Teachers’ Learning of Fraction Division With Area Models

Michael Leitch
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ABSTRACT

Teachers’ Learning of Fractions Division with Area Models

by

Michael Leitch, Doctor of Philosophy
Utah State University, 2023

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Department: Mathematics Education and Leadership

This qualitative, multiple case study provided comprehensive descriptions of the conceptual difficulties and learning experiences of in-service teachers as they improved their ability to effectively model fraction division with pictorial diagrams. Video data were collected on eight teachers as they individually progressed through a professional development (PD) program. The data were used to generate case-based descriptions and to conduct a cross-case synthesis.

Ma’s Knowledge Package for Understanding the Meaning of Division by Fractions and Vergnaud’s Multiplicative Conceptual Field provided the conceptual framework for the PD tasks. The Knowledge in Pieces (KiP) epistemological framework provided a lens for the analysis of the participants’ engagement in the tasks.

Results from the analysis of interview data contributed to the literature by identifying participants’ underlying conceptual resources. Some of these conceptual resources were idiosyncratic, such as conceptualizing partitive division with fractions as a type of density. Other conceptual resources and their functions were common, such as
knowing that the quotient is based on scaling the divisor to a value of one yet being unable to identify it in some contexts.

Additionally, distinct psychological structures emerged that might be common among learners when engaging in partitive division with fractional divisors. The participants in this study exhibited multiple models of partitive division that generalized into two distinct structures of partitive division with fractional divisors. These models generalized into part-whole models and unit-rate models. Part-whole models attended to a single referent. The referent was seen as a quantity, part of which was known; or a process, part of which was completed. Unit-rate models attended to separate referents for the dividend and divisor. This finding extends the research literature as the structures and their variants seen in the present study do not appear to have received much attention.

Results of this study can be leveraged in curriculum design for teacher education on the subject of division with fractions. Results suggest that the KiP epistemological framework is a productive analytical framework for future research on learners’ connections between partitive division and other mathematics topics to which it is foundational, such as rate, intensive quantity, proportion, derivatives, probability, and statistics.

(287 pages)
PUBLIC ABSTRACT

Teachers’ Learning of Fractions Division with Area Models

Michael Leitch

Research shows that fractions concepts play an essential role in the learning of later mathematics. However, fractions are notoriously difficult to learn and difficult to teach. Division with fractions is a frequent subject in mathematics education research because division is the most conceptually difficult of the four basic arithmetic operations and rational numbers are the most conceptually difficult numbers in K-12 mathematics curricula.

In the U.S., teachers are generally proficient with mathematical procedures, but often have difficulty explaining the concepts underlying the procedures. Research indicates a positive association between student learning and teachers’ depth of conceptual understanding of mathematics. Thus, it is important to ensure that future and practicing teachers are competent with fractions operations at a deep, conceptual level.

In order to gain a better understanding of teachers’ conceptions of division with fractions, this study engaged teachers in a 4-hour professional development program designed to deepen the teachers’ understanding of fractions and their ability to represent fraction operations through the construction of rectangular area models. Eight teachers were given one-to-one professional development. Analysis of these videos showed that teachers constructed idiosyncratic conceptions yet faced some common challenges. One common challenge was that a central part of the division concept was readily visible to
the teachers in some contexts but not in other contexts. Another common difficulty teachers experienced was conceptually explaining why the quotient to a fraction division problem should be based on a whole unit of the divisor.

Additionally, teachers constructed different modules of division arising from the structure of the situation in which the division was conceptualized. Models of partitive division with one apparent referent were easier to conceptualize and represent. Models of partitive division with two apparent referents were more difficult to conceptualize and represent. Two-referent models of partitive division with fractions are fundamental to rate and intensive quantity, and directly relate to other topics in mathematics, such as proportion and derivatives. Results of this study shed light on potentially common conceptual difficulties as well as suggest ways that learners can facilitate a conceptual understanding and representational fluency with fractions division.
I first want to thank Drs. Patricia Moyer-Packenham and Beth MacDonald for years of mentorship and support in addition to their leadership roles on my committee. I would also like to thank Drs. Marla Robertson and Colby Tofel-Grehl for their suggestions, time, and patience as they served on my committee. I am grateful for Dr. Mariana Levin’s participation as the outside committee member, providing background that gave me the confidence to conduct this study. Dr. Levin’s Knowledge in Pieces working group was a valuable community throughout the data collection and analysis for this dissertation. I give special thanks to Maya for her patience and support through many years of coursework and the process of writing this dissertation. Last, I want to thank my doctoral student buddies, Jill and Angie, for countless hours of discussing each other’s writing and ideas. Their support over the years got me through the most difficult times.

Michael Leitch
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CHAPTER 1
INTRODUCTION

Background

Research shows that fraction concepts play an essential role in the learning of later mathematics (Hackenberg & Lee, 2016; Matthews & Ziols, 2019; Siegler et al., 2012). Early success or failure with fractions skills and concepts correlates positively with later mathematics achievement (Siegler & Lortie-Forgues, 2014). However, fractions are notoriously difficult to learn (e.g., Newton, 2008) and difficult to teach (e.g., Lamon, 2007). The U.S. Department of Education National Mathematics Advisory Panel (2008) concluded that proficiency with fractions is a foundational skill that “seems to be severely underdeveloped” (p. xvii) in meeting mathematics education goals. Despite the attention given to fractions concepts and skills in elementary and middle school, students’ difficulties with fractions often persist into adulthood (Jordan et al., 2013; Siegler & Lortie-Forgues, 2015).

Division with Fractions

Division with fractions is a frequent subject in mathematics education research because division is the most conceptually difficult of the four basic arithmetic operations and rational numbers are the most conceptually difficult numbers in K-12 mathematics curricula (Clarke, 2011; Lamon, 2007; Ma, 2010). While division of fractions is a conceptually rich and difficult topic for students (S. J. Lee et al., 2011; Sidney & Alibali, 2017; Siegler & Lortie-Forgues, 2015) it is also a particularly fruitful area for examining
preservice teachers’ (PSTs’) and inservice teachers’ (ISTs’) conceptual understanding of fundamental mathematics (Flores, 2002; Izsák et al., 2012; Jansen & Hohensee, 2016; M. Y. Lee, 2017; Lo & Luo, 2012; Olanoff et al., 2014; Newton, 2008; Nillas, 2003; Tirosh, 2000). Despite the importance of division with fractions, many mathematics textbooks in the U.S. overlook the distinctions between the partitive model of division (fair sharing) (Jansen & Hohensee, 2016) and the measurement model of division (repeated subtraction; Siebert, 2002) or present the two different models of division in the context of whole numbers only. Even textbooks designed for PSTs’ coursework can describe the difference between partitive and measurement division for whole numbers without extending the concepts to fractions (Stevens et al., 2020).

**Teacher Effectiveness in Fractions Instruction**

Research indicates a positive association between student learning and teachers’ depth of conceptual understanding of mathematics (Hill et al., 2005; Ma, 2010; Newton, 2008). Concomitantly, Lo and Luo (2012) suggest a vicious cycle in the U.S. through which “poor K-12 mathematics education leads to underprepared teachers with insufficient mathematics knowledge for teaching” (p. 497). This description is supported by research indicating that U.S. students’ low achievement in mathematics may be due, in part, to variability in teachers’ conceptual understanding of fundamental mathematics (Lo & Lou, 2012; Ma, 2010; Olanoff et al., 2014; Stevens et al., 2020; Torbeyns et al., 2015; Zhou, et al. 2006) and variability in variability in teachers’ abilities with multiple representations (Izsák et al., 2012; Jansen & Hohensee, 2016; J. E. Lee & Lee, 2020). In the U.S., teachers are generally proficient with mathematical procedures, but often have
difficulty explaining the concepts underlying the procedures (Ball, 1990; Olanoff et al., 2014; Stigler et al., 1999; Whitacre & Nickerson, 2016; Wilhelm, 2014). Ma (2010) concluded that U.S. teachers’ comprehension of fractions is missing “necessary concepts for understanding and their connections” (p. 82). Pertinent to this study, many U.S. teachers cannot explain why the Invert-and-Multiply (IM) rule is taught, have difficulty conceptualizing the process (e.g., Borko et al., 1992; Ma, 2010; Siebert, 2002), and struggle with representing the procedure (M. Y. Lee, 2017; Lo & Luo, 2012; Ma, 2010; Siebert, 2002).

**Representations and Fraction Instruction**

Mathematics education standards such as the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (NCTM, 2000) emphasize the role of multiple representations in mathematical understanding and problem solving. Fluency with multiple representations is important for effective fractions instructions (e.g., Stevens et al., 2020). Yet many U.S. teachers exhibit persistent difficulties developing pedagogically effective representations of multiplication and division with fractions (Baek et al., 2017; Izsák et al., 2012; Lo & Lou, 2012; Simon, 1993; Tirosh & Graeber, 1989) and many inservice teachers simply do not use multiple representations to promote fraction understanding (Izsák, 2008; Lee, 2017, Lee et al., 2011). Lo and Luo (2012) showed that even mathematically competent PSTs in Taiwan found it challenging to represent fraction division in word problems and pictorial diagrams. Izsák et al. found that when ISTs’ ability to use drawings to model fraction multiplication and division concepts was diminished, so was their effective
implementation of standards-based curriculum. These authors concluded that, given the importance of reform-oriented fractions instruction, our lack of understanding of teachers’ use of drawings is a “significant omission in research on teacher’s knowledge” (p. 140).

Recent Research on Teacher’s Conceptual Understanding and Representation of Fractions

Recent research has moved from identifying that PSTs and ISTs experience difficulties with conceptualizing and representing fractions operations to examining what those difficulties might be (e.g., Hohensee & Jansen, 2017; Jansen & Hohensee, 2016; M. Y. Lee, 2017; Stevens et al., 2020). Convergent findings from this research show that disconnected concepts and inflexible or missing awareness of the referent unit commonly occur when PSTs and ISTs exhibit difficulty with fraction division concepts and representations. Most relevant to this dissertation, ISTs exhibit difficulty with the referent unit when representing both types of division with fractions and when representing multiplication of fractions (Izsák, 2008; M. Y. Lee, 2017; S. J. Lee et al., 2011).

Research examining the difficulties teachers exhibit with division of fractions generally use task-based pen-and-paper tests, and researchers analyze performance based on written responses (e.g., Jansen & Hohensee, 2016; M. Y. Lee, 2017; Lo & Luo, 2012). These analyses often tabulate correct or incorrect performance and their associated types of productive strategies or misconceptions. These studies are also focused on content, for example, examining only partitive division or measurement division, or multiplication, or focusing only on the coordination of meaning across representations. The few studies that
include interviews focus on specific tasks (e.g., S. J. Lee et al., 2011, conducted 1-hour interviews to validate a task). Thus, recent research identified in this section commonly calls for future studies to provide more “in-depth examination of thinking processes” (M. Y. Lee, 2017, p. 346) to gain a better understanding of the connections needed between concepts and the role that referent units play in using multiple representations for effective instruction of multiplication and division of fractions.

**Statement of the Problem**

Improving teacher preparation and professional development (PD) requires a deeper understanding of how teachers think, and more reliance on theory for understanding the connections between components of teachers’ knowledge (e.g., S. J. Lee et al., 2011). This increase in depth of understanding of the difficulties teachers experience is necessary before research results can be implemented into effective professional development materials and practices (Jansen & Hohensee, 2016; M. Y. Lee, 2017; Lo & Luo, 2012). In order to make progress in this direction research needs to extend examinations of teachers’ performance on individual pencil and paper tasks by collecting interview data across tasks and develop conceptual and theoretical tools for analysis of the potentially rich and complex data that deeper examinations may generate (S. J. Lee et al., 2011). In sum, the research described above has moved from identifying that teachers experience difficulties to examining what those difficulties might be. A better understanding of the nature of teachers’ conceptual difficulties is needed to inform teacher preparation and professional development that more effectively address the
difficulties and support teachers in their knowledge construction.

**Purpose of the Study**

The purpose of this study is to provide comprehensive descriptions of the conceptual difficulties and learning experiences of ISTs as they construct interconnected understandings of fractions division, and to identify the role of the referent unit in the teachers’ learning as they improve their ability to effectively model fractions division with pictorial diagrams.

**Significance of the Study**

Professional development focused on improving teachers’ mathematical knowledge has become increasingly important in K-12 and undergraduate institutions (Reinholz et al., 2020; Sztajn et al., 2017). Toward the goal of improving teachers’ mathematical knowledge, this study aims to provide a deeper understanding of how ISTs construct their knowledge of fractions and develop representational fluency with their existing conceptual resources. The method employed in this study provides learning opportunities suggested by S. J. Lee et al. (2011) that acknowledge teachers’ expertise while “supporting them in making new connections among and adding to their knowledge pieces” (p. 218). A deeper and more comprehensive understanding of how ISTs use their existing conceptual resources to construct deeper understandings of division with fractions and referent units may contribute to the improvement of PD and teacher education.
Research Questions

This study employed a qualitative multiple case study design to conduct professional-development-based interviews with ISTs as they learned about a curriculum intervention on division with fractions. This qualitative, multiple case study was designed to answer the following research questions.

1. What are the conceptual resources exhibited by IST’s as they construct a deeper understanding of division with fractions and improve their ability to effectively model fractions division with pictorial diagrams.

2. What role does the teachers’ perceptions of the referent unit play in their knowledge construction across tasks involving multiplication and both types of division with fractions?

3. Do participants in this study who are actively teaching the PD content construct knowledge differently than participants for whom the content prepares for future teaching?

Summary of the Research Design

This study employed a qualitative, multiple case study design (Creswell, 2018; Merriam & Tisdell, 2016) to collect case-based data on individual teachers as they progressed through a PD program. This data was used to generate case-based descriptions (Creswell, 2018), and conduct a cross-case synthesis (Yin, 2018).

The qualitative multiple case study approach followed from the research purpose, allowing for depth of exploration (Creswell, 2018). In this study a case was defined as a single teacher. The primary source of data in this study was interviews with individual teachers, allowing for a higher density of observations (Siegler, 2006) without participants influencing each other’s responses (Tashakkori & Teddlie, 2003).
Definition of Terms

The following terms are used in this study according to the definitions described below.

*Conceptual resources:* Conceptual resources are components of knowledge systems, including mathematical concepts, procedures, rules, strategies, and intuitions.

*Cross-partitioning:* The procedure of partitioning a rectangular area diagram in both horizontal and vertical directions.

*Flexibility with the referent unit:* Flexibility with the referent unit is the ability to “keep track of the unit to which a fraction refers” and the ability to shift one’s “relative understanding of the quantities as the referent unit changes” (S. J. Lee et al., 2011, p. 204).

*Given rate:* The conceptualization of a division problem as a rate in the quantitative form \( \text{dividend per divisor} \), where the dividend and divisor quantities are taken directly from the numerical statement of the problem. Given rate is the precursor to *unit rate* (see below).

*Iterate:* Any mental or physical act (including drawing) that joins copies of a segment to recreate an established whole (Izsák et al., 2008). Partitioning and iterating are fundamental to fractions, with iterating being the action that creates a numerator.

*Measurement division:* A model of division based on equal-groups multiplication in which the size of the groups is known, and the quotient is interpreted as the number of groups that compose the dividend. Measurement division is also modeled as repeated subtraction (Siebert, 2002).
**Model:** In the noun form, a model is an instantiation (including drawings and digital images) that substitutes for an object or process. For example, there are circular area models of fractions and discrete set models of fractions. The term model is often used to indicate a mental image or internal state. In this dissertation, the internal state will be indicated by using *mental model*.

**Partition:** Any mental or physical act (including drawing) that divides a whole into equally sized parts. Partitioning and iterating are fundamental to fractions, with partitioning being the action that creates a denominator.

**Partitive division:** A model of division based on equal-groups multiplication in which the size of the groups is unknown, and the quotient is interpreted as the number of items per group. Partitive division is also modeled as fair sharing (Siebert, 2002).

**Numerical statement of the problem.** This terminology refers to a mathematical problem written as an arithmetic statement or equation (e.g., 1/2 ÷ 3/4).

**Rectangular area diagram:** A pictorial diagram (drawn or digital image) taking the form of a square or rectangle, the area of which represents a whole.

**Referent unit:** The whole to which a number or fraction refers.

**Representation:** In this study I use the word *representation* in the epistemological sense, meaning a description that one constructs to convey ideas. The description could be verbal or pictorial. In contrast, I use the word *model* in the ontological sense, meaning that a model is a form. For example, fractions can be represented with area models or discrete models. This study is concerned with how teachers learn to use rectangular area models to represent division with fractions.
Unit rate: The conceptualization of the quotient to a division problem as a rate resulting from the transformation of the given rate such that the divisor maps to one. The given rate and unit rate therefore express the same quantitative relationship, but the unit rate always has a value of one for the divisor.

Summary

Competency with fractions is a crucial component of mathematics achievement. Despite their importance, fractions concepts are difficult to teach and learn. Research shows a need to improve U.S. teachers’ understanding of fractions across K-12 and undergraduate levels. Recent research into the difficulties that PSTs and ISTs exhibit with division of fractions is beginning to identify the difficulties. However, further research is needed to gain an understanding of how teachers actually reason about fraction division that is comprehensive enough to inform more effective teacher education and PD programs. This study aimed to make progress in attaining that comprehensive understanding.
CHAPTER 2

LITERATURE REVIEW

This chapter describes the literature from which the conceptual framework was constructed, and the framework for the analyses applied in this study. The conceptual framework for this study integrates three lines of mathematics education research.

1. The Knowledge in Pieces (KiP) epistemological perspective (diSessa et al., 2018) applied to the learning of mathematics (Brown et al., 2019; diSessa & Wagner, 2005; Izsák & Jacobson, 2017; Weiland et al., 2020).


3. Representing fraction multiplication and division with pictorial diagrams (e.g., Izsák et al., 2012), the role of the referent unit (e.g., M. Y. Lee, 2017), and pictorial explanations of the invert-and-multiply (IM) rule for both measurement and partitive interpretations of fraction division (Flores, 2002; Siebert, 2002).

This study integrated the three lines of research listed above to further an understanding of in-service teachers’ (ISTs’) learning of division with fractions concepts by broadening the content domain, compared to previous studies, and including the development of representational fluency as a central goal in the tasks. While previous studies have investigated ISTs’ and pre-service teachers’ (PSTs’) challenges with particular tasks in the field of division with fractions (e.g., M. Y. Lee, 2017, investigated PSTs’ generated diagrams in solving a measurement division with fractions problem), the present study was designed to investigate conceptual resources that teachers activate and coordinate in moments of acquisition (or deepening) of connected concepts comprising the conceptual field of division with fractions. Vergnaud’s (1994) suggestion to not study
topics in isolation is supported from several directions. For example, NCTM’s (2000) *Principles and Standards for School Mathematics* includes *Connections* and *Representation* as two core standards for all K-12 grade levels. The empirical results from Ma (2010) and Lamon (2007) indicated that division with fractions (possibly the concept of fraction or division itself) cannot be defined as an isolated skill or concept.

Choosing the content domain of a study to be a conceptual field rather than a narrowly chosen area of specialized content knowledge (SCK; Hill et al., 2008) certainly involves complication and, one might argue, introduces too many variables. This study adopted KiP as an analytical framework and cross-case comparison as a methodology because they address complexity. The KiP epistemological framework views some concepts as complex systems that evolve through increasingly productive applications across varieties of contexts, involving the leveraging of perceptual and inferential resources. The simple concept “division” might be productively analyzed as such a complex system involving coordination clusters (A. diSessa, personal communication, January 25, 2020), described in the following section. Multiple-case study with cross-case synthesis is a method used to manage the complexity of real-world data (Yin, 2018).

Additionally, recent research in this area highlights the need for interview data and follow-up sessions to increase the credibility of analysis and inferences (e.g., M. Y. Lee, 2017). In this study, interviews supported ISTs in productive struggle so that moments of knowledge acquisition could be captured.
Conceptual Framework

The conceptual framework guiding the design, data collection, and analysis for this study is depicted in Figure 2.1. Task-based interviews in the context of professional development sessions scaffolded ISTs as they learned to construct pictorial diagrams representing multiplication and division of fractions.

In line with the research purpose, these task-based interviews “joined more directly with educational practice” (Goldin, 2000, p. 524) while providing the kind of data needed to inform the research questions.

Figure 2.1

*Conceptual Framework: Developing Conceptions of Division and Fractions*

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"IM" refers to the “invert-and-multiply” rule for dividing fractions.

The task sequence began with whole number measurement and partitive division applications followed by division with fractions applications. This task sequence aimed to
promote coherent mathematical meaning, a critical goal for the mathematical professional development of teachers (Thompson et al., 2007). Rectangular area diagrams were then introduced to support the development of teachers’ representational skills. However, introducing the task of representing division of fractions with pictorial diagrams also served as a motivator in the framework of adult education (Knowles et al., 2005). Specifically, the representation tasks presented disorienting experiences that perturbed the ISTs’ prior understandings. In resolving the disorientation, the ISTs had an opportunity to “relearn the mathematics they think they already know, in deeper, more connected ways” (Johnson & Olanoff, 2020, p. 725). The sections that follow present the areas of literature from which the conceptual framework was constructed.

**Knowledge in Pieces Framework**

This study was grounded in a constructivist orientation and theoretically situated in the KiP (diSessa et al., 2016) approach to conceptual change (Vosniadou, 2013). The KiP approach to conceptual change is an epistemological perspective (e.g., diSessa, 2018), sometimes referred to as a psychological perspective (e.g., Izsák & Jacobson, 2017). For the purpose of this study, I refer to KiP as an epistemological framework (Levin, 2018) because it played an integral role in the conceptual framework and functioned as a theoretical and analytical lens. This section orients the reader to KiP, describes applications to mathematics education, and shows how the use of KiP in this study was a natural extension of the current KiP literature.
An Orientation to Knowledge in Pieces

Knowledge in Pieces is an epistemological framework that allows for the modeling of knowledge elements, their relational structure, and restructuring processes at a microanalytic level in episodes of learning (diSessa, 2018). As such, it is a type of grounded theory, open to change and development, and accountable to empirical data. Introductions to KiP in the literature generally describe three central constructs: phenomenological primitives, coordination classes (which are models of knowledge), and concept projections.

Phenomenological Primitives

_Phenomenological primitives_ (p-prims; diSessa, 1993, 2018) are self-explanatory, atomistic and subconceptual forms of intuitive knowledge generated from experience with the physical world. Within a knowledge system, p-prims are assumed to be the smallest element, lacking substructure - hence _primitive_. A common example is Ohm's p-prim, which asserts that more effort begets more result. These knowledge elements do not disappear in the complex knowledge systems of experts. Rather, coordination of the complex knowledge systems of experts has served to all but eliminate their cueing when they are not appropriate.

In mathematics education, the role of p-prims is an open question and the likely equivalent form of knowledge in mathematics learning may result from instruction or class-based experiences (diSessa, 2018). Examples include the common notion that multiplication always results in a bigger number (Fischbein, 1987) or that the numerator in a fraction must be smaller than the denominator (Stafylidou & Vosniadou, 2004; Tzur,
1999). These notions can extend into adulthood (DeWolf & Vosniadou, 2015), a common property of p-prims.

**Coordination Class**

A *coordination class* models a certain type of concept, particularly scientific concepts for which an expert understanding often requires a difficult and protracted commitment (diSessa, 2004; diSessa et al., 2016). Unlike p-prims, a coordination class is a large and complicated knowledge system. It functions to discriminate among various perceptions of the world and select a particular class of information. That function is predicated on the assumption that information is not always transparently available in the world. “Instead, we have to learn how to access it” (diSessa, 2002, p. 43) in various contexts and for various reasons.

Results from pilot projects suggested that data analysis in this study would make productive use of two central properties of coordination class development. First, because a coordination class is an evolving system, states of partial development may exhibit productive use in some situations but not others. A concept “may be composed of a complex variety of knowledge resources” (Wagner, 2010, p. 450). A *concept projection* (diSessa, 2004; diSessa & Wagner, 2005) is a set of resources instantiated in a particular context. Accommodation can be viewed as the construction of new concept projections that productively function across more contexts. *Span* refers to the extent to which the concept projection may be productively applied across a variety of contexts (diSessa et al., 2016). In mathematics education research, Wagner (2006) applied the concept projections construct to describe the learning of an undergraduate student as she
constructed an understanding of the Law of Large Numbers. Wagner showed how, as the student developed a more complex understanding of the concept, she was able to recognize and apply the concept in increasingly varied situations. In the present study, concept projections operationalized the epistemological commitment expressed by Vergnaud, Lamon, and Ma that one characteristic of learning in the MCF domain is the development of increasing complexity of a concept.

The second central property of coordination class development informing the analysis of this study is the dynamic interplay between perceptual and inferential elements (Levin & diSessa, 2016). In common vernacular, there is a relationship between what we see and what we know. The dynamic interplay between perceptual and inferential elements and processes into mathematics education research has been described by Levin (2018) in developing the strategy system construct. Levin’s strategy system extended the construct of coordination class to solving problems that may require a diverse set of resources not necessarily related to determining a single concept-specific class of information.

**Knowledge in Pieces Applied to the Multiplicative Conceptual Field**

Izsák and Jacobson (2017) applied KiP to account for the diverse knowledge resources PSTs exhibited in determining whether two variables in a mathematics task varied in direct proportion or had some other relationship. While Izsák and Jacobson chose not to venture into the question of whether determining proportional relationships constitutes a coordination class, these researchers provided a KiP account of task-based
interview data showing how and when conceptual resources (e.g., prior knowledge, inferences) were productive or counterproductive in making that determination.

Following Izsák and Jacobson’s (2017) study, Brown et al. (2019) and Weiland et al. (2020) conjectured that determining whether a relationship is proportional is a coordination class and empirically catalogued the conceptual resources inservice teachers used across a set of tasks to determine the existence of proportionality. In contrast to previous research focusing on a narrow selection of mathematical tasks, these researchers investigated a range of tasks covering the ratio and proportion subset of the MCF.

**Extending the Knowledge in Pieces Literature with the Present Study**

Noting the conceptual resources used across a set of concepts and tasks aligns with Vergnaud’s (1994) suggestions that researchers not study isolated situations or isolated concepts. However, my current reading of the literature has found no study aimed at identifying the conceptual resources that PSTs or ISTs use or need across the range of concepts comprising multiplication and division with fractions (a complimentary subset of the MCF to ratio and proportion).

Based on pilot projects, the interplay between perceptual and inferential elements and processes can play a role in the analysis of results. For example, teachers in pilot data who were unfamiliar with the practice of explicitly identifying and being flexible with referent units often made use their knowledge of percent in order to make sense of referent units in pictorial representations of fractions operations. These teachers knew that they were not yet able to see something that they were being asked to see. By
accessing an element in their inferential net (percent), they were able to leverage mathematics they already knew as a means to make new mathematics visibly apparent in a representation.

Ma’s Knowledge Package for Understanding the Meaning of Division by Fractions

This study was motivated by the potential of applying the KiP framework to Ma’s Knowledge Package for Understanding the Meaning of Division by Fractions. The following section situates the knowledge package within the mathematics education literature as it pertains to this study.

Situating Ma’s Knowledge Package in the Mathematics Education Literature

Vergnaud (1988) provided a model for mathematical concepts integrated with the proposition that concepts and situations interconnect to comprise conceptual fields. Vergnaud’s epistemological view was that content-free frameworks and general logical competencies cannot explain the development of complex mathematical understanding. Rather, cognitive development needs to be framed with reference to the content of the domain.

Vergnaud’s (1994) MCF includes two central features. First is the content of the domain, second is the principle that concepts are interdependent and situation specific. With respect to content analysis in this dissertation, the relevant domain is the proportional reasoning and rational number concepts ubiquitous in middle school
curricula (Lamon, 2007). With respect to the epistemological orientation and purpose of this study, the importance of interdependency of concepts is made clear by Vergnaud’s statement that “researchers should study conceptual fields and not isolated situations or isolated concepts” (p. 47). As described in Chapter 3, this dissertation study aligns with Vergnaud’s suggestion through the selection of tasks, data collection, and analytic framework.

Lamon conducted extensive research on children’s conceptual development in the domain of proportional reasoning (1994), the role of unitizing (1996) and rational number (2007). After a 5-year longitudinal study comparing curricula that each begin fractions instruction from a different fractions subconstruct (Doyle et al., 2016; Kieren, 1993), Lamon (2007) published professional development (PD) materials for the teaching of ratio, proportion, and rational number concepts (2008, 2020). In these materials, Lamon proposed a web of topics and ideas essential for instruction that is designed to foster conceptual understanding of rational number and proportional reasoning (see Figure 2.2). Research Question Three addresses the potential relationship between knowledge acquisition within a system of interrelated elements in terms of common content knowledge (CCK; Ball et al., 2008) of the MCF, and pedagogical content knowledge for teaching (PCK; Ball et al., 2008).

Similar to Vergnaud’s description of the MCF, Lamon (2008) describes the learning of these interrelated components as occurring in a web-like fashion, in which the learning of any element has “repercussions throughout the web” (p. 10). Another observation central to this dissertation is that the elements comprising the web are
ontologically diverse; Lamon refers to them as “ideas and processes and representations” (p. 10). As described in the following sections, Ma’s knowledge package and the mathematical tasks in this dissertation share these aspects of interconnectedness and ontological diversity; features that motivated the choice of KiP as the framework for analysis in this study.

Ma’s Construct of Profound Understanding of Fundamental Mathematics

Curious about the differential in mathematics achievement between U.S. and Chinese students, Ma (2010) investigated the mathematics knowledge of U.S. and Chinese elementary school teachers and interviewed selected teachers after observing them in the classroom. Ma observed that while many U.S. teachers were comfortable with procedures, they could not explain the reasons for the procedures and were prone to
making mistakes. The Chinese teachers’ understanding of mathematical topics was connected to other pieces of mathematical knowledge. Ma described the rich and interconnected knowledge exhibited by the Chinese teachers as profound understanding of fundamental mathematics (PUFM).

**Understanding the Meaning of Division by Fractions**

In Ma’s (2010) terminology, “generating a representation” amounted to the teacher creating a story problem, or situation for which a given symbolic arithmetic problem provides the answer to a question. Ma asked participating teachers to calculate the answer to the division problem, \( \frac{3}{4} \div \frac{1}{2} = ? \) and then generate a representation for the division problem by constructing a story problem that could be solved by that division problem. In her sample only 4% of the U.S. teachers were able to generate a correct representation whereas 90% of the Chinese teachers generated correct representations. There were qualitative differences as well. The Chinese teachers explicitly differentiated between the partitive and measurement models of division.

Partitive division can be modeled as equal sharing (Flores, 2002), in which a quantity is distributed equally among groups (Siebert, 2002). The partitive quotient quantifies the amount in each group. Measurement division can be modeled as repeated subtraction (Siebert, 2002) to find the number of groups of a known size are contained in another quantity (Lo & Luo, 2012). The measurement quotient quantifies the dividend when it is measured in units of the divisor. These models will be elaborated in a later section on pictorial diagrams. Here, it is important to note the difference in SCK between
the two groups of teachers in Ma’s study. That a significant portion of practicing teachers in the U.S. do not exhibit a high level of SCK motivates the purpose of this study, but also suggests that conceptual change issues are involved in the underlying source of the problem.

Ma’s (2010) discussions of the meaning of division by fractions with the Chinese teachers revealed a network of common SCK that Ma describes as a knowledge package. Ma found that the Chinese teachers had a common knowledge package for each of the tasks in her interviews (subtraction with regrouping, multidigit multiplication, division by fractions, and the relationship between perimeter and area). The knowledge package for the meaning of division by fractions is reproduced in Figure 2.3.

**Figure 2.3**

*The Knowledge Package for the Meaning of Division by Fractions*

![Diagram](Image)

*Note.* Reproduced from Ma (2010).

Ma (2010) constructed knowledge packages to represent the SCK of the Chinese
teachers in her study for each of the four interview tasks. Two central features of the
teachers’ knowledge were common to all four knowledge packages. First, the web-like
nature of the packages was most valued and discussed by the more experienced teachers.
These teachers emphasized not only the interrelatedness of concepts, but further noted the
bootstrapping nature of learning within the domain. For example, the learning of partitive
and measurement models for the division of fractions will make the learner’s
understanding of the models of division (introduced with whole numbers) more
comprehensive and intensify previous basic concepts of division and multiplication. To
illustrate this epistemological orientation, Ma quotes one teacher as saying, “Learning is a
back and forth procedure” (p. 77).

The second central feature of the teachers’ knowledge, common to all four
knowledge packages, was an unexpected phenomenon: in each package, the concept that
the teachers considered central was not the target concept in the interview task. In the
case of division by fractions, the teachers considered the meaning of multiplication by
fractions to be most important because it was the “intersection” of mathematical
concepts, also referred to as a “knot” (Ma, 2010, p. 78).

**Extending Ma’s Knowledge Package**

Though Ma’s (2010) knowledge package for effective instruction of fraction
division gained currency in the literature on teaching rational number (e.g., Lo & Luo,
2012; NCTM, 2000), it may be clarified in several ways. For example, Ma does not
discuss the meaning or nature of the arrows illustrating the relationships among the
components in the figure of the knowledge package. This is also true for Lamon’s web of
essential ideas for instruction of rational number and proportional reasoning. In both
cases the figures are graphical representations of the researchers’ empirical results, but
they are not accompanied by descriptions of how the graphical elements were
determined. If interdependency plays a primary role in these SCK conceptual fields for
effective instruction, then the nature of the relationships is an empirical question worth
pursuing. How to operationalize Ma’s knowledge package as a framework for further
empirical research and how to characterize the relationships among the components are
open questions in the literature. Izsák (2008) suggested finer grain-size is needed. The
present study aimed to shed light on these questions.

Ma’s knowledge package for the meaning of division by fractions provided an
empirically based expectation of the concepts that teachers in this study may connect and
develop through the task sequence. However, Ma’s research does not explicitly suggest
tasks or methods for improving the PUFM model, and there are stark differences in what
constitutes a representation in Ma (2010) and various studies (e.g., M. Y. Lee, 2017). For
example, in much of the foundational research on teachers’ understanding of division
with fractions (e.g., Ball, 1990; Ma, 2010; Tirosh, 2000), teachers were asked to represent
division problems by creating a word problem that represented a division expression. In
this sense, a representation is a contextualized narrative for a symbolic mathematical
expression. Although it takes work to understand them (NCTM, 2000), multiple
representations are increasingly being emphasized in the classroom learning of
mathematics in a variety of forms (e.g., pictorial, physical manipulatives, digital-
interactive applications). However, much of the recent research that does emphasize
representational fluency in task design involving division with fractions continues to focus on cataloging the types of representations participants generated (e.g., Jansen & Hohensee, 2016; Lo & Luo, 2010) rather than focusing on the conceptual resources teachers exhibit as they developed their abilities with representations.

The following section briefly describes types of representations used in the recent literature on division with fractions, and the representations that will be central to the tasks in this study.

Representing Fraction Division and the Role of the Referent Unit

The representation of fractional values, equivalency of fractions, and operations with fractions has a long history and a large literature. This section highlights a few points about the representation of fractions relevant to this study.

Common Fraction Models in Mathematics Education Literature

Three types of models are commonly used to represent fractions as entities, or values: linear, area, and set (or discrete; see Figure 2.4).

Figure 2.4

Linear, Area, and Set (or Discrete) Models for the Fractional Value $\frac{3}{4}$

Note. Reproduced from Watanabe (2002).
Children’s early exposure to fraction models is usually in the form of circular areas. Unfortunately, some of the research comparing the effectiveness of different representations of fractions use the term “area model” without distinguishing between circular and rectangular area models. In this study, all area models are rectangular area models. Rectangular area models have the affordance of being partitionable in the vertical and horizontal directions in the same diagram. The process of orthogonally partitioning a diagram is referred to as cross-partitioning (e.g., Izsák, 2008; see Figure 2.5).

**Figure 2.5**

*Cross-Partitioned Rectangular Area Modeling 3/8 x 4/9*

![Image](image_url)

Figure 2.5 shows a representation of fraction multiplication by cross-partitioning a rectangular area according to the denominators of the fractions, and then by shading the resulting squares according to the numerators. This representation for fraction multiplication makes visible (though not necessarily apparent) the product of the two fractions and the commutative property of multiplication. This representation for fraction multiplication also makes visible (though not necessarily apparent) multiple quantities...
that could be referred to as a unit at any given moment in discourse or individual thinking about the meaning of the representation. The issue of referent units is addressed in the next section.

Representing division of fractions with pictorial diagrams is complicated by the fact that measurement division and partitive division are two different processes. Figure 2.6 shows cross-partitioning of a rectangular area to model the measurement division of $3/4 \div 2/3$. First, the area is cross-partitioned according to the denominators of the dividend and divisor. Then the dividend, $3/4$, is shaded. Next, the divisor, $2/3$, is identified as a quantity of small squares and subtracted from the quantity of squares comprising the dividend.

**Figure 2.6**

*Cross-Partitioned Rectangular Area Representing Measurement Division of $3/4 \div 2/3$*

Figure 2.7 shows a common representation of partitive division with fractions (e.g., Lamon, 2008). In this diagram, the partitive division of $3/4 \div 2/3$ is represented by sharing, or distributing the dividend into $2/3$ of a share. This representation makes visible two related conceptual difficulties with the partitive model of division when the dividend is a fraction. First, the notion of *a fraction of a share* is not intuitive because common experiences of fair sharing involve distributing a quantity among whole numbers of
recipients such as whole people (e.g., Jansen & Hohensee, 2016; S. J. Lee et al., 2011). Second, it is often difficult to see why, after the distribution to a fractional share has been represented, iterating is necessary to complete the division (e.g., Siebert & Gaskin, 2006).

Figure 2.7

*Figure 2.7*  
A Diagram Modeling Partitive Division of 3/4 ÷ 2/3

However, the rectangular area model of partitive division (see Figure 2.7) supports conceptual understanding of the invert-and-multiply (IM) rule (e.g., Siebert, 2002). In the problem above, the IM procedure amounts to performing 3/4 x 3/2, which is frequently accomplished algorithmically without conceptual understanding. Figure 2.7 illustrates that the dividend, 3/4, is partitioned into two equal quantities and distributed (indicated by the arrows) to 2/3 of a share (the divisor). Then one of the dividend partitions is iterated until a full share has been generated. Together, these two steps
represent multiplication by $3/2$.

