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A Classification of Tensors in ECSK Theory

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A CLASSIFICATION OF TENSORS IN ECSK THEORY

by

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ABSTRACT

A Classification of Tensors in ECSK Theory

by

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This dissertation presents a Petrov/Plebanski/Segre/Algebraic (PPSA) classification scheme for both the curvature and torsion tensors in Einstein-Cartan-Sciama-Kibble theory (ECSK); additionally the work includes a software package in Maple with computational and PPSA classification tools. Six different solutions are classified using the new software tools developed. We also present a new boundary term for ECSK-NMC scalar theory to ensure Dirichlet boundary conditions. New work completed includes the decomposition of an arbitrary 4th rank tensor under $SO(p,q)$, equivalent $SL(2,\mathbb{C})$ irreducible spinor decompositions of arbitrary 3rd and 4th rank tensors. We provide new proofs that the corresponding spinors are the irreducible components in the $SL(2,\mathbb{C})$ decomposition of 3rd and 4th rank tensors using Maple. Additionally, we provide an algebraic decomposition of the torsion spinors, an algorithm determining their algebraic decomposition, an ECSK-PPSA classification for the curvature tensor, and finally the Gibbons-Hawking-York (GHY) - ECSK - Non-Minimally-Coupled (NMC) scalar field boundary term such that the metric variation vanishes on the boundary without any need to constrain the normal derivatives of the variation of the metric.

(292 pages)

PUBLIC ABSTRACT

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Joshua James Leiter

You might have heard of Einstein's theory of General relativity (GR): it is the one where mass and energy curve the fabric of spacetime to create gravity. This is the major theory which allows communication through satellites and our GPS to work too! Wormholes have interested me, but there are some issues about forming them in GR. Interestingly enough, elementary particles are also characterized by their spin in the standard model. However, intrinsic spin is nowhere geometrically coupled to the geometry of spacetime in Einstein's theory. Later, Élie Cartan, Dennis Sciama, and Tom Kibble all flushed out adding different aspects of Spin into GR making a new theory called Einstein-Cartan-Sciama-Kibble (ECSK) theory where spin is linked to the torsion tensor of Cartan. This addition of spin according to several articles allows for wormholes without any invocation of exotic matter (negative mass). There's the background! This dissertation breaks apart ECSK theory into observable through the use of the Lorentz group, encompassing time dilation and rotations. The consequences are that we can find new physics through the use of these tools which correspond to structures in spacetime. Then by forming combinations of these objects (think x^2) we can further analyze the geometrical structures and get a handle on what is happening physically! Computer tools in the Maple software package have been developed to expedite calculation on several ECSK problems. Together these tools form an ECSK toolkit which corresponds to the ideas used by Petrov, Plebanski, Segre, and Penrose (PPSP) to classify structures in GR.

To my love for wormholes and all the limitless possibilities brought forth.
To my overflowing heart as a gift to the world from its depths.
To all the great adventures that blossom in my soul.

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Joshua James Leiter

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ACRONYMS

ECSK	Einstein-Cartan-Sciama-Kibble Theory
EHCB	Einstein Hilbert Contorsion Boundary
GHY	Gibbons-Hawking-York
GR	General Relativity
MC	Minimally Coupled
NMC	Non-Minimally Coupled
PPSA	Petrov, Plebanski, Segre, Algebraic
SR	Special Relativity
NP	Newman-Penrose

NOTATION

Tensors

Contorsion Contortion[1], the tensor: $C^a{}_{bc}$.

Variation

δ_g a variation with respect to the field g.

E_ψ the field equation for ψ.

$\Theta^a\{\delta\psi\}$ the boundary term related to ψ.

Invariants and Covariants

co(in)variant Covariant or Invariant.

[1](Contortion & Contorsion: Both of these terms are used in the literature. Hehl (F. Hehl et al., 1976) would call this object the Contortion)

CHAPTER 1

INTRODUCTION

This dissertation presents a Petrov/Plebanski/Segre/Algebraic (PPSA) classification scheme for the curvature and torsion tensors in Einstein-Cartan-Sciama-Kibble theory (ECSK); additionally, it includes a related software package in Maple with computational and PPSA classification tools. Six different solutions given in references (Chen et al., 2018), (Platania & Rosania, 1997), and (Bronnikov & Galiakhmetov, 2015) are classified using the new software tools developed. We also determine boundary terms for the Einstein-Cartan-Sciama-Kibble, and Non-Minimally Coupled scalar field action, which we call ECSK-NMC, such that the boundary conditions needed to get the field equations are Dirichlet conditions; the normal derivative of the metric at the boundary is unconstrained. We decompose an arbitrary 4th rank tensor under $SO\left(p,q\right)$. Then we for the $SL\left(2,\mathbb{C}\right)$ irreducible spinor decompositions for arbitrary 3rd and 4th rank tensors. We provide new proofs that show these spinors are the irreducible counterparts using Maple. After that, we develop an algebraic decomposition of the torsion spinors, and an algorithm to determine their algebraic decomposition. Moving forward we provide an ECSK-PPSA classification for the curvature tensor. Finally, we present a new GHY boundary term for ECSK-NMC such that the metric variation vanishes on the boundary without any need to constrain the normal derivatives of the variation of the metric.

1.1 The History of ECSK Theory

ECSK theory got its name from Albert Einstein, Élie Cartan, Dennis Sciama, and Thomas Kibble. The gravity theory includes Einstein's name because he originally developed the theory of General Relativity (GR). Many of the mathematical parts of GR come from differential geometry. In GR, the gravitational field is described by the curvature of spacetime; spacetime curves because of the matter and energy in it. However, in the development of

GR, the torsion tensor which naturally occurs in differential geometry was left out.

In 1922 Cartan (Cartan, 1922) began working on a generalization of Riemannian geometry which included the torsion tensor. In 1923 Cartan noticed (Cartan, 1923) that GR could be generalized with the addition of torsion. This was the birth of ECSK theory; Cartan is the "C" in the theory.

Cartan theorized torsion would couple to the same spin that was just discovered in the Stern-Gerlach experiment; it is interesting as a historical note that Cartan's paper generalizing Riemannian geometry (Cartan, 1922) was published the same year, just a bit after the Stern-Gerlach experiment took place. By including torsion in spacetime, the geometry is augmented even further, leading to new physical phenomena. Einstein did not add torsion into GR in 1922, but over the next three years Cartan further developed the theory, (Cartan, 1924), (Cartan, 1925). It was not until 1928, when Einstein tried to match torsion with electromagnetism, that he became affiliated with the theory; Einstein's name is represented by the "E". The first one to view gravity as a gauge theory was Uityama (Utiyama, 1956) in 1956. In the 1960s, ECSK theory took off. Sciama (Sciama, 1962) and Kibble (Kibble, 1961) were the revisers of ECSK theory with both of their papers; they were the "S" and "K" respectively. Then came the great compilation review paper by Hehl (F. Hehl et al., 1976) in 1976 which tied the theory together in a beautiful physical package. Most recently, Hehl (F. W. Hehl, 2023) has presented ECSK theory from a Poincaré gauge theory viewpoint.

1.2 Significance of this Work

The fundamental part of ECSK that is different from General Relativity (GR) is that ECSK naturally couples spin to the geometry through torsion. In the standard model of particle physics, particles are characterized by their mass and spin. Mass/energy generated the gravitational field in GR, and spin generates a related contorsion gravitational field in ECSK. It is natural to consider ECSK as an extension to GR to include spin effects. As an aside, these ECSK tools will help with the development of a wormhole. Stated in (Mehdizadeh & Ziaie, 2017), wormholes without exotic matter (think negative energy density) are possible in ECSK theory; torsion allows this to happen. Wormholes would be incredibly useful for space exploration, and thus are of physical importance; not to mention mathematically beautiful.

There are not many experimental results available which constrain torsion, and to the author's knowledge, only one of the three possible irreducible representations has an experimental constraint. The axial torsion has been constrained by Lämmerzahl (Lämmerzahl, 1997) to:

$$K_z \leq 1.5 \times 10^{-15}\ m^{-1}$$

$$K_{(0)} \leq 10^{-2}\ m^{-1}$$

where K_z is the z-component of the axial torsion vector in 4 dimensions, and $K_{(0)}$ is the time component of the axial torsion vector in 4 dimensions.

1.3 Mission Statement

One of the difficulties in GR is the coordinate freedom that solutions to the field equations have; it is difficult to tell if two solutions are equivalent up to coordinate transformations just by looking at them. The Cartan-Karlhede algorithm (Karlhede, 1980) is one method used to distinguish these solutions. To run the algorithm, however, it is important to have scalars which provide a unique local characterization (See Stephani (Stephani et al., 2003) Pg. 116). These same scalars are used in the Petrov and Segre classifications of spacetime and are intimately related to them. There is currently no corresponding classification of solutions, or equivalence method yet for ECSK Theory; this is mainly due to the inclusion of the torsion tensor. This dissertation aims to provide several tools for classifications in ECSK theory. We examine and classify several known solutions in chapters 6, 7, and 8.

1.4 Survey of the Dissertation

In this section we give an overview and roadmap of the entire dissertation. We discuss each chapter and its contents with hopefully enough detail to direct the reader to sections they will find most interesting for further study. We begin by stating the classification task we wish to accomplish, and then walk the reader through the relevant chapters relating to this task.

In our task to classify the curvature and torsion tensors in ECSK theory, we begin by first finding irreducible representations of arbitrary second, third, and fourth rank tensors under $GL(N)$. We then proceed to decompose these representations further under the subgroup $SO(p,q)$ and find the irreducible decomposition of these tensors under this group. For our classification of the curvature and torsion tensors, we use $SL(2,\mathbb{C})$ spinors; for this, we also specialize to four dimensions with signature $[+,-,-,-]$.

Once we have all the $SL(2,\mathbb{C})$ irreducible spinors which correspond to second, third, and fourth rank tensors, we specialize to the cases of the curvature and torsion tensors. Using the tools we developed in the general case, we find the $SO(3,1)$ irreducible tensors which compose both the curvature and torsion tensors. We then turn these tensors into spinors. There are three spinors which completely determine the torsion tensor which we call Θ, Ξ, and Ω. Likewise, there are six spinors which completely determine the curvature tensor. We call these spinors Ψ, Φ, Λ, Ж, Ю, and $\aleph$.

We classify the curvature and torsion tensors by classifying these nine spinors. For the curvature spinors we use Petrov-Plebanski-Segre-Algebraic (PPSA) tools, and for the torsion we use analogous algebraic tools. In both cases, we provide a list of invariants and covariants to determine the PPSA type. We use the word covariant here in the same fashion as Olver (Olver, 2003). In this case, a covariant is similar to an invariant other than the fact that it transforms tensorially.

