$$c_{U\otimes V,W}(u \otimes v \otimes w) = c_{U,W} \otimes \mathrm{id}_V \circ \mathrm{id}_U \otimes c_{V,W}(u \otimes v \otimes w). \tag{6.10}$$

We will show Equation (6.9). The calculation for (6.10) is similar. On the left, we have

$c_{U,V\otimes W}(u\otimes v\otimes w)$:

$$\begin{aligned} u\otimes v\otimes w \xmapsto{c_{U,V\otimes W}} & A(-1)^{|u||v|+|u||w|} v\otimes w\otimes u \\ & + A(q-q^{-1})q^{-\langle h_1+h_2,\nu\rangle}(-1)^{|u||v|+|v|+|w|} Ev\otimes w\otimes Fu \\ & + A(q-q^{-1})(-1)^{|u||v|+|u||w|+|w|+1} v\otimes Ew\otimes Fu \end{aligned}$$

where $A = q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu\rangle+\langle h_1,\lambda\rangle\langle h_1,\nu\rangle-\langle h_2,\lambda\rangle\langle h_2,\nu\rangle}$. On the right, we have

$$\begin{aligned} u\otimes v\otimes w \xmapsto{c_{U,V}\otimes \mathrm{id}_W} & q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(-1)^{|u||v|} v\otimes u\otimes w \\ & + q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(q-q^{-1})(-1)^{|u|+(|u|+1)(|v|+1)} Ev\otimes Fu\otimes w \\ \xmapsto{\mathrm{id}_V\otimes c_{U,W}} & A(-1)^{|u||v|+|u||w|} v\otimes w\otimes u \\ & + A(q-q^{-1})(-1)^{|u|+(|u|+1)(|v|+|w|+1)} q^{-\langle h_1+h_2,\nu\rangle} Ev\otimes w\otimes Fu \\ & + A(q-q^{-1})(-1)^{|u||v|+|u||w|+|w|+1} v\otimes Ew\otimes Fu. \end{aligned}$$

Direct comparison of the coefficients of each term shows that the two expressions are equal. □

Lemma 6.4.6 and Lemma 6.4.7 together imply that $c_{V,W}$ is a braiding on the category of weight $U_q(\mathfrak{gl}(1|1))$-modules.

6.4.3

Rigid Category

Duality is given by the right and left evaluation and coevaluation maps. The evaluation maps on a $\mathbb{Z}_2$-graded vector space W are defined by

$$\begin{aligned} \mathrm{ev}_W : & W^\vee \otimes W \to \mathbb{C}, \\ & \phi\otimes w \mapsto \phi(w) \end{aligned}$$

and

$$\begin{aligned} \widehat{\mathrm{ev}}_W : & W\otimes W^\vee \to \mathbb{C}, \\ & w\otimes \phi \mapsto (-1)^{|\phi||w|}\phi(w). \end{aligned}$$

Let $\{w_i\}$ be a basis of W and let $\{w_i^\vee\}$ be the corresponding dual basis of $W^\vee$. The coevaluation maps are defined by

$$\begin{aligned}\mathrm{coev}_W :&\mathbb{C} \to W \otimes W^\vee, \\ &1 \mapsto \sum_{i=1}^{n} w_i \otimes w_i^\vee\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathrm{coev}}_W :&\mathbb{C} \to W^\vee \otimes W, \\ &1 \mapsto \sum_{i=1}^{n} (-1)^{|w_i|} w_i^\vee \otimes w_i.\end{aligned}$$

Fortunately, the maps $\widehat{\mathrm{ev}}$ and $\widehat{\mathrm{coev}}$ are $U_q(\mathfrak{gl}(1|1))$-linear, so these maps give us a right evaluation and coevaluation on the category of $U_q(\mathfrak{gl}(1|1))$-modules. Unfortunately, ev and coev are not $U_q(\mathfrak{gl}(1|1))$-linear. We can fix this with a simple change. Define

$$\begin{aligned}\mathrm{ev}_W :&W^\vee \otimes W \to \mathbb{C}, \\ &\phi \otimes w \mapsto \phi(K^{-1} w)\end{aligned}$$

and

$$\begin{aligned}\mathrm{coev}_W :&\mathbb{C} \to W \otimes W^\vee, \\ &1 \mapsto \sum_{i=1}^{n} K w_i \otimes w_i^\vee.\end{aligned}$$

We verify that these two maps are $U_q(\mathfrak{gl}(1|1))$-linear on generators. We will work out the details for K and E; the rest are similar. Let $\phi \otimes v \in V^\vee \otimes V$. For K, we have

$$\begin{aligned}\mathrm{ev}(K \cdot \phi \otimes v) &= \mathrm{ev}(K\phi \otimes Kv) \\ &= (K\phi)(K^{-1}Kv) \\ &= \phi(S(K)v) \\ &= \phi(K^{-1}v).\end{aligned}$$

Meanwhile,

$$\begin{aligned} K \cdot ev(\phi \otimes v) &= K \cdot \phi(K^{-1}v) \\ &= \phi(K^{-1}v) \end{aligned}$$

because $\phi(K^{-1}v) \in \mathbb{C}$, where $\mathbb{C}$ is the trivial module, on which K acts by 1.

We now check for linearity with E. We have

$$\begin{aligned} E \cdot \mathrm{ev}(\phi \otimes v) &= E \cdot \phi(K^{-1}v) \\ &= 0, \end{aligned}$$

because E acts by 0 on the trivial module. Meanwhile,

$$\begin{aligned} \mathrm{ev}(E \cdot \phi \otimes v) &= \mathrm{ev}(E\phi \otimes K^{-1}v + (-1)^{|\phi|}\phi \otimes Ev \\ &= (E\phi)(K^{-2}v) + (-1)^{|\phi|}\phi(K^{-1}Ev) \\ &= (-1)^{|\phi|}\phi(S(E)K^{-2}v) + (-1)^{|\phi|}\phi(K^{-1}Ev) \\ &= (-1)^{|\phi|+1}\phi(EK^{-1}v) + (-1)^{|\phi|}\phi(EK^{-1}v) \\ &= 0. \end{aligned}$$

We also verify that ev and coev satisfy the snake relations. We only need to check the relations involving these two maps, Equations (4.5) and (6.12). We suppress the associators on these relations, so that we check the equality of the following maps:

$$V \xrightarrow{\mathrm{coev}_V \otimes \mathrm{id}_V} V \otimes V^\vee \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} V = V \xrightarrow{\mathrm{id}_V} V \tag{6.11}$$

$$V^\vee \xrightarrow{\mathrm{id}_{V^\vee} \otimes \mathrm{coev}_V} V^\vee \otimes V \otimes V^\vee \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_V} V^\vee = V^\vee \xrightarrow{\mathrm{id}_{V^\vee}} V^\vee. \tag{6.12}$$

Let $\{v_1, \ldots, v_n\}$ be a weight basis of V and let $\{v_1^\vee, \ldots, v_n^\vee\}$ be the corresponding dual basis. Let λ_i be the weight of v_i. We verify Equation (6.11) by checking that it sends our chosen basis of V to itself. On v_j, we have

$$\begin{aligned} v_j &\xrightarrow{\mathrm{coev}_V \otimes \mathrm{id}_V} \sum_{i=1}^{n} Kv_i \otimes v_i^{\vee} \otimes v_j \\ &\xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} \sum_{i=1}^{n} Kv_i \otimes v_i^{\vee}(K^{-1}v_j) \\ &= \sum_{i=1}^{n} q^{\langle h_1+h_2, \lambda_i \rangle} v_i \otimes v_i^{\vee}(q^{-\langle h_1+h_2, \lambda_j \rangle} v_j) \\ &= \sum_{i=1}^{n} q^{\langle h_1+h_2, \lambda_i - \lambda_j \rangle} v_i \otimes v_i^{\vee}(v_j) \\ &= q^{\langle h_1+h_2, \lambda_j - \lambda_j \rangle} v_j \otimes 1 \\ &= v_j. \end{aligned}$$

Now we check Equation (6.12). Again, we check that the basis is sent to itself. For $v_j^{\vee}$, we have

$$\begin{aligned} v_j^{\vee} &\xrightarrow{\mathrm{id}_{V^{\vee}} \otimes \mathrm{coev}_V} \sum_{i=1}^{n} v_j^{\vee} \otimes Kv_i \otimes v_i^{\vee} \\ &\xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_V} \sum_{i=1}^{n} v_j^{\vee}(K^{-1}Kv_i) \otimes v_i^{\vee} \\ &= v_j^{\vee}. \end{aligned}$$

The left and right evaluation and coevaluation maps make the category of weight $U_q(\mathfrak{gl}(1|1))$-modules (left and right) rigid.

6.4.4

Pivotal Category

We check that the category of weight $U_q(\mathfrak{gl}(1|1))$-modules is pivotal. We already know that it is rigid and that the left and right dual objects are the same.

We check that the category is pivotal.

Definition 6.4.8 (Pivotal Category)**.** *Let $\mathcal{C}$ be a rigid category. Given a morphism $f : V \to W$, define $f_L^{\vee} : W^{\vee} \to V^{\vee}$ as*

$$f_L^{\vee} = \mathrm{ev}_W \otimes \mathrm{id}_{V^{\vee}} \circ \mathrm{id}_W \otimes f \otimes \mathrm{id}_{V^{\vee}} \circ \mathrm{id}_{W^{\vee}} \otimes \mathrm{coev}_V \tag{6.13}$$

and define $f_R^\vee : W^\vee \to V^\vee$ *as*

$$f_R^\vee = \mathrm{id}_{V^\vee} \otimes \widehat{\mathrm{ev}}_W \circ \mathrm{id}_{V^\vee} \otimes f \otimes \mathrm{id}_{W^\vee} \circ \widehat{\mathrm{coev}}_V \otimes \mathrm{id}_{W^\vee}. \tag{6.14}$$

Moreover, we define $\gamma^L_{V,W} : W^\vee \otimes V^\vee \to (V \otimes W)^\vee$ *as*

$$\gamma^L_{V,W} = \mathrm{ev}_W \otimes \mathrm{id}_{(V\otimes W)^\vee} \circ \mathrm{id}_{W^\vee} \otimes \mathrm{ev}_V \otimes \mathrm{id}_W \otimes \mathrm{id}_{(V\otimes W)^\vee} \circ \mathrm{id}_{W^\vee} \otimes \mathrm{id}_{V^\vee} \otimes \mathrm{coev}_{V\otimes W} \tag{6.15}$$

and $\gamma^R_{V,W} : W^\vee \otimes V^\vee \to (V \otimes W)^\vee$ *as*

$$\gamma^R_{V,W} = \mathrm{id}_{(V\otimes W)^\vee} \otimes \widehat{\mathrm{ev}}_V \circ \mathrm{id}_{(V\otimes W)^\vee} \otimes \mathrm{id}_V \otimes \widehat{\mathrm{ev}}_W \otimes \mathrm{id}_{V^\vee} \circ \widehat{\mathrm{coev}}_{V\otimes W} \otimes \mathrm{id}_{W^\vee} \otimes \mathrm{id}_{V^\vee}. \tag{6.16}$$

