A Frobenius-Schur Extension for Real Projective Representation

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A FROBENIUS-SCHUR EXTENSION FOR REAL PROJECTIVE REPRESENTATION
THEORY

by

Levi Gagnon-Ririe

A thesis submitted in partial fulfillment
of the requirements for the degree

of

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in

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ABSTRACT

A Frobenius-Schur extension for Real projective representation theory

by

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Utah State University, 2023

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Representation theory of finite groups over \( \mathbb{C} \) is a field of mathematics that allows us to study the symmetries of a mathematical object by using their local linearizations and the induced group actions on these linearizations. In the classical setting, representation theory of finite groups over \( \mathbb{C} \) is complete, and with the addition of the Frobenius-Schur indicator representation theory of finite groups over \( \mathbb{R} \) is also complete. In some situations, studying representation theory of finite groups over \( \mathbb{R} \) is too weak. In this thesis we generalize the representation theory of finite groups over \( \mathbb{R} \) to Real projective representations. The main result of the thesis is an extension of the Frobenius-Schur indicator into Real projective representations. This yields an algorithm by which we can completely determine the Real projective representation theory of a finite \( \mathbb{Z}_2 \)-graded group.

(92 pages)
PUBLIC ABSTRACT

A Frobenius-Schur extension for Real projective representation theory

Levi Gagnon-Ririe

Many problems in physics have explicit mathematical descriptions. This thesis aims to provide the mathematical tools for a particular problem in physics, that of Quantum Mechanical symmetries. In essence, we extend the known mathematics to a more general setting and provide a wider view of Real projective representation theory. The work done in this thesis contributes to the subfield of mathematics known as representation theory and to the subfield of physics concerned with time reversal symmetry.
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CHAPTER 1
INTRODUCTION

In trying to understand the symmetries of mathematical objects such as sets, vector spaces, geometric objects, and manifolds, or the symmetries of physical systems such as classical mechanical systems and quantum mechanical systems, we look at groups and group actions. Some of these mathematical objects can be complicated to understand globally, but computations can be simplified by considering them locally, i.e. considering local linearizations. This leads to viewing global group actions on an object through the lens of group actions on its local linearizations. For example, given a manifold $M$ with an action of a group $G_\alpha$, we can consider a fixed point $p \in M$ under the action of $G_\alpha$ and linearize the manifold at that point. This induces a linear action of $G_\alpha$ on the tangent space $T_p M$. The study of group actions on a linear space such as $T_p M$ is the field of mathematics known as representation theory, a field that allows us to approach the potentially complicated world of group actions via the context of well-understood linear algebra. Representation theory of finite groups on complex vector spaces in particular, is a rich and well-studied field whose theory is complete in many regards. There are analogues of representation theory for infinite groups as well, but in this thesis we concern ourselves with finite groups.

A representation of a finite group $G_\alpha$ is a group homomorphism $G_\alpha \to \text{GL}(V)$ where $V$ is a vector space over some fixed field $k$. Such a homomorphism contains at least some of the information of the group, but it also encodes a group action on $V$ since each group element is mapped to a linear map on $V$. We could take this vector space to be over any field and to have any dimension, but considering finite dimensional vector spaces over $\mathbb{C}$ is a great way to understand many known linearizations of mathematical objects. Choosing $\mathbb{C}$ as our field is due to two prominent features. The first feature is that $\mathbb{C}$ is a field of characteristic zero, which means the order of each group is invertible in $\mathbb{C}$ and therefore implies the existence of Reynolds operators for all finite groups. The second feature is that $\mathbb{C}$ is algebraically closed, which means that linear algebra is easy. Such a choice of ground field allows us to complete the theory for representations of finite groups.
1.1

Motivation

We now briefly discuss representation theory and two of its extensions.

1.1.1

Representation theory of finite groups over $\mathbb{C}$

One of the key pieces to completing the representation theory of finite groups over $\mathbb{C}$ is Maschke’s Lemma, which states that every representation can be written as a direct sum of irreducible representations [11]. Thus, in order to understand the representation theory of a finite group, we need only understand the irreducible representations of that group. Another key piece to the completion of the theory is that there are only finitely many irreducible representations, and furthermore that each irreducible appears in the regular representation [11]. This provides us with an algorithm by which to find the irreducible representations of a finite group.

A key tool in decomposing representations into their irreducible parts is character theory. The character $\chi_V$ of a representation $V$ is a function $G \rightarrow \mathbb{C}$ obtained by tracing over the linear maps of the representation. We can also use characters to distinguish representations, viewing a particular representation as a function with finitely many values, rather than some collection of matrices (that is, a collection of potentially large matrices). Furthermore, the study of character theory shows that irreducible characters form an orthonormal basis for the space of class functions on $G$ [11], thereby allowing us to study group homomorphisms of the form $G \rightarrow GL(V)$ through the lens of functions of the form $G \rightarrow \mathbb{C}$.

We will hereafter refer to representation theory of finite groups over $\mathbb{C}$ as classical representation theory.

1.1.2

Projective representation theory of finite groups over $\mathbb{C}$

Here we consider an extension of classical representation theory, known as projective representation theory over $\mathbb{C}$. A mathematical model for a quantum mechanical system is a pair $(V, \langle -, - \rangle)$ where $V$ is a complex vector space (assumed to be finite dimensional here) and $\langle -, - \rangle$ is a Hermitian inner product on $V$ [20]. It is well known that the normalized inner product given by

$$\frac{\langle \phi, \psi \rangle}{\|\phi\| \|\psi\|}$$
for $\phi, \psi \in V \setminus \{0\}$, is one of the observables in quantum mechanics [20]. This normalization means that particular distances are irrelevant, since $\phi$ and $\lambda \phi$ for $\lambda \in \mathbb{C}$ have the same physics. The resulting mathematics treats each line in $V$ as a unique element rather than each vector in $V$ as a unique element, or in other words we analyze $V$ through its 1-dimensional subspaces. In mathematics, analyzing a vector space $V$ through its 1-dimensional subspaces is precisely the same as analyzing the projective space $\mathbb{P}V$. Therefore, it is natural to define the quantum mechanical symmetries of $(V, \langle - , - \rangle)$ as linear symmetries of $\mathbb{P}V$ that preserve the inner product. In this thesis, we don’t concern ourselves with the inner product condition but instead consider symmetries of a projective space given by group homomorphisms $G_\circ \to \mathbb{P}GL(V)$, as opposed to classical representation theory which considers symmetries of the state space given by group homomorphisms $G_\circ \to GL(V)$ [8]. Explicitly, a projective representation $\rho_V : G_\circ \to \mathbb{P}GL(V)$ is the data of a map $\rho_V : G_\circ \to GL(V)$ along with a degree 2 group cocycle $\theta_\circ \in \mathbb{Z}^2(G_\circ; U(1))$ such that for all $g, h \in G_\circ$ we have

$$\rho_V(g)\rho_V(h) = \theta_\circ(g, h)\rho_V(gh)$$

where $\theta_\circ(g, h) \in \mathbb{C}$. Projective representation theory of finite groups over $\mathbb{C}$ has analogues of Maschke’s Lemma and character theory, and is therefore similarly complete to classical representation theory of finite groups over $\mathbb{C}$ [16].

### 1.1.3 Representation theory of finite groups over $\mathbb{R}$ and Real representations

We now consider a separate extension of classical representation theory, called Real representation theory, and discuss its origin. We know that representation theory of finite groups is completely determined over $\mathbb{C}$, but there are situations where using $\mathbb{C}$ is not desirable. Consider an $n$-dimensional manifold $M$ for example, and linearize it at a point $p \in M$. What we find is that the resulting tangent space $T_pM$ is isomorphic to $\mathbb{R}^n$ and not $\mathbb{C}^n$. This motivates the study of representation theory for finite groups over $\mathbb{R}$. Such representation theory is harder since $\mathbb{R}$ is not algebraically closed. Surprisingly, representation theory for finite groups over $\mathbb{R}$ is complete as well, due to one extra key piece: the Frobenius-Schur indicator [10]. In particular, we consider the standard Hermitian inner product on class functions of $G_\circ$ given by $\langle \phi, \psi \rangle = \frac{1}{|G_\circ|} \sum_{g \in G_\circ} \phi(g)\overline{\psi(g)}$, and let the Frobenius-Schur indicator be the class function $\nu_2 = \sum_{g \in G_\circ} \ell_{g^2} \in \mathbb{C}[G_\circ]$. We then find the key result for representation theory over $\mathbb{R}$:

**Theorem** (Frobenius-Schur [10]). *Let $V$ be a finite dimensional, complex, irreducible representation...*
of $G_\alpha$ and let $\chi_V$ be its character. Then we have

$$\langle \chi_V, \nu_2 \rangle = \begin{cases} 
0 & V \text{ can only be realized over } \mathbb{C} \\
1 & V \text{ can be realized over } \mathbb{R} \\
-1 & V \text{ can be realized over } \mathbb{H}.
\end{cases}$$

We already know where to find each irreducible representation (the regular representation, or equivalently the group algebra $\mathbb{C}[G_\alpha]$), but this indicator allows us to determine whether an irreducible can be realized over $\mathbb{R}$. The interesting thing to note is that the quaternions $\mathbb{H}$ show up in this indicator as well. This comes from the algebraic result that $\mathbb{R}, \mathbb{H},$ and $\mathbb{C}$ are the only normed division algebras over $\mathbb{R}$, but we can also understand this geometrically. Consider the classification of complex simple Lie algebras, which can be viewed as linearizations of Lie groups. We know that there are four infinite families of Lie algebras that show up. One family is given by $\mathfrak{sl}_n(\mathbb{C})$, which are the general traceless symmetries of a complex vector space. Another family is $\mathfrak{so}_n(\mathbb{C})$, which really yields two infinite families depending on the parity of $n$. These families correspond to the isometries of a complex vector space with a non-degenerate symmetric bilinear form. The last family is $\mathfrak{sp}_n(\mathbb{C})$, which corresponds to the isometries of a symplectic form on a complex vector space. We find that $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, and $\mathfrak{sp}_n(\mathbb{C})$ relates to $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{H}$ respectively.

Hidden behind the computation of the Frobenius-Schur indicator and its values is the existence of a non-degenerate, $G_\alpha$-invariant bilinear form. If a complex representation $V$ of a finite group $G_\alpha$ can be realized over $\mathbb{R}$, this is equivalent to the existence of a symmetric, non-degenerate, $G_\alpha$-invariant bilinear form on $V$, and if it can instead be realized over $\mathbb{H}$, then this is equivalent to the existence of a skew-symmetric, non-degenerate, $G_\alpha$-invariant bilinear form on $V$. The classical Frobenius-Schur indicator utilizes this $G_\alpha$-invariant bilinear form along with a non-degenerate $G_\alpha$-invariant Hermitian form to construct a linear isomorphism $\mathbb{V} \rightarrow V$ where $\mathbb{V}$ is the vector space $V$ with scalar multiplication given by $\lambda \cdot v = \overline{\lambda} v$. This naturally leads us to consider the group of all linear and anti-linear isomorphisms on $V$, which we call $\text{GL}(\mathbb{V})$. In particular, given a representation $G_\alpha \rightarrow \text{GL}(V)$, we can add a second copy of $G_\alpha$ that maps strictly to anti-linear isomorphisms. Therefore we consider group homomorphisms $\rho$ that make the following diagram commute.

$$
\begin{array}{ccc}
G_\alpha \times \mathbb{Z}_2 & \longrightarrow & \text{GL}(\mathbb{V}) \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{Z}_2 & \rightarrow & \mathbb{Z}_2
\end{array}
$$
where $\pi$ is projection to the second factor and $\bar{\pi}$ indicates whether $\phi \in \overline{GL}(V)$ is linear or anti-linear, the former being projected to the identity of $\mathbb{Z}_2$ and the latter being projected to the generator of $\mathbb{Z}_2$. If $V$ is invariant under both copies of $G_o$, then $V$ is invariant under conjugation and can therefore be realized over $\mathbb{R}$. Thus, representation theory of finite groups over $\mathbb{R}$ and over $\mathbb{H}$ is equivalent to the representation theory given by $G_o \times \mathbb{Z}_2 \to \overline{GL}(V)$.

Though we may enjoy the simplicity given by the trivial $\mathbb{Z}_2$-grading $G_o \times \mathbb{Z}_2$, it is natural to ask whether more general $\mathbb{Z}_2$-gradings give a similar theory. In order to generalize this, we switch our perspective from linear or anti-linear maps on $V$, given by $GL(V)$, to the group of linear isomorphisms of the form $V \to V$ or of the form $V^\vee \to V$, which we denote by $GL^\vee(V)$. (We denote by $V^\vee$ the linear dual $\text{Hom}_\mathbb{C}(V, \mathbb{C})$ of $V$.) This switch in perspective utilizes the non-degenerate, $G_o$-invariant bilinear forms that exist if a representation can be realized over $\mathbb{R}$ or $\mathbb{H}$. In particular, if $(-,-)$ is a non-degenerate, $G_o$-invariant bilinear form then we consider the mapping given by $x \mapsto (x, -)$ for all $x \in V$, which gives a linear isomorphism between $V$ and $V^\vee$. Therefore, given a finite $\mathbb{Z}_2$-graded group $\pi : G \to \mathbb{Z}_2$, we consider maps $\rho$ that make the following diagram commute

\[
\begin{array}{ccc}
G & \xrightarrow{\rho} & GL^\vee(V) \\
\downarrow{\pi} & & \downarrow{\bar{\pi}} \\
\mathbb{Z}_2 & & \\
\end{array}
\]

where $\bar{\pi}$ indicates what form an isomorphism $\phi \in GL^\vee(V)$ takes, projecting automorphisms on $V$ to the identity of $\mathbb{Z}_2$ and projecting isomorphisms of the form $V^\vee \to V$ to the generator of $\mathbb{Z}_2$. We call such a map $\rho$ a Real representation, where the capital 'R' here is reminiscent of—but separate from—representations over the real numbers, as discussed by Atiyah [1]. This generalization makes sense because it restricts to the classical case when $G$ is replaced with $G_o \times \mathbb{Z}_2$ and $GL^\vee(V)$ is replaced with $\overline{GL}(V)$ through use of the non-degenerate, $G_o$-invariant bilinear and Hermitian forms. The interesting thing about generalizing to any $\mathbb{Z}_2$-graded group is that this takes us away from representation theory over $\mathbb{R}$. However, this generalization appears in many places throughout mathematics and is therefore meaningful to study. In representation theory, the generalization to an arbitrary $\mathbb{Z}_2$-graded group in [17], [18], and [23]. In geometry, we see this generalization in [1], [2], [0], and [7]. In physics, we see this generalization in [5], [8], [13], [14], [19], and [22].
1.2

Main results

The main results of this paper include completing Real representation theory of finite groups while simultaneously including projective representation theory of finite groups over \( \mathbb{C} \), unifying both theories into one. We call this Real projective representation theory of finite groups. We will develop an analogue of Maschke’s Lemma (Theorem 4.7.6), determine that there are only finitely many irreducibles (Theorem 4.5.1), and give an algorithm to find each irreducible (Theorem 5.0.1).

One application of this type of theory is that which Wigner discussed, referring to Real representations as corepresentations [21]. Similar concepts have been discussed by Freed and Moore [8] as well as Ichikawa and Tachikawa [13]. Other sources on this topic include papers by Young [22] and by Noohi and Young [17]. A potential motivation for such a study is considering the time evolutions of a quantum mechanical system. In particular, group elements that map into \( \text{GL}^\vee(V) \setminus \text{GL}(V) \) give maps that reverse the time evolution. Therefore considering Real representations in a projective space can be thought of as analyzing the symmetries of a quantum mechanical system with potential time reversals.

We give an overview of the remaining paper. In chapter 2 we give a brief overview of classical representation theory following [11] as well as some miscellaneous results from linear algebra. Chapter 3 then gives motivation for studying projective representation theory, a brief discussion of group cohomology, and an overview of some main results following [16] and [4]. We then introduce Real projective representations in chapter 4. In that chapter we define Real projective representations, give an analog of Maschke’s Lemma, discuss what irreducible Real projective representations look like, and give a notion of character theory for Real projective representations. The main result of this paper lies in chapter 5 where we give an extension of the Frobenius-Schur indicator:

**Theorem** (Theorem 5.0.1). Let \( \pi : G \to \mathbb{Z}_2 \) be a surjective group homomorphism for a finite group \( G \) and let \( G_0 = \ker(\pi) \). Let \( \theta \in \mathbb{Z}^2 + \pi(G; \mathbb{U}(1)) \) be a normalized cocycle. Let \( \theta_0 \in \mathbb{Z}^2(G_0; \mathbb{U}(1)) \) be the restriction of \( \theta \) to the subgroup \( G_0 \). Let \( (V, \rho_V) \) be an irreducible \( \theta_0 \)-projective representation of \( G_0 \). Then the element \( \nu_2 := \sum_{\xi \in G \setminus G_0} \theta^{-1}(\xi, \xi) \xi \xi^2 \in \mathbb{Z}[^\theta_0^{-1}|G_0|] \) gives the following result:

\[
\langle \chi_V, \nu_2 \rangle = \begin{cases} 
0 & V \text{ cannot be realized as a Real representation of } G \\
1 & V \text{ admits a Real } \theta \text{-projective representation of } G \\
-1 & V \text{ admits a Real } \alpha \theta \text{-projective representation of } G 
\end{cases}
\]
where $\alpha \in \mathbb{Z}^{2+\pi}(G;\mathbb{U}(1))$ is defined by

$$
\alpha(g, h) = \begin{cases} 
1 & \text{either } g \text{ or } h \text{ is an element of } G_o \\
-1 & g, h \in G \setminus G_o.
\end{cases}
$$

This result finishes the theory of Real projective representations, similar to how the classical Frobenius-Schur indicator finished the theory for representations over $\mathbb{R}$. In particular, if we take our group to be the trivial product $G_o \times \mathbb{Z}_2$ and we take our cocycle twist $\theta$ to be trivial as well, then this recovers the classical Frobenius-Schur indicator for representations over $\mathbb{R}$ as in Theorem 2.2.4. If the cocycle twist is $\alpha$, then this recovers the classical indicator for representations over $\mathbb{H}$. Next, if we consider $G$ to be any $\mathbb{Z}_2$-graded group with a trivial multiplier $\theta$, then we recover Gow’s work [12]. Lastly, if we take $G$ to be the trivial product $G_o \times \mathbb{Z}_2$ with some multiplier $\theta_o \in \mathbb{Z}^2(G_o;\mathbb{Z}_2) \subset \mathbb{Z}^2(G_o;\mathbb{U}(1))$, then we recover the work of Ichikawa and Tachikawa [13].

1.3 Conventions

Throughout the thesis, we assume the reader has a general knowledge of representation theory of finite groups at the level of Fulton-Harris [11]. We will occasionally refer to the representation theory of finite groups over $\mathbb{C}$ as classical representation theory, and will also occasionally refer to a complex representation of a finite group as a classical representation. Unless otherwise noted, we take all groups to be finite and all vector spaces to be finite dimensional over $\mathbb{C}$. We will discuss Hermitian forms on a vector space $V$, which for this thesis has the convention of anti-linearity in the second argument, or in other words if $H$ is a Hermitian form on $V$ then for all $v, u \in V$ and $\lambda \in \mathbb{C}$ we have

$$
H(\lambda v, u) = \lambda H(v, u) \quad \text{and} \quad H(v, \lambda u) = \bar{\lambda} H(v, u).
$$

We also denote

$$
\mathbb{U}(1) := \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \}.
$$

Lastly, we use the symbol $(\cdot)^\vee$ to denote the linear dual, so $V^\vee := \text{Hom}_\mathbb{C}(V, \mathbb{C})$ and if $\phi : V \to W$ is a linear map, then $\phi^\vee : W^\vee \to V^\vee$ is the dual map. In situations where both an inverse and a dual are being applied to a mapping, we will write $\phi^{-\vee}$ rather than $(\phi^\vee)^{-1}$ or $(\phi^{-1})^\vee$. 
CHAPTER 2
BACKGROUND

2.1

Important concepts from linear algebra

We assume the reader knows the theory of linear algebra at the level of [9], but cover a few concepts that will be seen throughout the paper. We first discuss dual spaces, with a particular focus on the double dual of a vector space. Following that, we mention the minimal polynomial of a square matrix and focus on its application to two results: (1) the relationship between roots of the minimal polynomial and diagonalizability of a matrix, and (2) the relationship between a projection map and the dimension of the subspace being projected onto. As stated in section 1.3, we assume all vector spaces are finite dimensional over $\mathbb{C}$.

2.1.1

Duality

Let $V$ be a vector space and denote by $V^\vee$ the linear dual $\text{Hom}_\mathbb{C}(V, \mathbb{C})$. There is an isomorphism $V \cong V^\vee$. Let $\{v_i\}_{i=1}^n$ be a chosen basis for $V$ and consider the map

$(-)^\vee : V \rightarrow V^\vee$

$v_i \mapsto v_i^\vee$

where $v_i^\vee : V \rightarrow \mathbb{C}$ is the linear map determined by

$v_i^\vee(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$

Note that this isomorphism depends on a choice of basis and is therefore not canonical. On the other hand, there is a canonical vector space isomorphism from $V$ to its double dual $V^{\vee\vee} := \text{Hom}(V^\vee, \mathbb{C})$. 
This isomorphism is

\[ \text{ev}_V : V \rightarrow V^{\vee\vee} \]
\[ v \mapsto v^{\vee\vee} \]

where the double dual vector \( v^{\vee\vee} \) is determined by

\[ v^{\vee\vee} : V^\vee \rightarrow \mathbb{C} \]
\[ f \mapsto f(v). \]

We can also dualize linear mappings. Given \( \phi : V \rightarrow U \), the dual map \( \phi^\vee : U^\vee \rightarrow V^\vee \) is given by \( \phi^\vee (f) = f \circ \phi \) for all \( f \in U^\vee \). Notice that taking the dual of a linear map both dualizes the vector spaces and swaps the direction in which it travels. Thus, dualizing a linear map twice will give a map that travels in the same direction as its original mapping, and since there is a canonical isomorphism between a vector space and its double dual, we would expect to see a relationship between \( \phi^{\vee\vee} : V^{\vee\vee} \rightarrow U^{\vee\vee} \) and \( \phi : V \rightarrow U \). This relationship is described in the following

**Lemma 2.1.1.** Given any linear map \( \phi : V \rightarrow U \), the diagram below commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\text{ev}_V} & V^{\vee\vee} \\
\downarrow{\phi} & & \downarrow{\phi^{\vee\vee}} \\
U & \xrightarrow{\text{ev}_U} & U^{\vee\vee}
\end{array}
\]

Equivalently, the assignment \( v \mapsto \text{ev}_V(v) \) defines a natural isomorphism \( \text{ev} : \text{id}_{\text{Vect}_\mathbb{C}} \Rightarrow (-)^\vee \circ (-)^{\vee} \).

**Proof.** Let \( v \in V \) and \( f \in U^\vee \). Let \( \phi \in \text{Hom}(V,U) \). Then we have

\[
\left( \phi^{\vee\vee} \circ \text{ev}_V \right)(v)(f) = \left( \phi^{\vee\vee} (v^{\vee\vee}) \right)(f) = v^{\vee\vee} (\phi^\vee (f)) = v^{\vee\vee} (f \circ \phi) = (f \circ \phi)(v) = (\phi(v))^{\vee\vee}(f) = \left( \text{ev}_U \circ \phi \right)(v)(f)
\]
as needed.

The commuting diagram of Lemma 2.1.1 allows us to swap between double duals of a linear map and the original linear map. In particular, Lemma 2.1.1 implies $ev_U^{-1} \circ \phi^\vee \circ ev_V = \phi$ for all linear maps $\phi : V \to U$. We can likewise construct a relationship between $ev_V$ and $ev_V^\vee$.

**Lemma 2.1.2.** $ev_V^\vee = (ev_V)^{-1}$.

**Proof.** Let $f \in V^\vee$ and $v \in V$. We have

$$
\left( ev_V^\vee \circ ev_V^\vee \right)(f)(v) = ev_V^\vee (ev_V^\vee(f))(v) \\
= ev_V^\vee (f)(ev_V(v)) \\
= ev_V(v)(f) \\
= f(v)
$$

and similarly

$$
\left( ev_V^\vee \circ ev_V^\vee \right)(f^\vee)(v^\vee) = ev_V^\vee (f^\vee \circ ev_V)(v^\vee) \\
= v^\vee (f^\vee \circ ev_V) \\
= (f^\vee \circ ev_V)(v) \\
= f^\vee (v^\vee)
$$

Therefore $ev_V^\vee$ is the left and right inverse of $ev_V$. □

We will frequently see the operation $((-\cdot)^\vee)^{-1}$ throughout this paper, and denote this by $(-)^{-\vee}$. Another important piece we will mention throughout are mappings from $V^\vee$ into $V$. We can again use evaluation maps to relate such mappings to their double duals.

**Lemma 2.1.3.** Let $\phi : V^\vee \to V$ be a linear map. Then $ev_V^{-1} \circ \phi^\vee \circ ev_V^\vee = \phi$, or equivalently $ev_V^{-1} \circ \phi^\vee \circ ev_V^\vee = \phi$.

**Proof.** This follows directly from the preceding two lemmas. □

### 2.1.2

**Minimal polynomials**

Recall that each linear map has a matrix representation under a chosen basis.
Definition (minimal polynomial of a matrix). Let $A$ be an $n \times n$ complex matrix. We say a polynomial $m_A \in \mathbb{C}[x]$ is the minimal polynomial of $A$ if $m_A$ satisfies the following:

1. $m_A(A) = 0$
2. $m_A$ is monic (i.e. the coefficient of the highest degree is 1)
3. $m_A$ divides all other polynomials $Q \in \mathbb{C}[x]$ such that $Q(A) = 0$.

Lemma 2.1.4. A matrix $A$ is diagonalizable if and only if the minimal polynomial $m_A$ has distinct roots.

Proof. Note that because $\mathbb{C}$ is algebraically closed, every matrix has at least one complex eigenvalue. Let $A$ be an $n \times n$ matrix and let $\{\lambda_i\}^k_i$ be the distinct eigenvalues of $A$ for $1 \leq k \leq n$. Recall that each matrix is similar to a matrix $J$ in Jordan normal form. Let $k_i$ be the maximum size of the Jordan block corresponding to the distinct eigenvalue $\lambda_i$. Then the minimal polynomial of $A$ is given by

$$m_A = \prod_{i=1}^{k} (x - \lambda_i)^{k_i}.$$ 

If $A$ is diagonalizable, then $k_i = 1$ for all $i$, so $m_A$ has distinct roots. If $m_A$ has distinct roots, then $k_i = 1$ for all $i$, so $A$ is diagonalizable.