The three preliminary chapters of this work discuss rank 2, rank 3, and rank 4 tensors and how to decompose them under $GL(N)$, and $SO(p,q)$. Additionally, these chapters analyze the $SL(2,\mathbb{C})$ irreducible spinors which correspond to $SO(3,1)$ irreducible tensors; there is a 1 to 1 relationship between these irreducible spinors and tensors. In our work, we discuss this correspondence in signature $[+,-,-,-]$. Finally, in the last sections of these chapters PPSA classification tools are discussed with regard to the curvature and torsion tensors.

For the later chapters, chapter 5 and onward, we employ variational calculus heavily and keep the boundary terms the whole way through in the ECSK-Non-Minimally-Coupled (NMC) scalar field case. The terms we modify the ECSK-NMC Scalar field action guarantee that the boundary conditions are Dirichlet. For the Dirac field, and the spinning fluid/generalized ECSK theory chapters, we do not include all the boundary terms. This is an interesting direction for future work. If this route of research is pursued, we encourage the researcher to work with the vielbein $e_a{}^{\mu}$ and spin connection $\omega^{\mu}{}_{\nu a}$ as their field variables.

Chapter 2 is the rank 2 tensor decomposition section and walks through the Young, $SO\,(p,q)$, and $SL\,(2,\mathbb{C})$ decompositions. Chapters 3 and 4 apply the same decompositions to 3rd and 4th rank tensors as well. We use this correspondence to turn the tensors into spinors because spinors are easier to classify and may reduce further under epsilon spinor traces than their corresponding tensors would. This is due to how, when irreducible, they are always symmetric in their unprimed and primed indices respectively. Stewart (Stewart, 1993) states that every $SL\,(2,\mathbb{C})$ irreducible spinor is totally symmetric in its indices; we use this fact throughout our work. Another direction for future work would be to accomplish this classification in N dimensions. The work of Brauer and Weyl (Brauer & Weyl, 1935) may be of interest in this regard.

Chapter 3 presents a classification of the torsion tensor in terms of the algebraic reducibility of the $SL\,(2,\mathbb{C})$ irreducible torsion spinors which we call Θ, Ξ, and Ω. We present a new algorithm to classify the torsion tensor's structural reducibility, usually called algebraic irreducibility in the literature. The algorithm calculates several co(in)variants for this purpose and is found at the end of chapter 3 in figures (3.1) and (3.2).

Chapter 4 presents decompositions of an arbitrary rank 4 tensor under $GL\,(N)$, $SO\,(p,q)$, and $SL\,(2,\mathbb{C})$. Many of the explicit formulas we use and derive here can be found in the appendix due to there being 25 $SO\,(p,q)$ irreducible subspaces for rank 4 tensors. Additionally, later in the chapter we present the PPSA classification of the curvature tensor in terms of the $SL\,(2,\mathbb{C})$ irreducible curvature spinors which we call Ψ, Φ, Λ, Ж, Ю, and $\aleph$.

Each of the above classifications examines a tensor in a different $SO(p, q)$ invariant subspace of the space of curvature tensor. To make an analogy, decomposing the curvature tensor, in GR, into irreducible components in these subspaces is called the Ricci decomposition, i.e. $Riemann \cong Weyl + Trace\ Free\ Ricci + Ricci\ Scalar$. The Weyl tensor is classified via its Petrov classification. The trace-free Ricci tensor is classified through the Plebanski and Segre Classifications. The Ricci Scalar is classified at a point by whether it is positive, negative, or zero; we choose this classification because it holds in an open set around any point chosen, with a small caveat for exceptional points which may change type.

In ECSK there are three more $SO(p, q)$ irreducible representations appearing in the curvature tensor than in GR because of the addition of the torsion tensor. We present the PPSA classification of each of the six different tensors resulting from the $SO(p, q)$ decomposition of the curvature tensor. All the additional irreducible elements of the curvature tensor not found in GR, but found in ECSK, can be classified using the PPSA like methods presented. However, determining these elements of the curvature tensor in the first place is non-trivial; it also requires heavy representation theory and tools from computer algebra to not be cumbersome.

The idea for the PPSA classification in ECSK theory is to use the spinor methods along with Young tableaux in references (Penrose & Rindler, 1987a), (Stewart, 1993), (J. T. Wheeler, unpublished), (Penrose & Rindler, 1987b) to complete the classification of the curvature and torsion tensors. Maple procedures have been implemented to work with the *DifferentialGeometry* package described in (Anderson & Torre, 2012). For the mathematical background on the Petrov classification and Segre classification, there are good explanations given by Penrose and Rindler in (Penrose & Rindler, 1987a), (Penrose & Rindler, 1987b). The Petrov classification asks about the multiplicity of eigenvalues of the Weyl tensor (from GR) acting on the space of bivectors; this can be reformatted as finding the multiplicities of the eigenvalues/eigenvectors of the totally symmetric Weyl Spinor (which is a spinor irreducible representation of the Weyl part of the curvature tensor) acting on the space of rank 2 symmetric spinors. The Segre classification asks for the Jordan normal form of the trace-free Ricci tensor. The Petrov and Segre classifications have been implemented in Maple for GR by (Anderson & Torre, 2012).

Chapter 5 presents a background on the ECSK field equations, and defines several new tensors in analogy to GR, most notably we define a "Cartan" tensor labeled by $\mathscr{C}^{a}\ bc$ which is analogous to the Einstein tensor but for the second field equation arising in ECSK; just as the Einstein tensor is the left-hand side to the metric g_{ab} field equations, the Cartan tensor is the left-hand side to the contorsion/spin-connection field equations $C^{a}{}_{bc}/\omega^{\mu}{}_{\nu a}$.

Chapter 6 walks through ECSK-NMC scalar field theory through a comparison to Bronnikov and Galiakhmetov's (Bronnikov & Galiakhmetov, 2015) work. We present an additional Gibbons-Hawking-York (GHY)-like boundary term in the action which prevents the variational principle from over-constraining the metric variation on the boundary, and we classify Bronnikov and Galiakhmetov's solution according to the classification tools presented in chapters 3 and 4.

Chapter 7 presents a walkthrough on ECSK-Dirac theory, at the end of which we classify a solution presented by Platania and Rosania (Platania & Rosania, 1997). Chapter 8 gives a short note on spinning fluids, as they generate the last irreducible representation of the torsion tensor not covered by the NMC or Dirac fields.

Chapter 8 presents Chen, Zhang, and Jing's (Chen et al., 2018) extension to ECSK (cubic torsion action) of which we classify three of the solutions they present. As a note here, our algorithm does not provide enough information to determine that these solutions are distinct; to create a Cartan-Karlhede (Karlhede, 1980) algorithm for ECSK we would need more information; this is an interesting and fruitful direction for future work.

Finally, in chapter 9 we end with a short conclusion of all the new work we have accomplished.

There are several appendices. The first of these is appendix A which compares the decomposition of the torsion tensor we present to that in the literature; see Shapiro (Shapiro, 2001) for an example. The next of these, appendix B) formulas for the $GL(N)$ decomposition of an arbitrary rank 4 tensor; these were developed using Young tableaux tools. We continue this in appendix C where we give the formulas for the $SO\left(p,q\right)$ decomposition of an arbitrary rank 4 tensor. Appendix D presents proofs of the $SO\left(p,q\right)$ decomposition of a 4th rank tensor. Appendix E presents the formulas for the number of independent components/degrees of freedom for $SO\left(p,q\right)$ irreducible rank 4 tensors and their corresponding $SL\left(2,\mathbb{C}\right)$ irreducible spinors when $p=3,q=1$. Appendix F presents formulas for all 25 $SL\left(2,\mathbb{C}\right)$ irreducible spinors in terms of their $SO\left(3,1\right)$ irreducible rank 4 tensors. Finally, appendix G is our last appendix and discusses the Belinfante-Rosenfeld relation. This relation is incredibly important for relating ECSK theories with differing field variables. Finally, there are several Maple worksheets and modules which will be available digitally. The worksheets include: Rank 3 Spinor Proofs, Rank 4 Spinor Proofs, Ricci Canonical Forms, Platania-Rosania Classification, Chen Classification, Bronnikov Classification, and Alpha(ABC)A'. The last of these is an explanation on the classification of the torsion Ω spinor.

The modules include: ECSKModule, Rank2TensorModule, Rank3TensorModule, Rank4TensorModule, SegreInvariantsModule, and TorsionInvariantsModule. The ECSK module is the fundamental module and also includes several new Maple procedures for calculating the contorsion tensor, a metric compatible torsion-full connection, and the torsion-only part of the ECSK curvature tensor which we call the Alphonse tensor. These are only a few of the new commands, however. There are many others, all of which are sufficiently compatible with the *DifferentialGeometry* package.

CHAPTER 2

RANK 2 TENSOR DECOMPOSITIONS

This section details: how to decompose rank 2 tensors using Young tableaux, how to perform a $SO(p,q)$ decomposition of rank 2 tensors in any dimension, how to find the $SL(2,\mathbb{C})$ irreducible spinors of a rank 2 tensor in dimension 4, and provides Maple procedures to calculate each of these. Although we could give a comprehensive overview of Young tableaux in all generality, we will point the reader to the wonderful maple documentation for tableaux in *DifferentialGeometry* (Anderson & Torre, 2012); additionally, Young tableaux are dissected in (Fulton & Harris, 1991) pages 44-52/62.

To begin, we will discuss Young tableaux in terms of how matrices are usually decomposed. Young tableaux give the irreducible elements of vector spaces under the general linear group $GL(N)$. This is related to symmetric and antisymmetric/skew-symmetric matrices, for they are the two $GL(N)$ irreducible elements of rank 2 tensors under that group. We will then show how the notation used in Young diagrams quickly illustrates this in a computationally and conceptually useful way. We will next show how to count the independent components with Young diagrams; to this end, the idea of the hook length product is introduced. The hook length product will be increasingly important for higher rank tensors.

After that, we will introduce how further specializing the group to $SO(p,q)$, a subgroup of $GL(N)$, refines the number of irreducible elements we are able to construct from two to three. Following this, we will give a few short examples of Maple procedures which will do the same calculation in the Rank2TensorModule module.