If $f_L^\vee = f_R^\vee$ *and* $\gamma^L_{V,W} = \gamma^R_{V,W}$*, then* $\mathcal{C}$ *is pivotal.*

Lemma 6.4.9. *Let* V *and* W *be* $U_q(\mathfrak{gl}(1|1))$*-modules and* $f : V \to W$ *a* $U_q(\mathfrak{gl}(1|1))$*-module homomorphism. Then* $f_L^\vee = f_R^\vee$*.*

Proof. As a $U_q(\mathfrak{gl}(1|1))$-module homomorphism, f is $U_q(\mathfrak{gl}(1|1))$-linear and degree zero. Let $w^\vee \in W^\vee$. We compute $f_L^\vee(w^\vee)$. We have

$$\begin{aligned} w^\vee &\xmapsto{\mathrm{id}_{W^\vee}\otimes \mathrm{coev}_V} \sum_{i=1}^n w^\vee \otimes K v_i \otimes v_i^\vee \\ &\xmapsto{\mathrm{id}_W \otimes f \otimes \mathrm{id}_{V^\vee}} \sum_{i=1}^n w^\vee \otimes f(K v_i) \otimes v_i^\vee \\ &\xmapsto{\mathrm{ev}_W \otimes \mathrm{id}_{V^\vee}} \sum_{i=1}^n w^\vee(K^{-1} f(K v_i)) \otimes v_i^\vee = \sum_{i=1}^n w^\vee(f(v_i)) v_i \end{aligned}$$

where the final equality holds because f is $U_q(\mathfrak{gl}(1|1))$-linear. We now compute $f_R^\vee(w^\vee)$:

$$\begin{aligned} w^\vee &\xmapsto{\widehat{\mathrm{coev}}_V \otimes \mathrm{id}_{W^\vee}} \sum_{i=1}^n (-1)^{|v_i|} v_i^\vee \otimes v_i \otimes w^\vee \\ &\xmapsto{\mathrm{id}_{V^\vee}\otimes f \otimes \mathrm{id}_{W^\vee}} \sum_{i=1}^n (-1)^{|v_i|} v_i^\vee \otimes f(v_i) \otimes w^\vee \\ &\xmapsto{\mathrm{id}_{V^\vee}\otimes \widehat{\mathrm{ev}}_W} \sum_{i=1}^n (-1)^{|v_i| + |w^\vee||f(v_i)|} v_i^\vee \otimes w^\vee(f(v_i)). \end{aligned}$$

Now, in each term where $w^\vee(f(v_i)) \neq 0$, $|w^\vee| = |f(v_i)| = |v_i|$, so $(-1)^{|v_i| + |w^\vee||f(v_i)|} = (-1)^{|v_i| + |v_i|^2} =$

$(-1)^{2|v_i|} = 1$. Therefore,

$$\sum_{i=1}^{n} (-1)^{|v_i|+|w^\vee||f(v_i)|} v_i^\vee \otimes w^\vee(f(v_i)) = \sum_{i=1}^{n} w^\vee(f(v_i))v_i.$$

The signs on the terms of the sums of $f_L^\vee(w^\vee)$ and $f_R^\vee(w^\vee)$ agree. □

Lemma 6.4.10. *Let* V *and* W *be* $U_q(\mathfrak{gl}(1|1))$*-modules of dimensions* n *and* m*, respectively. Then* $\gamma^L_{V,W} = \gamma^R_{V,W}$.

Proof. Let $v^\vee \in V^\vee$ and $w^\vee \in W^\vee$. We compute $\gamma^L_{V,W}(w^\vee \otimes v^\vee)$:

$$\begin{aligned} w^\vee \otimes v^\vee &\xmapsto{\mathrm{id}_{W^\vee}\otimes \mathrm{id}_{V^\vee}\otimes \mathrm{coev}_{V\otimes W}} \sum_{i=1}^{n}\sum_{j=1}^{m} w^\vee \otimes v^\vee \otimes Kv_i \otimes Kw_j \otimes (v_i \otimes w_j)^\vee \\ &\xmapsto{\mathrm{id}_{W^\vee}\otimes \mathrm{ev}_V \otimes \mathrm{id}_W \otimes \mathrm{id}_{(V\otimes W)^\vee}} \sum_{i=1}^{n}\sum_{j=1}^{m} w^\vee \otimes v^\vee(v_i) \otimes Kw_j \otimes (v_i \otimes w_j)^\vee \\ &\xmapsto{\mathrm{ev}_W \otimes \mathrm{id}_{(V\otimes W)^\vee}} \sum_{i=1}^{n}\sum_{j=1}^{m} v^\vee(v_i) w^\vee(w_j)(v_i \otimes w_j)^\vee. \end{aligned}$$

We compute $\gamma^R_{V,W}(w^\vee \otimes v^\vee)$:

$$\begin{aligned} w^\vee \otimes v^\vee &\xmapsto{\widehat{\mathrm{coev}}_{V\otimes W}\otimes \mathrm{id}_{W^\vee}\otimes \mathrm{id}_{V^\vee}} \sum_{i=1}^{n}\sum_{j=1}^{m} (-1)^{|v_i\otimes w_j|}(v_i \otimes w_j)^\vee \otimes v_i \otimes w_j \otimes w^\vee \otimes v^\vee \\ &\xmapsto{\mathrm{id}_{(V\otimes W)^\vee}\otimes \mathrm{id}_V \otimes \widehat{\mathrm{ev}}_W \otimes \mathrm{id}_{V^\vee}} \sum_{i=1}^{n}\sum_{j=1}^{m} (-1)^{|v_i\otimes w_j|+|w_j||w^\vee|}(v_i \otimes w_j)^\vee \otimes v_i \otimes w^\vee(w_j) v^\vee \\ &\xmapsto{\mathrm{id}_{(V\otimes W)^\vee}\otimes \widehat{\mathrm{ev}}_V} \sum_{i=1}^{n}\sum_{j=1}^{m} (-1)^{|v_i\otimes w_j|+|w_j||w^\vee|+|v_i||v^\vee|} w^\vee(w_j) v^\vee(v_i)(v_i \otimes w_j)^\vee. \end{aligned}$$

Now, when $w^\vee(w_j) \neq 0$, then $|w^\vee| = |w_i|$. Likewise with $v^\vee$ and v_i. Therefore, the exponent of (-1) in each term of the sum becomes

$$\begin{aligned} |v_i \otimes w_j| + |w_j|^2 + |v_i|^2 &= |v_i| + |w_j| + |w_j| + |v_i| \\ &= 2|v_i| + 2|w_j| \\ &= 0 \pmod 2. \end{aligned}$$

Thus, the sum simplifies to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w^\vee(w_j) v^\vee(v_i)(v_i \otimes w_j)^\vee.$$

The signs of the terms in the sums of $\gamma^L_{V,W}(w^\vee \otimes v^\vee)$ and $\gamma^R_{V,W}(w^\vee \otimes v^\vee)$ agree. □

Corollary 6.4.11. *The category of weight* $U_q(\mathfrak{gl}(1|1))$*-modules is pivotal.*

6.4.5

Generically Semisimple

We now show that the category $\mathcal{C}$ of weight $U_q(\mathfrak{gl}(1|1))$-modules is generically semisimple. The category has a grading by $\mathbb{Z}$ given by the action of K. In other words, for $n \in \mathbb{Z}$, $\mathcal{C}_n$ is the subcategory of modules V for which $Kv = q^n v$ for all $v \in V$. In this grading, the following properties hold:

1. For $V \in \mathcal{C}_\alpha$ and $W \in \mathcal{C}_\beta$, $V \otimes W \in \mathcal{C}_{\alpha+\beta}$, because $\Delta(K) = K \otimes K$.

2. For $V \in \mathcal{C}_\alpha$, $V^\vee \in \mathcal{C}_{-\alpha}$, because $S(K) = K^{-1}$.

Let $K(\lambda)$ be the Kac module as defined in Section 6.3. Then $K(\lambda) \in \mathcal{C}_{\langle h_1+h_2,\lambda\rangle}$ Recall that if λ is typical, then $K(\lambda)$ is simple.

Let $X = \{0\} \subset \mathbb{Z}$. This set has the following properties:

1. Symmetry, $X = -X$, and

2. Finite unions of the form $\bigcup_i (n_i + X) \neq \mathbb{Z}$ for all $n_i \in \mathbb{Z}$.

Lemma 6.4.12. *If* $n \notin X$*, then* $\mathcal{C}_n$ *is semisimple.*

Proof. Let $V \in \mathcal{C}_n$ for $n \neq 0$ be nonzero. Recall that V has a highest weight vector, which we label v with weight λ. Let m record the action of q^{h_1} on v; i.e. $q^{h_1}v = q^m v$. Because $n \neq 0$, $[\lambda] = \frac{q^n - q^{-n}}{q-q^{-1}} \neq 0$, so $Fv \neq 0$. Also, $EFv = \frac{K-K^{-1}}{q-q^{-1}}v$, which is also nonzero. Therefore, v generates a simple submodule of V which, by the universal property of Kac modules, is isomorphic to a Kac module determined by m and n. Label this Kac module $\langle v\rangle$.

Now, by [5, Chapter 10, Proposition 34], if $\langle v\rangle$ is injective, then $\langle v\rangle$ is a direct summand of V. Moreover, by [7, Proposition 6.1.3] and Corollary 6.4.11, the set of injective modules in $\mathcal{C}$ is exactly the set of projective modules. Therefore, it is sufficient to show that all simple Kac modules are projective.

Define $U_q^{\geq 0}(\mathfrak{gl}(1|1))$ to be the subalgebra of $U_q(\mathfrak{gl}(1|1))$ generated by q^{h_1}, K, and E. Let $\mathbb{C}(\lambda)$ be the one-dimensional $U_q^{\geq 0}(\mathfrak{gl}(1|1))$-module with $U_q^{\geq 0}(\mathfrak{gl}(1|1))$ action defined on $1 \in \mathbb{C}(\lambda)$ by $K \cdot 1 = q^{\langle h_1+h_2,\lambda\rangle}1$, $q^{h_1} \cdot 1 = q^{\langle h_1,\lambda\rangle}1$, and $E \cdot 1 = 0$. Kac modules have a universal property which gives the a $U_q(\mathfrak{gl}(1|1))$-module isomorphism

$$K(\lambda) \simeq U_q(\mathfrak{gl}(1|1)) \otimes_{U_q^{\geq 0}(\mathfrak{gl}(1|1))} \mathbb{C}(\lambda).$$

Let

$$V \xrightarrow{f} W \to 0$$

be an exact sequence in $\mathcal{C}$ and let $\phi : K(\lambda) \to W$ be a nonzero morphism. There is an isomorphism $\mathrm{Hom}_{U_q(\mathfrak{gl}(1|1))}(K(\lambda), W) \simeq \mathrm{Hom}_{U_q^{\geq 0}(\mathfrak{gl}(1|1))}(\mathbb{C}(\lambda), W)$, see [5, Section 10.5, Theorem 43].