This process of representing partitive division of fractions also provides a model for creating unit rates, an important part of middle-school mathematics and basic science (Jansen & Hohensee, 2016; Lamon, 2020; Siebert, 2002). The literature on division with fractions has long recognized that representing partitive division with fractions, as shown in Figure 2.7, is difficult to master for students and teachers alike. Recent research indicates that flexible reasoning about the referent unit may be prerequisite to conceptual understanding and an ability to represent partitive division with fractions.

**Flexibility with the Referent Unit**

Flexibility with the referent unit appears to be a critical part of understanding and generating correct and pedagogically useful representations of multiplication and division (e.g., Izsák, 2008; Jansen & Hohensee, 2016; Lee & Sztajn, 2008; M. Y. Lee et al., 2011; Lo & Luo, 2012; Zhang et al., 2015). Representational fluency with pictorial diagrams of multiplication and division with fractions requires an understanding of the units to which any given number refers (S. J. Lee, 2017). Flexibility with the referent unit can be defined as the ability to “keep track of the unit to which a fraction refers” and the ability to shift one’s “relative understanding of the quantities as the referent unit changes” (M. Y. Lee et al., 2011, p. 204). For example, as shown in Figure 2.7, there are multiple quantities that one might, at any given moment in discourse or thinking, consider to be “the whole.” The three shaded blue boxes at the top of the diagram, representing the dividend $\frac{3}{4}$, has meaning in reference to the whole blue box composed of four smaller boxes. Yet, when considering the divisor, $\frac{2}{3}$, as acting on the dividend, the three blue
boxes can be thought of as the whole being acted upon. With respect to the purple boxes in the bottom half of the diagram, $\frac{3}{8}$ can be considered a whole share – a share that each one-third partition of the bottom rectangle receives. When that share is iterated, it can be thought of as the whole being acted upon. Yet, the iteration produces another whole, $\frac{9}{8}$ (the quotient), when the iteration fills the whole purple box composed of three boxes to create a whole share.

Pedagogically, flexibility with the referent unit can be introduced through explicit attention to units in descriptions of measurement and partitive division with whole numbers. Pertinent to the task sequence in this study, a whole-number introduction to measurement and partitive division has the potential to introduce flexibility in referent units to the participants (S. J. Lee, 2017). This approach (see Table 2.1) makes explicit the four components of whole number division and the units associated with each component.

| Table 2.1 |
| Referent Units Associated with the Components of Whole-Number Division |
| Division type | Dividend | Divisor | Quotient | Remainder |
| Partitive | Same | Different | Same | Same |
| Measurement | Same | Same | Different | Same |

In this study, the mathematical concepts needed for fluency with representations of division were first introduced in the context of whole numbers. The same concepts were then applied to division with fractions, creating an opportunity for participants to construct concept projections as they applied knowledge they had constructed in a whole
number context to the context of fractions.

**Summary**

Ma’s (2010) Knowledge Package for Understanding the Meaning of Division by Fractions provides an empirical basis for treating division with fractions as a conceptual field. Conceptualizing the knowledge package as a form of MKT arising from the MCF may further integrate research on teacher learning, representational fluency, and KiP as an epistemological framework for the analysis of knowledge construction. This application of the KiP perspective allows Ma’s model for the conceptual development of fractions to be operationalized more flexibly and at a finer grain size.
CHAPTER 3

METHODS

The purpose of this study was to provide comprehensive descriptions of the conceptual difficulties and learning experiences of ISTs as they constructed interconnected understandings of fractions division, and to identify the role of the referent unit in the teachers’ learning as they improved their ability to effectively model fractions division with pictorial diagrams.

This study employed a qualitative multiple case study design to conduct professional-development-based interviews with ISTs as they learned about a curriculum intervention on division with fractions. This qualitative multiple case study was designed to answer the following research questions.

1. What are the conceptual resources exhibited by IST’s as they construct a deeper understanding of division with fractions and improve their ability to effectively model fractions division with pictorial diagrams.

2. What role does the teachers’ perceptions of the referent unit play in their knowledge construction across tasks involving multiplication and both types of division with fractions?

3. Do participants in this study who are actively teaching the professional development (PD) content construct knowledge differently than participants for whom the content prepares for future teaching?

This chapter describes the study’s research design, participants and setting, data collection procedures and analysis.

Research Design

This study employed a qualitative multiple case study design (Creswell, 2018;
Merriam & Tisdell, 2016) to collect case-based data on individual teachers as they progressed through a PD program on division with fractions. The data was used to generate case-based descriptions (Creswell, 2018) in which a case was defined as a single teacher. Case-based descriptions were used to conduct a cross-case synthesis (Yin, 2018; see Figure 3.1).

**Figure 3.1**

_Multiple Case Study Design for this Qualitative Study_

![Diagram of multiple case study design](image)

*Note.* Adapted from Yin (2018).

This qualitative multiple case study design followed from the research purpose, allowing for an extended exploration of conceptual change processes (Vosniadou, 2013) beyond pencil-and-paper surveys (diSessa, 2018) or single interviews. The primary source of data in this study was interviews with individual teachers as they progressed through a series of tasks, allowing for a high density of observations (Siegler, 2006) without the participants influencing each other’s responses (Tashakkori & Teddlie, 2003). Structured interviews (Goldin, 2000) and semistructured interviews (Merriam & Tisdell,
followed from Research Questions 1 and 2, allowing for a consistent line of inquiry to be pursued with some fluidity (Yin, 2018) while building the individual cases. To address Research Question 3, data was collected from two groups of participants. Each group consisted of four teachers. The first group of participants was teaching the LA 255 course during the data collection period. This group included two teachers in the spring 2021 semester and two teachers in the summer 2021 semester. Each of these teachers received two, two-hour, one-to-one PD sessions with the researcher during the semester they are teaching the course. The second group of participants consisted of teachers who had experience teaching the LA 255 course but were not teaching the course during the semester in which they participated in the study. These teachers received the same two, 2-hour, one-to-one PD sessions with the researcher during the spring and summer of 2021. The following sections describe the implementation of this qualitative multiple case study design.

Setting and Participants

This section describes the setting and participants of the present study. A brief, one-paragraph biography of each participant is provided.

Setting

This study was conducted at a private university for the arts located in the Western United States. Degrees can be completed online, on-campus, or through hybrid enrollment. Due to COVID-19, all classes related to this study were conducted online through the school’s proprietary learning management system (LMS).
Curriculum and Instruction Background to the Study

All participants in this study were teachers of College Mathematics, a basic mathematics course for undergraduate students who are not prepared for coursework in analytic geometry and trigonometry. This course has been conducted in a traditional classroom setting and in an online course format since 2002. The curriculum, textbook (Basic College Mathematics: An Applied Approach, Aufmann & Lockwood, 2013), assignments and assessments are identical in both onsite and online courses, and the faculty have put extended effort into achieving parity between the two campuses. All of the teachers in the study had previous experience teaching this course online. Thus, the shift to conducting all classes online in response to COVID-19 was a relatively seamless transition and did not introduce the kind of variability experienced at institutions less prepared for a complete transition to LMS-based delivery of courses.

A quantitative analysis conducted in 2020 found that students taking this course demonstrated high proficiency with fractions arithmetic questions, but significantly lower proficiency (with large effect size) when fractions questions involved a conceptual component (e.g., identifying the referent unit, conceiving of a fraction as an operator, or interpreting a fraction as a measurement). A review of the textbook and online course pages found no mention of the following mathematics topics, which appear frequently in the mathematics education research literature: the difference between partitive and measurement division, the actions of partitioning and iterating, the concept of referent unit, and the use of area models to model division of fractions processes. The math-science department director (author of this dissertation) developed eight pages of new
online course content and two new assignments to introduce and develop these concepts in the course (see Appendix A). The math-science department director also created a one-to-one professional development program to (a) provide instruction on the concepts to the teachers, (b) support the teachers in their first experience teaching the concepts, and (c) engage the teachers in a post hoc assessment of the new content and assignments.

**Participants**

All recruited participants in this study hold a masters’ degree in a STEM field. All of the participants have working experience in a STEM field or hold a secondary credential in a STEM field and have teaching experience in public high schools. Teaching experience among this population ranges from 5 to 20 years. A brief biography for Group 1 participants is provided below. All names are gender-preserving pseudonyms.

Ivy is a middle-aged woman, born and educated in China. After her K-12 education in China, Ivy moved to the U.S. where she received her undergraduate education. Among the teachers in this study, Ivy had the strongest mathematics background. In addition to teaching the Basic College Mathematics course referenced in this study, Ivy also taught upper-division mathematics courses for computer science majors. At the time of this study, Ivy was actively enrolled in an Ed.D. program and actively teaching fractions to her fifth-grade daughter using multiple representations.

Evelyn is a middle-aged woman with 20 years of teaching experience in public and private schools, at both the high school and undergraduate levels. She is a secondary-credentialed teacher in life sciences with a master’s degree in STEM education. At the time of this study, Evelyn had been teaching the Basic College Mathematics course as an
adjunct faculty member for 12 years.

Hannah is a middle-aged woman with a master’s degree in the life sciences. At the time of the study, she had been teaching the Basic College mathematics course online for 7 years.

Rachel is a middle-aged woman with a master’s degree in education and a secondary teaching credential in life sciences. At the time of the study, she had been teaching the Basic College Mathematics course on campus and online for 6 years. She was currently teaching the class at the time of this study.

A brief biography for Group 2 participants is provided below. All names are gender-preserving pseudonyms.

Audrey works full-time as an accountant. At the time of this study, she had taught the Basic College Mathematics course several times in the classroom on a part-time basis, but had no other background, academically or professionally, in education. In this study, she was one of the participants not currently teaching the Basic College Mathematics course. Previous to this study, I observed Audrey’s classroom when she was teaching the regular course content on fractions. I was impressed by the constructivist character of her pedagogy. She solicited multiple ways of seeing, approaching, and solving problems.

Mark is a middle-aged man with an academic and professional background in physics. Mark immigrated to the U.S. as an adult from Eastern Europe. At the time of this study, he had been teaching the Basic College Mathematics course on campus and online for 12 years as a part-time faculty member. Mark’s regular course assignment was a physics class. He did not teach the Basic College Mathematics course every semester,
Malory is a middle-aged woman with an IT background. At the time of this study, she had been teaching the Basic College Mathematics course on campus and online for eight years as a part-time faculty member. Malory did not teach the Basic College Mathematics course every semester, and she was not teaching the course at the time of this study.

Zoey is a woman in her 60s, with a background in physics and computer science. She grew up in Eastern Europe and received all of her education there. At the time of this study, she had been teaching the Basic College Mathematics course on campus and online for 7 years as a part-time faculty member. Zoey did not teach the Basic College Mathematics course every semester, and she was not teaching the course at the time of this study.

**Sampling**

All recruited participants were purposively selected (Creswell, 2018) as teachers who have CCK (Ball et al., 2008) of the multiplicative conceptual field (MCF; Vergnaud, 1988, 1994), and some experience teaching this content (1 to 10 years), but little or no training in the mathematics topics identified above as missing from the course curriculum and course textbook. By participating in the PD sessions as individuals, this sample of recruited participants provided an opportunity for data collection on teachers’ construction of more integrated understandings of division with fractions. Additionally, these recruited participants, in this setting, allowed for two, 2-hour, one-to-one sessions with the researcher, extending the scope of tasks beyond the data collection methods in
previous research identified in the literature review. Implications from this sample of
teachers may be relevant to PST’s as well as in-service K-12 teachers with little or no
professional development on division of fractions concepts and associated
representations. Generalizing the findings of this study addresses the problem statement
described in Chapter 1.

**Demographics**

Recruited participants in this study were predominantly female (80%) and
Caucasian (80%); several recruited participants are Asian-American or mixed race
(20%).

**Consent and Institutional Support**

Institutional support is documented in the form of a Letter of Support from the
institutional Vice President of the college in which the study took place (Appendix B).
The Letter of Support approved the components and activities of the study and
underscores the institution’s commitment to supporting any teacher who chooses to
decline participation in this study.

The principal investigator contacted prospective participants by email, inviting
them to participate in the study. The email described the expectations and compensation
for participation (see Appendix C for Teacher Recruitment: Description of Activities and
Appendix D for Informed Consent form).
Data Collection and Procedures

The primary data for this study was 32 hours of video of one-to-one PD sessions. Data also included the pen-and-paper artifacts from the PD sessions in the form of a four-page worksheet (see Appendix E) that was the focus of the first PD meeting, and teacher-selected samples of student work on the new assignments (see Appendix A).

Case Study Logic

In a multiple case study, each case can be considered as an experiment. Case study logic refers to the way the individual cases relate to each other as replications of an experiment (Yin, 2018). Literal replications confirm similar results under similar circumstances; theoretical replications predict different results under different circumstances (Yin, 2018).

In this study, participants within the same group created a replication logic. The two groups of participants created a theoretical replication to address Research Question 3: Do participants in this study, who are actively teaching the PD content, construct knowledge differently than participants for whom the content prepares for future teaching? As discussed in the literature review, there may be qualitative differences in constructing MCF concepts when constructed as CCK compared to constructing the same mathematical concepts as PCK (Ball et al., 2008). Having a somewhat different relationship to the PD sessions, the two groups may construct knowledge differently (i.e., PCK for Group One and CCK for Group Two). Results regarding replication logic are presented and discussed at the level of cross-case synthesis.
Zoom-Based Recording Methods

The PD sessions were conducted remotely and recorded using Zoom. An external webcam and tripod were supplied to each participant and used to video-capture the area in which the participant was drawing and writing. The participants drew and wrote directly on the worksheet within view of the camera throughout the PD sessions. The worksheets included enough space for extemporaneous calculations and drawings. While the interviewer and participant communicated verbally in real time, the participant was able to see the shared screen of the interviewer, and the interviewer had a view of the participant’s working area as the participant drew and wrote on paper (see Figure 3.2).

Figure 3.2

Interviewer’s View of Shared Screen and Participant’s Working Area
While the interviewer was able to toggle views within the Zoom application during the interview, the view configuration shown in Figure 3.2 above allows the researcher to simultaneously see the shared screen (the view seen by the participant) and the participant’s working area (the window in the lower left of Figure 3.2). This allowed for the same kind of coordination of task and performance that would occur in a traditional, one-to-one, interview or workshop environment. However, the Zoom application does not record the composed view in Figure 3.2. The Zoom application provides separate, full-screen recordings titled *Shared Screen* and *Speaker* views. *Shared Screen* view shows only the screen that is shared to the members of the meeting. In Figure 3.2, that view is the larger window showing Question 7 on the PD worksheet. *Speaker* view shows the view from the camera of the person speaking at a given moment, and toggles between active speakers as the meeting’s participants converse. Thus, Zoom does not provide a recording of an individual participant’s camera view. In order to capture continuous video from the participant’s camera view a third recording was made on a separate laptop. The third view was created by the interviewer logging into the meeting as a third member of the meeting, pinning the video of the study participant, and video-capturing that view using Camtasia©. This pinned view was expanded to full screen and provided a continuous view from the participant’s camera. In sum, this method of video recording captured (a) the shared screen of the researcher, (b) the active speaker at any given time, and (c) a dedicated view from the participant’s camera. The videos from each of these three views included their own audio.

Legal-size return envelopes were provided to the teachers so that they could
return all artifacts generated during the session. These artifacts included the completed worksheets and drawings on additional pieces of paper when that occurred.

Data Collection for Group One

The primary source of data for Group One was the video-recorded, Zoom-based, one-to-one PD sessions. The PD program consisted of two separate two-hour meetings, described in the next section. Table 3.1 summarizes the data sources for each of these two meetings. These PD meetings resulted in 4 hours of video-recorded data with each individual participant, for a total of 16 hours of video for Group One.

Table 3.1

Data Sources for Each Meeting

<table>
<thead>
<tr>
<th>Data source</th>
<th>Description</th>
<th>Data type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Professional development</td>
<td>Two-hour, one-to-one instructional meeting on new course content. Structured interview format (see Appendix E for data collection instrument).</td>
<td>Video and audio recording; pen-and-paper artifacts.</td>
</tr>
<tr>
<td>Meeting One</td>
<td>Two-hour, one-to-one meetings on (a) assessing student work (see Appendix F for data collection instrument) and (b) assessing the curriculum additions (i.e., new content and assignments, and clarifying any remaining questions, see Appendix G for data collection instrument).</td>
<td>Video and audio recording; pen-and-paper artifacts.</td>
</tr>
<tr>
<td>Professional development Meeting Twoa</td>
<td>This meeting was a time commitment outside the teachers’ regular job tasks and required consent and compensation.</td>
<td></td>
</tr>
</tbody>
</table>

Professional Development Meeting One

Professional Development Meeting One was a source for collecting structured, task-based interview data (Goldin, 2000) from tasks embedded in the PD session (see Appendix E). The structured, task-based interview allowed the research to focus on the “subjects’ process of addressing mathematical tasks, rather than just on the patterns of
correct and incorrect answers” (Goldin, 2000, p. 520). This PD meeting was based on a worksheet (see Appendix E) provided to the teachers in both color-print and digital formats. The worksheet had been revised through several pilot projects to focus the content and tasks as outlined in the conceptual framework, and to create a task sequence that could be completed by an individual teacher in a two-hour session. The task sequence aimed to promote coherent mathematical meaning, a critical goal for the mathematical professional development of teachers (Thompson et al., 2007). Introducing the task of representing partitive and measurement division with area diagrams into the task sequence provided a disorienting dilemma, compelling participating teachers to “relearn the mathematics they think they already know in deeper, more connected ways” (Johnson & Olanoff, 2020, p. 725).

Professional Development Meeting One was planned for two hours total time, including a 10-minute break halfway through the meeting. The interview format adopted a constructivist mentoring approach in which the participating teacher was placed in the role of a student (Loucks-Horsley et al., 2010). All PD meetings were conducted with individual teachers with the aim of generating data that included moments of rapidly changing competence (Siegler, 2006) and an extended focus on a single participant’s construction of knowledge. Table 3.2 summarizes the components of the Professional Development Meeting One worksheet, identifying the tasks and the data collection goals for each section.

**Professional Development Meeting Two**

Professional Development Meeting Two was a source for collecting
Table 3.2

Overview of the Professional Development Meeting One Worksheet Components

<table>
<thead>
<tr>
<th>PD worksheet section</th>
<th>Tasks and goals</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Welcome and introduction</td>
<td>Task a: Provide participant with an overview of the PD program. Task b: Discuss the participant’s conception of fractions, as well as the meaning of fractions that the participant generally teaches.</td>
</tr>
<tr>
<td>2. Partitive and measurement models of division with whole numbers</td>
<td>Task a: Introduce partitive and measurement models of division with whole numbers with bowls and cookies. Task b: Introduce and practice partitive and measurement models of division with whole numbers with story problems.</td>
</tr>
<tr>
<td>3. Partitive and measurement models of division with fractions</td>
<td>Task a: Introduce partitive and measurement models of division of fractions with bowls and cookies. Task b: Introduce and practice partitive and measurement models of division of fractions with story problems.</td>
</tr>
<tr>
<td>4. Flexible conception of the referent unit</td>
<td>Task a: Introduce the concept of referent unit. Task b: Practice flexibility with the referent unit.</td>
</tr>
<tr>
<td>8. Partitive division of fractions with rectangular area models</td>
<td>Task a: Compare measurement and partitive models of division with rectangular area diagrams. Task b: Introduce and practice partitive division of fractions with rectangular area models as delineated in Siebert (2002).</td>
</tr>
</tbody>
</table>

Ten-minute break Ten-minute break

semistructured interview data with individual participants through Zoom with the same equipment and procedures described above for Meeting One. These meetings served the general purpose of collecting data on the participants’ construction of knowledge over an extended period of time. The semistructured interview format allowed for pursuing
questions developed in the analysis of Meeting One, while at the same time allowing the interviewer to respond to new ideas as they occurred in the interview (Merriam & Tisdell, 2016). Professional Development Meeting Two was also two hours in total, including a ten-minute break half-way through the meeting.

There were two objectives in Professional Development Meeting Two. The first objective was to support the participants in evaluating student work on the two assignments in the new curriculum (see Appendix F for the interview protocol). The researcher prepared for this objective by assuring that six samples of student work on the two new assignments were available for discussion. Before the interview, the researcher asked the participants to select a minimum of three samples of student work from each of the two new assignments. Participants were instructed to remove all identifying information from the samples and send them as email attachments to the interviewer at least two days prior to meeting. The participants were instructed to provide samples of student work for which the participant had conceptual or pedagogical questions. Contingency and summary questions (see Appendix F for the protocol) were used when the participant’s questions did not carry the session through to the ten-minute break.

The second objective was to discuss the participant’s assessment of the curriculum and assignment additions, assess any potential effect that these additions may have made on students’ mathematical achievement and class participation, and address any remaining questions that the participants had (see Appendix G for the protocol).

Data Collection for Group Two

Group Two participants were not teaching the LA 255 course concurrent with
their participation in the PD program. Data collection for Group Two was identical to that described above for Group One with the exception that I provided the de-identified samples of student work.

**Data Storage**

Cloud-based Zoom recordings were made in real time, downloaded as MP4 files. These cloud-based recordings were deleted at the end of the semester, as per university policy where the data collection took place. Camtasia® recordings of the pinned view generated MP4 files. All researcher-constructed databases were saved on the researcher’s Box account at Utah State University, with sharing permission to the Principal Investigator.

**Data Analysis**

This section describes the analysis of all data outlined in the previous section. As a qualitative multiple case study, the first task was the building of cases of individual participants (Byrne, 2009; Merriam & Tisdell, 2016). A within-task analysis was conducted for each task, to assist the development of the within-case analyses. The within-case analyses were followed by a cross-case synthesis, comparing “within-case patterns across the cases” (Yin, 2018, p. 196). The cross-case synthesis refrained from considering the eight cases as a sample from which statistical inferences could be made. Rather, the eight individual cases created a space in which analytic contrasts were made visible.
Procedures for Video Data Analysis

The first stage in the analysis of video recordings was cleaning and organizing the raw data. Cleaning the video data was accomplished by viewing the complete video data corpus and identifying relevant and non-relevant moments of data. Non-relevant moments were moments in which the participants began off-topic conversation or experienced momentary interruptions in their environment, or moments in which the interviewer took time out to manage technical difficulties. These moments were edited out of the video data corpus. Organizing the video data was accomplished by viewing the cleaned video data corpus and creating a log of the content. The cleaned video data corpus was then edited into sections according to participant and content. For PD Meeting One, the cleaned video data was edited into separate videos for each task on the PD worksheet for each participant. This resulted in 64 separate videos with a median duration of 13 minutes and total time of 17.2 hours. For PD Meeting Two, the cleaned video data was edited into separate videos for the review of student work, and the review of new course content for each participant. This resulted in 16 separate videos with a median duration of 38 minutes and totaling 10.2 hours. These cleaned and organized videos were the unit of analysis for the next stage.

The next stage in the analysis of video data was to view the organized videos by task and create memoing of themes (Merriam & Tisdale, 2016) with respect to the research questions on a 14”x 17” sketchpad. This type of note-taking allowed for timestamps, sketches, and analytic notes, with remaining space on the page for later additions throughout recursive analyses (Parnafes & diSessa, 2013; Simon, 2019). After
the first viewing of the complete video data corpus, I decided that exploring the data through memoing and theming by task would be necessary before productively viewing the video data by case. This stage of viewing the data by task promoted conceptual organization, surfaced themes, and identified connections and inferences that were closely tied to the data (Simon, 2019). Figure 3.3 shows a sample page from this memoing and theming in a 14”x 17” sketchpad.

Figure 3.3

Example of Large-Format Memoing of Video Data by Task

The large-format notes provided the initial structure for the next stage of analysis in which observations were made regarding the participants’ language, actions, and inferred thinking of the mathematical content (Simon, 2019). In this stage, I entered observations into each participants “portrait” (Lightfoot, 1983, as cited in Merriam & Tisdell, 2016) in a Word document. Themes within the portraits organized preliminary
answers to the research questions. I then completed the portraits by transcribing the video data that appeared relevant to the research questions and integrating the transcriptions with my notes. Transcriptions across all eight participants totaled approximately 200 pages at 1.15 line spacing of 12-point, Times New Roman font.

**Within-Case Analyses**

Within-case analyses developed case descriptions (Yin, 2018) of each participant through recursive writing that saturated all of the identified themes (Creswell & Plano Clark, 2018). Recursive writing began with the case portraits. Multiple revisions and expansions were boot-strapped by coordinating the emerging findings between drafts of Chapter 4 and Chapter 5, integrating comments provided by the committee chair, and self-reflection. These revisions often involved a time-consuming return to the research literature and to the video data.

**Cross-Case Synthesis**

The cross-case synthesis “conceptualized the data from all the cases” (Merriam & Tisdell, 2016, p. 234) through the description of themes across the participants and across the tasks. Yin (2018) underscored the principle that analytic generalizations from case studies “generalize from the case study, not from the cases” (p. 38) and explicitly states that the sample size is too low to draw statistical inferences. That perspective justified analysis of behaviors that were sometimes exhibited by only one or only several participants. In other words, the data points from all the participants created a data space in which contrasts could arise without statistical criteria for inclusion (e.g., At least half
the participants evidenced $X$).

**Analysis of Pen-and-Paper Artifacts**

Pen-and-paper artifacts were generated in the PD meetings and therefore made visible in the video. However, the physical artifacts were also collected, reviewed and analyzed. Analysis of pen-and-paper artifacts was conducted in parallel with the analysis of video data and did not contribute information above and beyond the information in the video data. The pen-and-paper artifacts were at times helpful in clarifying images that were difficult to see in the video.

**Table 3.3**

*Organization of Task Analyses, Case Analyses, and Cross-Case Syntheses*

<table>
<thead>
<tr>
<th>Participant</th>
<th>Task 1…</th>
<th>…Task n</th>
<th>Within-Case analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher 1</td>
<td>$All$ $themes$ $relating$ $to$: 1. What are the conceptual resources exhibited by participants as they construct a deeper understanding of division with fractions and improve their ability to productively represent division with fractions using rectangular area models? 2. What role does the teachers’ perceptions of the referent unit play in their knowledge construction across tasks involving division with fractions?</td>
<td>$→$ Within-Case analysis: Teacher 1</td>
<td>$→$ Within-Case analysis: Teacher 2</td>
</tr>
<tr>
<td>Teacher 2</td>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>…</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teacher $N$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.3 summarizes the analysis process resulting in the final cross-case synthesis. Note that while the task analyses informed the within-case analyses, the cross-case synthesis was generated from comparative analysis of the cases, not the tasks.
Pilot Projects

I conducted two pilot projects prior to the study presented in this dissertation. First, I designed a 90-minute workshop for middle school teachers on representing multiplication and division of fractions using rectangular area models and conducted the workshop at the NCTM 2019 National Conference in San Diego. The participants’ conceptual difficulties and questions provided the initial inquiry for this dissertation study.

Second, I piloted an initial version of the PD sessions with three teachers who had similar backgrounds and similar teaching experience as the teachers in this dissertation study. These pilot sessions identified the most productive content areas to focus on and informed ways to coordinate the sessions in a more coherent fashion. In the first pilot session I provided the PD session to a pair of teachers. In the next pilot session, I provided a one-to-one PD session. I found the data from the one-to-one session to be easier to manage and analyze. Moreover, the data produced by the individual teacher was more useful in addressing the research questions because it provided longitudinal coherence of an individual’s knowledge construction. Teachers in the paired session influenced each other’s attention and train of thought. Therefore, I decided to conduct all eight of the PD sessions in this dissertation study as one-to-one sessions.

The pilot sessions were conducted in-person and two video recordings were made: one of the drawing areas as the teacher(s) drew on the worksheet, and a second camera captured a larger view of the participant(s), allowing for the recording of gestures. Compared to the remote sessions in this dissertation study, the in-person pilot
sessions allowed for more fluidity and immediacy, and also allowed me, as the interviewer, to better focus on the teachers’ knowledge construction in the moment.

Validity and Reliability

In this section I describe three types of validity that applied to this study: internal, external, and construct validity. I also describe two practices that provided reliability.

Internal Validity

In case studies internal validity primarily concerns the making of inferences (Yin, 2018). Inferences in this multiple case study were kept close to the data by using two levels of analysis. Following Parnafes and diSessa’s (2013) discipline, the first level was a literal description which includes contextual details and avoided theoretical interpretation. At this level multiple viewings of the raw data in short segments produced a first level of local inferences that are necessary because participants’ verbalizations do not directly indicate their mental states (Simon, 2019). These inferences did not answer the research questions.

Parnafes and diSessa (2013) describe the second level of analysis as a restating of the literal level in theoretical terms relevant to the research questions. Simon (2019) describes this restating in terms of using the results of the first level of analysis as the data for the second level. While the two levels can be boot-strapping and recursive (Parnafes & diSessa, 2013; Simon, 2019; Yin, 2018), I generally conducted the first level on large format paper and moved to the second level of analysis in creating the digital case profiles. These case profiles combined the transcripts with my first-level analysis to
infer the second level of inferences aimed at answering the research questions in terms of theoretical constructs.

**External Validity**

Two practices increased external validity in this study. First, the within-case results provided thick descriptions of the participants’ conceptual difficulties and evidence of knowledge-construction. These descriptions were considered “thick” because they were drawn from multiple data points and multiple sessions within each case.

Second, cross-case analyses allow for only the major themes that arise from a comparison of the within-case results to be described. Both literal and theoretical replication logic (Yin, 2018) was applied to surface these themes.

**Construct Validity**

The constructs applied in this study are established in the KiP research literature on mathematics education and science education. The application of concept projections to knowledge construction of models of division is novel. However, it was first described by Wagner (2010).

Additionally, the development of the PD worksheet as well as the development of the writing of results was peer-reviewed. Second, to some degree I was able to triangulate the primary results from the PD worksheet tasks with data from the later interviews. In some cases, I was able to negotiate the interview session protocol so as to provide triangulation for the main results of the PD task-based sessions.
Reliability

Two practices contributed to the reliability of his study. First, the same PD worksheet was implemented in all eight cases. This insured that all participants engaged in the same tasks and in the same order. These tasks are included in Appendix E. The two levels of analysis described above produced a chain of evidence (Yin, 2018) in the form of large format paper notes and digital profiles of each participant.

Second, a colleague with a Ph.D. in mathematics education and 21 years of experience teaching mathematics in elementary and middle school contributed to the reliability of this study in all phases. This colleague reviewed the study proposal, revisions of the PD worksheets that I made after each pilot study and discussed with me some of the pilot study results, this dissertation rough draft and this dissertation final draft.

Ethical Considerations

The primary ethical consideration in this study was that I was the researcher and professionally related to the participants. As an administrator at the school in which the study took place, I had a mentorship role with the participants in their regular work as teachers. This incurred two considerations. First, performance in the task-based interviews may have been considered as an assessment of the participants’ professional abilities. Therefore, I regularly reminded participants that they were participating in research designed to improve curriculum and instruction, and as such, any answer or comment or question was both valid and invited. A second, related consideration was the
overlap between the research activities and the regular activities the participants engaged in during the course of teaching. I made the difference in these two activities clear to all participants and clearly delineated these activities for the IRB review. These activities are delineated as *regular activities* and *research activities* in Appendix D.

**Summary**

This study employed a qualitative multiple case study design (Creswell, 2018; Merriam & Tisdell, 2016) to collect case-based data on individual teachers as they progressed through a PD program composed of two, 2-hour, one-to-one meetings. Individual case summaries and a cross-case synthesis informed results for Research Questions 1 and 2. Only the cross-case synthesis informed results for Research Question 3. This study included two groups of participants, allowing the cross-case synthesis to address Research Question 3 through theoretical replication logic (Yin, 2018). The primary source of data was video recordings of Zoom-based PD meeting with individual participants, generating 27.4 hours of cleaned video data.
CHAPTER 4
RESULTS

Improving teacher preparation and PD requires a deeper understanding of how teachers think, and more reliance on theory for understanding the connections between the components of teachers’ knowledge (Hohensee & Jansen, 2017; Izsák & Jacobson, 2017). A better understanding of the particular difficulties teachers experience will help in developing and implementing more effective professional development materials and practices (Jansen & Hohensee, 2016; M. Y. Lee, 2017; Lo & Luo, 2012). To make progress in this direction, research needs to extend examinations of teachers’ performance on individual pencil and paper tasks through the analysis of potentially rich and complex interview data across tasks (e.g., S. J. Lee et al., 2011). In sum, a better understanding of the nature of teachers’ conceptual difficulties can improve teacher preparation and professional development by more effectively addressing teachers’ difficulties and supporting teachers in their knowledge construction.

The purpose of this qualitative, multiple case study was to provide descriptions of the conceptual difficulties and learning experiences of inservice teachers as they construct deeper understandings of division with fractions through the representation of division with fractions using rectangular area models.

This chapter is organized into two main sections. The first section presents within-case results for each of the eight teachers participating in this study. The within-case results are limited to the participants’ engagement in four tasks (described below) in the PD task sequence (see Appendix E). The second section describes the cross-case
findings most relevant to the research questions. Data for the cross-case findings were drawn from the within-case findings as well as from participants’ interviews in the second PD meeting (see Appendices F and G for interview protocols). Some of the cross-case findings were common to many of the participants; some cross-case findings emerged from the contrast between the participants. Finally, cross-case results are discussed with respect to Research Question 3.

**Within-Case Results**

This section presents within-case results for each of the eight teachers participating in this study. Each case is presented as an individual subsection. The case descriptions are limited to portions of the participant’s engagement in the PD task sequence that most informed the following research questions:

1. What are the conceptual resources exhibited by inservice teachers as they construct a deeper understanding of division with fractions and improve their ability to productively represent division with fractions using rectangular area models?

2. What role does the teachers’ perceptions of the referent unit play in their knowledge construction across tasks involving division with fractions?

Transcriptions of the video data use the following protocol: All instances of italics indicate the original vocal stress of the speaker. Commas and ellipses are used to indicate the speaker’s own cadence. Instances of “–” indicate one speaker interrupting the other. Author’s comments are in parentheses. Square brackets indicate edits, such as capitalization and replacing prepositions with their referents for clarity. An ellipse appearing between lines indicates the removal of a portion of the transcript. This was done for brevity (removal of redundant sections) and clarity when a conversation
diverged from and then returned to the subject being presented in this dissertation. Transcript excerpts are titled by the participant’s name followed by the chapter and number in which they appear.

Case 1: Audrey

Audrey had a positive attitude throughout this study and was diligent about learning the new content. She was particularly adept with think-aloud communication of her ideas in real time. Thus, Audrey’s engagement generated informative data with respect to the conceptual resources involved in her construction of deeper understandings. The following sections describe selected results from Audrey’s engagement in the PD task sequence (see Appendix E). I chose to present Audrey’s case first because it covers many of the common findings with clarity, laying the groundwork for the other cases. Given that Audrey’s case is presented as an exemplar, it is the longest case.

Audrey’s engagement in the PD task sequence totaled 3.5 hours over two separate meetings. The following sections provide observed development of Audrey’s conception of division as she completed four task sets: (a) describing her intuitive ideas about fractions and the meaning of division, (b) learning to distinguish between measurement and partitive division with whole numbers, (c) discussing the meaning of partitive division when the divisor is a fraction, and (d) representing partitive division with fractions using rectangular area models. These sections appear in chronological order.

Audrey’s Initial Descriptions of Fractions and Division

The first task in the PD task sequence (see Appendix E) asked the teachers to
informally describe their conceptions of fractions as well as their intuitive ideas of the meaning of division. These questions served to surface some of the elements in the teachers’ conceptual ecologies. The teachers’ answers were taken to be reference points and possible subjects of later discussion rather than constituting a pre-test of knowledge.

Audrey’s initial description of fractions was situated in “tangible items” and oriented toward part-whole thinking.

_Audrey 4.1_

Interviewer (Int): What are your own ideas about what a fraction is?

A: What I start thinking is, when something is split into pieces, then fraction relates to the individual pieces of a whole. (Audrey writes “Pieces of a whole.” on her worksheet), so, like a pizza pie, and the slices of that, as an example. I always – when I think of math – a lot of math functions, I think of, um, converting sizes related to cooking…and possibly sewing – so that would be a, um, practical application of fractions.

Int: And how do fractions come up in cooking and sewing?

A: So say you’re, you have a recipe for two people but you only want to make enough for one person (rising tone). You would be (pause) dividing it, so, you know, or multiplying it by one-half. And then that would be your fraction.

Int: What are your intuitive ideas about what division is?

A: I think…so, it’s uh…splitting, I was going to say ‘dividing something’ but ‘splitting’ is another word. Um, division, you know, it’s interesting because division I think of totally related to math. But a fraction, I don’t think of it so directly related to math, although, probably it uh…it is. I think of a fraction being part of something visible (rising tone), whereas division, I think of more as a math concept (rising tone).

One minute later, Audrey reiterated her conception of division as an operation on numbers and a fraction as the tangible product of the operation:
Audrey 4.2 A: …again, for division I think of a number, so it’s splitting a number…splitting one number by another number. I think of differentiating the two, division versus fraction: I think of division as the, um, method (rising tone) or tool, and I think of fraction more as the result. A fraction…like I said, more likely as related to a tangible thing (rising tone).

In this initial PD task, Audrey’s part-whole thinking was evidenced by her description of splitting an object, referring to slices of pizza, and by her writing, “Pieces of a whole” on her worksheet. Additionally, she concluded that a fraction is “related to a tangible thing.”

Audrey used the term “splitting” to describe her conception of fractions as well as her conception of division. However, she described fractions as the result of splitting using the passive voice, for example, “When something is split into pieces.” Audrey used the active voice when describing division as a “math concept” consisting of “splitting one number by another number.” Her choice of passive and active tense is consistent with her description of a fraction as an object and division as a process.

Notably, Audrey did not mention that a fraction is a number, or that the value of a fraction could be one or greater than one. Audrey’s only use of a numerical value was in her cooking example where a recipe was multiplied by one half. One interpretation of the above excerpt, consistent with later results, is that Audrey primarily uses fractions for quantifying part-whole situations.

Audrey’s Engagement in the Introduction to Partitive and Measurement Models of Division Task

This task began with Audrey viewing Video 1: Partitive and Measurement Division (see Appendix H). The 2-minute video used animated balls to represent 15÷5
using the partitive model and then again using the measurement model (hereafter called the $15 \div 5$ Ball video). The interviewer then asked Audrey if she had any clarifying questions about the content of the video. The fact that measurement division is sometimes called repeated subtraction was stated in the video narration. Audrey attended to that comment as it was not, at first, a sense-making conceptual resource for her:

*Audrey 4.3*  
**Int:** What are your questions or thoughts about the difference between these two models of division?