Following that we move to spinor calculations, beginning with the same Young calculation but on a spinor; this example shows how spinors further reduce rank, making their classification easier, due to the symplectic form ϵ_{AB}. Recall that ϵ_{AB} is the object left invariant under the group $Sp(2)$; additionally the symplectic group of dimension 2 is isomorphic to the special linear group of dimension 2 over $\mathbb{C}$ i.e., $Sp(2) \cong SL(2,\mathbb{C})$; see Stewart (Stewart, 1993).

After being warmed up by that example, we move to the hermitian spinor, $\theta_{ABA'B'}$ which corresponds to a real rank 2 tensor M_{ab}. Following Penrose and Rindler (Penrose & Rindler, 1987a) we decompose $\theta_{ABA'B'}$ into its irreducible spinor elements and relate the idea back to what we did with $SO\left(p,q\right)$. We summarize the decomposition of a tensor under $SO\left(3,1\right)$ and how it corresponds to irreducible two-component spinor elements in 4 dimensions; the notation $\mathfrak{b}_{ab} \cong \mathfrak{b}_{ABA'B'}$ refers to the relation $\mathfrak{b}_{ab} = \mathfrak{b}_{ABA'B'}\sigma_a{}^{AA'}\sigma_b{}^{BB'}$ and will be used in the same way as Penrose and Rindler, and Stewart (Penrose & Rindler, 1987a), (Stewart, 1993). This notation means that the expressions are the same one we apply the Infeld–Van der Waerden symbols $\sigma_a{}^{AA'}$. We note that $\mathfrak{b}_{ab}$ is technically not equal to $\mathfrak{b}_{ABA'B'}$ but the two are equal once $\sigma_a{}^{AA'}$ is applied. Finally, we illustrate Maple procedures to generate these irreducible spinors given a rank 2 tensor and the Infeld–Van der Waerden symbols.

2.1 Rank 2 Young Tableaux

Let us review some elementary facts about matrices in linear algebra. Recall that we could split a matrix $\boldsymbol{M}$ into its symmetric, $\boldsymbol{M}^S$, and antisymmetric (skew-symmetric), $\boldsymbol{M}^A$, parts by taking combinations of that matrix with its transpose, $\boldsymbol{M}^T$, as shown in the following formulas:

$$\boldsymbol{M} = \boldsymbol{M}^S + \boldsymbol{M}^A \tag{2.1}$$

$$\boldsymbol{M}^S = \frac{1}{2}\left(\boldsymbol{M} + \boldsymbol{M}^T\right) \tag{2.2}$$

$$\boldsymbol{M}^A = \frac{1}{2}\left(\boldsymbol{M} - \boldsymbol{M}^T\right) \tag{2.3}$$

This is a well known fact, but maybe less well known is that it directly corresponds to what happens in Young tableaux. In Young tableaux terminology, we have boxes. It is these boxes which correspond to the symmetry of the indices. Horizontally connected blocks correspond to symmetric indices, and vertical blocks correspond to skew-symmetric indices. If we write

$\boldsymbol{M}^S$ and $\boldsymbol{M}^A$ using the Young blocks, which are called Young diagrams, then we would find that $\boldsymbol{M}^S$ corresponds to the horizontal diagram, and $\boldsymbol{M}^A$ corresponds to the vertical diagram:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \tag{2.4}$$

We express the decomposition given in equation (2.1) as

$$M_{ab} = Y1\left(M\right)_{ab} + Y2\left(M\right)_{ab}, \tag{2.5}$$

Here $Y1\left(M\right)_{ab}$ corresponds to applying the Young projection defined by the box:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

to $\boldsymbol{M}$; this would give us $\boldsymbol{M}^S$ as in equation (2.2). Likewise, $Y2\left(M\right)_{ab}$ corresponds to applying the Young projection:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

to $\boldsymbol{M}$; this would give us $\boldsymbol{M}^A$ as in equation (2.3). Right now, there is no ambiguity in what these mean; later for rank 3 and higher tensors we will find that these projections are not unique as we can either symmetrize first or skew-symmetrize first. Additionally, we can check that the Young symmetrizers are projections by applying the same projector twice:

$$Y1\left(Y1\left(M\right)\right)_{ab} = Y1\left(M\right)_{ab}$$
$$Y2\left(Y2\left(M\right)\right)_{ab} = Y2\left(M\right)_{ab}$$

$$Y1\left(Y2\left(M\right)\right)_{ab} = Y2\left(Y1\left(M\right)\right)_{ab} = 0$$

Let $\mathcal{A}_{ab}$ be the symmetric part of M_{ab} and let $\mathcal{B}_{ab}$ be the skew part of M_{ab}. We will use calligraphic script to designate $GL(N)$ irreducible tensor elements.

$$\mathcal{A}_{ab} = Y1\,(M)_{ab}\,, \quad \mathcal{B}_{ab} = Y2\,(M)_{ab} \tag{2.6}$$

The 1 and 2 inside the boxes correspond to the indices. For example, in

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

this means that the first and second index are symmetric. We always start with 1 in the top left-most corner notationally and proceed increasing the number as we move down and move to the side.

It is also important for us to discuss Ferrers Diagrams. They are a step before Young tableaux, which we get from adding on blocks. For example, we build the 2-dimensional skew and symmetric blocks by starting with 1 dimensional blocks and putting them on the top and bottom as follows in equation (2.7) which we show below.

$$\begin{array}{|c|}\hline \ \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \end{array} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array} \;,\; \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \tag{2.7}$$

We have two Ferrers sectors given by

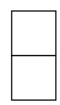
$$\begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array}$$

and

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$$

which we then populate with 1 and 2 which correspond to the indices to make the irreducible Young sectors. The Ferrers diagrams are a useful way to construct all the necessary shapes of the boxes before enumerating the possible Young tableaux. Here the Ferrers diagrams are trivially the same as the Young tableaux, but notice the lack of numbers in the boxes; this will become important later for higher rank tensors.

The Young sectors are independent and they do not change type under a $GL(N)$ transformation i.e., an antisymmetric tensor will not become symmetric under a $GL(N)$ transformation. The factor of a half at the very beginning in our matrix transpose formula comes from the division by the hook length product. There are formulas to describe this in complete generality, as in Fulton and Harris (Fulton & Harris, 1991), but we find it easier to describe the process with some examples. In each box, count that box and then add the number of the boxes to the right of it, and the number of boxes below it. Then do this for each box and multiply the result in each box together. We use the diagram:



as our example. For the above diagram, we would take these steps.

Step 1: Count the top box ($a = 1$) and add the boxes to the left of it (there are none) so ($b = 0$). Then add the boxes below it (there is only 1) so ($c = 1$). Then count the second box ($a = 1$) and add the boxes to the left of it (there are none) so, ($d = 0$). Finally, add the boxes below it (there are none) so ($e = 0$). Do this for each box. To help the reader see where the variables are, we have labeled them.

$$\begin{bmatrix} a+b+c \\ a+d+e \end{bmatrix}$$

Step 2: Here we have filled in the numbers from before.

$$\begin{bmatrix} 1+0+1 \\ 1+0+0 \end{bmatrix}$$

Step 3: Multiply the result in each box together.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow 2 * 1 = 2$$

That is how we get the $\frac{1}{2}$ in the matrix formula above. One over the hook length product, which in this case is 2. Therefore, we could write the formula:

$$HookLengthProduct \left(\begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} \right) = 2$$

Often times the hook length product is denoted μ as in Fulton and Harris (Fulton & Harris, 1991). Again, an excellent description is found in the help files of Anderson and Torre (Anderson & Torre, 2012) in the Young Tableaux section. This calculation of course becomes more difficult and interesting for higher rank tensors.

Similarly, we can also count the number of independent components in each of these tableaux, as we could for the matrix example above. We will use the terms degrees of freedom and independent number of components interchangeably throughout this work. Recall that the formulas yielding the number of independent components for N-dimensional symmetric, and antisymmetric matrices are:

$$number\ of\ components\left(\boldsymbol{M}^S\right) = \frac{1}{2}N\left(N+1\right)$$

$$number\ of\ components\left(\boldsymbol{M}^A\right) = \frac{1}{2}N\left(N-1\right)$$

We can get similar formulas with a quick Young tableaux trick. We just put N in the first box, and then we subtract each time we go down, and add each time we go over, again dividing by the hook length product.

$$number\ of\ components\left(\boldsymbol{M}^S\right) = \begin{bmatrix} N & N+1 \end{bmatrix}$$

$$\textit{number of components}\left(\boldsymbol{M}^{A}\right)=\begin{bmatrix} N \\ N-1 \end{bmatrix}$$

This can again be extended to higher rank tensors, and makes counting degrees of freedom conceptually simple. The lower rank tensor, 1st rank, is trivial; it is a vector, which will always have dimension N. To summarize, we decomposed the space of our covariant tensor M_{ab} as in equation (2.1) into two other $GL\left(N\right)$ irreducible tensors: $\mathcal{A}$, and $\mathcal{B}$ as given by equation (2.6). Again, we repeat that we will always use calligraphic script to represent $GL\left(N\right)$ irreducible tensors. In our notation here, we write:

$$M_{ab}=\mathcal{A}_{ab}+\mathcal{B}_{ab} \tag{2.8}$$

as the $GL\left(N\right)$ decomposition of a totally covariant second rank tensor.

2.2 Rank 2 $SO\left(p,q\right)$ Trace Decomposition

Our next goal is to find the $SO\left(p,q\right)$ irreducible decomposition. In the process of finding this decomposition, we take traces with the metric tensor g_{ab}, the object left invariant under $SO\left(p,q\right)$. Recall from linear algebra, that we can further break apart the symmetric matrices under $SO\left(p,q\right)$ into a trace free piece which we call $\mathring{M}_{(ab)}$, and a trace-full piece M. Recall from earlier how we talked about $GL\left(N\right)$; these trace pieces do not suddenly switch sectors under a $SO\left(p,q\right)$ transformation. The symmetric trace free element in the irreducible decomposition will always stay symmetric trace free under a $SO\left(p,q\right)$ transformation. However, under a $GL\left(N\right)$ transformation, a trace free piece may become trace-full.