Therefore, there is a unique $\phi_0 \in \mathrm{Hom}_{U_q^{\geq 0}(\mathfrak{gl}(1|1))}(\mathbb{C}(\lambda), \mathbb{W})$ which is the image of ϕ under this isomorphism. Now, $\phi_0(1) \in W$ determines ϕ_0 which in turn determines ϕ. Because f is $U_q(\mathfrak{gl}(1|1))$-linear, there is a highest weight vector $v \in V$ of weight λ such that $f(v) = \phi_0(1)$. Running this process in reverse, each highest weight vector $v \in V$ of weight λ identifies a unique $\phi' \in \mathrm{Hom}_{U_q(\mathfrak{gl}(1|1))}(K(\lambda), V)$. This ϕ' is a lift of ϕ, making $K(\lambda)$ projective.

Since, $\langle v \rangle$ is projective, it is a direct summand of V. We write the decomposition as $V = \langle v \rangle \oplus V'$. Since $\mathcal{C}_n$ is additive, $\langle v \rangle$ and V' are in $\mathcal{C}_n$. Repeat this process inductively on V'. Since V is finite dimensional and the dimension of V' is $\dim(V)-2$, the process will eventually terminate. □

Corollary 6.4.13. *The category $\mathcal{C}$ is generically semisimple.*

6.4.6

Ribbon Category

Let $\mathcal{C}$ be the category of weight $U_q(\mathfrak{gl}(1|1))$-modules. For $V \in \mathcal{C}$, define $\theta_V : V \to V$ by

$$\theta_V = \mathrm{id}_V \otimes \widehat{\mathrm{ev}}_V \circ c_{V,V} \otimes \mathrm{id}_{V^\vee} \circ \mathrm{id}_V \otimes \mathrm{coev}_V. \tag{6.17}$$

To prove that $\mathcal{C}$ is ribbon, we show that it satisfies the conditions of the following theorem:

Theorem 6.4.14 ([8, Theorem 9])**.** *Let $\mathcal{C}$ be a generically semisimple pivotal braided category. If $\theta_{V^\vee} = \theta_V^\vee$ for all simple V, then $\{\theta_V\}$ is a twist on $\mathcal{C}$, and $\mathcal{C}$ is ribbon.*

We need to check that $\theta_{V^\vee} = \theta_V^\vee$ for all simple modules with typical weights. Fortunately, it is sufficient to check on the simple modules, on which θ acts by a scalar.

Let $K(\lambda)$ have typical weight (so it is simple). Because $\theta_{K(\lambda)}$ acts by a scalar on $K(\lambda)$, we only need to compute this scalar for a single vector. We choose $v \in K(\lambda)$ to be the highest weight vector. We have

$$v \xrightarrow{\theta_V} q^{\langle h_1, \lambda-\alpha\rangle^2 - \langle h_2, \lambda-\alpha\rangle^2 + \langle h_1+h_2, \lambda\rangle} v.$$

Now, the scalar produced by $\theta_V^\vee$ is

$$q^{\langle h_1,\alpha-\lambda-\alpha\rangle^2-\langle h_2,\lambda-\alpha\rangle^2-\langle h_1+h_2,\lambda\rangle} = q^{\langle h_1,\lambda\rangle^2\langle h_2,\lambda\rangle^2-\langle h_1+h_2,\lambda\rangle}.$$

By direct inspection, these two scalars are equal. Therefore, $\theta_{V^\vee} = \theta_V^\vee$. We have proved the following theorem:

Theorem 6.4.15. *The category $\mathcal{C}$ is ribbon.*

6.5
Modified Reshetikhin–Turaev Link Invariant Approach

Now, given a ribbon category, the Theorem 4.3.1 gives a link invariant.

Unfortunately, if we restrict the domain of F to the set of link diagrams, our invariant is useless. We prove below that $F(D) = 0$ for any link diagram D colored by a two-dimensional simple module. However, the invariant can be modified by a cutting procedure that we describe later on.

Before we prove the theorem, we need a simple lemma.

Lemma 6.5.1. *Let M and N be modules over a ring. Assume that $\phi : M \to N$ is an inclusion and that there is a module homomorphism $\psi : N \to M$ such that $\psi \circ \phi$ is the identity map. Then M is a direct summand of N.*

Proof. Since M is a submodule of N, there is a short exact sequence

$$0 \to M \xrightarrow{\phi} N \xrightarrow{\pi} N/M \to 0.$$

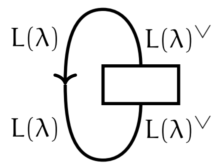
Because ψ is a retraction, the following is an isomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\phi} & N & \xrightarrow{\pi} & N/M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \psi\oplus\pi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & M & \xrightarrow[i_M]{} & M \oplus N/M & \xrightarrow[\pi_{N/M}]{} & N/M & \longrightarrow & 0 \end{array}$$

where i_M and $\pi_{N/M}$ are the obvious inclusion and projection. □

Theorem 6.5.2. *Let L be a link with diagram D labeled by at least one simple module of $U_q(\mathfrak{gl}(1|1))$ with typical weight. Then $F(D) = 0$.*

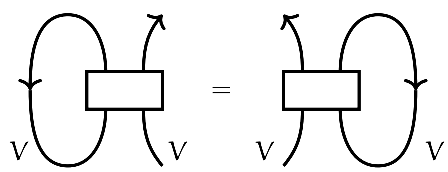
Proof. Because D has no source or target points, the invariant $F(D)$ is an endomorphism ϕ of $\mathbb{C}$. By planar isotopy we can move strands until the diagram is



for λ a typical weight. Let $\phi = \phi_2 \circ \phi_1$ where $\phi_1 : \mathbb{C} \to L(\lambda) \otimes L(\lambda)^\vee$ is coev and $\phi_2 : L(\lambda) \otimes L(\lambda)^\vee \to \mathbb{C}$ is $\widehat{\mathrm{ev}} \circ (uv^{-1} \otimes \mathrm{id}) \circ \mathrm{id} \otimes T$, where $T : L(\lambda)^\vee \to L(\lambda)^\vee$ is the map constructed from the data in the coupon (the square in the diagram).

Suppose, by way of contradiction, that ϕ is nonzero. Then ϕ_1 gives an inclusion of $\mathbb{C}$ in $L(\lambda) \otimes L(\lambda)^\vee$ and $\phi_2/\langle \phi \rangle$ is a module homomorphism from $L(\lambda) \otimes L(\lambda)^\vee$ onto $\mathbb{C}$ such that $\phi_2 \circ \phi_1 = \mathrm{id}_{\mathbb{C}}$. By Lemma 6.5.1, $\mathbb{C}$ is a direct summand of $L(\lambda) \otimes L(\lambda)^\vee$. However, because $\lambda - (\lambda - \alpha)$ is atypical, $L(\lambda) \otimes L(\lambda)^\vee$ is indecomposable by Theorem 6.3.2. This is a contradiction. □

Definition 6.5.3 (Ambidextrous Representation)**.** *An ambidextrous representation* V *is a representation for which the following formula holds for all coupons:*

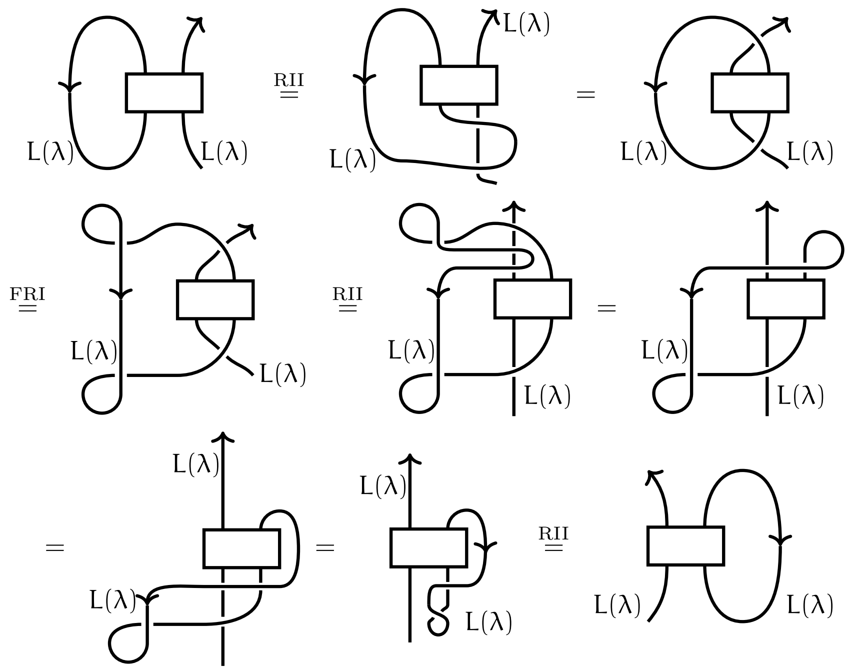


(6.18)

where each strand in both sides of the equality is labeled by V*.*

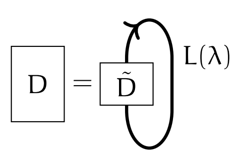
Theorem 6.5.4. *Equation* (6.18) *holds for all typical modules* $L(\lambda)$*; i.e. all typical modules are ambidextrous.*

Proof. This proof is a sequence of framed Reidemeister moves as given in Theorem 2.2.3. We have

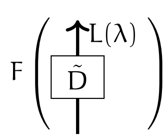


The second equality holds because the braiding commutes with endomorphisms of $L(\lambda) \otimes L(\lambda)$, because λ is typical and the sixth equality holds by a combination of Reidemeister moves that depends on the interior of the coupon. The fifth and seventh equalities hold by applying first RIII and then RI. □

We now describe a cutting procedure that gives a nontrivial link invariant, see [9, Theorem 3]. See also [24, Section 4.3] in the case of $U_q(\mathfrak{gl}(1|1))$. Let D be any link diagram with all strands labeled by $L(\lambda)$ for some typical λ. Under isotopy, we can pull a strand to the right and write the diagram as



where $\tilde{D}$ is a $(1,1)$-tangle. Define $\langle D^\lambda \rangle \in \mathbb{C}$ as

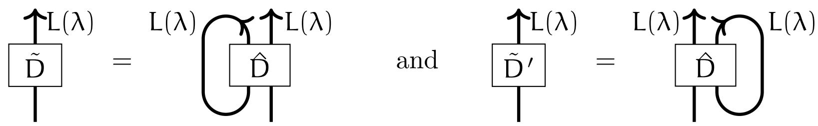


$$= \langle D^\lambda \rangle \cdot \mathrm{id}_{L(\lambda)}$$

Theorem 6.5.5. *The map* $D \mapsto \langle D^\lambda \rangle$ *is an invariant of framed oriented links.*

Proof. We need only show that the assignment is independent of the choice of the strand cut.