**A:** Um – what was it? They said another word for the measurement division was ‘something subtraction?’

**Int:** Mm, “repeated subtraction.”

**A:** (Audrey noted this on her worksheet). I think that, let’s see, that concept, or phrase doesn’t uh, ring (Audrey hand-gestures as if she’s grasping an object) with me… Let’s see, ‘divided by five.’ Well, maybe it does. Because you’re taking the…divisor (rising tone) – is five the divisor?

**Int:** Yes.

**A:** You’re, you’re subtracting that repeatedly. (To herself) Equal sharing. (Aloud) I guess I can relate more to partitive division is the approach I would be more familiar with at this point. Uh…I don’t think I ever really thought of division and fractions as related to this quotitive approach.

**Int:** To your point about repeated subtraction…I think of it more as repeated addition because the divisor becomes the measuring unit, so you’re measuring off, so you’re counting, really --

**A:** (Audrey reacted to the word “counting,” talking over the interviewer.) Yeah, by fives.

**Int:** I’m sorry?

**A:** By five. (She gestured with her hand as if placing groups adjacently) Counting by five. Five plus five plus five.

In the excerpt above, the notion of “repeated subtraction” changed from a source
of cognitive conflict to a productive conceptual resource. The interviewer’s suggestion that repeated subtraction might also be viewed as addition may have cued Audrey’s prior knowledge of skip counting. It may have been simply fortuitous that the divisor was five in this case, as five is a commonly used unit for skip counting. Even though the measurement interpretation of division did not seem familiar to her initially, she constructed an original situation for measurement division in the next task:

*Audrey 4.4  Int:  Could you come up with an example of measurement division?*

A: (Confidently) Yeah, actually the picture on your Zoom screen kinda fits in with what I’m thinking. So a board and it’s like a certain length. And saying um, so I can picture with that say laying a 12-inch ruler against that, um, or even something less directly related to measurement. Say, say you have a board, that you need to set up against a brick wall. So the units or the, the quotient, I guess, would be the, uh, length of the brick, so this board is equal to how many bricks, you know. So the division, it would be one board length divided by one brick length equals how many units.

The shared screen was displaying an image of a board that was to be cut into shelves. However, there were no bricks in the image. Audrey spontaneously constructed an original application of measurement division by integrating the board that she observed externally with contextually relevant prior knowledge. Her substitution of a ruler with a brick was explicit. She constructed a concept projection of measurement division in which she could “see” measurement division in the context of boards and bricks and operationalize the length of the brick as the measuring unit. In this concept projection, Audrey may have integrated a new conceptual resource, her *repeated subtraction* image, with existing conceptual resources, prior knowledge of skip-counting, boards, bricks, and rulers. Her language evidenced that her coordination of conceptual resources was nascent in that moment. For example, her statement, “So the units or the,
the quotient, I guess,” indicated that she was new to coordinating the components of a measurement division concept projection, or at least she was new to articulating them.

**Audrey’s Engagement in the Partitive Model of Division with Fractions Task**

This task began with Audrey viewing *Video 5: Crackers and Bowls* (see Appendix H). This 2-minute video extended the partitive interpretation of division from whole numbers to fractions with a live-action demonstration modeling the dividend as crackers and modeling the divisor as bowls. The demonstration ended by modeling $\frac{3}{4} \div \frac{1}{2}$ with three quarters of a cracker placed in one half of a bowl. While Audrey could easily determine the quotient of $\frac{3}{4} \div \frac{1}{2}$ by applying the invert-and-multiply (IM) procedure to the numerical statement of the division problem, she was unable to determine the quotient when the division problem was physically represented with crackers-and-bowls (see Figure 4.1).

**Figure 4.1**

*Partitive Division $\frac{3}{4} \div \frac{1}{2}$ with Crackers and Bowls*

In particular, Audrey was unable to coordinate the procedural step, *dividing by*
one-half is the same as multiplying by two with the elements comprising the physical situation. She was asked to comment on the video immediately after watching it:

*Audrey 4.5*  
Int: What are your initial thoughts about this video?

A: That the idea of having half a bowl is confusing. And I guess (long pause) I guess I’m having trouble going, getting from figuring the equation…so I think the example was showing it visually (long pause), and my approach just from my history is doing, you know, of having it equal, uh -- flipping the one-half so it’s three-quarters times two and multiplying that out to six-quarters. So, (long pause) so I’m trying to um (to self) three-quarters divided by half. (Aloud) Somehow, I’m wanting to see your half a bowl there with the three quarters, to see two of those together and then I’d be able to (pause while she taps the table) count the six quarters.

Audrey’s statements exhibited an understanding that the physical demonstration of crackers and bowls should coordinate with the IM procedure even though she did not see how that coordination was taking place. In other words, Audrey appeared to have inferred that ‘flip-and-multiply’ should be present in the physical situation, but she simply could not see it yet. The dynamic relationship between perceptions and inferences is central to concept projection development in the KiP framework of conceptual change. Through inference, Audrey may have anticipated seeing the doubling process at the end of the video demonstration. Audrey continued:

*Audrey 4.6*  
A: I guess part of where I’m stumbling with it, also, is…the tangible item (she places her palm on the worksheet) and seeing that divided by half, seems like it almost should be multiplied by half (laughing). But of course you’d get a different answer. I can’t quite um, picture…how dividing by half is different than multiplying by half, if I just think of a physical bowl and crackers there (rising tone).

The interviewer believed that Audrey might have momentarily constructed the final step of the physical representation (missing in the video presentation) that would correspond to the final IM procedural step of multiplying by two. The interviewer tried to
redirect Audrey’s attention to that point by suggesting that the video could end by showing the second half of the bowl, or by drawing it. She responded:

_Audrey 4.7_  
A: So it’s harder to picture with physical things that, the idea of dividing by half. I mean, even what I’m seeing in front of me, that half a bowl, I’m seeing three-quarters, three quarters, but that’s the starting point not the ending point (rising tone).

Int: And to get to the ending point you multiply by 2?

A: Yeah, It’s hard for me to make that leap (laughing) with physical things that dividing by half is the same as multiplying by 2. I mean I can do it with the numbers and a piece of paper, and that, that works out.

In the _Audrey 4.5_ excerpt, Audrey stated “I’m wanting to see two [half-bowls] together and then I’d be able to count the six quarters.” The interviewer was unsuccessful in getting Audrey to re-instantiate and develop the perceptions and inferences that had prompted her statement. However, Audrey’s engagement in this task did evidence the important relationship between perception and inference. Specifically, Audrey appeared to infer that the IM rule was taking place in the physical demonstration of the problem. Thus, she may have had a sense of what to look for; she knew she was not seeing something that was potentially visible. While the data does not show what specifically happened to develop Audrey’s competence, Audrey did identify the IM rule happening in an abstract representation of partitive division with fractional divisors in the next task set.

_Audrey’s Engagement in the Representing Partitive Division of Two Fractions with Rectangular Area Models Task_  

This final task set of the PD task sequence (see Appendix E, part 8) was conducted in Audrey’s second meeting, five days after the first meeting. In this 58-
minute task set, Audrey (a) discussed examples of area models of division, (b) watched a one-and-a-half-minute video (Video 6, see Appendix H) showing a method to represent partitive division with fractions using rectangular area models (hereafter called the Partitive Division with Fractions Using Area Models video), and (c) constructed two of her own rectangular area models of partitive division with fractions. This section describes Audrey’s developing understanding of partitive division as she developed an ability to represent partitive division with fractions and exhibited the meaning of the partitive quotient with increasing fluency, less scaffolding, and in increasingly complicated contexts. Her concept projections of the partitive quotient ultimately approached an understanding of the partitive quotient as a unit rate. Her progression included a parallel to the progression from a part-whole interpretation of fractions to a measurement interpretation of fractions.

**Crackers and bowls as a bridging analogy.** After a 20-minute review of the tasks Audrey had completed in the first meeting, Audrey engaged in the task of explaining a rectangular area model representing $1/2 ÷ 1/3$ (see Figure 4.2). Initially Audrey referred to the components of rectangular area models in terms of her prior knowledge of bowls and crackers.

**Audrey 4.8**

A: This is half shared among a third of a bowl?

Int: Yeah.

A: Okay, I’m definitely stuck here. Let’s see, half…if I think back to the bowls, I might be able to make it make sense. Because this (hovering her pen over the $1/2$ dividend in the drawing), to draw the parallel, this is one-third of a bowl. And half of a unit fits within that bowl. And so, if you look at all three (touching the other two one-third partitions), if you want to say, compare one-third of a bowl, to…get it to equal, not ‘equal,’ to correspond to a full bowl, you can say each of
these of these half units go in each of the third of the bowl (touching the other two one-third partitions). So it’s one half, two halves, three halves. Now, I guess, I guess you could think of it the same way without relating it necessarily to the crackers and bowls (she starts laughing). But that helps for me to do.

In the excerpt above, Audrey used two iterations of the given dividend in the rectangular area model to construct a partitive quotient using a full-bowl conceptual resource rooted in her prior knowledge of the crackers and bowls video. She recognized the meaning of the 1-rectangle as one whole share through analogy with a full bowl. Thus, crackers and bowls served as a bridging analogy from the conceptions of division that Audrey brought to this study, to a concept projection that could explain this partitive division with fractions represented with a rectangular area model. After the analogy had served its purpose, Audrey recognized that the analogy was not strictly necessary.

Audrey exhibited a simple, yet productive type of conceptual resource by attending to $\frac{1}{2}$ as the unit of her counting. She also appeared to be aware, to some degree, that proportional scaling was happening: “And so, if you look at all three
(touching the other two one-third partitions), if you want to say, compare one-third of a bowl, to...get it to equal, not ‘equal,’ to correspond to a full bowl, you can say each of these of these half units go in each of the third of the bowl (touching the other two one-third partitions).”

**Audrey “sees” flip-and-multiply.** To further illicit Audrey’s understanding of partitive division with the representation above, the interviewer took the role of a confused student and asked her to explain why the dividend gets iterated two times after being placed into one-third of the divisor’s 1-rectangle. Audrey seamlessly stated that dividing by one-third is the same as multiplying by three in her explanation:

**Audrey 4.9**  
Int: [The divisor] is only there once in the problem, so where do we get the extras from?  

A: (Laughing) So, we’re visually picturing the two pieces of the problem. The one-half, which you see there, and the one-third is the second part of the equation or problem. So the dotted lines show the remaining thirds that when you put it all together equals one full unit. So you have half in the first third, but because you’re dividing it by thirds, or same thing as multiplying it by three, you need to put that share into the other pieces, the other thirds to get a full unit.

Audrey’s concept projections of partitive division, at first situated in the physical crackers-and-bowls context, became capable of conceptualizing the partitive quotient as the result of partitioning and iterating in the context of rectangular area models. Five days earlier, Audrey was unable to recognize the IM procedure when it was instantiated in the crackers-and-bowls context. Here, she was able to recognize the IM procedure in the context of rectangular area models without being prompted. By the end of this task, her conceptualization included a coordination between the quantity of the dividend and divisor in the process of transforming the given rate to a unit rate (i.e., determining the
Audrey spans numerical contexts. There was a 10-minute break half-way through this second meeting. During the break, Audrey came up with an idea that allowed her concept projections of partitive division to further progress toward a unit rate conception of the partitive quotient. Her idea paralleled a progression from a part-whole conception of fractions to a measurement conception of fractions in that for the first time (at least in the data for this study) Audrey applied partitioning and iterating in a larger-than-one numerical context (see Figure 4.3). Resuming from the break.

*Audrey 4.10*  
A: When I was away I was thinking of the problem of five halves divided by five thirds. So the divisor is more than one…you’re still drawing five partitions but it’s – to show five thirds – but you have to go back, to get your answer you have to go back to the full share which is less than the picture you drew.

Int: Could you write that out? I want to make sure I understand your point.

A: So the divisor is going to determine how I draw things here (laughing). (She drew a 1-rectangle with five vertical partitions (see Figure 4.3). So we have one, two, three, four, five…and each of these are a third, so our actual, full share, as I’m starting to call it, is this (picking up a different color, she drew a perimeter around the left-most three partitions to identify one share), it’s smaller than the entire picture. This is three thirds (she drew a bracket under the left-most three partitions in purple). This is five thirds (she drew a bracket under all the partitions in blue). And so I have to put these five halves into the five portions, so a half each (she wrote “1/2” in each partition). So it’s just that um, it does work where you’re talking back about a full share…or a full unit or whatever you want to call it. That’s what I was thinking of when I was gone.

Int: That’s really neat. So your point is, or the interesting thought you had is, that in this case you have to go back down…you have to make it go smaller to get to your whole share? …

A: Right. And it still works.
The data does not show how or why Audrey thought of using an improper fraction as the divisor, and the interviewer did not ask her what cued her idea or motivated her imagination. Nonetheless, Audrey generated a concept projection that determined the partitive quotient in a new context. She extended the process of determining the quotient by partitioning and then iterating up to a full share into a context where the divisor was larger than one. This required a *subtractive iteration* to determine the “full share.” She iterated backwards from the fifth partition to “go back to the full share.” She seemed pleasantly surprised to find that the representation process she had just learned “still works” in the novel context she had constructed.

**Evidence of a deeper understanding of partitive division.** In the final task of
the PD sequence Audrey was prompted to represent $1/2 \div 3/4$ using a rectangular area model. She completed the task without difficulty while providing a think-aloud narration. Audrey exhibited several actions that evidenced she had developed a deeper understanding of partitive division with fractions.

First, Audrey used a three-color scheme to distinguish three levels of units: the dividend as a unit, the partitioned components of the dividend as a second unit, and the full share as a third unit (see Figure 4.4). Additionally, she distinguished the single iteration of $1/6$ beyond the given dividend by writing that “$1/6$” at the bottom of that partition.

**Figure 4.4**

*Audrey’s Representation of $1/2 \div 3/4$*

Audrey’s color coding evidenced a well-coordinated concept projection of partitive division with fractions that included the mental actions of unitizing and units.
coordination, as well as the coordination of partitioning and iterating to determining the quotient.

Audrey indicated (with her pen) the fact that she had drawn the three instances of \( \frac{1}{6} \) comprising the divisor spatially higher than the one iteration of \( \frac{1}{6} \) needed to construct a full share, commenting:

*Audrey 4.11* A: I think this is handy, to show what’s in the divisor, versus what’s in the ghost.

Audrey and the interviewer had independently coined the term *ghost* to refer to an iteration that was not part of the given divisor. As seen in later cases, this term became a useful conceptual resource for some of the teachers.

Audrey’s language and actions at one point in her construction of this representation indicated that she may have mentally reframed, or normalized the division process by conceptually operating on the fractional partitions as if they were whole-number units and distributing them as if she were fair-sharing among whole number recipients:

*Audrey 4.12* A: I have one half that I have to equally share among these three slices…so I wanna be able to share it equally, I need to um (she picked up a third color) I have to come up with the least common denominator. I have to have something that can be split three ways equally, so one sixth plus one sixth plus one sixth (she drew ”1/6” in the three partitions she identified as the divisor-share of the one-rectangle).

While Audrey may have momentarily reframed the division as a whole-number process, she was still emulating the steps presented in the *Partitive Division with Fractions Using Area Models* video. In particular, the first step in constructing her representation was to draw a large, blank 1-rectangle to represent one “full share” of the
divisor. The second step was to partition that rectangle in accordance with the divisor’s denominator. Next, the divisor’s numerator was identified as a second unit, coordinating the space into which the dividend was to be distributed. Unless the divisor is a unit fraction, the dividend needs to be partitioned before it can be distributed. This partition-and-distribute step of the representation thus has the affordance of being mapped to a whole number fair-sharing image, making productive use of prior knowledge of whole-number counting and of the fair-sharing psychological situation.

**The End-State of Audrey’s Partitive Division Conception**

The interviewer had hoped that Audrey’s conceptual development over the two meetings would end with her articulating a unit-rate conception of partitive division with fractions. Her concept projections of partitive division of fractions had become capable of independently determining the quotient, as demonstrated in the two previous examples.

In the final twenty minutes of the second meeting, the interviewer asked:

*Audrey 4.13*  
Int: Does unit rate make sense to you…for the, as the answer to a partitive division problem?  
A: No (she laughs heartily). Well, the reason I say that is because “rate” hasn’t been defined yet.  
Int: Oh, right.  
A: And um, I kind of think of rate as like a speed…in a way (rising tone)  
Int: Yeah.  
A: And putting that with worms and chicks is like, ‘Whaaaat?’ Um, but, that was a thought I had, in looking at [the units chart], the quotient isn’t exactly the same unit, but, I mean it was close enough for this, for me.
While Audrey was able to recognize that the physical units of the partitive quotient are not “exactly the same” as the dividend, at the end of this study she was apparently unable to articulate a concept projection of partitive division as a unit rate. She did acknowledge a conceptual resource of speed as an example of a rate. But in the remaining few minutes she did not appear to integrate her intuitive notion of rate-as-speed with the conceptual resources she had used to construct representations of partitive division with fractions.

In the final few minutes, the interviewer asked if the PD meetings had changed Audrey’s conceptions of fractions and division:

*audrey 4.14* A: Yeah, the whole concept of dividing by a fraction, I couldn’t, I didn’t relate to that at the beginning, but after a while, and seeing the course material, and bringing it to bed with me, so to speak; yeah, I could relate to that. I think for me it was an expanding way of learning or thinking about fractions to have it so much related to pictures, because in general I’m just more comfortable doing the algebra to get an answer. So this was a good little stretch on my brain cells.

Int: When you said “dividing fractions” did you mean before looking at these pictures and thinking about the conceptualization process, you were looking at dividing by fractions as basically the same as any other division problem?

A: I wasn’t thinking of a real-life context for them (rising tone). You know, it’s like, “Divide by half, what does that mean?” (Audrey starts laughing.) You know, so of course I couldn’t use it dividing by persons, but like the crackers and the bowls, and later I thought of pills, and uh different things that you could divide by fractions. I guess I don’t really think of it in my daily life, but now I can see how it might come up.

**Summary of Case 1: Audrey**

Audrey began the PD task sequence without an articulated understanding of the
difference between measurement and partitive division. She also had no experience with representing division with fractions using area models. After the two interpretations of division were presented to her, she integrated her newly-learned repeated subtraction image with prior knowledge of skip-counting, boards, bricks, and rulers to create a concept projection of measurement division of whole numbers.

When introduced to partitive division with a fractional divisor in a physical context, Audrey was unable to see how the IM rule applied in the physical situation. She was, however, aware that she should be able to see the IM rule applied the physical situation. When working with rectangular area models in the second meeting, Audrey was able to identify the IM rule taking place without being prompted. Audrey may have been able to identify the IM rule taking place in the later task because (a) she had already developed a sense of what to look for and (b) the affordances of the rectangular area models somehow made the process more visible.

Audrey also tested the partition-and-iterate method she had just learned by imagining applying it to a novel (for her) case where the divisor was an improper fraction. She was able to construct that representation without scaffolding and determine the quotient. By testing her idea through a representation of her own construction, Audrey exhibited a concept projection of division that spanned a new context.

By the end of the PD task sequence Audrey was able to independently construct a representation of partitive division with a unit fraction, proper fraction, and an improper fraction as the divisor. Audrey did not immediately connect her intuitive ideas of rate to the representations of partitive division that she had just made. However, had the
interviewer asked her to consider her last task as representing walking half a mile in three-quarters of an hour, she might have made that connection.

**Case 2: Ivy**

Among the eight participants in this study, Ivy was the best prepared to accomplish the PD tasks. She enjoyed engaging in the tasks and discussing them. Throughout the four hours of video-recorded PD Ivy was communicative and comfortable, and her engagement generated much useful information. Ivy’s tone was consistently confident; she was committed to her mathematical knowledge. Although Ivy was quite capable of completing the PD tasks, she also exhibited some inflexibility in the final task of representing partitive division with fractions using rectangular area diagrams.

Ivy’s engagement in the PD task sequence (see Appendix E) totaled one hour and forty minutes and took place in one meeting. The following sections provide observed development of Ivy’s conception of division as she completed four task sets: (a) describing her intuitive ideas about fractions and the meaning of division, (b) learning to distinguish between measurement and partitive division with whole numbers, (c) discussing the meaning of partitive division when the divisor is a fraction, and (d) representing partitive division with fractions using rectangular area models. These sections appear in chronological order.

**Ivy’s Initial Descriptions of Fractions and Division**

The first task in the PD task sequence asked Ivy to informally describe her
conceptions of fractions as well as her intuitive ideas of the meaning of division. Her description of fractions exhibited a sophisticated understanding of fractions as rational numbers. During the four hours of video data, Ivy never seemed to be influenced by part-whole models:

*Ivy 4.1*  
Int: What are your own ideas about what a fraction is?

I: A fraction is some number that describe like, not an exact integer, but in between integers, between whole numbers. I mean it could also be integer, but the main purpose of fraction is like what is a number between one and two?

She continued her description of fractions without mentioning a value less than one, noting the relationship between fractions, whole numbers, division and decimal representation. It is notable that her first numerical example of a fraction was greater than one:

*Ivy 4.2*  
I: But fraction can also be a whole number like six divided by two. That is a fraction, but that’s equal to three. But fraction is more efficient when you try to talk about numbers, those in-between numbers. Yeah, like one-point-five, one-point-two.

Ivy also noted that a fraction may also represent a ratio. She concluded that she mostly used fractions as a way to describe non-integer values. Thus, Ivy evidenced a conception of fractions that was not restricted by part-whole thinking and not tethered to quantifying tangible items. Rather, Ivy appears to conceive of fractions as “numbers in their own right” (Lamon, 2001).

*Ivy’s Engagement in the Introduction to Partitive and Measurement Models of Division Task*

This task began with Ivy viewing the $15 \div 5$ Ball video (see Appendix H). When the video ended, the interviewer asked Ivy if she had any clarifying questions about the
video. At first she conflated measurement division with a method of answer-checking she practiced with her daughter, that of using multiplication to check a division problem.

*Ivy 4.3*  
I: That's similar to what I taught my daughter, so it's like 15 divided by 3, so I taught her two different ways, right...I could also ask her to double-check the answer like draw three circles and then she has to put five things into those three circles.

Int: The second type of division, which is measurement division –

I: Is a totally different idea; yeah that's really good, yeah.

Int: – the divisor becomes like a measuring unit and we just measure how much stuff we’ve got.

I: How many units. We have three units, three such units. Yeah.

Ivy’s response indicated that she was accustomed to using drawings as a form of mathematical sense-making. She was also apparently unfamiliar with the difference between the two models of division as a mathematical construct. However, she recognized the underlying structure and quickly corrected her first impression of measurement division being a way to check the answer to a partitive division problem.

*Ivy’s Engagement in the Partitive Model of Division with Fractions Task*

This task began with Ivy viewing the *Crackers and Bowls* video (see Appendix H). When the video ended, the interviewer asked Ivy if she had any questions about the content. Ivy asked to see the last division problem a second time. Her immediate response focused on the point that the quotient is found by constructing a whole share:

*Ivy 4.4*  
I: Okay, I got it. The confusing part is that the answer we are looking for is to, “How many things are in one plate.” So that’s the answer; not how many things in half a plate. So the answer is, “How many things are in one plate.” So right now there are three items in half a plate so that means there are six items in one plate. So the answer is
six of those things and those things is a quarter, right? So six quarter. So that’s the question in division is the final answer should, should answer the question, how many items in one whole plate. Even though we see half a plate, at the very end you have to think about the answer in terms of the entire plate. So there should be six items in the entire plate, if we, we, we um, scale it like, up.

Ivy’s response evidenced a concept projection strongly driven by a conceptual resource that division is based on one. This excerpt alone does not provide enough data to indicate whether her concept projection was driven by a single heuristic kind of knowledge or if Ivy was expressing a deeper understanding of the meaning of division. In either case, Ivy knew what to look for. She stated that even though we see half a plate, we need to think in terms of a whole plate, because that’s the question division answers. The physical context of crackers and half a bowl did not hinder Ivy from determining the quotient.

Ivy’s “scaling up” comment evidenced another conceptual resource. Namely, how to get to the whole bowl once the need for it was determined. “Scaling up” indicates an understanding that the given rate of three quarters of a cracker in half a bowl must be maintained. However, Ivy did not use language such as “to maintain the proportion” or “to create a unit rate.”

The interviewer asked Ivy what she thought about the idea of ending the video by drawing the other half of the plate. She replied:

*Ivy 4.5* I: Yeah, because that’s what the question in division’s asking you, “What’s in one plate?” Even if you have three plates, even a hundred plates, even half a plate. It doesn’t matter what numbers are there, how many plates are there, answer always represents how many items in one whole plate.

In listing examples of values that the divisor could take, Ivy constructed a list that
included a number greater than one, a large number, and then a fraction. This evidenced Ivy’s concept projections of division to be invariant with respect to whether the divisor is greater than one or less than one. In other words, from the start of the study Ivy’s concept projections of partitive division spanned the contexts of whole number divisors and fractional divisors. The interviewer wanted to know how Ivy had developed her conception of partitive division:

*Ivy 4.6*  
*Int:* And that seems so clear to you; is that the way you think of division? Or fractions or…

*I:* I think because that’s how I learn it in China, I guess. I did not learn of it as a share, I did not learn it as “equal share,” I only divide like that when I taught my daughter -- I don’t know how I came up with that. I think I’ve been here for too long or something.

The interviewer did not follow up on these statements. However, Ivy’s statements indicated that the fair-sharing image of partitive division was likely not a conceptual resource that she acquired in her childhood education.

*Ivy’s Engagement in the Representing Partitive Division of Two Fractions with Rectangular Area Task*

In this final task set of the PD task sequence Ivy (a) discussed examples of area models of division, (b) watched the *Partitive Division with Fractions Using Area Models* video (see Appendix H), and (c) constructed her own rectangular area models of partitive division with fractions. Ivy’s engagement in these tasks evidenced a different interpretation of partitive division with fractions than the interpretation suggested by the worksheet examples and video demonstrations in the PD content. This section describes Ivy’s engagement with the tasks and the interpretation of partitive division with fractions that she brought to the tasks.
The first task in this segment was to make sense of a representation of the partitive interpretation of $1/2 \div 1/3$ (see Figure 4.5), ostensibly constructed by a student, and then explain it. Ivy did not immediately make sense of the representation. The interviewer-oriented Ivy by likening the outer-most rectangle to a full bowl.

**Figure 4.5**

*A Rectangular Area Model of $1/2 \div 1/3$*

*Ivy 4.7*

I: What are you doing, divided by one-third right there...you just multiply by three? Or what is, looks to me just make three copies of that one-half (laughing).

Int: It is three copies. It’s iterated three times. Remember the crackers and the bowls?

I: Mhm.

Int: Here’s a whole bowl (indicating the outer rectangle) --

I: (Immediately) Oh, that’s the one-third, that’s one-third ‘a whole thing. One-half, you are willing one-half as a third of something then you have to draw the whole thing. That’s the whole thing right there. Okay. I see. You think of one-half as one-third of something.

Ivy’s language evidenced an interpretation of partitive division with fractions that is consistent with the interpretation described by the Chinese teachers in Ma’s (2010) research comparing American and Chinese teachers. This interpretation views the
dividend as an incomplete instantiation, the progress of which is quantified by the divisor. I refer to this interpretation as the *in-progress interpretation*. Ivy’s statement, “You think of one-half as one-third of something” may evidence that Ivy was considering the dividend and the divisor to have the same referent. This interpretation is distinct from the *unit rate interpretation*, in which the divisor and the dividend have different units.

Ivy evidenced the in-progress interpretation of partitive division while engaging in the rest of the tasks in this section as well. In the next task, Ivy was asked to interpret and explain another example of a partitive division representation, this time with a mixed number as the dividend and a proper fraction (not a unit fraction) as the divisor (see Figure 4.6).

**Figure 4.6**

*A Rectangular Area Model Representing 1 1/2 ÷ 3/5.*

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*Ivy 4.8  Int: Can you describe what’s happening?*

*I:  So three one-half and now you have to think about three one-half, those yellow rectangles, you have to think about is part of something. It’s three parts of five thing. It’s three out of five things. You have to think of it three out of five things, that means two things missing.*
Ivy’s language describing “part of something” and “three out of five things” implied that the dividend and divisor have the same referent. Additionally, she framed the construction of a whole share as the resolution to her perception that there were “two things missing.” These points suggest that Ivy considered this representation as a part-whole situation.

In the next task Ivy was prompted to draw \( \frac{3}{4} \div \frac{1}{2} \) with partitive interpretation. This was the first time in the PD sequence that she constructed her own representation of partitive division with fractions. She began by representing the dividend without reference to the divisor (see Figure 4.7).

**Figure 4.7**

*Ivy Begins by Representing the Dividend*

I: So I’m going to here’s one-quarter, one-quarter, one-quarter. So I begin with three quarter. (Pointing her pen to the equation) So that’s the dividend, three-quarter. Now I have to think about, this, this three quarter right here (hovering her pen over her drawing) is actually…*half* of something. It’s one-half of something. What is that something.

Ivy continued by extending the outer rectangle to represent the “something” she wanted to quantify (see Figure 4.8). She then used the term “fill in” to describe the process of iterating the one-quarter units to construct a whole unit of the divisor.
Ivy distinguished between iterations that were part of the given dividend, and iterations that were part of the scaling by using dashed lines for the scaling iterations (see Figure 4.9). Ivy’s “fill in” language and dashed-lines representation of the scaled iterations could have evidenced a rate-based conception of partitive division. However, they may also have been heuristics that Ivy employed, consistent with her perceived affordances of the representation at that moment rather than reflecting her conception of partitive division.

**Figure 4.9**

*Ivy Quantifies the “Something” with Ghost Partitions.*
In the next task, Ivy represented $\frac{1}{2} \div \frac{3}{4}$, repeating her method of starting with a representation of the dividend as a single entity and ascribing its meaning as “part of something”:

*Ivy 4.10*  
I: I am going to draw one-half. This is one-half. Now we have to think about this dividend, one-half, as part of something. This one-half is three quarter of something, um, let me think, three-quarter of something.

Because the divisor was not a unit fraction in this division problem, the dividend needed to be partitioned before it could be distributed into the partitions of the divisor. But Ivy did not see that at first. Rather, she represented the dividend as a single rectangle (see Figure 4.10). When she began her next step of scaling the divisor partitions to represent the “something,” she realized that the single rectangle representing the dividend did not have the affordance of being easily scaled by one-third of its own size. She scratched out the single rectangle and started over, representing the dividend as three, one-sixth partitions (see Figure 4.10).

*Figure 4.10*

*Ivy Represents $1/2 \div \frac{3}{4}$.*
In constructing her representation for this task, Ivy did not take the step of qualitatively representing the whole “something” as a blank rectangle before quantifying it. Rather, she simply tacked on an additional one-sixth rectangle to the existing three one-sixth rectangles, and used dashed lines to indicate that it was an addition to the given divisor. This may seem trivial, because it was easy for Ivy to see that only one more one-sixth was needed to construct a whole share in the second example. However, the steps she took to construct her representation indicated context dependency. One explanation for the difference in her construction of the two representations above is that her primary focus was on the dividend, not the relationship between the dividend and divisor. If true, then whatever steps she would take to scale the dividend would be determined on a case-by-case basis according to her judgment of what mathematical steps would be most expeditious given the specific numbers involved in the problem. This explanation seemed to be true in later tasks as well, some of which will be described in the cross-case analysis.

**Summary of Case 2: Ivy**

Ivy began this study with previous experience using drawings as part of her mathematical sense-making. Learning how to represent division of fractions using rectangular area models did not seem to require notable restructuring of her conceptual resources. Rather, the constructs that Ivy learned in her participation with the PD task set appeared to be additive conceptual change. For example, she learned the names *measurement* and *partitive*, but she was already familiar with the mathematical structures and activities underlying these two models of division. In representing partitive division
with fractions, Ivy learned that the dividend might need to be partitioned. This became an important piece of additive knowledge for Ivy because of her practice of starting off with a representation of the dividend. Ivy also learned the terms *partition* and *iterate*, though she was familiar with the underlying actions. However, she did not embrace these terms; she did not incorporate them into her representations or verbal descriptions while engaged in the tasks.

Although the PD examples of representing partitive division with fractions began with a representation of the divisor in its whole-unit state, Ivy began her representations with the dividend. Moreover, Ivy did not seem to resonate with the intensive-quantity aspect of the representations of partitive division with fractions demonstrated in the PD sequence. In other words, she did not seem to attend to the relationship between the dividend and divisor as her primary focus in the division tasks. The invariant across her representations was a commitment to her *this is part of something, what is the thing* conceptual resource. That is, Ivy repeatedly evidenced a conception of the partitive division problems from a part-whole perspective in which the dividend and divisor had the same referent unit. This point will be presented in more detail in the cross-case analysis. In contrast, the following case highlights a conception of partitive division in which the intensive-quantity aspect of a rate is visible in the teacher’s representations.

**Case 3: Evelyn**

Evelyn was skilled at explaining her thinking in real time. She made an effort to describe anything that might be relevant to the study such as comments a student might make or past teaching experiences that related to the task at hand. Her engagement
generated much informative data. Evelyn’s engagement in the PD task sequence totaled 2 hours and 15 minutes and was completed in a single meeting. The following sections provide observed development of Evelyn’s conception of division as she completed four task sets: (a) describing her intuitive ideas about fractions and the meaning of division, (b) learning to distinguish between measurement and partitive division with whole numbers, (c) discussing the meaning of partitive division when the divisor is a fraction, and (d) representing partitive division with fractions using rectangular area models. These sections appear in chronological order.

**Evelyn’s Initial Descriptions of Fractions and Division Task**

The first task in the PD task sequence (see Appendix E) asked Evelyn to informally describe her conceptions of fractions as well as her intuitive ideas of the meaning of division. Evelyn expressed a number-oriented conception of fractions:

*Evelyn 4.1*  
**Int:** What are your own ideas about what a fraction is?

**E:** A fraction is a way of representing a number. And then you could say a way of representing a number that is not necessarily a whole number (rising tone). I can do a mental dump, but if I’m defining it I’ll say it’s a way of representing a number; typically numbers that are not whole numbers.

Evelyn began describing division as an operation with numbers. She then used physical metaphors of chopping and breaking to describe the operation:

*Evelyn 4.2*  
**Int:** What are your intuitive ideas about what a division is?

**E:** Uh, division…is when you divide up one number or quantity by another one (rising tone)…into pieces (rising tone).

**Int:** I’m wondering if there’s a way to say what division is without using the word ‘division.’
E: Oh, oh, I used the word ‘divide.’ Let me think: Okay, division is when you take a number…well, I’ll just substitute a word like ‘chop’ or ‘break into pieces’; you take a number and break it into pieces, by, another number…by a certain defined amount…which could be a number, it could be an amount, a quantity…but anything that’s quantifiable…and you break it into pieces.

This excerpt established that viewing a fraction as a number, as well as thinking of division as an operation with numbers were conceptual resources available to her. However, Evelyn relied on physical images for making sense of fractions and division throughout her four hours of engagement in this study.

Evelyn’s Engagement in the Introduction to Partitive and Measurement Models of Division Task

This task began with Evelyn viewing the 15÷5 Ball video (see Appendix H). When the video ended, the interviewer asked Evelyn if she had any questions about the content of the video. Evelyn had viewed the video several days earlier. This allowed her to discuss both her current and past thoughts about the content:

*Evelyn 4.3*  
**Int:** Do you have questions about the difference between measurement and partitive division?

**E:** The response I had when I first looked at this stuff is simply, “Huh, I never thought of division that way.” I just thought of it as a number going into another number.

**Int:** When you said “a number going into another number,” were you imagining the procedure -- the algorithm of doing the division problem?

**E:** Yes.

**Int:** As opposed to visualizing these two physical examples of what the process might represent.

**E:** Right.
As mathematical constructs, the terms measurement and partitive division were new to Evelyn at the start of the study. In the next task she demonstrated familiarity with the underlying mathematical activity by constructing examples of both interpretations using whole numbers. However, the underlying mathematical activity of the measurement interpretation seemed less familiar to her:

_Evelyn 4.4_  
**Int:** Could you come up with an example of a partitive division problem and a measurement division problem with whole numbers?

**E:** Okay I have uh, three dogs and fifteen dog treats…this would be partitive. If each dog gets the same numbers of treats, how many treats does each dog get?

**Int:** Could you write down the division problem – just the numbers.

**E:** Got it. (She writes $15 \div 3 = 5$)

**Int:** Using that same division problem, could you make a measurement example?

**E:** Okay, hold on…that’s going to take a second, ‘cause that’s a little harder with that specific…(laughing) example. A measurement, so I gotta create a _unit_…um, out of dog treats, so….I got fifteen dog treats, if every dog gets…five treats, how many dogs can I give treats to.

**Int:** That’s great. So what is your measuring unit?

**E:** My measuring unit is five treats. (She writes on her worksheet while saying aloud) “Five treats is a unit.”

Although Evelyn had to pause momentarily, she constructed a measurement example using the same elements of the physical situation she had just used in her partitive example. Her language evidenced that thinking of the divisor as a measurement unit was not a regular practice for her. Nonetheless, the concept projection Evelyn had just constructed evidenced coordination between perceptual and inferential components.
Specifically, she could perceive the dividend as a measuring unit, but more importantly, her statement, “I gotta create a unit” indicated that she knew what to look for in order to determine if the division problem was a measurement division problem. Thus, Evelyn constructed a concept projection of division that distinguished between the two division interpretations in the context of whole numbers.

**Evelyn’s Engagement in the Partitive Model of Division with Fractions Task**

This task began with Evelyn viewing the Crackers and Bowls video (see Appendix H). As they began to discuss the content of the video, the interviewer asked Evelyn to construct an example of $1 \frac{3}{4} \div \frac{1}{2}$ using the partitive interpretation. This section presents the conceptual difficulties Evelyn experienced in that discussion and with constructing her example.

**Making sense of division in a physical context.** Mentally constructing a whole bowl from the half bowl in the video perturbed Evelyn’s conception of division. After watching the video, Evelyn expressed difficulty conceptualizing this step in the physical context, despite understanding the need for it:

_Evelyn 4.5  
Int: What do you think of the math in this video?  
E: I think the big jump for me was that your final answer is…it’s always based on…one…on a whole, but in reality, you’ve only got a half. So that throws—that threw me off. I couldn’t make sense of that [at first] because I was looking at the visual aide. And I did not have one whole bowl. So how could that possibly be the answer.

...  
Int: If it were up to you to improve this video, is there a way that would make that more clear to students?
E: If you thought about your wording, and say, “The answer’s based on how much fits into one bowl. Not half a bowl; one bowl. So imagine if you had a whole bowl: how many would be in there.”