Under $SO\left(p,q\right)$, only $\boldsymbol{M}^{S}$ decomposes further. The matrix $\boldsymbol{M}^{A}$ is already irreducible under $SO\left(p,q\right)$. A formula for how $\boldsymbol{M}^{S}$ decomposes further is shown below.

$$M_{(ab)}=\mathring{M}_{(ab)}+\frac{1}{N}Mg_{ab} \tag{2.9}$$

In terms of our tensors, $\mathcal{A}$, and $\mathcal{B}$ in equation (2.8) we find that only $\mathcal{A}$ becomes further reducible when we change the group to $SO\left(p,q\right)$. This is due to the skew tensor $\mathcal{B}$ not having any trace portion. We label each of the three $SO(p,q)$ irreducible representations by: $\mathfrak{A}$, $\mathfrak{a}$, and $\mathfrak{b}$; when viewed as tensors, these will be defined shortly. The tensors represented by $\mathfrak{A}$ and $\mathfrak{a}$ result from the decomposition of $\mathcal{A}$ under $SO(p,q)$. The $\mathfrak{A}$ tensor will be the trace-full part and $\mathfrak{a}$ will be the trace free part. Additionally, we now label $\mathcal{B}$ by $\mathfrak{b}$ to mirror our notation above.

Similarly to how we wrote $GL(N)$ irreducible tensors in equation (2.8), with calligraphic script, we also use special script to distinguish $SO(p,q)$ irreducible tensors. For totally trace-free tensors, we always use lowercase Fraktur script. For trace-full tensors, we always use uppercase Fraktur script. The $SO(p,q)$ decomposition of M_{ab} is given by:

$$M_{ab} = \mathring{M}_{(ab)} + \frac{1}{N} M g_{ab} + M_{[ab]} \tag{2.10}$$

$$M_{ab} = \mathfrak{a}_{ab} + \mathfrak{A}_{ab} + \mathfrak{b}_{ab} \tag{2.11}$$

with both equations representing the same decomposition. We use the Fraktur notation for the tensors $\mathfrak{a}$, $\mathfrak{A}$, and $\mathfrak{b}$. The tensors $\mathfrak{a}$, and $\mathfrak{b}$ are defined as follows.

$$\mathfrak{a}_{ab} = \mathcal{A}_{ab} - \mathfrak{A}_{ab} \tag{2.12}$$

$$\mathfrak{b}_{ab} = \mathcal{B}_{ab} \tag{2.13}$$

We now define the tensor $\mathfrak{A}$ in equation (2.12). We refer to tensors of even rank (tensors of odd rank) constructed from a scalar (vector) and the metric tensor g_{ab} as trace-full; here we are only concerned for even rank. This comment is used to differentiate the case for a rank 3 tensor as in chapter 3. We also define N to be the dimension of the vector space we are working with. Following this definition, the tensor $\mathfrak{A}$ is defined by:

$$\mathfrak{A}_{ab} = g_{ab} \left(\frac{1}{N} g^{cd} \mathcal{A}_{cd} \right) \tag{2.14}$$

In index notation, this calculation is made easier. Here we find that $M_{(ab)} g^{ab} = M$, and $\mathring{M}_{(ab)} = \mathfrak{a}_{ab}$. We needed the $\frac{1}{N}$ factor in the $\mathfrak{A}_{ab}$ equation because $g_{ab} g^{ab} = N$. Next, we present Maple procedures which calculate the objects in equation (2.11).

2.3 Maple Procedures for the Decomposition of Rank 2 Tensors

The code follows the decomposition given in equation (2.11). There are three commands which each take a rank 2 tensor as the first argument, and a metric tensor as the second argument. The command *Rank2TraceA* makes the trace-full tensor $\mathfrak{A}$ given by equation (2.12). The command *Rank2FrankA* makes the totally trace-free symmetric tensor $\mathfrak{a}$. The last command *Rank2FrankB* only needs a tensor as its first argument, but can take a metric tensor as the second argument to match with the prior defined codes. This command makes the skew-symmetric tensor $\mathfrak{b}$ given by equation (2.13).

2.4 Summary of $SO\left(p,q\right)$ Tensor Decomposition

In the last three sections, we introduced techniques on how to decompose rank two tensors in terms of $GL\left(N\right)$ and $SO\left(p,q\right)$ in arbitrary dimensions. Then, for the $GL\left(N\right)$ decomposition, we introduced Young tableaux, which enumerate and break apart a rank two tensor into its irreducible elements. Specializing further, we chose the subgroup $SO\left(p,q\right)$ of $GL\left(N\right)$ and asked for the irreducible elements of it. This lead to one more irreducible sector than we had before. Following this, we introduced three Maple procedures to compute all of the above, the rank two commands being new, and the Young tableaux commands being already in the software. These three commands can be found in the Rank2TensorModule Maple module.

2.5 Spinor Correspondence to Rank 2 Tensors in 4D

Next we will move to the spinor decomposition of rank two tensors in 4D with signature $[+,-,-,-]$ and talk about some new interesting phenomena as a result. Spinors are well explained in Stewart (Stewart, 1993), and Penrose (Penrose & Rindler, 1987a); we follow their notation. Recall that for two component spinors, we naturally have the group $Spin\left(3,1\right)$ which is isomorphic to two copies of $SL\left(2,\mathbb{C}\right)$; we note that $Spin\left(3,1\right)$ is a double cover of the group $SO\left(3,1\right)$; each of these three groups has real dimension 6. The next paragraph is adapted from Stewart.

To discuss spinors, we define the two-dimensional symplectic vector space $\mathcal{S}$. The dual of this space we denote by $\mathcal{S}^*$. A symplectic linear structure on an even dimensional vector space $\mathcal{S}$ is a non-degenerate bilinear skew-symmetric 2-form in $\mathcal{S}$; we call this 2-form ϵ_{AB}. A linear transformation $Q : \mathcal{S} \rightarrow \mathcal{S}$ is called symplectic if it preserves ϵ_{AB}. The set of symplectic transformations forms the symplectic group (of appropriate dimension). The symplectic group of dimension two, $Sp\,(2,\mathbb{C})$, is isomorphic to the special linear group of dimension two over $\mathbb{C}$, i.e., $SL\,(2,\mathbb{C})$. Let $\lambda \in SL\,(2,\mathbb{C})$ and $\iota^A \in \mathcal{S}$ be a spinor. Then we can represent the action of $SL\,(2,\mathbb{C})$ on $\mathcal{S}$ by:

$$\rho(\lambda) : \mathcal{S} \rightarrow \mathcal{S}$$

$$\rho(\lambda)(\iota^A) = \lambda^A{}_B \iota^B$$

It is useful to define the idea of a p, q spinor. We use capital Latin (A) indices to denote the parts of the p, q spinor living on $\mathcal{S}$, and capital Latin prime ($A^{'}$) indices to denote the parts of the p, q spinor living on $\overline{\mathcal{S}}$. For example, $\alpha_{ABCD'}$ would be a 3,1 spinor. We also define $\overline{\mathcal{S}}$ as the complex conjugate of the space $\mathcal{S}$. Complex conjugation takes a spinor in $\mathcal{S}$ and maps it to $\overline{\mathcal{S}}$. Like before, we can represent the action of $SL\,(2,\mathbb{C})$ on a spinor $o^{A^{'}} \in \overline{\mathcal{S}}$, then we can represent the action of $SL\,(2,\mathbb{C})$ on $\overline{\mathcal{S}}$ by:

$$\rho(\lambda) : \overline{\mathcal{S}} \rightarrow \overline{\mathcal{S}}$$

$$\rho(\lambda)(o^{A^{'}}) = \overline{\lambda}^{A^{'}}{}_{B'}\; o^{B^{'}}$$

Recall, from Stewart (Stewart, 1993), that spinors living on the vector space $\mathcal{S} \otimes \overline{\mathcal{S}}$ are isomorphic to vectors on the tangent bundle TM of our spacetime manifold M through the Infeld–Van der Waerden symbols $\sigma^a{}_{AA'}$.

$$\alpha^A \rightarrow \overline{\alpha^A} = \overline{\alpha}^{A^{'}}$$

where similar results hold for lowered indices. Note further that $\overline{\epsilon_{AB}} = \epsilon_{A'B'}$. Following the wording of Stewart: "We may now build a grand tensor algebra out of $\mathcal{S}$, $\mathcal{S}^*$, $\overline{\mathcal{S}}$, $\overline{\mathcal{S}^*}$." We provide a few more technical details. Since $\mathcal{S}$, and $\overline{\mathcal{S}}$ are different vector spaces, we do not need to distinguish between spaces like $\mathcal{S} \times \overline{\mathcal{S}}$ and $\overline{\mathcal{S}} \times \mathcal{S}$. The means we can shuffle primed indices through unprimed indices.

$$\alpha_{AB'} = \alpha_{B'A}$$

The price to be paid for this notational convenience is that we cannot regard $\alpha_{AB'}$ as a 2×2 matrix.

With that established, we begin by applying our spinor decomposition to a doubly covariant spinor τ_{AB} using Young tableaux.

$$\tau_{AB} = Y1\left(\tau\right)_{AB} + Y2\left(\tau\right)_{AB} \tag{2.15}$$

We can use the epsilon tensor to decompose the second tensor in equation (2.15) further by taking traces. To do so we define the tensor ϵ^{AB} with the following relation of Stewart (this is his equation 2.2.4):

$$\epsilon^{AB}\epsilon_{CB} = \epsilon_C{}^A = \delta_C{}^A = -\epsilon^A{}_C$$

A pedagogical idea goes as follows "raise on the right, lower on the left." This means that when we raise an index with ϵ^{AB}, that the B index gets contracted, and the A index is leftover. Similarly, when we lower an index with ϵ_{AB} the A index gets contracted, and the B index is leftover. This is what allows us to further decompose some tensors when we could not before. In this case when we take the subgroup $Sp\left(2\right)$ of $GL\left(N\right)$ we find that the $Y2$ sector decomposed further because it is skew. This results in the sector being fully determined by a scalar τ.

$$Y2\left(\tau\right)_{AB} = \frac{1}{2}\tau\epsilon_{AB}$$

$$Y2(\tau)_{AB}\,\epsilon^{AB} = \tau$$

Thus, our decomposition of τ_{AB} is given in terms of a symmetric spinor and a scalar times the epsilon tensor.

$$\tau_{AB} = \tau_{(AB)} + \frac{1}{2}\tau\epsilon_{AB} \tag{2.16}$$

We find spinors to be useful here because of a theorem from Stewart (Stewart, 1993) that states any symmetric spinor can be decomposed in terms of its principal spinors:

$$\upsilon_{(AB\ldots C)} = \alpha_{(A}\beta_{B}\ldots\gamma_{C)} \tag{2.17}$$

Equation (2.17) will become exceedingly important once we examine the classification of tensors using spinors. We note that symmetric spinors are irreducible under $SL(2,\mathbb{C})$. Now that we have established the background of what a spinor is, we discuss which of the rank 2 spinors correspond to real rank two tensors.