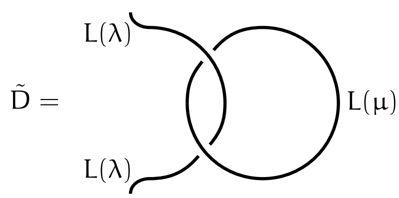
Suppose two different strands are cut to produce two different diagrams, $\tilde{D}$ and $\tilde{D}'$. Then, by isotopy,



for some tangle $\hat{D}$. By Theorem 6.5.4, $F(\tilde{D}) = F(\tilde{D}')$. □

If two strands are labeled by simple modules of different weights λ and μ, then we can construct an invariant by rescaling, see Theorem 7.4.6.

Example 6.5.6. *To see that this invariant is nontrivial, we apply it to the Hopf link, see Figure 1.1c, with the left strand labeled by* $L(\lambda)$ *and the right strand labeled by* $L(\mu)$*, for* λ *and* μ *typical. We cut the left strand as shown in the diagram below.*



Now, $F(\tilde{D}) = \mathrm{id}_{L(\lambda)} \otimes \widehat{\mathrm{ev}}_{L(\mu)} \circ R^{-1}_{L(\mu)\otimes L(\lambda)} \otimes \mathrm{id}_{L(\mu)^\vee} \circ R^{-1}_{L(\lambda)\otimes L(\mu)} \otimes \mathrm{id}_{L(\mu)^\vee} \circ \mathrm{id}_{L(\lambda)} \otimes \mathrm{coev}_{L(\mu)}$*. We compute this function on the basis vector* $v_0 \in L(\lambda)$ *(recall the structure of* $L(\lambda)$ *given in Equation (6.2)). Let* $\{w_0, w_1\}$ *be a weight basis of* $L(\mu)$ *with* w_0 *a highest weight vector. We have*

$$\begin{aligned} v_0 &\xmapsto{\mathrm{id}_{L(\lambda)}\otimes \mathrm{coev}_{L(\mu)}} v_0 \otimes K w_1 \otimes w_1^\vee + v_0 \otimes K w_0 \otimes w_0^\vee \\ &\xmapsto{\mathbb{R}^{-1}_{L(\lambda)\otimes L(\mu)}\otimes \mathrm{id}_{L(\mu)^\vee}} (-1)^{|w_1||v_0|} q^{-\langle h_1, \mu-\alpha\rangle\langle h_1,\lambda\rangle + \langle h_2,\mu-\alpha\rangle\langle h_2,\lambda\rangle + \langle h_1+h_2,\mu-\alpha\rangle} w_1 \otimes v_0 \otimes w_1^\vee \\ &\qquad + (-1)^{|w_0||v_0|} q^{-\langle h_1,\mu\rangle\langle h_1,\lambda\rangle + \langle h_2,\mu\rangle\langle h_2,\lambda\rangle + \langle h_1+h_2,\mu\rangle} w_0 \otimes v_0 \otimes w_0^\vee \\ &\xmapsto{\mathrm{id}_{L(\lambda)}\otimes \widehat{\mathrm{ev}}_{L(\mu)} \circ R^{-1}_{L(\mu)\otimes L(\lambda)} \otimes \mathrm{id}_{L(\mu)^\vee}} (-1)^{|w_0|} q^{-2(\langle h_1,\mu\rangle\langle h_1,\lambda\rangle - \langle h_2,\mu\rangle\langle h_2,\lambda\rangle) + \langle h_1+h_2,\mu\rangle} (-q^{-\langle h_1+h_2,\lambda\rangle} + 1) v_0. \end{aligned}$$

Because λ *is typical,* $q^{-\langle h_1+h_2,\lambda\rangle} \neq 1$*, so*

$$c = (-1)^{|w_0|} q^{-2(\langle h_1,\mu\rangle\langle h_1,\lambda\rangle - \langle h_2,\mu\rangle\langle h_2,\lambda\rangle) + \langle h_1+h_2,\mu\rangle} (-q^{-\langle h_1+h_2,\lambda\rangle} + 1)$$

is nonzero.

We now have our first example of a nontrivial link invariant constructed from the modified Reshetikhin–Turaev approach using a non-semisimple category.

Chapter 7
THE UNROLLED QUANTUM GROUP OF $\mathfrak{sl}_2(\mathbb{C})$

In this chapter we define the unrolled quantum group of $\mathfrak{sl}_2(\mathbb{C})$ and study its category of modules. We construct a link invariant using the modified Reshetikhin–Turaev approach, see [9]. Additional detail about the unrolled quantum group can be found in [3].

7.1
Definition

Fix a positive integer $r' > 1$ and set $r = r'$ if r' is odd and $r = \frac{r'}{2}$ if r' is even. Fix $q = e^{\frac{\pi\sqrt{-1}}{r}}$. Note that q is a primitive $2r$-th root of unity. Define $q^x = e^{\frac{x\pi\sqrt{-1}}{r}}$ and define $\{x\} = q^x - q^{-x}$ for $x \in \mathbb{C}$.

Definition 7.1.1. *The unrolled quantum group, denoted* $U_q^H(\mathfrak{sl}_2(\mathbb{C}))$ *is the* $\mathbb{C}$*-algebra generated by* E, F, K, K^{-1}, *and* H *with relations*

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \quad && KE = q^2EK, \quad && KF = q^{-2}FK, \\ &EF - FE = \tfrac{K-K^{-1}}{q-q^{-1}}, \quad && HK = KH, \quad && HE = EH + 2E, \\ &HF = FH - 2F \end{aligned}$$

and coproduct, counit, and antipode defined by

$$\begin{aligned} &\Delta(E) = 1 \otimes E + E \otimes K, \quad && \varepsilon(E) = 0, \quad && S(E) = -EK^{-1}, \\ &\Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad && \varepsilon(F) = 0, \quad && S(F) = -KF, \\ &\Delta(K) = K \otimes K, \quad && \varepsilon(K) = 1, \quad && S(K) = K^{-1} \\ &\Delta(H) = H \otimes 1 + 1 \otimes H, \quad && \varepsilon(H) = 0, \quad && S(H) = -H. \end{aligned}$$

This structure makes $U_q^H(\mathfrak{sl}_2(\mathbb{C}))$ into a Hopf algebra. It is straightforward to check that the ideal of $U_q^H(\mathfrak{sl}_2(\mathbb{C}))$ generated by E^r and F^r is a Hopf ideal.

Definition 7.1.2. *The restricted unrolled quantum group* $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ *is* $\frac{U_q^H\mathfrak{sl}_2(\mathbb{C})}{R}$, *where* R *is the Hopf ideal generated by* E^r *and* F^r.

Throughout this paper, we will only ever consider the restricted unrolled quantum group. We therefore drop the prefix "restricted" when we refer to it. We are interested in studying modules of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$. We begin by proving a fact about the center of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$.

Definition 7.1.3. *The quadratic Casimir* C *is*

$$C = EF + \frac{Kq^{-1} + K^{-1}q}{\{1\}^2}.$$

Note that using the relation $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, we can rewrite C as

$$FE + \frac{Kq + K^{-1}q^{-1}}{\{1\}^2}.$$

Let $\mathcal{T}_r$ be the rth Chebyshev polynomial,

$$\mathcal{T}_r\left(\frac{X + X^{-1}}{2}\right) = \frac{X^r + X^{-r}}{2}.$$

Theorem 7.1.4 ([3, Proposition 4.1])**.** *The center of* $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ *contains the subalgebra generated by* C *and* $K^{\pm r}$ *with the relation* $\mathcal{T}_r\left(\frac{\{1\}^2}{2}C\right) = -\frac{K^r+K^{-r}}{2}$.

Proof. It is easy to check that $K^{\pm r}$ is central by checking on the generators of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ and using the defining relations. For example,

$$K^rE = q^{2r}EK^r = EK^r \quad \text{(since } q \text{ is a } 2r\text{th root of unity)},$$

so K^r commutes with E.

Now, the DeConcini-Kac quantum group is generated by E, F, K, and K^{-1} with the same relations as $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$, deleting those that involve H. A proof that C is central in $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is given in [12, Section 2.7], so it commutes with E, F, K, and K^{-1}. It is only necessary to show that C commutes with H. We have

$$
\begin{aligned}
CH &= \left(FE + \frac{Kq + K^{-1}q^{-1}}{\{1\}^2}\right)H \\
&= FEH + \frac{KHq + K^{-1}Hq^{-1}}{\{1\}^2} \\
&= F(HE - 2E) + \frac{HKq + HK^{-1}q^{-1}}{\{1\}^2} \\
&= (HF + 2F)E - 2FE + H\frac{Kq + K^{-1}q^{-1}}{\{1\}^2} \\
&= HFE + H\frac{Kq + K^{-1}q^{-1}}{\{1\}^2} \\
&= HC.
\end{aligned}
$$

So the $\mathbb{C}$-subalgebra generated by C and $K^{\pm r}$ is contained in the center of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$. It remains to show that the relation $\mathcal{T}_r\left(\frac{\{1\}^2}{2}C\right) = -\frac{K^r + K^{-r}}{2}$ holds. We prove this relation in parts:

Part 1. We show by induction on $k \in \mathbb{N}$ that

$$
\prod_{i=0}^{k-1}\left(C - \frac{q^{-2i-1}K + q^{2i+1}K^{-1}}{\{1\}^2}\right) = E^kF^k. \tag{7.1}
$$

For $k = 1$, we have

$$
\begin{aligned}
C - \frac{q^{-1}K + qK^{-1}}{\{1\}^2} &= EF + \frac{Kq^{-1} + K^{-1}q}{\{1\}^2} - \frac{q^{-1}K + qK^{-1}}{\{1\}^2} \\
&= EF.
\end{aligned}
$$

Now, assume that the statement holds for k. We show that it holds for $k + 1$. Define $[K; k] =$

$\frac{q^kK+q^{-k}K^{-1}}{\{1\}^2}$. We have

$$\begin{aligned}
&\prod_{i=0}^{k}\left(C-\frac{q^{-2i-1}K+q^{2i+1}K^{-1}}{\{1\}^2}\right)\\
&=E^kF^k\left(C-\frac{q^{-2k-1}K+q^{2k+1}K^{-1}}{\{1\}^2}\right)\\
&=E^kF^k\left(FE+\frac{qK+q^{-1}K^{-1}}{\{1\}^2}-\frac{q^{-2k-1}K+q^{2k+1}K^{-1}}{\{1\}^2}\right)\\
&=E^{k+1}F^{k+1}-[k+1]E^kF^k[K;-k]+E^kF^k\frac{qK+q^{-1}K^{-1}-q^{-2k-1}K-q^{2k+1}K^{-1}}{\{1\}^2}\\
&=E^{k+1}F^{k+1}+E^kF^k\frac{(-q^{k+1}+q^{-k-1})(Kq^{-k}-K^{-1}q^k)+qK+q^{-1}K^{-1}-q^{-2k-1}K-q^{2k+1}K^{-1}}{\{1\}^2}\\
&=E^{k+1}F^{k+1}+E^kF^k\frac{Kq+K^{-1}q^{2k+1}+Kq^{-2k-1}+K^{-1}q^{-1}-qK-q^{-1}K^{-1}-q^{-2k-1}K-q^{2k+1}K^{-1}}{\{1\}^2}\\
&=E^{k+1}F^{k+1}.
\end{aligned}$$