Conjecturing that Evelyn was mentally scaling the denominator up to one as a heuristic rather than as a sense-making operation, the interviewer attempted to surface a conceptualization for the scaling process.

Evelyn 4.6  Int: What if a student says to you, “What does a whole bowl have to do with it if you’re dividing by a half?”

E: Yeah, well that was my problem, wasn’t it. Um, and I still kind of wonder that. In a sense. Now I know: it’s because every division problem comes with a presupposed unit. And unless stated otherwise, the presupposed unit is one.

**Evelyn constructs an example of partitive division with fractions.** The interviewer proceeded to the next task on the worksheet, asking Evelyn to construct an example of $1 \frac{3}{4} \div \frac{1}{2}$ using the partitive interpretation. This section presents key moments of the 22-minute process Evelyn underwent to construct her example and determine the quotient.

Evelyn 4.7  E: Let me think for a second. In what case would I have one and three quarters of something that I would need to divide by half? Can’t be people because you don’t have half a person (laughing).

Int: You could make an animation character or something.

E: Yeah, but I’m trying to make it a little more realistic. It helps me.

Evelyn mentally and visually searched for a realistic context in which to construct her story problem. She noticed a Tupperware-type food container in the next room and decided to model the division problem with a container-filling situation, admitting that she was emulating the crackers and bowls video.
Evelyn 4.8  E: So I’ve got five food containers and I’m going to be filling each one half full with…desert.

Int: Where does the five come from?

E: I just made it up. Oh, I can’t make it up. Half. (Drawing a rectangle around the 1/2 in the problem statement) Half full. So uh, that’s where the half comes from. So that means I have one and three-quarters (tapping the dividend in the problem statement). Got it. I have one and three-quarters, (speaking slowly) of a pie. And I need to create lunches. And I wanna put…some pie, into half of each container (rising tone). How many containers can I fill…with pie. Does that work?

Int: Um, so you’re doing partitive right now?

E: Yeah, I’m doing the partitive…(Looking at the tasks on the worksheet) Oh, I see measurement was first.

Int: That’s okay.

E: That was partitive. Absolutely.

Because Evelyn’s first statements regarding the dividend and the divisor appeared to be incorporating components of measurement division, the interviewer asked Evelyn to repeat her example. She again phrased the question as “How many containers can I fill?” She began to doubt that her example would work. The interviewer asked if she could draw her example. Evelyn drew a circle diagram for the pie, and a single oval for the container, partitioning it into halves (see Figure 4.11).

Evelyn 4.9  E: So I got one and three-quarters of a pie. And then I’ve got these containers. And I want to put even amounts of pie into half of each container (Evelyn shaded one half of the container using a separate color). That’s as far as I get. I’ve got different units here. These are pies; these are containers. The answer will be in pies per half-container (tapping the filled half-container in her drawing).

Evelyn’s 20-year background as a science teacher likely provided her with the ability to productively extract the quotient’s physical units of the situation even though
(at this point) she could not determine the quotient as a quantity. While pies and containers are not standard physical units, her use of the word “in” exhibited fluency with thinking about the physical units of a problem independently from the numerical values of a problem. Based on her teaching experience, the language she chose and her vocal tone and pen gesture, when Evelyn said, “The answer will be in pies per half-container” it seemed apparent that she was analyzing the situation in the same way she would with standard physical units. In other words, the idea she was expressing at that moment had the same sense as “The answer will be in grams per liter.”

Despite identifying the physical units of the situation, Evelyn was having difficulty constructing a specific question that she felt correctly coordinated the numbers in the problem statement to her pies and containers situation. Because she felt stuck at this point, Evelyn asked if she could “do the math” to determine the numerical value of the quotient. The interviewer asked if she could instead use her drawing to solve the
problem.

_Evelyn 4.10_ E: Yeah. So I’ve got one and three-quarters pie. That means, if I’m going to disseminate them evenly (she partitioned the circles into fourths), I could disseminate them in so many different ways. But let’s say I stick with quarters, because that’s the unit I’ve been given (she counted and labeled them 1-7) (see Figure 4.12). So I have seven quarters, and that means I can fill – see the problem is this half isn’t well represented (she hovered her pen over the 1/2 in the numerical statement of the division problem).

In this excerpt Evelyn appeared to use the term “unit” in a different sense than she did just moments earlier. In the previous excerpt (_Evelyn 4.9_) she was explicitly referring to the physical attributes of the situation (pies and containers). In this excerpt, she was referring to quarters as units to “disseminate.” After identifying quarters as a unit, she counted them, labeled them, and acknowledged that she had seven of them. Thus, in this excerpt she appears to have been using the term _unit_ in the sense of a unit of mathematical structure rather than a physical unit.

The excerpt above also shows that Evelyn was using a _disseminate evenly_ conceptual resource and seemed to have clarified the partitive nature of the task. She had represented the dividend and partitioned it. She knew it needed to be “disseminated” into containers somehow, but she was not sure how to coordinate the dissemination. She expressed uncertainty (for the third time) about how to coordinate the value of the divisor (1/2), both within the numerical statement of the problem and within her representation. She began to verbally express several concept projections that appeared to incorporate the following conceptual resources in various ways: (a) _repeated subtraction_ relating back to the measurement interpretation, (b) _fair sharing_ related to the partitive interpretation, and (c) _answer is based on one_ related to partitive interpretation. First, she considered placing
one quarter piece of pie in each half of the container. She then considered placing the
seven quarters into the whole container and phrasing the division problem as “How much
would be in half a container?” She then considered placing a quarter piece of pie in
“each” container and drew a piece of pie straddling the container’s partition (see Figure
4.12).

Figure 4.12

Evelyn Partitions the Dividend and Centers a Piece in a Partitioned Container.

Evelyn 4.11 E: So if I put in each container a quarter piece of pie, and I can do that
Seven times, then…I ask, “How much is in half a container (rising
tone). That would be…an eighth.

Evelyn then stated that she did not think she could represent the “actual math
problem.” The interviewer asked her to calculate the answer, hoping that she might notice
how the steps of the IM procedure were components of the various solutions she had just
considered. She determined the quotient with the IM procedure (14/4) but was unable to
coordinate the calculation with her representation.

Evelyn 4.12 E: Fourteen over four is the answer. So, I don’t really see how
that helps me with my pies.

Int: I see you have seven quarters because that’s what’s given.

E: Yep.

Int: So how does seven quarters become fourteen quarters?
E: This one-half, I inverted and multiplied straight across. So I got fourteen over four.

Int: And in terms of your drawing, how would that correspond…

E: I really don’t know, that’s why I think it’s not working, ‘cause I don’t know any more if this is pies (circling her numerical answer, 14/4) or containers or what.

Int: Okay, well let’s see if we can figure that part out.

E: I think it’s pies, based on my story problem, it’s got to be pies.

Int: So you start off with seven-fourths pie.

E: Yup.

Int: Divided into what?

E: (Pause) What do you mean, “Divided into what?” (Emphatically) Divided by one half.

Int: Okay, one half of…what

E: A container?

Int: Then what are the units of the answer?

E: Pieces of pie per half container? (She starts laughing) And I realize, that it fully brings me back to the thing we were talking about in the video. Which is, the answer’s based on one…yet I now want to base my answer on half a container. So I’m reverting, I guess…I’m looking at your story problem ‘cause it’s the same problem, it’s the exact – instead of crackers and bowls it’s pies and Tupperware. And yet I can’t do it. Right now.

Calculating the correct answer did not appear to cue productive conceptual resources for Evelyn. However, focusing her attention on the units of the dividend and divisor may have helped reorganize her conceptual resources. In the Evelyn 4.11 excerpt and the moments just before that excerpt, Evelyn considered three different ways to coordinate the dividend and divisor to determine the quotient. Each of them involved the
whole container in some way, but none of them were correctly coordinating the given dividend to the value of the dividend associated with one whole share of the divisor. In other words, she was constructing concept projections that included an answer is based on one conceptual resource, but that resource was somehow not productively coordinating the dividend to determine the quotient. Another feature of these concept projections was that Evelyn was fairly sure none of them were correct; she had a sense that she would know it when she finally saw the right answer. At the end of the excerpt above, Evelyn became aware of the fact that she was thinking in terms of “per half container” when she knew the answer should be “based on one.” This reminded her of the crackers and half-bowl example. However, she was not seeing, or at least not articulating, the need to scale the divisor up to a value of 1 in this context.

The interviewer conjectured that the features of the context, as Evelyn was imagining it, might have made the correct solution implausible to her. Specifically, that she might be imagining pieces of pie that were too large to fit in the containers. The interviewer’s next questions, about the relative size of the pie and the container, may have cued a change in her perception of the problem.

_Evelyn 4.13_ Int: How big are you imagining your container to be?  
E: Average container size (rising tone). Bigger than a quarter – bigger than a piece of pie (rising tone).  
Int: Your container is bigger than a piece of pie?  
E: Yes.  
Int: Is it bigger than a pie?  
E: No.
Int: Your container is smaller than a pie?

E: Definitely. I don’t know where you’re going with this line of questioning. (They both laugh). All I know, is if I have one and three-quarters of a pie…Ohhh, wait a second, the way this, this problem is phrased, shall we say, is, the given is, I only have half a container. And that’s my problem; I didn’t think about that. I only have half a container. (Long pause.) So what? Now I have to go back and -- I just don’t understand how to turn this into the story problem…I don’t know how to make this a reasonable question…I only have half a container, so…

The interviewer then asked her to draw arrows to indicate the pieces of the dividend going into half a container. Evelyn drew a new rectangle and partitioned it into two halves with a dashed line. She then drew seven pieces in one half of the new container and verbally identified them as quarters. She then drew an arrow from the seven pieces she just drew to the circle diagram she originally made to represent the dividend (see Figure 4.13). Evelyn was finally able to correctly conceptualize the problem by coordinating her new representation with her calculation.

**Figure 4.13**

*Evelyn Draws Seven Pieces of Pie in Half a Container.*
Int: Based on this [new drawing], what’s the answer to the division problem?

E: Seven. (Long pause) I mean if I’m looking at the picture, if I’m not thinking about it, but it’s not, it’s clearly 3 and 1/2 (indicating the results of her calculation).

Int: Or fourteen fourths.

E: Or fourteen fourths.

Int: And what are the units on that fourteen fourths?

E: Let’s see, I have one and three-quarter pies divided by half a container. So it would be pieces of pie per half a container.

Int: So let’s look at your [calculation]. Your answer is saying fourteen quarters per container.

E: (Pause) Oh yeah. I couldn’t see that. But I see it now. “Cause I got seven on one side so it’s fourteen quarters per container. Yeah. I realize why I had such a hard time with my story problem. It’s because, back to the given (indicating the problem statement) I didn’t conceptualize it properly to begin with. I was unable to think of a story that fit, trying to cram a bunch of stuff into half a thing. And I didn’t think about this (indicating the divisor in the problem statement) as half a thing. It makes no sense to put seven quarters of pie into half of one container. And because of that, I could not finish this problem. But if I had said instead, for example, I’m packing for a trip, and I filled half my suitcase, I only have half a suitcase left. And I have, I don’t know what I’d come up with for one and three quarters, but the point is that the story has to fit the numbers. And I have to see these two parts of the division problem as being meaningful within the story.

The concept projections Evelyn constructed in this section included knowledge, exhibited at in the previous section and at the beginning of this section, that the determination of the quotient based on a whole unit of the divisor. While she could express that knowledge in discussing the crackers and bowl example in the video, she did not readily transfer that knowledge to her own pies and container example. This was true even though she recognized that the two examples were similar.
Evelyn’s own analysis of the conceptual difficulty she experienced was that she did not see the divisor as “one half of a thing.” This prevented her from constructing a story problem that made sense to her. When asked to identify the physical units of the quotient, Evelyn replied, “pies per half container.” This point corroborates her analysis. Specifically, there was some reason that Evelyn was not scaling the given divisor (1/2) to a whole unit even though she had access to a quotient is based on a l-unit of the divisor conceptual resource. Evelyn had chosen a viable situation in which a story problem could have been constructed for the given quantities. Yet she experienced difficulty constructing a partitive division example with the given division problem.

Evelyn’s concept projections, at least at first, appeared to include components of measurement and partitive division. One possible source of the difficulty Evelyn experienced in constructing a partitive example for this division problem may have been an inability to align her previous conceptions of partitive division in the whole number context to this context in which the divisor was less than one. Additionally, she stated that “the story has to fit the numbers.” It is possible Evelyn’s conceptual resources needed to be reorganized before constructing a question that would motivate the need to find the unit rate in this context. The in-progress example of packing a suitcase that she extemporaneously constructed as a sense-making example appears to have made the meaning of a whole unit of the divisor more visible to her. She was able to recognize the quotient in her pies and containers context only after constructing a sufficiently detailed representation of the problem and coordinating that representation to the numerical value of the quotient.
Evelyn’s Engagement in the Representing Partitive Division of Two Fractions with Rectangular Area Task

In this final task set of the PD task sequence Evelyn (a) discussed examples of area models of division, (b) watched the *Partitive Division with Fractions Using Area Models* video (see Appendix H), and (c) constructed her own rectangular area models of partitive division with fractions. For Evelyn, this task set began one hour after the task set described in the section above. During the hour between these task sets, Evelyn engaged in representing multiplication with fractions as well as measurement division with fractions using rectangular area models. Evelyn’s engagement with the tasks in this section evidenced a developing ability to construct partitive division representations, determine the quotient, and articulate the correspondence between the mathematical statement of the division problem and her representations. This section briefly highlights Evelyn’s developing competence during the 30-minutes she engaged in the task set.

**Evelyn explained examples of partitive division with fractions.** The first task in this task set was to make sense of and explain sample representations of partitive division with fractions. Evelyn evidenced a developing understanding of partitive division as a unit rate in the following ways that persisted into the tasks that followed.

1. Evelyn described the partitive division process as one in which “we allocate things to other things” and began to look for features in the representations that would indicate what is being allocated and where it “fits into.” Evelyn stated that she was “crystal clear” about the dividend and divisor having different referents and different physical units. She explained that when she was having trouble with previous partitive division tasks, she was “stuck on the units should be the same for the dividend and divisor.”

2. Evelyn strengthened her *quotient based on a 1-unit of the divisor* conceptual resource. By *strengthened* I mean that the resource appeared to be cued more frequently and applied more productively. As an example of productive
application, Evelyn was consistently able to scale the given rate to a unit rate while using language such as “how much would fit into one,” and “at the same rate.” This conceptual resource for converting the given rate to the unit rate was not cued and used productively in the 22-minute pies-and-containers episode described in the previous section.

Evelyn demonstrated representational competence with partitive division. The last task of this final PD task set was for Evelyn to represent $1/2 ÷ 3/4$ with any kind of drawing she wanted to use. The following description of Evelyn’s engagement on this task demonstrates the depth of understanding she had developed for the unit-rate interpretation of partitive division and her ability to coordinate the IM procedure with a visual representation of the division process.

Underscoring her conception that the dividend and divisor “are different things,” Evelyn represented the dividend in blue using a circle diagram and the divisor in orange as a rectangular diagram (see Figure 4.14). Her construction began by drawing a divisor rectangle that established the value of one. She then partitioned the divisor rectangle vertically into quarters and, switching to a different color, identified the left-most three quarters as the space that the dividend will occupy.

Arguably, these are procedural steps that Evelyn was emulating from the PD examples and video demonstrations. In fact, Evelyn stated, “I’m just following the steps I learned in all this.” However, Evelyn’s language and several of her actions indicated that she was not simply following a formula. The novel aspects of her representation evidenced a unit-rate conception of division. While the use of different colors to distinguish the dividend and divisor was a feature of the PD examples, there was no use of different shapes in any of the drawn representations of operations with fractions in the PD content.
When Evelyn began to draw the dividend (as a circle) outside the divisor rectangle she stated, “and then I have some stuff over here.” She then tapped her pen on the dividend in the problem statement. After finishing the circle representing the dividend, she stated, “and I’m going to take half and stick it in here,” indicating the three partitions of the divisor rectangle that she had outlined in blue (see Figure 4.14). She then drew a half-circle centered within the perimeter of the three selected partitions of the divisor rectangle. These actions and language evidenced a concept projection with two important features: First Evelyn distinguished the dividend and divisor as separate entities with different attributes. Second, in coordinating the dividend and divisor, she evidenced a type of **inserting** activity.

Evelyn also departed from the procedure presented in the PD session by placing the dividend in the divisor prior to partitioning it into thirds. Perhaps she did not think about the need to partition the dividend when she drew the half-circle dividend in the divisor rectangle. However, Evelyn was able to recognize the problem, self-correct, and
refine her representation. Specifically, after drawing the half-circle in the rectangle she noticed something didn’t seem right.

_Evelyn 4.15_ E: That is weird, it doesn’t look right. One half divided by three quarters. I’m a little lost, on how this represents that, but anyway I’m just gonna keep moving forward with it. So I’ve got half a circle stuck into my three quarters and the question is how much would fit, _at the same rate_, how much would fit into a whole square.

Despite being somewhat disoriented by her representation, she was confident enough in her understanding to continue. Evelyn recognized that she needed to “take a third” of the half circle and add it to the fourth partition of the divisor rectangle (see Figure 4.14). She then questioned the origin of thirds as the partition size.

_Evelyn 4.16_ E: How did I get to a third, wow, okay. So I have, it looks like I have four thirds, but I don’t know; nobody’s asked me about thirds. I might’ve just messed up…but...All right, well let’s see. (Evelyn used the IM procedure to calculate the answer 4/6.) How did I get thirds instead of sixths. Dang it. (Evelyn mentally re-traced her steps.) Oh, I know how: Bam Bam Bam Bam. (Partitioning the circle she first drew as the original representation of the dividend, Evelyn exclaimed “Bam” as she drew each partitioning line.) Because there’s six pieces.

Partitioning the dividend before distributing it into the partitions of the divisor has the affordance of creating a one-to-one correspondence between the partitions of the dividend and the partitions of the divisor. By skipping that step, Evelyn made the coordination of the components of the problem more difficult because partitioning the half-circle generated pieces that were thirds with respect to the half-circle but were in fact sixths of the dividend. She was momentarily unaware of the origin of the thirds/sixths discrepancy, yet she was able to recognize the discrepancy, evaluate her representation, locate the problem, and back-fill the missing step. Evelyn completed the entire
representation process above with no comments or scaffolding from the interviewer.

**Summary of Case 3: Evelyn**

At the beginning of the PD task sequence Evelyn was able to distinguish between measurement and partitive division with whole numbers. She also recognized that the quotient is based on the divisor having a value of 1. However, her expressions of that knowledge were tentative, using qualifiers such as “presupposed” and “unless otherwise stated.” Evelyn’s conceptual distinction between measurement and partitive division appeared to be perturbed in the context of fractions. Evelyn also experienced difficulty applying her knowledge that the quotient is based on a whole when first constructing an example of partitive division with fractions. Her initial difficulty appeared to be related to coordinating the dividend and divisor as two different physical quantities. She was able to construct an *in-progress* example of the same division problem effortlessly.

Evelyn expressed multiple meanings of the term *unit*. As a science teacher, working with physical units was natural for Evelyn. However, she was also adept at using the term *unit* as a tool to communicate mathematical structure. She expressed flexibility with the referent unit in her communication as well as using the term measurement unit to clarify the role of the divisor in measurement division. With some scaffolding, Evelyn developed the ability to independently represent partitive division with fractions using rectangular area models by the end of the PD task sequence.

**Case 4: Mark**

Mark seemed to be indifferent to the PD content and made several comments
indicating that he was not sure if it was worth teaching. However, he understood the importance of discussing his thought-process while completing the task sequence. He was a willing participant in the study in that regard, and his particular perspective provided valuable data.

Mark’s engagement in the PD task sequence totaled two and a half hours over two separate meetings. The following sections provide observed development of Mark’s conception of division as he completed four task sets: (a) describing her intuitive ideas about fractions and the meaning of division, (b) learning to distinguish between measurement and partitive division with whole numbers, (c) discussing the meaning of partitive division when the divisor is a fraction, and (d) representing partitive division with fractions using rectangular area models. The following sections describe purposively selected moments in Mark’s engagement in the tasks and appear in chronological order.

**Mark’s Initial Conceptions of Fractions and Division**

This first task prompted Mark to express his informal ideas about fractions and division.

*M.4.1*  
Int: If a student said, “What is a fraction?” What is your own version of what a fraction is?

M: There’s different ways of introducing it. One would be you know, showing whole and then showing the pizza with a slice taken out, and that slice is a fraction of the whole. So a partial representation of a whole. And of course, I would mention fractions could be larger than the whole, meaning taking the unit several times. Another way I’ve explained what a fraction is — and this might come from [the online course], but it’s a division problem that you just, aren’t — you’re not doing long division on it yet, so it’s like a division problem that uh, we haven’t finished yet. // I guess [that description] meshed with how I thought of fractions as a child, so I liked that a lot, so I have been using that
Int: Okay, also, you said, about pizzas that you’d show them more than one.

M: Yeah, I mean, because, you know, one-fourth, that’s a fraction, but what about five fourths (falling tone). That’s also a fraction. And I don’t like showing just one kind of a thing because then, if it’s the first time somebody sees it, sometimes gets pigeon-holed as to their understanding and then so they’re just thinking of fractions as something that’s less than a whole.

Mark’s statements evidenced a multi-faceted conception of fractions. He also exhibited a pedagogical commitment to teaching multiple interpretations of fractions. The interviewer then asked Mark about division.

Mark 4.2

Int: Let me ask you the same question about division: What if a student asked you what division is; what would you say?

M: Um. I would say…oh, that’s a good one; I haven’t been asked that. It’s just (laughing) yeah, it’s kind of a very basic thing. So it’s an operation – I wouldn’t use the word ‘operation’ if I’m talking with a kid, um (long pause). I guess one way of explaining it would be saying, if you wanna see how much does a…Hm. Yeah, you kinda got me on the spot. I’m trying to think of an easy way to explain it. I’m drawing a blank right now.

Int: I’m not assuming it’s easy.

M: Yeah. Um. I mean, you know, it’s an operation that you can do with two numbers to see what part of a, one number – that the other number takes up. It’s the inverse of multiplication. Um…or another way to think of it is uh, multiplying by a fraction…where it’s a unit fraction, you know with the other number on the bottom. None of these are very good. So…

Int: I think they’re fine.

M: Yeah, but I would not use any of them to explain what division is to a kid. I would have to think about it.

Framing the question as if he were tasked with explaining division to a child, Mark seemed to be in search of an age-appropriate grounding metaphor but could not think of one. In expressing division as a numerical operation, Mark’s statement “what
part of a number another number takes up” implied a measurement or renormalization conception. Notably, he did not use metaphors such as break up or split into pieces, nor did he reference the action of sharing. While Mark might express part-whole conceptions at other times, the excerpt above indicates that he is at least tacitly familiar with multiple interpretations of fractions and thinks of them flexibly.

*Mark’s Engagement in the Introduction to Partitive and Measurement Models of Division Task*

This task began with Mark viewing the 15÷5 Ball video (see Appendix H). When the video ended, the interviewer asked Mark if he had any questions about the content of the video. Mark’s initial reaction was that he would not teach both interpretations at the same time. He found the mathematics content “straight-forward” and wondered if one interpretation might be better than the other for dealing with remainders. Mark noted that we all have “natural experiences” with partitive division since childhood but that students at the college should be able to apply measurement division in their work.

*Mark’s Engagement in the Partitive Model of Division with Fractions Task*

This task began with Mark viewing the Crackers and Bowls video (see Appendix H). When the video ended, the interviewer asked Mark if he had any questions or comments about mathematics or the presentation. The interviewer then attempted to engage Mark in discussing why he perceived the determination of what’s in one bowl to be simple in the case of multiple bowls, but problematic in the case of a fraction of a bowl:
**Mark 4.3**

*Int:* Okay, questions or comments about this video?

*M:* Definitely the last problem is hard to get. And that’s because in the previous ones you see the quotient in the bowl. But here, I don’t see it, right? I see three-quarters still. Visually, I wish there was a different way to represent that. In the [previous example in the video] you could focus on what is the solution to the problem; the solution is whatever’s in the bowl.

*Int:* Even though there are three bowls here (referring the previous example).

*M:* (Confidently) The solution is whatever is in one bowl. This [previous example] is like three people, so we divided it for everybody. But here the solution is – I can only see half a bowl (rising tone). So I don’t know…I’d have to make like a mental leap and imagine doubling it.

The interviewer attempted to surface Mark’s thinking about that mental leap by suggesting a revision to clarify the video.

**Mark 4.4**

*Int:* Some of the teachers suggested, well why don’t you just draw the rest of the bowl and three more quarters in there to illustrate it.

*M:* But then, where does it come from? (Laughing) This is a hard one, ‘cause you do need to imagine it. If you just put the other half of a bowl there suddenly with three more quarters in it (rising tone) then the question becomes like, “Where did this come from?” Maybe the answer is that there needs to be an extra step. It’s fine to say you have three quarters in half a bowl but most people will want to know, what does, if we could relate that to what’s in an entire bowl, because that’s what we’re using for other simpler problems. Which is where you come to the problem of just multiplying it by two. So I don’t know if it’s even worth it teaching it this way, instead of just, rote, um, forcing everybody to when they see division by a fraction to just flip and multiply.

In reference to the three-bowl divisor, Mark was quick to state that the quotient was whatever is in one bowl. When the divisor was half a bowl, Mark knew that the quotient was still whatever is in one bowl. His statement, “Which is where you come to the problem of just multiplying it by two” indicates that he was mentally able to
coordinate the IM procedure with the physical representation of the problem. The conceptual difficulty he evidenced appeared to be in finding a satisfactory explanation for students as to why the half-bowl gets doubled. This was true despite the fact that “The solution is whatever is in one bowl” was a trivial statement for him in the context of whole bowls.

Mark’s Engagement in the Representing Partitive Division of Two Fractions with Rectangular Area Models Task

In this final task set of the PD task sequence Mark (a) discussed examples of area models of division, (b) watched the Partitive Division with Fractions Using Area Models video (see Appendix H), and (c) constructed his own rectangular area models of partitive division with fractions.

For the first task, the interviewer asked Mark to look at a rectangular area model representing $1 \frac{1}{2} \div \frac{3}{5}$ (see Figure 4.15) and see if he could explain it. Mark exhibited a conceptual resource that indicated his concept projection of partitive division (at least in this task) was based on conceptualizing the quotient as an intensive (or derived) quantity:

Mark 4.5  Int: Now the dividend is a mixed number. Does this representation make sense?

M: Um, yeah, we have three fifths of something, and we need to put one and a half into that three-fifths of something. So, it kind of makes sense but it’s a difficult way of – it makes sense if I follow along from examples of the cracker and the bowl. I have three fifths, and I need to div -- place this evenly into three fifths of something, and then I need to figure out, if I kept doing it until I had five fifths, um, what would be my, I guess density or…thing per whole.

Because of his background in physics, Mark is likely able to construct concept
Figure 4.15

*A Rectangular Area Model Representing $1 \frac{1}{2} \div \frac{3}{5}$.

projections of density across a wide variety of contexts. Here Mark used density as a conceptual resource in making sense of what he previously described as “the leap” one needs to take in order to construct the unit rate from a given rate.

Mark and the interviewer watched the *Partitive Division with Fractions Using Area Models* video. The interviewer asked Mark to comment on it. Mark asked to view the video a second time before commenting:

*Mark 4.6*  

*Int:* What strikes you about the math and the representation here?

*M:* I get it now because I saw the other one. Had I not seen the one with the crackers and bowl I’d be more confused by this one. It’s a little hard to think about throwing three fourths into one half. In the bowl video you went through steps until you got to the difficult problem. I think it would make sense to do that with area also.

*Int:* Yeah, the crackers went into *three* bowls at first.

*M:* But you could even do three quarters into *one* before doing this one.

*Int:* Is there something – what’s pivotal about that to understanding
how this division problem works?

M: I think it’s because we’re now using like an abstract number but we’re doing something physical with it. It would make more sense if I was putting that something into a whole square, which is easier to conceptualize (pause) and then, okay what if I only put it in half of that square. Then how many – it makes it an easier leap if I first have three fourths in a whole square: What do I have? Well I have three fourths per share.

Mark again referred to the transformation of the given rate to the unit rate as a “leap.” He suggested a pedagogical strategy for making that leap easier by adding an intermediate example, in which the divisor is one, to a sequence of examples from whole number divisors to fractional divisors.

Mark’s first two-hour PD meeting ended here. Returning to this point in the task sequence 7 days later, the interviewer prompted Mark to construct rectangular area models representing $\frac{3}{4} ÷ \frac{1}{2}$ and then $\frac{1}{2} ÷ \frac{3}{4}$ using partitive interpretation and think-aloud protocol. In his first construction, Mark said he was going to “try to match” the example above (see Figure 4.15). He drew a rectangle and partitioned it into fourths. He then drew “1/4” in the first three partitions, inadvertently representing $\frac{3}{4} ÷ \frac{3}{4}$ (see Figure 4.16) and explained.

Mark 4.8 M: I’m going to start with my whole. I’m going to split it into fourths so I could start with the three fourths (long pause) of something. Okay, so I’m dividing $\frac{3}{4}$ into $\frac{1}{2}$. So, what I care about is, if three fourths goes into a half of something, how much would go into a whole…(long pause) it looks like I need to draw two more partitions here…(long pause).

Int: I can’t quite see the detail. Are those one-quarters? Do you have three instances of quarters in there?

M: Yeah.

Int: Okay. Can you tell me what thinking of, is happening there?
Figure 4.16

Mark Inadvertently Represents $\frac{3}{4} \div \frac{3}{4}$.

M: At first I wanted to see, um, what a whole looks like with my three-quarters…split…uh… I guess what I was trying to look at is the problem above where the three halves were just drawn by themselves. But I don’t know if that’s necessary. I mean, um, logically I can talk through this because I’m saying if I have three quarters of something in a half, then how much do I have in a whole. That would be the answer to the problem, right? So I don’t need this (Mark places his pen over the fourth partition of his divisor rectangle to block it (see Figure 4.17). If I had just started with this (meaning the three visible partitions), with just my three quarters partitioned like this and then I said, Okay, well now I’m gonna double it, and then count the number of quarters, and that would be my answer of how much is in a whole.

It had been a week since Mark had seen the demonstration video and discussed the previous example (see Figure 4.15). The interviewer did not review the previous material. Additionally, this was the first time Mark had constructed his own representation of partitive division in the PD tasks. So it is not surprising that he drew his initial unit rectangle as the referent unit for the dividend instead of the divisor. Picking up
at this point in the task sequence with fresh eyes, Mark apparently attended to the dividend quantity first. His representation did not provide the quotient, but he was able to mentally construct the quotient. Mark stated, “logically I can talk through this” and then proceeded to revise his representation so that the dividend quantity could simply be doubled. This approach was consistent with the in-progress interpretation in that he did not refer to the divisor either in speech or in his drawn representation. In fact Mark stated that it would have been easier if he had just started with the dividend (meaning, starting without drawing the referent unit for the dividend quantity) and doubled it. This is opposite to Mark’s exhibited perception of the previous example (see Figure 4.15) a week earlier. Mark’s first comment in that task was “we have three fifths of something and we need to put one and a half into that three fifths of something.” This statement indicated that Mark attended to the divisor (possibly first) and perceived the dividend and divisor as composite quantities. He also coordinated the dividend and divisor quantities with a
kind of *inserting* mental action, stating “place this evenly into three-fifths of something.”
The data does not indicate why he took a different approach to this task, but it is possible
that dividing by 1/2 simply invoked doubling as the solution, obviating the need to
coordinate quantities, partition, and iterate.

In the next and final task of the PD sequence, representing $1/2 ÷ 3/4$ with partitive
interpretation, Mark began by representing the dividend in isolation as a single rectangle,
stating, “I start with a half.” Similar to Ivy, he could not proceed from that starting point
and scratched it out.

*Mark 4.9*

M: So a half is going into three fourths of something, so I need to
draw that something before I – maybe that’s what the problem is.
That’s what I did wrong.

Int: In the first one?

M: Yeah

Int: Could you explain a little more…

M: So, uh, what I’d like to see is three fourths of something (rising
tone) and then I could put that half in there. Would that still be
partitive, though? I don’t know (laughing); I’m confusing
myself…What’s left for me to figure out is if I have half of
something put into three quarters of a unit, how much would I have
in the whole unit.

Mark repeated his last statement several times before starting a new drawing. In
his second drawing he represented a 1-unit of the divisor with a rectangle, partitioned it
into fourths, and drew “1/2” centered in the left-most three partitions with a new color.
Mark drew an arrow from the dividend in the problem statement to his representation (see
Figure 4.18). He was then unsure how to proceed at first, but then partitioned the 1/2 into
sixths and “added one more” to complete the problem.
Figure 4.18

*Mark’s Revised Representation of 1/2 ÷ 3/4.*

Mark’s statement, “what I’d like to see is three fourths of something and then I could put that half in there” is consistent with a conceptualization of the quotient as a unit rate and possibly as an intensive quantity (such as density). However, Mark’s language may have been simply directed at the affordances of the representation, rather than his conceptualization of the quotient. The interviewer wondered if Mark was thinking in terms of an in-progress interpretation or a unit-rate interpretation. Attempting to surface Mark’s interpretation, the interviewer asked Mark to recall that the word *of* can represent *multiply*, and asked if he could think of a word that would represent partitive division.

*Mark 4.10*  
Int: Is there a word you can think of that explains or models the process?  

M: Um. Into? I’m not sure if I can come up with something clear, and even if you do, I think with these factional division problems it’s hard to conceptualize in your head. It’s harder than multiplication because here we have to imagine something that’s not in front of us. With multiplication you start with the whole and you take a part of it. So you can imagine one to begin with. Whereas *here*, you start with…something, and then you take another part of it but then you have to imagine this imaginary whole unit and then
answer the question about the whole unit.

Mark and the interviewer discussed the unit-rate interpretation later in this meeting. By the end of the PD task sequence, however, it was unclear if Mark was primarily thinking in terms of the in-progress or the unit-rate interpretation. Given comments such as “it’s hard to conceptualize in your head,” it may be that Mark’s concept projections of partitive division with fractions in this task set were not clearly organized around one interpretation or the other.

**Summary of Case 4: Mark**

Mark exhibited a flexible conception of fractions and had no difficulty distinguishing between measurement and partitive division with whole numbers. Engaging in partitive division with fractional divisors, Mark acknowledged that determining the quotient required “a leap” because transforming the given divisor to 1 required imagining something that wasn’t there. Mark’s concept projection of the partitive quotient as a *density* may have been context-sensitive. He constructed it in the context of physical crackers and bowls but did not later reconstruct it in the context of drawn area models. In the context of area models, Mark exhibited an understanding of how the model coordinated with the numerical statement of the problem at the end of the first meeting. He also evidenced a *unit rate* or possibly *intensive quantity* conception of the partitive quotient at the end of the first meeting. One week later, however, his concept projections of partitive division with fractions appeared to be more attentive to an in-progress interpretation of the quotient. Consistent throughout all of Mark’s concept projections of division was the conceptual resource that the quotient is based on one
whole unit of the divisor.

**Case 5: Malory**

Malory was comfortable with think-aloud communication and was willing to express her questions and ideas extemporaneously. Many of the tasks took considerable time for Malory to complete. Completing the PD task sequence took three hours in total. In part that was because she experienced difficulties with some of the tasks; in part it was because she was communicative and discussed her mathematical sense-making in detail.

Malory’s engagement in the PD task sequence totaled two and three-quarter hours over two separate meetings. The following sections provide observed development of Malory’s conception of division as he completed four task sets: (a) describing her intuitive ideas about fractions and the meaning of division, (b) learning to distinguish between measurement and partitive division with whole numbers, (c) discussing the meaning of partitive division when the divisor is a fraction, and (d) representing partitive division with fractions using rectangular area models. These sections appear in chronological order.

**Malory’s Initial Conceptions of Fractions and Division**

This first task in the PD task sequence (see Appendix E) asked Malory to describe her informal ideas about fractions and division. Here and throughout the study, Malory expressed fractions as being a form of proportion.

*Malory 4.1*  
Int: What are your own ideas about what a fraction is?  
M: I would say a fraction is a proportion when you do not have a whole number. Whole numbers can be made to be fractions…but a fraction I would say is a proportion comparing the numerator to the
denominator.

Int: You’re thinking a proportion of the numerator to the denominator, so how would that be different from a ratio?

M: Well ratios can be considered a form – I mean they can be made into a fraction. So they’re again, comparing -- ratios can be part of a whole or two different wholes being compared. Whereas a fraction, if you do it graphically (rising tone), I would say you can see it as the same thing, but when you write it down…the numerator over the denominator…it is still a ratio.

Int: And you also said something about how a whole number can be…turned into a fraction; did I hear that right?

M: Right. So, a whole number such as ten over one is a fraction. Just - - there’s an assumption made that it is over one…when you see it, because it’s a division problem.

Int: So, if a student says, um, ‘Well what is division?’ what would you say?

M: Uh, division is dividing…some numerical…number or some numerical compound, by another numerical number or numerical compound. Meaning, it’s breaking up the original.

Int: Breaking up the –

M: Breaking up the original or breaking up the dividend.

Malory expressed a variety of interpretations for the meaning of a fraction. While it’s not clear from these data what conceptions Malory had for the terms ratio and proportion, it is clear that her conception of fractions, as expressed here, attended to the relationship between the numerator and denominator. Ten was the only numerical example that she gave, implying that Malory’s conception of fractions was not strongly associated with part-whole thinking. Last, Malory’s description of division as “breaking up” included the specific point that the dividend was the object upon which the operation acted.
Malory’s Engagement in the Introduction to Partitive and Measurement Models of Division Task

This task began with Malory viewing the 15÷5 Ball video (See Appendix H). When the video ended, the interviewer asked Malory if she had any questions about the content of the video. Malory had no questions about the difference between partitive and measurement division as presented in the video:

Malory 4.2  Int: Focusing on the math content, do you have any questions?
M: This is very straight-forward to me. It’s either dividing the fifteen pieces up, or it’s how many 5’s can go into 15.

Malory’s language describing partitive division as “dividing up” and measurement division as “go into” suggested that she was not explicitly aware of the two interpretations of division as mathematical constructs at the start of the PD sequence. The interviewer asked how she might describe the two interpretations in a classroom:

Malory 4.3  M: Um, I think I would start very similar to what you did. Fifteen divided by five, and the way you showed it, and then I’d do one that isn’t as clear with a remainder.