Lemma 2.1.5. For a given matrix $A$ each eigenvalue is a root of $m_A$.

Proof. Let $v_i$ be an eigenvector of $A$ associated with the eigenvalue $\lambda_i$. Then we have $Av_i = \lambda_i v_i$ and similarly for all $k \in \mathbb{Z}_{\geq 0}$ we have $A^k v_i = \lambda_i^k v_i$. This implies $f(A)v_i = f(\lambda_i)v_i$ for all polynomials $f \in \mathbb{C}[x]$. In particular, we have $0 = m_A(A)v_i = m_A(\lambda_i)v_i$. Therefore $\lambda_i$ is a root of $m_A$.

The preceding two lemmas allow us to prove a fact about projection matrices, or those matrices $P$ that satisfy $P^2 = P$.

Lemma 2.1.6. Let $V$ be a finite dimensional complex vector space with subspace $U$. Let $P : V \to V$ be a projection onto $U$. Then $\text{tr}(P) = \dim(U)$.

Proof. Let $U \subset V$ be a subspace with projection map $P : V \to U$. Note that $P^2 = P$. Thus for the polynomial $q(x) = x^2 - x$ we have $q(P) = 0$, which implies that $m_P$ divides $q$. Note that $q(x) = x(x-1)$, so $q(x)$ has distinct roots and therefore $m_P$ has distinct roots. Thus by Lemma 2.1.4 we know $P$ is diagonalizable. We have three cases:
Case 1. $m_P = x$. In this case the only root of $m_P$ is $x = 0$, so by Lemma 2.1.5 we know each eigenvalue of $P$ is $\lambda = 0$. In particular, since $P$ is diagonalizable we have $P = ADA^{-1}$ for invertible matrix $A$ and $D = 0$. Therefore $P = 0$. Thus $U = \{0\}$ and we have $\text{tr}(P) = 0 = \dim(U)$.

Case 2. $m_P = x - 1$. Here we know that the only root of $m_P$ is $x = 1$. Thus by Lemma 2.1.5 each eigenvalue of $P$ is $\lambda = 1$. In particular, since $P$ is diagonalizable we have $P = ADA^{-1}$ for invertible matrix $A$ and $D = \text{id}$. Therefore $P = \text{id}$. Thus $U = V$ and we have $\text{tr}(P) = \text{tr}(\text{id}) = \dim(V) = \dim(U)$.

Case 3. $m_P = x(x - 1)$. In this case we know $m_P$ has roots $x = 0$ and $x = 1$. By Lemma 2.1.5 we know $P$ has eigenvalues $\lambda_0 = 0$ or $\lambda_1 = 1$. If every eigenvalue is $\lambda_0$, then we reduce to Case 1, whereas if every eigenvalue is $\lambda_1$, then we reduce to Case 2. Recall $\text{tr}(P) = \text{tr}(\sum \lambda_i)$. Therefore $\text{tr}(P)$ is the multiplicity of $\lambda_1$. In particular, since $P$ is diagonalizable the multiplicity of $\lambda_1$ is equivalent to the number of linearly independent eigenvectors with eigenvalue $\lambda_1$. Therefore $\text{im}(P) = U$ has dimension equal to the multiplicity of $\lambda_1$, and the multiplicity of $\lambda_1$ is equal to $\text{tr}(P)$. Thus $\text{tr}(P) = \dim(U)$.

\[ \square \]

2.2

Representation theory of finite groups over $\mathbb{C}$

Here we quickly discuss a few key concepts and results from the representation theory of finite groups over $\mathbb{C}$. The next section discuss further concepts and miscellaneous results with proof, but this section leaves proof out in order to give a quick birds-eye view. The major discussion points include the definition of a representation along with duals, tensors, and direct sums, then Schur’s Lemma and Maschke’s Lemma, followed by character theory. As a final note, we mention the Frobenius-Schur indicator and its important role in representation theory over $\mathbb{R}$. For more on representation theory of finite groups see Fulton-Harris [11].

2.2.1

Basic definitions

**Definition** (representation of $G$). Let $G$ be a group and let $V$ be a vector space. Let $\rho_V : G \rightarrow \text{GL}(V)$ be a group homomorphism. We say the pair $(V, \rho_V)$ is a representation of $G$.

It is common practice to simply say $V$ is a representation of $G$ without addressing the group homomorphism $\rho_V$. We find that the linear dual $V^\vee$ inherits a representation of $G$ under the
definition $\rho_{V^\vee}(g) := \rho_V(g^{-1})^\vee$. This definition respects the natural pairing of $V^\vee$ and $V$, since for $f \in V^\vee$ and $v \in V$ we have

$$\rho_{V^\vee}(g)f(\rho_V(g)v) = f(\rho_V(g^{-1})\rho_V(g)v) = f(v).$$

In particular, we have $\rho_V(g^{-1}) = \rho_V(g)^{-1}$ so the definition of a dual representation can be written as $\rho_{V^\vee}(g) = \rho_V(g)^{-\vee}$ instead. We also find that the tensor product and direct sum of two representations are again a representation under the definitions

$$\rho_{V \otimes U}(g) := \rho_V(g) \otimes \rho_U(g) \quad \text{and} \quad \rho_{V \oplus U}(g) := \rho_V(g) \oplus \rho_U(g).$$

These three representations give rise to many possible representations that we can construct and explore.

We next consider a map between two representations of $G$.

**Definition** (morphism of representations). We say the $\mathbb{C}$-linear map $\psi : V \to U$ is a morphism of representations of $G$ if for all $g \in G$ the following diagram commutes

We will often say $\psi$ is $G$-equivariant, which describes the idea that $\rho_V(g)$ can be factored through $\psi$ for all $g \in G$. From here we consider subspaces $U \subset V$ of a vector space and whether the natural inclusion map is $G$-equivariant. This gives the concept of a subrepresentation, which we denote as a subspace of $V$ such that $\rho_V(g)u \in U$ for all $u \in U$ and $g \in G$. One example of a subrepresentation is the kernel of a morphism of representations. In particular, if $\psi : V \to U$ is a morphism of representations then for all $v \in \ker(\psi)$ we have $\psi(\rho_V(g)v) = \rho_U(g)\psi(v) = 0$.

**Definition** (irreducible representations). A representation $V$ of $G$ is called irreducible if every subrepresentation is either $\{0\}$ or $V$. Equivalently, an irreducible representation is one that does not contain any non-trivial subrepresentations.

**Lemma 2.2.1.** Let $V$ be a representation of $G$. Then $V$ is irreducible if and only if every surjective morphism of representations $\psi : V \to U$ is either zero or an isomorphism.

**Proof.** Suppose $V$ is irreducible and let $\psi : V \to U$ be a surjective morphism of representations.
Then we have the following exact sequence:

\[ 0 \rightarrow \ker(\psi) \xrightarrow{\iota} V \xrightarrow{\psi} U \rightarrow 0 \]

Since \( \ker \psi \) is a subrepresentation, we know \( \ker(\psi) = \{0\} \) or \( \ker(\psi) = V \). In the former case, \( \psi \) is an isomorphism and in the latter case \( \psi \) is the zero map.

**Theorem 2.2.2.** *(Schur’s Lemma)* Let \( V \) and \( W \) be irreducible representations of \( G \). Then a morphism between \( V \) and \( W \) is either zero or an isomorphism.

Schur’s Lemma allows us to classify irreducible representations into isomorphism classes, freeing us from having to distinguish irreducibles by the chosen basis, by the vector space presentation, or by other non-canonical differences. Schur’s Lemma also allows us to explore how irreducibles show up in their non-irreducible counterparts.

**Theorem 2.2.3.** *(Maschke’s Lemma)* Every representation of \( G \) is isomorphic to a direct sum of irreducible representations of \( G \).

A key concept in proving Maschke’s Lemma is the linear algebra fact that non-degenerate Hermitian forms split a vector space into a direct sum of subspaces. Given a representation \( V \) of \( G \), we can force a non-degenerate Hermitian form to be \( G \)-invariant. Let \( H : V \times V \rightarrow \mathbb{C} \) be a non-degenerate Hermitian form and define a new Hermitian form by

\[
\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} H(\rho_V(g)v, \rho_V(g)w).
\]

(Note that we say Hermitian forms are conjugate-linear in the second argument, but other sources may use a different convention.) This yields a non-degenerate, \( G \)-invariant Hermitian form, which can then be used to break \( V \) up into a direct sum of \( G \)-invariant subspaces. We can inductively do this on each subrepresentation until each direct summand is an irreducible representation. Note that the above Hermitian form \( \langle v, w \rangle \) is defined by averaging the non-degenerate Hermitian form over the group. In particular, we rely on the fact that \( \mathbb{C} \) is a field of characteristic 0, guaranteeing that \( |G| \) is invertible.

The power of Maschke’s Lemma is that we know every possible representation is completely determined by its irreducible summands. Therefore, if we understand the irreducible representations of a group, then we essentially understand the entire representation theory of that group.
2.2.2

Character theory

A key component to understanding how a representation $(V, \rho_V)$ decomposes into irreducibles is its character $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) := \text{tr}_V(\rho_V(g))$. The cyclicity of trace allows to conclude that characters are class functions, that is for all $g, h \in G$ we have $\chi_V(hgh^{-1}) = \chi_V(g)$, so characters are constant on conjugacy classes of $G$. Characters also satisfy some convenient properties:

$$
\chi_{V \vee} = \chi_V \quad \chi_{V \otimes U} = \chi_V \chi_U \quad \chi_{V \oplus U} = \chi_V + \chi_U
$$

Moreover, with respect to the non-degenerate Hermitian inner product on the vector space $\text{Fun}(G, \mathbb{C})$ given by

$$
\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}
$$

characters of irreducible representations form an orthonormal basis of $\text{Fun}(G, \mathbb{C})$. Further study of character theory gives the surprising result that the number of conjugacy classes of $G$ is equivalent to the number of irreducible representations of $G$. Thus for a finite group $G$, we can be assured that there are only finitely many irreducibles (up to isomorphism, of course).

The natural question then is, where can we look for irreducibles? By Maschke’s Lemma we know that irreducibles completely determine the theory, and by character theory we know there are finitely many irreducibles, so what are they? This question can be answered by studying the the group algebra $\mathbb{C}[G]$.

**Definition** (the group algebra). $\mathbb{C}[G]$ is a vector space with basis $\{\ell_g\}_{g \in G}$ and multiplication given by $\ell_g \ell_h = \ell_{gh}$ extended linearly over $\mathbb{C}$.

Note that the group algebra is isomorphic to the regular representation, which is a linearization of the left action of $G$ on itself. The Peter-Weyl theorem asserts that, for each irreducible $V_i$, the group algebra (equivalently, the regular representation) contains $\dim(V_i)$ copies of $V_i$. Therefore, the group algebra contains at least one copy of each irreducible, providing the perfect place for us to search for them. We even find that the group algebra and the matrix algebra given by $\bigoplus_{V_i \in \text{Irr}(G)} \text{End}(V_i)$ are isomorphic as algebras. (Note that $\text{Irr}(G)$ is the collection of irreducible representations of $G$ up to isomorphism.) Two final things about $\mathbb{C}[G]$ are worth mentioning: (1) that modules over $\mathbb{C}[G]$ are equivalent to representations of $G$ and (2) that by identifying $\ell_g$ with the indicator function $\delta_g$ for each $g$, there is an isomorphism $\mathbb{C}[G] \simeq \text{Fun}(G, \mathbb{C})$. 
2.2.3

Frobenius-Schur

Lastly, we discuss the Frobenius-Schur indicator. Many of the important results in representation theory rely on the field \( \mathbb{C} \) being algebraically closed, yet we can ask whether a representation can be realized over a sub-field of \( \mathbb{C} \) such as \( \mathbb{R} \).

**Theorem 2.2.4. (Frobenius-Schur)** Let \( V \) be an irreducible representation of a group \( G \). The element

\[
\nu_2 = \sum_{g \in G} \ell_{g^2} \in \mathbb{C}[G]
\]

is in the center of \( \mathbb{C}[G] \) and moreover

\[
\langle \chi_V, \nu_2 \rangle = \begin{cases} 
0 & \text{if } V \text{ can only be realized over } \mathbb{C} \\
1 & \text{if } V \text{ can be realized over } \mathbb{R} \\
-1 & \text{if } V \text{ can be realized over } \mathbb{H},
\end{cases}
\]

where \( \mathbb{H} \) is the quaternions and \( \langle -,- \rangle \) is the non-degenerate Hermitian inner product on the vector space of class functions of \( G \).

Therefore, if we apply the Frobenius-Schur indicator \( \nu_2 \) to irreducible representations, we can completely determine the representation theory of finite groups over \( \mathbb{R} \) or \( \mathbb{H} \). (For more details and proofs, see section 3.5 in Fulton-Harris [11].) We will see in chapter 5 that this indicator has an extension for Real representations, and similarly determines representation theory there.

As part of the proof of Theorem 2.2.4, there are two important pieces. One is the existence of a non-degenerate \( G \)-invariant Hermitian form on \( V \). We mentioned this earlier, stating that such a form always exists by averaging some preexistent non-degenerate Hermitian form over \( G \). The second important piece is whether a non-degenerate, \( G \)-invariant bilinear form exists on the vector space. Since \( V \) is irreducible, such a bilinear form is either symmetric or skew-symmetric. If such a bilinear form \( B \) exists, then there is an induced isomorphism \( V \simeq V^\vee \) via the map \( v \mapsto B(v,-) \), which we can pair with an induced isomorphism \( V^\vee \simeq \overline{V} \) via the map \( v \mapsto H(-,v) \) for the \( G \)-invariant Hermitian form \( H \). Together, these induce a \( G \)-equivariant isomorphism \( V \simeq \overline{V} \). This isomorphism shows that the representation \( V \) is invariant under conjugation, and therefore can be realized over the real numbers if \( B \) is symmetric or can be realized over the quaternions if \( B \) is skew-symmetric.

We therefore find that the important piece for representation theory of finite groups over \( \mathbb{R} \)...
is a symmetric bilinear form, viewed as a linear isomorphism $V \cong V^\vee$. This idea of maps between a vector space and its dual is one of the key concepts we will explore in chapters 4 and 5 and while extending representation theory over $\mathbb{R}$ to what we call a Real representation.

2.3 Miscellaneous results and further representation theory

Here we comment on some results in classical representation theory that will be used throughout the paper but may not be covered in an introductory course. We first show that for a representation $V$ of $G$, each element contained in $\text{im} (\rho_V)$ is diagonalizable. This is followed by an exploration of a particular subrepresentation, the subspace of $G$-invariants, which yields a quotient representation and has an interesting projection map. After that, we make one final comment regarding how a group homomorphism paired with a representation of the codomain induces a representation on the domain of the homomorphism.

**Lemma 2.3.1.** Any square matrix satisfying $A^N = \text{id}$ for some $N \geq 1$ is diagonalizable. Moreover, if $\rho_V : G \to \text{GL}(V)$ is a group homomorphism, then $\rho_V(g)$ is diagonalizable for each $g \in G$.

**Proof.** Suppose $A$ is an $n \times n$ matrix over $(\mathbb{C})$ that satisfies $A^k = \text{id}$ for some $k \geq 1$. Note that $A^k - \text{id} = 0$. Thus $\text{m}_A$ divides the polynomial $Q := x^k - 1$. The roots of $Q$ are the $k$th roots of unity since

$$x^k - 1 = \prod_{j=0}^{k-1}(x - e^{\frac{2\pi i j}{k}}).$$

Note that these roots are distinct. Therefore, because $\text{m}_A$ divides $Q$ we know $\text{m}_A$ must also have distinct roots. Thus by Lemma 2.1.4 we know $A$ is diagonalizable. For the next portion we let $g \in G$ and consider $\rho_V(g)$ in some chosen basis of $V$. If $n = |G|$, then we have

$$\rho_V(g)^n = \rho_V(g^n) = \rho_V(e) = \text{id}_V.$$ 

The result follows.

One corollary of Lemma 2.3.1 is that for each fixed element $g \in G$, an eigenbasis of $\rho_V(g)$ exists. This particular idea is one that is used during the proof of the classical Frobenius-Schur indicator and will be extended to projective representations and used in chapter 5. We now look at a particular subrepresentation.
**Definition.** \((V^G, \text{the subspace of } G\text{-invariants})\) Let \((V, \rho_V)\) be a representation of \(G\). We denote

\[
V^G = \{ v \in V \mid \rho_V(g)v = v \text{ for all } g \in G \}
\]
as the subspace of \(G\)-invariants.

The subspace \(V^G\) holds an important piece of information about the representation \(V\): the dimension of \(V^G\) is the multiplicity of the trivial representation \(\mathbb{C}_{\text{triv}}\) in \(V\). This can easily be seen by determining the dimension of \(V^G\) and then recognizing that \(G\) acts trivially on \(V^G\). We can find the dimension of \(V^G\) by taking the trace of a projection map onto \(V^G\).

**Lemma 2.3.2.** The mapping \(P : V \to V\) defined by

\[
P(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)v.
\]

is a projection mapping onto \(V^G\), the space of \(G\)-invariants.

**Proof.** In order to show that \(P\) is a projection mapping, we must show three things, namely (1) that \(P\) acts on \(V^G\) as the identity, (2) that \(\text{im}(P) \subset V^G\), and (3) that \(P\) is idempotent \((P^2 = P)\). First, let \(v \in V^G\) and \(g \in G\). We have

\[
P(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)v = \frac{1}{|G|} \sum_{g \in G} v = v.
\]

Thus (1) is satisfied. Now let \(v \in V\) and \(h \in G\). We have

\[
\rho_V(h) P(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(hg)v = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)v = P(v).
\]

Therefore \(\text{im}(P) \subset V^G\), so (2) is satisfied. Lastly, for \(v \in V\) we compute

\[
P^2(v) = \frac{1}{|G|^2} \sum_{g \in G} \sum_{h \in G} \rho_V(gh)v
\]

\[
= \frac{1}{|G|^2} \sum_{g \in G} \sum_{h} \rho_V(h)v
\]

\[
= \frac{|G|}{|G|^2} \sum_{h \in G} \rho_V(h)v
\]

\[
= P(v).
\]
Thus $P$ is idempotent, so (3) is satisfied.

From Lemma 2.1.6 and Lemma 2.3.2 it directly follows that

$$\dim(V^G) = \text{tr}(P) = \text{tr} \left( \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \right) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

This is an interesting result, since the above quantity is equivalent to $\langle \chi_V, \chi_{\text{triv}} \rangle$, and therefore tells us how many copies of the trivial representation $\mathbb{C}_{\text{triv}}$ are present in $V$.

**Lemma 2.3.3.** Let $G$ be a group with normal subgroup $H$. Let $V$ be a representation of $G$. Then $V^H$ inherits a representation structure on $G/H$ with actions given by $\rho_{V^H}(g_iH) := \rho_V(g_i)$ for chosen coset representatives $\{g_i\}$ of $G/H$.

**Proof.** Let $n = |G/H|$ and choose coset representatives $\{e = g_0, g_1, ..., g_n\}$ for $G/H$. Fix an index $i \in \{0, ..., n\}$ and define $\rho_{V^H}(g_iH) : V^H \to V^H$ by

$$\rho_{V^H}(g_iH) = \rho_V(g_i)$$

We first show that our definition satisfies $\text{im}(\rho_{V^H}) \subset \text{GL}(V^H)$. Fix an index $i$, then pick $h \in H$ and let $v \in V^H$. Recall that because $H$ is a normal subgroup, we have $(hg_ih^{-1})H = g_iH$. Therefore we have

$$\rho_V(h) [\rho_{V^H}(g_iH)v] = \rho_V(h) \rho_V(g_i)v = \rho_V(hg_i)v = \rho_V(hg_ih^{-1}) \rho_V(h)v = \rho_{V^H}((hg_ih^{-1})H)v = \rho_{V^H}(g_iH)v.$$  

Note $\rho_{V^H}(g_iH)$ inherits linearity from $\rho_V(g_i)$. Therefore $\rho_{V^H}(g_iH) \in \text{GL}(V^H)$ for each coset $g_iH \in G/H$. Next we show our definition is well-defined. Let $x, y \in G$ be such that $xH = yH$. Thus there exists some $h \in H$ such that $x = yh$. Therefore given $v \in V^H$ we have

$$\rho_{V^H}(xH)v = \rho_V(x)v = \rho_V(yh)v = \rho_V(y)\rho_V(h)v = \rho_V(y)v = \rho_{V^H}(yH).$$

Thus the choice of coset representative does not change $\rho_{V^H}$. Lastly we check $\rho_{V^H}$ is a group
We have
\[
\rho_{V^H}(g_i H) \rho_{V^H}(g_j H) = \rho_V(g_i) \rho_{V^H}(g_j) = \rho_V(g_i g_j H) = \rho_{V^H}(g_i H g_j H).
\]

Therefore \( V^H \) is a representation of \( G/H \) with actions defined above.

In chapter 5, we use the above result for a group \( G \) paired with a surjective group homomorphism \( \pi \), thereby considering \( V^{\ker(\pi)} \) as a representation of \( G/\ker(\pi) \).

**Definition** (pullback of \( V \) by \( f \)). Let \( f : H \to G \) be a group homomorphism and let \( V \) be a representation of \( G \). Denote by \( f^*V \) the vector space \( V \) together with the mapping \( \rho_{f^*V} := \rho_V \circ f \).

We call this the pullback of \( V \) along \( f \).

**Lemma 2.3.4.** Let \( f : H \to G \) and let \( V \) be a representation of \( G \). Then \( f^*V \) is a representation of \( H \).

**Proof.** Note that the composition of group homomorphisms is a group homomorphism. Therefore \( \rho_{f^*V} = \rho_V \circ f : H \to \text{GL}(V) \) is a group homomorphism.

**Lemma 2.3.5.** Let \( f : H \to G \) be a surjective group homomorphism and \( V \) be an irreducible representation of \( G \). Then \( f^*V \) is an irreducible representation of \( H \).

**Proof.** Let \( U \subset V \) be a subrepresentation of \( G \), i.e. either \( \{0\} \) or \( V \). Pick \( g \in G \) and let \( h \in H \) be such that \( f(h) = g \). Thus for \( u \in U \) we have
\[
\rho_{f^*V}(h)u = (\rho_V \circ f)(h)u = \rho_V(g)u \in U
\]

Therefore \( U \) is \( H \)-invariant. Note that the above argument also shows every subrepresentation of \( G \) corresponds to a subrepresentation of \( H \). Thus \( f^*V \) is an irreducible representation of \( H \) because the only subrepresentations of \( H \) are \( U = \{0\} \) and \( U = V \).

In this paper, we will concern ourselves with inner automorphisms of \( G \). As an abuse of notation, for each \( h \in G \) we denote the group homomorphism
\[
h : G \to G
\]
\[
g \mapsto h^{-1}gh.
\]

The nice thing about these inner automorphisms is that they are surjective, so by Lemma 2.3.5 they preserve irreducibility of \( G \) representations.
**Lemma 2.3.6.** Let $V$ be a representation of $G$. Then for all $h \in G$, we have $h^*V \simeq V$ where $h : G \rightarrow G$ is the mapping $g \mapsto h^{-1}gh$.

**Proof.** By the definition of a pullback we have $\rho_{h^*V} = \rho_V \circ h$. Therefore for all $g \in G$ we have

$$
\rho_V(h)\rho_{h^*V}(g) = \rho_V(h)\rho_V(h^{-1}gh)
$$

$$
= \rho_V(h)\rho_V(h^{-1})\rho_V(g)\rho_V(h)
$$

$$
= \rho_V(g)\rho_V(h).
$$

Therefore, the following diagram commutes:

$$
\begin{array}{ccc}
\text{h}^*V & \xrightarrow{\rho_V(h)} & V \\
\downarrow{\rho_{h^*V}(g)} & & \downarrow{\rho_V(g)} \\
\text{h}^*V & \xrightarrow{\rho_V(h)} & V
\end{array}
$$

$\square$
CHAPTER 3
PROJECTIVE REPRESENTATION THEORY

3.1 Motivation

The motivation for projective representation theory is tied to the complex projective space \( \mathbb{CP}^n \). Fix some positive integer \( n \) and define an equivalence relation \( \sim \) on \( \mathbb{C}^n \setminus \{0\} \) such that \( v \sim \lambda v \) for all non-zero \( v \in \mathbb{C}^n \) and \( \lambda \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\} \). Define the complex projective space

\[
\mathbb{CP}^{n-1} := \left( \mathbb{C}^n \setminus \{0\} \right) / \sim
\]

We want to consider the symmetries of \( \mathbb{CP}^{n-1} \). Note that \( \mathbb{C}^\times \cdot \text{id}_n = \{ \lambda \cdot \text{id}_n \mid \lambda \in \mathbb{C}^\times \} \) is a normal subgroup of \( \text{GL}_n(\mathbb{C}) \) since it commutes with each element. We define the projective linear group

\[
\mathbb{PGL}_n(\mathbb{C}) := \left( \text{GL}_n(\mathbb{C}) \right) / \left( \mathbb{C}^\times \cdot \text{id}_n \right).
\]

Thus for any two mappings \( \phi, \psi \in \text{GL}_n(\mathbb{C}) \) if there exists some \( \lambda \in \mathbb{C}^\times \) such that \( \phi = \lambda \psi \) then both maps lie in the same equivalence class \( [\phi] = [\psi] \in \mathbb{PGL}_n(\mathbb{C}) \). In particular, the elements in \( \mathbb{PGL}_n(\mathbb{C}) \) are equivalence classes of linear isomorphisms of \( \mathbb{C}^n \) such that multiplication by a non-zero constant is trivial. This is analogous for vectors in \( \mathbb{C}^n \) and their equivalence classes in \( \mathbb{CP}^{n-1} \), where multiplication by a non-zero constant is trivial.