Part 2. We have

$$\begin{aligned}
2\left(\mathcal{T}_r\left(\frac{X+X^{-1}}{2}\right)-\mathcal{T}_r\left(\frac{Y+Y^{-1}}{2}\right)\right)&=(X^r+X^{-r})-(Y^r+Y^{-r})\\
&=X^{-r}(X^r-Y^r)(X^r-Y^{-r})\\
&=\prod_{i=0}^{r-1}X^{-1}(X-q^{2i}Y)(X-q^{-2i}Y^{-1})\\
&=\prod_{i=0}^{r-1}(X+X^{-1}-Yq^{2i}-Y^{-1}q^{-2i}).
\end{aligned}$$

Part 3. Choose X to satisfy $X+X^{-1}=\frac{\{1\}^2}{2}C$ and $Y=qK^{-1}$. Then, Part 2),

$$\begin{aligned}
2\mathcal{T}_r\left(\frac{\{1\}^2}{2}C\right)-2\mathcal{T}_r\left(\frac{qK^{-1}+q^{-1}K}{2}\right)&=\prod_{i=0}^{r-1}\left(C-\frac{q^{2i+1}K^{-1}+q^{-2i-1}K}{\{1\}^2}\right)\\
&=E^rF^r\\
&=0.
\end{aligned}$$

Thus, $\mathcal{T}_r\left(\frac{\{1\}^2}{2}C\right)=-\frac{K^r+K^{-r}}{2}$ holds. □

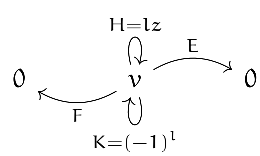
7.2
Simple Weight Modules

Definition 7.2.1 (Highest Weight Module)**.** *A module* V *of* $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ *is called a weight module if it is a direct sum of* H*-eigenspaces. The corresponding* H*-eigenvalues are called weights. A weight vector* v *is a highest weight vector if it is a weight vector and if* $Ev = 0$*. If* V *has a highest weight vector, then* V *is a highest weight module.*

In this section, we only consider the weight modules of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$. Because all modules are assumed to be finite dimensional, every weight module is also a highest weight module. Assume that $K = q^H$ as operators on all weight modules.

We classify the simple weight modules of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ into three types according to their dimension.

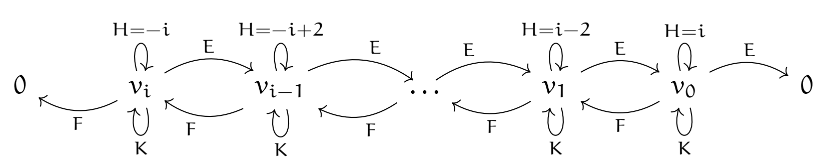
1. One-dimensional modules. We label these modules $\mathbb{C}_{lr}$ for $l \in \mathbb{Z}$. The action of H on $v \in V$ is $Hv = lrv$. This means that K acts on v by $(q^r)^l = (-1)^l$. Both E and F act by zero. The following diagram captures the data of this module:



   If $l \neq l'$, then the action of H on the corresponding modules is not equivalent; therefore, the two modules are not isomorphic. There are $|\mathbb{Z}|$ modules of this type.

   The trivial module is $\mathbb{C}_0$.

2. Modules with dimension in $D = \{2, \ldots, r-1\}$. Let $i \in D$. Consider the weight module S_i of dimension $i+1$. Label the highest weight vector v_0. The action of H on v_0 is $Hv_0 = iv_0$, so $Kv_0 = q^i v_0$. A weight basis of S_i is given by $\{v_0, v_1, \ldots, v_{i-1}\}$ where $v_j = F^j v_0$ for $0 \leq j \leq i$. The action of H on v_j is $Hv_j = (n-2j)v_j$. E acts on v_j by $Ev_j = [j][n+1-j]v_{j-1}$. The following diagram captures the data of this module:



   This module is simple because on $\{v_1, \ldots, v_i\}$ the action of E is nonzero and on $\{v_0, \ldots, v_{i-1}\}$

the action of F is nonzero. The S_i's are the usual simple modules for the quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$, see [12, Proposition 2.12].

Tensor products $S_i \otimes \mathbb{C}_{lr}$ are also irreducible.

3. Modules with dimension r. For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, let V_α be the weight module with highest weight $\alpha + r - 1$. As in type (2), label the highest weight vector v_0 and take the weight basis $\{v_0, v_1, \ldots, v_{r-1}\}$ where $v_j = F^j v_0$. The action of H on v_j is $Hv_j = (\alpha + r - 1 - 2j)$. The action of E is $Ev_j = \frac{\{j\}\{\alpha - j\}}{\{1\}^2}$.

   Because the action of H depends on α, if $\alpha \neq \alpha'$, then V_α and $V_{\alpha'}$ are not isomorphic modules. There are uncountably many modules of this type.

The fact that this list is a complete summary of the simple $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$-weight modules is a corollary of the following theorem. See also [3, Lemma 5.3].

Theorem 7.2.2. *Each simple weight module is uniquely identified by its highest weight.*

Proof. Let V be a simple weight module of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ and let $v \in V$ be the highest weight H-eigenvector with eigenvalue λ.

Now, consider the action of F on v. The vector Fv is an H-eigenvector with eigenvalue $\lambda - 2$. Act on v repeatedly by F to produce a set of distinct H-eigenvectors. Since $F^r = 0$, there is some nonnegative power $j < r$ such that $F^j v \neq 0$ while $F^{j+1} v = 0$.

Now, we compute the action of E on each of the F^is identified above. For $i \in \{1, \ldots, j\}$, pulling E past F repeatedly yields the following equation:

$$\begin{aligned} EF^i v &= F^i E v + \left(\sum_{j=i}^{i} \frac{(q^{2(i-j)} - q^{-2(i-j)})(q^\lambda - q^{-\lambda})}{q - q^{-1}} \right) F^{i-1} v \\ &= \left(\sum_{j=i}^{i} \frac{(q^{2(i-j)} - q^{-2(i-j)})(q^\lambda - q^{-\lambda})}{q - q^{-1}} \right) F^{i-1} v. \end{aligned}$$

Thus, $EF^i v \in \mathrm{span}(F^{i-1} v)$.

Now, the span of the vectors $\{F^i v \,|\, i \in \{0, \ldots, j\}\}$ is a vector subspace of V. We have shown, using the generators E, F, and H, that it is also a submodule. Since V is irreducible, it must equal the span.

By the action of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$, if two modules have the same highest weight, they must be isomorphic.

We note that every highest weight lives in $\mathbb{C}$. We argue that each $\lambda \in \mathbb{C}$ corresponds to a unique simple module.

First, we note that there are no duplicated highest weights among the three types. In type (1), all highest weights are in $r\mathbb{Z}$. In type (2), all highest weights are in $\mathbb{Z} \setminus r\mathbb{Z}$. In type (3), all highest weights are in $\mathbb{C} \setminus \mathbb{Z}$. Therefore, the only possibility that some $\lambda \in \mathbb{C}$ could be the highest weight of two distinct simple modules is if both modules belong to the same type.

We argue that there are no simple modules unaccounted for in the list above. Because every highest weight lies in $\mathbb{C}$, the number of simple modules is at most the order of $\mathbb{C}$. We claim that every $\lambda \in \mathbb{C}$ is the highest weight of at least one simple module in the list above.

If $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, then it is the highest weight of $V_{\lambda-r+1}$ from type (3). If $\lambda \in r\mathbb{Z}$, then it is the highest weight of a $\mathbb{C}_{lz}$ module from type (1). If $\lambda \in \mathbb{Z} \setminus r\mathbb{Z}$, then it is in type (2a). Specifically, if $\lambda \in \{1, \ldots, r-1\}$, then it is the highest weight of some $S_i \otimes \mathbb{C}_0 \simeq S_i$. If $\lambda \in \{r+1, \ldots, 2r-1\}$, then it is the highest weight of some $S_i \otimes \mathbb{C}_r$, and so on. □

Let $\mathcal{C}$ be the category of finite dimensional weight modules of $\overline{U}_q^H \mathfrak{sl}_2(\mathbb{C})$.

Definition 7.2.3. *Let $\alpha \in \mathbb{C}/2\mathbb{Z}$. Define $\mathcal{C}_\alpha$ as the full sub-category of weight modules whose weights are in the class α.*

Example 7.2.4. *Let $[\,] : \mathbb{C} \to \mathbb{C}/2\mathbb{Z}$ be the quotient group homomorphism. Consider the type (1) module $\mathbb{C}_{lr}$. It lies in $\mathcal{C}_{[lr]}$. If lr is even, then $\mathbb{C}_{lr} \in \mathcal{C}_0$. If lr is odd, then $\mathbb{C}_{lr} \in \mathcal{C}_1$.*

The type (2) module S_i lies in $\mathcal{C}_{[i]}$. If i is even, $S_i \in \mathcal{C}_1$; if odd, $S_i \in \mathcal{C}_0$.

The type (3) module V_α has highest weight $\alpha + r - 1$, so $V_\alpha \in \mathcal{C}_{[\alpha+r-1]}$. Recall that $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. If $\alpha = r$, then $V_r \in \mathcal{C}_1$.

The subcategories $\mathcal{C}_\alpha$ give a grading for $\mathcal{C}$ by the group $\mathbb{C}/2\mathbb{Z}$. For $\alpha, \alpha' \in \mathbb{C}/2\mathbb{Z}$, let $V \in \mathcal{C}_\alpha$ and $V' \in \mathcal{C}_{\alpha'}$. The tensor product $V \otimes V'$ has weights congruent to $\alpha + \alpha'$. Thus, $V \otimes V' \in \mathcal{C}_{\alpha+\alpha'}$.

The dual of a module in $\mathcal{C}_\alpha$ for $\alpha \in \mathbb{C}/2\mathbb{Z}$ is in $\mathcal{C}_{-\alpha}$ because, for $f \in V^\vee$ and eigenvector v, H acts on f by

$$(Hf)(v) = f(s(H)v) = -f(Hv) = -f(av) = -af(v)$$

where $a \in \mathbb{C}$ is a lift of α.