Malory did not construct a different description for the two types of division. The interviewer conjectured that Malory’s cuing of the remainder might have been an indication that she had a basic understanding of the two interpretations but needed more examples. The interviewer therefore proceeded to the next task.

Malory’s Engagement in the Partitive Model of Division with Fractions Task

This task began with Malory viewing the Crackers and Bowls video (see Appendix H). In the discussion that followed, the interviewer and Malory had difficulty
understanding each other. The interviewer began by asking the following.

**Malory 4.4**

Int: What do you think about the math content here as it’s unfolded?

M: I think it’s really simple until you get to the last one, and I understand how you got six. The four was hard to understand.

Int: The four…

M: Six over four. So if you double it I understand that, you have six. How do you get the four? …you know what I mean? I see four and I see two, which one do I choose, but it doesn’t have anything to do with that top equation it has to do with the half portion…doubling the half portion.

The interviewer tried to understand the source of Malory’s questions. Being unsuccessful, he invited her to express her question on the worksheet (see Figure 4.19):

**Figure 4.19**

*Malory’s Revised Representation of 1/2 ÷ 3/4.*

**Malory 4.5**

M: Okay if I have three over two is what it looks like I have there. That’s my answer. But if you double it, are you saying three over two plus three over two? Or three over two times two?

Time constraints concluded this section with the interviewer still unsure of the nature and source of Malory’s questions. Multiple viewings of the video suggested it is likely that Malory had interpreted the information presented in the video in a way that
was unintentional. Specifically, the quotient \( \frac{3}{2} \), the answer to the problem, was written in a central location on the screen. This location was, without design, proximal to the half bowl containing three quarter-crackers (see Figure 4.20).

**Figure 4.20**

*Malory Misinterprets the Meaning of the Quotient.*

It is possible that Malory interpreted the \( \frac{3}{2} \) as representing the given rate (the half bowl containing three quarter-crackers) instead of the unit rate (the quotient).

Specifically, she may have interpreted the 3 in the numerator of the quotient as indicating the three quarter-pieces of cracker in the half-bowl, and the 2 in the denominator as a 1/2, indicating that there was only half a bowl. This would explain why she was trying to double \( \frac{3}{2} \) instead of \( \frac{3}{4} \), and why she could not see “where the four came from.”

**Malory’s Engagement in the Representing**

**Partitive Division of Two Fractions with Rectangular Area Models Task**

In this task set of the PD task sequence Malory (a) discussed examples of area models of division, (b) watched the *Partitive Division with Fractions Using Area Models* video (see Appendix H), and (c) constructed her own rectangular area models of partitive
division with fractions. This 49-minute task set occurred at the start of Malory’s second meeting, 12 days after the first meeting. By the end of this task set Malory was able to construct a representation of partitive division with fractions, but developing that ability required scaffolding through all the tasks. The following sections highlight the challenges Malory experienced and the revisions to Malory’s representations that led to a correct representation by the end of the task set.

**Malory perceives randomness in distributing the dividend.** While watching the *Partitive Division with Fractions Using Area Models* video, Malory was perturbed when the dividend was represented as originating from outside the divisor rectangle. She referred to the placing of the dividend into the divisor rectangle as seeming “random.” Lacking a sense-making coordination between the representation of the dividend and the divisor, Malory was unable to see why the dividend was iterated after being placed into the divisor. She was also unable to coordinate the initial distribution of the dividend into the divisor. Immediately after watching the video, Malory remarked.

*Malory 4.6*  
M: How is it that three fourths just…gets plopped into here. Because what if it was something like five sixteenths?  
Int: It would still get plopped in there.  
M: But it would be five sixteenths and five sixteenths, correct?  
Int: Yeah, that’s right.  
M: See what still doesn’t make sense is it looks like you randomly assigned three fourths at the top and three fourths at the bottom.

The interviewer decided to proceed to the next task (see Appendix E, 8b) hoping that the mapping feature of that example would address her perception of randomness. In this representation, yellow arrows indicate the relationship between the dividend in the
problem statement and the dividend as it is placed into the area representation (see Figure 4.21). However, she still asked, “How do you randomly assign each one-half?”

Figure 4.21

*A Rectangular Representation of 1 1/2 ÷ 3/5.*

In her think-aloud of what she saw when looking at this example Malory fleetingly mentioned, “two and a half fit into one” but then seemed to change her mind. Trying to capitalize on that moment, the interviewer asked her to identify one in the representation.

*Malory 4.7: *Int: I just heard you say, “fit into one.” Can you show me with a pen where one is in the diagram?

M: One would be here. (Malory started to draw a bold perimeter around the left-most four partitions and then stopped.) Sorry, I’m looking at something else. Um, so one would be here. (Malory then drew a perimeter around the left-most two partitions.) One-half plus one-half equals one. And here’s another one (drawing a perimeter around the third and fourth partitions), and a half so that would give me two and a half. Two and a half fit into five fifths, so you’re asking how many one-halves fit in three fifths.

Malory identified the referent unit of the dividend rather than the divisor, but the question was not properly worded with specific reference to the divisor. She framed the
representation as asking how many one-halves fit in three fifths. While this is not the representation of the quotient, her framing evidenced a coordination between the problem statement and the elements of the area model.

In order to help Malory see the quotient in the area model, the interviewer asked her to describe the partitive interpretation of division. Malory repeated, “how many one-halves fit in three fifths.”

The interviewer then hoped to improve her coordination of the dividend and divisor by noting that the left-most three one-halves were yellow, and the right-most one-halves were magenta. He asked if she could explain the reason for the difference in colors. Though only one and a half minutes had elapsed since her previous response, her response in this moment evidenced that she was correctly interpreting the drawing.

Malory 4.8:

Int: Can you explain what that color coding is trying to show?

M: That’s what the blue shows you – the yellows are the three fifths, the…divisor (rising tone) and the one half is what else fits into, um, five over five or into a one unit. The one unit thing I think is the key point.

Malory may have meant to say dividend when she said divisor. She referred to “the yellows” which comprised the given dividend and “three fifths” which comprised the given divisor. In either case, she appeared to be coordinating the problem statement and the elements of the representation instead of seeing it as a random assignment.

Malory exhibits difficulty coordinating the dividend and divisor. In the next task, Malory was prompted to represent 1/2÷3/4 using partitive interpretation. Malory began by drawing a 1-rectangle and partitioning it into fourths. She drew a perimeter around the top three partitions while saying, “So what we’re working with I believe
would be the three out of four.” These steps were productive, and she performed them with a sense of routine. She then picked up a different color and wrote “1/2” in all four partitions, while stating, “and then you’re distributing one half” (see Figure 4.22, left panel). She then realized something was wrong.

Malory 4.9  M: But in the three fourths, that would give me…This is where it gets confusing, because logically what it says is that I’m looking at three out of four (rising tone). So I should have one, two, three halves…in three fourths. But in the one, I have four halves.

Figure 4.22

Malory’s Initial and Final Representation of $\frac{1}{2} \div \frac{3}{4}$ Using Partitive Interpretation.

Malory summed the three instances of one half within the perimeter that she had shaded and wrote “3/2 in 3/4” next to the representation. Her placement of “3/2 in 3/4” seemed to spatially relate to her shaded perimeter indicating 3/4. She then wrote “4/2 in 1” in a location that seemed to spatially relate to the whole rectangle (see Figure 4.22, left panel).

Malory had just constructed two answers in the same representation. She may
have been relying, in part, on emulating the steps in the previous example. For instance, she iterated “1/2” in each partition, as in the previous example, but that was not the correct value to iterate in this case. However, she did correctly indicate the given divisor as a subset (3/4) of the 1-rectangle, even though the divisor had a different value than it did in the previous example. This indicates that she was not simply copying the previous example.

The interviewer directed Malory’s attention to the previous example and noted that the dividend had been partitioned before it was distributed. Malory replied:

_Malory 4.10_ M: I see what I did. Does that mean I have to divide…see I’m trying to mimic this one (indicating a representation of 3/4÷1/2, which did not require partitioning of the dividend) ‘cause it wasn’t an improper fraction. Does that mean I have to divide one half by three?

Malory then corrected her representation (see Figure 4.22, right panel). Her statement, “‘cause it wasn’t an improper fraction” suggested that she had interpreted the color-coding feature of the previous example in a way that was not intended. Specifically, she may have interpreted the yellow circle around the dividend in the problem statement and the yellow lines connecting that dividend to its partitioned segments in the representation as indicating that the dividend had been partitioned because it was an improper fraction. In fact, the dividend required partitioning because the divisor was not a proper fraction. The fact that 1 1/2 can be represented, and sometimes needs to be represented, as 3/2 may have been a familiar resource in Malory’s conceptual ecology; a resource that spuriously explained the yellow features of the representation.

At this point, the interviewer wanted to test Malory’s knowledge that the quotient
is determined by the iterations of the scaled divisor, not the given divisor. Playing the role of a confused student he said:

_Malory 4.11_ Int: I see you took that one-half, which is the dividend, and you cut it into three pieces so you could share it among the three fourths of something. My question is, Why do you have to continue and do that one more?

M: That was my question last time (laughing). Um, because you’re looking at a proportion of one (rising tone). So you always have to go back – proportion back to the one. That would be my only explanation. It doesn’t make a lot of sense visually, but…according to this method, that’s what we’re always looking at. Fractions are proportion to one.

Malory’s knowledge that the quotient is determined by the iterations of the scaled divisor, not the given divisor, was present but seemed to lack conviction. Her concept projection in the excerpt above appeared to incorporate an _answer-is-based-on-one_ conceptual resource, but in a way that evidenced rule-based thinking. Her statement, “according to this method” implied that her _answer-is-based-on-one_ conceptual resource was not playing a strong inferential role. Also, her statement that, “It doesn’t make a lot of sense visually” implies that her concept projection was only loosely coordinating with the representation. In other words, her perceptions of the components of the representation were not productively associating or integrating with inferences of the conceptual resources she was activating.

**Malory coordinates the dividend and divisor.** Malory had reached the end of the PD sequence. Given that Malory was showing progress, the interviewer wanted to continue with one more task. He suggested Malory try a problem that she had brought with her to this second meeting, that of representing \( \frac{3}{4} \div \frac{2}{3} \).

Malory began with a 1-rectangle, partitioned it into thirds, and identified two of
the thirds in her drawing while saying, “I have three thirds and I’m dealing with two thirds.” Thinking that the dividend should be placed into all three partitions, she partitioned the dividend by three and wrote “1/4” in all three partitions (see Figure 4.23, left panel).

Malory 4.12  

M: So in one…how many…I would have to divide the three fourths, now, divided by three (rising tone). So that would be one fourth and one fourth and one fourth. And that gives me three fourths.

Int: You lost me there.

M: Well last time I had to divide the…(she checked her work from the previous segment)…oh, the one-half by three (rising tone). But I guess it’d be by two (indicating the two partitions she had labeled as comprising “2/3”) because I’m…(rising tone) See that’s where it gets confusing, are you doing it between one and two but you’re only dealing with two here, so then it would be three fourths divided by two (rising tone) (Malory indicated the two partitions she had labeled as comprising “2/3.” See Figure 23, center panel), ‘cause you’re…but you’re actually putting it in three slots. Do you see why that’s confusing? (Malory re-partitioned the dividend into two units of 3/8 and started a new representation. See Figure 23, right panel). Wow. That’s the step I was missing, I think.

Figure 4.23

Malory’s Initial, Intermediate, and Final Representation of $3/4 ÷ 2/3$ Using Partitive Interpretation.

Note. Left panel: Malory iterated an incorrectly partitioned dividend. Center panel: Malory revised the quantity to which the dividend should be partitioned. Right panel: Malory constructed a new representation after correctly partitioning the dividend.
In this final task, Malory improved her ability to represent partitive division with a proper fraction as the divisor by partitioning the dividend before distributing it into the divisor partitions. However, at the start of this representation she still evidenced tentative coordination between the given divisor and the scaled divisor. She initially partitioned the dividend with respect to all three partitions of the scaled divisor, but then self-corrected. While the correction was spurred by the interviewer’s comment, “You lost me there,” she independently realized that the dividend should be partitioned and distributed in coordination with the given divisor. She then revised both her calculation of the partitioning and her representation with no further comments from the interviewer.

**Summary of Case 5: Malory**

Malory had a unique perspective on fractions, referring to them as proportions throughout her engagement with the tasks. She experienced many of the same conceptual difficulties described in the previous cases as the teachers were introduced to new concepts. Malory also experienced unique challenges engaging in the tasks. She made connections between task features that were visible to her but not intended. Sometimes this was productive and sometimes it was unproductive. In representing partitive division with fractions, Malory experienced difficulty making sense of the representational structures presented in the new course content (see Appendix A) and PD task sequence (see Appendix E). At the end of the PD task sequence Malory was able to represent partitive division with fractions using the method presented to her. However, even then Malory appeared to be following procedural steps in her construction of the representations rather than engaging in them as a sense-making activity.
Case 6: Hannah

Hannah kept her verbal engagement limited to short responses to the interviewer’s questions and did not engage in think-aloud communication. She completed the tasks without apparent conceptual difficulty, exhibiting only brief moments of perturbation. Therefore, Hannah’s case did not generate much data for this study. The following sections describe purposively selected transcripts from Hannah’s engagement in the PD task sequence.

Hannah’s engagement in the PD task sequence totaled two hours and comprised the first meeting. The following sections provide observed development of Hannah’s conception of division as she completed four task sets: (a) describing her intuitive ideas about fractions and the meaning of division, (b) learning to distinguish between measurement and partitive division with whole numbers, (c) discussing the meaning of partitive division when the divisor is a fraction, and (d) representing partitive division with fractions using rectangular area models. These sections appear in chronological order.

Hannah’s Initial Conceptions of Fractions and Division

This first task in the PD task sequence asked Hannah for her informal ideas about fractions and division. Her responses were brief and ended in long pauses.

Hannah 4.1

Int: What are your own ideas about what a fractions is?

H: Um, I would tell them a fraction is a portion of a whole (rising tone)…and maybe an equal portion of a whole (rising tone). (Long pause)

Int: So, it’s got to be equal. What about division; what’s your intuitive idea about what division is?
H: Um, breaking something down into smaller, equal portions (rising tone). (Long pause)

Int: That kinda sounds like what you said a fraction is. Do you see a fraction as being a division problem…or related to it?

H: Yes. (Long pause)

Int: So how do you see it related, or…

H: Well, I mean a fraction divides a whole…into equal pieces (rising tone). I don’t know if I’m doing…if I’m using circular reasoning right here but…(she laughs).

Hannah expressed part-whole language throughout this segment. She did not provide quantitative examples of fractions. Her description of division in this segment was similar to her description of fractions, and she did not appear to be interested in expounding upon any of her initial responses.

**Hannah’s Engagement in the Introduction to Partitive and Measurement Models of Division Task**

This task began with Hannah viewing the $15 \div 5$ Ball video (see Appendix H). When the video ended, the interviewer asked Hannah if she had any questions about the content of the video. She immediately picked up and correctly used the language “measurement tool” and “unit rate” to distinguish the two interpretations. Additionally, she evidenced a conception of the two interpretations as the inverse of the multiplicative relationship between the number of equal-sized groups and the size of the groups.

**Hannah 4.2**

H: It’s the same problem, essentially, like fifteen divided by five equals three, but, um, are you just basically switching around the group and the unit rate? – I don’t know if that makes sense.

Int: When you say “switching around,” what are you switching?

H: For partitive I think of a story problem like the teacher has fifteen
and she wants to give it, she has five groups and so there’s gonna be three units per group. When you switch it around to measurement there won’t be a unit on that, it’s just a number, right?

Although Hannah seemed to understand the two interpretations, she wondered about their application.

_Hannah 4.3_ H: Is this just basically like for word problems? I mean I’m just trying to think why I’d look at it – it would only be word problems, right? I mean if I were to look at a division problem, just straight-up numbers, I wouldn’t look at it like, “I could solve this…partitive” - - I’d just do algebra straight across.

**Hannah’s Engagement in the Partitive Model of Division with Fractions Task**

This task began with Hannah viewing the *Crackers and Bowls* video (see Appendix H). The interviewer then asked if she had any comments or questions. She asked to see the last example again, that of dividing by one-half. Hannah was able to see that the half bowl in the physical situation was the divisor in the numerical statement of the division problem.

_Hannah 4.4_ H: (To herself) Three fourths divided by half. (Aloud) So you always want to know what it would be per full share?

Int: Yeah. What makes you say that?

H: Well I’m not even trying to think about dividing the numbers in my mind, but if you have the three that are divided into the half…I was like, Okay, why would you use the six? But like why would you multiply by two, but then it’s because, oh, that half was only a half share and I’m just assuming we have to know this out of what it would be out of a full share.

The interviewer attempted to surface some of the conceptual resources that Hannah might be using to construct her understanding.
Hannah 4.5  Int: Let’s say you were to remake this video. Can you think of a way to make it better?

H: Showing it more clearly, or something?

Int: Yeah, is there a better way to show it?

H: It makes sense to me the way it is.

Int: Wouldn’t it be better if at the end of the video we brought in a second half a bowl and joined it with this one to make a whole so you could see there’s six pieces in the whole bowl.

H: I mean I did that in my head, but yeah.

Hannah’s Engagement in the Representing Partitive Division of Two Fractions with Rectangular Area Models Task

In this final task set of the PD task sequence Hannah (a) discussed examples of area models of division, (b) watched the Partitive Division with Fractions Using Area Models video (see Appendix H), and (c) constructed her own rectangular area models of partitive division with fractions. Hannah was able to view Example 8b and describe all the steps. Her description included the partitioning of the dividend, the distribution of the dividend into the given divisor, an awareness that the quotient was represented by the scaled partitions, and that the large rectangle composed of five partitions represented “the whole.”

After watching the Partitive Division with Fractions Using Area Models video Hannah remarked that it was the same as the Crackers and Bowls video because “you have to figure it out for the whole share.” She had no questions about the video.

The time period for the first meeting ended before Hannah started the final task of constructing her own representations of division with fractions using rectangular area
models. She completed this task on her own. A review of her representations indicated that she constructed them without apparent difficulty. She used color to distinguish iterations of the given dividend and iterations associated with scaling the given rate to a unit rate. She also used the term iterate in her written description of her representations. She determined the quotients correctly.

Summary of Case 6: Hannah

Hannah’s engagement in this study did not generate much data because she understood much of the PD content without apparent difficulty, asked few questions, and did not engage in think-aloud communication. She could see that the measurement and partitive interpretations of division derived from equal-sized groups multiplication, but she did not see much utility in that knowledge. Hannah’s conception of the quotient as being “out of a whole share” was consistent throughout her engagement in the tasks.

Case 7: Rachel

Rachel kept her verbal engagement limited to short responses to the interviewer’s questions and did not engage in think-aloud communication. Rachel’s engagement in the PD task sequence totaled two hours and comprised the first meeting. The following sections provide observed development of Rachel’s conception of division as she completed four task sets: (a) describing her intuitive ideas about fractions and the meaning of division, (b) learning to distinguish between measurement and partitive division with whole numbers, (c) discussing the meaning of partitive division when the divisor is a fraction, and (d) representing partitive division with fractions using
rectangular area models. These sections appear in chronological order.

**Rachel’s Initial Description of Fractions and Division**

In describing her informal ideas about fractions and division, Rachel expressed part-whole conceptions of fractions:

**Rachel 4.1**  
**Int:** What if a student and said, ‘What is a fraction.’ What would you say?  
**R:** Uh, it’s a way to look at pieces of a whole…like pieces of a — slices of a pizza or a pie…is what I would say.  
**Int:** Well, in teaching this class, have you ever come across any other definitions of fractions, or other ideas about what a fraction is?  
**R:** No, and you know, especially in person, introducing fractions…I never had to say much more than that… so any other thing I’d say would just go deeper in to describe the details. But just ‘what is a fraction?’ no, I can’t say I’ve come up with any other way to say it…that’s more relatable or better.  
**Int:** And when you say ‘go into the details,’ that would be details about what?  
**R:** Like, uh, one-third versus two-thirds versus three-thirds, and what that looks like.  

Rachel’s comments indicated that she had not experienced a pedagogical need to explore the nature of fractions beyond part-whole interpretations. While there was no other data point in this study to corroborate or disconfirm a pattern of thinking, By “go deeper in to describe the details” Rachel seemed to be referring to process and notation.

Rachel’s described division from the measurement interpretation and quickly invoked a conceptual resource of division relating to multiplication. She stated that division is the opposite of multiplication and that it followed multiplication.
Rachel 4.2  Int: Okay. What are your intuitive ideas about what division is?

R: Um, figuring out how many times something goes in to something else. Like how many times does one-fifth go into one; how many times does, you know, two go into ten. So, in terms of relating it to multiplication, I guess.

Int: How’s it related to multiplication?

R: When you – you know, like, how many times – using the word ‘times,’ brings up multiplication, and that division is doing the opposite function of, multiplying. So you multiply numbers together, and then you divide those numbers, and you get what you multiplied with.

Int: So you said ‘how many times does two go into ten,’ as an example…

R: Yeah.

Int: Like multiplication and division are the same thing? Or you can’t…look at one without thinking about the other?

R: No (rising tone). I just think that learning division often follows multiplication. So, trying to understand what you’re doing when you’re dividing, relating that to multiplying. You’re doing the opposite of multiplying. So the terms I would use to describe division include multiplication terms to relate the two together.

It is notable that Rachel did not use physical situations as metaphors, such as break or split up. Rather, her initial description was one of mathematical structure incorporating fractions as well as whole numbers.

Rachel’s Engagement in the Introduction to Partitive and Measurement Models of Division Task

This task began with Rachel viewing the 15÷5 Ball video (see Appendix H). Before the interviewer could ask Rachel if she had any questions about the content of the video, she began describing how the video reminded her of a particular division problem
in the textbook regarding the length of the board. Rachel explained that many students had trouble with that particular problem, and she would explain the solution in class by physically demonstrating a repeated subtraction process with her hands. The textbook problem asked for the length of the remaining piece of a board after it was cut into shelves of a certain size. In Rachel’s explanation to her students, she had cast the length of an individual shelf as a measuring unit. However, prior to this study, she did not have language to express that action.

Rachel had no questions about the content of the video. Rachel and the interviewer spent several minutes discussing other applications of a measuring unit in division such as tiling the floor or working with fabric.

Rachel’s Engagement in the Partitive Model of Division with Fractions Task

This task began with Rachel viewing the Crackers and Bowls video (see Appendix H). Before the interviewer could ask Rachel if she had any questions about the content of the video, she began describing how the video could be improved by bringing another half a bowl containing three quarter crackers into the scene to illustrate the quotient. The interviewer asked her to state her ideas as if she were talking to a video production team receiving instructions for her revision of the video.

Rachel 4.3 R: Your visual is three fourths in half a bowl but your answer is six fourths on the top. So, your picture should represent six fourths. And to show that, you know your answer when you do that problem, is giving you, like, the whole answer; you want to have the whole bowl represented with all the pieces that would’ve gone into it.

Int: So just add a second half there and you could see there should be
six pieces there –

R: Yeah. You know what you could also do, rather than having the bowl be physically split, just draw a line down the middle and mark one half, one half, and then you don’t have to worry about “Why are we talking about half a whole?” “Cause a whole will just be there already.

Int: That’s even better. What makes that um, how would you explain how that’s more clear to you?

R: Its, its, more clear because (pause) having the students, you could divide that bowl into any number of fractions, by just drawing some lines on it and then writing in, you know, maybe one-eighth, one-eighth, one-eighth, and all the different parts and you could have your crackers or whatever it is an put them into each of those and point to the answer above and say, “See, we have this many pieces in our bowl.”

Int: Okay, let’s say we do that with this example. So we draw a line down the middle and we put three pieces in one half –

R: Yes. And we say so now we have put three fourths into half. And then, take another three pieces and say well, we have to represent the entire answer. This, what we’re looking at is not a number value we can work with. So, here’s the rest of the answer. Something like that. That probably wasn’t the best way to explain it.

Int: But what if a student says, “Why do we do the whole bowl ‘cause it’s telling me half a bowl?”

R: Hm (long pause). I don’t –

Int: “How come the answer’s not three? ‘Cause it told me to put three quarters in half a bowl and I did and there’s three quarters sitting there in the half a bowl. So why do I need to make a whole bowl?”

R: I don’t know how to answer that.

Rachel exhibited an understanding that the partitive quotient was the amount associated with a whole bowl. She suggested two ways to make the whole bowl visible at the end of the video demonstration. However, she was unable to construct an explanation.
for the need to transform a partial bowl to a whole bowl.

Rachel’s Engagement in the Representing Partitive Division of Two Fractions with Rectangular Area Models Task

In this final task set of the PD task sequence Rachel (a) discussed examples of area models of division and (b) watched the *Partitive Division with Fractions Using Area Models* video (see Appendix H). The two hours allotted for the first meeting ended before Rachel could engage in the final task of constructing her own rectangular area models of partitive division with fractions. She completed that portion of the PD worksheet on her own time, so there is no video data of her conceptual processes as she completed that task.

The interviewer began the task by asking Rachel to look at a rectangular area model representing the partitive division problem $1/2 \div 1/3$ (see Figure 4.24) and see if she could make sense of it.

**Figure 4.24**

*A Rectangular Area Model of $1/2 \div 1/3$.*
Rachel 4.4

Int: Could you tell me what makes sense or what doesn’t make sense there?

R: Okay so we have the rectangle divided into thirds and, the one half is filled in here. I’m not sure why only this portion (indicating the leftmost, shaded partition) is filled in and labeled as one-half. I don’t understand that.

Int: Well, what is partitive division doing? What happens in partitive division?

R: Just using the words, you’re taking the whole and breaking it apart, right? So one-half is our whole, and we’re breaking it into thirds. Or we’re seeing how many thirds fit into a half. Something that was clicking last time is escaping me at the moment, with this.

Int: “How many thirds fits into a half” sounds like measurement.

R: Yeah. That’s just where my mind goes. (Long pause.)

Rachel was unable to coordinate the components of the representation with her conceptual resources at that moment to construct a partitive division concept projection.

Rachel exhibited a conceptual resource of partitive division as “breaking apart.” Her description of the 1/2 as “our whole” indicated that she conceived of it as the to-be-broken-apart dividend. Thus, Rachel began with the right image of the partitive interpretation and knew that the dividend should be partitioned, presumably so that it could be distributed. However, the divisor in this case was a unit fraction, thus obviating the need to partition the dividend before distributing it. Also, the dividend is a unit fraction, which arguably obscures it’s affordance for decomposition. One explanation for Rachel’s conceptual difficulty is that Rachel could not coordinate her inferences and perceptions. Specifically, the inference (from conceptual resources that functioned productively in the whole number context) that partitive division is a breaking apart and distributing did not appear to be represented in the model. This is consistent with her
statement, “Something that was clicking last time is escaping me at the moment, with this.” Thus, her impasse might have been due to the fact that the dividend was not partitioned in the representation (being a unit fraction), yet the inferences within her concept projection in that moment had her looking for something to be “broken apart.” She probably had not yet confronted the situation in which an unpartitioned dividend gets distributed into a portion of a whole share.

Rachel also evidenced an inference of measurement division when she stated, “Or we’re seeing how many thirds fit into a half.” The model can be seen as a representation of the measurement division problem, \( \frac{3}{2} \div \frac{1}{2} \).

Attempting to address the potential conflict between an expectation that the \( \frac{1}{2} \) should be seen as “broken apart,” or breakable, and the perception that it was an intact unit, the interviewer suggested they discuss the next representation on the worksheet. That representation (see Figure 4.25) had a proper fraction as the dividend, thus making visible the “breaking apart” she may have been looking for.

\[ \text{Rachel 4.5} \quad \text{Int:} \quad \text{Does [that one] make sense at all?} \]

\[ \text{R:} \quad \text{I guess where I’m struggling, I get the three-fifths part (indicating Figure 4.25) and the one-third part (indicating Figure 4.24) in both models. I get the dividing up into thirds (indicating Figure 4.24) and dividing into fifths (indicating Figure 4.25). I get that. What I don’t get is how you decide how to draw the dividend in the model. What I don’t get is why you have 1/2 in each of these the sections (indicating Figure 4.25) and 1/2 in this section (indicating Figure 4.24).} \]

At this point the interviewer wasn’t sure exactly what Rachel meant by “I get” the way the divisors were partitioned. She was indicating a feature common to both representations; she could recognize it in both contexts. The interviewer noticed that the
Figure 4.25

A Rectangular Representation of $1 \frac{1}{2} \div \frac{3}{5}$.

“three-fifths part” in Figure 4.25 was labeled, but the “one-third part” in Figure 4.24 was not labeled. The feature she was attending to in both models were likely perceivable to her as an inference from the conceptual resources in her developing concept projection. It was clear that Rachel was closer to understanding these representations and probably had a productive understanding of how the divisors were being represented but could not yet coordinate the dividend partitions with the divisor partitions. The interviewer decided to show Rachel the *Partitive Division with Fractions Using Area Models* video because it animated partitive division of two-unit fractions, potentially providing a model in which Rachel’s difficulty coordinating the unit fraction dividend could be surfaced. They had technical problems and spent several minutes unsuccessfully resolving a lack of clear audio on Rachel’s end of the remote meeting. Instead, the interviewer verbally described the points made in the video as they watched it without audio.

Rachel 4.6 R: So in creating this model (referring to Figure 4.25), you would start by, separating one and a half into manageable parts, which, to the person creating the model is, “Oh, well, if we say three halves,
that’s a reasonable way to think of one and a half.” Right? Like one and a half bowls, from the other model earlier. So, we draw (next to Figure 4.25 on her worksheet, Rachel drew three adjacent rectangles and drew “1/2” in each) I’m just trying to recreate. So this is one and a half, and we want to know (she touched the divisor in the numerical statement of the problem)…how to characterize that as three fifths (rising tone). Or, or, (touching the three shaded fifths in the representation) did you choose to do three halves because it’s three fifths? Is that something that is considered?

Int: Yes.

R: Okay. I see. Alright. So, if this one and a half represents three fifths (she drew a bracket under the three rectangles she had just drawn and labeled it “3/5”) then we have to consider what’s five fifths? In terms of the one and a half. Does that make sense, what I said?

Int: It does.

…

R: So if we turn this three fifths into five fifths (indicating here drawing) that means we have to add two more of these (indicating one of the three rectangles she drew) so that there’s five of them.

Rachel had considered that the dividend was divided so as to make it “manageable.” She then spontaneously constructed her own representation as a sense-making tool in her developing understanding. She exhibited an understanding that the coordination between the representation of the dividend partitions and the divisor partitions was the missing part of her comprehension of these representations. She constructed her own representation of the dividend and wondered how that coordinated with the given divisor (3/5) in the numerical statement of the problem. It appeared as though she needed to work with her own ideas, free of all the features visible in Figure 4.25. The data does not show why, but she suddenly saw how the dividend partitions
coordinated with the given divisor partitions.

Although Rachel used the language, “add two more,” she evidenced a conception of the solution to the division problem as iterating the partitions until the divisor partitions had been filled. Moreover, her language, “if we turn this three fifths into five fifths” indicates that she conceived of the filling process as meaningful; as the transformation of the given divisor to a whole share of the divisor. Time constraints did not permit follow-up questions to see if scaling or the leap were now explainable with Rachel’s current concept projection.

Rachel completed the last section of the worksheet on her own, constructing two prompted representations of partitive division with fractions. Artifacts from her independent work indicated several features of her concept projections at the time she constructed the representations (see Figure 4.26).

First, in representing $3/4 ÷ 1/2$ (left panel) Rachel partitioned the dividend even though that step was not necessary. The arrows she drew from the dividend to the dividend partitions in her representation indicate that she was emulating the features of the example shown in Figure 4.25. Second, in representing $1/2 ÷ 3/4$ (right panel), Rachel appears to have represented $3/4$ and then doubled it. Noticing that this was incorrect, she created a second representation. In both representations, Rachel made clear the meaning of the whole unit of the divisor by labeling it as “1” with a large pink bracket. Last, her use of colors and the plus sign indicate that she was attending to the given rate as a before state and the unit rate as an after state.
Summary of Case 7: Rachel

Rachel began this study with no explicit knowledge of measurement and partitive division as mathematical constructs. Throughout the PD task set measurement division situations and actions appeared to be natural for Rachel. She did not “connect” with the fair sharing language to describe partitive division in the context of fractional divisors. In learning to represent partitive division with fractions using rectangular area models, Rachel developed the ability to productively represent the dividend and the divisor as composite and partitioned quantities. The primary difficulty she experienced was in coordinating the two. She was finally able to construct that coordination through a process that began with her own drawing. Her drawing was a replication of the model she was trying to understand, but possibly isolating the features she was trying to coordinate. Her independent work constructing prompted representations at the end of the worksheet
indicated that Rachel productively understood the representations, as well as productively conceived of the partitive quotient as a form of unit rate. She explicitly represented the given rate as a beginning state and an iterated whole unit of the divisor as the end state. Time constraints prevented further discussions that may have provided information about Rachel’s conception of the transformation, as well as her conception of the relationship between dividend and divisor had physical units been brought into consideration.

Case 8: Zoey

Zoey was willing to express her thoughts, reactions, and questions. She also provided pedagogically-oriented comments and suggestions as if she was always thinking about how to present mathematics content to students. However, this kind of engagement often side-tracked the communication away from the primary focus of the PD task sequence. Language difficulties also made the communication process more time-consuming. Zoey’s engagement in the PD task sequence totaled three and a half hours over two separate meetings.

The following sections provide observed development of Zoey’s conception of division as she completed four task sets: (a) describing her intuitive ideas about fractions and the meaning of division, (b) learning to distinguish between measurement and partitive division with whole numbers, (c) discussing the meaning of partitive division when the divisor is a fraction, and (d) representing partitive division with fractions using rectangular area models. The following sections describe purposively selected transcripts from Zoey’s engagement in the PD task sequence in chronological order.
Zoey’s Initial Descriptions of Fractions and Division

Zoey began with a part-whole description of fractions but quickly related fractions to percentages.

Zoey 4.1

Int: Can you tell me in your own words, what is a fraction? What does a fraction mean to you?

Z: Well to me a fraction is a part of something. Part of…the whole, whatever whole is; it doesn’t need to be one. It’s part of something. Fraction of money, fraction of salary, fraction of, I don’t know, your food, which you use for a party or whatever. It’s just part of something.

Int: You said was, it doesn’t have to be of one.

Z: Yeah, it doesn’t have to, because if I think of…money I made, it’s not one dollar, or a particular amount. In my case it’s variable every month. But I can say, no more than ten or twenty percent of it can go for entertaining or eating out, whatever.

Int: Percent?

Z: Well, yeah, we often think about fraction related to percent. At least I do; I’m not sure if students have the same image. For me, it’s just a part; can be expressed as a fraction, as a decimal or percent…I was thinking about it, when we learn, actually or we teach fractions, maybe it would not be a bad idea to do it simultaneously, by the way, fractions can be represented as decimals or percent. ’just to point out this is not the only way to represent a part. Our brain is sometimes already wired to think about percentages.

Int: So can you tell me your own personal conception of what does division do; why do we use it?

Z: Well I always think how many times divisor fits into something. How many times can you fit it into. So for me, it’s the first notion I get. How many times one thing fits into another. And it can be even less than one, if the thing trying to fit is bigger, so, but, it’s how I personally feel about it. Or you can think about it’s like using measuring tape, you know, you have unit of measurement. And unit can be anything, possibly. I’m a physicist talking.

Int: You mentioned the divisor might be bigger that the dividend,
which is confusing.

Z: It is confusing. Because what it means is you can’t fit it; only part of it will go. I try to think about possibly having a box and you’re trying to fit something which is bigger than the box. You can possibly fit part of it or half of it but the rest of it will still be outside.

Zoey’s part-whole conception of fractions quickly connected to other concepts, such as the conception of fractions as an operator. She also stated that fractions can be represented as decimals, expressing a fluidity and connectedness of mathematical constructs of rational numbers. Her fluidity and connectedness of mathematical constructs was again demonstrated in her description of division, which quickly included measurement and the possibility that the divisor could be bigger than the dividend.

**Zoey’s Engagement in the Introduction to Partitive and Measurement Models of Division Task**

This task began with Zoey viewing the 15÷5 Ball video. Zoey’s first comment was that students might be confused by the terms quotitive and partitive. However, she quickly exhibited an understanding of the difference between the two interpretations of division by constructing an example in which she took students out to “play computer games” and she had a certain number of tokens. Before she constructed that example, she asked if she and the interviewer could bring fractions into the conversation about the two different division interpretations. This again indicated Zoey’s fluency and connectedness of mathematical constructs.
**Zoey’s Engagement in the Partitive Model of Division with Fractions Task**

This task began with Zoey viewing the *Crackers and Bowls* video. The interviewer then asked Zoey what she thought about the demonstration. She suggested that using slices of pizza as the dividend instead of portions of a broken cracker would make the demonstration “more tangible.” Zoey constructed an example of three divided by one-half using pizza that she referred to as “fair sharing” but was in fact measurement.

Zoey 4.2 Z: You have three pizzas and you give half a pizza to each student. How many students will get pizza?

The interviewer pointed out that she had constructed a measurement example. She quickly recognized that point. The interviewer asked Zoey if she could use pizza to create an example of $3/4 \div 1/2$ with partitive interpretation. In the 3-minutes that followed, Zoey exhibited several conceptual resources related to each other: The quotient is the amount in the whole bowl, but we can’t model the divisor as half a student. After three minutes of conversation Zoey constructed “a more tangible” example of putting the pizza in “a box which has two smaller boxes in it.”

**Zoey’s Engagement in the Representing Partitive Division of Two Fractions with Rectangular Area Models Task**

In this task set of the PD task sequence Zoey (a) discussed examples of area models of division, (b) watched the *Partitive Division with Fractions Using Area Models* video, and (c) constructed her own rectangular area models of partitive division with fractions.

In the first task, Zoey was asked to look at a representation of the division
problem, $1/2 \div 1/3$ (see Figure 4.27). She was told that it represented partitive division and she was asked if the representation made any sense to her. At first, she replied, “Not really.” The interviewer noted that the whole area contained by outer perimeter represented one. She mentioned that the answer was one and a half, although she didn’t explain how she attained that answer.

**Figure 4.27**

*A Rectangular Area Model of $1/2 \div 1/3$.*

In an attempt to surface her conceptualization, the interviewer suggested that this representation as an example of sharing and he engaged Zoey in discussing how this representation compared to the crackers and bowls representation. This appeared to perturb her conceptualization of the representation.