For a finite group \( G \), consider a group homomorphism \( \tilde{\rho} : G \to \mathbb{PGL}_n(\mathbb{C}) \). We ask whether \( \tilde{\rho} \) can be lifted to a representation of \( G \) or in other words, we ask whether there exists a map \( \rho \) (not necessarily a group homomorphism) such that the following diagram commutes:
for canonical projection homomorphism $P : \text{GL}_n(\mathbb{C}) \to \text{PGL}_n(\mathbb{C})$. If such a map exists then we have

$$P \left( \rho(g_2) \rho(g_1) \right) = (P \circ \rho)(g_2)[(P \circ \rho)(g_1)] = \tilde{\rho}(g_2) \tilde{\rho}(g_1) = \tilde{\rho}(c_{g_2, g_1}) = P \left( \rho(g_2 g_1) \right).$$

Therefore we know $\rho(g_2) \rho(g_1)$ and $\rho(g_2 g_1)$ lie in the same equivalence class of $\text{PGL}_n(\mathbb{C})$. The two mappings need not be equal on the nose, but we do know there is a constant $c_{g_2, g_1} \in \mathbb{C}^\times$ such that $\rho(g_2) \rho(g_1) = c_{g_2, g_1} \rho(g_2 g_1)$. Any choice of two elements $g_1, g_2 \in G$ thus yields a (possibly different) constant $c_{g_2, g_1}$, giving rise to a collection of constants $\{c_{g_2, g_1}\}_{g_1, g_2 \in G}$ that describes how the mappings in $\text{im}(\rho)$ are related. Moreover this collection of constants must satisfy certain equalities since for $g_1, g_2, g_3 \in G$ we have

$$c_{g_3, g_2} c_{g_2, g_1} \rho(g_3 g_2 g_1) = c_{g_3, g_2} \rho(g_3) \rho(g_2) \rho(g_1)$$

$$= \rho(g_3) \rho(g_2) \rho(g_1)$$

$$= c_{g_2, g_1} \rho(g_3) \rho(g_2 g_1)$$

$$= c_{g_2, g_1} c_{g_3, g_2} \rho(g_3 g_2 g_1).$$

Therefore, given any three elements in $G$, we know $\{c_{g_2, g_1}\}_{g_1, g_2 \in G}$ must satisfy

$$c_{g_3, g_2} c_{g_2, g_1} = c_{g_3, g_2} c_{g_2, g_1}. $$

Alternatively, if we begin with a map $\rho : G \to \text{GL}_n(\mathbb{C})$ and a collection of constants $\{c_{g_2, g_1}\}_{g_1, g_2 \in G}$ such that $\rho(g_2) \rho(g_1) = c_{g_2, g_1} \rho(g_2 g_1)$ and $c_{g_3, g_2} c_{g_2, g_1} = c_{g_3, g_2} c_{g_2, g_1}$ for all $g_3, g_2, g_1 \in G$ we can ask whether this induces a group homomorphism $\tilde{\rho} : G \to \text{PGL}_n(\mathbb{C})$. Beginning with this type of data, we define $\tilde{\rho} := P \circ \rho$, giving

$$\tilde{\rho}(g_2) \tilde{\rho}(g_1) = (P \circ \rho)(g_2)[(P \circ \rho)(g_1)] = P \left( \rho(g_2) \rho(g_1) \right) = P \left( \rho(g_2) \rho(g_1) \right).$$

Recalling that $P$ projects $\lambda \phi$ and $\phi$ to the same equivalence class for all $\lambda \in \mathbb{C}^\times$ and $\phi \in \text{GL}_n(\mathbb{C})$, we conclude

$$\tilde{\rho}(g_2) \tilde{\rho}(g_1) = P \left( c_{g_2, g_1} \rho(g_2 g_1) \right) = P \left( \rho(g_2 g_1) \right) = P \left( \rho(g_2 g_1) \right).$$

Thus $\tilde{\rho}$ is a group homomorphism. Therefore, we can either begin with a group homomorphism $\tilde{\rho} : G \to \text{PGL}_n(\mathbb{C})$ and ask for a lift $\rho : G \to \text{GL}_n(\mathbb{C})$ paired with a collection of constants, or we can begin with a mapping $\rho : G \to \text{GL}_n(\mathbb{C})$ paired with a collection of constants satisfying certain
conditions and get a group homomorphism $\bar{\rho}$. These ideas motivate the following definition.

**Definition** (projective representation). Let $G$ be a group and let $V$ be a complex vector space. Let $\rho_V : G \to \text{GL}(V)$ be a mapping and $\{c_{g_2,g_1}\}_{g_1, g_2 \in G} \subset U(1)$ be a collection of constants such that for all $g_3, g_2, g_1 \in G$ we have

$$\rho_V(g_2)\rho_V(g_1) = c_{g_2, g_1} \rho_V(g_2g_1) \quad \text{and} \quad c_{g_3g_2, g_1} = c_{g_3, g_2}c_{g_2, g_1}.$$ 

We also require that $\rho_V(e) = \text{id}_V$. We call $\rho$ a projective representation of $G$ with multiplier $\{c_{g_2, g_1}\}_{g_1, g_2 \in G}$.

### 3.2 Projective multipliers and group cohomology

The definition of a projective representation of a group $G$ requires a collection of constants satisfying certain relations. In this section we give an explanation for what these constants are as mathematical objects and where they come from, the answer to both being group cohomology. For more on group cohomology see Brown’s textbook [3].

#### 3.2.1 Group cohomology

**Definition** (the chain complex $C^*(G;M)$). Let $G$ be a group and let $M$ be a $G$-module. Here we denote the commutative binary operation on $M$ as multiplication, and we denote the action of $G$ on $M$ as multiplication. For each $n \geq 0$ we define $C^n(G;M) := \text{Fun}(G^n, M)$. For $\theta, \alpha \in C^n$ define $\theta \alpha$ as the point-wise product of $\theta$ and $\alpha$ in $M$. Therefore $C^n(G;M)$ is an abelian group.

**Lemma 3.2.1.** For each $n \geq 0$ define mappings $d : C^n(G;M) \to C^{n+1}(G;M)$ by

$$(d\theta)(g_{n+1}, g_n, ..., g_1) = g_{n+1} \cdot \theta(g_n, ..., g_1) \left( \prod_{i=1}^n \theta(g_{n+1}, ..., g_{i+1}g_i, ..., g_1) (-1)^{n-i+1} \right) \theta(g_{n+1}, ..., g_2) (-1)^{n+1}$$

for $\theta \in C^n$. Then $\{C^n(G;M)\}_{n \geq 0}$ together with $\{d\}$ makes a chain complex which we denote by $C^*(G;M)$.

**Proof.** We need only show that $d \circ d = 0$, where $0$ denotes the map defined by $(g_{n+2}, ..., g_1) \mapsto 1 \in M$ for all $g_i \in G$. Here, we show this computation for $n = 1$, but leave the general case to the reader.
Let $g_3, g_2, g_1 \in G$ and let $\theta \in C^1(G;M)$. Then we have

\[
(dd\theta)(g_3, g_2, g_1) = \frac{g_3 \cdot \theta(g_1) \theta(g_2)}{\theta(g_2 g_1)} \frac{g_3 g_2 \cdot \theta(g_1) \theta(g_3 g_2)}{\theta(g_3 g_2 g_1)} \frac{g_3 \cdot \theta(g_2 g_1) \theta(g_3)}{\theta(g_3 g_2)} \frac{g_3 g_2 g_1 \cdot \theta(g_1)}{\theta(g_3 g_2)}
\]

\[
= 1
\]

Therefore $d \circ d = 0$.  

**Definition** (group cohomology). Let $G$ be a group and $M$ be a $G$-module. Consider the chain complex $C^\ast(G;M)$ as in Lemma 3.2.1. For all $n \geq 0$, we denote cocycles as $Z^n(G;M) := \ker(d : C^n \to C^{n+1})$, coboundaries as $B^n(G;M) := \text{im}(d : C^{n-1} \to C^n)$ and the $n$th-cohomology as

\[
H^n(G;M) := Z^n(G;M) / B^n(G;M).
\]

Note that for each $n \geq 0$, the set $Z^n(G;M)$ is an abelian group with point-wise multiplication inherited from $C^n(G;M)$. Furthermore, we find $H^n(G;M)$ is an abelian group under the multiplication $[\theta][\alpha] = [\theta \alpha]$.

3.2.2

**Examples**

We concern ourselves with two examples of this cohomology. The first is where we take $M = U(1)$ to be the trivial $G$-module. Therefore, given $\theta \in Z^2(G;U(1))$ we have

\[
1 = (d\theta)(g_3, g_2, g_1) = \frac{\theta(g_2, g_1) \theta(g_3, g_2 g_1)}{\theta(g_3 g_2, g_1) \theta(g_3, g_2)}.
\]

Notice that the above equality implies

\[
\theta(g_3, g_2) \theta(g_3 g_2, g_1) = \theta(g_2, g_1) \theta(g_3, g_2 g_1),
\]

which is the same condition that a collection of constants for a projective representation of $G$ must satisfy. Therefore, we can write an equivalent definition for projective representations.

**Definition** (projective representation). Let $G$ be a group and consider a cocyle $\theta \in Z^2(G;U(1))$
where \( U(1) \) is the trivial \( G \)-module. Then a \( \theta \)-projective representation of \( G \) is a vector space \( V \) with a map \( \rho_V : G \to GL(V) \) such that \( \rho_V(e) = \text{id}_V \) and for all \( g, h \in G \) we have

\[
\rho_V(g) \rho_V(h) = \theta(g, h) \rho_V(gh).
\]

We will equivalently say that \( V \) is a projective representation of \( G \) with multiplier \( \theta \).

We note that any \( \theta \in Z^2(G; U(1)) \) is cohomologous to a normalized cocycle where \( 1 = \theta(g, e) = \theta(e, g) \) for all \( g \in G \). In this paper we will only concern ourselves with normalized cocycles, but there are similar calculations for un-normalized cocycles. We further note that if \( \theta \) is the trivial multiplier, \( 1 \in Z^2(G; U(1)) \), then the projective representation is a classical representation in the sense of section 2.2. This indicates that projective representation theory is a good extension of representation theory of finite groups over \( \mathbb{C} \). In section 3.5 we will show that the projective representation theory of a group is the same for two cohomologous multipliers.

The second example of group cohomology that we will concern ourselves with is when \( M = U(1) \) with a non-trivial \( G \)-action. Let \( \pi : G \to \mathbb{Z}_2 \) be a surjective group homomorphism for some group \( G \) and denote \( G_\alpha := \ker(\pi) \). We denote \( U(1)_\pi \) as the abelian group \( U(1) \) with \( G \)-module structure given by

\[
g \cdot \lambda = \lambda \quad \text{and} \quad \xi \cdot \lambda = \bar{\lambda}
\]

for all \( g \in G_\alpha \), \( \xi \in G \setminus G_\alpha \), and \( \lambda \in U(1) \). In particular, since we have \( \lambda \in U(1) \), we know \( \bar{\lambda} = \lambda^{-1} \), so the action of \( G \setminus G_\alpha \) inverts \( \lambda \). Therefore, if we consider a cocycle \( \theta \in Z^2(G; U(1)_\pi) \), then we have

\[
1 = (d\theta)(g_3, g_2, g_1) = \frac{g_3 \cdot \theta(g_2, g_1) \theta(g_3, g_2 g_1)}{\theta(g_3 g_2, g_1) \theta(g_3, g_2)}.
\]

This implies

\[
\theta(g_3, g_2) \theta(g_3 g_2, g_1) = \theta(g_2, g_1)^{\pi(g_3)} \theta(g_3, g_2 g_1),
\]

which is a condition that we will discuss in section 4.3. In order to denote the cohomology theory
for this non-trivial $G$-module $U(1)_\pi$, we will write the following for all $n \geq 0$:

\[
\begin{align*}
C^{n+\pi}(G; U(1)) & := C^n(G; U(1)_\pi) \\
Z^{n+\pi}(G; U(1)) & := Z^n(G; U(1)_\pi) \\
B^{n+\pi}(G; U(1)) & := B^n(G; U(1)_\pi) \\
H^{n+\pi}(G; U(1)) & := Z^{n+\pi}(G; U(1)) / B^{n+\pi}(G; U(1)).
\end{align*}
\]

**Lemma 3.2.2.** Let $\pi : \mathbb{Z}_2 \to \mathbb{Z}_2$ be the identity map. Then we have $H^{2+\pi}(\mathbb{Z}_2; U(1)) \simeq \mathbb{Z}_2$.

Moreover, the non-trivial element of $H^{2+\pi}(G; U(1))$ is given by $[\bar{\alpha}]$ where

\[
\bar{\alpha}(a, b) = \begin{cases} 
1 & \text{either } a \text{ or } b \text{ is the trivial element of } \mathbb{Z}_2 \\
-1 & a = b = -1.
\end{cases}
\]

**Proof.** By definition $B^{2+\pi}(\mathbb{Z}_2; U(1))$ is the set of $\theta \in C^{2+\pi}(\mathbb{Z}_2; U(1))$ such that $\theta = d\lambda$ for some $\lambda \in C^{1+\pi}(\mathbb{Z}_2; U(1))$. Let $\theta \in B^{2+\pi}(\mathbb{Z}_2; U(1))$. Thus for $a, b \in \mathbb{Z}_2$ we have

\[
\theta(a, b) = (d\lambda)(a, b) = \frac{\lambda(a)^{\pi(b)}\lambda(b)}{\lambda(ab)}.
\]

**Case 1.** $a = 1$. We have $\theta(a, b) = \frac{\lambda(b)}{\lambda(b)} = 1$.

**Case 2.** $b = 1$. We have $\theta(a, b) = \frac{\lambda(a)}{\lambda(a)} = 1$.

**Case 3.** $a = b = -1$. We have $\theta(a, b) = \frac{\lambda(-1)^{-1}\lambda(-1)}{\lambda(1)} = 1$.

Therefore $B^{2+\pi}(\mathbb{Z}_2; U(1))$ is trivial. Thus $H^{2+\pi}(\mathbb{Z}_2; U(1)) = Z^{2+\pi}(\mathbb{Z}_2; U(1))$. Let $\theta \in Z^{2+\pi}(G; U(1))$. Since we are assuming $\theta$ is normalized, the only value that could potentially be non-trivial is $\theta(-1, -1)$. Recall that $\theta \in Z^{2+\pi}(G; U(1))$ is equivalent to the condition

\[
\theta(a, b)\theta(ab, c) = \theta(b, c)^{\pi(a)}\theta(a, bc)
\]

for all $a, b, c \in \mathbb{Z}_2$. In particular, we have

\[
\theta(-1, -1)\theta(1, -1) = \theta(-1, -1)^{-1}\theta(-1, 1)
\]

which implies $\theta(-1, -1)^2 = 1$. Therefore we have either $\theta(-1, -1) = 1$ or $\theta(-1, -1) = -1$. In the former case, $\theta$ is trivial. In the latter case, define $\bar{\alpha} := \theta$. The result follows. ☐
3.2.3

Basic functorality

It is important to understand how group homomorphisms interact with the chain complexes, as later we will be considering surjective homomorphisms $\pi : G \to \mathbb{Z}_2$ and the inclusion $\ker(\pi) \to G$.

**Lemma 3.2.3.** Let $G$ and $H$ be groups and let $M$ be an $H$-module. Let $\phi : G \to H$ be a group homomorphism. Then the pullback $\phi^* M$ is a $G$-module and $\phi$ induces a morphism of chain complexes $\phi^* : C^*(H; M) \to C^*(G; M)$.

**Proof.** For all $g \in G$ define the action of $g$ on $M$ by $\phi(g) \cdot m$ for all $m \in M$. This is a group action since the group homomorphism structure is preserved and since the action of $H$ on $M$ behaves well with respect to the multiplication in $M$. We now check that $\phi$ induces a morphism of chain complexes. Let $\phi$ act diagonally on $G^n$. For $n \geq 0$ define

$$\phi^* : C^n(H; M) \to C^n(G; M)$$

$$f \mapsto f \circ \phi.$$  

We now show $d_G \circ \phi^* = \phi^* \circ d_H$. Let $\theta \in C^n(H; M)$ and $g_i \in G$ for $1 \leq i \leq n + 1$. We compute

$$(d_G \phi^* \theta)(g_{n+1}, \ldots, g_1) = (d_G (\theta \circ \phi))(g_{n+1}, \ldots, g_1)$$

$$= (d_H \theta)(\phi(g_{n+1}), \ldots, \phi(g_1))$$

$$= ((d_H \theta) \circ \phi)(g_{n+1}, \ldots, g_1)$$

$$= (\phi^* d_H \theta)(g_{n+1}, \ldots, g_1).$$

Therefore, $\phi^*$ commutes with the boundary mappings. Thus $\phi$ induces a morphism of chain complexes. \qed

In particular, let $\pi : G \to \mathbb{Z}_2$ be a surjective group homomorphism and define $G_\alpha = \ker(\pi)$. Then by Lemma [3.2.3] the inclusion map $\iota_\alpha : G_\alpha \to G$ induces a morphism of chain complexes

$$\iota_\alpha^* : C^2(G; U(1)) \to C^2(G_\alpha; U(1)).$$

We then denote $\theta_\alpha := \iota_\alpha^* \theta$ for any $\theta \in C^2(G; U(1))$. Notice that $\iota_\alpha^*$ effectively "forgets" $G \setminus G_\alpha$. In particular, because $G_\alpha$ acts trivially on $U(1)_\pi$ and the image of $\iota_\alpha^*$ only concerns itself with $G_\alpha$, we can equivalently define $\theta_\alpha := \iota_\alpha^* \theta$ for $\theta \in C^2(G; U(1)_\pi)$. 
**Definition (the universal multiplier \( \alpha \)).** Let \( \alpha := \pi^* \tilde{\alpha} \in \mathbb{Z}^2 + \pi(G; \mathbb{U}(1)) \) where \( \tilde{\alpha} \) is the multiplier from Lemma 3.2.2. We call \( \alpha \) the universal multiplier.

We use the term "universal" in the above definition because any \( \mathbb{Z}_2 \)-grading of a group \( G \) will induce the same function \( \alpha \). We will see the importance of this universal multiplier in chapter 5.

**Lemma 3.2.4.**

\[
\alpha(g_1, g_2) = \begin{cases} 
1 & \text{either } g_1 \text{ or } g_2 \text{ is an element of } G_o \\
-1 & g_1, g_2 \in G \setminus G_o.
\end{cases}
\]

*Proof.* Let \( g_1, g_2 \in G \). We have

\[
\alpha(g_1, g_2) = \pi^* \tilde{\alpha}(g_1, g_2) = \tilde{\alpha}(\pi(g_1), \pi(g_2)).
\]

The result follows from the definition of \( \tilde{\alpha} \). \( \square \)

### 3.3 Further projective representations

As with classical representation theory, there are further projective representations that can be induced from one or two existing projective representations. Here we comment specifically on duals, tensor products, and direct sums of projective representations.

**Lemma 3.3.1.** Let \( \mathcal{V} \) be a finite dimensional \( \theta \)-projective representation of \( G \) for some \( \theta \in \mathbb{Z}^2(G; \mathbb{U}(1)) \).

For each \( g \in G \) define

\[
\rho_{\mathcal{V}^\vee}(g) := \lambda_\theta^{-1}(g) \rho_{\mathcal{V}}(g^{-1})^\vee
\]

where \( \lambda_\theta : G \to \mathbb{C} \) is defined by \( \lambda_\theta(g) = \theta(g, g^{-1}) \) (see the appendix for more on \( \lambda_\theta \)). Then \( \mathcal{V}^\vee \) is a \( \theta^{-1} \)-projective representation of \( G \).
Proof. Let \( g, h \in G \). We compute

\[
\rho_{V^\vee}(g)\rho_{V^\vee}(h) = \lambda_{\theta}^{-1}(g)\rho_{V}(g^{-1})^\vee \lambda_{\theta}^{-1}(h)\rho_{V}(h^{-1})^\vee \\
= \lambda_{\theta}^{-1}(g)\lambda_{\theta}^{-1}(h)\left(\rho_{V}(h^{-1})\rho_{V}(g^{-1})\right)^\vee \\
= \lambda_{\theta}^{-1}(g)\lambda_{\theta}^{-1}(h)\left(\theta(h^{-1}, g^{-1})\rho_{V}(h^{-1}g^{-1})\right)^\vee \\
= \theta(h^{-1}, g^{-1})\frac{\lambda_{\theta}^{-1}(g)\lambda_{\theta}^{-1}(h)}{\lambda_{\theta}^{-1}(gh)}\rho_{V}(gh) \\
= \theta^{-1}(g,h)\rho_{V}(g,h)^{-1}(g,h)\rho_{V}(gh) \quad \text{identity A.1} \\
= \theta^{-1}(g,h)\rho_{V^\vee}(gh).
\]

Note that we have \( \theta^{-1} \in Z^2(G; U(1)) \). Thus with this definition of \( \rho_{V^\vee}(g) \) we find \( V^\vee \) is a \( \theta^{-1} \)-projective representation.

Lemma 3.3.2. For a \( \theta \)-projective representation \( V \) of \( G \) we have \( \rho_{V^\vee}(g^{-1})^\vee = \lambda_{\theta}(g)\rho_{V}(g)^{-\vee} \).

Proof. We compute

\[
\rho_{V^\vee}(g^{-1})^\vee \rho_{V}(g)^\vee = \lambda_{\theta}(g)\rho_{V^\vee}(g)\lambda_{\theta}(g^{-1})\rho_{V^\vee}(g^{-1}) \\
= \lambda_{\theta}(g)\lambda_{\theta}(g^{-1})\theta^{-1}(g,g^{-1})\rho_{V^\vee}(gg^{-1}) \\
= \lambda_{\theta}(g)\lambda_{\theta}(g^{-1})\lambda_{\theta}^{-1}(g)\rho_{V^\vee}(gg^{-1}) \\
= \lambda_{\theta}(g)\text{id}_{V^\vee}.
\]

Multiplying in the other order yields the same result.

From the previous two lemmas, we see that

\[
\rho_{V^\vee}(g) = \lambda_{\theta}^{-1}(g)\rho_{V}(g^{-1})^\vee = \lambda_{\theta}^{-1}(g)\lambda_{\theta}(g)(\rho_{V}(g)^\vee)^{-1} = \rho_{V}(g)^{-\vee},
\]

so in particular, we can define \( \rho_{V^\vee}(g) := \rho_{V}(g)^{-\vee} \), and this matches the same result from Lemma 3.3.1. We also see that just as in classical representation theory, this definition preserves the natural pairing of \( V^\vee \) and \( V \):

\[
\rho_{V^\vee}(g)f(\rho_{V}(g)v) = f(v).
\]

Lemma 3.3.3. Let \( V \) be a \( \theta \)-projective representation of \( G \) and let \( U \) be a \( \beta \)-projective representation.
of $G$ for $\theta, \beta \in Z^2(G; U(1))$. For each $g \in G$ define

$$\rho_{V \otimes U}(g) := \rho_V(g) \otimes \rho_U(g).$$

Then $V \otimes U$ is a $\theta\beta$-projective representation of $G$.

**Proof.** For $g, h \in G$ we have

$$\rho_{V \otimes U}(g)\rho_{V \otimes U}(h)(v \otimes u) = \rho_V(g)\rho_V(h)\rho_U(g)\rho_U(h)u$$

$$= \theta(g, h) \rho_V(gh)\rho_U(gh)u$$

$$= \theta(g, h) \beta(g, h) \rho_V(gh)\rho_U(gh)u$$

$$= \theta(g, h) \rho_{V \otimes U}(gh)(v \otimes u).$$

**Lemma 3.3.4.** Let $V$ be a $\theta$-projective representation of $G$ and let $U$ be an $\beta$-projective representation of $G$ for $\theta, \beta \in Z^2(G; U(1))$. For each $g \in G$ define

$$\rho_{V \oplus U}(g) := \rho_V(g) \oplus \rho_U(g).$$

Then $V \oplus U$ is a projective representation of $G$ if and only if $\theta = \beta$, in which case the multiplier is $\theta$.

**Proof.** Suppose $\theta = \beta$ and let $g, h \in G$. Then we have

$$\rho_{V \oplus U}(g)\rho_{V \oplus U}(h)(v \oplus u) = \rho_V(g)\rho_U(h)v \oplus \rho_U(g)\rho_U(h)u$$

$$= \theta(g, h) \rho_V(gh)\rho_U(gh)u$$

$$= \theta(g, h) \rho_{V \oplus U}(gh)(v \oplus u).$$

If we instead suppose that $V \oplus U$ is a projective representation then we have

$$\rho_{V \oplus U}(g)\rho_{V \oplus U}(h)(v \oplus u) = \theta(g, h)\rho_{V \oplus U}(g)(v \oplus u)$$

which implies that $\theta(g, h)$ was factored out of the direct sum, so $\theta = \beta$. 

We find a nice result from classical representation theory extends to projective representation...
theory: the diagonalizability of $\rho_V(g)$ for each $g \in G$. This is an analogous result to Lemma 2.3.1 and will be used in chapter 5.

**Lemma 3.3.5.** Let $V$ be a $\theta$-projective representation of $G$. Then for each $g \in G$ the linear map $\rho_V(g)$ is diagonalizable.

**Proof.** Let $n = |G|$ and fix some $g \in G$. Therefore we have

$$\rho_V(g)^n = \left( \prod_{i=1}^{n-1} \theta(g, g^i) \right) \rho_V(g^n) = \left( \prod_{i=1}^{n-1} \theta(g, g^i) \right) \text{id}_V.$$

Let $\beta_g = \prod_{i=1}^{n-1} \theta(g, g^i)$. Thus

$$\left( \frac{1}{\beta_g^{1/n}} \rho_V(g) \right)^n = \frac{1}{\beta_g} \rho_V(g)^n = \text{id}_V.$$

Therefore, by Lemma 2.3.1 we know $\frac{1}{\beta_g^{1/n}} \rho_V(g)$ is diagonalizable. Let $P$ be an invertible matrix and $D$ be a diagonal matrix such that $\frac{1}{\beta_g^{1/n}} \rho_V(g) = PDP^{-1}$. Thus we have

$$\rho_V(g) = \beta_g^{1/n} PDP^{-1} = P(\beta_g^{1/n} D)P^{-1},$$

so $\rho_V(g)$ is diagonalizable. \hfill \Box

### 3.4

Irreducible projective representations and Schur’s Lemma

**Definition** (morphism of projective representations). Let $V$ and $U$ be projective representations of $G$. Let $\psi: V \to U$ be a $\mathbb{C}$-linear map. We say $\psi$ is a morphism of projective representations if the following diagram commutes for all $g \in G$:

$$
\begin{array}{ccc}
V & \xrightarrow{\psi} & U \\
\downarrow{\rho_V(g)} & & \downarrow{\rho_U(g)} \\
V & \xrightarrow{\psi} & U
\end{array}
$$

We note that this definition restricts to a morphism of representations if the multiplier of both representations is trivial. If both projective representations $V$ and $U$ have the same multiplier $\theta$ and $\psi: V \to U$ is a morphism of projective representations we will say that $\psi$ is a morphism of $\theta$-projective representations.
Lemma 3.4.1. Let $V$ be a $\theta$-projective representation of $G$ and let $W$ be $\beta$-projective representation of $G$. Let $\psi : V \to W$ be a morphism of representations. If $\theta \neq \beta$ then $\psi$ is the zero map.

Proof. Suppose by way of contradiction that $\theta \neq \beta$ and $\psi$ is not the zero map. Because $\psi$ is a morphism of representations, for all $g, h \in G$ we have $\rho_V(gh) \circ \psi = \psi \circ \rho_W(gh)$. We also have

$$
\theta(g, h) \rho_V(gh) \circ \psi = \rho_V(g) \rho_V(h) \circ \psi = \psi \circ \rho_W(g) \rho_W(h) = \beta(g, h) \psi \circ \rho_W(gh).
$$

This implies $\theta(g, h) = \beta(g, h)$ for all $g, h \in G$. Thus we have $\theta = \beta$, a contradiction. Therefore if $\theta \neq \beta$ then $\psi$ must be the zero map.