Lemma 7.2.5. *Let $a \in \mathbb{C} \setminus \mathbb{Z}$ be in the preimage of α under the map $\mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$ and define $c_a = \frac{q^{a+r}+q^{-a-r}}{\{1\}^2}$. If $i \neq j$ (mod $2r$), then $c_{-2i-1+a+2z-r} - c_{-2j-1+a+2z-r}$ is nonzero.*

Proof. We have

$$c_{a-r-1+2i} - c_{a-r-1+2j} = \frac{q^{a-1}(q^{-2i} - q^{-2j})}{\{1\}^2},$$

which cannot be zero. □

Theorem 7.2.6 ([3, Theorem 5.2])**.** *If $\alpha \in \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$ then $\mathcal{C}_\alpha$ is semi-simple.*

Proof. Let $W \in \mathcal{C}_\alpha$. Notice that $c_{\alpha+2rk} = c_\alpha$ for all $k \in \mathbb{Z}$.

Let w^* be a weight vector in W. By the definition of $\mathcal{C}_\alpha$, there exists a unique integer z such that $Hw^* = (a+2z)w^*$, which implies that $Kw^* = q^{a+2z}w^*$. We now compute how each term in the left hand side of Equation (7.1) acts on w^*. For each i, we have

$$\begin{aligned}\left(C - \frac{q^{-2i-1}K + q^{2i+1}K^{-1}}{\{1\}^2}\right) w^* &= Cw^* - \left(\frac{q^{-2i-1+a+2z} + q^{2i+1-a-2z}}{\{1\}^2}\right) w^* \\ &= (C - c_{a-r-1-2(i-z)})w^*.\end{aligned}$$

Since $c_{\alpha+2rk} = c_\alpha$, then by equation (7.1),

$$\prod_{i=0}^{r-1}(C - c_{a-r-1-2(i-z)})w^* = \prod_{i=0}^{r-1}(C - c_{a-r-1-2i})w^* = E^rF^rw^* = 0. \tag{7.2}$$

Because our choice of weight vector in W was arbitrary and W is a direct sum of weight spaces, it follows that

$$\prod_{i=0}^{r-1}(C - c_{a-r-1-2i})w = 0$$

for all $w \in W$.

By lemma 7.2.5 the polynomial

$$\prod_{i=0}^{r-1}(x - c_{-2i-1+a+2z-r}) \tag{7.3}$$

has only simple roots. Because the minimal polynomial for the action of C on W divides (7.3), it must also have only simple roots. This implies that the action of C on W is diagonalizable.

Assume that C acts on W by a single scalar; otherwise, C acts by a block diagonal matrix and we could consider a direct sum of W. Let j be chosen so that C acts on W by the scalar $c_{a-r-1+2j}$. Let $V \subset W$ be the maximal semi-simple submodule and assume that $V \neq W$. Because the weights of W are ordered (by the ordering on $2\mathbb{Z}$), there is some weight vector in $W \setminus V$, which we denote w, of maximal weight, which we denote λ (i.e. if $v' \in V$ is a weight vector of weight λ', then $\lambda - \lambda' \geq 0$).

Suppose then that $Ew = 0$. Then w generates a simple submodule of W (see the proof of Theorem 7.2.2) which is a direct sum complement of V. This contradicts the maximality condition on V. Suppose then that $Ew \in W \setminus V$. Then the weight of Ew is $\lambda + 2$, which contradicts the maximality assumption on w.

We assume that $Ew \in V$ and compute how F acts on this vector. We have

$$\begin{aligned}
F(Ew) &= (FE)w \\
&= (C - \frac{Kq + K^{-1}q^{-1}}{\{1\}^2})w \\
&= c_{a-r-1-2j}w - \frac{Kq + K^{-1}q^{-1}}{\{1\}^2}w \\
&= \left(\frac{q^{a-1-2j} + q^{-1+1+2j}}{\{1\}^2} - \frac{q^{\lambda+1} - q^{-\lambda-1}}{\{1\}^2}\right)w.
\end{aligned}$$

Now, FEw is in V and in the submodule generated by w, so $FEw = 0$. Hence, $c_{a-r-1+2j} = c_{\lambda+1-r}$. By lemma 7.2.5, $\lambda = a + 2j - 2 \pmod{2r\mathbb{Z}}$.

We compute

$$\begin{aligned}
E^{r-1}F^{r-1}w &= \prod_{i=0}^{r-2}(C - \frac{q^{-2i-1}K + q^{2i+1}K^{-1}}{\{1\}^2})w \\
&= \prod_{i=0}^{r-2}(c_{a-r-1+2j} - \frac{q^{-2i-1+\lambda} + q^{2i+1-\lambda}}{\{1\}^2})w \\
&= \prod_{i=0}^{r-2}(c_{a-r-1+2j} - c_{\lambda-r-1-2i})w \\
&= \prod_{i=0}^{r-2}(c_{a-r-1+2j} - c_{a-r-1+2j-2i-2})w \quad \text{(because } \lambda = a + 2j - 2 \pmod{2r\mathbb{Z})).}
\end{aligned}$$

The zero term in the product is removed, so by lemma 7.2.5, the product is nonzero. Let $p = \prod_{i=0}^{r-2}(c_{a-r-1+2j} - c_{a-r-1+2j-2i-2})$. Then $Ew = \frac{1}{p}E^rF^{r-1}w = 0$. Then, as in the case above, w generates the simple module V_α, which is a submodule of W and is a direct sum complement of V. This contradicts the maximality assumption on V. Therefore, $W = V$. □

7.3

Ribbon Structure on the Category of Modules

The Hopf algebra structure on $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ makes $\mathcal{C}$ into a monoidal category with unit the trivial module, $\mathbb{C}_0$.

Let V, W be objects in $\mathcal{C}$. Let $\{v_i\}$ be a basis of V and $\{v_i^\vee\}$ be a dual basis of $V^\vee$. Define the map

$$\mathrm{coev} : \mathbb{C} \to V \otimes V^\vee$$

as $\mathrm{coev}(1) = \sum_i v_i \otimes v_i^\vee$ and the map

$$\mathrm{ev} : V^\vee \otimes V \to \mathbb{C}$$

as $\mathrm{ev}(f \otimes v) = f(v)$. These are left duality morphisms of $\mathcal{C}$. The right duality maps are given by

$$\widehat{\mathrm{ev}} : V^\vee \otimes V \to \mathbb{C}$$

where $\widehat{\mathrm{ev}}(f \otimes v) = f(K^{1-r}v)$ and

$$\widehat{\mathrm{coev}} : \mathbb{C} \to V \otimes V^\vee$$

where $\widehat{\mathrm{coev}}(1) = \sum_i K^{r-1}v_i \otimes v_i^\vee$. It is straightforward to check that these maps are $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$-linear and that they satisfy the snake relations.

As in the $\mathfrak{gl}(1|1)$ example, we cannot construct a ribbon Hopf algebra structure on $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$. Instead, use the known R matrix on the h-adic quantum group of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ to motivate a braiding on $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$-modules. In [20, Section 4.4] the necessary R-matrix operator is defined on the tensor product of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ modules $V \otimes W$ by

$$R = q^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n,$$

where $q^{H\otimes H/2}$ is the operator given by

$$q^{H\otimes H/2}(v \otimes v') = q^{\lambda\lambda'/2} v \otimes v'$$

where $v \in V$ has weight λ and $v' \in V$ has weight λ'. R gives a braiding $C_{V,W} : V \otimes W \to W \otimes V$ defined by $C_{V,W}(v \otimes w) = \tau(R(w \otimes v))$.

The category of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$ weight modules is also a generically semisimple category, as can be shown by adapting the proof from the $\mathfrak{gl}(1|1)$ and using Theorem 7.2.6. Define

$$\theta_V = \mathrm{id}_V \otimes \widehat{\mathrm{ev}}_V \circ C_{V,V} \otimes \mathrm{id}_{V^\vee} \circ \mathrm{id}_V \otimes \mathrm{coev}_V. \tag{7.4}$$

It is left to the reader to verify that $\theta_{V^\vee} = \theta_V^\vee$. Then, by Theorem 6.4.14, θ_V is a twist on the category of $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$-modules.

Corollary 7.3.1. *The category of modules of the unrolled quantum group is ribbon.*

7.4
Modified Reshetikhin–Turaev Link Invariant Approach

As in the $\mathfrak{gl}(1|1)$ example, the Reshetikhin–Turaev link invariant defined using the modules of the unrolled quantum group gives the zero map on each link colored by V_α because the category of modules is non-semisimple. Therefore, applying the same procedure as in $\mathfrak{gl}(1|1)$, we can try to cut a strand of the link in order to get an endomorphism of a simple module.

Lemma 7.4.1. *Let V and W be irreducible $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$-modules. If $V \otimes W$ is completely reducible and multiplicity free, then the algebra $End_{\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})}(V \otimes W)$ is isomorphic to a direct sum of copies of $\mathbb{C}$. In particular, endomorphisms of $V \otimes W$ commute with braidings.*

Proof. Since $V \otimes W$ is completely reducible and without multiplicity, we have

$$\begin{aligned} \mathrm{End}(V \otimes W) &\simeq \mathrm{End}(\bigoplus_i U_i) \\ &\simeq \mathrm{Hom}(\bigoplus_i U_i, \bigoplus_j U_j) \\ &\simeq \bigoplus_{i,j} \mathrm{Hom}(U_i, U_j) \\ &\simeq \bigoplus_{i,j} \mathbb{C} \end{aligned}$$

where $U_1, \dots, U_n$ are all irreducible and pairwise nonisomorphic. □

Lemma 7.4.2. *Assume that $\eta \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$. Then $V_\eta \otimes V_\eta$ is a semisimple $\overline{U}_q^H\mathfrak{sl}_2(\mathbb{C})$-module without multiplicities.*

Proof. The simple module V_η is of type (3), since $\eta \notin \mathbb{Z}$, and has highest weight $\eta + r - 1$. Furthermore, the module $V_\eta \otimes V_\eta \in \mathcal{C}_{[2\eta]}$.

Since $2\eta \notin \mathbb{Z}$, by Theorem 7.2.6 $V_\eta \otimes V_\eta$ is semisimple. We know from Example 7.2.4 that the following set contains all the simple modules in $\mathcal{C}_{[2\eta]}$:

$$\{V_{2\eta+r-1}, V_{2\eta+r-2}, \dots V_{2\eta-r+2}, V_{2\eta-r+1}\}.$$

Therefore,

$$V_\eta \otimes V_\eta \cong V_{2\eta-r+1}^{m_{2\eta-r+1}} \oplus V_{2\eta-r+3}^{m_{2\eta-r+3}} \oplus \cdots \oplus V_{2\eta+r-3}^{m_{2\eta+r-3}} \oplus V_{2\eta+r-1}^{m_{2\eta+r-1}}.$$

where the m_j's are the multiplicities of the simple modules.