2.6 Hermitian Rank Two Spinor Decomposition

Given a real rank two tensor M_{ab}, once we apply the solder form ${\sigma^a}_{AA'}$ we obtain the hermitian spinor $\theta_{ABA'B'}$:

$$\theta_{ABA'B'} = M_{ab}{\sigma^a}_{AA'}{\sigma^b}_{BB'} \tag{2.18}$$

The spinor $\theta_{ABA'B'}$ need not be hermitian if M_{ab} is not real. Before examining Young tableaux and their usefulness in this context, we could have found the decomposition of $\theta_{ABA'B'}$ in Penrose & Rindler (Penrose & Rindler, 1987a) (PG 141).

$$\theta_{ABA'B'} = \theta_{(AB)(A'B')} + \frac{1}{2}\epsilon_{AB}{\theta_C}^{C}{}_{(A'B')} + \frac{1}{2}\epsilon_{A'B'}{\theta_{(AB)C'}}^{C'} + \frac{1}{4}\epsilon_{AB}\epsilon_{A'B'}{\theta_C}^{C}{}_{C'}{}^{C'} \tag{2.19}$$

Later, we will provide another form of this decomposition. Upon closer inspection, we can associate each of these pieces to tensors we already know. For instance, the spinor $\theta_{(AB)(A'B')}$ corresponds to the $\mathfrak{a}$ tensor from equation (2.12). The spinors ${\theta_C}^{C}{}_{(A'B')}$ and ${\theta_{(AB)C'}}^{C'}$ correspond to the $\mathfrak{b}$ tensor from equation (2.13). Lastly, the spinor ${\theta_C}^{C}{}_{C'}{}^{C'}$ corresponds to the $\mathfrak{A}$ tensor from equation (2.14); we will show how exactly this happens soon. To begin the thought process, we know that that symmetric spinors are irreducible are already. We want to decompose a spinor into its symmetric parts by taking epsilon traces; this is shown in equations (2.15) and (2.16). So we apply this to our spinor $\theta_{ABA'B'}$ using Young tableaux and traces. To start applying the trace decomposition in equation (2.16), we begin with $\theta_{ABA'B'}$ and take an epsilon trace on the AB indices.

$$\theta_{ABA'B'} = \theta_{(AB)A'B'} + \frac{1}{2}\theta_{C}{}^{C}{}_{A'B'}\epsilon_{AB}$$

Next, we take a second trace on the $A'B'$ indices. Simplifying the resulting expression gives us Penrose's formula (2.19). Now to condense this and make it match Penrose and Rindler's (Penrose & Rindler, 1987a) PGs 148-149, we define the following spinors:

$$\theta_{(AB)(A'B')} = S_{(AB)(A'B')}$$

$$\frac{1}{4}\theta_{C}{}^{C}{}_{C'}{}^{C'} = \tau$$

$$\frac{1}{2}\theta_{C}{}^{C}{}_{(A'B')} = \psi_{(A'B')}$$

$$\frac{1}{2}\theta_{(AB)C'}{}^{C'} = \mu_{(AB)}$$

We have labeled Penrose and Rindler's θ in the corresponding reference as μ here to avoid confusion). Then we obtain a condensed formula as in equation (2.20.

$$\theta_{ABA'B'} = S_{(AB)(A'B')} + \epsilon_{AB}\psi_{(A'B')} + \epsilon_{A'B'}\mu_{(AB)} + \epsilon_{AB}\epsilon_{A'B'}\tau \tag{2.20}$$

When we apply the reality condition $\theta_{ABA'B'} = \overline{\theta}_{ABA'B'}$ (hermiticity) we find that $\overline{S}_{(AB)(A'B')} = S_{(AB)(A'B')}$, $\overline{\mu}_{(A'B')} = \psi_{(A'B')}$, and $\overline{\tau} = \tau$. Thus, we know that S, τ are real, and ψ, μ are complex conjugates. Therefore, our hermitian spinor is given by the decomposition:

$$\theta_{ABA'B'} = S_{(AB)(A'B')} + \epsilon_{AB}\psi_{(A'B')} + \epsilon_{A'B'}\overline{\psi}_{(AB)} + \epsilon_{AB}\epsilon_{A'B'}\tau \tag{2.21}$$

The above method produced the decomposition that we were looking for, but there is a more efficient way to form this decomposition. In our approach, we use Young tableaux and then specialize to traces. This decomposes each sector under $GL(N)$. Each of these decomposes further when we look at $SL(2,\mathbb{C})$ irreducibility.

2.6.1 Counting Degrees of Freedom for Rank 2 Tensors and Spinors

We are interested in counting the degrees of freedom of $SL(2,\mathbb{C})$ irreducible spinors because of the correspondence between them and the $SO(3,1)$ irreducible tensors. The easiest way to perform spinor decompositions is to count the independent components for M_{ab} in terms of its constituent pieces $\mathfrak{A}_{ab}$, $\mathfrak{a}_{ab}$, and $\mathfrak{b}_{ab}$ (note these tensors are defined in equation (2.11)). The number of independent components (degrees of freedom) of the tensor M_{ab} is N^2; this is the dimension of the tensor space. Similarly, we can use Young tableaux to count the number of components of $\mathcal{A}_{ab}$ and $\mathfrak{b}_{ab}$. Recall that $\mathcal{A}_{ab}$ is the symmetric piece of the Young/$GL(N)$ decomposition. The tensor $\mathcal{A}_{ab}$ has $\frac{1}{2}N(N+1)$ number of independent components, and $\mathfrak{b}_{ab}$ has $\frac{1}{2}N(N-1)$ number of independent components. When we break $\mathcal{A}_{ab}$ into $\mathfrak{A}_{ab}$ and $\mathfrak{a}_{ab}$, we find that $\mathfrak{A}_{ab}$ has 1 independent component because it is constructed from a scalar, and that $\mathfrak{a}_{ab}$ as a result has $\frac{1}{2}(N+2)(N-1)$ independent components. The results are repeated below:

$$\deg(\mathfrak{A}_{ab}) = 1 \tag{2.22}$$

$$\deg(\mathfrak{a}_{ab}) = \frac{1}{2}(N+2)(N-1) \tag{2.23}$$

$$\deg(\mathfrak{b}_{ab}) = \frac{1}{2}N(N-1) \tag{2.24}$$

All spinors in this category must be symmetric, because we can take off epsilon spinors to ensure so. We find four different possibilities:

$$S_{(AB)(A'B')},\quad \psi_{(AB)},\quad \mu_{(A'B')},\quad \tau$$

Penrose and Rindler (Penrose & Rindler, 1987a) list these spinors as corresponding to the $SO(3,1)$ irreducible tensor components of a rank 2 tensor. Since we are interested in real tensors we also have a reality condition. This means that p, q spinor, where $p = q$, is hermitian. Likewise, if a p, q spinor, where $p \neq q$, is present, then so must its complex conjugate spinor be present. In this case we see that $\psi_{AB} = \bar{\mu}_{AB}$ following the decomposition of Penrose and Rindler for a skew-symmetric second rank tensor. The most useful way to examine this correspondence is to count the degrees of freedom. Upon counting the degrees of freedom of these spinors, we find that $S_{(AB)(A'B')}$ has 9 degrees of freedom.

$$\deg\left(S_{(AB)(A'B')}\right) = \left(\frac{1}{2}N\left(N+1\right)\right)\left(\frac{1}{2}N\left(N+1\right)\right)\Big|_{N=2} = 9$$

Next, $\psi_{(AB)}$ has $2\left(\frac{1}{2}N\left(N+1\right)\right)\big|_{N=2} = 6$ degrees of freedom; one may think this should be 3, but recall that $\psi_{(AB)}$ and $\mu_{(A'B')}$ are complex; $S_{(AB)(A'B')}$ has only 9 degrees of freedom because of the reality condition. As a general statement we can say whenever the number of $\mathcal{S}$ indices does not match the number of $\overline{\mathcal{S}}$ then the spinor's degrees of freedom are doubled (because of the complex nature). Finally, τ has only 1 degree of freedom. All of this together tells us that $S_{(AB)(A'B')}$ will correspond to $\mathfrak{a}_{ab}$. That $\psi_{(AB)}$, and likewise $\bar{\psi}_{(AB)}$ correspond to $\mathfrak{b}_{ab}$. Finally, that τ will correspond to $\mathfrak{A}_{ab}$. Now we move to finding explicit formulas for these spinors. We will begin with the $\mathfrak{A}_{ab}$ spinor.

2.6.2 Spinor Decomposition of $\mathfrak{A}$

Recall that $\mathfrak{A}_{ab}$ is given by $\mathfrak{A}_{ab} = g_{ab}\left(\frac{1}{N}g^{cd}\mathcal{A}_{cd}\right)$ in equation (2.14). We know that the metric g_{ab} is given in terms of epsilon spinors as $g_{ab} \cong \epsilon_{AB}\epsilon_{A'B'}$ once the solder form is applied. Additionally, $\frac{1}{N}g^{cd}\mathcal{A}_{cd}$ is a scalar, which we will define to be τ. Thus, in this case, we may write $\mathfrak{A}$ in terms of τ and epsilons. The only extra thing we need to remember is that for our case $N = 4$.

$$\mathfrak{A}_{ab} \cong \tau\epsilon_{AB}\epsilon_{A'B'}, \quad \tau = \frac{1}{4}g^{cd}\mathcal{A}_{cd} \tag{2.25}$$

Next we move onto the $\mathfrak{a}_{ab}$ spinor.

2.6.3 Spinor Decomposition of $\mathfrak{a}$

Recall that $\mathfrak{a}_{ab}$ is given by $\mathfrak{a}_{ab} = \mathcal{A}_{ab} - \mathfrak{A}_{ab}$ in equation (2.12). We found that $S_{(AB)(A'B')}$ should correspond to $\mathfrak{a}_{ab}$, and indeed we find that it does in equation (2.26).

$$\mathfrak{a}_{ab} \cong S_{(AB)(A'B')} \tag{2.26}$$

We can also easily see that $\mathfrak{a}_{ab}$ is trace free from the form of $S_{(AB)(A'B')}$. There are no epsilon traces we can take off that do not result in 0 because of the symmetry.