*Zoey 4.3*  
Int: This is the same thing but now the divisor’s a third. So we have to cut the bowl into thirds, and then we put the dividend into that third, which is why there’s one-half put into a one-third space.

Z: But at first we have to divide one-half by three. So one-half divided by three would be one-sixth, right? I couldn’t imagine, my brain doesn’t…

Despite understanding the crackers and bowls example and having just
determined one and a half as the quotient in this example three minutes earlier, Zoey was at an impasse. Their conversation went off-topic. To bring the focus back to the tasks, the interviewer showed Zoey the *Partitive Division with Fractions Using Area Models* video. Immediately after viewing the video Zoey stated the following.

*Zoey 4.4*  

**Z:** Well I cannot say it makes any – we don’t have any additional partition (the example in the video used a unit fraction for the divisor, thus the dividend required no partitioning). So we’re just distributing three-quarter into half. So our task is basically to find, uh, how many will be in the whole thing. Indeed, three-quarter goes into half, right, well we have *another* half so we have to put another three-quarter there.

**Int:** Exactly. So with that background, does this representation (see Figure 4.27) make more sense?

**Z:** No. I would say no because I don’t see why we need this third partition (indicating the perimeter of the representation).

**Int:** It’s not a partition; imagine that’s the whole box.

**Z:** Ah! I see now. The whole box is divided into three parts and one-half is placed into one third of it.

Zoey’s concept projection apparently included all the conceptual resources she needed. What was missing was the correct perception of the representation that would allow her conceptual resources to do the inferential work of determining the quotient.

The interviewer and Zoey briefly discussed words that would be better than “sharing” to describe the relationship between the dividend and divisor in the case where the divisor is a fraction.

*Zoey 4.5*  

**Int:** When the divisor is a fraction it doesn’t make sense because now we’re distributing it to part of a person.

**Z:** Well, you know, if you’re not thinking about a person, but you’re thinking about space. Then indeed it may have real physical sense.
Zoey began to describe several situations involving remodeling a house. Her situations appeared to be measurement applications, such as tiling a floor. The interviewer then directed Zoey’s attention to Figure 4.28.

**Figure 4.28**

*An Rectangular Representation of $1 \frac{1}{2} \div 3/5$.*

Zoey 4.6  
Int: Take a second and look at this division problem and the representation and let me know if it makes any sense to you.

Z: So, one and a half goes into three-fifths of something. Oh yes, I can immediately say, okay think about backyard. For example your buying fertilizer or pesticide right? And I would think one and half can of pesticide is enough to spray three-fifth of back yard. So if you want to spray the whole backyard, how much pesticide do you need.

The interviewer was impressed with Zoey’s back yard example as a unit rate conception of division and moved on to the final task of representing $3/4 \div 1/2$ and then $1/2 \div 3/4$ using partitive interpretation. Zoey read the instructions and began thinking about the problem silently. Before starting her representation on the worksheet, she remarked, “This is the most confusing thing I ever did. Calculus is much easier.” The interviewer prompted her to begin by drawing a rectangle that represents one. She
constructed a large rectangle on the worksheet and stated, “Three-quarter goes into one-half. Basically, it’s what we did before.” She partitioned the rectangle vertically into two partitions. She then labeled each partition “1/2,” drew “3/4” in each partition, and said, “I can just add it together to get six quarters.”

Zoey moved on to the last task of representing $1/2 ÷ 3/4$ using partitive interpretation. Without prompting, she drew a large rectangle, partitioned it into fourths, and began shading three of the partitions (see Figure 4.29).

Zoey 4.7

Z: I sort of got this, again, very confusing concept. Now one-half is going into three quarter. And this is my three quarter; I can possibly shade it, which will accommodate one half. So if one half is just thing which is my three quarter. So the question is, How much do I need to fill the whole thing?

Figure 4.29

Zoey Begins Representing $1/2 ÷ 3/4$.

Somehow Zoey then stated that the answer would be more than one and wrote “1” to the right of her representation (see Figure 4.30).

Zoey 4.8

Z: And one-half is definitely more than one, so I can put 1
immediately. And to fill remaining quarter I will need, um, one-third of one-half…which is one-sixth. So I get one and one-sixth. Is that correct?

**Figure 4.30**

*Zoey Completes Her Representation of $1/2 ÷ 3/4$.*

The interviewer responded by saying that she might have done some measurement thinking. Zoey retraced her steps and clarified her representation by drawing “1/6” in each of the three shaded partitions. This prompted her to say, “I just need one more sixth to fill in. Oh, yes. All together I get four sixths.”

**Summary of Case 8: Zoey**

Zoey exhibited mathematical fluency and connectedness throughout her engagement in the PD tasks. She suggested that the partitive division with fractions representation were so confusing that she could not imagine students “getting into it.” Despite her conviction that the representations were confusing, she was able to learn how to independently construct them. She also exhibited a strong understanding of the difference between measurement and partitive division, though sometimes momentarily mixing components of the two. Zoey generated many “tangible” situations in which both
measurement and partitive division examples could be constructed. The difficulties she experienced mostly appeared to be perceptual. In other words, her conceptual resources appeared to provide her with the correct inferences across different contexts; she often just needed to see some aspect of a representation to properly frame it.

**Cross-Case Results**

I maintained a list of observations and themes that emerged in the process of analyzing and writing the within-case sections. Findings emerged in two ways: First, some observations became thematic as they repeated among the cases. Second, some observations were idiosyncratic to individual participants but had relevancy with respect to the broader context established by the other participants. In other words, while much of the data generated by the participants did not stand out as a finding, in total the data established a space in which contrasts could emerge.

As the list of observations and themes developed, it largely coalesced around five main categories: Diversity of conceptions of fractions and division, multiple models of partitive division with fractions, difficulty conceptualizing partitive division with fractional divisors, the role of prior knowledge, and managing the meanings of the term “unit.” This section describes the cross-case findings organized around these five categories, followed by findings related to Research Question 3.

**Diversity of Conceptions of Fractions and Division**

The participants in this study were a group of eight teachers who had been teaching the same mathematics course to the same student population, using the same
Despite their common teaching experience, they expressed both within-case and cross-case variability in their conceptions of fractions. The conceptions of fractions expressed by the teachers can be grouped into three subcategories: part-whole, fraction-as-number, and fraction-as-operation. Comparatively, there was less diversity in the teachers’ interpretations of division.

**Part-Whole Descriptions of Fractions**

Part-whole descriptions of fractions were tied to physical objects. Throughout the task sequence Audrey and Evelyn in particular expressed a need to see fractions as tangible things. Audrey, Evelyn, and Malory expressed moments in which they needed to know what the fraction was physically referring to, using language such as, “One half of what?” These moments in which the meaning of a fraction lost its physical grounding were associated with moments in which the teacher could no longer proceed coordinating the components of a mathematical situation as sense-making activity. For example, in her pies and containers segment, Evelyn lost track of the meaning of the numbers and was unable to determine the quotient. Her ability to regain coordination of the division problem and determine the quotient coincided with the moment she was able to see that the divisor was “one half of a thing.”

**Fraction-as-Number**

Fraction-as-number conceptions were consistently exhibited by Ivy, Mark, and Zoey, the three teachers who did not grow up in the United States. They did not privilege the use of fractions to values less that one. Rather, they viewed fractions as a way to refer
to values in between whole numbers. These teachers appeared to be more comfortable operating with fractions without a tangible reference or a physical meaning attached to them. For example, Mark solved a division problem numerically and then used physical units to confirm his answer in a second step, remarking that “the units check out.” This evidences a mathematical practice in which Mark is accustomed to attaching or detaching physical units to numbers. Ivy, for example, constructed a solution to the Flour Problem by coordinating the ratio of two sets of boxes. This coordination required several instances of arithmetic with fractions. When she identified the quotient in her representation, she exclaimed, “Here is my answer.” Her next statement was, “What is my answer?” meaning that she had to bring the physical units back into her thinking in order to make sense of the quantity of the quotient with respect to the physical situation.

**Fraction-as-Operation**

Fraction-as-operation conceptions included viewing a fraction as a quotient and viewing a fraction as a scaling factor or proxy for “percent of.” Evelyn and Zoey remarked while solving some problems that converting the fraction to a percent would make the solution easier.

**Interpretations of Division**

There was less diversity in the teachers’ interpretations of division. However, there was also less specificity in their descriptions of the meaning of division. All the teachers described division as a kind of operation. These descriptions included physically grounded metaphors such as “breaking something up,” and situation-based descriptions
such as “how many times the divisor fits into something.” More abstract descriptions included “inverse of multiplication” and “an operation that you can do with two numbers to see what part of one number that the other number takes up.”

When expressing their intuitive conceptions of division at the start of the PD task sequence, none of the teachers articulated a conception of division as a renormalization; that is, as a change of base transformation. However, as the task sequences progressed, teachers began to express conceptual resources that identified the result of division as being “per one” or “per one whole” or “per share.” This was most notable with Ivy, who was quite animate about the need to see division as producing a “per one” quantity. Ivy did not consider the fair-sharing image of division to be a productive conceptual resource. Nor did her representations indicate that she was conceptualizing partitive division as resulting in an intensive quantity.

Prior to this study none of the teachers had been exposed to the measurement and partitive interpretations of division. While they were all familiar with the underlying mathematical processes, none of the teachers were familiar with the terms, measurement and partitive division. Similarly, at the start of the study, none of the teachers appeared to be aware that partitive division could be interpreted in several ways (described in the next section). The teachers exhibited several different interpretations of partitive division while engaged in tasks, but without articulating or recognizing their differences.

**Multiple Models of Partitive Division**

Participants in this study constructed concept projections of partitive division with fractional divisors that reflected two distinct situational structures. In this study, I refer to
these structures as the *part-whole structure* and the *unit-rate structure*. Each structure had two versions: one being more conceptually complicated than the other. These structures and their versions are described and compared in the following sections.

**Part-Whole Structure of Partitive Division**

In my analysis, the *part-whole* structure of partitive division occurred when a participant considered the dividend to be part of a whole. In this structure, the divisor quantified the portion of the whole represented by the dividend. The central conceptual feature of this structure was that participants evidenced attending to one referent. The quotient to the division problem quantified the whole amount.

This structure was evidenced in two versions. The simpler version attended to the dividend as an amount. This version was most frequently seen in Ivy’s approach to partitive division problems. As Ivy phrased it, “This is part of something; what is that thing?” I called this version the *partial amount* version of the part-whole structure. I called the second version the *in-progress* structure. The *in-progress* structure described a process that was not finished, such as being part-way through an image scan (see Figure 4.31). In this version of part-whole partitive division, the divisor established the portion of progress made as a fraction, with *completion* as the referent unit for that fractional divisor. The dividend quantified the progress in units relevant to the situation. When participants engaged in this conception of partitive division, time was arguably a second referent unit, but it appeared to be conceptually subsumed or chunked, possibly simplifying the cognitive processing.
**Unit-Rate Structure of Partitive Division**

In the *unit-rate* structure, the dividend and divisor established a rate, and the quotient took the form of a unit rate. When teachers engaged in this type of thinking about partitive division, the distinction between the physical units of the dividend and of the divisor was a salient aspect of their mathematical activity. In other words, the fact that the dividend and divisor had separate referents was a critical part of the problem.

Participants evidenced two versions of the unit-rate structure. In one version, participants attended to the dividend quantity with more priority than the divisor. This structure addressed an amount question with reference to the dividend and placed the divisor quantity in a supporting role. I refer to this as the *unit-amount* version of the unit-rate structure of partitive division. This version was exhibited more frequently by the participants in this study.

The second version of unit-rate partitive division attended to the separate referents of the dividend and divisor as co-constructors of a third (intensive) quantity. For example, Ivy constructed a concept projection of partitive division as speed; Mark constructed a concept projection of partitive division as density. Speed and density are
examples of intensive quantities, also called derived quantities (discussed in Chapter 5). I refer to this as the intensive quantity version if the unit-rate structure of partitive division. This version was exhibited less frequently by the participants in this study.

**A Structure that Happened Once**

One of Hannah’s students posed the following problem: “A toy train travels 12 cm/second. How long does it take to travel 20 cm?” Hannah and the interviewer debated about whether this example was a measurement or partitive division problem. As will be discussed in Chapter 5, it is neither because it is not based on the equal groups model of multiplication.

**Part-Whole and Unit-Rate Structures of Partitive Division**

In the new course curriculum (see Appendix A), the Maple Trees example represented the in-progress version of the part-whole structure. Given that the landscaper had planted six trees before taking her break, the quotient to $6 \div 2/3$ determined that there were nine total trees to plant for that job. The Flour example represented a unit-rate structure by establishing that half a pound of flour costs three-quarters of a dollar. The quotient to $1/2 \div 3/4$ determined the cost of flour as the unit rate, $2/3$ pounds per dollar. This distinction, that the quotient in the unit-rate structure embodies two different physical units, whereas the quotient in the part-whole structure describes a quantity attending to a single physical unit, was never made explicit in the course textbook, new or old course content, or in the PD tasks. This distinction emerged from analysis of the data.
When the participants constructed an example, the structure of partitive division seemed to appear organically from the situation in which the participants conceptualized the task. Evelyn constructed examples of part-whole and unit-rate structures. For instance, her pies and containers situation represented unit-rate partitive division because the dividend and divisor had explicitly different referent units. On the other hand, Evelyn’s extemporaneous suitcase example typified the in-progress version of the part-whole structure, with completion as the salient referent unit.

One structure or the other might be invoked in a given situation by the particular construction of the question. For example, Evelyn’s suitcase situation could be cast as a unit-rate problem by constructing a question for which the answer was in terms of items per suitcase or pounds per luggage.

Because of the research focus, the PD task sets generally framed partitive division within the unit-rate structure (see Chapter 5). This was particularly true in representing partitive division with fractions using rectangular area models. Examples of representations of partitive division with fractions using rectangular area models in the PD worksheet described the representational process as beginning with a large 1-rectangle that represented the divisor scaled to a value of one. This was often referred to, by the interviewer and teachers alike, as one whole share. The next step was to distribute the dividend into the portion of the 1-rectangle quantified by the divisor. The last step was to iterate the partitions of the dividend until the partitions of the 1-rectangle had been “filled.” However, Ivy’s construction and discussion of her representations evidenced that she was accustomed to conceiving of partitive division with fractional divisors using the
in-progress interpretation. For example, Ivy began all but one of her representations of partitive division in this study by representing the dividend. That one exception was, according to her, a mistake. After representing the dividend, Ivy constructed a representation of the divisor in a separate space, essentially constructing a strip diagram (Siy, 2018). Ivy’s representations of partitive division with fractions relied on a strategy of coordinating partitions of the two strips (representing the dividend and the divisor). For instance, in the interview, Ivy constructed her own representation of the Flour example, $\frac{3}{4} \div \frac{1}{2}$ (see Figure 4.32). This was the only time Ivy started by representing the divisor first. She commented that she was starting off wrong and that it might not work.

**Figure 4.32**

*Ivy’s Representation of $\frac{3}{4} \div \frac{1}{2}$ for the Flour Example.*

When asked to describe her representation, Ivy made several remarks evidencing an in-progress interpretation.

*Ivy 4.12 I:* I have three-quarter dollar and then I, I got one half a pound. So I drew half a pound. And three-quarter dollar, identical to each other. And then if I add another quarter dollar I have to draw another square. Have to add an extra square. So adding an extra square.
Ivy’s last step of “adding an extra square” evidenced that she was conceptualizing an amount rather than a rate.

Another contrast between part-whole and unit-rate conceptions of partitive division occurred as Malory and Ivy independently expressed difficulty understanding a representation of $3/4 ÷ 2/3$ in the new course content (see Figure 4.33).

**Figure 4.33**

*A Representation of $3/4 ÷ 2/3$ in the New Course Content.*

Malory expressed difficulty interpreting Figure 4.33 because the difference in heights between the blue rectangles and the purple rectangles gave her the sense that something was wrong with the representation. Ivy also said that the representation did not make sense because the height of the rectangles representing the dividend was different than the height of the rectangles representing the divisor. However, the height difference is of no consequence when conceptualizing the problem in the unit-rate interpretation because the dividend and divisor have different referents and, therefore, different
physical units. In order to make sense of the problem in Figure 4.33, Ivy had to make her own representation using the equal-height strip diagram method that she had developed before entering this study (see Figure 4.34).

**Figure 4.34**

*Ivy’s Representation of $\frac{3}{4} ÷ \frac{2}{3}$.*

The in-progress interpretation was evidenced at times by Audrey, Evelyn, Hannah, Malory, and Mark. For example, Mark’s representations of partitive division with fractions in the PD task sequence began with a representation of the dividend. This was similar to Ivy’s procedure, though Mark did not exhibit the same commitment to his representations that Ivy exhibited. His first representations appeared to include elements of both interpretations. In the interview, Mark was able to conceptualize and represent the unit-rate interpretation after some discussion.

During Hannah’s interview, the interviewer mentioned that most partitive division examples constructed by the Chinese teachers in Ma’s (2010) book were of the in-progress interpretation. Hannah quickly noted, “like the Maple Tree example,” and asked,
“Why don’t we teach it that way?” Hannah’s tone evidenced that she thought the part-whole structure was more accessible and should be taught to students in the U.S. or in the Basic Mathematics course.

Evidence of the part-whole structure being conceptually less challenging occurred at the end of Evelyn’s 22-minute pies and containers episode. After determining the unit-rate quotient in the pies and containers context, Evelyn constructed her suitcase example apparently effortlessly to illustrate the same numerical division problem, but in a representation that readily facilitated sense-making for her.

### A Contrast between Part-Whole and Unit-Rate Structures of Partitive Division

An incisive summary of the two structures of partitive division can be made by contrasting the representations constructed by Ivy and Evelyn for the same final task in the PD task sequence. In this case, the division problem was $1/2 \div 3/4$ and there was no story problem associated with the numerical representation of the problem. The participants were asked to represent the division problem any way they chose to. Ivy’s representation indicated that she was conceptualizing a part-whole structure. Evelyn’s representation indicated that she was conceptualizing a unit-rate structure.

Ivy’s representation began with the dividend as a single, non-composite unit (an amount). She realized that she needed to re-conceptualize it as a composite in order to operate on it. She then drew a second, composite version of the dividend (see Figure 4.35, left panel). Ivy represented the division process as an operation that transformed the dividend from its initial, partial state to its whole completed state (the “something”).
When Evelyn represented the same problem, she began by representing the divisor. She constructed a rectangle representing the divisor, scaled to a value of 1. She then partitioned that 1-rectangle into fourths and identified three of those partitions (see Figure 4.36). Thus, she had represented both the given and the scaled divisor before representing the dividend.

In Evelyn’s representation, the dividend appeared to play an equal role with the divisor. Evelyn represented the dividend as having originated from outside of the divisor.
and distinguished it from the divisor by representing it as a circle diagram. This feature of Evelyn’s representation is consistent with a conception of the dividend and divisor referring to distinct entities. Another feature of Evelyn’s representation is that she inserted the dividend quantity into the divisor space. After constructing the separate representations of the dividend and divisor as circular and rectangular, respectively, Evelyn represented the given rate by re-representing the half-circle inside the representation of the given divisor. She then represented the unit rate by drawing a one-sixth circle segment inside the fourth partition of the divisor, the partition associated with the transformation from given rate to unit rate. In other words, her representation evidences that to whatever degree she considered the fourth partition of the divisor to be an iteration, she considered it an iteration of a rate relationship between the dividend and divisor.

**Difficulty Conceptualizing Partitive Division with Fractional Divisors**

For all teachers in this study, the most difficult tasks were those involving...
partitive division when the divisor was a fraction. Compared to their engagement in other tasks, the teachers’ engagement in partitive division with fractional divisors included more instances of being unable to complete a task or not being able to complete the task correctly. There were several common difficulties expressed by the teachers: First, the teachers lacked experiences that could provide conceptual resources for making sense of fractional divisors in physical situations. Second, in working with representations of the division process, the quotient-determining transformation of the given rate to a unit rate required a “leap.” Last, the teachers found it difficult to coordinate the components of their representations, particularly partitions, iterations, and unitizations of the dividend and divisor. They found it difficult to coordinate their representations with the IM procedure, a form of knowledge with which they could readily solve a division problem. These three difficulties are described in the following sections.

A Lack of Experiences with Fractional Divisors

One of the conceptual challenges involved in representing partitive division with a fractional divisor may be that people generally lack experiences of sharing among non-whole-numbers of recipients, and therefore do not have prior knowledge to activate as a conceptual resource. All the teachers except Ivy and Mark stated that sharing among fractions of people or bowls did not make sense.

Because sharing among non-whole-numbers of recipients did not make sense, teachers were unable to productively apply their fair-sharing conceptual resource. The interviewer explored solutions to this difficulty in two ways. First, by soliciting alternate conceptual resources to the fair-share resource by asking the teachers if they
could think of language other than “share” to describe the division process. Teachers in this study productively used the terms “distribute” and “disseminate.” By productive I mean that using these terms was associated with teachers’ progress in their ability to construct representations of partitive division with rectangular area models and to analyze such representations. Second, the interviewer elicited alternative ideas of the recipient of the shares, hoping to find situations in which the fair-sharing resource could be sensibly extended to fractional quantities. This approach appeared to be less productive. One example of a productive alternative recipient was Evelyn’s choice of Tupperware. Even in this example, however, Evelyn’s experiences using food containers only made the situation plausible. Her experiences did not include enough use of fractions of containers to provide her with prior knowledge that, along with other conceptual resources, could readily construct a quotient-determining concept projection. In other words, while knowledge that food containers can be partitioned was a productive part of her ultimately determining the quotient, it required some work with her conceptual resources to accomplish that determination. In particular, Evelyn’s ability to productively use one half of a container (in that moment) relied on directing her attending to the representation she made and possibly scaffolding an inferential connection between that representation and the physical units in the problem.

**Getting to 1 Requires a Leap**

Determining the quotient when the divisor is less than one required a “leap,” as described by Audrey, Evelyn, and Mark. All teachers in this study exhibited difficulty transforming the divisor to one in their representations in the form of drawings as well as
in their representations in the form of speech.

When the divisor was a whole number, the teachers conceptualized the partitive quotient as the amount per one share so easily that it seemed at times sub-conceptual. Mark evidenced this point while discussing the crackers and bowls video. Referring to the demonstration with three bowls, Mark said, “You see the quotient in the bowl.” Hoping to surface a distinction in Mark’s conception between a divisor larger than one and smaller than one, the interviewer inquired, “Even though there are three bowls here.” With a matter-of-fact tone, Mark replied, “The solution is whatever is in one bowl.”

In reference to constructing one bowl from a half bowl, Mark conjectured that the difficulty lied in the fact that it requires “imagining the whole.” Yet when presented with the option of alleviating the need to “imagine the whole” by redesigning the video such that a second half bowl was brought in and joined to the existing half bowl, Evelyn reacted unfavorably. She stated that it would be confusing to see another half bowl because “it’s not there.” Evelyn did respond favorably to the option of drawing the other half of the bowl as the conclusion to the video. She stated that “If you draw it in, it implies you’re just thinking about it as a concept.”

Ivy was the only teacher in this study to construct a description of the process by referring to it as “scaling up.” Notably, scaling can be continuous, but a leap is discrete. Arguably, the partition and iterate procedure in the method of representing partitive division with fractional divisors as presented in the PD materials is somewhere in between continuous and discrete. While the iterations are discrete, most problems require more than one iteration. However, the data in this study does not include clear evidence
that the teachers experienced the iterations as a bridging analogy to scaling. Perhaps this is because the teachers experienced the iteration process itself to require cognitive load (described in the next section).

**Coordinating All the Components is Challenging**

The Within-Case Results section describes specific instances of the difficulties that teachers in this study exhibited in learning to represent fraction operations with rectangular area models. From the cross-case view, these instances can be commonly described as challenges presented by the need to coordinate multiple components of the representations.

**Dividend and divisor.** In this study, the teachers’ engagement with the PD tasks consistently provided evidence that coordinating the dividend and divisor in the same representation was a central difficulty in representing measurement division with fractions, and both models of partitive division with fractions. In representing division with fractions operations, the dividend and divisor are composite units, requiring a *within-space* coordination of units as well as coordinating their respective unitizations. Thus, coordinating representations of the dividend and divisor requires coordinating levels of units.

The teachers in this study identified another complication with the dividend and divisor in partitive division: That is the perception that the dividend and divisor have a before state and an after state, with a “leap” in between. The within-case results evidence that this conceptual difficulty cannot be ameliorated by the simple introduction of declarative and procedural knowledge. For example, Evelyn was able to state that the
result of division is always “per one whole thing.” Yet, directly after making that statement, she spent 22 minutes making sense of a partitive division example that she created. Despite knowing that the result of division is always “per one whole thing,” Evelyn had to reflect on her representation for a considerable amount of time before seeing how that declarative knowledge applied to her example. One comment that Evelyn made at the end of that episode was, “The numbers have to fit the story.” One interpretation of her comment is that she needed a narrative to her example that would provide meaning to the mathematical task of transforming the given rate to a unit rate.

**Difficulty coordinating the IM procedure and the representation.** Teachers in this study did not experience difficulty in using the IM procedure to determine the quotient of division problems. With the exceptions of Ivy, the teachers did experience difficulty coordinating the IM procedure with the components of their representations. In discussing Example 8b for example (see Figure 4.37), the interviewer asked Ivy to consider decomposing the “multiply by 5/3” into two steps and compare to the area diagram. She responded,

*Ivy 4.13*  
I: Oh, (indicating the dividend portion of the area drawing with her pen) that’s the divided by three, that’s the division right there into three yellow thing and then (pointing to the 5/3 in her algebra) and then times by five that means (indicating the two ghost fifths with her pen) you fill up the two more things? Oh. That looks very nice, yes.

In summary, conceptualizing partitive division with fractions may be difficult for multiple reasons, including a lack of experiences sharing among fractional recipients, conceptualizing the “leap” from the given rate to the unit rate, and coordinating several levels of units in the representations.
The Role of Prior Knowledge

Teachers in this study incorporated prior knowledge as productive resources in their concept projections of division and operations with fractions. In addition to the examples already described (e.g., Audrey’s use of skip counting and bricks to construct meaning for measurement division, and Evelyn’s suitcase), several other examples strongly informed the first research question for this study. These examples are described in this section.

Crackers and Bowls

The Crackers and Bowls video demonstration, which occurred early in the PD task sequence, initially presented conceptual challenges for the teachers in this study. However, it became a productive resource in later segments. (The question of what kind of prior knowledge this demonstrates is discussed in Chapter 5.) Crackers and bowls became a productive resource particularly when the teachers were first learning to
represent partitive division with fractions using rectangular area models. All of the
teachers leveraged their knowledge of crackers and bowls through analogy to make sense
of the abstract context of rectangles, sometimes explicitly referring to the 1-rectangle as
the bowl, and the partitioned units of the divisor as crackers.

**Rates and Density**

Given that the unit-rate interpretation is based on rates, it is not surprising that the
teachers’ prior knowledge about rates played a role in the concept projections regarding
partitive division with fractions. Two of the teachers in this study had a physics
background and three had a life sciences background. These teachers were comfortable
with rates and some of them used the term “at the same rate” to describe the
transformation of the given rate to the unit rate, or to describe the meaning of partitioning
and iterating in their representations. Audrey and Malory, on the other hand, had no
science background and did not exhibit confidence in the discussions about rates and
physical units.

Mark incorporated prior knowledge in a novel way in making sense of the bowls
and crackers video. In making the “leap” from a half bowl to a full bowl, Mark
incorporated his prior knowledge of density. During the interview he again referred to
density in the context of crackers and bowls during a discussion about the two
interpretations of partitive division. Mark noted the unit-rate interpretation (shown in the
crackers and bowls video) establishes a density that needs to be maintained in the
transformation from a fractional divisor to a 1-unit divisor.

Mark further noted that the crackers and bowls context could be used to
demonstrate the difference between the two interpretations of partitive division. The same division problem, \( \frac{3}{4} \div \frac{1}{2} \), could be represented as an in-progress division problem by filling one whole bowl with three quarter-crackers and stating that we are half-way done filling the bowl. That in-progress representation is physically distinct from the unit-rate representation in that it does not establish a density. Rather, with the in-progress interpretation the density changes each time a quarter-cracker is added to the bowl.

**Managing the Meanings of “Unit”**

Discussion of the meaning of the terms *unitizing*, and *referent unit*, were included in the PD task sets. In addition to these two terms, describing the teachers’ activity in this study requires several different meanings of the term *unit*. The teachers adopted the term *unit* and used it liberally in this study. This section describes the ways in which these terms were integral to the teachers’ conceptions, representations, and verbal communication of their mathematical activity.

**Unitizing**

In this study, the term *unitizing* was used to describe the activity of grouping objects together and referring to the group as if it was a single object. This activity could be mental activity or representational activity in the form of verbal communication or representation with area models or drawings. The teachers were introduced to the term unitizing in the PD task sequence. Unitizing also appeared in the new course content.

The PD tasks sets involving the representation of fractions operations with rectangular area models required the teachers to unitize as a reflective practice. Thus, at
the end of the study teachers were able to discuss unitizing as a mathematical activity. While discussing the term unitizing in the interview, Hannah said that at first the term seemed strange to her, but “after practice it got easier.” During the interview, Ivy provided her own definition of unitizing:

Ivy 4.13  I: [Y]ou could be talking about one family, but, or you could be talking about individual family member. So in the can example it could be one box or one batch or one individual can. So one, it change depending on the situation and depending on the context.

Referent Unit

In this study, the term referent unit was used to indicate the object (mathematical or physical) to which a given number was representing. It was briefly presented in the PD content. The researcher expected this term would be useful in describing the whole to which a fraction referred. However, the term referent unit was not adopted by the teachers. Rather, the teachers simply used the term unit. The teachers sometimes used the term unit to mean referent unit, but they also used the term unit in a variety of ways. One complication with using the term unit is that there are several different types of units. The section below describes the different types of units that occurred in this study.

Types of Units

Physical units. Physical units are part of quantifying physical contexts. In this study, physical units were integral in constructing unit-rate quotients. Given that much of the activity in this study involved the representation of operations with fractions, physical units were a conceptual resource teachers incorporated in concept projections of operations with fractions. One topic of discussion in the interviews was a chart that
attempted to clarify the difference between measurement and partitive division through an analysis of the physical units of the dividend, divisor, and quotient (discussed in Chapter 5).

Audrey, Evelyn, Ivy, Malory, and Mark stated that physical units and their role in defining a rate needed to be explained in the new course content. In particular, Evelyn and Mark noted that in general students are notorious for lacking knowledge and skills regarding physical units. None of the teachers in this study exhibited difficulty in conceptualizing what a physical unit was. However, Audrey and Malory were less confident than the teachers with a science background in discussing physical units and less fluent with incorporating physical units in their descriptions.

**Structural versus physical units.** Another distinction in the use of the term unit emerged from the use of units of mathematical structure. Contrasting the *Evelyn 4.9* and *Evelyn 4.10* excerpts provides an example of how these two different uses of the term unit could quickly change, depending on the context. While Evelyn used the term *unit* to refer to physical units in the *Evelyn 4.9* excerpt, she used the term to refer to her thinking of quarters (1/4) as a unit of mathematical structure in the *Evelyn 4.10* excerpt.

**Measuring unit.** Teachers in this study frequently made use of the term unit in the context of describing the divisor in representations of measurement division. This could be a specific example of a structural unit, described above. In this context, the teachers sometimes used the terms *unit* and *measuring unit* interchangeably. For example, if the numerical statement of a measurement division problem was $2/5 ÷ 1/3$, teachers would typically read the problem statement and then say, “Okay, one-third is my
unit.” During her interview, Ivy provided a definition of measurement division that included flexibility in conceptualizing the measurement unit as part of representational competence.

*Ivy 4.14* I: Or in another language, four feet is what’s considered one unit, and how many units are there? So measurement model is we have to think about...we kind of change the unit. Is three apples one unit? In this case four feet of wood is one unit. In other examples, maybe three candies are considered one unit and you have to think about how many batches of candies can we make.

**Ghost units versus real units.** As described in the Within-Case Results section, teachers in this study acknowledged an ontological difference between iterations of the given dividend and the following iterations that established a unit rate in representations of partitive division with fractions. The ontological distinction was represented in two ways: first, by drawing dashed lines instead of solid lines for iterations that were part of the scaling process, and second, by verbally referring to those iterations as “ghosts.” This ontological distinction became a conceptual resource for some teachers as they made the “leap” between the representation of the given rate and the unit rate in partitive division with fractions. For example, Malory applied the distinction between “ghost units” and “real units” as a conceptual resource in distinguishing between unit-rate and in-progress interpretations of partitive division. While discussing the Maple Tree example in Malory’s interview, she noted that because this was an in-progress partitive division example the iterations in the representation (see Figure 4.38) were not ghost iterations, stating “the phantoms are real.”

**Coordinated Counting**

Teachers in this study made use of whole number counting in the tasks on
representing fractions operations with rectangular area models. Their use of whole number counting of groups (counting unitizations) was not surprising. However, all the teachers exhibited flexibility in the referent of the numbers they counted in that they regularly counted composite units as well as fractional units within the same representation. In a sense, when the teachers constructed units to represent fractions and then counted those units as if they were whole numbers, they were exhibiting a version of unitizing. Ivy in particular was adept at counting composite units and fractional units in short spans of time. In other words, she exhibited flexibility with the referent of her whole number counting. This finding parallels the finding that Ivy’s conceptualization of division appears to be invariant with respect to whether the divisor is a whole number or a fraction.

Flexibility with the referent of whole number counting involved coordinating
levels of units, and at times the teachers lost track of the referent to their whole number counting. One productive application of this coordinated counting was that teachers were able to extend their fair-sharing conceptual resource to the representation of partitive division with fractions. Specifically, considering the fractional partitions of the dividend and divisor to be whole numbers assisted the teachers in the process of correctly partitioning the dividend and distributing it into the divisor. For example, Audrey stated,

*Audrey 4.14  A:  I have one half that I have to equally share among these three slices…so I wanna be able to share it equally, I need to um (she picked up a third color) I have to come up with the least common denominator. I have to have something that can be split three ways equally, so one sixth plus one sixth plus one sixth (she drew ”1/6” in the three partitions she identified as the divisor-share of the 1-rectangle).*

The whole-number-to-whole-number correspondence became an affordance of the representation once the teacher had made fractional units the referent to their whole number counting.

**Research Question 3**

Results from the cross-case analysis did not inform Research Question 3: *Do participants in this study who are actively teaching the PD content construct knowledge differently than participants for whom the content prepares for future teaching?*

Between-Case variability in knowledge construction was visibly dependent on individual participants’ conceptual resources. None of the participants exhibited knowledge construction that I could see as being framed, motivated, informed, or constrained by the participant’s teaching status.

Between-Case variability in the participants’ willingness to engage in the tasks
and participate in think-aloud protocol appeared to be related to the participant’s personality, not teaching status. Of the four participants actively teaching the course during the data collection, Ivy and Evelyn were engaged throughout the four hours of PD. Rachel and Hannah were comparatively reserved. Of the four participants not actively teaching the course during the data collection, Audrey and Malory were engaged throughout the four hours of PD. Mark and Zoey were less engaged. There was no data in the video corpus that led me to infer a participant’s behavior was contingent on their teaching status.

Summary

This chapter presented within-case results for each of the eight teachers participating in this study, followed by cross-case results. Within-case results were limited to portions of the teachers’ engagement in the PD task sequence (see Appendix E) that most informed Research Questions 1 and 2. Cross-Case results emerged in five categories: Diversity of conceptions of fractions and division, multiple interpretations of partitive division with fractions, difficulty conceptualizing partitive division with fractional divisors, the role of prior knowledge, and managing the meanings of the term “unit.”

Participants in this study provided diverse descriptions of fractions, though the descriptions could be categorized as either part-whole instantiations of tangible items, or as rational numbers. There was less variability in the participants’ descriptions of division. Participants generally stated that division is a process or operation on numbers.
The process was qualitatively described as “breaking up” or “splitting.”

In addition to the measurement and partitive interpretations of division, participants in this study constructed multiple models of partitive division. These models generalized into part-whole models and unit-rate models. Part-whole models attended to a single referent. The referent was seen as a quantity, part of which was known; or a process, part of which was completed. Unit-rate models attended to separate referents for the dividend and divisor.

While the measurement and partitive interpretations did not pose challenges in the context of whole numbers, participants in this study experienced conceptual difficulties when the divisor was a fraction. A common difficulty was that the mental model of sharing could no longer be productively applied to partitive division with fractional divisors. Another common difficulty in conceptualizing partitive division with fractional divisors was making sense of the transformation of the fractional divisor to a value of one. This difficulty was in sharp contrast to the ease that participants exhibited in transformation a whole number divisor to a value of one. Participants also found it difficult to coordinate the components of the IM rule with a physical demonstration of the same division problem. However, while developing the ability to construct representations of partitive division with fractional divisors using rectangular area models, Audrey, Ivy, and Mark spontaneously coordinated components of the IM rule with their representations.

Knowledge construction was dependent on conceptual resources. Frequently, prior knowledge was an influential conceptual resource. Prior knowledge was exhibited
in two forms: First, prior knowledge had origins rooted in a participant’s experiences, academic or otherwise, prior to the study. Examples included participants’ images of fair-sharing or Mark’s description of unit-rate as density. The second form of prior knowledge seemed to have been constructed during the participants’ engagement in the tasks. In particular, participants’ experienced conceptual difficulties at first with partitive division with fractional divisors, as presented in the Crackers and Bowls video. However, their productive struggle resulted in a form of prior knowledge that was productively applied when the participants began to construct representations of partitive division with fractions using rectangular area models.

Participants in this study made productive use of the term unit, quickly adapting it into their specialized knowledge structures and their pedagogical thinking. The term afforded specificity in the communication of mathematical ideas during the PD meetings. However, broad use of the term sometimes indicated that the participants could have been even more effective in their communication of mathematical ideas had they been provided with even more specific terms.
CHAPTER 5
DISCUSSION

This qualitative, multiple case study compared the conceptual difficulties and experiences of eight in-service teachers as they learned to use rectangular area models to represent partitive division operations with fractional operands. These tasks provided the teachers with an opportunity to apply their existing understandings of fractions operations in a new context. This study focused on the teachers’ experiences learning to represent partitive division with fractions because partitive division with fractions is the most difficult fractions operation (e.g., Hohensee & Jansen, 2017; Lo & Luo, 2012; Ma, 2010, 2010; Wahyu et al., 2020). Data collected from the teachers’ engagement in these tasks was analyzed using the Knowledge in Pieces (KiP) epistemological framework (diSessa, 1993, 2004) to inform the following research questions.