For a module V, the character of V is

$$\mathrm{ch}(V) = \sum \dim(V_\lambda)x^\lambda \in \mathbb{Q}[\mathbb{C}]$$

where the sum runs over the H-weight decomposition and $V = \bigoplus_\lambda V_\lambda$. The character of V_α is

$$\mathrm{ch}(V_\alpha) = x^{\alpha+r-1} + x^{\alpha+r-2} + \cdots + x^{\alpha-r+2} + x^{\alpha-r+1}.$$

We can write the character more simply as

$$\mathrm{ch}(V_\alpha) = x^\alpha[r]_x$$

where $[r]_x = x^{r-1} + x^{r-3} + \cdots + x^{-r+3} + x^{-r+1}$. We check that the set

$$\{\mathrm{ch}(V_{2\eta-r+2i+1}) \,|\, 0 \le i \le r-1\}$$

is linearly independent in the group algebra $\mathbb{Q}[\mathbb{C}]$. Consider the equation,

$$\sum_{i=0}^{r-1} a_i \mathrm{ch}(V_{2\eta-r+2i+1}) = 0. \tag{7.5}$$

All the powers of x lie on the same affine line in the complex plane. On this line, the powers differ from each other by integers; therefore, there is an ordering of the powers given by their positions on the line. The largest power of x in the sum is $x^{2\eta+r+2(r-1)+1}$. There is only one term of this degree. In order for (7.5) hold, it must be the case that $a_{r-1} = 0$.

Now consider the next largest power of x, $x^{2\eta+r+2(r-2)+1}$. Because the previous term has coefficient zero, there is only one term of this degree. Therefore, $a_{r-2} = 0$. Continuing in this way, we argue that every $a_i = 0$.

The character of $V_\eta \otimes V_\eta$ is

$$\mathrm{ch}(V_\eta) \cdot \mathrm{ch}(V_\eta) = x^{2\eta}[r]_x[r]_x.$$

On the other hand,

$$\mathrm{ch}(V_\eta)\cdot\mathrm{ch}(V_\eta) = m_{2\eta-r+1}x^{2\eta-r+1}[r]_x + m_{2\eta-r+3}x^{2\eta-r+3}[r]_x + \cdots + m_{2\eta+r-3}x^{2\eta+r-3}[r]_x + m_{2\eta+r-1}x^{2\eta+r-1}[r]_x.$$

We check that setting each m_i equal to 1 is a solution. We have

$$x^{2\eta-r+1}[r]_x + x^{2\eta-r+3}[r]_x + \cdots + x^{2\eta+r-3}[r]_x + x^{2\eta+r-1}[r]_x = x^{2\eta}[r]_x[r]_x = \mathrm{ch}(V_\eta)\cdot\mathrm{ch}(V_\eta).$$

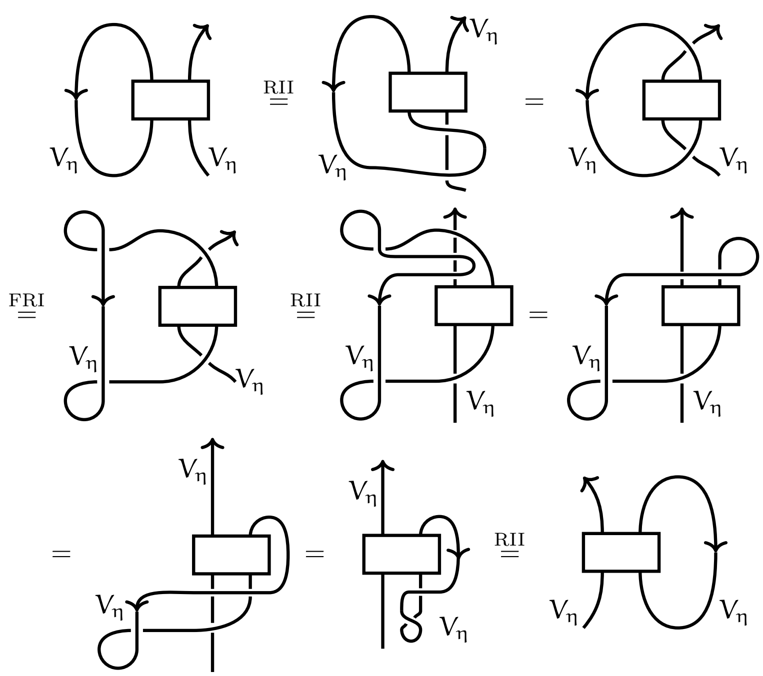
By linear independence, setting the m_is equal to 1 must be the only solution. Thus,

$$V_\eta \otimes V_\eta \cong V_{2\eta-r+1} \oplus V_{2\eta-r+3} \oplus \cdots \oplus V_{2\eta+r-3} \oplus V_{2\eta+r-1},$$

as desired. □

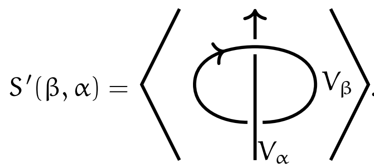
Theorem 7.4.3. *Let* $\eta \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$*. Then* V_η *is ambidextrous.*

Proof. Because the endomorphism algebra $\mathrm{End}(V_\eta \otimes V_\eta)$ is commutative, we have the following sequence of equalities, where we apply the functor F to every tangle.



The second equality is implied by the commutativity of endomorphisms. The fifth and seventh equalities are each a combination of framed RIII and RII. The sixth equality holds by a combination of framed Reidemeister moves that depends on the coupon, but holds because the horizontal strand crosses over both of the strands that feed into and emerge from the coupon. □

Definition 7.4.4 ($S'(\beta, \alpha)$)**.** *Let* $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}$*. Define*

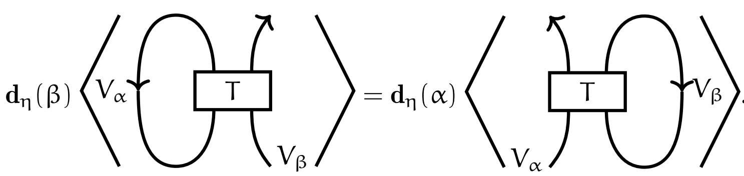


It is a straightforward calculation to check that $S'(\beta, \alpha)$ is nonzero for all $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z} \cup r\mathbb{Z}$.

Definition 7.4.5 ($\mathbf{d}_\eta(\alpha)$)**.** *Let* $\eta \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$*. Define* $\mathbf{d}_\eta : \mathbb{C} \setminus \mathbb{Z} \cup r\mathbb{Z} \to \mathbb{C}$ *by*

$$\mathbf{d}_\eta(\alpha) = \frac{S'(\alpha, \eta)}{S'(\eta, \alpha)}.$$

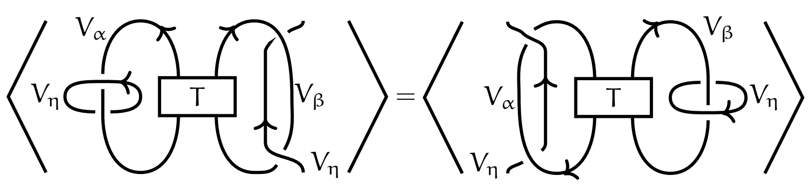
Theorem 7.4.6. *Let* $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}$ *and* $\eta \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$*. Then*



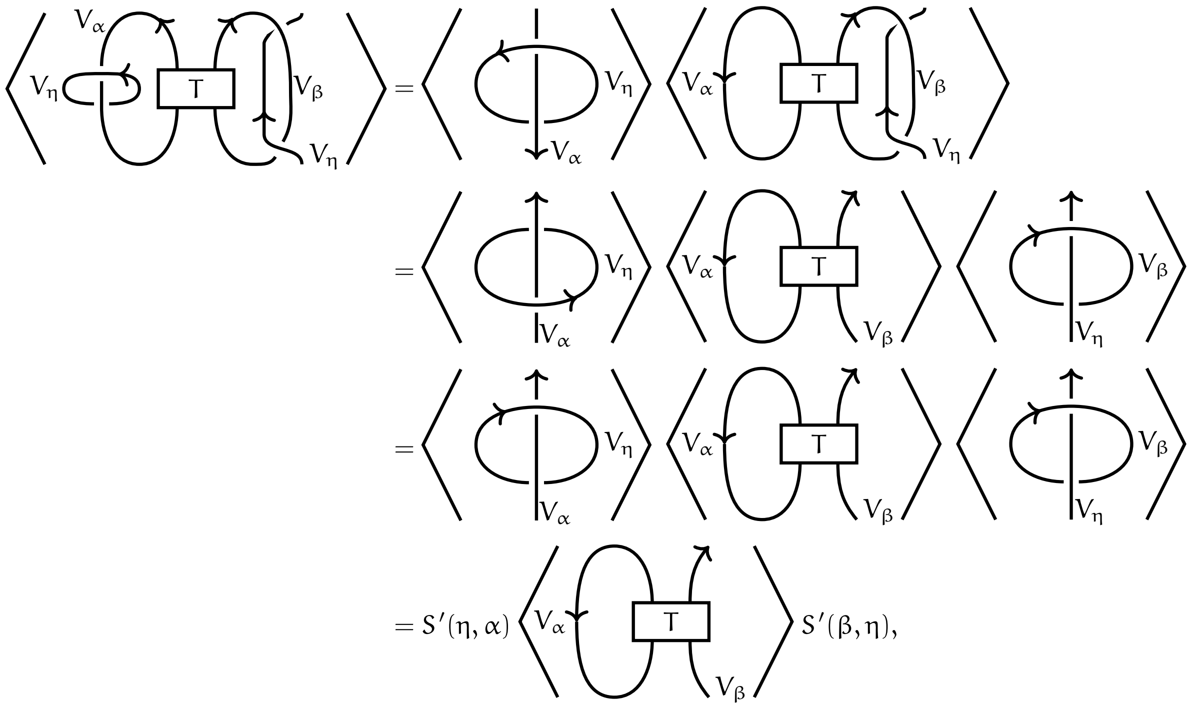
(7.6)