Lemma 3.4.1 tells us that the space of $G$-equivariant linear maps between two projective representations $V$ and $W$ of $G$ only ever has a chance to be non-zero when $V$ and $W$ have the same multiplier. This result distinguishes the projective representation theory of $G$ based on the multiplier $\theta \in Z^2(G; U(1))$. Despite this, we will see in the next section that the projective representation theory of two cohomologous multipliers is the same.

Definition (projective subrepresentation). Let $V$ be a projective representation of $G$. We say a subspace $U \subset V$ is a projective subrepresentation of $V$ if for all $u \in U$ and $g \in G$ we have $\rho_V(g)u \in U$.

We note that projective subrepresentations inherit the projective multiplier, so if $U \subset V$ is a projective subrepresentation of $\theta$-projective representation $V$, then $U$ is also $\theta$-projective.

Definition (irreducible projective representation). We say a projective representation $V$ of $G$ is irreducible if the only projective subrepresentations are $\{0\}$ and $V$.

As to be expected, the above two definitions recover the classical representation theory definitions when $\theta$ is trivial.

Theorem 3.4.2 (Schur’s Lemma, [4, Lemma 2.1]). If $V$ and $W$ are irreducible $\theta$-projective representations of $G$ and $\psi : V \to W$ is a morphism of $\theta$-projective representations then $\psi$ is either an isomorphism or the zero map.
3.5

The twisted group algebra and projective characters

**Definition (\(C^\theta[G]\), the twisted group algebra).** Let \(G\) be a finite group and \(\theta \in \mathbb{Z}^2(\mathbb{G}; 	ext{U}(1))\). Let \(C^\theta[G]\) be the complex vector space with basis \((\ell_g)_{g \in G}\) and for all \(g, h \in G\) define multiplication by

\[
\ell_g \ell_h = \theta(g, h) \ell_{gh}
\]

then extend bilinearly over \(C\).

The cocycle condition of \(\theta\) makes the multiplication of \(C^\theta[G]\) associative. Therefore, \(C^\theta[G]\) is an associative algebra over \(C\) with unit \(\ell_e\).

**Definition (mod \(C^\theta[G]\)).** Denote by \(\text{mod } C^\theta[G]\) the category whose objects are finite dimensional (left) \(C^\theta[G]\) modules and whose morphisms are module homomorphisms.

**Definition (Rep\(^\theta\)(\(G\))).** Denote by \(\text{Rep}^\theta(G)\) the category whose objects are \(\theta\)-projective representations of \(G\) and whose morphisms are morphisms of representations.

We note that Lemma 3.3.4 and section 3.4 together imply that \(\text{Rep}^\theta(G)\) is an abelian category. Furthermore, Lemma 3.3.3 implies that \(\text{Rep}^\theta(G)\) is a monoidal category if and only if \(\theta\) is the trivial multiplier.

**Lemma 3.5.1.** There is an equivalence of categories between \(\text{Rep}^\theta(G)\) and \(\text{mod } C^\theta[G]\).

**Proof.** First consider a left \(C^\theta[G]\)-module

\[
\phi_V : C^\theta[G] \to \text{End}_C(V).
\]

Thus for each \(a, b \in C^\theta[G]\) and \(\lambda \in C\) we know \(\phi_V(a)\) is a linear map on \(V\) satisfying \(\phi_V(ab) = \phi_V(a)\phi_V(b)\) and \(\phi_V(\lambda a) = \lambda \phi_V(a)\). Note that for each \(g \in G\), the element \(\phi_V(\ell_g) \in \text{End}(V)\) is invertible with inverse \(\theta(g, g^{-1})^{-1}\phi_V(\ell_{g^{-1}})\). We then define the functor on objects \(\phi_V\) and morphisms \(\gamma\) as

\[
F : \text{mod } C^\theta[G] \to \text{Rep}^\theta(G)
\]

\[
\phi_V \mapsto \rho_{V,\phi}
\]

\[
\gamma \mapsto \gamma
\]
where $\rho_{V,\phi}(g) := \phi_V(\ell_g)$ for all $g \in G$. Note this is well-defined on the objects because each $\phi_V(\ell_g)$ is an invertible linear map on $V$ and

$$\rho_{V,\phi}(g)\rho_{V,\phi}(h) = \phi_V(\ell_g)\phi_V(\ell_h) = \phi_V(\ell_g\ell_h) = \theta(g,h)\phi_V(\ell_{gh}) = \theta(g,h)\rho_{V,\phi}(gh).$$

We also define a functor on objects $\rho_V$ and morphisms $\mu$ by

$$H : \text{Rep}^\theta(G) \to \text{mod} \mathbb{C}[G]^\theta$$

$$\rho_V \mapsto \phi_{V,\rho}$$

$$\mu \mapsto \mu$$

where $\phi_{V,\rho}(\ell_g) := \rho_V(g)$ for all $g \in G$ and everything is extended by $\mathbb{C}$-linearity. Note this is well-defined because we extended linearly and

$$\phi_{V,\rho}(\ell_g\ell_h) = \phi_{V,\rho}(\ell_g)\phi_{V,\rho}(\ell_h) = \theta(g,h)\phi_{V,\rho}(\ell_{gh}) = \theta(g,h)\phi_{V,\rho}(gh) = \rho_V(g)\rho_V(h) = \phi_{V,\rho}(\ell_g)\phi_{V,\rho}(\ell_h),$$

so $\phi_{V,\rho}$ is a group homomorphism. We claim that $F$ and $H$ are inverse functors. We have

$$F \circ H(\rho_V)(g) = F(\phi_{V,\rho})(g) = \rho_{V,\phi}(g) = \phi_{V,\rho}(\ell_g) = \rho_V(g)$$

and similarly

$$H \circ F(\phi_V)(\ell_g) = H(\rho_{V,\phi})(\ell_g) = \phi_{V,\rho}(\ell_g) = \rho_{V,\phi}(g) = \phi_V(\ell_g).$$

Clearly, the compositions are equal to their respective identity functors.

**Lemma 3.5.2.** Let $\theta, \beta \in \mathbb{Z}^2(G;\mathbb{U}(1))$ such that $[\theta] = [\beta] \in H^2(G;\mathbb{U}(1))$. Then $\text{Rep}^\theta(G) = \text{Rep}^\beta(G)$.

**Proof.** To show that the projective representation theory with these two multipliers is the same, we employ Lemma 3.5.1 and instead consider modules over $\mathbb{C}[G]$ and $\mathbb{C}[G]$ respectively. With this perspective, we need only show there is an isomorphism between these two algebras. Since $[\theta] = [\beta]$,
we know there exists some \( \lambda \in C^1(G;U(1)) \) such that \( \theta = \beta d^1 \lambda \). Define a map

\[
\phi : C^0[G] \to C^\beta[G]
\]

\[
\ell_g \mapsto \lambda(g) \ell_g
\]

for each \( g \in G \) and extend it linearly. Clearly, this is a vector space isomorphism. We next check that \( \phi \) respects the algebra multiplication. We have

\[
\phi(\ell_g \ell_h) = \phi(\theta(g,h) \ell_{gh})
\]

\[
= \theta(g,h)\lambda(gh) \ell_{gh}
\]

\[
= \beta(g,h) \lambda(g)\lambda(h) \lambda(gh) \ell_{gh}
\]

\[
= \lambda(g)\lambda(h) \ell_g \ell_h
\]

\[
= \phi(\ell_g) \phi(\ell_h).
\]

Therefore \( \phi \) is an algebra isomorphism, so in particular the module theory over \( C^0[G] \) is equivalent to the module theory over \( C^\beta[G] \). Thus by Lemma 3.5.1 we have \( \text{Rep}^\theta(G) = \text{Rep}^\beta(G) \). \( \square \)

We next concern ourselves with the center \( Z(C^\theta[G]) \).

**Definition** (\( \theta \)-class function). We say a function \( f : G \to \mathbb{C} \) is a \( \theta \)-class function if for all \( g, h \in G \) we have

\[
f(gh^{-1}) = \frac{\theta(gh^{-1})}{\theta(h,gh^{-1})\theta(h^{-1},g)} f(h)
\]

**Lemma 3.5.3.** The center of the twisted group algebra \( C^\theta[G] \) corresponds to \( \theta^{-1} \)-class functions on \( G \). In other words, we have

\[
Z\left(C^\theta[G]\right) = \left\{ \sum f(g)\ell_g \mid f \text{ is a } \theta^{-1} \text{-class function on } G \right\}
\]

**Proof.** Define

\[
\alpha : C^\theta[G] \to \text{Fun}(G,\mathbb{C})
\]

\[
\ell_g \mapsto \delta_g
\]

and extend it linearly (note \( \delta_g \) is the Dirac-delta function for \( g \in G \)). In particular, each \( v \in C^\theta[G] \) can be written in terms of the function it maps to. Let \( v \in C^\theta[G] \), let \( f = \alpha(v) \), and let \( a_g \in \mathbb{C} \) be
such that \( v = \sum_g a_g \ell_g \). Thus we have

\[
f(h) = \alpha(v)(h) = \alpha \left( \sum_g a_g \ell_g \right)(h) = \sum_g a_g \alpha(\ell_g)(h) = \sum_g a_g \delta_g(h) = a_h.
\]

In other words, we can write \( v = \sum_g f(g)\ell_g \) where \( f = \alpha(v) \). We know \( v \in Z(\mathbb{C}^θ(G)) \) if and only if \( v\ell_g = \ell_g v \) for all \( g \in G \). Let \( v \in Z(\mathbb{C}^θ(G)) \) and let \( f = \alpha(v) \) so \( v = \sum_g f(g)\ell_g \). Thus we have

\[
\sum_g \theta(h, g)f(g)\ell_{hg} = \sum_g f(g)\ell_h\ell_g
\]

\[
= \ell_h v
\]

\[
= v\ell_h
\]

\[
= \sum_g f(g)\ell_g\ell_h
\]

\[
= \sum_g \theta(g, h)f(g)\ell_{gh}.
\]

We equate the coefficient of \( \ell_k \) for some \( k \in G \), so for \( hg_1 = k \) and \( g_2h = k \) we have

\[
\theta(h, g_1)f(g_1) = \theta(g_2, h)f(g_2).
\]

Note \( g_2 = hg_1h^{-1} \). Thus we have

\[
f(hg_1h^{-1}) = \frac{\theta(h, g_1)}{\theta(hg_1h^{-1}, h)} f(g_1)
\]

for all \( g_1, h \in G \).

\[\square\]

Given a \( \theta \)-projective representation \( (V, \rho) \) of \( G \), we observe

\[
\chi_V(hg^{-1}) = \text{tr}(\rho(hg^{-1}))
\]

\[
= \text{tr} \left( \frac{1}{\theta(h, gh^{-1})\theta(g, h^{-1})} \rho(h)\rho(g)\rho(h^{-1}) \right)
\]

\[
= \frac{1}{\theta(h, gh^{-1})\theta(g, h^{-1})} \text{tr}(\rho(g)\rho(h^{-1})\rho(h))
\]

\[
= \frac{\theta(h^{-1}, h)}{\theta(h, gh^{-1})\theta(g, h^{-1})} \text{tr}(\rho(g))
\]

\[
= \frac{\theta(hg^{-1}, h)}{\theta(h, g)} \chi_V(g).
\]
Therefore characters of \( \theta \)-projective representations of \( G \) are in \( \mathbb{Z}(\mathbb{C}^{\theta^{-1}}[G]) \). Furthermore, by [4, Proposition 2.6] we know the characters of irreducible \( \theta \)-projective representations form an orthonormal basis for \( \theta \)-class functions on \( G \) with standard Hermitian inner product on \( \mathbb{Z}(\mathbb{C}^{\theta^{-1}}[G]) \) given by
\[
\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}.
\]
Lastly, we have an analogue of Maschke’s Lemma for projective representations.

**Theorem 3.5.4 ([4, Proposition 2.3]).** Every \( \theta \)-projective representation decomposes into a direct sum of irreducible \( \theta \)-projective representations.
CHAPTER 4
REAL REPRESENTATIONS

4.1

The group $GL^\vee(V)$

**Definition** ($GL^\vee(V)$). Let $V$ be a finite dimensional complex vector space. Define $GL^\vee(V)$ as the set of all linear isomorphisms of the form $V \to V$ and $V^\vee \to V$, where $V^\vee = \text{Hom}_\mathbb{C}(V, \mathbb{C})$ is the linear dual of $V$.

**Lemma 4.1.1.** Let $A_1, A_2 : V \to V$ and $B_1, B_2 : V^\vee \to V$ be linear isomorphisms. The binary operation $(- \cdot -)$ on $GL^\vee(V)$ defined by

$$A_1 \cdot A_2 := A_1 \circ A_2 \quad \quad A_1 \cdot B_1 := A_1 \circ B_1$$
$$B_1 \cdot A_1 := B_1 \circ A_1^{-\vee} \quad \quad B_1 \cdot B_2 := B_1 \circ B_2^{-\vee} \circ \text{ev}_V$$

gives $GL^\vee(V)$ the structure of a group. Moreover, the inverse of $B : V^\vee \to V$ under this operation is given by $\text{ev}_V^{-1} \circ B^\vee$.

**Proof.** First note that each multiplication defines an isomorphism $V \to V$ or $V^\vee \to V$. We must show that $(- \cdot -)$ is associative, has a multiplicative identity, and has inverses. For associativity, let $B_1, B_2, B_3 \in GL^\vee(V) \setminus GL(V)$ and note that $B_1 \cdot B_j \in GL(V)$ for all $i$ and $j$. We have

$$\begin{align*}
(B_1 \cdot B_2) \cdot B_3 &= (B_1 \circ B_2^{-\vee} \circ \text{ev}_V) \cdot B_3 \\
&= B_1 \circ B_2^{-\vee} \circ \text{ev}_V \circ B_3 \\
&= B_1 \circ (B_2 \circ B_3^{-\vee} \circ \text{ev}_V) \text{, by Lemma 2.1.2} \\
&= B_1 \cdot (B_2 \cdot B_3)
\end{align*}$$

The computations for the other seven combinations of mappings $A_i \in GL(V)$ and mappings $B_i \in$...
GL\(^\vee\)(V) \setminus GL(V) are similar. We next find that id\(_V\) : V \to V is the multiplicative identity under \((- \cdot -\)) since id\(_{-1}^V = id\(_V\) and for f \in V\(^\vee\) we have id\(_\wedge\)(f) = f \circ id\(_V\) = f, so id\(_{-1}^\wedge\)(f) = id\(_\wedge\). Next, we note by function composition in GL(V) the inverse of A \in GL(V) under \((- \cdot -\)) is A\(^{-1}\), the linear inverse of A. We claim that the inverse of B \in GL\(^\vee\)(V) \setminus GL(V) is given by ev\(_{-1}^V\) \circ B\(^\wedge\). We have

\[
B \cdot (ev\(_{-1}^V\) \circ B\(^\wedge\)) = B \circ (ev\(_{-1}^V\) \circ (B\(^\wedge\))\(^{-1}\)) \circ ev\(_V\)
= B \circ (B\(^\wedge\) \circ (ev\(_{-1}^V\))\(^{-1}\)) \circ ev\(_V\)
= B \circ ev\(_V\) \circ (B\(^\wedge\))\(^{-1}\) \circ ev\(_V\)
= B \circ (ev\(_{-1}^V\) \circ B\(^\wedge\) \circ (ev\(_{-1}^V\))\(^{-1}\))\(^{-1}\)
= B \circ (B\(^{-1}\)) \quad \text{by Lemma 2.1.3}
= id\(_V\),
\]

and similarly

\[
(ev\(_{-1}^V\) \circ B\(^\wedge\)) \cdot B = ev\(_{-1}^V\) \circ B\(^\wedge\) \circ (B\(^{-1}\)) \circ ev\(_V\)
= ev\(_{-1}^V\) \circ ev\(_V\)
= id\(_V\).
\]

Therefore ev\(_{-1}^V\) \circ B\(^\wedge\) is the left and right inverse of B under \((- \cdot -\)).

\[\square\]

**Lemma 4.1.2.** Let V be a finite dimensional complex vector space. The map

\[
\pi : GL\(^\vee\)(V) \to \mathbb{Z}_2
\]

\[
A \mapsto 1
\]

\[
B \mapsto -1
\]

where A \in GL(V) and B \in GL\(^\vee\)(V) \setminus GL(V) is a group homomorphism.

**Proof.** Writing out the compositions given by the multiplication \((- \cdot -\)), we find that A \cdot A' and B \cdot B' are both elements in GL(V) for A, A' \in GL(V) and B, B' \in GL\(^\vee\)(V) \setminus GL(V). Under \(\pi\), the same maps would be sent to \(1(1) = 1\) and \((-1)(-1) = 1\) respectively. We also see that B \cdot A and A' \cdot B' are both isomorphisms of the form V\(^\vee\) \to V, which match the group structure of \(\mathbb{Z}_2\) since \((-1)(1) = (1)(-1) = -1.\) \(\square\)
4.2

Real representations

We now define Real representations, following the work of Atiyah [2] and Karoubi [15].

**Definition** (Real representation). Let \( \pi : G \rightarrow \mathbb{Z}_2 \) be a surjective group homomorphism for some finite group \( G \). A Real representation of \( G \) is a group homomorphism \( \rho_V \) such that the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\rho_V} & \text{GL}^\vee(V) \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{Z}_2 & & \\
\end{array}
\]

Note that the data of a Real representation takes as input a finite \( \mathbb{Z}_2 \)-graded group \( G \). This differs slightly from classical representation theory which takes as input any finite group \( G \). We note that if \( V \) is a Real representation of \( G \), then \( V \) restricted to \( G_o = \ker(\pi) \) is a classical representation of \( G_o \). This follows from the commutativity of the above diagram, since \( \rho_V(G_o) \subset \text{GL}(V) \).

As an example, consider \( G = G_o \times \mathbb{Z}_2 \). In this case, we recover classical representation theory, which takes as input any group \( G_o \). The extra data attached to \( V \), a Real representation of \( G \), is an isomorphism \( \rho_V((e,-1)) : V^\vee \rightarrow V \). In particular, this isomorphism commutes with \( G_o \), so we can define a bilinear form

\[
B(v, u) := \rho_V((e,-1))^{-1}(v)(u)
\]

that is non-degenerate and \( G_o \)-invariant. Furthermore, we find that this bilinear form is symmetric (see section 4.7). The data of a representation of \( G_o \) with a non-degenerate, \( G_o \)-invariant, symmetric bilinear form is equivalent to a representation over \( \mathbb{R} \) if the representation \( V \) of \( G_o \) is irreducible (see section 5.7).

Another example we consider is the degenerate case \( G = \mathbb{Z}_2 \). In this case, we find that \( G_o \) acts trivially, which recovers linear algebra. The extra data included with a Real representation of \( \mathbb{Z}_2 \) is simply a non-degenerate bilinear form on \( V \). Therefore, Real representation theory can be viewed as an extension of the theory for bilinear forms on finite-dimensional vector spaces.

A key difference between Real representations and classical representations is that in the former there is a parity associated with each element \( g \in G \). This parity determines how a particular map \( \rho(g) \) interacts with other maps \( \rho(h) \) as defined by the multiplication on \( \text{GL}^\vee(\mathbb{C}) \) (see Lemma 4.1.1). In particular, let \( V \) be a Real representation of \( G \) and consider \( \xi \in G \setminus G_o \) and
$g \in G_\circ$. Then for a constant $\lambda \in U(1)$ we have

$$
\rho_V(\xi) \cdot (\lambda \rho_V(g)) = \rho_V(\xi) \circ \left(\lambda \rho_V(g)\right)^{-\vee}
$$

$$
= \rho_V(\xi) \circ \lambda^{-1}(\rho_V(g))^{-\vee}
$$

$$
= \lambda^{-1} \rho_V(\xi) \circ (\rho_V(g))^{-\vee}
$$

$$
= \lambda^{-1} \rho_V(\xi) \cdot \rho_V(g)
$$

Therefore the mappings $\rho_V(\xi)$ for each $\xi \in G \setminus G_\circ$ are $\mathbb{C}$ antilinear on $U(1)$.

4.3

Real projective representations

We wish to use the above ideas to generalize Real representations to Real projective representations. Consider a multiplier $\theta \in C^2(G; U(1))$ such that $\rho_V(g) \cdot \rho_V(h) = \theta(g,h) \rho_V(gh)$ for all $g, h \in G$. Thus for $g_1, g_2, g_3 \in G$ we have

$$
\theta(g_3, g_2) \theta(g_3 g_2, g_1) \rho(g_3 g_2 g_1) = \theta(g_3, g_2) \rho(g_3 g_2) \cdot \rho(g_1)
$$

$$
= \rho(g_3) \cdot \rho(g_2) \cdot \rho(g_1)
$$

$$
= \rho(g_3) \cdot \left(\theta(g_2, g_1) \rho(g_2 g_1)\right)
$$

$$
= \theta(g_2, g_1)^{\pi(g_3)} \rho(g_3) \cdot \rho(g_2 g_1)
$$

$$
= \theta(g_2, g_1)^{\pi(g_3)} \theta(g_3, g_2 g_1) \rho(g_3 g_2 g_1)
$$

Thus, in order to have associativity, we require $\theta$ to satisfy the equality

$$
\theta(g_3, g_2) \theta(g_3 g_2, g_1) = \theta(g_2, g_1)^{\pi(g_3)} \theta(g_3, g_2 g_1).
$$

In section 3.2 we saw that this equality is equivalent to the condition $\theta \in Z^{2+\pi}(G; U(1))$.

Definition (Real projective representation). For a finite group $G$ with a surjective group homomorphism $\pi : G \to \mathbb{Z}_2$ and a finite dimensional complex vector space $V$ let $\rho : G \to GL^\vee(V)$ be a
mapping such that the diagram below commutes

\[
\begin{array}{ccc}
G & \xrightarrow{\rho_V} & \text{GL}^\vee(V) \\
\downarrow{\pi} & & \downarrow{\tilde{\pi}} \\
\mathbb{Z}_2 & & \\
\end{array}
\]

and let \( \theta \in Z^{2+\pi}(G; U(1)) \) be such that for all \( g_1, g_2 \in G \) we have

\[
\rho(g_2) \cdot \rho(g_1) = \theta(g_2, g_1) \rho(g_2 g_1).
\]

Note that \((- \cdot -)\) is the multiplication on \( \text{GL}^\vee(\mathbb{C}) \). We call \((V, \rho)\) a Real projective representation of \( G \) with multiplier \( \theta \), or equivalently a Real \( \theta \)-projective representation of \( G \).

Note that the restriction of \( \theta \) to \( G_o \) given by \( \theta_o := \iota^* \theta \) (where \( \iota^* \) is the chain complex map induced by the inclusion \( \iota : G_o \to G \)) is a multiplier in \( Z^2(G_o; U(1)) \). Thus, if a Real representation \( V \) has a multiplier \( \theta \in Z^{2+\pi}(G_o; U(1)) \), then the restriction of this representation to \( G_o \) is a projective representation of \( G_o \) with multiplier \( \theta_o \in Z^2(G_o; U(1)) \).

One example of a Real projective representation is when \( G = G_o \times \mathbb{Z}_2 \) and the multiplier is \( \alpha \), the universal multiplier as defined in section 3.2 In this case, we recover the representation theory of \( G_o \) and have a non-degenerate, \( G_o \)-invariant, skew-symmetric bilinear form on \( V \). This is the same data as a representation of \( G \) over the quaternions \( \mathbb{H} \).

Lastly, we make a note regarding inverses in Real projective representation theory. By Lemma 4.1.1 we know the inverse of a map \( B : V^\vee \to V \) under the group multiplication on \( \text{GL}^\vee(V) \) is given by \( \text{ev}^{-1}_V \circ B^\vee \). This means that if \( V \) is a Real representation of \( G \) and \( \omega \in G \setminus G_o \) then we have \( \rho_V(\omega^{-1}) = \text{ev}^{-1}_V \circ \rho_V(\omega)^\vee \). In order to relate \( \rho_V(\omega) \) and \( \rho_V(\omega^{-1}) \) in a Real projective representation, we need to account for the projective multiplier. This is similar to our work in Lemma 3.3.2.

**Lemma 4.3.1.** Let \( V \) be a Real \( \theta \)-projective representation of \( G \). For all \( g \in G_o \) and \( \omega \in G \setminus G_o \) we have

\[
\rho_V(g^{-1}) = \theta(g^{-1}, g) \rho_V(g)^{-1}
\]

\[
\rho_V(\omega^{-1}) = \theta(\omega^{-1}, \omega) \text{ev}^{-1}_V \circ \rho_V(\omega)^\vee
\]
Proof. The first equality follows from lemma 3.3.2. The second equality follows from our definition of a Real $\theta$-projective representation: we have

$$\theta(\omega^{-1}, \omega) \text{id}_V = \rho_\theta(\omega^{-1}) \cdot \rho_\theta(\omega)$$

$$= \rho_\theta(\omega^{-1}) \circ \rho_\theta(\omega)^{-\vee} \circ \text{ev}_V.$$

\[\square\]

4.4

Further Real projective representations

We next discuss duals, tensor products, and direct sums of Real projective representations. Let $G$ be a finite group with chosen surjective group homomorphism $\pi : G \to \mathbb{Z}_2$. Let $\theta, \beta \in \mathbb{Z}_2^\pi(G; U(1))$. Let $V$ be a finite dimensional Real $\theta$-projective representation and let $U$ be a finite dimensional Real $\beta$-projective representation.

**Lemma 4.4.1.** For each $g \in G$ define

$$\rho_{V^\vee}(g) := \rho_V(g)^{-\vee}.$$  

Then $V^\vee$ is a Real $\theta^{-1}$-projective representation of $G$.