2.6.4 Spinor Decomposition of $\mathfrak{b}$

Recall that $\mathfrak{b}_{ab}$ is given by $\mathfrak{b}_{ab} = M_{[ab]}$ in equation (2.13). Because of the reality condition $\overline{\mathfrak{b}}_{ab} = \mathfrak{b}_{ab}$. We can write out equation (2.28). We can tell that we can remove a single epsilon from $M_{[ab]}$ because of the skew tableaux structure:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \tag{2.27}$$

Whenever we see this, we can always remove at least one epsilon from the defined tensor. We can see this as follows.

$$\mathfrak{b}_{ab} \cong \psi_{(AB)}\epsilon_{A'B'} + \epsilon_{AB}\overline{\psi}_{(A'B')} \tag{2.28}$$

$$\psi_{(AB)} = \frac{1}{2}\mathfrak{b}_{ABA'B'}\epsilon^{A'B'}$$

$$\overline{\psi}_{(A'B')} = \frac{1}{2}\mathfrak{b}_{ABA'B'}\epsilon^{AB}$$

There is no reference to the $\mu_{(A'B')}$ spinor from earlier because of the reality condition which forces $\mu_{(A'B')} = \overline{\psi}_{(A'B')}$. Since we have explicit formulas for the spinors, we can now discuss Maple procedures to generate these objects for us.

2.7 Maple Procedures for $SL(2,\mathbb{C})$ Irreducible Spinors

There are six different new commands to construct $SL(2,\mathbb{C})$ irreducible spinors, and to produce tensors from these spinors. They all require a spinor and a solder form as input. Furthermore, the ψ spinor commands can take a third optional argument. The argument takes either "spinor" or "barspinor" and will return either the ψ spinor for the "spinor" argument, or the conjugate spinor $\overline{\psi}$ for the "barspinor" argument. We recommend using the optional arguments when using the package.

The command *Rank2SSpinor* when given a rank 2 totally covariant tensor, and a solder form, will produce the spinor $S_{(AB)(A'B')}$ in equation (2.26). The command *Rank2PsiSpinor* with the same inputs as before, will produce either the spinor $\psi_{(AB)}$ or the spinor $\overline{\psi}_{(AB)}$ as in equation (2.28). The last of these three *Rank2TauSpinor*, again with the same inputs at the S spinor, produces the spinor τ from equation (2.25).

The next set of three equations takes as input a spinor of type S, ψ, or τ respectively, and a solder form. It then creates the corresponding $SO(3,1)$ irreducible tensor from the spinor. In this way, an arbitrary real totally covariant rank 2 tensor can be constructed from these three irreducible spinors. We hope these commands will be useful for future research.

The command *Rank2GenerateSTensor* takes a S type spinor from equation (2.26) and constructs the corresponding totally symmetric trace-free tensor from it; here this tensor would be $\mathfrak{a}$ from equation (2.12). The command *Rank2GeneratePsiTensor* takes the ψ spinor, and optionally the $\overline{\psi}$ spinor as the third argument, and constructs the corresponding trace-free totally skew tensor $\mathfrak{b}$ from it; see equation (2.13). Finally, the last command *Rank2GenerateTauTensor* takes a real scalar, (the tau spinor), and constructs its corresponding trace-full tensor; that tensor for us is $\mathfrak{A}$ from equation (2.14).

These are all the commands to calculate the $SL(2,\mathbb{C})$ irreducible spinors and utilize them.

2.8 Summary of $Spin\,(3,1) \cong SL\,(2,\mathbb{C})$ Spinor decomposition

In the last three sections we have described: how a rank 2 spinor decomposes, how a rank 4 hermitian spinor decomposes following (Penrose & Rindler, 1987a), how to count the independent components of rank 2 irreducible tensors and spinors, presented the decomposition of $\mathfrak{A}$, $\mathfrak{a}$, and $\mathfrak{b}$ in terms of spinors, and finally constructed Maple procedures to generate each of the spinors given a rank 2 tensor and solder form, and make a tensor when given an irreducible spinor and a solder form. All this together builds a solid foundation on which to examine rank 3 and later rank 4 tensors in terms of $SO\,(p,q)$ irreducibility and the corresponding spinors in terms of $SL\,(2,\mathbb{C})$.

CHAPTER 3

RANK 3 TENSOR DECOMPOSITIONS AND THE TORSION TENSOR

This section explains how to decompose rank 3 tensors under various groups. Tools from representation theory are heavily used, and to this end we make clear what a representation is. A representation on a vector space of dimension N is a map ρ from some group G to $GL(N)$.

$$\rho : G \to GL(N) \tag{3.1}$$

We will not often use explicit mappings, but they can be inferred without much effort. Here we remind the reader of the definition of an irreducible representation.

A representation of a group is called irreducible if it cannot be decomposed into smaller, non-trivial subrepresentations. In other words, there are no proper, non-zero subspaces that are invariant under the action of the group.

Mathematically, let G be a group, and let V be a vector space over a field F (we choose $F = \mathbb{R}$ for tensors and $F = \mathbb{C}$ for spinors). A representation of G on V is a group homomorphism $\rho : G \to GL(V)$, where $GL(V)$ is the general linear group of invertible linear transformations on V. The representation ρ is called irreducible if the only subspaces $W \subseteq V$ that are invariant under the action of $\rho(g)$ for all $g \in G$ are the trivial subspaces $\{0\}$ and V itself.

In the context of matrix representations, a representation is irreducible if it cannot be brought into a block-diagonal form by a similarity transformation (a change of basis). In other words, there are no non-trivial, simultaneous block-diagonalizations for all the matrices representing the group elements.

We examine the vector space V of rank 3 tensors over TM, the tangent bundle of our spacetime manifold M. We then examine how our group of interest G acts on V, and we use this information to decompose V into irreducible subspaces. For the case of the irreducible representations of $GL(N)$, we look at the map:

$$\rho : GL(N) \to GL(V) \tag{3.2}$$

and how this map translates into invariant subspaces. We will use the Young tableaux methods established in chapter 2, and in Fulton and Harris (Fulton & Harris, 1991) to generate the four $GL(N)$ irreducible representations of a 3rd rank tensor. We build on the Young decomposition using the tools of Hammermesh (Hammermesh, 1962/1989). This follows similarly to what we did in chapter 2. These tools allow us to build the seven $SO(p,q)$ irreducible representations of a 3rd rank tensor. We then build upon the ideas of Penrose and Rindler to find the $SL(2,\mathbb{C})$ irreducible spinors. In this case, the representation would be given by:

$$\rho : SL(2,\mathbb{C}) \to GL(V) \tag{3.3}$$

where V is now the "grand tensor algebra" of Stewart (Stewart, 1993) (see page 71) built from $\mathcal{S}$, $\mathcal{S}^*$, $\overline{\mathcal{S}}$, and $\overline{\mathcal{S}^*}$. In the case of rank 3 tensors, we will often apply (3.3) when looking at a hermitian spinor of type $a_{ABCA'B'C'}$.

Several times throughout this section and in later sections, we use bold font for tensors and spinors, suppressing the indices to save space. We hope the context the objects are used in will make this clear.

This section details: how to decompose rank 3 tensors into $GL(N)$ irreducible components using Young tableaux, how to decompose rank 3 tensors in any dimension with respect to $SO(p,q)$, how to find the $SL(2,\mathbb{C})$ irreducible spinors of a rank 3 tensor in dimension 4 with signature $[+,-,-,-]$, and how to use the corresponding Maple procedures to determine each of these.

We expand upon the work done in the "Rank 2 Tensor Decomposition" section (see chapter 2) and apply the same Young tableaux tools to rank 3 tensors.

We also apply this decomposition to the torsion tensor from differential geometry. We take a moment here to define and describe the torsion tensor when viewed through various lenses. In differential geometry, the torsion tensor is an object used to describe the non-commutativity of the covariant derivative, which is a generalization of the ordinary derivative for curved spaces. It is an important concept in the study of the geometry and topology of manifolds, and it plays a key role in GR and ECSK. The torsion tensor is defined as:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

where T is the torsion tensor, ∇ is the covariant derivative, X and Y are vector fields on the manifold, and $[X,Y]$ is the Lie bracket of the vector fields (also known as the commutator). The Lie bracket measures the difference between the actions of X and Y when applied in different orders.

The torsion tensor is antisymmetric, meaning that $T(X,Y) = -T(Y,X)$. This property follows directly from the definition.

In coordinate notation, the torsion tensor can be expressed the skew part of the connection:

$$T^a_{bc} = 2\omega^a{}_{[bc]}$$

where T^a_{bc} are the components of the torsion tensor, and $\omega^a{}_{bc}$ are the connection coefficients, which describe the affine connection on the manifold.

In a manifold with a torsion-free connection, the torsion tensor vanishes identically. This is the case for Riemannian manifolds, which are equipped with a Levi-Civita connection. In general relativity, the Levi-Civita connection is used because it is torsion-free and metric-compatible, meaning that it preserves the metric tensor under parallel transport.

We can also view the torsion tensor from a Cartan geometry view point (Cartan, 1922) (Sharpe, 1997). Cartan geometry is a generalization of Riemannian geometry that unifies the notions of curvature and torsion. It is based on the concept of a principal fiber bundle with a connection. In Cartan geometry, the geometry of a manifold is described using a

G-structure, where G is a Lie group acting on the tangent spaces. This G-structure defines a set of preferred frames (or bases) for the tangent spaces, and the connection is given by a Cartan connection, which is a Lie algebra-valued 1-form on the manifold.

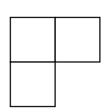
The curvature (which we will examine in chapter 4) and torsion tensors in Cartan geometry are defined in terms of the Cartan connection. The curvature tensor measures the non-integrability of the G-structure, and the torsion tensor measures the failure of the Cartan connection to be compatible with the G-structure. Both curvature and torsion are essential features of the geometry in Cartan's framework.

To relate the torsion tensor in Cartan geometry to the torsion tensor in differential geometry, let's consider a specific case: when the G-structure is associated with a linear connection on the tangent bundle. In this case, the Cartan connection can be identified with the connection 1-forms in the usual differential geometry setting.

In summary, the torsion tensor in Cartan geometry is a generalization of the torsion tensor in differential geometry, as it incorporates the failure of the Cartan connection to be compatible with the G-structure. This allows for a more general framework that can accommodate both curvature and torsion as fundamental aspects of the geometry. We find that both viewpoints are useful when discussing torsion and point the reader to Hehl et al. (F. Hehl et al., 1976) for more discussion on how torsion enters into physics.

There are some new complexities which arise as soon as we try to apply Young tableaux here that make the decomposition more interesting. The decomposition is no longer unique because of the choice of either symmetrizing or skew-symmetrizing indices first. This occurs in both the Y2a and Y2b sectors and both ways result in a different, but likewise irreducible, decomposition.