for all $(2, 2)$*-tangles* T*.*

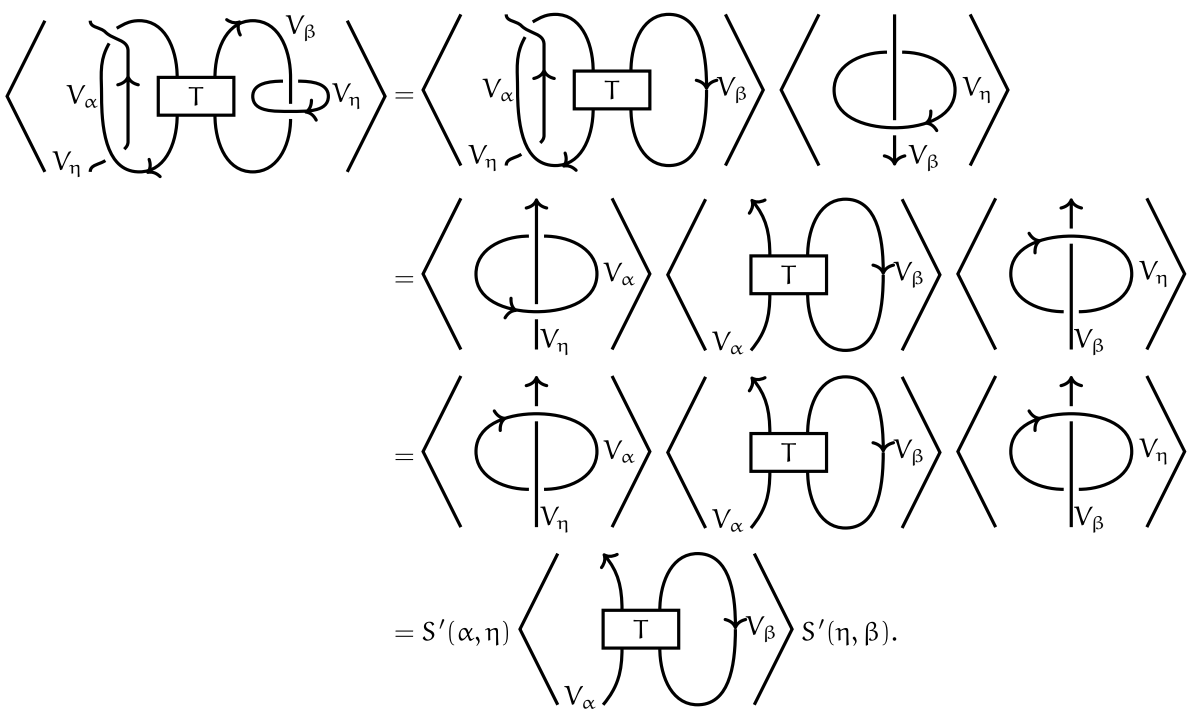
Proof. Because V_η is ambidextrous, it is the case that



We expand both sides of this equality. On the left, we have

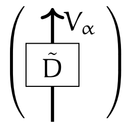


On the right, we have



Setting these two sides equal and dividing out the S's yields relation 7.6. □

Corollary 7.4.7. *Every simple module of type (3) is ambidextrous. Moreover, we can modify the proof of Theorem 6.5.5 to construct a link invariant*

F 

$= c \cdot \mathrm{id}_{V_\alpha}$

Computing the morphism for the unknot colored by V gives nq^{α}, where n is the dimension of V. Thus, the invariant is nonzero.

The unrolled quantum group at various roots of unity leads to the construction of a variety of non-semisimple categories. Although some of the invariants that come from various small roots of unity are equivalent to classical invariants. For example, at 4th root of unity the invariant is the same as the one constructed by $\mathfrak{gl}(1|1)$ in Chapter 6. At other roots of unity, the invariant seem to be previously undiscovered, see [9], [2], and [1].

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APPENDIX

Let V and W be weight $U_q(\mathfrak{gl}(1|1))$-modules. In section 6.4, lemma 6.4.6, we claim that the operator $c_{V,W}$ is $U_q(\mathfrak{gl}(1|1))$-linear. Here we provide the detailed calculations in the proof of this claim.

We recall that $c_{V,W} = \tau \circ \Theta_{V,W}\Upsilon_{V,W}$, where

$$\Theta_{V,W} = 1 + (q - q^{-1})F \otimes E$$
$$\Upsilon_{V,W} = q^{h_1 \otimes h_1 - h_2 \otimes h_2}.$$

Let $v \in V$ and $w \in W$ be weight vectors with weights λ and μ, respectively. The equation that we need to verify to check that $c_{V,W}$ is $U_q(\mathfrak{gl}(1|1))$-linear is

$$c_{V,W}(x \cdot v \otimes w) = x c_{V,W} \cdot (v \otimes w) \tag{7.7}$$

for all $x \in U_q(\mathfrak{gl}(1|1))$. It suffices to check this equation on generators E, F, q^{h_1}, and $K^{\pm 1}$.

Check E first. The left side of (7.7) is

$$\begin{aligned}
c_{V,W}E(v \otimes w) &= c_{V,W}\Delta(E)(v \otimes w) \\
&= c_{V,W}(E \otimes K^{-1} + 1 \otimes E)(v \otimes w) \\
&= \tau \circ \Theta_{V,W}\Upsilon_{V,W}(Ev \otimes q^{-\langle h_1+h_2,\mu\rangle}w + (-1)^{|v|}v \otimes Ew) \\
&= \tau \circ \Theta_{V,W}(q^{\langle h_1,\lambda+\alpha\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda+\alpha\rangle\langle h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle}Ev \otimes w \\
&\quad + (-1)^{|v|}q^{\langle h_1,\lambda\rangle\langle h_1,\mu+\alpha\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu+\alpha\rangle}v \otimes Ew) \\
&= \tau(q^{\langle h_1,\lambda+\alpha\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda+\alpha\rangle\langle h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle}Ev \otimes w \\
&\quad + (-1)^{|v|}q^{\langle h_1,\lambda\rangle\langle h_1,\mu+\alpha\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu+\alpha\rangle}v \otimes Ew \\
&\quad + (q-q^{-1})(-1)^{|v|+1}q^{\langle h_1,\lambda+\alpha\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda+\alpha\rangle\langle h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle}FEv \otimes Ew) \\
&= (-1)^{(|v|+1)|w|}q^{\langle h_1,\lambda+\alpha\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda+\alpha\rangle\langle h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle}w \otimes Ev \\
&\quad + (-1)^{|v|+|v|(|w|+1)}q^{\langle h_1,\lambda\rangle\langle h_1,\mu+\alpha\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu+\alpha\rangle}Ew \otimes v \\
&\quad + (q-q^{-1})(-1)^{|v|+1+(|w|+1)|v|}q^{\langle h_1,\lambda+\alpha\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda+\alpha\rangle\langle h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle}Ew \otimes FEv.
\end{aligned}$$

The fourth term in the expression after the fifth equality is zero because $E^2 = 0$. We simplify the coefficients in front of each term. The coefficient of $w \otimes Ev$ is

$$\begin{aligned}
&(-1)^{(|v|+1)|w|}q^{\langle h_1,\lambda+\alpha\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda+\alpha\rangle\langle h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle} \\
&= (-1)^{(|v|+1)|w|}q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu\rangle+\langle h_1,\alpha\rangle\langle h_1,\mu\rangle-\langle h_2,\alpha\rangle\langle h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle} \\
&= (-1)^{(|v|+1)|w|}q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu\rangle+\langle h_1+h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle} \\
&= (-1)^{|v||w|+|w|}q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu\rangle}.
\end{aligned}$$

The second equality holds because $\langle h_1, \alpha\rangle = 1$ and $\langle h_2, \alpha\rangle = -1$.

The coefficient of $Ew \otimes FEv$ is

$$\begin{aligned}
&(q-q^{-1})(-1)^{|v|+1+(|w|+1)|v|}q^{\langle h_1,\lambda+\alpha\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda+\alpha\rangle\langle h_2,\mu\rangle-\langle h_1+h_2,\mu\rangle} \\
&= (q-q^{-1})(-1)^{|v||w|+1}q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle-\langle h_2,\lambda\rangle\langle h_2,\mu\rangle}.
\end{aligned}$$

The coefficient of $Ew \otimes v$ is

$$\begin{aligned}
&(-1)^{|v|+|v|(|w|+1)} q^{\langle h_1,\lambda\rangle\langle h_1,\mu+\alpha\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu+\alpha\rangle} \\
&= (-1)^{|v||w|} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle + \langle h_1,\lambda\rangle\langle h_1,\alpha\rangle - \langle h_2,\lambda\rangle\langle h_2,\alpha\rangle} \\
&= (-1)^{|v||w|} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle + \langle h_1+h_2,\lambda\rangle}.
\end{aligned}$$

The right side of equation (7.7) is

$$\begin{aligned}
Ec_{V,W}(v \otimes w) &= E\tau \circ \Theta_{V,W} \Upsilon_{V,W}(v \otimes w) \\
&= E\tau \circ \Theta_{V,W}(q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle} v \otimes w) \\
&= E\tau(q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle} v \otimes w \\
&\quad + (q - q^{-1})(-1)^{|v|} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle} Fv \otimes Ew) \\
&= \Delta(E)(q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(-1)^{|v||w|} w \otimes v \\
&\quad + (q - q^{-1})(-1)^{|v|+(|v|+1)(|w|+1)} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle} Ew \otimes Fv \\
&= q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle - \langle h_1+h_2,\lambda\rangle}(-1)^{|v||w|} Ew \otimes v \\
&\quad + q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(-1)^{|v||w|+|w|} w \otimes Ev \\
&\quad + (q - q^{-1}) q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(-1)^{|v||w|} Ew \otimes EFv.
\end{aligned}$$

We use the relation $EF = -FE + \frac{K-K^{-1}}{q-q^{-1}}$ on the last term. We have

$$\begin{aligned}
Ec_{V,W}(v \otimes w) &= q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle - \langle h_1+h_2,\lambda\rangle}(-1)^{|v||w|} Ew \otimes v \\
&\quad + q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(-1)^{|v||w|+|w|} w \otimes Ev \\
&\quad + (q - q^{-1}) q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(-1)^{|v||w|+1} Ew \otimes FEv \\
&\quad + q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(-1)^{|v||w|}(q^{\langle h_1+h_2,\lambda\rangle} - q^{-\langle h_1+h_2,\lambda\rangle}) Ew \otimes v \\
&= (-1)^{|v||w|+|w|} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle} w \otimes Ev \\
&\quad + (q - q^{-1})(-1)^{|v||w|+1} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle} Ew \otimes FEv \\
&\quad + (-1)^{|v||w|} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle + \langle h_1+h_2,\lambda\rangle} Ew \otimes v.
\end{aligned}$$

The coefficients of the terms on the right side agree with the coefficients on the left, so the equality holds.

The calculations to check equation (7.7) on F is similar. For K, we have

$$\begin{aligned}
c_{V,W}K(v \otimes w) &= c_{V,W}(Kv \otimes Kw) \\
&= (-1)^{|v||w|} q^{\langle h_1+h_2,\lambda+\mu\rangle + \langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle} w \otimes v \\
&\quad + (-1)^{(|v|+1)(|w|+1)+|v|} q^{\langle h_1+h_2,\lambda+\mu\rangle + \langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(q - q^{-1}) Ew \otimes Fv \\
&= K((-1)^{|v||w|} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle} w \otimes v \\
&\quad + (-1)^{(|v|+1)(|w|+1)+|v|} q^{\langle h_1,\lambda\rangle\langle h_1,\mu\rangle - \langle h_2,\lambda\rangle\langle h_2,\mu\rangle}(q - q^{-1}) Ew \otimes Fv) \\
&= Kc_{V,W}(v \otimes w).
\end{aligned}$$

The calculations to check K^{-1} and q^{h_1} are similar.