Proof. Note that by our definition, for each $g \in G_o$ we have $\rho_{V^\vee}(g) : V^\vee \to V^\vee$ and for each $\xi \in G \setminus G_o$ we have $\rho_{V^\vee}(\xi) : V^{\vee \vee} \to V^\vee$. By Lemma 3.3.1, we know that $\rho_{V^\vee}$ restricted to $G_o$ is a $\theta_o^{-1}$-projective representation. Fix some $\omega \in G \setminus G_o$. Note that for all $\xi \in G \setminus G_o$, there is a unique $g \in G_o$ such that $\xi = g\omega$. Therefore, we need only show that the definitions of $\rho_{V^\vee}$ on $G_o$ and on $\omega$ have group homomorphism structure. For a fixed $g \in G_o$ we have

$$\rho_{V^\vee}(g) \cdot \rho_{V^\vee}(\omega) = \rho_V(g)^{-\vee} \circ \rho_V(\omega)^{-\vee}$$

$$= \left(\rho_V(g) \circ \rho_V(\omega)\right)^{-\vee}$$

$$= \left(\theta(g, \omega)p_V(g\omega)\right)^{-\vee}$$

$$= \theta^{-1}(g, \omega)p_{V^\vee}(g\omega).$$
Similarly, we compute

\[ \rho_V(\omega) \cdot \rho_V(g) = \rho_V(\omega)^{-\vee} \circ (\rho_V(g)^{-\vee})^{-\vee} \]
\[ = (\rho_V(g) \circ \rho_V(\omega))^{-\vee} \]
\[ = (\theta(\omega, g) \rho_V(\omega g))^{-\vee} \]
\[ = \theta^{-1}(\omega, g) \rho_V(\omega g). \]

Lastly, recall from Lemma 2.1.2 that \( ev_V = ev^{-\vee}_V \). We have

\[ \rho_V(\omega) \cdot \rho_V(\omega) = \rho_V(\omega)^{-\vee} \circ (\rho_V(\omega)^{-\vee})^{-\vee} \circ ev_V \]
\[ = (\rho_V(\omega) \circ \rho_V(\omega)^{-\vee} \circ ev_V)^{-\vee} \]
\[ = (\theta(\omega, \omega) \rho_V(\omega^2))^{-\vee} \]
\[ = \theta^{-1}(\omega, \omega) \rho_V(\omega^2). \]

Lemma 4.4.2. The space \( V \otimes U \) is a Real \( \theta \beta \)-projective representation with

\[ \rho_{V \otimes U}(g) := \rho_V(g) \otimes \rho_U(g) \]

for all \( g \in G \).

Proof. Note that there is a canonical isomorphism \( (V \otimes U)^\vee \simeq V^\vee \otimes U^\vee \). Thus, for \( \xi \in G \setminus G_o \), the mapping \( \rho_{V \otimes U}(\xi) \) is well-defined. The proof then follows as in Lemma 3.3.3.

Lemma 4.4.3. The space \( V \oplus U \) has a Real projective representation structure with

\[ \rho_{V \oplus U} := \rho_V \oplus \rho_U \]

if and only if \( \theta = \beta \), in which case its twist is \( \beta \).

Proof. This follows from the canonical isomorphism \( (V \oplus U)^\vee \simeq V^\vee \oplus U^\vee \) so that \( \rho_{V \oplus U}(\xi) \) is well-defined for all \( \xi \in G \setminus G_o \). We also see as in Lemma 3.3.4 that if \( \theta \neq \beta \) then the multipliers cannot be factored out of the direct sum:

\[ \rho_{V \oplus U}(g)\rho_{V \oplus U}(h) = \theta(g, h) \rho_V(gh) \oplus \beta(g, h) \rho_U(gh). \]
Inherent to the definition of Real projective representations of $G$ is the $\mathbb{Z}_2$-grading given by $\pi$. This separates $G$ into two distinct parts: $G_o$ and $G \setminus G_o$, where each element in the latter is assigned an isomorphism $V^\vee \to V$. We fix some $\omega \in G \setminus G_o$ and consider the surjective map $\omega : G \to G$ given by $g \mapsto \omega^{-1} g \omega$. We then claim $\omega^* V$ is again a Real projective representation (see section 2.3 for more on pullbacks). Define a function $\mu_{\theta, \omega} : G \to U(1)$ by $\mu_{\theta, \omega}(g) = \frac{\theta(g, \omega)}{\theta(\omega^{-1} g \omega)}$ for all $g \in G$. See the appendix for more on $\mu_{\theta, \omega}$.

**Lemma 4.4.4.** The definition

$$\rho_{\omega^* V}(g) := \mu_{\theta, \omega}^{-1}(g) \rho_V(\omega^{-1} g \omega) = \theta(\omega, \omega^{-1}) \rho_V(\omega^{-1}) \cdot \rho_V(g) \cdot \rho_V(\omega),$$

for all $g \in G$ makes the pullback $\omega^* V$ a Real $\theta^{-1}$-projective representation of $G$.

**Proof.** Given $g, h \in G$ and using identity A.2 in the appendix, we have

$$\rho_{\omega^* V}(g) \cdot \rho_{\omega^* V}(h) = \mu_{\theta, \omega}^{-1}(g) \rho_V(\omega^{-1} g \omega) \cdot \mu_{\theta, \omega}^{-1}(h) \rho_V(\omega^{-1} h \omega)$$

$$= \mu_{\theta, \omega}^{-1}(g) \mu_{\theta, \omega}(h) \rho_V(\omega^{-1} g \omega) \cdot \rho_V(\omega^{-1} h \omega)$$

$$= \mu_{\theta, \omega}^{-1}(g) \theta(\omega^{-1} g \omega, \omega^{-1} h \omega) \rho_V(\omega^{-1} g \omega) \rho_V(\omega^{-1} h \omega)$$

$$= \mu_{\theta, \omega}^{-1}(g) \mu_{\theta, \omega}(h) \left( \theta^{-1}(g, h) \mu_{\theta, \omega}(g) \mu_{\theta, \omega}(h) \right) \rho_V(\omega^{-1} g \omega)$$

$$= \theta^{-1}(g, h) \rho_{\omega^* V}(gh).$$

In particular, we find that taking the pullback along $\omega$ commutes with taking duals since for all $g \in G$ we have

$$\rho_{(\omega^* V)^\vee}(g) = \left( \rho_{\omega^* V}(g) \right)^\vee$$

$$= \left( \mu_{\theta, \omega}^{-1}(g) \rho_V(\omega^{-1} g \omega) \right)^\vee$$

$$= \mu_{\theta, \omega}(g) \rho_V(\omega^{-1} g \omega)^\vee$$

$$= \mu_{\theta^{-1}, \omega}(g) \rho_V(\omega^{-1} g \omega)$$

$$= \rho_{(\omega^* V)^\vee}(g).$$

We will therefore simply write $\rho_{\omega^* V^\vee}$ to denote the pullback of $V^\vee$ by $\omega$, since commutativity of
this action leaves no ambiguity. Note that Lemma 4.4.1 and Lemma 4.4.4 together imply \( \mathcal{V} \) and \( \omega^*\mathcal{V}^\vee \) have the same projective multiplier, since dualizing will invert the multiplier and then pulling back by \( \omega \) will invert it again.

We now take a look at the restriction of a Real projective representation of \( G \) to its normal subgroup \( G_o \) and note that this is a projective representation of \( G_o \) with multiplier \( \theta_o \), in the sense of chapter 3. This will help us determine the nature of Real projective representations in the next section, but we state some preliminary results.

**Lemma 4.4.5.** Let \( \mathcal{V} \) be a Real \( \theta \)-projective representation of \( G \) with chosen \( \mathbb{Z}_2 \)-grading and fix some \( \omega \in G \setminus G_o \). Then \( \mathcal{V} \simeq \omega^*\mathcal{V}^\vee \) as \( G_o \) representations and moreover \( \rho_V(\omega) : \omega^*\mathcal{V}^\vee \to \mathcal{V} \) is a \( G_o \)-equivariant isomorphism.

**Proof.** Let \( g \in G_o \) and recall the definitions in Lemma 4.4.1 and Lemma 4.4.4. We compute

\[
\rho_V(\omega) \circ \rho_{\omega^*\mathcal{V}^\vee}(g) = \rho_V(\omega) \circ \left[ \theta^{-1}(\omega, \omega^{-1}) \rho_{\mathcal{V}^\vee}(\omega^{-1}) \cdot \rho_{\mathcal{V}^\vee}(g) \cdot \rho_{\mathcal{V}^\vee}(\omega) \right]
\]

\[
= \theta^{-1}(\omega, \omega^{-1}) \rho_V(\omega) \circ \rho_{\mathcal{V}^\vee}(\omega^{-1}) \circ \rho_{\mathcal{V}^\vee}(g)^{-\vee} \circ \rho_{\mathcal{V}^\vee}(\omega)^{-\vee} \circ \text{ev}_{\mathcal{V}^\vee}
\]

\[
= \theta^{-1}(\omega, \omega^{-1}) \rho_V(\omega) \circ \left( \theta^{-1}(\omega^{-1}, \omega) \text{ ev}_{\mathcal{V}^\vee}^{-1} \circ \rho_{\mathcal{V}^\vee}(\omega) \right) \circ \rho_{\mathcal{V}^\vee}(g)^{-\vee} \circ \rho_{\mathcal{V}^\vee}(\omega)^{-\vee} \circ \text{ev}_{\mathcal{V}^\vee}
\]

\[
= \theta^{-1}(\omega, \omega^{-1}) \theta^{-1}(\omega^{-1}, \omega) \rho_V(\omega) \circ \text{ ev}_{\mathcal{V}^\vee}^{-1} \circ \rho_{\mathcal{V}^\vee}(\omega) \circ \rho_{\mathcal{V}^\vee}(g)^{-\vee} \circ \rho_{\mathcal{V}^\vee}(\omega)^{-\vee} \circ \text{ev}_{\mathcal{V}^\vee}
\]

\[
= \rho_V(\omega) \circ \text{ ev}_{\mathcal{V}^\vee}^{-1} \circ \left( \rho_V(\omega)^{-1} \circ \rho_V(g) \cdot \rho_V(\omega) \right) \circ \text{ev}_{\mathcal{V}^\vee}
\]

\[
= \rho_V(\omega) \circ \rho_V(g) \circ \rho_V(\omega)
\]

as needed. \( \square \)

**Lemma 4.4.6.** For a chosen \( \mathbb{Z}_2 \)-grading of \( G \) fix some \( \omega \in G \setminus G_o \). Then the action of \( \omega \) given by \( \rho_V(\omega)^{-1} : \mathcal{V} \to \omega^*\mathcal{V}^\vee \) induces an involution on \( \text{Irr}^{\theta_o}(G_o) \), the set of \( \theta_o \)-projective irreducible representations of \( G_o \).

**Proof.** For each \( \mathcal{U}_i \in \text{Irr}^{\theta_o}(G_o) \), we know \( \mathcal{U}_i^\vee \in \text{Irr}(G_o) \) as well, and by Lemma 2.3.5 we know \( \omega^*\mathcal{U}_i \in \text{Irr}(G_o) \) too. Therefore \( \omega^*\mathcal{U}_i^\vee \in \text{Irr}(G_o) \). We write \( \mathcal{U}_{\omega^{-1}} := \omega^*\mathcal{U}_i^\vee \). Recall there is a canonical isomorphism \( \mathcal{U}_i^\vee \simeq \mathcal{U}_i \), and by Lemma 2.3.6 that \( g^*\mathcal{U}_i \simeq \mathcal{U}_i \) for all \( g \in G_o \). Note \( \omega^2 \in G_o \) because \( G_o \) is of index 2. Thus we have

\[
\mathcal{U}_{\omega^2(\omega^{-1})} = \omega^*\left(\omega^*\mathcal{U}_i^\vee\right)^\vee \simeq (\omega^2)^*\mathcal{U}_i \simeq \mathcal{U}_i.
\]
Schur’s Lemma for Real projective representations

As with classical representation theory, we wish to distinguish representations up to isomorphism, but this requires a notion of Schur’s Lemma in the Real projective case. In order to talk about this, we need the concept of a map between Real projective representations and we need the concept of irreducibility.

**Definition** (morphism of Real projective representations). Let $V$ and $U$ be Real $\theta$-projective representations of $G$. Let $\psi : V \to U$ be a $\mathbb{C}$-linear map. We say $\psi : V \to U$ is a morphism of Real projective representations if the following diagram commutes for all $g \in G_o$ and $\xi \in G \setminus G_o$:

\[
\begin{array}{ccc}
V & \xrightarrow{\psi} & U \\
\downarrow{\rho_V(g)} & & \downarrow{\rho_U(g)} \\
V & \xrightarrow{\rho_V(\xi)} & U \\
\uparrow{\psi^\vee} & & \uparrow{\rho_U(\xi)} \\
V^\vee & & U^\vee
\end{array}
\]

Recall that $V$ and $U$ restricted to $G_o$ are $\theta_o$-projective representations of $G_o$. Thus restricting everything above to $G_o$ makes $\psi : V \to U$ a map of projective representations of $G_o$. We also note that this definition implies $\psi$ is surjective. This is because the commuting diagram is equivalent to $\rho_U(\xi) = \psi \circ \rho_V(\xi) \circ \psi^\vee$, and since $\rho_U(\xi)$ is an isomorphism, we must have that $\psi$ is surjective. (We also see that the definition implies $\psi^\vee$ is injective, but this is equivalent to $\psi$ being surjective.) Note that we defined morphisms of Real projective representations in such a way that implies surjectivity. We could have instead defined it in such a way that implies injectivity, but regardless of this choice we find that Real projective representations do not form an abelian category. Lastly, note that, similar to Lemma 3.4.1, morphisms of Real projective representations only exist when $V$ and $U$ have the same projective multiplier.

**Definition** (irreducible Real projective representation). Let $V$ be a Real $\theta$-projective representation of $G$. We say $V$ is irreducible if every morphism of Real projective representations $\psi : V \to U$ for $U \neq 0$ is an isomorphism.
Theorem 4.5.1. Let $V$ be an irreducible Real $\theta$-projective representation of $G$. Fix some $\omega \in G \setminus G_\theta$. Let $\{U_i\}_{i \in \text{Irr}^{G_\theta}(G_\theta)}$ be the collection of distinct irreducible $\theta_\omega$-projective representations of $G_\theta$ up to isomorphism. Then for some index $i$ we have either $V \simeq U_i$ or $V \simeq U_i \oplus \omega^*U_i^\vee$ as $G_\theta$ representations.

Proof. By Maschke’s Lemma for $\theta_\omega$-projective representations of $G_\theta$ (Theorem 3.5.4), we know $V \simeq \bigoplus_{i \in \text{Irr}^{G_\theta}(G_\theta)} U_i^{\oplus m_i}$ with multiplicities $m_i$. If this direct sum happens to only contain one irreducible $U_i$, then we are done. We then show that there cannot be more than two irreducibles in the direct sum. Suppose there are more than two summands in $\bigoplus_{i \in \text{Irr}^{G_\theta}(G_\theta)} U_i^{\oplus m_i}$. Then by Lemma 4.4.6 we have the $G_\theta$-equivariant map

$$\rho_V(\omega)^{-1} : V \simeq \left( \bigoplus_{i \in \text{Irr}^{G_\theta}(G_\theta)} U_i^{\oplus m_i} \right) \rightarrow \left( \bigoplus_{i \in \text{Irr}^{G_\theta}(G_\theta)} (\omega^*U_i^\vee)^{\oplus m_i} \right) \simeq \omega^*V^\vee.$$ 

Lemma 4.4.6 tells us that this mapping induces an involution on the indexing set $\text{Irr}^{G_\theta}(G_\theta)$. Thus if we apply the map again, then each summand returns to its original position. Therefore, an orbit of $\omega$ contains at most two irreducibles within the summand. Given the assumption that there are more than two summands in the direct sum above, we know there must be at least two separate orbits under the action of $\omega$. Let $\{U_1, U_2\}$ be an orbit under $\omega$, so in particular we have $U_1 \simeq \omega^*U_2^\vee \simeq U_{\omega \cdot 1}$.

We have three cases.

Case 1. $U_i \not\simeq U_{\omega \cdot 1}$. Therefore, both irreducibles are contained in $\bigoplus_{i \in \text{Irr}^{G_\theta}(G_\theta)} U_i^{\oplus m_i}$, so consider the projection map

$$p_{U_i \oplus U_{\omega \cdot 1}} : V \rightarrow U_i \oplus U_{\omega \cdot 1}.$$ 

Clearly, this is a map of projective $G_\theta$-representations since $U_i \oplus U_{\omega \cdot 1}$ is a subrepresentation of $V$.

We claim this is also a map of Real projective $G$-representations. Using Lemma 4.4.6, note that

$$\omega^*\left(U_i \oplus U_{\omega \cdot 1}\right)^\vee \simeq \omega^*U_{\omega \cdot 1}^\vee \oplus \omega^*U_i^\vee \simeq U_{\omega^-(\omega \cdot 1)} \oplus U_{\omega \cdot 1} \simeq U_i \oplus U_{\omega \cdot 1}.$$ 

Therefore $\rho_V(\omega)\left(U_i \oplus U_{\omega \cdot 1}\right) \simeq U_i \oplus U_{\omega \cdot 1}$. We also note that by Schur’s Lemma for $G_\theta$ representations, the map $\rho_V(\omega)$ splits into blocks, so there is a block $\rho_V(\omega)_i : U_i \oplus U_{\omega \cdot 1} \rightarrow U_i \oplus U_{\omega \cdot 1}$. Therefore we have

$$\rho_{U_i \oplus U_{\omega \cdot 1}}(\omega) = \rho_V(\omega)_i = p_{U_i \oplus U_{\omega \cdot 1}} \circ \rho_V(\omega) \circ p_{U_i \oplus U_{\omega \cdot 1}}^\vee.$$ 

Thus, the following diagram commutes

\[
\begin{array}{c}
V \\ \downarrow \rho_V(g) \\
\rho_V(\omega) \downarrow \rho_V(\omega) \\
V^\vee \\
\end{array}
\quad \begin{array}{c}
\stackrel{p_{U_i \oplus U_{w_i}}}{\longrightarrow} \\
\downarrow \rho_U(g) \\
\downarrow \rho_U(\omega) \\
\stackrel{p_{U_i \oplus U_{w_i}}}{\leftarrow} \end{array}
\quad \begin{array}{c}
U \\
\rho_U(g) \downarrow \\
\rho_U(\omega) \downarrow \\
U^\vee \\
\end{array}
\]

Therefore \( p_{U_i \oplus U_{w_i}} \) is a map of Real \( \theta \)-projective representations of \( G \), but clearly it is not an isomorphism since there are more than two summands in the direct sum. This contradicts the assumption that \( V \) is irreducible.

Case 2. \( U_i \simeq U_{w_i} \) and \( m_i \geq 2 \). Here we consider the projection map with only two factors:

\[
p_{U_i \oplus U_i} : V \to U_i \oplus U_i.
\]

Then follow a similar argument to Case 1.

Case 3. \( U_i \simeq U_{w_i} \) and \( m_i = 1 \). Here we consider the projection map

\[
p_{U_i} : V \to U_i
\]

then follow a similar argument to Case 1.

Therefore, if there are more than two factors in the direct sum \( \bigoplus_{i \in \text{Irr}^\theta(G_\alpha)} U_i^{\oplus m_i} \simeq V \), then the irreducibility of \( V \) as a Real representation of \( G \) is contradicted. Thus there are at most two factors, and in each case we have either \( V \simeq U_i \) or \( V \simeq U_i \oplus U_{w_i} \).

\[\square\]

**Theorem 4.5.2 (Schur’s Lemma).** Let \( V \) and \( W \) be irreducible Real \( \theta \)-projective representations of \( G \). Then either there exists some \( G \)-equivariant isomorphism \( \phi : V \to W \) or there are no morphisms of Real projective representations between \( V \) and \( W \).

**Proof.** Let \( \{U_i\}_{i \in \text{Irr}(G_\alpha)} \) be the set of irreducible \( \theta_\alpha \)-projective representations of \( G_\alpha \). By Theorem 4.5.1 we know \( V \simeq U_i \) or \( V \simeq U_i \oplus \omega^*U_i^\vee \) for some irreducible \( U_i \) and similarly for \( W \) with some irreducible \( U_j \).

Case 1. \( V \simeq U_i \) and \( W \simeq U_j \). Then either \( U_i \simeq U_j \) or \( U_i \not\simeq U_j \) and by Schur’s Lemma for \( G_\alpha \) representations we know the latter case implies the only morphism between them is the zero map. Note that the zero map is not surjective, but morphisms of Real representations must be surjective.
\textit{Case 2.} $V \simeq U_i$ and $W \simeq U_j \oplus \omega^* U_j^\vee$. By assumption, we know $U_i \simeq U_{\omega \cdot i}$ but $U_j \not\simeq U_{\omega \cdot i}$. Clearly, this means $U_i \not\simeq U_j$ and $U_i \not\simeq U_{\omega \cdot j}$. Thus, by Schur’s Lemma for $G_\circ$ representations we know the only map of $G_\circ$ representations must be the zero map, which cannot be a map of Real representations of $G$ since it is not surjective.

\textit{Case 3.} $V \simeq U_i \oplus \omega^* U_i^\vee$ and $W \simeq U_j$. The argument here is similar to that of Case 2.

\textit{Case 4.} $V \simeq U_i \oplus \omega^* U_i^\vee$ and $W \simeq U_j \oplus \omega^* U_j^\vee$. If $U_i \simeq U_j$, then clearly $V \simeq W$. If $U_i \simeq \omega^* U_j^\vee$, then by identifying $\omega^* U_i$ with $U_j$, we have an isomorphism of $V$ and $W$. Lastly, if neither of those two are true, then we are left with the zero map, which cannot be a morphism of Real representations.

4.6

Characters of Real projective representations

Let $\pi : G \to \mathbb{Z}_2$ be a surjective group homomorphism for a finite group $G$ and denote $G_\circ = \ker(\pi)$. Let $\theta \in Z^2(\pi(G; U(1)))$ and denote by $\theta_\circ \in Z^2(G_\circ; U(1))$ the restriction of $\theta$ to $G_\circ$ (see section 3.2 for more on $\theta_\circ$). Let $V$ be a finite dimensional Real $\theta$-projective representation of $G$.

\textbf{Lemma 4.6.1.} \textit{In the above setting and for $g \in G_\circ$ and $h \in G \setminus G_\circ$ we have}

$$\text{tr}_V\left(\rho_V(h) \cdot \rho_V(g^{-1}) \cdot \rho_V(h^{-1})\right) = \theta^{-1}(h^{-1}, h) \theta^{-1}(g^{-1}, g) \text{ tr}_V(\rho_V(g)).$$
Proof. We compute

\[
\text{tr}_V\left(\rho_V(h) \cdot \rho_V(g^{-1}) \cdot \rho_V(h^{-1})\right)
\]

\[= \text{tr}_V\left(\rho_V(h) \circ \rho_V(g^{-1}) \circ \rho_V(h^{-1}) \circ \text{ev}_V\right)\]

\[= \text{tr}_V\left(\rho_V(g^{-1}) \circ \rho_V(h^{-1}) \circ \text{ev}_V \circ \rho_V(h)\right)\]

\[= \text{tr}_V\left(\rho_V(g^{-1}) \circ (\theta(h^{-1}, h) \circ \rho_V(h)^{-1}) \circ \text{ev}_V \circ \rho_V(h)\right)\quad \text{Lemma 4.3.1}\]

\[= \theta^{-1}(h^{-1}, h) \text{tr}_V\left(\rho_V(g^{-1}) \circ \text{ev} \circ \rho_V(h)^{-1} \circ \rho_V(h)\right)\]

\[= \theta^{-1}(h^{-1}, h) \text{tr}_V\left((\theta^{-1} \circ \rho_V(h)^{-1} \circ \rho_V(h)) \circ \rho_V(h)\right)\quad \text{Lemma 2.1.2}\]

\[= \theta^{-1}(h^{-1}, h) \text{tr}_V\left(\rho_V(g^{-1})\right)\]

\[= \theta^{-1}(h^{-1}, h) \theta^{-1}(g^{-1}, g) \text{tr}_V\left(\rho_V(g)\right)\quad \text{Lemma 4.3.1}\]

\[\square\]

Definition (Real θ-projective class function). We say a θ_o-class function \(f : G_o \to \mathbb{C}\) is a Real θ-class function if for all \(g \in G_o\) and \(h \in G\) we have

\[f(hg^{\pi(h)}h^{-1}) = \theta(g^{-1}, g) \frac{\theta(hg^{\pi(h)}h^{-1}, h)}{\theta(h, g^{\pi(h)})} f(g)\]

See [17] or [23] for more on this definition.

Lemma 4.6.2. Let \(V\) be a Real θ-projective representation of \(G\). Then the character \(\chi_V\) of the θ_o-projective representation restricted to \(G_o\) is a Real θ-class function.

Proof. Let \(g \in G_o\) and \(h \in G\). If \(h \in G_o\) then we can use that \(\chi_V\) is a θ_o-class function which
\[ \chi_V(hg^{-1}) = \frac{\theta(hg^{-1}h)}{\theta(h,g)} \chi_V(g). \]

Now suppose \( h \in G \setminus G_\theta \). Thus we have

\[ \chi_V(hg^{\pi(h)}h^{-1}) = \text{tr}_V \left( \rho_V(hg^{-1}h^{-1}) \right) \]

\[ = \theta^{-1}(h,g^{-1}h^{-1}) \text{tr}_V \left( \rho_V(h) \cdot \rho_V(g^{-1}h^{-1}) \right) \]

\[ = \theta^{-1}(h,g^{-1}h^{-1}) \text{tr}_V \left( \rho_V(h) \cdot \theta^{-1}(g^{-1}, h^{-1}) \rho_V(g^{-1}) \cdot \rho_V(h^{-1}) \right) \]

\[ = \theta^{-1}(h,g^{-1}h^{-1}) \theta(g^{-1}, h^{-1}) \theta^{-1}(h^{-1}, h) \theta^{-1}(g^{-1}, g) \text{tr}_V \left( \rho_V(g) \right) \]

\[ = \theta^{-1}(g^{-1}, g) \frac{\theta(h^{-1}, h)^{\pi(h)}}{\theta(h, g^{\pi(h)}h^{-1}) \theta(g^{\pi(h)}, h^{-1})^{\pi(h)} \chi_V(g)} \]

\[ = \theta^{-1}(g^{-1}, g) \frac{\theta(h^{-1}, h)^{\pi(h)}}{\theta(h, g^{\pi(h)}h^{-1})} \frac{\theta(h^{\pi(h)}, h)}{\chi_V(g)} \]

Recall the non-degenerate Hermitian form on \( \theta_\alpha \)-class functions of \( G_\alpha \) given by:

\[ \langle \phi, \psi \rangle = \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \phi(g) \overline{\psi(g)}. \]

**Lemma 4.6.3.** The non-degenerate Hermitian form \( \langle \cdot, \cdot \rangle \) on \( \theta_\alpha \)-class functions of \( G_\alpha \) restricts to a non-degenerate Hermitian form on Real \( \theta \)-class functions of \( G \).

**Proof.** Suppose \( \phi \) is a Real \( \theta \)-class function of \( G \) such that \( \langle \phi, \psi \rangle = 0 \) for all Real \( \theta \)-class functions \( \psi \). Then in particular \( \langle \phi, \phi \rangle = 0 \), but by non-degeneracy of the Hermitian form on \( \theta_\alpha \)-class functions on \( G_\alpha \), we know that this implies \( \phi = 0 \). Therefore the Hermitian form is non-degenerate on Real \( \theta \)-class functions as well. \( \square \)

**Lemma 4.6.4.** The characters of irreducible Real \( \theta \)-projective representations form an orthogonal basis for Real \( \theta \)-class functions.