1. What are the conceptual resources exhibited by in-service teachers as they construct a deeper understanding of division with fractions and improve their ability to productively represent division with fractions using rectangular area models?

2. What role does the teachers’ perceptions of the referent unit play in their knowledge construction across tasks involving division with fractions?

3. Do participants in this study who are actively teaching the PD content construct knowledge differently than participants for whom the content prepares for future teaching?

This chapter is organized in the following sections: (1) multiple models of division, (2) findings related to the KiP framework, (3) future research and educational implications, and (4) conclusion of the study.
Multiple Models of Division

Research on in-service teachers’ (ISTs’) and pre-service teachers’ (PSTs’) learning of division with fractions widely attends to the distinction between measurement and partitive models of division (Lamon, 2007; Lee & Sztajn, 2008; Lo & Luo, 2012; Luo et al., 2011; Ma, 2010; Siy, 2018). These two models are often called interpretations (e.g., Flores, 2002; Siebert, 2002; Sinicrope et al., 2002). Some studies focus on measurement division (e.g., M. Y. Lee, 2017) and some studies focus on partitive division (e.g., Hohensee & Jansen, 2017; Wahyu et al., 2020). Research shows that these models of division become challenging when the divisor is a fraction (Lo & Luo, 2012; Simon, 1993). Teachers in the present study were introduced to both models of division and tasked with representing and discussing them. As the teachers engaged with these two models, their concept projections of partitive division reflected several distinct situational structures when the divisor was a fraction. In Chapter Four I described the concept projections of partitive division that the teachers exhibited and categorized them based on their distinct structures. In this section I summarize these structures and discuss similar findings in the literature.

Participants’ Structures of Partitive Division

In generating examples of partitive division with fractional divisors, teachers in this study constructed concept projections reflecting two distinct structures. I called the first structure part-whole because concept projections of this structure attended to the dividend as part of a whole and identified finding the whole as the goal of division. This
was Ivy’s general approach to representing division problems with rectangular area models. Ivy would state the problem, $1 \frac{3}{4} \div \frac{1}{2}$ as “one and three-fourths is half of something; what’s that something.” Ma (2010) defined partitive division with fractions using this numerical example stating that it is “finding a number such that $\frac{1}{2}$ of it is $1 \frac{3}{4}$” (p. 74). Story problems of this structure attended to one referent quantity. A teacher-generated example from Ma is finding the length of a rope if part of it is known.

Teachers in this study also constructed a time-situated version of part-whole structure applied to partitive division with fractional divisors in which time was the factor explaining the incomplete state. In Chapter 4, I called this the in-progress structure. Evelyn’s suitcase situation was a solution to $1 \frac{3}{4} \div \frac{1}{2}$ where the divisor quantified the amount of progress made toward completing the task of packing a suitcase. An example from the test instrument in Lo and Luo’s (2012) study is finding the miles a jogger plans to jog in a week if he jogged $1 \frac{1}{2}$ miles yesterday, and that was $\frac{3}{8}$ of his weekly goal.

The second distinct structure of partitive division with fractional divisors was the unit-rate structure. Evelyn’s pies and containers situation structured $1 \frac{3}{4} \div \frac{1}{2}$ as a unit rate because the quotient quantified the amount of pie per one container. Story problems or situations of the unit rate structure attended to two referent units (e.g., pies and containers). However, the unit-rate structure usually prioritized attention to the dividend quantity. Similar examples are common in the literature because situating partitive division with fractions as a unit rate is a developing line of research on the learning of division with fractions (e.g., Hohensee & Jansen, 2017; Jansen & Hohensee, 2016; Lo & Luo, 2012; Wayhu et al., 2020). For instance, M. Y. Lee’s (2017) Flour example (used in
the present study) states that half a pound of flour can be purchased with three-quarters of a dollar and asks how much flour can be purchased with one dollar. Many unit-rate partitive division situations in the literature are time rates. For instance, Sinicrope et al. (2002) give an example of a printer that prints 20 pages in two and a half minutes, asking how many pages it prints per minute.

A time-rate example may be conflated with the in-progress structure because it is situated in time. Time-rate is not an in-progress structure in my analysis because time-rate quantifies time in terms of physical units as a second referent. Despite the existence of a second referent, unit-rate applications of partitive division with fractions in the literature appear to focus on the dividend quantity. For example, two common definitions of partitive division of fractions as unit rate are: (a) how much of the dividend should be associated with one whole unit of the divisor (Jansen & Hohensee, 2016; Siebert, 2002), and (b) finding the amount in one whole set or the amount per unit (e.g., Sinicrope et al., 2002).

In the present study, unit rates were sometimes conceptualized as intensive quantities (Greer, 1992). An intensive quantity is one in which two (or more) units have equal importance and function together to describe a physical quality that is not reducible to one of the units. In the context of rational number learning, Lamon (2008) described intensive quantities as “ratios that are formed by comparing two other quantities” (p. 7). Some researchers (e.g., Beckman & Izsák, 2015) use the term derived instead of intensive. Giancoli (2013) stated that there are seven base quantities (units of measurement), calling “all other quantities derived quantities” (pp. 10-11) because they
are defined in terms of the base quantities. In the present study, Ivy explained unit rate in
terms of speed (discussed below). Mark’s concept projection of partitive division of
fractions as density was a novel construction of the partitive quotient as intensive
quantity.

Table 5.1 summarizes the two distinct structures of partitive division with
fractional divisors observed in this study. As shown in Table 5.1, each structure has two
versions. Because the literature does not yet appear to have developed a common
terminology, locating the findings summarized by Table 5.1 is not straight-forward. The
following section discusses the diversity of terminology in the literature on division with
fractions.

Table 5.1

*Structures of Partitive Division with Fractional Divisors Observed in This Study*

<table>
<thead>
<tr>
<th>Structure variations</th>
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</thead>
<tbody>
<tr>
<td>Part-Whole</td>
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<tr>
<td>Partial amount</td>
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<tr>
<td>In-Progress</td>
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<tr>
<td>Unit-Rate</td>
</tr>
<tr>
<td>Unit amount</td>
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<tr>
<td>Intensive quantity</td>
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</table>

**Diverse Terminologies of Division with Fractions**

The literature on division with fractions shows a diversity of terminology (Wahyu et al., 2020). Greer (1992) described the psychological complexity of multiplication and division as models of situations and categorized them in terms of classes. Greer’s classes of situations include seventeen classes of division. In addition to the measurement and partitive models of division common in the literature, Greer’s classes include the part-
whole model evidenced in the present study.

Participants in Lo and Luo’s (2012) study exhibited part-whole “strategies” (e.g., p. 491) in their pencil-and-paper responses to division word problems. These researchers developed a coding that was constructed to be broad enough to capture models beyond the measurement and partitive models commonly attended to in previous studies. One out of the 45 PSTs in their study constructed a rate “structure” (p. 493) in generating an example of division with fractions. Rate is one of Greer’s (1992) classes of division. It occurred once in the present study as well, in the form of the train example that one of Hannah’s students had constructed as a homework assignment.

In summary, the literature does not yet have a standard terminology to describe the various constructions of division seen across multiple studies. In this study I used the term models in the general sense for lack of a better term. However, I used the term structures for the specific findings in the present study (see Table 5.1) because, as researchers have noted, when the operands are fractions, division problems become conceptually difficult, and constructions of their solutions become tied to structures of the context.

**Differences Between the Structures in the Present Study and Models in Ma (2010)**

The partitive division with fractions story problems described in Ma (2010, pp. 74-76) include classes of division described by Greer (1992) that are not based on the equal groups model of multiplication. Some of these story problems are based on the *part-whole* and *rectangular area* classes of division. However, the unit rate model is
missing from models of division in Ma’s study. Neither Ma nor the teachers in her study refer to partitive division as finding a unit rate. Given that Ma’s findings contributed to the conceptual framework for the present study, the contrasting results deserve discussion.

One possible explanation for the difference between Ma’s (2010) results and the results of the present study is the difference in the participants. Ma’s participants were elementary school teachers. Thus, the concepts of unit rate and intensive quantity may have been beyond the content that these teachers regularly taught. The teachers in Ma’s study may have chosen to make partitive division with fractional divisors accessible to their elementary school students by focusing on the simpler, part-whole interpretations. Similarly, recent studies on PSTs abilities with and conceptions of partitive division with fractions do not generally include participants who are mathematics or science majors. In other words, the participants in the present study were a novel population, given that half of them were experienced science teachers at the high school and college levels. The background of the participants in the present study, along with the multi-hour interview format, likely explains the larger occurrence of unit rate conceptions compared to other studies on the learning of partitive division with fractions.

In summary, different models of division are described together under the partitive division umbrella in Ma’s (2010) results, while the unit-rate model is missing. In the present study, the unit-rate model was a common mathematical structure that emerged in two versions. The following section describes the two versions of the unit-rate structure.
Comparing the Unit Amount and Intensive Quantity Versions of the Unit Rate Structure

As findings from this study indicated, the unit-rate structure of partitive division with fractions appeared to have two versions. In the unit-amount version of unit-rate, the quotient is the answer to a question attending to a single quantity represented by the dividend. In the intensive quantity version of unit-rate, the dividend and divisor quantities have equal status, and the quotient is recognized as an intensive quantity. This distinction between unit-amount and intensive quantity versions of unit-rate structure of partitive division with fractions does not seem to be discussed in the literature. In addition to the potential lack of relevance to elementary school teaching, mentioned above, another possible reason is that the two versions are often conflated or that their distinctions are overlooked or considered unimportant. This might happen when either version could be constructed from the same example, representation, or analysis. For example, Seibert (2002) presented the following story problem:

Melissa used 2/3 of a can of frosting to frost 1/2 a cake. How much of the cake could she frost with the whole can of frosting? (p. 251)

If the focus is on Melissa’s use of a can of frosting, then the amount frosted by that can define the per-group amount that is to be found (3/4 of the cake). In terms of unit-amount unit-rate, this can be stated as 3/4 cake for one can. If the focus is on the frosting capacity of the product, then the quotient expresses an intensive quantity unit-rate (3/4 cakes per can). While the later focus seems less likely to arise in everyday activity, it illustrates the point that the difference between the two unit-rate structures emerging in this study involves only a modest shift in perspective.
Beckmann and Izsák’s (2015) description of the two different ways that “measurement units can be associated with quantities” (p. 20) quantitatively describes this shift of perspective. Their analysis is based on the equal groups model of multiplication:

\[ M \text{ (# of groups)} \cdot N \text{ (# of units in a whole group)} = P \text{ (# of units in M groups)} \]

They give an example of making three cakes, each of which requires four cups of flour. In one perspective, the quotient is recognized as an intensive quantity (Lamon, 2007). In Izsák and Beckmann’s terms, it is a “derived unit of the form ‘units for P per 1 unit for M’ which expresses a unit rate” (p. 20). This perspective can be written as:

\[(3 \text{ cakes}) \cdot (4 \text{ cups of flour/cake}) = (12 \text{ cups of flour})\]

The second perspective is a type of multiplicative comparison in which \( N \) and \( P \) have the same units, and the multiplier, \( M \), “may be viewed as a unitless scale factor” (Beckmann & Izsák, 2015, p. 20). This second perspective can be written as:

\[3 \cdot (4 \text{ cups of flour}) = (12 \text{ cups of flour})\]

This second perspective reduces the complexity of the coordination of the physical units. Many partitive division situations appear to be reducible in this way, particularly when the question is framed as seeking one quantity in a two-quantity situation. For example, M. Y. Lee’s (2017) Flour example (see Figure 5.1), used in the present study, asks how much flour can be purchased with a dollar (unit-amount unit-rate).

M. Y. Lee’s (2017) representation illustrates the partitioning and iterating involved in determining the quotient. However, only one of the units composing the unit rate is illustrated. Each bar refers to an amount of flour, and the total length represents
one pound of flour. The second unit, *dollars*, does not explicitly appear. In describing the representation, Lee is explicit that “the referent unit for 1/2 and 4/6 is the whole, but the referent unit for 3/4 is 4/6” (p. 332-333). However, an intensive quantity version of the same structure can be constructed from the same situation by asking for the cost of flour. In that case, the referent for 3/4 would be one whole dollar rather than 4/6 pounds of flour. 

In the present study, Ivy expressed an understanding of this shift in perspective that produces the two versions of unit rate in partitive division. The interviewer asked Ivy if she could explain how the referent units for the quotient of a partitive division problem are the same or different from the referent unit for the dividend. She replied:

*Ivy 5.1  I:* If we think about the dividend is apples and the divisor is the number of people. Let’s say twenty apples and four students. The quotient is same thing, it’s also in apples, but the quotient is also a rate, it could be apples per person. So a rate is actually a little bit different than the dividend. If we are talking about speed, like the distance and the time, and the speed, right. The speed and the distance are two different things. So it’s the same thing but it’s also a different thing if you’re talking about getting to detail because
the quotient is actually apples per person. So it’s almost like 60 miles per hour. So 120 miles and 60 miles per hour are two different things because one is just a straight distance; the other one is a rate. So I think of the quotient as a rate…more than just apples.

Thus, she exhibited an understanding that the partitive quotient is a unit rate even when a simple answer such as *five apples* might do. At the same time, she exhibited an understanding that the unit rate may be an intrinsic quantity, such as speed.

To summarize this section on models of division, the research literature on the learning of division with fractions widely describes two models (or interpretations) of division based on equal-measures multiplication: measurement and partitive. However, findings from the present study and other studies (e.g., Lo & Luo, 2012; Ma, 2010) indicate that when participants are asked to construct their own examples, multiple models based on various situations and structures emerge. Researchers do not yet appear to have a common terminology for describing the full range of these results. Additionally, the types of multiple models that emerge may largely depend on the conceptual resources of the participant and the context in which the concept projections were constructed. Findings in the present study regarding the unit-rate structure of partitive division contrast with the findings of Ma (2010), in which participants did not construct unit-rate examples of partitive division. The difference in findings likely resulted from the difference in the type and level of the subject matter taught by the teachers in the two studies.

**Knowledge in Pieces Framework**

According to Wagner (2010), each different model of division “entails a different
construal (one might say “structure”) of the situation…so each entails a different concept projection” (p. 451). Multiple concept projections are needed to span the division concept. The present study used the KiP epistemology as a framework for analyzing participants’ engagement in tasks that deepened their understanding of division with fractions. In this section I discuss selected topics within the findings that were informed by the KiP framework. These topics are organized by findings related to contextuality, findings related to types of knowledge, and findings that align with the findings of Hohensee and Jansen (2017).

Contextuality

Contextuality is one of the core phenomena that KiP attends to. Analysis of knowledge as a complex system of conceptual resources shows that the coordination of the resources depends on the context in which a concept is applied (e.g., diSessa, 2004). Findings from the present study support the notion that, in the learning of mathematics, numbers can constitute a context. For example, while *fair-sharing* is an intuitive and productive conceptual resource in the context of partitive division with whole numbers (Empson, 1995), it does not make sense when applied to fractional divisors (e.g., Lo & Luo, 2012). Participants in the present study experienced conceptual difficulties when trying to understand area models of partitive division with a fractional divisor, even though they could explain partitive division with whole numbers.

Concept projections are tied to the particulars of situations (e.g., diSessa & Sherin, 1998; Wagner, 2006, 2010). Several findings in the present study highlighted the role of context. For example, Audrey constructed her first concept projection of
measurement division as she looked at a picture of a board and imagined using bricks to measure its length.

In their discussion of transfer from a KiP perspective, diSessa and Wagner (2005) describe how these context-dependent particulars can facilitate transfer rather than hinder it. In the present study, the participants made use of the Bowls and Crackers video to transfer their newly constructed understanding of partitive division with fractional divisors from the context of the physical example to the context of rectangular area models. This finding highlights several KiP features. First, what kind of knowledge did the teachers use when referring to the Bowls and Crackers video as they learned to represent partitive division with fractional divisors in the context of rectangular area models? Empirical results show that prior knowledge plays a critical role as a conceptual resource in constructing concept projections (diSessa, 2004; diSessa & Sherin, 1998; Wagner, 2010). The teachers’ knowledge developed from the Bowls and Crackers video functioned as a form of prior knowledge. However, it was newly developed knowledge. Moreover, it was generally knowledge that the teachers acquired through productive struggle. The data does not settle the question of whether the teachers’ productive struggle built up a structure that facilitated the transfer or whether the productive struggle developed conceptual resources that the teachers applied in later concept projections. The data does show that the participants engaged in analogical thinking, for example mapping the divisor rectangle to the bowl and partitions of the dividend to pieces of crackers.

The second KiP feature invoked by the participants’ engagement with the Bowls and Crackers video is that learning can be described as the span of a concept (e.g.,
diSessa et al., 2016). Participants exhibited span of their division concept as they learned to represent partitive division with fractional divisors in the context of rectangular area models by making productive use of knowledge they constructed from the Bowls and Crackers video. Audrey and Mark, for example, exhibited increased span when learning to represent partitive division with fractional divisors in the context of rectangular area models by coordinating the steps of the invert-and-multiply (IM) procedure with their representations. Previously, during their productive struggle with the Crackers and Bowls video, they were unable to coordinate the steps in the IM procedure with the physical demonstration.

A novel example of span in the present study was the idea that Audrey had while on break. Audrey wondered if the method of representation she was learning would “still work” if the divisor was a mixed number instead of a proper fraction. Her conceptual resources in that moment provided her with an inference that it might work. Without scaffolding, she was able to use her newly developed representational skills to test the idea and construct an informal mathematical proof.

Another context-related finding was the various situation-dependent mathematical structures that participants exhibited in their concept projections of division with fractions. Evelyn provided a stark example of how concept projections function with respect to context as her 22-minute productive struggle with pies and containers ended with her extemporaneous construction of the suitcase context for partitive division with fractions. As described in the previous section, the suitcase context may have been conceptually simpler because of the reduced complexity of the part-whole structure.
Nonetheless, the excerpts from that episode illustrated how the mathematical structures of the concept projections of partitive division reflected the constraints and affordances of the situation.

Findings from the present study indicate that different work must be done to span different contexts with different conceptual resources. For example, Evelyn’s concept of partitive division with fractional divisors spanned the pies and containers situation to the suitcase situation apparently effortlessly (in that moment). Distinguishing between the unit-amount and the intensive quantity versions of the unit-rate structure appeared to be more difficult to recognize or articulate for Audrey, who was not confident in her knowledge of intensive quantity. Mark, on the other hand, readily constructed a concept projection of partitive division as intensive quantity by applying his well-developed prior knowledge of density.

**Types of Knowledge**

As described in Chapter 2, KiP takes a grounded theory approach to types of knowledge and has empirically identified two distinct forms of knowledge (i.e., p-prims and coordination classes). These forms of knowledge are ubiquitous in learning Newtonian physics and thermodynamics (diSessa, 2018). An empirical question for KiP epistemology is whether these forms of knowledge exist in the learning of other domains, such as mathematics. The data and analysis from the present study indicates that at least the coordination of conceptual resources described in previous studies in physics learning applies to the learning of division concepts. Additionally, participants in the present study exhibited *situational-primitive* knowledge that aligned with findings from previous
studies. For example, the sharing image is commonly invoked in conceptions of partitive division (Empson, 1995; Lamon, 1996, 2007, 2008). The completing image was exhibited in the present study and ubiquitous in Ma (2010). Additionally, spreading appears to be a productive example of primitive knowledge in constructing concept projections of partitive division as intensive quantity (Hohensee & Jansen, 2017; Siebert, 2002, Wahyu et al., 2020).

diSessa (2018) suggested that in the learning of mathematics, much prior knowledge is more likely to come from previous instruction. In the present study, participants incorporated prior knowledge of at-the-same-rate as a conceptual resource in transforming the given rate to a unit rate. However, instructed prior knowledge can precipitate challenges when not coordinated to function in a new context. For example, the notion that “multiplication makes things bigger” (p. 70) may present a challenge when first constructing concept projections of multiplication with rational numbers. In the present study, Malory’s prior knowledge that mixed numbers must sometimes be represented as improper fractions presented a challenge for her as she first learned to represent partitive division with fractions using rectangular area models.

In the construction of concept projections there is a dynamic interplay between perceptions and inferences (e.g., Levin & diSessa, 2016). Existence (diSessa & Wagner, 2005, p. 131) is one type of inference exhibited in the present study. diSessa (2004) described this happening when inferences “let us know that the scene before us contains an exemplar of the right type. It must be there” (p. 142, italics in original). Findings from the present study demonstrated existence when participants inferred something important
was there, but they just could not see it. For example, in discussing the *Crackers and Bowls* video Audrey stated “I’m wanting to see another half bowl there” as she productively struggled to coordinate her perceptions in that moment with her prior knowledge that the quotient is based on one. Evelyn exhibited a similar experience in her pies and containers episode: Evelyn knew that the quotient must be based on one. She also knew that her example was the same numerical division problem as in the *Crackers and Bowls* video. Yet she could not see the one in her example (for quite a while). The data from this study doesn’t tell us what exactly changed in Audrey’s or Evelyn’s concept projections, but they were ultimately able to see the one in the quotient.

**Transitional Conceptions**

Findings from the present study support findings from Hohensee and Jansen’s (2017) study on PSTs’ conceptions of partitive division of fractions. Hohensee and Jansen applied a theoretical framework that shares some common roots with KiP. These researchers adopted the *transitional conceptions* construct (Moschkovich, 1999) because their earlier research suggested that “PST’s conceptions offered potential for fine-grained refinement” (Hohensee & Jansen, 2017, p. 212). This perspective differs from previous research on partitive division with fractions that focused on identifying PST’s and IST’s misconceptions and inabilities (e.g., Ball, 1990; Ma, 2010; Tirosh, 2000). The *transitional conceptions* construct aligns with the KiP view that conceptual change involves restructuring prior knowledge in productive ways.

Hohensee and Jansen identified an initial transitional conception and two refinements that explained PST’s conceptions with partitive division when solving
partitive division problems with proper fractions as divisors. One of the criteria for coding participants as exhibiting the initial transitional conception was whether PSTs included iteration in their solutions to the partitive division tasks. Findings from the present study align with Hohensee and Jansen’s (2017) finding that participants could correctly solve and generate a representation of partitive division problems with proper fraction divisors without evidencing iteration as a part of their conception of partitive division.

Similar findings between Hohensee and Jansen (2017) and the present study are interpreted somewhat differently. For example, Hohensee and Jansen found it interesting that several PSTs “did not fully interpret iterating as part of division despite using iterating to solve the problem” (p. 226). These PST’s were considered to have a conception of division based on partitioning; a conception that fell short of the target conception of partitive division. For example, when asked to identify where division was taking place in her representation (of \(18 \div \frac{1}{3}\)), one PST replied,

> It kind of already has it divided because you’re looking for one whole and it’s been divided into one third and eighteen pieces. So it’s like you’re kind of adding it back up but it was already divided because it’s not a full it’s a piece of the whole. (Hohensee & Jansen, 2017, p. 226)

From the perspective of the present study, this PST evidenced a concept projection in which the proportional relationship between the dividend and divisor in the problem statement may have been a knowledge resource that was more prominent than knowledge of the iteration process needed to determine the quotient. This was a common finding in the present study. For example, Mark explicitly stated that iteration “didn’t feel natural” and Ivy referred to it as “a detail.” This disinterest in iteration, however, did not
mean that these participants were unaware of, or unconcerned with the multiplicative relationship between the dividend and divisor and the essential fact that the multiplicative relationship needs to be preserved. For these participants, iteration was a conceptual resource, but not strongly cued or incorporated in their concept projections. At least in the PD tasks these participants engaged with, the multiplication process necessary to scale the given rate to the unit rate did not appear to be based on a repeated addition conception of multiplication, which iteration models, but rather on what Thompson and Saldanha (2003) referred to as conceptualized multiplication (p. 116); that is, seeing multiplication as a continuous scaling. These participants were fully aware of the goal of determining the amount of the dividend associated with the divisor when the divisor is scaled to a value of one, but iteration was not a necessary component of reaching that goal. They might not have been considered by Hohensee and Jansen (2017) as having the initial transitional conception of partitive division. However, from a KiP view, mathematical competence is often exhibited by a “remarkable range of strategies adapted rather precisely to particulars of the problems posed” (diSessa, 2018, p. 78). Mark and Ivy were quite capable of determining the partitive quotient in a variety of contexts by choosing efficient strategies based on the particular numbers involved rather than relying on partitioning and iterating as a heuristic. It was in fact their competence that led them to devalue the use of partitioning and iterating.

Moments of perturbation in PST’s and IST’s thinking was another common finding between the present study and Hohensee and Jansen (2017). Perturbation was a primary subject for the analysis in both studies, and both studies relied on interviews as a
data source to capture these moments. For example, in Hohensee and Jansen (2017) Kay had correctly solved a partitive division problem with a proper fraction divisor but felt as though her solution was not right because it involved multiplication, stating, “I keep thinking at it from like multiplication point of view…because when I see division, I assume that you’re breaking them down” (p. 228). From a KiP view, the assumption that division results in smaller numbers (“breaking down”) and multiplication results in larger numbers may be a type of primitive knowledge. That prior knowledge may cause conflict when its inference is not aligned with other inferences or perceptions in a situation.

In transitional conceptions terms, this conflict of inference might be a criterion for coding a particular transitional conception that has the potential for refinement. In KiP terms, the refinement happens at the level of components of a concept projection, allowing the components to span more contexts and situations. For example, Ivy knew that division is always finding a value “per one,” whether the divisor is bigger or smaller than one. In the right discussion and context, Ivy might evidence that she still holds a multiplication makes numbers bigger primitive knowledge resource, but that knowledge is integrated into a knowledge system that cues the multiplication make numbers bigger primitive knowledge resource more judiciously. Ivy developed that knowledge system before participating in the present study in what was likely a long process.

In sum, the transitional conception construct aligns with the KiP framework and findings from the present study in terms of framing knowledge and learning of division with fractions as changes in an initial structure that need refinement rather than replacement, and that the changes can take place in identifiable pieces.
Research Implications and Future Research

Recent research has been moving from finding that PSTs and ISTs experience difficulties with understanding and representing division with fractions to observing what the difficulties are (Hohensee & Jansen, 2017; Izsák, 2008; M. Y. Lee, 2017; Stevens et al., 2020). Future research could evaluate the effectiveness of teacher education curricula and pedagogies that have been developed based on the difficulties identified by recent research. Cross-case findings from the present study indicate that developing such curricula and pedagogy may face challenges. For example, with respect to conceptualizing the partitive quotient when the divisor is a fraction, participants in the present study exhibited variability in the ways that they ultimately made sense of transforming the given divisor to a value of one. Further studies focused on that one conceptual difficulty could be valuable, since results from the present study suggest that it poses challenges even for experienced teachers. Another difficulty in developing PSTs and ISTs understanding of division of fractions is the fact that there are multiple models of division, and that the division examples teachers encounter in a classroom may be varied in structure and bound to a variety of situations. Future research might find ways that teachers manage that complexity when it arises in the classroom.

Another line of research could address PSTs and ISTs learning of unit rate and the intensive quantities that unit rates can form. Siebert (2002) closed his chapter on the IM rule by stating that children should “be able to quantify intensive quantities such as speed and slope” which “arise from sharing situations” (p. 256). Future research could address how learners construct conceptions of partitive division as intensive quantity. This
mathematics topic is important because students’ understanding of measurement and physical units needs improvement in the U.S. (Smith & Barrett, 2017).

**Educational Implications**

In this section I describe four educational implications of the findings from the present study. The educational implications are (a) knowledge at the level of conceptual resources, (b) use of the term *units*, (c) that unit-rate division is more difficult but more realistic, and (d) the use of rectangular area models for representing division with fractions.

**Knowledge at the Level of Conceptual Resources**

Arguably, focusing on the level of conceptual resources may be too detail-oriented to be of interest to many teachers and researchers. However, it may be that this level of detail accounts for phenomena such as changes between transitional conceptions and diversity in learning trajectories. For example, at the end of the bowls and crackers video, Rachel and Mark said that physically bringing in another half bowl would provide a sense making representation. However, Evelyn said that would be confusing because “it’s not really there.” For Evelyn and Rachel, *drawing* the other half of the bowl along with its contents was preferable because it demonstrated that that part of the process was, in Evelyn’s words, “just a concept.” In the present study, these types of idiosyncrasies seemed particularly evident when the participants had just been presented with a new representation or new idea. Expecting that students may react idiosyncratically to new material at the level of individual conceptual resources while also expecting that
classrooms are likely to experience a predictable set of transitional conceptions can leverage the findings of the present study as well as Hohensee and Jansen (2017). While this approach has been developed in the teaching and learning of Newtonian mechanics (diSessa & Minstrell, 1998; Kapon & diSessa, 2012; Minstrell, 1992), findings from the present study suggest this may be a productive approach in the teaching and learning of operations with fractions. In other words, instruction may be more effective when teachers expect a set of transitional conceptions to recur in multiple classrooms but at the same time are willing to attend to the idiosyncratic differences in individual students’ conceptual resources.

**Use of the Term *Units***

The issue of units of mathematical structure (e.g., composite units, ratios or proportions, quantities in terms of fractions such “quarters”) having a separate psychological sense from physical units likely emerged in this study because half of the participants had a science background. Compared to other studies on ISTs’ or PSTs’ conceptions of division with fractions, the demographics of the participants in the present study likely explained their attention to physical units and their ability to construct concept projections of division with fractions entailing physical units and intensive quantities. However, k-12 students learning about partitive division and being exposed to *unit* concepts in mathematics classes will also be exposed to the term *unit* in their science classes as well. A lack of attention to unit concepts and the distinction between uses of the term *unit* is likely to promote confusion.
Unit-Rate Division is More Difficult but More Realistic

Teachers in the present study noted that part-whole partitive division with fractional divisor situations seemed contrived. Evelyn, Hannah, and Rachel commented that story problems in which a portion of a quantity is known don’t come up very often. The elementary school teachers in Ma’s (2010) study may have chosen to use them for their conceptual simplicity. Unit rate situations appeared more realistic to the teachers in the present study. The intensive unit rate model of partitive division invokes the physical sciences at a level that may not be accessible to students until middle school or high school (Lamon, 2007). For example, 75 lumens fall on a 3/4 square-meter surface, what is the illuminance? (100 lux). Or, 75 pounds of force are applied to a 3/4 square-centimeter surface, what is the pressure? (100 PSI). Findings from the present study indicate that intensive unit rate lies at the difficult end of the spectrum of structures and conceptions that students will encounter in learning partitive division with fractional divisors. Competence with intensive quantity division may require several phases of conceptual change, involving changes in psychological activity structures from discrete whole number sharing to discrete fractional sharing to continuous spreading (e.g., Hohensee & Jansen, 2017; Siebert, 2002).

Use of Rectangular Area Models for Representing Division with Fractions

Finally, findings from the present study indicate that representing partitive division with fractions using rectangular area models provides productive struggle that can surface conceptual resources and provide opportunities to develop concept
projections of division. Partitive division with fractional divisors does not make practical sense at first because learners find it challenging to conceptualize a fraction of a share (e.g., Lo & Luo, 2012). However, rectangular area representations of partitive division with fractions can tangibly illustrate the partitioning and distributing actions needed in the case of a fractional share without raising concern for a situational meaning. Rectangular area representations of partitive division with fractions also explicate the actions of the IM rule (Siebert, 2002). Additionally, rectangular area representations of partitive division provide a model in which partitioning and iterating can be practiced. For example, Audrey’s representation of fractional division with a dividend larger than one, and Rachel’s unnecessary partitioning of the dividend in her representation of the last problem in the PD task set illustrated teachers using the model as a tool in their own mathematical sense-making activity.

Conclusions

In this section I describe conclusions with respect to the foci of the research questions, followed by general conclusions with respect to the literature.

Conceptual Resources Exhibited by ISTs When Constructing Division of Fractions

Research question 1 asked, What are the conceptual resources exhibited by ISTs as they construct a deeper understanding of division with fractions and improve their ability to effectively model fractions division with pictorial diagrams? Conceptual resources exhibited by the ISTs as they constructed a deeper understanding of division
with fractions included prior knowledge as well as newly formed knowledge. These conceptual resources included common, normative knowledge as well as idiosyncratic knowledge. After watching the Crackers and Bowls video, participants experienced a perturbation referred to as the leap. The participants knew that the IM rule applied to the division problem but could not see it happening in the representation. As a type of procedural knowledge, the IM rule did not at first function as a productive conceptual resource. This was also true for the division is based on one conceptual resource that all the participants brought to the study. Through productive struggle on later tasks, the participants began to use the Crackers and Bowls video as a conceptual resource to productively represent division with fractions problems and identify the IM rule. This conceptual resource afforded analogical reasoning. It mapped a whole bowl to the one in the division is based on one conceptual resource, also referred to as the scaled denominator; also represented by the 1-rectangle in the rectangular area models. All participants evidenced this mapping, and it recurred as a conceptual resource. This analogy may have been such a productive conceptual resource because, at least in part, it might access a more primitive form of knowledge: that of filling containers (M. Levin, personal communication). Audrey, for example, transitioned her language across tasks from full bowl to full share, evidencing productive use of filling containers, sharing, and a concept of one, as interacting conceptual resources.

Whole number counting was a conceptual resource used by all the participants. Coordinated counting was exhibited across the tasks by all participants as a general skill. In a more particular way, whole number counting combined with a fair sharing
A notable result relating to conceptual resources was variation of the productive conceptual resources that were employed idiosyncratically across the participants. All participants exhibited an informal version of a mathematically normative concept: that division is based on one. This concept has been referred to renormalization (Lamon, 1994 M. Y. Lee, 2017) and I referred to it as the transformation of the given rate to the unit rate. This transformation was described with idiosyncratic conceptual resources, such as Ivy using *scale it up*, Evelyn using *at the same rate*, Malory stating that fractions are based on proportions, Mark’s use of density, and Audrey’s description of adding *ghost units*.

**ISTs’ Context and Models of Division When Constructing the Referent Unit**

Research question 2 asked, *What role does the ISTs’ perceptions of the referent unit play in their knowledge construction across tasks involving multiplication and both types of division with fractions?* Participants’ perceptions of the referent unit were context-sensitive and had a visible effect on the participants’ knowledge construction, particularly with respect to models of division that included two referents.

Contextuality was evidenced by the fact that all participants except Ivy experienced *the leap* perturbation. In the physical context crackers and bowls, the referent unit was not at first evident leaving the participants unable to apply their knowledge of
Multiple models of division emerged in this study. Participants found it easier to construct productive concept projections when the division model had only one referent. This finding may explain why previous researchers have found that when dividing fractions, the measurement model appears to be easier. Participants in this study first learned the difference between measurement and partitive division using whole numbers. When learning to represent division with fractions using rectangular area models, the participants learned to recognize that the dividend and divisor have the same referent in the measurement model. Additionally, partitive division models emerged for which part-whole concept projections could be constructed, effectively attending to only one referent.

Ivy and Malory could not at first interpret a rectangular area representation of partitive division because they assumed that the referent unit for the dividend and divisor needed to be the same. For Ivy, this was consistent with her part of something concept projection of partitive division that she brought to the study. Despite her mathematical skills and knowledge, applying a unit rate concept projection of partitive division in rectangular area models required scaffolding. Once she was able to recognize the two-referent-unit nature of partitive division in rectangular area models, she articulated examples based on her prior knowledge, such as speed. Participants found it more challenging to construct productive concept projections of division with fractions when attending equally to two different referents. Participants with experience teaching science appeared to be better prepared to construct productive concept projections of two-referent
models of division with fractions.

**Differences in Knowledge Construction When Actively or Not Actively Teaching**

Research question 3 asked, *Do participants in this study who are actively teaching the PD content construct knowledge differently than participants for whom the content prepares for future teaching?* None of the participants exhibited knowledge construction that I could see as being framed, motivated, informed, or constrained by the participant’s teaching status. While there was between-case variability in the participants’ engagement in the tasks and use of conceptual resources, nothing in the data corpus led me to infer a participant’s behavior was contingent on their teaching status.

**Next Steps and Research Contributions**

The KiP epistemological framework has been applied by mathematics education researchers studying PSTs’ and ISTs’ understanding of proportional reasoning (e.g., Izsák & Jacobson, 2017; Weiland et al., 2020) and multiplication with fractions (e.g., Izsák, 2008; Izsák & Beckmann, 2018). The present study extended previous research on ISTs’ learning of division with fractions by applying KiP to the learning of partitive division with fractions.

The KiP epistemological perspective applied to the analysis of interview data contributed to the identification of underlying conceptual resources. The identification of conceptual resources resulted from longer interview time periods and analysis at a finer grain-size than previous studies on PST’s or IST’s conceptions of partitive division with fractions that relied on survey instruments. Distinct psychological structures emerged that
may be common among learners when engaging in partitive division with fractional divisors. The participants in this study exhibited two distinct structures of partitive division with fractional divisors: one focusing on a single referent quantity and the other focusing on two distinct referent quantities. This finding extends the research literature as these structures and their variants seen in the present study do not appear to have received much attention (e.g., Flores, 2002; H. S. Lee & Sztajn, 2008; Lo & Luo, 2012; Ma, 2010; Nillas, 2003; Siebert, 2002; Tirosh, 2000). Multiple models of partitive division may be in part responsible for some of the difficulties educational researchers have met in understanding the conceptual challenges learners experience with partitive division when the divisor is a fraction.

Findings from the present study support Hohensee and Jansen’s (2017) contribution to the research on learning partitive division with fractions based on the *transitional conception* construct (Moschkovich, 1999). The present study extended the results of Hohensee and Jansen (2017) by identifying conceptual resources (e.g., density, based on one, and the leap) and the construct of concept projections that potentially describe the construction of transitional conceptions.

Research indicates that partitive division with fractions is an important topic in mathematics because it is a foundational to more complex topics such as rate, intensive quantity, proportion, derivatives, probability, and statistics (e.g., Hohensee & Jansen, 2107; Lo & Luo, 2012; Siebert, 2002, Simon, 2003). However, there is little research on how an understanding of partitive division evolves or supports these important mathematics topics. Results of the present study demonstrate that the KiP epistemology
can be a productive framework for future research on learners’ development of partitive division models and connections between partitive division and these other mathematics topics.
REFERENCES


Byrne, D. (2009). Case-based methods: Why we need them; what they are; how to do them. In D. Byrne & C. C. Ragin (Eds.), *The Sage handbook of case-based methods* (pp. 1-10). SAGE Publications.


Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). NCTM.


Appendix A

*College Mathematics* Curriculum Additions
New Page: Two Types of Division

Quantitative literacy is not just about math skills and knowledge; it’s also about conceptual understanding and communication. This first topic and the first assignment in this course provide examples of what is meant by conceptual understanding and communication in quantitative literacy:

There are two very different models for division. We use the word “model” because the mathematics is a model for a physical phenomenon or process. In order to understand division conceptually, we first need to identify the anatomy of a multiplication and division statement. One way to think about multiplication is:

\[(\text{Number of Groups}) \times (\text{Size of Each Group}) = \text{Product}\]

If we know the product and one of the other quantities, we can calculate the missing quantity with division. When we want to know the size of each group, we use partitive division. Partitive division is also called “fair sharing” because it models the process of evenly distributing an amount among recipients. When we want to know the number of groups, we use measurement division. Let’s look at some examples. Basic division has the form:

\[\text{Dividend} \div \text{Divisor} = \text{Quotient}\]

Consider the simple problem, \(8 \div 4 = ?\) An example of partitive division is a Raven who gathered eight worms and has four chicks to feed. How many worms does each chick get?
Of course the chicks get equal (fair) shares. Notice that the quotient, 2, refers to “two worms per chick.” In partitive division, the quotient has the form of a unit rate, as in kilometers per hour or dollars per pound. An example of measurement division is that you have a board that is eight feet long and you want to make shelves that are four feet long. How many shelves can you make?

The reason this is called measurement division (also called quotitive division) is that we used one shelf as a unit of measurement. Yes, we want to know how many shelves we can make, but the quotient, 2, tells us the length of the board in units of the shelves we’re making.

Please watch the animated video before going to the next page. [1 Partitive and
New Page: Referent Units

Units are common in mathematics and for many majors at this university units are part of daily practice. We use the term *referent unit* simply to identify what a particular number is referring to. In the last two examples of $8 \div 4 = 2$ the numbers referred to different kinds of things. In the partitive example the dividend, 8, referred to worms and the divisor, 4, referred to chicks. The quotient, 2, referred again to worms (per chick). In the measurement example, both the dividend and the divisor referred to feet. The quotient, however, referred to number of shelves.

*Referent Units Associated with the Components of Division*

<table>
<thead>
<tr>
<th>Division Type</th>
<th>Dividend</th>
<th>Divisor</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partitive</td>
<td>Same</td>
<td>Different</td>
<td>Same</td>
</tr>
<tr>
<td>Measurement</td>
<td>Same</td>
<td>Same</td>
<td>Different</td>
</tr>
</tbody>
</table>

The table above shows how the referent units compare between the two models of division. Thinking about the referent units can help clarify the difference between the two models of division. Another way to distinguish between the two is to ask the question: Are you trying to find the number of items in each share, or are you trying to find the number of times a fixed share (measurement unit) goes into the amount you have?

New Page: Equivalent Fractions

We saw before that units can differ in the kinds of things that they are (e.g., worms, chicks, shelves). Units can also differ in size, but be the same kind of thing. Sometimes it’s possible to create the exact same amount of something in different ways,
For example, three-eighths of a pizza can be thought of (and mathematically worked with) in different ways. The most common way to think of three-eighths of a pizza is shown on the left of the image above. Here, three-eighths of a pizza is conceptualized as one unit. In the middle illustration, three-eighths of a pizza is conceptualized as three instances of one-eighth of a pizza. The illustration on the right is really fancy: three-eighths of a pizza is conceptualized as one and a half instances of one-fourth of a pizza. Similarly, a fractional value can be conceptualized, written, and drawn an infinite number of ways, depending on what the unit is. Please watch the animated video before going to the next page. [2 Intro to Area Models.MP4] It shows a visualization of how 2/3 is equal to 8/12 using a rectangular area model.

**Assignment 1.1 Two Meanings of Division: Part 1**

**Description**

Take out some paper and write out an example of partitive division and example of measurement division. Including illustrations or sketches is fine, especially if that helps you think things through. You will be presenting your paper in class, so try to make your drawings and sentences clear enough for another person to review. Ideally, present us with examples that relate to your major or something you are interested in. If you can’t think of examples of division, you can always look at examples in the textbook (especially the last few pages of Section 1.5 Division of Whole Numbers, or the last few
pages of Section 2.7 Division of Fractions and Mixed Numbers). Do not copy text or images from the Internet, though you may refer to web sources for ideas.

**Purpose**

To develop your understanding the differences between partitive and measurement division.

**Tools**

- Paper and colored pens or pencils
- Digital camera, smartphone, or scanner

**Due Date**

This exercise is due by the end of the module.

**Submission Directions**

Please display your work as an image file in your post and select a large display size so that it is readily visible to the class. Please do not simply attach a pdf or doc file that needs to be downloaded.

**New Page: Flexibility with Referent Units**

Welcome to Module 2. The textbook is designed to demonstrate procedures for solving problems. By now you’ve seen this fact in the reading assignments. This feature makes the textbook helpful in completing the exercise set. Often you simply need to review the examples in the reading assignment, and that will show you how to solve the problems in the exercise set. However, sometimes there will be a problem in the homework for which the textbook does not provide an identical example to follow. Rather than follow a given procedure, you will sometimes need to think about a problem
conceptually and try to be creative. Much of the content in this module gives you thinking tools to be creative with.

As we saw in the last module, there are units in mathematics and the numbers involved in any given math problem can refer to a variety of things. Additionally, what we think of as “one thing” can shift around as we work. For example, take a look at the photo below:

![Beans in Packs](image)

The photo shows one case of beans, of course. But it is made up of six packs, and each pack is made up of fours cans. This means we have a multi-level unit structure. Depending on what we’re doing or saying at any moment of time (especially depending on the scale of the cooking project) what we call “one” could be the case, a pack, or a can. Over the course of a project or conversation or set of calculations, what we call “one of them” may shift up or down in the unit structure. In other words, the meaning of the word “one” depends on the situation and the task and the strategy at hand.

**New Page: Partitioning and Iterating**

When it comes to working with fractions, the most powerful thinking tools are partitioning and iterating. *Partitioning* is the process of subdividing an area or space or quantity into equal portions. *Iterating* is literally a copy-and-paste process that produces
duplicates. However, what gets iterated might be thought of as one thing, but at the same time have unit structure. Here’s an example: A landscaper is planting a group of maple tree saplings on a jobsite.

She planted a few saplings and then decided to take a photo of the rest before planting them. The saplings in this photo are two thirds of the total she’s planting for this job. How many total maple tree saplings will she be planting for this job? One way to answer this question is to translate the situation into an algebraic sentence and solve the equation through algebraic manipulations (we’ll do that later in this course). Another way to answer the question is to use partitioning and iterating as follows:
Imagine six trees that are two thirds of the total.

Partition them into two groups.
Each group is one third.

Place one group into one third of a box.

Then iterate that group.

Then iterate that group again.

There are nine trees in total.

New Page: Multiplying Fractions

As we saw at the end of Module 1, a square area can be partitioned in the horizontal and vertical directions and then shaded to represent two fractions at on the same square. This is a particularly powerful way to represent multiplication with fractions because we can see all of the parts of the process at once.

The image above shows a not-so-simple example. The big blue square represents
one. The square has been partitioned into eighths in the horizontal direction, and three
eighths are shaded in light blue. The square has also been partitioned vertically into
ninths with an orange color. Four of those ninths are shaded. We say that the square has
been \textit{cross-partitioned}, and in this case the cross-partitioning produces 72 smaller boxes.
The 12 boxes that are double-shaded make up the product of $\frac{3}{8}$ and $\frac{4}{9}$. This
representation also illustrates why the word “of” is often used to mean “multiplied by.” In
this case we say that the product is three-eighths of four ninths, or we can say that the
product is four-ninths of three eighths.

Please watch the animated video before going to the next page. [3 Fraction
Multiplication.MP4] It walks through a simpler example of multiplication with fractions
using a rectangular area model.

New Page: Measurement Division of Fractions with Area Models

As you’ve seen, measurement and partitive division are very different-looking
processes and must be represented differently. The animated video [4 Fraction
Measurement Division.MP4] shows an example of modeling measurement division with
fractions using the square area model. As you’re watching it, think of the following
questions.

1. Are there any strong points about this area model for representing measurement
division with fractions?
2. What other ways might work for representing measurement division with
fractions?
3. Toward the end of the video you’ll see that the division problem has a remainder.
In that context, pause the video as soon as you see the number 2 appear, and ask
yourself, “What is the unit to which that 2 refers? Give it a try!
New Page: Partitive Division of Fractions with Area Models

The first video on this page [5 Crackers and Bowls.MP4] shows a physical demonstration of partitive division with fractions. Most people find that representing partitive division with fractions using area models is not as clear or powerful as representing multiplication and measurement division with fractions using area models. However, area models for partitive division do have one strong point: they help to visualize the “flip-and-multiply” algorithm for division with fractions. For example, \( \frac{3}{4} \div \frac{2}{3} \) can be represented as:

Three fourths of one unit (shaded in blue) gets distributed (shared) into two thirds of another unit (in purple). Partitive division is hard to think about with fractions because we don’t usually think of a fraction of a share; we don’t share between fractions of people. Nonetheless, the three-quarters gets partitioned in half so that it can fit evenly into two of the thirds of the purple rectangle. That is the first step in the algorithm: to divide the dividend by 2. The next point is also hard to think about: once the distribution
is done, what does it mean to iterate to fill the bottom rectangle? The goal is the same as division with whole numbers: we’re finding the amount that one whole share receives. And that is the next step in the algorithm: to multiply the dividend by 3.

Please watch this animated video before going to the next page. [6 Fraction Partitive Division.MP4]

**New Page: The Invert-and-Multiply Rule**

Here is a slightly different way to represent partitive division with fractions. Notice that in the example on the previous page we had both eighths and thirds happening at once, and that was dealt with by actually writing out the fractions to show “where they go” in the diagram. In the example below, there are no fractions written out in the diagram. The representation shows the process of solving the question, “Half a pound of flour costs three-quarters of a dollar. How much flour can be purchased with one dollar?” The question is asking for the unit rate resulting from $\frac{1}{2} \text{ lb flour} \div \frac{3}{4}$.

The steps are described in text to the right of each step in the diagram. Notice that the partition and iteration steps visualize the steps taken in the algorithm for dividing the fractions: divide the dividend by 3 and multiply the dividend by 4.
Assignment 2.1 Two Meanings of Division: Part 2

Description

Choose one person’s post in Assignment 1.1: Two Meanings of Division: Part 1 and critique their work. The critique should include 1) a summary of what you like about the work, 2) a check on the accuracy, and 3) any suggestions you have for extending or revising the work. If you have no suggestions, you can tell us any thoughts, questions, or realizations the work inspired for you. In this assignment topic, please post the name of the person whose work you are critiquing. Post your critique as a reply to the post in which the work was submitted.

Purpose

To develop your understanding the differences between partitive and measurement division.

Tools

Paper and colored pens or pencils

Digital camera, smartphone, or scanner

Due Date

This exercise is due by the end of the module.

Submission Directions

Please post your critique as text in a reply to the post you are critiquing. If you choose to include an image, please display the image as an image file in your post and select a large display size so that it is readily visible to the class. Please do not simply attach a pdf or doc file that needs to be downloaded.
Also, in this assignment topic, please post the name of the person whose work you are critiquing.

**Assignment 2.2 Representing Division with Fractions: Part 1**

**Description**

Please take out some paper and create a simple division problem between two fractions (one of both fractions could be mixed numbers). Show us the division problem and calculate the quotient. For example, \( \frac{9}{2} \div \frac{3}{4} = 6 \). Then make two drawings: one showing the partitive division model and the other showing the measurement division model. Feel free to simply emulate the rectangular area representations shown in this module, or find another way of illustrating these processes on the Internet, or invent your own. You will be presenting your paper in class, so try to make your drawings and sentences clear enough for another person to review.

**Purpose**

To develop your skills in representing and communicating challenging mathematical concepts.

**Tools**

- Paper and colored pens or pencils
- Digital camera, smartphone, or scanner

**Due Date**

This exercise is due by the end of the module.

**Submission Directions**

Please display your work as an image file in your post and select a large display
size so that it is readily visible to the class. Please do not simply attach a pdf or doc file that needs to be downloaded.

**Assignment 3.1: Representing Division with Fractions: Part 2**

**Description**

Choose one person’s post in Assignment 2.2: Representing Division with Fractions: Part 1 and critique their work. The critique should include 1) a summary of what you like about the work, 2) a check on the accuracy, and 3) any suggestions you have for extending or revising the work. If you have no suggestions, you can tell us any thoughts, questions, or realizations the work inspired for you.

In this assignment topic, please post the name of the person whose work you are critiquing. Post your critique as a reply to the post in which the work was submitted.

**Purpose**

To develop your skills in representing and communicating challenging mathematical concepts.

**Tools**

Pencil and colored pens or pencils

Digital camera, smartphone, or scanner

**Due Date**

This exercise is due by the end of the module.

**Submission Directions**

Please post your critique as text in a reply to the post you are critiquing. If you choose to include an image, please display the image as an image file in your post and
select a large display size so that it is readily visible to the class. Please do not simply attach a pdf or doc file that needs to be downloaded. Also, in *this assignment topic*, please post the name of the person whose work you are critiquing.
Appendix B

Letter of Institutional Support
Dear Dr. MacDonald,

January 4, 2021

As the Academic Director of Liberal Arts and Michael’s immediate supervisor, I approve of Michael Leitch conducting research for his dissertation on the learning of fractions at the College of Mathematics, and Zoom-based professional development sessions with consenting instructors as they review, discuss and practice curriculum content, instruction, and pedagogy in the same course. I approve of Michael communicating with consenting instructors as part of his dissertation research while maintaining his professional role at the College. I also support any instructor who declines to consent to participate in Michael’s dissertation study. Nonconsenting instructors will not be asked to participate in Michael’s research and none of Michael’s professional interactions with nonconsenting instructors may be used as data in Michael’s dissertation.

Sincerely,

Academic Vice President of Liberal Arts
Appendix C

Teacher Recruitment: Description of Activities
Teacher Recruitment: Description of Activities

Dr. Beth MacDonald, P.I. for this study, will recruit the participants through an invitational email. A critical point of communication in that email will be distinguishing tasks and activities that incur by participating in this study from those that occur as regular employment tasks and activities. The table below delineates activities that are part of the teachers’ regular job tasks, and activities that incur solely in the process of participating in this study.

Table E.1

Tasks and Activities in which Teachers will Engage and be Compensated

<table>
<thead>
<tr>
<th>Regular activities</th>
<th>Research activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Teach College Mathematics online in the SP21 semester or planning to teach College Mathematics in a following semester.</td>
<td>• Attend an additional one-hour DAT meeting via Zoom to review student work submitted in the two assignments added to College Mathematics.</td>
</tr>
<tr>
<td>• Attend a two-hour DAT meeting via Zoom to go over new content and assignments added to College Mathematics.</td>
<td>• Attend an additional one-hour DAT meeting via Zoom to assess the content and assignments added to College Mathematics.</td>
</tr>
<tr>
<td>• Make use of the Faculty Lounge topic to share questions and concerns with other teachers.</td>
<td>• Maintain an open channel of communication with Michael Leitch regarding any questions about the content and assignments.</td>
</tr>
<tr>
<td>• Make use of email to discuss questions or concerns with Michael Leitch.</td>
<td>• Maintain an open channel of communication with Michael Leitch regarding any questions or concerns relating to the research activities.</td>
</tr>
</tbody>
</table>
Appendix D

Letter of Informed Consent
Teachers’ Conceptions of Division with Fractions

Introduction
You are invited to participate in a research study conducted by Beth MacDonald, assistant professor in Teacher Education and Leadership at Utah State University. The purpose of this research is to help teachers improve their teaching of fractions operations. Your participation is entirely voluntary.

This form includes detailed information on the research to help you decide whether to participate. Please read it carefully and ask any questions you have before you agree to participate.

Procedures
Your participation will involve three Zoom-based professional development sessions, conducted as one-to-one meetings with Michael Leitch. The first session will introduce you to new content implemented in the LA 255: College Mathematics course. This first professional development session is planned two hours, and approved as a regular DAT meeting for spring of 2021. The second professional development session is planned for one hour, supporting you in assessing student work in two new assignments to the course. The third professional development session is planned for one hour, discussing your assessment of the course content and assignments and answering any content questions you may have. We anticipate that 6-8 people will participate in this research study.

Risks
This is a minimal risk research study. That means that the risks of participating are no more likely or serious than those you encounter in everyday activities. The foreseeable risks or discomforts include being observed teaching in the classroom as you implement ideas about fractions from the professional development sessions, and being video-taped in professional development sessions and in interviews. In order to minimize those risks and discomforts, the researchers will only record audio of your observed lessons and have no expectations regarding your implementation of the professional development content.

Benefits
Although you will not directly benefit from this study, it has been designed to learn more about improving the effectiveness of teaching fraction multiplication and division.

Confidentiality
The researchers will make every effort to ensure that the information you provide as part of this study remains confidential. Your identity will not be revealed in any publications, presentations, or reports resulting from this research study. However, it may be possible for someone to recognize your particular story/situation/response. While we will ask all group members to keep the information they hear in this group confidential, we cannot guarantee that everyone will do so.

We will collect your information through video recordings, audio recordings, interviews, a one-page questionnaire, and possibly brief notes that you keep in a journal. This information will be kept in a restricted-access office. Within one year of recording, all audio and video tape will be transcribed without identifying personal information and erased. Within one year of collection, all written documents and artifacts that you generate (i.e., the questionnaire, worksheets from the professional development sessions, drawings made during interviews) will be copied or transcribed without identifying personal information and destroyed. This form will be kept for five years after the study is complete, and then it will be destroyed.
It is unlikely, but possible, that others (Utah State University) may require us to share the information you give us from the study to ensure that the research was conducted safely and appropriately. We will only share your information if law or policy requires us to do so. State law requires that the researchers report abusive behavior to the authorities.

**Voluntary Participation & Withdrawal**

Your participation in this research is completely voluntary. If you agree to participate now and change your mind later, you may withdraw at any time by contacting your principal or the Principal Investigator for this study, Beth MacDonal (435-797-1097; beth.macdonald@usu.edu). If you choose to withdraw after we have already collected information about you, that information will be destroyed with the exception of video recordings of the professional development sessions with your peers.

**Compensation & Costs**

Your participation in this research study will be compensated at the Academy of Art’s standard DAT rate of $40/hour for DAT sessions and the current Other Work rate for all other activities incurred by your participation.

**Findings [& Future Participation]**

Identifiers will be removed from your information. These de-identified data may be used or distributed for future research without additional consent from you. If you do not wish for us to use your information in this way, please state so below.

- [ ] I do not agree to allow my de-identified information to be used or shared for future research.

**IRB Review**

The Institutional Review Board (IRB) for the protection of human research participants at Utah State University has reviewed and approved this study. If you have questions about the research study itself, please contact the Principal Investigator at (435) 797-1097; beth.macdonald@usu.edu. If you have questions about your rights or would simply like to speak with someone other than the research team about questions or concerns, please contact the IRB Director at (435) 797-0567 or irb@usu.edu. The signature blocks below look funny now but will sort themselves out once information is filled in and deleted.

---

Principal Investigator  
(435) 797-1097; beth.macdonald@usu.edu

Student Investigator  
(435) 817-6769; michaelleitch@aggiemail.usu.edu

**Informed Consent**

By signing below, you agree to participate in this study. You indicate that you understand the risks and benefits of participation, and that you know what you will be asked to do. You also agree that you have asked any questions you might have, and are clear on how to stop your participation in the study if you choose to do so. Please be sure to retain a copy of this form for your records.

---

Participant’s Signature  
Participant’s Name, Printed  
Date

| Teacher Education and Leadership  | 435-797-0389  | 2805 Old Main Hill  | Logan, UT 84322 |
Appendix E

Professional Development Meeting One Worksheet
Professional Development Meeting One Worksheet

Teachers’ Workshop: Additions to the SP21 Curriculum

1. a. This DAT will take two hours. We will take a ten-minute break in the middle.
   
   In this DAT we will discuss and practice the following topics:
   
   - Measurement and partitive models of division with whole numbers
   - Partitioning and iterating
   - Referent units
   - Measurement and partitive models of division with fractions
   - Area models of multiplication and division with fractions

   b. “What is a fraction?”

2. a. Partitive and measurement models of division: whole numbers. [VIDEO 1]

   Measurement division asks how many times the divisor “tiles” the dividend.
   Partitive division is the result of equal sharing.

   b. Create a story problem with the partitive model of division.

   Create a story problem with the measurement model of division.

3. a. Partitive and measurement models of division with fractions. [VIDEO 5]

   b. From Ma (2010): \(1 \frac{3}{4} \div \frac{1}{2}\)

   Create a story problem with the measurement model of division.

   Create a story problem with the partitive model of division.

4. a. Draw \(\frac{3}{8}\) of a pizza in the remaining space on this page.
4. b. How much pizza do you have? What fraction is represented here?

5. Core concepts: **Partitioning** and **iterating**. Use these terms in class!

a. How many lady bugs are on my tree if this picture shows 4/5 of them?

b. is \( \frac{3}{4} \) of something. How much is \( \frac{1}{2} \)?

c. is 3 of something. How much is \( \frac{1}{3} \)?
d. What number does X represent?

Ten-Minute Break

6. a. Introduction to area models. Label the two fractions. [VIDEO 2]

[Cross-partitioning]

\[ \frac{2}{3} \times \frac{3}{5} = \]

\[ \underline{\phantom{\frac{2}{3}}} = \underline{\phantom{\frac{3}{5}}} \]

b. [VIDEO 3] Draw the product \( \frac{2}{3} \times \frac{3}{5} \).
7. a. Use an area model to find $\frac{3}{4} \div \frac{2}{3}$ [VIDEO 4] (How many $\frac{2}{3}$ are in $\frac{3}{4}$?)

($\frac{2}{3}$ is the measuring unit)

One

1. Shade $\frac{3}{4}$ in one direction 3. How many squares is $\frac{2}{3}$?

2. Identify $\frac{2}{3}$ in the other 4. How many $\frac{2}{3}$ are in $\frac{3}{4}$?

8. Partitive division of fractions with area models.

a. Draw $\frac{1}{2} \div \frac{1}{3}$ for both measurement and partitive cases. How do they compare?
Draw $1\frac{1}{2} \div \frac{3}{5}$ using partitive interpretation.

Draw $\frac{3}{4} \div \frac{1}{2}$ using partitive division below. Draw $\frac{1}{2} \div \frac{3}{4}$ using partitive division below.
<table>
<thead>
<tr>
<th>Rational Number Interpretations of $\frac{3}{4}$</th>
<th>Meaning</th>
<th>Selected Classroom Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part–Whole Comparisons With Unitizing</td>
<td>$\frac{3}{4}$ means three parts out of four equal parts of the unit, with equivalent fractions found by thinking of the parts in terms of larger or smaller chunks.</td>
<td>Unitizing to produce equivalent fractions and to compare fractions</td>
</tr>
<tr>
<td>“3 parts out of 4 equal parts”</td>
<td>$\frac{3}{4} = \frac{12}{16} = \frac{1}{2}$ (pair of pies)</td>
<td>Successive partitioning of a number line; reading meters and gauges</td>
</tr>
<tr>
<td>Measure</td>
<td>$\frac{3}{4}$ means a distance of 3 (1/4-units) from 0 on the number line or 3 (1/4-units) of a given area.</td>
<td>Machines, paper folding, xeroxing, discounting, area models for multiplication and division</td>
</tr>
<tr>
<td>“3 (1/4-units)”</td>
<td>$\frac{3}{4}$ gives a rule that tells how to operate on a unit (or on the result of a previous operation); multiply by 3 and divide your result by 4 or divide by 4 and multiply the result by 3. This results in multiple meanings for $\frac{3}{4}$: 3 (1/4-units), 1 (3/4-unit) and $\frac{1}{4}$ (3-unit)</td>
<td></td>
</tr>
<tr>
<td>Operator</td>
<td>$\frac{3}{4}$ is the amount each person receives when 4 people share a 3-unit of something.</td>
<td>Partitioning</td>
</tr>
<tr>
<td>“3 divided by 4”</td>
<td>$\frac{3}{4}$ is a relationship in which there are 3 A’s compared, in a multiplicative rather than an additive sense, to 4 B’s.</td>
<td>Bi-color chip activities</td>
</tr>
<tr>
<td>Ratios</td>
<td>“3 to 4”</td>
<td></td>
</tr>
</tbody>
</table>

Lamon’s (2008) Fraction “Personalities”
<table>
<thead>
<tr>
<th>Situations</th>
<th>Measurement</th>
<th>Sharing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joel is walking around a circular path in a park that is 3/4 miles long. If he walks 2 1/2 miles before he rests, how many times around the path did he travel?</td>
<td>Joel is walking around a circular path in a park. If he can walk 2 1/2 miles in 3/4 of an hour, how far can he walk in an hour, assuming he walks at the same speed?</td>
<td></td>
</tr>
<tr>
<td>Guiding question for interpreting 2 1/2 ÷ 3/4</td>
<td>How many groups of 3/4 are in 2 1/2?</td>
<td>If 3/4 of a group gets 2 1/2, how much does a whole group get?</td>
</tr>
<tr>
<td>Meaning of reciprocal</td>
<td>The reciprocal 4/3 means there are 4/3 groups of 3/4 in 1.</td>
<td>The reciprocal 4/3 is the operator necessary to shrink 3/4 to 1/4 and then expand 1/4 to 1.</td>
</tr>
<tr>
<td>Reason for multiplying the dividend by the reciprocal of the divisor</td>
<td>There are 4/3 groups of 3/4 in 1. There are 2 1/2 times as many groups of 3/4 in 2 1/2 as there are in 1. Thus, there are 2 1/2 × 4/3 groups of 3/4 in 2 1/2.</td>
<td>Since we shrink/expand 3/4 by 4/3 to get 1 whole group, we have to shrink/expand 2 1/2 by 3/4 in order to find out how much the whole group gets.</td>
</tr>
</tbody>
</table>

Appendix F

Semistructured Interview Protocol for Professional Development Meeting Two:

Evaluating Student Work
Semistructured Interview Protocol for Professional Development Meeting Two: Evaluating Student Work

**Group One Preparation**

Teachers collect a minimum of three de-identified samples of student work from each of the two new assignments and send them to the researcher.

**Group Two Preparation**

Open folder of pre-selected de-identified samples (provided by Group One teachers) of student work.

**Preparation for Both Groups**

For both groups, teachers will be prepared to meet via Zoom as in PD Session One, with the camera capturing a drawing area. Teachers will be asked to prepare blank paper and colored pens.

**Group One Session**

Group One teachers will be given the opportunity to present the samples for whatever reason they chose the samples. Reasons may include exemplary work, a novel way of representing the mathematics, or a question about the correctness of a representation.

**Group Two Session**

Group Two teachers will be presented with de-identified samples collected by the researcher from Group One sessions. Reasons adding a sample to the Group Two collection include exemplary work, a novel way of representing the mathematics, or the opportunity to pose a question about the correctness of a representation.
Contingency Questions for Both Groups

Where there any common themes in their questions?

In what ways did students benefit from these assignments?

Were there common difficulties that students had to address?

Do you have suggestions for improving the assignments?

Summary Questions For Both Group

Did you learn any mathematics points evaluating these assignments?

Did you learn any mathematics points from communicating with a student?
Appendix G

Semistructured Interview Protocol for Professional Development Meeting Two:

Conclusion and Assessment of Curriculum
Semistructured Interview Protocol for Professional Development Meeting Two:

Conclusion and Assessment of Curriculum

Preparation for Both Groups

Teachers will be prepared to meet via Zoom as in PD Session One, with the camera capturing a drawing area. Teachers will be asked to prepare blank paper and colored pens.

Questions for Both Groups

First, do you have any questions that you want to make sure we address?

Do you have any questions about the mathematics presented in the added curriculum or the assignments?

Generally speaking, did the curriculum and assignment additions appear to have any effect on your students’ mathematical achievement and class participation?

• If so, describe some examples.

• If not, why do you suppose the curriculum and assignment additions had no impact on your students?

What is division?

What is a fraction?

• Have your ideas about what division is or what a fraction is changed since the first session?

• Since the first session, have your ideas changed about any component of basic math?

Is there anything you learned in working with this new content that you wish you had
learned earlier?

What was that hardest part to learn, in your experience?

What would have made it easier to learn?

What suggestions do you have for improving the presentation of the content?

Do you have any ideas for improving the assignments?

How would you respond to a student who asks why we “flip and multiply?”
Appendix H

Links and Transcripts for the Videos in the PD Task Set
**Video 1 Partitive and Measurement Division**

This video was referred to as the 15÷5 Ball video. The following is a transcript of the voice-over in Video 1:

This simple division problem, fifteen divided by five equals three, can be modeled with two different types of division: partitive and quotitive. Let’s take a look at partitive division first. Partitive division is also called fair sharing, because that’s the activity that were modeling. The dividend, fifteen, tells us how much stuff we’ve got. And the divisor, five, tells us how many people or containers we’ve got to distribute that stuff equally among. So we’ve got fifteen things and were going to distribute them equally among five recipients. And we can make sure we do this fairly just by going one by one until we run out of stuff -- which in this case works out evenly, that’s nice. The quotient, the answer to the division problem, three, tells us how much each share is worth; how many of these things each person got. That’s a unit rate, in this case, three items per share.

Another way to look at division is quotitive, or measurement, and in this class will use the term measurement because it’s easier to think about and easier to say. In measurement division were actually creating a unit out of the divisor. The divisor, in this case five, becomes a unit of measurement, and we use that unit to measure off the total amount of the dividend. Sometimes we think of this as repeated subtraction because we have the size of the group fixed and we keep subtracting out that same-sized group. Either way, whether you look at it as repeated subtraction or measurement, we have three times. So that quotient, three, tells us that the measurement unit goes into the dividend three times.

**Video 2 Intro to Area Models**

The following is a transcript of the voice-over in Video 2:

In this video we’re going to start working with area models. Area models can be any shape, but we’re going to use squares in these videos. The meaning of the square is the value one. The first thing were going to do to this square is partitioned it into thirds. I’m going to partition it horizontally in thirds, and then choose two of those thirds. As you’ll see in later videos, it’s a good idea to use different colors represent different fractions. So I’m going to use green to represent thirds and I’ve shaded two of those thirds to indicate two-thirds of one.

Next, I’m going to partition this unit square again, but vertically instead of horizontally. This is called cross partitioning. We can see that this second partition has created twelve boxes out of our original unit square. That visually shows that the fraction two-thirds is equivalent to the fraction eight-twelfths. That two thirds of the square we originally selected is made up of eight out of twelve boxes.
**Video 3 Fraction Multiplication**

The following is a transcript of the voice-over in Video 3:

In this video were going to multiply fractions using square area models. Let’s multiply two thirds by three quarters. First, I’m going to partition this one-square into thirds. Then I’m going to identify two of those thirds, and I’m going to pick a unique color to do that. Just quickly making lines through both of those rows to indicate that I have shaded two thirds of that one square. Next, I’m going to partition the one-square into quarters and the other direction. Using glue, I’m going to shade in three quarters of that one square using this crosshatch pattern to distinguish the two. Let me make this clear: Here I’ve shaded two thirds of the one square with green, and then vertically I shaded three quarters of the one square with blue.

Now, notice that there are six boxes that are double-shaded. That double-shaded area is the product of the two fractions. And it’s not a coincidence that there are six boxes double shaded. If we go to the procedure for multiplying fractions, we go to times three and six in the numerator, and three times four is twelve in the denominator. So we have a product of six out of twelve boxes, and of course we can reduce that to one-half. We can look at this as two-thirds times three-quarters, or three-quarters times two thirds. Either way, the product is one-half.

**Video 4 Fraction Measurement Division**

The following is a transcript of the voice-over in Video 4:

In this video we’re going to do a measurement division with fractions using an area model. We’ve got three-quarters divided by one-half, so I’m going to start by partitioning the one square vertically into quarters. I’m just quickly drawn lines across three of the quarters. You could shade in each quarter individually, or use squiggly lines, or whatever you want to indicate your three quarters. Next, I’m going to cross partitions the one square into halves and pick a color for that. I’ll pick the screen. Now I’m going to indicate one half of the one square; I’m not going to shade it in yet. One half turns out to be four boxes. I’m going to write that down, because that’s going to be important. So I’m going to make a note that one-half, which is the divisor, which is my measuring unit now, is equal to four boxes.

Now the question is how many times does one half go into three quarters. Well, that’s how many times four boxes goes into six boxes. I can’t use his geometry, I have to make something up. So I’m just going to fill in these four boxes, because that’s the measuring unit, and it goes in at least once. I’ve got my measuring unit of four boxes that goes in once and there are these two boxes left over. I’m going to write that down: one-half goes into three-quarters one time and there’s a remainder of two boxes so I’ll write down two over -- well two of what? You might think eight because there were a total boxes in the one-square. But keep in mind that the measuring unit, the divisor, is for
boxes. So the answer to the question two of what is two of four boxes. So I’ll put for in the denominator of the remainder which gives us to fourths. And of course reduce that the one half. So one-half goes into three-quarters one and a half times.

**Video 5 Partitive Division Physical Model**

The following is a transcript of the voice-over in Video 5:

In this video we’ll look at partitive division modelled physically, first with whole numbers, and then with fractions. We’ll start with an easy problem, six divided into three because we’re using partitive division. In this case the dividend is six whole crackers and the divisor is three whole bowls. So, we just fair share until the dividend has been exhausted. The quotient of this division problem simply tells us the number associated with one whole share, which is obviously two.

Next, let’s try this again but with the dividend as a fraction. I split the cracker into quarters and took one way, so now the dividend is three quarters. But the process is the same. I’m distributing the dividend into the divisor until it’s exhausted, and the quotient tells us how many crackers are associated with one whole share.

Last, is change this so that all the numbers are fractions. I’ll stick with the dividend of three-quarters that we had before, but now, instead of using three whole bowls for the divisor let’s use half a bowl. That’s half a share right there. So we have three quarters divided into half a share. That’s half a share there. So I’ll take the cracker, split it into quarters and remove a quarter. Now I have a dividend of three quarters, and if I distribute that evenly into half a share, that means, if half a share has three quarters then a whole share would have six quarters. That’s the quotient. We should reduce it to three halves, or one and a half in mixed number form.

**Video 6: Partitive Division with Fractions Using Area Models**

The following is a transcript of the voice-over in Video 6:

In this video, will look at a drawn area model of that same division problem we looked at physically at the end of the last video: Three-quarters divided into a half. Again, the area of the square represents one, and I’m going to use different colors to distinguish the roles of the dividend and the divisor. Since the divisor is half, we can partition the square in half and indicate half a share. We’ve got the divisor share of three quarters, and before I throw that divisor of three quarters into the half share I want to identify the half share with magenta dashed lines to indicate were dealing with a fractional share, not a whole share. Basically were done but the calculation is not done: we distributed the three quarters into half a share, but we still need to find out what is associated with the whole share. We can just copy and paste that half share with three quarters in it, and move it down so that it occupies the whole one-square. We can see that six quarters is a full share. So I’ll write that down and of course reduce it to three halves.
CURRICULUM VITAE

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EDUCATION

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Concentration: Mathematics Education and Leadership

M.Ed. May 2007
Master of Education, University of California, Berkeley.

B.S. May 1986
Focus on math and physics, Evergreen State College, Olympia Washington.

EMPLOYMENT HISTORY

Academy of Art University, San Francisco, California

Associate Director, Liberal Arts. (2010-present).

Quantitative Literacy Coordinator and Professor. (1998-present).

Responsibilities include directing the Quantitative Literacy program, developing courses, reporting to accreditors, hiring and training teachers, and teaching quantitative courses.

University of California, Berkeley, California

Graduate Student Instructor. (2005-07).

Physics for Future Presidents. (Dr. Richard Muller)

Responsibilities include teaching recitation, coordinating curriculum, editing textbook.
The Crowden School, Berkeley, California


Responsibilities include developing and teaching math/science curricula for grades 4 through 8

RESEARCH

Journal Articles (Coauthored)


Research Interests:
- K-8 mathematics learning and instruction with technology
- Teacher development and instructional design
- Knowledge in Pieces epistemology

PUBLICATIONS

Curriculum Development: Onsite Courses and Online Course Authorship

1. College Mathematics (2001)
3. Human-Centered Design (2005)
5. Topics in Modern Science (2007)
7. College Algebra and Geometry (2013)

Unpublished Manuscripts


UNIVERSITY TEACHING

Online and Onsite (1998-present)

*Conceptual Physics*
This course exposes students to the principles that underlie complex motion found in the real world. Topics covered include motion, matter, sound, light and heat, with specific emphasis on the role of physics in photography, product design, architecture, animation and visual effects.

*Astronomy in the New Millennium*

This course introduces students to the structure and evolution of planets, stars, galaxies, and our current understanding of cosmology. Contemporary topics such as black holes, relativity, planets outside of our solar system, and the possibility of life on other planets will also be covered. This course meets the quantitative literacy requirement.

*Topics in Modern Science*

In this survey course, students explore the main topics in modern science as they appear in current publications, such as trade journals and popular magazines. Students will strengthen their understanding of scientific concepts through discussion of seminal works by Nobel and Pulitzer Laureates.

*Human-Centered Design*

This course introduces students to the science of ergonomics, providing them with an awareness of how to make products that satisfy the physical, physiological and psychological needs of consumers

*College Mathematics*

This course provides the opportunity for the artists to strengthen basic math skills. Math skills are applied to personal finance, accounting and investing. Topics include fractions, percentages, ratio and proportion, probability, converting units of measurement, and fundamentals of algebra and geometry.

*Contemporary Topics in Mathematics*

In this course, the perspectives of algebra and geometry are combined in a study of golden ratio and fractals in art and nature, non-Euclidean forms of geometry, and introductions to calculus and statistics.

*College Algebra and Geometry*

This course provides an introduction to linear systems, algebraic modeling of lines and curves, and applications including angles, triangles, area, and volume.

*Discrete Mathematics*

This course provides students an introduction to the mathematics common to computer
science. Topics include logic, sets, algorithms, Boolean algebra, number theory, counting techniques, recurrence, graph theory, and trees.

PRESENTATIONS


PROFESSIONAL DEVELOPMENT

WASC Conferences:

*Assessment of Quantitative Literacy in the Majors*, Pomona, California (2012, November).

*Academic Resource Conference*, Los Angeles, California (2014, April)