The Y2a and Y2b sectors both fall into the Ferrers diagram:



,

which splits into the Y2a and Y2b Young tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Although we have said before that the horizontal indices are symmetric, and the vertical ones are skew, there is now ambiguity on whether the tensors corresponding to these objects are symmetric or skew-symmetric.

For instance, in the Y2a section we could symmetrize the indices 1,2 first then skew-symmetrize the indices 1,3. This is the convention we will use; "*DifferentialGeometry*" in Maple also follows this by default. The skew-symmetrization on the indices 1,3 breaks the 1,2 symmetry but guarantees that the resulting tensor is an irreducible representation. This representation comes out on the space of 1,3 skew tensors. The other option we could choose is to skew 1,3 first and then symmetrize 1,2 resulting in a 1,2 symmetric irreducible tensor element. These representations are not unique, and by a theorem of Young we know that they represent the same irreducible element in the decomposition under $GL(N)$; see (Penrose & Rindler, 1987b) for more details. For third rank and higher-order tensors, the non-uniqueness of the irreducible decomposition is a well known fact, see Landsberg (Landsberg, 2012).

Following this, we present the Maple procedures in the "Rank3TensorModule" module. This module feeds into our ECSK module to apply the same calculations to the torsion tensor for classification purposes.

Next we move to spinor calculations for 3rd rank tensors. For the total decomposition of an arbitrary rank 3 tensor in 4-dimensions, we find that the tensor is completely characterized by 7 spinors. For the Y1 sector, these spinors are: $S_{(ABC)(A'B'C')}$ for the totally trace free part which corresponds to the tensor $\mathfrak{a}_{abc}$, and , $\omega_{AA'}$ for the trace-full part which corresponds to the tensor $\mathfrak{A}_{abc}$. For the Y2a sector, we find the spinors: $\eta_{(ABC)B'}$ which corresponds to the tensor $\mathfrak{b}_{abc}$, and $\tau_{AA'}$ which corresponds to the tensor $\mathfrak{B}_{abc}$. For the Y2b sector, we find the spinors: $\sigma_{(ABC)B'}$ which corresponds to the tensor $\mathfrak{c}_{abc}$, and $\psi_{AA'}$ which corresponds to the tensor $\mathfrak{C}_{abc}$. Lastly, for the Y3 sector, we find the spinor $\Xi_{DD'}$ corresponds to the tensor

$\mathfrak{d}_{abc}$.

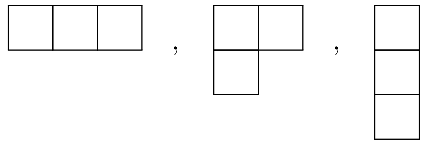
We can then use these spinors to classify the torsion tensor in 4-D. The torsion tensor decomposes into 3 irreducible tensors. The first of which is the trace-full tensor, which we call $\mathfrak{Q}_{a[bc]}$. It turns out this tensor equals $\mathfrak{B}_{abc} + \mathfrak{C}_{abc}$. The second of which is the trace-free, not-totally-skew tensor, which we call the leftovers piece. We label the leftovers part of the torsion tensor by $\mathfrak{q}_{a[bc]}$. This tensor equals $\mathfrak{b}_{abc} + \mathfrak{c}_{abc}$. Finally, the third irreducible part of the torsion tensor is given by the tensor $\mathfrak{d}_{[abc]}$. This is the totally skew part of the torsion tensor. These tensors are then turned into spinors and classified algebraically.

The irreducible tensor elements of the torsion tensor decompose into the spinors: $\Theta_{CC'}$, $\Omega_{(ABC)A'}$, and $\Xi_{DD'}$. These correspond to the trace-full, leftovers, and totally skew parts respectively. An entirely new classification algorithm is used for $\Omega_{(ABC)A'}$ in terms of irreducible spinors (see subsection (3.8.1)). This classification method was inspired by Penrose and Rindler (Penrose & Rindler, 1987b).

The algebraic/structural classification we present can be refined using the tools of Penrose (Penrose, 1972) with algebraic geometry. We can examine the curve on $\mathbb{CP}^1 \times \mathbb{CP}^1$ in the complex case, again see (Penrose, 1972) for more details. Furthermore, see Crade and Hall (Crade & Hall, 1982) for a survey of several classification schemes and how they compare. In this reference, Ludwig and Scanlan's (Ludwig & Scanlan, 1971) classification also appears, which is yet another refinement of our classification. Although we will not go into great depth on these considerations, they are placed here as possible future work. The other two spinors $\Theta_{CC'}$, and $\Xi_{DD'}$ have a simpler classification which tells us whether they correspond to a null vector, or are zero.

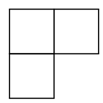
3.1 Rank 3 $GL(N)$ Young Decomposition

We begin with an arbitrary rank 3 tensor B_{abc}. We will then decompose B_{abc} using Young tableaux. The rank 3 Ferrers diagrams are given in 3.4.



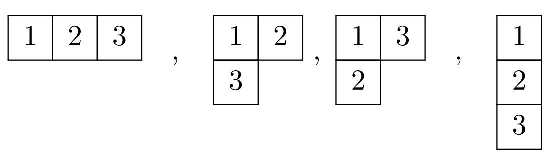
(3.4)

We have the usual totally symmetric and totally skew diagrams, which we had for rank 2 tensors. However, there is now a sector in between. This is given by the Ferrers diagram:



It is this sector which is not unique. This is due to the fact we could either first symmetrize, then second skew-symmetrize or first skew-symmetrize, then second symmetrize. Although they result in different irreducible representations, they are both subspaces, and irreducible. Either choice results in an irreducible subspace under $GL(N)$. Recall that we will always skew-symmetrize last in our convention.

Another interesting phenomenon happens in the rank 3 case that doesn't happen in the rank 2 case. The middle Young tableaux splits from the Ferrers diagram into two Young tableaux. They are different because of the way we can arrange the numbers in each section. This is seen in equation (3.5).



(3.5)

Recall that we always start with the number 1 in the top left most corner and proceed increasingly down and to the right. Because of this, there are options where we put the numbers 2 and 3 in the middle $Y2$ diagram. Another difference is that there is no longer a general way to write these decompositions like in the rank 2 tensor case. We could have written $M_{ab} = M_{(ab)} + M_{[ab]}$ in that case. If we try to do that here, we would wind up with the opaque and nonsensical notation $B_{(a|[b|c])}$ or things like it. To adopt a convention, we can find inspiration from Penrose and Rindler (Penrose & Rindler, 1987a). Their notation can be written as in equation 3.6.

$$B_{abc} = B_{\begin{smallmatrix} a & b & c \end{smallmatrix}} + B_{\begin{smallmatrix} a & b \\ c & \end{smallmatrix}} + B_{\begin{smallmatrix} a & c \\ b & \end{smallmatrix}} + B_{\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}} \tag{3.6}$$

However, eventually, we determined that the notation, which we will soon present, better represents the Young decomposition, as it matches the idea of Young projectors. Since we can look at Young tableaux as projection operators, we use $Y1, Y2a, Y2b$, and $Y3$ to denote the same objects as in equation (3.6).

$$B_{abc} = Y1\,(B)_{abc} + Y2a\,(B)_{abc} + Y2b\,(B)_{abc} + Y3\,(B)_{abc} \tag{3.7}$$

In equation (3.7) the $Y1, Y2a, Y2b$, and $Y3$ all correspond to the Young diagrams in (3.5). For example and clarity: $Y1$ means symmetrize on a, b, c; $Y2a$ means symmetrize on a, b, then skew-symmetrize on a, c; $Y2b$ means symmetrize on a, c, then skew-symmetrize on a, b; finally, $Y3$ mean skew-symmetrize on a, b, c.

Although the notation in equation (3.7) is incredibly clear, it feels cumbersome. To the end of ameliorating the notation, we developed another notation. This new notation consists of calligraphic letters for the Young decomposition: i.e., $\mathcal{A}, \ldots, \mathcal{D}$ which we will soon define. This is the same as how we used calligraphic script in the $GL\,(N)$ decomposition of rank two tensors in chapter 2.

In our case for B_{abc} we have 4 tensors under this decomposition: $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, and $\boldsymbol{\mathcal{D}}$. These are defined as follows:

$$\begin{aligned} \mathcal{A}_{abc} &= Y1\,(B)_{abc}\,, \quad \mathcal{B}_{abc} = Y2a\,(B)_{abc} \\ \mathcal{C}_{abc} &= Y2b\,(B)_{abc}\,, \quad \mathcal{D}_{abc} = Y3\,(B)_{abc} \end{aligned} \tag{3.8}$$

We use equation (3.8) to write the $GL\left(N\right)$ decomposition of a rank 3 tensor as follows:

$$B_{abc} = \mathcal{A}_{abc} + \mathcal{B}_{abc} + \mathcal{C}_{abc} + \mathcal{D}_{abc} \tag{3.9}$$

and this completes the Young analysis of the tensor B_{abc}. Although we present the $GL\left(N\right)$ irreducible tensor elements in equation (3.8), we still need a way to write out constructively what each of these tensors is, i.e., a formula. Normally, this calculation would be incredibly difficult and cumbersome, but the tools of "*DifferentialGeometry*" have greatly simplified these calculations, making it possible to work with these objects without lengthy pen and paper calculations. To be clear, when we see "symmetrize on 1,2 first then skew-symmetrize on 1,3 next" we do not swap positions, we swap indices. The following example is useful to describe this. We examine the tensor $\mathcal{B}_{abc}$ generated by the

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Young diagram to illustrate this. Then we proceed with the following steps:

$$\begin{aligned} \mathcal{B}_{abc} &\to \frac{1}{3} B_{abc}, \\ &\to \frac{1}{3}\left(B_{abc} + B_{bac}\right), \\ &= \frac{1}{3}\left(B_{abc} + B_{bac} - B_{cba} - B_{bca}\right) \end{aligned}$$

to get the proper information out of the Young tableaux. Now we write out how to get each of these sectors. Formulas are given below in (3.10) by applying the Young tableaux idea. These formulas also work in Maple. First we have the $Y1$ tableaux.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

$$\mathcal{A}_{(abc)} = \frac{1}{6}\left(B_{abc} + B_{bca} + B_{cab} + B_{acb} + B_{cba} + B_{bac}\right) \tag{3.10}$$

Next the $Y2$ pieces. First with the $Y2a$ tableaux.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\mathcal{B}_{[a|b|c]} = \frac{1}{3}\left(B_{abc} + B_{bac} - B_{cba} - B_{bca}\right) \tag{3.11}$$

Then we have the $Y2b$ tableaux.