**Proof.** Let \( V \) and \( W \) be irreducible Real \( \theta \)-projective representations of \( G \). By Theorem 4.5.1 we know that \( V|_{G_\alpha} \) and \( W|_{G_\alpha} \) are each either an irreducible \( \theta_\alpha \)-projective representation of \( G_\alpha \) or a direct sum of exactly two \( \theta_\alpha \)-projective representations of \( G_\alpha \). Also by Theorem 4.5.1 we know that if \( V|_{G_\alpha} \) is a single irreducible and \( W|_{G_\alpha} \) is a direct sum of two irreducibles, then \( V|_{G_\alpha} \) is not isomorphic to either summand of \( W|_{G_\alpha} \). Thus we have

\[ \langle X_V, X_W \rangle = \langle X_{U_1}, X_{U_1 \oplus U_2} \rangle = \langle X_{U_1}, X_{U_1} \rangle + \langle X_{U_1}, X_{U_2} \rangle = 0. \]
A similar result follows when $V|_{G_o}$ is a direct sum of two irreducibles and $W|_{G_o}$ is a single irreducible. If $V \not\simeq W$ as Real representations and either both are a single irreducible of $G_o$ or both are a direct sum of two irreducibles of $G_o$, then we can similarly split the characters over the direct sum and the Hermitian inner product over addition to give a result of 0, using that irreducible characters of $G_o$ are orthogonal. If $V \simeq W$ and $V|_{G_o}$ is a single irreducible of $G_o$, then we know $\langle \chi_V, \chi_W \rangle = 1$ since irreducible characters of $G_o$ are orthonormal. If $V \simeq W$ and $V|_{G_o}$ is a direct sum of two irreducibles of $G_o$ then we have

$$\langle \chi_V, \chi_W \rangle = \left\langle \chi_{U_i \oplus U_{\omega \cdot i}}, \chi_{U_j \oplus U_{\omega \cdot j}} \right\rangle = \left\langle \chi_{U_i}, \chi_{U_j} \right\rangle + \left\langle \chi_{U_{\omega \cdot i}}, \chi_{U_{\omega \cdot j}} \right\rangle = 2.$$ 

\[\square\]

4.7

Maschke’s Lemma for Real projective representations

4.7.1

An equivalent definition of Real projective representations

Lemma 4.7.1. Let $V$ be a Real $\theta$-projective representation of $G$ and fix $\omega \in G \setminus G_o$. Then

$$\langle -, - \rangle : V \times V \to \mathbb{C}$$

$$(v, u) \mapsto \rho_V(\omega)^{-1}(v)(u)$$

defines a non-degenerate bilinear form that satisfies the twisted $G_o$-invariance condition

$$\langle \rho_V(g)v, \rho_{\omega \cdot V}(g)u \rangle = \langle v, u \rangle$$

and $\omega^2$-symmetry

$$\langle v, u \rangle = \theta(\omega, \omega) \langle \rho_V(\omega^2)u, v \rangle.$$ 

Proof. Since $\rho_V(\omega)^{-1} : V \to V^\vee$ is a linear map into the linear dual, we know this defines a bilinear form. Non-degeneracy of the bilinear form follows from $\rho_V(\omega)^{-1}$ being an isomorphism. We next show the twisted $G_o$-invariance condition holds. Recall from Lemma 4.4.5 that $\rho_V(\omega) : \omega^* V^\vee \to V$
is a $G_o$-equivariant map. Thus $\rho_V(\omega)^{-1}$ is $G_o$-equivariant and we compute

\[
\langle \rho_V(g)v, u \rangle = \rho_V(\omega)^{-1}(\rho_V(g)v)(u) \\
= \rho_{\omega \cdot V}(g) \circ \rho_V(\omega)^{-1}(v)(u) \\
= \rho_{\omega \cdot V}(g)^{-\vee} \circ \rho_V(\omega)^{-1}(v)(u) \\
= \rho_V(\omega)^{-1}(v)(\rho_{\omega \cdot V}(g)^{-1}u) \\
= \langle v, \rho_{\omega \cdot V}(g)^{-1}u \rangle.
\]

Therefore

\[
\langle \rho_V(g)v, \rho_{\omega \cdot V}(g)u \rangle = \langle v, \rho_{\omega \cdot V}(g)^{-1}\rho_{\omega \cdot V}(g)u \rangle = \langle v, u \rangle.
\]

Thus the twisted $G_o$-invariance condition holds. We now show that $\omega^2$-symmetry holds. Recall from Lemma 4.3.1 that $\rho_V(\omega) = \theta(\omega, \omega^{-1}) \, ev_{\omega^2}^{-1} \circ \rho_V(\omega^{-1})^{\vee}$. Thus we have

\[
\langle v, u \rangle = \rho_V(\omega)^{-1}(v)(u) \\
= \rho_V(\omega^2)^{-\vee} \circ \rho_V(\omega)^{-1}(v)(\rho_V(\omega^2)u) \\
= \rho_V(\omega^2)^{-\vee} \circ \left[ \theta(\omega, \omega^{-1}) \, ev_{\omega^{-1}}^{-1} \circ \rho_V(\omega^{-1})^{\vee} \right]^{-1}(v)(\rho_V(\omega^2)u) \\
= \theta^{-1}(\omega, \omega^{-1}) \left[ \rho_V(\omega^2) \circ \rho_V(\omega^{-1}) \right]^{-\vee} \circ ev_{\omega^{-1}}(v)(\rho_V(\omega^2)u) \\
= \theta^{-1}(\omega, \omega^{-1}) \, ev_{\omega^{-1}}(v)(\rho_V(\omega^2)u) \\
= \theta^{-1}(\omega, \omega^{-1}) \, \theta^{-1}(\omega^2, \omega^{-1}) \, \rho_V(\omega)(\rho_V(\omega^2)u)(v) \\
= \theta(\omega, \omega) \, \langle \rho_V(\omega^2)u, v \rangle.
\]

Note that the last equality uses the cocyle condition on $Z^{2+\pi}(G; U(1))$. \qed

We note that applying $\omega^2$-symmetry twice returns the bilinear form to itself since we have

\[
\theta(\omega, \omega) \, \langle \rho_V(\omega^2)u, v \rangle = \theta^2(\omega, \omega) \, \langle \rho_V(\omega^2)v, \rho_V(\omega^2)u \rangle \\
= \theta^2(\omega, \omega) \mu_{\omega, \omega}(\omega^2) \, \langle \rho_V(\omega^2)v, \rho_{\omega \cdot V}(\omega^2)u \rangle \quad \text{Lemma 4.4.4} \\
= \langle v, u \rangle \quad \text{identity A.3}
\]

**Lemma 4.7.2.** Let $V$ be a $\theta$-projective representation of $G_o$ and suppose $\langle -, - \rangle : V \times V \rightarrow \mathbb{C}$ is a non-degenerate bilinear form that satisfies the twisted $G_o$-invariant condition and $\omega^2$-symmetry.
Define $\rho_V(\omega) : V^\vee \to V$ by the equation

$$\rho_V(\omega)^{-1}(v) = \langle \nu, - \rangle.$$ 

Then the definition $\rho_V(g\omega) := \theta^{-1}(g, \omega) \rho_V(g) \circ \rho_V(\omega)$ for all $g \in G_o$ defines a Real $\theta$-projective representation of $G$ on $V$.

**Proof.** Let $g \in G_o$. We compute

\[
\rho_V(g\omega)^{-1}(v)(u) = \left[\theta^{-1}(g\omega\omega^{-1}, \omega) \rho_V(g\omega\omega^{-1} \circ \rho_V(\omega)\right]^{-1}(v)(u)
\]

\[
= \theta(g\omega\omega^{-1}, \omega) \langle \rho_V(g\omega\omega^{-1})^{-1}v, u \rangle
\]

\[
= \theta(g\omega\omega^{-1}, \omega) \langle v, \rho_{\omega \cdot V}(g\omega\omega^{-1})u \rangle
\]

\[
= \theta(g\omega\omega^{-1}, \omega) \mu_{\theta, \omega}(\omega) \rho_V(g)^\vee \circ \rho_V(\omega)^{-1}(v)(u) \quad \text{Lemma A.4.4}
\]

\[
= \theta(\omega, g) \left[\rho_V(\omega) \circ \rho_V(g)^\vee\right]^{-1}(v)(u) \quad \text{identity A.6}
\]

\[
= \left[\theta^{-1}(\omega, g) \rho_V(\omega) \cdot \rho_V(g)\right]^{-1}(v)(u).
\]

Therefore for all $g \in G_o$ we have

$$\rho_V(\omega) \cdot \rho_V(g) = \theta(\omega, g) \rho_V(g).$$

Next, we consider $\phi \in V^\vee$ and $v \in V$ and compute

\[
\rho_V(\omega)^\vee \circ \rho_V(\omega)^{-1}(v)(\phi) = \langle \nu, \rho_V(\omega)\phi \rangle
\]

\[
= \theta(\omega, \omega) \langle \rho_V(\omega^2) \circ \rho_V(\omega)\phi, \nu \rangle
\]

\[
= \theta(\omega, \omega) \langle \rho_V(\omega)\phi, \rho_{\omega \cdot V}(\omega^2)^{-1}v \rangle
\]

\[
= \theta(\omega, \omega) \rho_V(\omega)^{-1} \circ \rho_V(\omega)\phi(\rho_{\omega \cdot V}(\omega^2)^{-1}v)
\]

\[
= \theta(\omega, \omega) \phi \left([\mu_{\theta, \omega}(\omega^2) \rho_V(\omega^2)]^{-1}v\right)
\]

\[
= \theta(\omega, \omega) \mu_{\theta, \omega}(\omega^2) e_{V^\vee}(\rho_V(\omega)^{-1}v)(\phi).
\]
This gives an equality of mappings that we use in the next computation. We have

\[
[\rho_V(\omega) \cdot \rho_V(\omega)]^{-1}(v) = ev_V^{-1} \circ \rho_V(\omega) \circ \rho_V(\omega)^{-1}(v) \\
= \theta(\omega, \omega) \mu(\omega^2) \ ev_V^{-1}(ev_V(\rho_V(\omega^2)^{-1}(v))) \\
= \theta^{-1}(\omega, \omega) \rho_V(\omega^2)^{-1}(v) \quad \text{identity A.6} \\
= [\theta(\omega, \omega) \rho_V(\omega^2)]^{-1}(v).
\]

Therefore we have

\[
\rho_V(\omega) \cdot \rho_V(\omega) = \theta(\omega, \omega) \rho_V(\omega^2).
\]

\[\square\]

**Theorem 4.7.3.** A Real \( \theta \)-projective representation of \( G \) is equivalent to the data of a \( \theta_o \)-projective representation of \( G_o \) with a non-degenerate bilinear form that satisfies the twisted \( G_o \)-invariance condition and \( \omega^2 \)-symmetry.

**Proof.** This follows from Lemma 4.7.1 and Lemma 4.7.2 \[\square\]

### 4.7.2 Perpendicular subspaces

**Lemma 4.7.4.** Let \( V \) be a Real \( \theta \)-projective representation and let \( U \subset V \) be a \( G_o \)-invariant subspace such that the following diagram commutes for all \( \omega \in G \setminus G_o \)

\[
\begin{array}{ccc}
U & \xrightarrow{\rho_{U}^{-1}(\omega)} & V \\
\downarrow{\rho_V^{-1}(\omega)} & & \downarrow{\rho_V^{-1}(\omega)} \\
U^\vee & \xleftarrow{\rho_V^{-1}(\omega^2)} & V^\vee
\end{array}
\]

Let \( \langle -, - \rangle_V \) be the \( \omega^2 \)-symmetric, twisted \( G_o \)-invariant, non-degenerate bilinear form as in Lemma 4.7.1. Then the restriction of \( \langle -, - \rangle_V \) to \( U \) is non-degenerate.

**Proof.** This follows directly from the commutative diagram above and because \( \rho_V^{-1}(\omega) \) is an isomorphism.

\[\square\]

**Lemma 4.7.5.** Let \( V \) be a Real \( \theta \)-projective representation with the non-degenerate, \( \omega^2 \)-symmetric, twisted \( G_o \)-invariant bilinear form \( \langle -, - \rangle_V \) as defined in Lemma 4.7.1. Let \( U \subset V \) be a \( G_o \)-invariant subspace that also satisfies the condition of Lemma 4.7.4. Then the perpendicular subspace defined
by
\[U^\perp = \{v \in V \mid \langle v, u \rangle_V = 0 \text{ for all } u \in U\}\]
is a Real $\theta$-projective subrepresentation of $V$.

**Proof.** Let $v \in U^\perp$. Then for all $g \in G_\sigma$ and $u \in U$ we have
\[
\langle \rho_V(g)v, u \rangle = \langle v, \rho_{\omega^{-1}}(g)^{-1}u \rangle \nonumber
= \mu_{\theta,\omega}(g) \langle v, \rho_V(\omega^{-1}g\omega)^{-1}u \rangle \nonumber
= 0.
\]
For the last equality we use that $U$ is $G_\sigma$-invariant and hence $\rho_V(\omega^{-1}g\omega)u \in U$. Therefore $\rho_V(g)v \in U^\perp$, which means $U^\perp$ is $G_\sigma$-invariant. By Lemma 4.7.4 we know $\langle -, - \rangle_V$ restricts to a non-degenerate form on $U^\perp$. Furthermore, $\langle -, - \rangle_{U^\perp}$ inherits twisted $G_\sigma$-invariance and $\omega^2$-symmetry from $\langle -, - \rangle_V$. Thus by Theorem 4.7.3, we know $U^\perp$ is a Real $\theta$-projective representation of $G$. □

We can now prove the main result of this section, which allows us to split Real projective representations into irreducible parts.

**Theorem 4.7.6** (Maschke's Lemma for Real projective representations). Let $V$ be a Real $\theta$-projective representation of $G$. Then $V$ is a direct sum of irreducible Real $\theta$-projective representations of $G$.

**Proof.** We have shown in Lemma 4.7.5 that if $U \subset V$ is a Real $\theta$-projective subrepresentation, then $V = U \oplus U^\perp$ as Real $\theta$-projective representations of $G$. We then apply an inductive process on each direct summand until we are left with only irreducible parts. □
CHAPTER 5
EXTENDING THE FROBENIUS SCHUR INDICATOR

Let $\pi : G \to \mathbb{Z}_2$ be a surjective group homomorphism for a finite group $G$ and let $G_o = \ker(\pi)$. Let $\theta \in Z^{2+\pi}(G, U(1))$ be a normalized cocycle. Let $\theta_o \in Z^2(G_o, U(1))$ be the restriction of $\theta$ to the subgroup $G_o$. Let $(V, \rho_V)$ be an irreducible $\theta_o$-projective representation of $G_o$.

**Theorem 5.0.1.** In the above setting, we have

$$\nu_2 := \sum_{\xi \in G \setminus G_o} \theta^{-1}(\xi, \xi) \ell_\xi^2 \in Z(C^{\theta_o^{-1}}[G_o])$$

and

$$\langle \chi_V, \nu_2 \rangle = \begin{cases} 
0 & \text{V cannot be realized as a Real representation of } G \\
1 & \text{V admits a Real } \theta\text{-projective representation of } G \\
-1 & \text{V admits a Real } \alpha\theta\text{-projective representation of } G
\end{cases}$$

where $\alpha \in Z^{2+\pi}(G, U(1))$ is the universal multiplier defined by

$$\alpha(g, h) = \begin{cases} 
1 & \text{either } g \text{ or } h \text{ is an element of } G_o \\
-1 & g, h \in G \setminus G_o
\end{cases}$$

We will prove Theorem 5.0.1 in section 5.6 but must first build some necessary tools. We start by defining and exploring $\nu_2$. This is followed by a discussion about how $V$ admits a classical representation of $G$ on $V \otimes \omega^* V^\vee$, as well as $\text{Hom}(\omega^* V^\vee, V)$. We then determine how the subspace of $G_o$-invariants given by $\text{Hom}_{G_o}(\omega^* V^\vee, V)$ is a representation of $\mathbb{Z}_2 \simeq G/G_o$, which in-turn gives one of the three values above. Lastly, we show how $V$ can be extended to a Real $\theta$-projective representation under the right circumstances. These together will be the necessary tools that we need in order to prove Theorem 5.0.1. We then give a few examples.
5.1

The indicator $ν_2$

Define the Real projective Frobenius-Schur indicator $ν_2 \in \mathbb{C}\theta^{-1}[G_o]$ as

$$ν_2 = \sum_{ξ \in G \setminus G_o} θ^{-1}(ξ, ξ) \ell_{ξ^2} = \sum_{g \in G_o} θ^{-1}(gω, gω) \ell_{(gω)^2}.$$ 

Note that $ν_2$ depends on both $θ \in \mathbb{Z}^{2+π}(G; \mathbb{U}(1))$ and $G$.

**Lemma 5.1.1.** The element $ν_2$ is in the center of $\mathbb{C}\theta^{-1}[G_o]$. Equivalently, $ν_2$ is a $θ_o$-class function.

**Proof.** For all $k \in G_o$ we have

$$\ell_k \cdot ν_2 \cdot ℓ_{k^{-1}} = \ell_k \left( \sum_{g \in G_o} θ^{-1}(ξ, ξ) \ell_{ξ^2} \right) ℓ_{k^{-1}}$$

$$= \sum_{g \in G_o} θ^{-1}(ξ, ξ^2) ℓ_{kξ^2} ℓ_{k^{-1}}$$

$$= \sum_{g \in G_o} θ^{-1}(kξk^{-1}, kξ^2k^{-1}) \ell_{(kξk^{-1})^2}$$

$$= θ^{-1}(k^{-1}, k) \sum_{ξ \in G / G_o} θ(ξ, ξ) \ell_{ξ^2}$$

$$= θ^{-1}(k^{-1}, k) ν_2.$$ 

Therefore, we compute

$$\ell_k \cdot ν_2 = \ell_k \cdot ν_2 \cdot ℓ_{k^{-1}}\ell_k$$

$$= θ(k^{-1}, k) ⊺_2 \cdot ν_2 \cdot ℓ_{k^{-1}}\ell_k$$

$$= θ(k^{-1}, k) ν_2 \cdot ℓ_k$$

$$= ν_2 \cdot ℓ_k.$$ 

Thus $ν_2$ is in the center of $\mathbb{C}\theta^{-1}[G_o]$. Recall from Lemma 3.5.3 that the center of $\mathbb{C}\theta^{-1}[G_o]$ is correlated with $θ_o$-class functions, so we can equivalently conclude that $ν_2$ is a $θ_o$-class function. □

If we consider $G = G_o \times \mathbb{Z}_2$ with trivial multiplier $θ$, then the Real projective indicator is equivalent to the standard Frobenius-Schur indicator for representation theory of finite groups over
Lemma 5.1.2. Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian inner product on $\theta_o$-class functions as described in section 3.5. Then

$$\langle \chi_V, \nu_2 \rangle = \frac{1}{|G_o|} \sum_{\xi \in G \setminus G_o} \theta(\xi, \xi) \chi_V(\xi^2).$$

**Proof.** We have

$$\langle \chi_V, \nu_2 \rangle = \frac{1}{|G_o|} \sum_{g \in G_o} \chi_V(g) \bar{\nu_2(g)}$$

$$= \frac{1}{|G_o|} \sum_{g \in G_o} \chi_V(g) \sum_{\xi: \xi^2 = g} \theta^{-1}(\xi, \xi)$$

$$= \frac{1}{|G_o|} \sum_{g \in G_o} \sum_{\xi: \xi^2 = g} \theta(\xi, \xi) \chi_V(\xi^2)$$

$$= \frac{1}{|G_o|} \sum_{g \in G_o} \theta(\xi, \xi) \chi_V(\xi^2)$$

Note that the final equality uses that $G_o$ has index 2 in $G$, so $\xi^2 \in G_o$ for all $\xi \in G \setminus G_o$. 

5.2

Inducing a representation on $V \otimes \omega^*V$\textsuperscript{\textdagger}

In section 4.4 we defined $\omega^*V$ as the vector space $V$ with group action given by

$$\rho_{\omega^*V}(g) = \mu_{\omega^{-1}}^{-1}(g) \rho_V(\omega^{-1}g\omega)$$

and found that this made $\omega^*V$ a $\theta_o^{-1}$-projective representation of $G_o$. From Lemma 3.3.3 we know $V \otimes \omega^*V$ has multiplier $\theta_g \theta_o^{-1} = 1$. Thus for all $g \in G_o$ we have

$$\rho_{V \otimes \omega^*V}(g)(v_1 \otimes v_2) = \rho_V(g)v_1 \otimes \rho_{\omega^*V}(g)v_2.$$ 

Since the multiplier for the tensor product is trivial, we can view $V \otimes \omega^*V$ as a classical representation of $G_o$. We now wish to extend $V \otimes \omega^*V$ to a representation of $G$. 

Lemma 5.2.1. For each $g \in G_\omega$ and for fixed $\omega \in G \setminus G_\omega$ define

$$\rho_{V^\otimes \omega^*V}(\omega)(v_1 \otimes v_2) = \theta(\omega, \omega) \rho_V(\omega^2)v_2 \otimes v_1$$

$$\rho_{V^\otimes \omega^*V}(\omega) = \rho_{V^\otimes \omega^*V}(g) \circ \rho_{V^\otimes \omega^*V}(\omega).$$

Then $V^\otimes \omega^*V$ is a classical representation of $G$.

Proof. It is sufficient to show that (i) the action of $\omega^2$ is equivalent to acting by $\omega$ twice and that (ii) the action of $\omega g$ is equivalent to first acting by $g$ and then by $\omega$ for all $g \in G_\omega$. We have

$$\rho_{V^\otimes \omega^*V}(\omega)\rho_{V^\otimes \omega^*V}(\omega)(v_1 \otimes v_2) = \rho_{V^\otimes \omega^*V}(\omega)(\theta(\omega, \omega) \rho_V(\omega^2)v_2 \otimes v_1)$$

$$= \theta^2(\omega, \omega) \rho_V(\omega^2)v_1 \otimes \rho_V(\omega^2)v_2$$

$$= \theta^2(\omega, \omega) \rho_V(\omega^2)v_1 \otimes \mu_{\theta, \omega}(\omega^2) \mu_{\theta, \omega}^{-1}(\omega^2) \rho_V(\omega^{-1} \omega^2 \omega)v_2$$

$$= \theta^2(\omega, \omega) \mu_{\theta, \omega}(\omega^2) \left( \rho_V(\omega^2)v_1 \otimes \rho_{\omega^*V}(\omega^2)v_2 \right)$$

$$= \theta^2(\omega, \omega) \mu_{\theta, \omega}(\omega^2) \rho_{V^\otimes \omega^*V}(\omega^2)(v_1 \otimes v_2)$$

$$= \rho_{V^\otimes \omega^*V}(\omega^2)(v_1 \otimes v_2) \quad \text{(identity $\mathbf{A.3}$)}.$$ 

Similarly, for each $g \in G_\omega$ we have

$$\rho_{V^\otimes \omega^*V}(\omega)\rho_{V^\otimes \omega^*V}(g)(v_1 \otimes v_2)$$

$$= \rho_{V^\otimes \omega^*V}(\omega)\left[ \rho_V(g)v_1 \otimes \rho_{\omega^*V}(g)v_2 \right]$$

$$= \theta(\omega, \omega) \left[ \rho_V(\omega^2)\rho_{\omega^*V}(g)v_2 \otimes \rho_V(g)v_1 \right]$$

$$= \theta(\omega, \omega) \mu_{\theta, \omega}^{-1}(g) \mu_{\theta, \omega}(\omega g \omega^{-1}) \left[ \rho_V(\omega^2)\rho_V(\omega^{-1} g \omega)v_2 \otimes \rho_{\omega^*V}(\omega g \omega^{-1})v_1 \right]$$

$$= \theta(\omega, \omega) \theta(\omega^2, \omega^{-1} g \omega) \mu_{\theta, \omega}^{-1}(g) \mu_{\theta, \omega}(\omega g \omega^{-1}) \left[ \rho_V(\omega g \omega)v_2 \otimes \rho_{\omega^*V}(\omega g \omega^{-1})v_1 \right]$$

$$= \theta(\omega, \omega) \theta(\omega g \omega^{-1}, \omega^2) \left[ \rho_V(\omega g \omega)v_2 \otimes \rho_{\omega^*V}(\omega g \omega^{-1})v_1 \right] \quad \text{(identity $\mathbf{A.4}$)}$$

$$= \theta(\omega, \omega) \left[ \rho_V(\omega g \omega^{-1})\rho_V(\omega^2)v_2 \otimes \rho_{\omega^*V}(\omega g \omega^{-1})v_1 \right]$$

$$= \theta(\omega, \omega) \rho_{V^\otimes \omega^*V}(\omega g \omega^{-1}) \left[ \rho_V(\omega^2)v_2 \otimes v_1 \right]$$

$$= \rho_{V^\otimes \omega^*V}(\omega g)(v_1 \otimes v_2)$$

$$= \rho_{V^\otimes \omega^*V}(\omega g)(v_1 \otimes v_2).$$

Note that $\omega g \omega^{-1} \in G_\omega$, so the last equality comes from our definitions. \qed
Lemma 5.2.2. For all $\xi \in G_\circ$ we have $\chi_{V \otimes \omega^* V \otimes V}(\xi) = \theta(\xi, \xi) \chi_V(\xi^2)$.

Proof. By Lemma 3.3.5 we know the action of each $g \in G_\circ$ on $V$ is diagonalizable. Fix $\xi \in G \setminus G_\circ$ and let $\{v_i\} \subset V$ be an eigenbasis of $\rho_V(\xi^2)$ with eigenvalues $\{\lambda_i\}$. Define the representation $V \otimes \xi^* V$ of $G$ as in Lemma 5.2.1. We claim that the union of the following two sets is a basis for $V \otimes \omega^* V$:

$$P = \{ P_{ij} = v_i \otimes v_j + v_j \otimes v_i \mid i \leq j \}$$

$$N = \{ N_{ij} = v_i \otimes v_j - v_j \otimes v_i \mid i < j \}.$$ 

To show that $P \cup N$ is a basis for $V \otimes \omega^* V$, we first show $\dim(P) + \dim(N) = \dim(V \otimes \omega^* V)$ and then we show the collection of vectors $P \cup N$ are linearly independent. First note that, as vector spaces $V \otimes \omega^* V = V \otimes V$. Let $r = \dim(V)$. Then $\dim(V \otimes \omega^* V) = r^2$ and similarly

$$\dim(P) + \dim(N) = \sum_i r + \sum_i i = r(r + 1) + \frac{(r - 1)(r - 1 + 1)}{2} = r^2.$$

Therefore the dimension of $P \cup N$ is equivalent to the dimension of $V \otimes \omega^* V$. We now show that $P \cup N$ is a linearly independent set. We know $\{v_i\}$ is an eigenbasis of $V$, which implies $\{v_{ij} := v_i \otimes v_j\}_{1 \leq i,j \leq r}$ forms a basis for $V \otimes V$. Now suppose there are some constants $\{c_{ijP}\}_{i \leq j}$ and $\{c_{ijN}\}_{i < j}$ such that

$$\sum_{i \leq j} c_{ijP}P_{ij} + \sum_{i < j} c_{ijN}N_{ij} = 0.$$

Therefore we know

$$\sum_i 2c_{iip}v_{ii} + \sum_{i < j} (c_{ijP} + c_{ijN})v_{ij} + \sum_{i < j} (c_{ijP} - c_{ijN})v_{ji} = 0.$$

Since $\{v_{ij}\}_{1 \leq i,j \leq r}$ is a basis for $V \otimes V$, we know

$$2c_{iip} = 0, \quad c_{ijP} + c_{ijN} = 0, \quad c_{ijP} - c_{ijN} = 0.$$

Therefore $c_{ijP}$ and $c_{ijN}$ must be zero for all $i,j \in \{1, ..., r\}$. Thus $P \cup N$ is a collection of linearly independent vectors. Therefore $P \cup N$ is a basis of $V \otimes \omega^* V$.

We next analyze the action of $\xi$ on $V \otimes \omega^* V$ by explicitly computing it on each basis element.
Fix some indices $i$ and $j$ with $i \leq j$. We have

$$
\rho_{V \otimes \omega^* V}(\xi) P_{ij} = \rho_{V \otimes \omega^* V}(\xi) \left( v_i \otimes v_j + v_j \otimes v_i \right)
$$

$$
= \theta(\xi, \xi) \left( \rho_V(\xi^2) v_i \otimes v_i + \rho_V(\xi^2) v_j \otimes v_j \right)
$$

$$
= \theta(\xi, \xi) \left( \lambda_j(v_i \otimes v_i) + \lambda_i(v_i \otimes v_j) \right)
$$

$$
= \theta(\xi, \xi) \left( \lambda_j \left( \frac{P_{ij} - N_{ij}}{2} \right) + \lambda_i \left( \frac{P_{ij} + N_{ij}}{2} \right) \right)
$$

$$
= \theta(\xi, \xi) \left( \frac{\lambda_i + \lambda_j}{2} \right) P_{ij} + \left( \frac{\lambda_i - \lambda_j}{2} \right) N_{ij}.
$$

By similar calculation, we compute the action of $\rho_{V \otimes \omega^* V}(g \omega)$ on the basis elements in $\mathcal{N}$. For $i < j$

$$
\rho_{V \otimes \omega^* V}(\xi) N_{ij} = \theta(\xi, \xi) \left( \left( \frac{-\lambda_i + \lambda_j}{2} \right) P_{ij} - \left( \frac{\lambda_i + \lambda_j}{2} \right) N_{ij} \right).
$$

Therefore, writing $\rho_{V \otimes \omega^* V}(\xi)$ as a matrix under the basis $\mathcal{P} \cup \mathcal{N}$, the diagonal entries are $\frac{\lambda_i + \lambda_j}{2}$ for the $P_{ij}$’s and $-\frac{\lambda_i + \lambda_j}{2}$ for the $N_{ij}$’s. Since the trace is the sum of diagonal elements and is invariant under change of bases, we can compute the character of $\xi$ using these diagonal entries. Thus we have

$$
\chi_{V \otimes \omega^* V}(\xi) = \theta(\xi, \xi) \left( \sum_{i \leq j} \frac{\lambda_i + \lambda_j}{2} + \sum_{i < j} \frac{-\lambda_i - \lambda_j}{2} \right)
$$

$$
= \theta(\xi, \xi) \left( \sum_{i=j} \frac{\lambda_i + \lambda_j}{2} + \sum_{i < j} \left( \frac{\lambda_i + \lambda_j}{2} - \frac{\lambda_i - \lambda_j}{2} \right) \right)
$$

$$
= \theta(\xi, \xi) \sum_i \lambda_i
$$

$$
= \theta(\xi, \xi) \chi_V((\xi)^2).
$$

Lastly, we note $V \simeq V^{\vee\vee}$ as $\theta_o$-projective representations of $G_o$. Therefore we have $V \otimes \omega^* V \simeq V \otimes \omega^* V^{\vee\vee}$ as representations of $G$ and so the characters of these representations are equivalent.

5.3

Inducing a representation on $\text{Hom}(\omega^* V^{\vee}, V)$

Recall that for representations $V$ and $U$ of a finite group $G_o$, the canonical vector space isomorphism $V \otimes U^\vee \simeq \text{Hom}(U, V)$ is $G_o$-equivariant. We want to mimic this construction to build a $G$-equivariant isomorphism $V \otimes \omega^* V^{\vee\vee} \simeq \text{Hom}_C(\omega^* V^{\vee}, V)$. We first show that for a $\theta_o$-projective
representation $V$ of $G_\alpha$ there is an induced representation of $G$ on $\text{Hom}_\mathbb{C}(\omega^*V^\vee, V)$. For ease of notation throughout this section we denote $U := \text{Hom}_\mathbb{C}(\omega^*V^\vee, V)$.

**Lemma 5.3.1.** Let $\phi \in U$. For $g \in G_\alpha$ and fixed $\omega \in G \setminus G_\alpha$, define

$$
\rho_U(g) \phi = \rho_V(g) \circ \phi \circ \rho_{\omega \cdot V^\vee}(g)^{-1}
$$

$$
\rho_U(\omega) \phi = \theta(\omega, \omega) \rho_V(\omega^2) \circ \rho_{V^\vee}(\phi)^{-1}
$$

$$
\rho_U(g\omega) \phi = \rho_U(g) \circ \rho_U(\omega).
$$

Then $\text{Hom}_\mathbb{C}(\omega^*V^\vee, V) = U$ is a classical representation of $G$.

**Proof.** Recall from Lemma 3.3.1 that $\rho_{V^\vee}(g) = \rho_V(g)^{-\vee}$. Also recall from Lemma 4.4.4 that $\omega^*V$ is a $\theta_\omega^{-1}$-projective representation of $G_\alpha$. Therefore we know $\rho_{\omega \cdot V^\vee}(g)^{-1} = \rho_{\omega \cdot V^\vee}(g)^{\vee}$ and in particular for $g, h \in G_\alpha$ we have

$$
\rho_{\omega \cdot V^\vee}(h)^{-1}\rho_{\omega \cdot V^\vee}(g)^{-1} = \left(\rho_{\omega \cdot V^\vee}(g)\rho_{\omega \cdot V^\vee}(h)^{-1}\rho_{\omega \cdot V^\vee}(g)^{-1}\right)^{\vee}
$$

$$
= \theta^{-1}(g, h) \rho_{\omega \cdot V^\vee}(gh)^{\vee}
$$

$$
= \theta^{-1}(g, h) \rho_{\omega \cdot V^\vee}(gh)^{-1}.
$$

Thus we have

$$
\rho_U(g)\rho_U(h)\phi = \rho_V(g)\rho_V(h) \circ \phi \circ \rho_{\omega \cdot V^\vee}(h)^{-1}\rho_{\omega \cdot V^\vee}(g)^{-1}
$$

$$
= \theta(g, h)\theta^{-1}(g, h) \rho_V(gh) \circ \phi \circ \rho_{\omega \cdot V^\vee}(gh)^{-1}
$$

$$
= \rho_V(gh) \circ \phi \circ \rho_{\omega \cdot V^\vee}(gh)^{-1}
$$

$$
= \rho_U(gh)\phi,
$$

so $U$ is a representation of $G_\alpha$. For all $g \in G$ and using Lemma 3.3.1 and Lemma 4.4.4 we have

$$
\rho_{\omega \cdot V^\vee}(g) = \left(\mu_{\omega, \omega}^{-1}(g) \rho_V(\omega^{-1}g\omega)\right)^{-\vee}
$$

$$
= \mu_{\omega, \omega}(g) \rho_V(\omega^{-1}g\omega)^{-\vee}
$$

We use this in the next two computations. First, we show that acting by $\omega$ twice is the same as
acting by $\omega^2$. We compute

$$
\rho_U(\omega)\rho_U(\omega)\phi = \theta^2(\omega, \omega) \rho_V(\omega^2) \circ ev_V^{-1} \circ (\rho_V(\omega^2) \circ ev_V^{-1} \circ \phi^\vee)^\vee
$$

$$
= \theta^2(\omega, \omega) \rho_V(\omega^2) \circ ev_V^{-1} \circ \phi^\vee \circ ev_V^{-1} \circ \rho_V(\omega^2)^\vee
$$

$$
= \theta^2(\omega, \omega) \left( \rho_V(\omega^2) \circ \phi \circ \rho_V(\omega^2)^\vee \right) \quad \text{Lemma 2.1.3}
$$

$$
= \theta^2(\omega, \omega) \mu_{\theta, \omega}(\omega^2) \left( \rho_V(\omega^2) \circ \phi \circ \rho_{\omega^* \cdot V^\vee}(\omega^2)^{-1} \right)
$$

$$
= \rho_V(\omega^2) \circ \phi \circ \rho_{\omega^* \cdot V^\vee}(\omega^2)^{-1} \quad \text{identity A.3}
$$

$$
= \rho_U(\omega^2)\phi.
$$

Next we show that the action of $g$ for $g \in G_\alpha$ followed by the action of $\omega$ is the same as acting by $\omega g$. We compute

$$
\rho_U(\omega)\rho_U(g)\phi
$$

$$
= \theta(\omega, \omega) \rho_V(\omega^2) \circ ev_V^{-1} \circ \left( \rho_V(g) \circ \phi \circ \rho_{\omega^* \cdot V^\vee}(g)^{-1} \right)^\vee
$$

$$
= \theta(\omega, \omega) \left[ \rho_V(\omega^2) \circ ev_V^{-1} \circ \rho_{\omega^* \cdot V^\vee}(g)^{-1} \circ \phi^\vee \circ \rho_V(g)^\vee \right]
$$

$$
= \theta(\omega, \omega) \mu_{\theta, \omega}(\omega g\omega^{-1}) \left[ \rho_V(\omega^2) \circ ev_V^{-1} \circ \rho_V(\omega^{-1} g\omega) \circ ev_V^{-1} \circ \phi^\vee \circ \rho_{\omega^* \cdot V^\vee}(\omega g\omega^{-1})^{-1} \right]
$$

$$
= \theta(\omega, \omega) \frac{\mu_{\theta, \omega}(\omega g\omega^{-1})}{\mu_{\theta, \omega}(g)} \left[ \rho_V(\omega^2) \circ ev_V^{-1} \circ \rho_V(\omega^{-1} g\omega) \circ ev_V^{-1} \circ \phi^\vee \circ \rho_{\omega^* \cdot V^\vee}(\omega g\omega^{-1})^{-1} \right] \quad \text{identity A.4}
$$

$$
= \theta(\omega, \omega) \left[ \rho_V(\omega g\omega^{-1}) \circ \rho_V(\omega^2) \circ ev_V^{-1} \circ \phi^\vee \circ \rho_{\omega^* \cdot V^\vee}(\omega g\omega^{-1})^{-1} \right] \quad \text{Lemma 2.1.1}
$$

$$
= \rho_U(\omega g\omega^{-1}) \rho_U(\omega)\phi
$$

$$
= \rho_U(\omega g)\phi.
$$

Therefore the actions of $G$ on $U$ as defined above form a group homomorphism, thus defining a classical representation of $G$. \qed

In particular, we note that by Lemma 2.3.3 the quotient group $Z_2 \simeq G/G_\alpha$ acts on the space of $G_\alpha$-invariants $U^{G_\alpha} = \text{Hom}_{G_\alpha}(\omega^* V^\vee, V)$. The action of the generator $-1 \in Z_2$ is given by

$$
-1 \cdot \phi = \theta(\omega, \omega) \rho_V(\omega^2) \circ ev_V^{-1} \circ \phi^\vee.
$$
Lemma 5.3.2. The $G$-representations $\text{Hom}(\omega^*V^\vee, V) =: U$ and $V \otimes \omega^*V^\vee$ are isomorphic.

Proof. We consider the vector space isomorphism given by

$$\alpha : V \otimes \omega^*V^\vee \to \text{Hom}(\omega^*V^\vee, V)$$

$$(v \otimes u^\vee) \mapsto (f \mapsto u^\vee(f)v).$$

and show that it is $G$-invariant. Let $f \in \omega^*V^\vee$ and $g \in G_\alpha$. We compute

$$\alpha\left(\rho_{\otimes \omega^*V^\vee}(g)(v \otimes u^\vee)\right)(f) = \alpha\left(\rho_V(g)\alpha \rho_{\omega^*V^\vee}(g)u^\vee\right)(f)$$

$$= \rho_{\omega^*V^\vee}(g)u^\vee(f)\rho_V(g)v$$

$$= \rho_{\omega^*V^\vee}(g)^{-1}u^\vee(f)\rho_V(g)v$$

$$= u^\vee\rho_{\omega^*V^\vee}(g)^{-1}f\rho_V(g)v$$

$$= \left(\rho_V(g) \circ \alpha(v \otimes u^\vee) \circ \rho_{\omega^*V^\vee}(g)^{-1}\right)(f)$$

$$= \rho_U(g)\alpha(v \otimes u^\vee)(f)$$

Therefore $\alpha$ is $G_\alpha$-equivariant. Let $\omega \in G \setminus G_\alpha$. We compute

$$\alpha\left(\rho_{\otimes \omega^*V^\vee}(\omega)(v \otimes u^\vee)\right)(f) = \alpha\left(\theta(\omega, \omega) \rho_V(\omega^2)u \otimes v^\vee\right)(f)$$

$$= \theta(\omega, \omega) v^\vee(f)\rho_V(\omega^2)u$$

$$= \theta(\omega, \omega) f(v)\rho_V(\omega^2)u$$

$$= \theta(\omega, \omega) f(v)\rho_V(\omega^2)evV^{-1}(u^\vee)$$

$$= \theta(\omega, \omega) \rho_V(\omega^2) \circ evV^{-1} \circ f(v)u^\vee$$

$$= \theta(\omega, \omega) \rho_V(\omega^2) \circ evV^{-1} \circ f \circ \alpha(\omega \otimes u^\vee)$$

$$= \left(\theta(\omega, \omega) \rho_V(\omega^2) \circ evV^{-1} \circ \alpha(\omega \otimes u^\vee)\right)(f)$$

$$= \rho_U(\omega)\left(\alpha(\omega \otimes u^\vee)\right)(f)$$

Thus $\alpha$ is an $\omega$-equivariant map, and furthermore because all $\xi \in G \setminus G_\alpha$ can be written as $\xi = g\omega$ for a unique $g \in G_\alpha$ and because $\rho_{\otimes \omega^*V^\vee}$ and $\rho_U$ are both group homomorphisms, we conclude that $\alpha$ is $G$-equivariant. Thus $\alpha$ is an isomorphism of representations of $G$. \qed
5.4

Taking $G_\alpha$-invariants

As mentioned in section 2.3, given a representation $V$ of $G$, denote the space of $G_\alpha$-invariants by

$$V^{G_\alpha} = \{ v \in V \mid \rho_V(g)v = v \text{ for all } g \in G_\alpha \}.$$ 

By Lemma 2.3.3 we know $V^{G_\alpha}$ is a representation of the quotient group $G/G_\alpha \cong \mathbb{Z}_2$. By Lemma 2.3.2 we know that the map $P : V \to V$ given by

$$P(v) = \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \rho_V(g)v,$$

is a projection map onto the subspace $V^{G_\alpha}$. We also recall by Lemma 2.1.6 that for the identity $1 \in \mathbb{Z}_2$ we have

$$\frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \chi_V(g) = \text{tr}_V(P) = \dim(V^{G_\alpha})\text{tr}_{V^{G_\alpha}}(1) = \chi_{V^{G_\alpha}}(1).$$

In this way, averaging the character of $V$ over $G_\alpha$ is equivalent to the character of the non-trivial coset in the quotient $\mathbb{Z}_2 \cong G/G_\alpha$. This is related to the following

**Lemma 5.4.1.** For a representation $V$ of $G$ and some fixed $\omega \in G \setminus G_\alpha$, we have

$$\frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \chi_V(g\omega) = \text{tr}_{V^{G_\alpha}}(-1).$$

**Proof.** We compute

$$\frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \chi_U(g\omega) = \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \text{tr}_V(\rho_V(g\omega))$$

$$= \text{tr}_V \left( \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \rho_V(g\omega) \right)$$

$$= \text{tr}_V \left( \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \rho_V(g) \circ \rho_V(\omega) \right)$$

$$= \text{tr}_V (P \circ \rho_V(\omega))$$

Recall from Lemma 2.3.3 that the actions of $G/G_\alpha$ are determined by coset representatives. There-
fore we know $\rho_{V_{G_0}}(\omega G_0) = \rho_V(\omega)$, and this action is independent of the choice $\omega \in G / G_0$ because $G_0 \triangleleft G$ has index 2. Thus we have

$$\text{tr}_{V_{G_0}}(-1) = \text{tr}_{V_{G_0}}(\rho_{V_{G_0}}(\omega G_0)) = \text{tr}_{V_{G_0}}(\rho_V(\omega)).$$

In order to compute trace of $\omega$ over the subspace of $G_0$-invariants, we need to project $\rho_V(\omega)$ onto that subspace. Therefore with the projection map $P$ we know $\text{tr}_{V_{G_0}}(\rho_V(\omega)) = \text{tr}_V(\rho_V(\omega)).$ Thus we have

$$\frac{1}{|G_0|} \sum_{g \in G_0} \chi_V(g\omega) = \text{tr}_V(P \circ \omega)$$

$$= \text{tr}_{V_{G_0}}(\rho_V(\omega))$$

$$= \text{tr}_{V_{G_0}}(-1)$$

\[\square\]

**Lemma 5.4.2.** Let $V$ be an irreducible $\theta_\alpha$-projective representation of $G_0$. Let $-1 \in \mathbb{Z}_2 \simeq G / G_0$ be the non-trivial element. Thus we have

$$\text{tr}_{\text{Hom}_{G_0}(\omega^*V^\vee, V)}(-1) = \begin{cases} 
0 & \text{if } \text{Hom}_{G_0}(\omega^*V^\vee, V) = 0 \\
1 & \text{if } \text{Hom}_{G_0}(\omega^*V^\vee, V) = \mathbb{C}_{\text{triv}} \\
-1 & \text{if } \text{Hom}_{G_0}(\omega^*V^\vee, V) = \mathbb{C}_{\text{sign}} 
\end{cases}$$

**Proof.** Note that both $\omega^*V^\vee$ and $V$ are irreducible $\theta_\alpha$-projective representations of $G_0$, so we know $\text{Hom}_{G_0}(\omega^*V^\vee, V)$ has a chance to be non-zero. By Cheng [4], we can apply Schur’s Lemma to $\theta_\alpha$-projective representations, so $\dim(\text{Hom}_{G_0}(\omega^*V^\vee, V))$ is either 0 or 1, depending on whether $V$ is isomorphic to $\omega^*V^\vee$ as $\theta_\alpha$-projective representations of $G_0$. Taking $G_0$-invariants makes $\text{Hom}_{G_0}(\omega^*V^\vee)$ a representation of $\mathbb{Z}_2 \simeq G / G_0$. In the case where the dimension is zero, clearly the value of the trace is zero. In the case where $\dim(\text{Hom}_{G_0}(\omega^*V^\vee, V)) = 1$, the subspace of $G_0$-invariants is either the trivial representation of $\mathbb{Z}_2$, or the sign representation of $\mathbb{Z}_2$, which determines whether the trace of the generator $-1 \in \mathbb{Z}_2$ is positive or negative. \[\square\]
5.5

Extending $V$ to a Real representation of $G$

Recall the universal multiplier $\alpha \in \mathbb{Z}^2/\pi(K; \mathbb{U}(1))$ as defined in section 3.2.

Lemma 5.5.1. Let $\omega \in G \setminus G_\alpha$ and let $V$ be an irreducible $\theta_\alpha$-projective representation of $G_\alpha$. If $\text{tr}_{\text{Hom}_{G_\alpha}(\omega^* V^\vee, V)}(-1) = 1$ then $V$ can be extended to a Real $\theta$-projective representation of $G$. If $\text{tr}_{\text{Hom}_{G_\alpha}(\omega^* V^\vee, V)}(-1) = -1$ then $V$ can be extended to a Real $\alpha \theta$-projective representation of $G$.

Proof. First suppose $\text{tr}_{\text{Hom}_{G_\alpha}(\omega^* V^\vee, V)}(-1) = 1$. By Lemma 5.4.2 this means $\text{Hom}_{G_\alpha}(\omega^* V^\vee, V) = \mathbb{C}_{\text{triv}}$ as a representation of $\mathbb{Z}_2 \cong G/G_\alpha$. By Lemma 2.3.3 we know the cosets act by coset representatives, and the actions do not depend on the choice of coset representatives. Therefore, for $g \in G_\alpha$, some fixed $\omega \in G \setminus G_\alpha$, and all $\phi \in \text{Hom}(\omega^* V^\vee, V)$ we have

$$\phi = 1 \cdot \phi = \rho_{\text{Hom}(\omega^* V^\vee, V)}(g) \phi = \rho_V(g) \circ \phi \circ \rho_{\omega^* V^\vee}(g)^{-1} \quad (5.1)$$

and

$$\phi = -1 \cdot \phi = \rho_{\text{Hom}(\omega^* V^\vee, V)}(\omega) \phi = \theta(\omega, \omega) \rho_V(\omega^2) \circ \text{ev}^{-1}_V \circ \phi^\vee. \quad (5.2)$$

Note that equation (5.2) can be written as $\phi \circ \phi^{-\vee} \circ \text{ev}_V = \theta(\omega, \omega) \rho_V(\omega^2)$. Since $V$ is irreducible, Schur’s Lemma tells us $\dim(\text{Hom}_{G_\alpha}(\omega^* V^\vee, V)) = 1$. Thus there is an isomorphism $\phi: \omega^* V^\vee \rightarrow V$. Fix such a $\phi$ and define the following maps:

$$\rho_V(\omega) := \phi$$

$$\rho_V(g \omega) := \theta^{-1}(g, \omega) \rho_V(g) \cdot \rho_V(\omega).$$

We claim these definitions induce a Real $\theta$-projective representation of $G$. First, let $g \in G_\alpha$. We
compute

\[ \rho_V(\omega g) = \rho_V(\omega g \omega^{-1} \omega) \]

\[ = \theta^{-1}(\omega g \omega^{-1}, \omega) \rho_V(\omega g \omega^{-1}) \cdot \rho_V(\omega) \]

\[ = \theta^{-1}(\omega g \omega^{-1}, \omega) \rho_V(\omega) \circ \rho_{\omega \cdot V^\vee}(g) \]

\[ = \theta^{-1}(\omega g \omega^{-1}, \omega) \rho_V(\omega) \circ \left( \mu_{\omega, \omega}^{-1}(\omega g \omega^{-1}) \rho_V(g) \right)^{-\vee} \]

\[ = \theta^{-1}(\omega g \omega^{-1}, \omega) \mu_{\omega, \omega}(\omega g \omega^{-1}) \rho_V(\omega) \circ \rho_V(g)^{-\vee} \]

\[ = \theta^{-1}(\omega, g) \rho_V(\omega) \cdot \rho_V(g) \]

identity A.6

Lastly, we compute

\[ \rho_V(\omega) \cdot \rho_V(\omega) = \rho_V(\omega) \circ \rho_V(\omega)^{-\vee} \circ \mathrm{ev}_V \]

\[ = \theta(\omega, \omega) \rho_V(\omega^2) \]

equation 5.2

Therefore, the definitions above extend to a Real \( \theta \)-projective representation of \( G \).

Now suppose \( \text{tr}_{\text{Hom}_{G_\alpha}(V^\vee, V)}(1) = -1 \). Thus by Lemma 5.4.2 we know \( \text{Hom}_{G_\alpha}(\omega^*V^\vee, V) = \mathbb{C}_{\text{sign}} \) as a representation of \( \mathbb{Z}_2 \cong G/G_\alpha \). In particular, equation 5.1 is unaffected, but equation 5.2 becomes

\[ -\phi = -1 \cdot \phi = \rho_{\text{Hom}_{G_\alpha}(\omega^*V^\vee, V)}(\omega) \phi = \theta(\omega, \omega) \rho_V(\omega^2) \circ \text{ev}_V^{-1} \circ \phi^\vee. \] (5.3)

Re-writing equation 5.3 we have

\[ \phi \cdot \phi = -\theta(\omega, \omega) \rho_V(\omega^2), \]

which motivates the same definitions as the previous case, but adds a negative when \( \rho_V(\omega) := \phi \) acts twice. All other computations are unaffected, so the only difference is when we act by two elements in \( G \setminus G_\alpha \), which precisely describes the universal multiplier \( \alpha \).

The previous Lemma describes two of the three cases described by Lemma 5.4.2. In the third case, where \( \text{tr}_{\text{Hom}_{G_\alpha}(\omega^*V^\vee)}(1) = 0 \), we can instead consider the vector space \( V \oplus \omega^*V^\vee \). In particular, we have

\[ \omega^*(V \oplus \omega^*V^\vee)^\vee \simeq \omega^*(\omega^*V^\vee)^\vee \oplus \omega^*V^\vee \simeq V \oplus \omega^*V^\vee \]
as $G_\alpha$ representations, so we can then consider a similar notion as in Lemma 5.5.1 but for $V \oplus \omega^* V^\vee$. Note that this is related to the result from Theorem 4.5.1 describing irreducible Real representations of $G$.

5.6

Proof of Theorem 5.0.1

Let $\pi : G \to \mathbb{Z}_2$ be a surjective group homomorphism for a finite group $G$. Let $G_\alpha = \ker(\pi)$ and let $\theta \in \mathbb{Z}^{2+\pi}(G; U(1))$ be a normalized cocycle with restriction $\theta_\alpha$ to $G_\alpha$. Let $V$ be an irreducible $\theta_\alpha$-projective representation of $G_\alpha$. Recall the universal multiplier $\alpha$ from section 3.2. We have

$$
\langle \chi_V, \nu_2 \rangle = \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \chi_V(g) \bar{\nu}_2(g)
$$

$$
= \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \theta(g\omega, g\omega) \chi_V((g\omega)^2) \quad \text{Lemma 5.1.2}
$$

$$
= \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \chi_{V \otimes \omega^* V}(g\omega) \quad \text{Lemma 5.2.2}
$$

$$
= \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \chi_{V \otimes \omega^* V^\vee}(g\omega) \quad \text{Lemma 5.3.2}
$$

$$
= \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \chi_{\text{Hom}(\omega^* V^\vee, V)}(g\omega)
$$

$$
= \text{tr}_{\text{Hom}_{G_\alpha}(\omega^* V^\vee, V)}(\langle -1 \rangle) \quad \text{Lemma 5.4.1}
$$

$$
= \begin{cases} 
0 & V \text{ cannot be extended to a Real representation of } G \\
1 & V \text{ can be extended to a Real } \theta\text{-projective representation of } G \\
-1 & V \text{ can be extended to a Real } \alpha \theta\text{-projective representation of } G.
\end{cases}
$$

Note that the last equality follows from Lemma 5.4.2 and Lemma 5.5.1. We can also see what this particular extended Real representation is by following the proof of Lemma 5.5.1. As mentioned there, if this indicator gives a non-zero value, then we pick the $G_\alpha$-invariant map $\phi : \omega^* V^\vee \to V$ and define

$$
\rho_V(\omega) := \phi
$$

$$
\rho_V(g\omega) := \theta^{-1}(g, \omega) \rho_V(g) \cdot \rho_V(\omega).
$$
These definitions extend $V$ to a Real representation of $G$, whether that be a Real $\theta$-projective representation or a Real $\alpha\theta$-projective representation of $G$, depending on the sign of the indicator.

5.7

Examples

We begin by showing that the indicator above restricts to the classical Frobenius-Schur indicator. We then give examples using $\mathbb{Z}_{2^n}$ and the dihedral group $D_{2n}$.

5.7.1

$G_o \times \mathbb{Z}_2$

Let $G_o$ be a finite group. Consider the group homomorphism $\pi : G_o \times \mathbb{Z}_2 \to \mathbb{Z}_2$ given by projection to the second factor. Therefore we have $G_o := \ker(\pi) = G_o \times \{1\}$. Let $V$ be an irreducible, finite dimensional, complex representation of $G_o$ with trivial multiplier $\theta = 1 \in \mathbb{Z}^2(G_o; \mathbb{U}(1))$. Let $\omega = (e, -1)$. Note that $\omega$ commutes with every element of $G_o \times \mathbb{Z}_2$ and moreover $\omega^2$ is the identity in $G_o \times \mathbb{Z}_2$. Therefore we have

$$\nu_2 = \sum_{g \in G_o} \ell(g\omega^2) = \sum_{g \in G_o} \ell(g^2 \omega^2) = \sum_{g \in G_o} \ell(g^2),$$

which is the classical Frobenius-Schur indicator. Furthermore, since $\omega$ commutes with each element we have $\rho_{\omega^2} \cdot V^\vee (g) = \rho_{V^\vee} (g)$, or in other words $\omega^* V^\vee = V^\vee$ as representations of $G_o$.

Now suppose $\langle X_V, \nu_2 \rangle = 1$. By Theorem 5.0.1 this means $\text{Hom}_{G_o}(V^\vee, V) = \mathbb{C}^{\text{triv}}$ as a representation of $\mathbb{Z}_2 \cong (G_o \times \mathbb{Z}_2)/G_o$ and that $V$ can be extended to a Real representation of $G_o \times \mathbb{Z}_2$ with trivial multiplier. We then pick the $G_o$-equivariant map $\phi : V^\vee \to V$ and define $\rho_V(\omega) := \phi$. By Lemma 4.7.1, we know that

$$\langle -, - \rangle : V \times V \to \mathbb{C}$$

$$(v, u) \mapsto \rho_V(\omega)^{-1}(v)(u).$$

defines a non-degenerate, $\omega^2$-symmetric, twisted $G_o$-invariant bilinear form. Since $\omega^2$ is the identity in $G_o \times \mathbb{Z}_2$ and $\omega$ commutes with each element of $G_o \times \mathbb{Z}_2$, the bilinear form $\langle -, - \rangle$ is non-degenerate, symmetric, and $G_o$-invariant. This is precisely what the classical Frobenius-Schur indicator measures, and by following the proof in Fulton-Harris [11, Lemma 3.35], we see this implies $V$ can be
realized over $\mathbb{R}$ as a representation of $G_\alpha$.

Next suppose $\langle \chi_V, \nu_2 \rangle = -1$. Thus by Theorem 5.0.1 we know $V$ can be extended to a Real representation of $G_\alpha \times \mathbb{Z}_2$ with multiplier $\alpha$ given by

$$\alpha(g, h) = \begin{cases} 
1 & \text{either } g \text{ or } h \text{ are in } G_\alpha \times \{1\} \\
-1 & g, h \in G_\alpha \times \{-1\}.
\end{cases}$$

As with the previous case, we use Lemma 4.7.1 to construct a non-degenerate, $G$-invariant bilinear form. Unlike the previous case, though, the $\omega^2$-symmetry here makes the bilinear form skew-symmetric. We then see that this implies $V$ can be realized as a representation of $G_\alpha$ over $\mathbb{H}$.

Therefore, in the case where $G = G_\alpha \times \mathbb{Z}_2$ and $\theta = 1$, Theorem 5.0.1 recovers the classical Frobenius-Schur indicator: for an irreducible representation $V$ of $G$ and $\nu_2 = \sum_{g \in G_\alpha} \ell_g$, we have

$$\langle \chi_V, \nu_2 \rangle = \begin{cases} 
0 & \text{if } V \text{ can only be realized over } \mathbb{C} \\
1 & \text{if } V \text{ can be realized over } \mathbb{R} \\
-1 & \text{if } V \text{ can be realized over } \mathbb{H}.
\end{cases}$$

5.7.2

$\mathbb{Z}_{2n}$

Let $G = \mathbb{Z}_{2n}$ for some positive integer $n$ and let $\pi : G \rightarrow \mathbb{Z}_2$ be defined by $1 \mapsto -1$ where $\mathbb{Z}_{2n}$ is thought of as addition modulo $2n$ and $\mathbb{Z}_2$ is thought of as $\{1, -1\}$ with multiplication. Therefore we have $G_\alpha := \ker(\pi) = \{0, 2, 4, \ldots, 2(n-1)\} \simeq \mathbb{Z}_n$. Let $V$ be an irreducible representation of $G_\alpha$ with trivial multiplier $\theta = 1$. Let $\omega = 1 \in \mathbb{Z}_{2n}$. We know the irreducible representations of $\mathbb{Z}_n \simeq G_\alpha$ are 1-dimensional and parameterized by $n$th roots of unity, so for $1 \leq k \leq n$ let $\rho_k : G_\alpha \rightarrow \mathbb{C}^\times$ be defined by $2 \mapsto e^{\frac{2\pi i k}{n}}$. We have

$$\nu_2 = \sum_{g \in G_\alpha} \ell_{(g\omega)^2} = \sum_{g \in G_\alpha} \ell_2 \ell_{2g} = \ell_2 \left( \ell_0 + \ell_4 + \ell_8 + \cdots + \ell_{2(n-2)} \right).$$

Note that in $\mathbb{Z}_{2n}$, the group multiplication is given by integer addition mod $2n$, so we have $(g\omega)^2 = (g + \omega) + (g + \omega) = 2g + 2$. From here, we consider the pairity of $n$. 
First suppose \( n \) is even and let \( m \) be such that \( n = 2m \). Therefore we have

\[
\nu_2 = 2\ell_2 \left( \ell_0 + \ell_4 + \ell_8 + \ldots + \ell_{4(m-1)} \right) = 2 \sum_{j=0}^{m-1} \ell_2 \ell_{4j}.
\]

In particular, the above equality tells us that the value of \( \nu_2(i) \) is only non-zero when \( i \equiv 2 \mod 4 \).

Let \( \xi_k = e^{\frac{2\pi ik}{m}} \). Thus we have

\[
\langle \chi_k, \nu_2 \rangle = \frac{1}{|G_o|} \sum_{g \in G_o} \chi_k(g) \nu_2(g) \\
= \frac{1}{n} \left( \chi_k(0) \nu_2(0) + \chi_k(2) \nu_2(2) + \chi_k(4) \nu_2(4) + \ldots + \chi_k(2n-2) \nu_2(2n-2) \right) \\
= \frac{1}{n} \left( 1(0) + e^{\frac{2\pi ik}{n}}(2) + e^{\frac{4\pi ik}{n}}(0) + \ldots + e^{\frac{2(n-1)\pi ik}{n}}(2) \right) \\
= \frac{1}{m} \left( e^{\frac{2\pi ik}{m}} + e^{\frac{4\pi ik}{m}} + \ldots + e^{\frac{2(n-1)\pi ik}{m}} \right) \\
= \frac{1}{m} \sum_{j=0}^{m-1} \xi_k^{2j} + 1 \\
= \frac{\xi_k}{m} \sum_{j=0}^{m-1} (\xi_k^2)^j \\
= \begin{cases} 
1 & k = n \\
-1 & k = m \\
\frac{\xi_k}{m} \frac{1 - (\xi_k^2)^m}{1 - \xi_k} & \text{else}
\end{cases}
\]

Now suppose \( n \) is odd. Therefore

\[
\nu_2 = \ell_2 \left( \ell_0 + \ell_4 + \ldots + \ell_{2n-2} + \ell_2 + \ldots + \ell_{2n-4} \right) = \sum_{j=0}^{n-1} \ell_2 \ell_j \\
\]

In particular, the above equality tells us that \( \nu_2 \) hits every element in \( G_o \), so is non-zero for all
\( i \in \{0, 2, 4, \ldots, 2n - 2\} \). Let \( \xi_k = e^{\frac{2\pi i}{n}} \). Thus we have

\[
\langle X_k, \nu_2 \rangle = \frac{1}{|G_o|} \sum_{g \in G_o} X_V(g) \nu_2(g)
\]

\[
= \frac{1}{n} \left( X_k(0) \nu_2(0) + X_k(2) \nu_2(2) + X_k(4) \nu_2(4) + \ldots + X_k(2n - 2) \nu_2(2n - 2) \right)
\]

\[
= \frac{1}{n} \left( 1(1) + e^{\frac{2\pi i}{n}}(1) + e^{\frac{4\pi i}{n}}(1) + \ldots + e^{\frac{2(n-1)\pi i}{n}}(1) \right)
\]

\[
= \frac{1}{n} \sum_{j=0}^{n-1} \xi_k^j
\]

\[
= \begin{cases} 
1 & k = n \\
\frac{1}{n} \frac{1 - e^{n\pi i}}{1 - e^{2\pi i}} = 0 & \text{else}.
\end{cases}
\]

Writing our results together, we find

\[
\langle X_k, \nu_2 \rangle = \begin{cases} 
1 & k = n \\
-1 & n \text{ is even and } k = \frac{n}{2} \\
0 & \text{else}.
\end{cases}
\]

In the case where the indicator is 1, we have the trivial representation of \( G_o = \mathbb{Z}_n \) which extends to the trivial representation of \( \mathbb{Z}_{2n} \). In the case where the indicator is \(-1\), we know \( n \) is even and \( k = \frac{n}{2} \), so we have \( e^{\frac{2\pi i}{n}} = e^{\pi i} = -1 \). Thus \( V \) is the representation of \( G_o \simeq \mathbb{Z}_n \) given by

\[
\rho_k : G_o \rightarrow \mathbb{C}
\]

\[
2 \mapsto -1.
\]

We also know by Theorem 1 that this can be extended to a Real \( \alpha \)-projective representation of \( G \). With the choice of basis \( \{1\} \) for \( \mathbb{C} \), we have a \( G_o \)-equivariant map \( \phi : \mathbb{C}^\vee \rightarrow \mathbb{C} \) given by \( 1^\vee \mapsto 1 \). Therefore, the extension of this representation \( \rho : G \rightarrow \mathbb{C} \) is an \( \alpha \)-projective representation given by

\[
\begin{array}{cccccccc}
g & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & 2(n-1) & 2n - 1 \\
\hline
\rho_k(g) & 1 & 1 & -1 & -1 & 1 & 1 & \cdots & (-1)^{n-1} & (-1)^{n-1}
\end{array}
\]

with projective multiplier \( \alpha \) that takes value \(-1\) when both elements are in \( \mathbb{Z}_{2n} \setminus G_o \).
Let $G = D_{2n}$, the dihedral group on $n$ elements. Notationally, we write

$$D_{2n} = \langle r, s \mid r^n = s^2 = e, srs = r^{-1} \rangle$$

and note that $sr^k$ is a reflection with order 2 for each $k \in \{1, \ldots, n\}$. Let $\pi : G \rightarrow \mathbb{Z}_2$ be defined by $sr^k \mapsto -1$ for all $k$ such that $1 \leq k \leq n$. Therefore $G_o := \ker(\pi) = \{e, r, r^2, \ldots, r^{n-1}\} \cong \mathbb{Z}_n$. Fix some $\omega \in G \setminus G_o$. Note that for each $g \in G$, the element $g\omega$ is a reflection given by $sr^k$ for some $k \in \{1, \ldots, n\}$. Therefore we have

$$\nu_2 = \sum_{g \in G_o} \ell_{(g\omega)^2} = n \ell_e.$$

Let $V$ be a finite dimensional, irreducible, complex representation of $G_o$. Thus we compute

$$\langle \chi_V, \nu_2 \rangle = \frac{1}{|G_o|} \sum_{g \in G_o} \chi_V(g)\nu_2(g)$$

$$= \frac{1}{n} \left( \chi_k(e)\nu_2(e) + \chi_k(r)\nu_2(r) + \chi_k(r^2)\nu_2(r^2) + \ldots + \chi_k(r^{n-1})\nu_2(r^{n-1}) \right)$$

$$= \frac{1}{n} \left( \ell(n) + \chi_V(r)(0) + \ldots + \chi_V(r^{n-1})(0) \right)$$

$$= 1.$$

Thus every irreducible representation of $G_o \cong \mathbb{Z}_n$ can be extended to a Real representation of $D_{2n}$. This follows the results of [1].
REFERENCES


APPENDIX

Identities for Projective Multipliers.

Let \( \pi : G \to \mathbb{Z}_2 \) be a surjective group homomorphism and let \( G_\theta := \ker(\pi) \) We see in section 4.3 that \( \theta \in Z^{2+\pi}(G;U(1)) \) is equivalent to the identity

\[
\theta(g_3, g_2) \theta(g_2, g_1) = \theta(g_3, g_1)^{\pi(g_3)} \theta(g_3, g_2 g_1),
\]

for \( g_1, g_2, g_3 \in G \). Note that we assume \( \theta \) is normalized, so for \( g \in G \) and the identity \( e \in G \) we have \( \theta(e, g) = \theta(g, e) = 1 \). We also define two quantities that simplify certain computations. Let \( h \in G \) and define \( \lambda_\theta : G \to U(1) \) and \( \mu_{\theta, h} : G \to U(1) \) by

\[
\lambda_\theta(g) = \theta(g, g^{-1}) \quad \text{and} \quad \mu_{\theta, h}(g) = \frac{\theta(g, h)}{\theta(h, h^{-1}g)h}
\]

These two functions are elements in \( C^1(G;U(1)) \) from the chain complex defined in section 3.2. In particular, we can apply the boundary mapping \( d : C^1 \to C^2 \) given by

\[
(d\lambda_\theta)(g, h) = \frac{\lambda_\theta(g)\lambda_\theta(h)}{\lambda_\theta(gh)}
\]

and the analogous version for \( \mu_{\theta, h} \). We note that for all \( g \in G_\theta \) we have \( \lambda_\theta(g) = \lambda_\theta(g^{-1}) \). We also note that \( \mu_{\theta, h}(h) = 1 \) and if \( h \in G_\theta \) then \( \mu_{\theta, h}(h^{-1}) = 1 \), but if \( h \in G \setminus G_\theta \) then we have \( \mu_{\theta, h}(h^{-1}) = \theta^2(h^{-1}, h) \).

We now list seven identities that will be used in the main body of the paper.

**Lemma A.0.1.** Let \( \theta \in Z^{2+\pi}(G;U(1)) \). Let \( g, h, \omega \in G \). Let \( \omega, \omega \in G \). Then we have the following identities:

\[
\begin{align*}
\theta(h^{-1}, g^{-1}) &= \theta^{-1}(g, h)(d\lambda_\theta)(g, h) \quad \text{(A.1)} \\
\theta(x^{-1}y x, x^{-1}h x) &= \theta(y, h)^{\pi(x)}(\mu_{\theta, x}(y) \cdot \mu_{\theta^{-1, x}}(y) \cdot \mu_{\theta, x}(h))^{-\pi(x)} \quad \text{(A.2)} \\
\theta^2(\omega, \omega)\mu_{\theta, \omega}(\omega^2) &= 1 \quad \text{(A.3)} \\
\frac{\theta(\omega g \omega^{-1}, \omega^2)}{\theta(\omega^2, \omega^{-1} g \omega)} &= \mu_{\theta, \omega}(\omega g \omega^{-1}) \quad \text{(A.4)} \\
\theta(\omega h \omega^{-1}, \omega h \omega^{-1})\theta(h, h^{-1}) &= \theta(\omega, \omega)\theta(h, \omega^2)\theta(h \omega^{-1}, h^{-1}) \quad \text{(A.5)} \\
\theta^{-1}(\omega, g) &= \theta^{-1}(\omega g \omega^{-1}, \omega)\mu_{\theta, \omega}(\omega g \omega^{-1}) \quad \text{(A.6)} \\
\frac{\theta(x g x^{-1}, x)}{\theta(x, g)} &= \frac{\theta(x^{-1}, x)}{\theta(x, g^{-1})} \theta(g, x^{-1})^{-\pi(x)} \quad \text{(A.7)}
\end{align*}
\]

**Proof.** Here we list the computations, referring to equations that will be presented after the proof, but each equality follows from the associativity condition on \( Z^{2+\pi}(G;U(1)) \). Let \( g, h \in G_\theta \). Then
we have

\[
\theta(h^{-1}, g^{-1}) = \frac{\theta(h, h^{-1})}{\theta(h, h^{-1} g^{-1})}
\]

by equation A.8

\[
= \frac{\theta(g, g^{-1}) \theta(h, h^{-1})}{\theta(g, h) \theta(gh, h^{-1} g^{-1})}
\]

by equation A.9

\[
= \theta^{-1}(g, h) \cdot \frac{\theta(g, g^{-1}) \theta(h, h^{-1})}{\theta(gh, (gh)^{-1})}
\]

\[
= \theta^{-1}(g, h)(d\lambda_0)(g, h).
\]

Now suppose \( h \in G_\omega \) and \( x, y \in G \). Then we have

\[
\theta(x^{-1}yx, x^{-1}hx)
\]

\[
= \left( \frac{\theta(x, x^{-1}yx) \theta(yx, x^{-1}hx)}{\theta(x, x^{-1}hx)} \right)^{\pi(x)}
\]

by equation A.10

\[
= \left( \frac{\theta(x, x^{-1}yx)}{\theta(x, x^{-1}hx)} \cdot \frac{\theta(yx, x^{-1}hx)}{\theta(y, x)} \cdot \frac{\theta(\pi(x))}{\theta(\pi(y))} \right)^{\pi(x)}
\]

by equation A.11

\[
= \left( \frac{\theta(y, h) \cdot \theta(x, x^{-1}yx)}{\theta(y, x)} \cdot \left( \frac{\theta(x, x^{-1}hx)}{\theta(h, x)} \right)^{\pi(x)} \cdot \frac{\theta(yh, x)}{\theta(x, x^{-1}hx)} \right)^{\pi(x)}
\]

by equation A.12

\[
= \theta(y, h)^{\pi(x)} \left( \mu_{\theta, x}(y) \cdot \mu_{\theta, x}^{-1}(y) \cdot \mu_{\theta, x}(yh) \right)^{\pi(x)}
\]

\[
= \theta(y, h)^{\pi(x)} \left( \mu_{\theta, x}(y) \cdot \mu_{\theta, x}^{-1}(y) \cdot \mu_{\theta, x}(yh) \right)^{-\pi(x)}.
\]

For fixed \( \omega \in G \setminus G_\omega \) we have

\[
\theta(\omega, \omega)^2 \mu_{\theta, \omega}(\omega^2) = \theta(\omega, \omega)^2 \cdot \frac{\theta(\omega^2, \omega)}{\theta(\omega, \omega^2)}
\]

\[
= \theta(\omega, \omega)^2 \cdot \frac{1}{\theta(\omega, \omega^2)}
\]

by equation A.13

\[
= 1.
\]

Next we have

\[
\theta(\omega^2, \omega^{-1} g\omega) \mu_{\theta, \omega}^{-1}(g) \mu_{\theta, \omega}(\omega g\omega^{-1})
\]

\[
= \theta(\omega^2, \omega^{-1} g\omega) \theta(\omega, \omega^{-1} g\omega) \theta(\omega g\omega^{-1}, \omega)
\]

\[
= \theta(g, \omega) \theta(\omega, g) \cdot \theta(\omega, \omega) \theta(\omega g\omega^{-1}, \omega)
\]

by equation A.14

\[
= \theta(\omega g\omega^{-1}, \omega) \cdot \frac{\theta(\omega, \omega^{-1} g\omega)}{\theta(g, \omega) \theta(\omega, g)} \cdot \frac{\theta(\omega, \omega) \theta(\omega g\omega^{-1}, \omega)}{\theta(\omega, \omega^{-1} g\omega)}
\]

by equation A.15

\[
= \theta(\omega g\omega^{-1}, \omega^2) \cdot \frac{1}{\theta(g, \omega) \theta(\omega, g)} \cdot \frac{\theta(\omega, g) \theta(\omega g\omega^{-1}, \omega)}{1}
\]

by equation A.16

\[
= \theta(\omega g\omega^{-1}, \omega^2).
\]
Next we have
\[
\theta(h\omega^{-1}, h\omega^{-1})\theta(h, h^{-1})
\]
\[
= \frac{\theta(h\omega^{-1})}{\theta(h, h^{-1})}\theta(h^{-1}, h\omega^{-1})\theta(h^{-1}, h^{-1})
\]
by equation A.17
\[
= \frac{\theta(h\omega, \omega)\theta(\omega, h^{-1})\theta(h\omega^{2}, h^{-1})}{\theta(h\omega, h^{-1})}\theta(h^{-1}, h\omega^{-1})\theta(h, h^{-1})
\]
by equation A.18
\[
= \frac{\theta(\omega, \omega)\theta(h, h^{-1})\theta(h\omega^{2}, h^{-1})}{\theta(h, h^{-1})}\theta(h\omega^{-1}, h\omega^{-1})\theta(h\omega^{-1}, h^{-1})\theta(h, h^{-1})
\]
by equation A.19
\[
= \frac{\theta(\omega, \omega)\theta(h, h^{-1})\theta(h\omega^{2}, h^{-1})}{\theta(h, h^{-1})}\theta(h\omega^{-1}, h\omega^{-1})\theta(h, h^{-1})
\]
by equation A.20
\[
= \frac{\theta(\omega, \omega)\theta(h, \omega^{2})\theta(h\omega^{2}, h^{-1})}{\theta(h, h^{-1})}\theta(h\omega^{-1}, h\omega^{-1})\theta(h, h^{-1})
\]
by equation A.21
\[
= \frac{\theta(\omega, \omega)\theta(h, \omega^{2})\theta(h\omega^{2}, h^{-1})}{\theta(h, h^{-1})}\theta(h\omega^{-1}, h\omega^{-1})\theta(h, h^{-1})
\]
by equation A.22
\[
= \frac{\theta(\omega, \omega)\theta(h, \omega^{2})\theta(h\omega^{2}, h^{-1})}{\theta(h, h^{-1})}\theta(h\omega^{-1}, h\omega^{-1})\theta(h, h^{-1})
\]
by equation A.23

The following equations were used throughout the proof above, but each equation is simply writing down the associativity condition on $Z^{2+\pi}(G; U(1))$ with the particular elements that are
given. Let $g, h \in G$, let $\omega \in G \setminus G$, and let $x, y \in G$. Then we have

\[ \theta(h, h^{-1}) = \theta(h^{-1}, g^{-1})\theta(h, h^{-1}g^{-1}) \]  \hfil (A.8)
\[ \theta(h, h^{-1}g^{-1})\theta(g, g^{-1}) = \theta(g, h)\theta(gh, h^{-1}g^{-1}) \]  \hfil (A.9)
\[ \theta(x, x^{-1}yx)\theta(yx, x^{-1}hx) = \theta(x^{-1}yx, x^{-1}hx)\pi(x)\theta(x, x^{-1}yhx) \]  \hfil (A.10)
\[ \theta(y, x)\theta(yx, x^{-1}hx) = \theta(x, x^{-1}hx)\pi(y)\theta(y, hx) \]  \hfil (A.11)
\[ \theta(y, h)\theta(yh, x) = \theta(h, x^{-1}hx)\pi(y)\theta(y, hx) \]  \hfil (A.12)
\[ \theta(\omega, \omega)\theta(\omega^2, \omega) = \theta(\omega, \omega^{-1})\theta(\omega, \omega^2) \]  \hfil (A.13)
\[ \theta(\omega g\omega^{-1}, \omega g\omega) = \theta(\omega, \omega)\theta(\omega g\omega^{-1}, \omega^2) \]  \hfil (A.14)
\[ \theta(\omega, \omega)\theta(\omega^2, \omega^{-1}g\omega) = \theta(\omega, \omega^{-1}g\omega^{-1})\theta(\omega, g\omega) \]  \hfil (A.15)
\[ \theta(\omega, g)\theta(\omega g, \omega) = \theta(g, \omega^{-1})\theta(\omega, g\omega) \]  \hfil (A.16)
\[ \theta(h\omega, h^{-1})\theta(h\omega h^{-1}, h\omega h^{-1}) = \theta(h^{-1}, h\omega h^{-1})\theta(h\omega, \omega h^{-1}) \]  \hfil (A.17)
\[ \theta(h\omega, \omega)\theta(h\omega^2, h^{-1}) = \theta(\omega, h^{-1})\theta(h\omega, \omega h^{-1}) \]  \hfil (A.18)
\[ \theta(h, \omega)\theta(h\omega, \omega) = \theta(\omega, \omega)\theta(h, \omega^2) \]  \hfil (A.19)
\[ \theta(h, \omega)\theta(h\omega, h^{-1}) = \theta(\omega, h^{-1})\theta(h, \omega h^{-1}) \]  \hfil (A.20)
\[ \theta(h^{-1}, h) = \theta(h, \omega h^{-1})\theta(h^{-1}, h\omega h^{-1}) \]  \hfil (A.21)
\[ \theta(xg, x^{-1})\theta(xgx^{-1}, x) = \theta(x^{-1}, x)\pi(x) \]  \hfil (A.22)
\[ \theta(x, g)\theta(xg, x^{-1}) = \theta(g, x^{-1})\pi(x)\theta(x, gx^{-1}) \]  \hfil (A.23)