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\mathcal{C}_{[ab]c} = \frac{1}{3}\left(B_{abc} + B_{cba} - B_{bac} - B_{cab}\right) \tag{3.12}$$

Lastly, we have the $Y3$ tableaux.

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

$$\mathcal{D}_{[abc]} = \frac{1}{6}\left(B_{abc} + B_{bca} + B_{cab} - B_{acb} - B_{cba} - B_{bac}\right) \tag{3.13}$$

We can calculate the tensors $\mathcal{A}_{abc}, \ldots, \mathcal{D}_{abc}$ in the "*DifferentialGeometry*" software package more easily through the use of the "SymmetrizeIndices" command. This makes the calculation trivial and is what the author uses extensively. Furthermore, the commands: YoungTableauBasis, and YoungSymmetrizer satisfy this need comprehensively.

Furthermore, to explain the coefficients, recall the hook length product explained in the rank 2 tensors chapter (see chapter 2). Here we take the top left box and count all the boxes to the right and down from it plus one. Do this for each box, and then multiply the product together. For an example, in the Ferrers $Y2$ case, we find the hook lengths, and their product in 3.14.

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad , \quad HookLengthProduct\left(\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = 3 \tag{3.14}$$

Additionally, although these formulas work for an arbitrary tensor, the decomposition is not irreducible unless it has the fundamental symmetries of the original tensor. The torsion tensor which we will examine later runs into this issue once we apply our decomposition. Thus, its Young decomposition is smaller than the four subspaces which we have above; there are only two for the torsion tensor. Itin and Reches (Itin & Reches, 2021) give an excellent explanation of this phenomenon (see pages 25-26).

Since we have all the Young tableaux information, we can begin taking traces to build the $SO\left(p,q\right)$ irreducible elements of B_{abc}.

3.2 Rank 3 $SO\left(p,q\right)$ Trace Decomposition

In this section, we examine the irreducible representations of rank 3 tensors under $SO\left(p,q\right)$. The metric g_{ab} is a symmetric, non-degenerate, bilinear form which is preserved by the special orthogonal group $SO\left(p,q\right)$.

We use similar methods to those presented in chapter 2 to decompose rank 3 tensors. Hammermesh's tools (Hammermesh, 1962/1989) are incredibly useful as he explains how taking traces builds $SO\left(p,q\right)$ irreducible representations.

Let us first look at the general case to get a feeling for what happens: B_{abc}. Taking traces of B_{abc} results in equation (3.15) shown below.

$$B_{abc} = X_a g_{bc} + Y_b g_{ac} + Z_c g_{ab} + W_{abc} \tag{3.15}$$

Where each of the X_a, Y_b, Z_c, W_{abc} are to be determined; note this implies that W_{abc} is a totally trace free tensor. We are already done taking traces because we can't take traces of the vectors X_a. Now take and apply this idea to each of our Young sectors. This was an idea proposed by Wheeler (J. T. Wheeler, unpublished). We define several 1-forms to make later calculations easier. These are: P_a, R_b, and U_c:

$$g^{ab}B_{abc} = P_c, \quad , g^{ac}B_{abc} = R_b, \quad g^{bc}B_{abc} = U_a \tag{3.16}$$

The traces from before are defined in equation (3.16). With this idea established, we can decompose the tensors $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, and $\boldsymbol{\mathcal{D}}$ under $SO(p,q)$. Notationally, the 4 irreducible tensors: $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, and $\boldsymbol{\mathcal{D}}$ under $GL(N)$ (see equation 3.9) will decompose to 7 irreducible tensors: $\boldsymbol{\mathfrak{A}}$, $\boldsymbol{\mathfrak{a}}$, $\boldsymbol{\mathfrak{B}}$, $\boldsymbol{\mathfrak{b}}$, $\boldsymbol{\mathfrak{C}}$, $\boldsymbol{\mathfrak{c}}$, and $\boldsymbol{\mathfrak{d}}$ under $SO(p,q)$ because we can take traces of the tensors appearing in the $GL(N)$ decomposition. Recall that we use uppercase Fraktur script, i.e., $\boldsymbol{\mathfrak{A}}, \boldsymbol{\mathfrak{B}}, \boldsymbol{\mathfrak{C}}$ for trace-full tensors as we did in chapter 2.

We call the tensors $\boldsymbol{\mathfrak{A}}, \boldsymbol{\mathfrak{B}}, \boldsymbol{\mathfrak{C}}$ the trace tensors. We will use lowercase Fraktur script, i.e., $\boldsymbol{\mathfrak{a}}, \boldsymbol{\mathfrak{b}}, \boldsymbol{\mathfrak{c}}, \boldsymbol{\mathfrak{d}}$ for the totally-trace free tensors; we call these the frank tensors. This idea will be further extended and applied to rank 4 tensors later in chapter 4. As an example, the tensor $\boldsymbol{\mathcal{A}}$ will decompose into the tensors: $\boldsymbol{\mathfrak{A}}$, and $\boldsymbol{\mathfrak{a}}$.

We will also count the degrees of freedom, or the dimension of the vector subspace, of each of the $SO(p,q)$ irreducible tensors. This will be denoted by $\deg(\ )$. Young tableaux are easily suited for this type of counting, even in the $SO(p,q)$ case with a slight extension.

3.2.1 Decomposing $\mathcal{A}_{abc}$ Under $SO(p,q)$

We begin by decomposing the tensor $\mathcal{A}_{abc}$ under $SO(p,q)$. Recall that this tensor corresponds to the tableau:

1	2	3

which is the totally symmetric tableaux sector. Recall that this would give us the following relation: $\mathcal{A}_{abc} = \mathcal{A}_{(abc)}$. Upon decomposing $\mathcal{A}_{abc}$ under $SO(p,q)$ with the Hammermesh method (Hammermesh, 1962/1989) we produce the following decomposition:

$$\mathcal{A}_{abc} = \mathfrak{a}_{abc} + \mathfrak{A}_{abc} \tag{3.17}$$

$$\mathfrak{A}_{abc} = \frac{3}{(N+2)} P_{(a} g_{bc)} \tag{3.18}$$

$$P_c = g^{ab} \mathcal{A}_{abc} \tag{3.19}$$

where the tensor $\boldsymbol{\mathfrak{a}}$ is determined by equation (3.17).

Counting the degrees of freedom of $\boldsymbol{\mathcal{A}}, \mathfrak{A}$, and $\mathfrak{a}$

We can count the degrees of freedom of each of these tensors as well. For the tensor $\boldsymbol{\mathcal{A}}$, as before, we know that the degrees of freedom can be counted using Young tableaux with the same method presented in chapter 2. We place an N in the top leftmost box, and as we move right add one to it, as we move down subtract one. For the totally symmetric case this becomes, N for the top leftmost box, then $N+1$ for the middle box, and then $N+2$ for the final box. We also still divide by the hook length product, which is 6 in this case. To make it applicable to spacetime, the reader can choose $N=4$ as the dimension.

The results are given below. We find the degrees of freedom in $\mathfrak{a}_{abc}$ by subtracting the degrees of freedom of $\mathfrak{A}_{abc}$ from $\mathcal{A}_{abc}$.

$$\deg\left(\mathcal{A}_{abc}\right) = \frac{1}{6}N\left(N+1\right)\left(N+2\right) \tag{3.20}$$

$$\deg\left(\mathfrak{a}_{abc}\right) = \frac{1}{6}N\left(N+4\right)\left(N-1\right)$$

$$\deg\left(\mathfrak{A}_{abc}\right) = N$$

3.2.2 Decomposing $\mathcal{B}_{abc}$, and $\mathcal{C}_{abc}$ Under $SO\left(p,q\right)$

We will begin with decomposing the $Y2a$ tensor. The $Y2a$ Young sector is given by the tableau:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\;.$$

We will label the tensor which represents this sector as $\mathcal{B}_{abc}$. This tensor is equivalent to applying the $Y2a$ projection to $\boldsymbol{B}$ i.e. $Y2a\left(B\right)_{abc} = \mathcal{B}_{abc}$. Our convention tells us to first symmetrize on the a,b indices, and then skew-symmetrize on the a,c indices. This results in $\boldsymbol{\mathcal{B}}$ having the symmetry $\mathcal{B}_{abc} = \mathcal{B}_{[a|b|c]}$. Taking traces, we find the following decomposition for $\boldsymbol{\mathcal{B}}$:

$$\mathcal{B}_{abc} = \mathfrak{b}_{abc} + \mathfrak{B}_{abc} \tag{3.21}$$

$$\mathfrak{B}_{abc} = -\frac{2}{(N-1)} P_{[a} g_{|b|c]} \tag{3.22}$$

$$P_c = g^{ab} \mathcal{B}_{abc} \tag{3.23}$$

Now we move onto the $Y2b$ sector, which is represented by the tableau

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

which we represent by the tensor $\mathcal{C}_{abc}$. This follows similar logic as what we did for $\boldsymbol{\mathcal{B}}$; this is $Y2b\,(B)_{abc} = \mathcal{C}_{abc}$. We have the symmetry $\mathcal{C}_{abc} = \mathcal{C}_{[ab]c}$. Using the trace decomposition, we arrive at the following three equations.

$$\mathcal{C}_{abc} = \mathfrak{c}_{abc} + \mathfrak{C}_{abc} \tag{3.24}$$

$$\mathfrak{C}_{abc} = -\frac{2}{(N-1)} R_{[a} g_{b]c} \tag{3.25}$$

$$R_b = g^{ac} \mathcal{C}_{abc} \tag{3.26}$$

Counting the degrees of freedom of $\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \boldsymbol{\mathfrak{B}}, \boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{b}}$, and $\boldsymbol{\mathfrak{c}}$

Because $Y2a$, and $Y2b$ have the same Ferrers sectors, they have the same degrees of freedom, i.e., $deg\,(\mathcal{B}_{abc}) = deg\,(\mathcal{C}_{abc})$. The degrees of freedom are:

$$\deg\left(\mathcal{B}_{abc}\right) = \frac{1}{3} N\,(N+1)\,(N-1) \tag{3.27}$$

$$\deg\left(\mathfrak{b}_{abc}\right) = \frac{1}{3} N\,(N+2)\,(N-2)$$

$$\deg\left(\mathfrak{B}_{abc}\right) = N$$

3.2.3 Decomposing $\mathcal{D}_{abc}$ Under $SO(p,q)$

Since the $Y3$ sector is given by the tableaux

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

the tensor $\mathcal{D}_{abc}$ is totally skew, hence there are no traces we can take. Therefore, it is already irreducible under $SO(p,q)$. To follow our notation from earlier, we define $\boldsymbol{\mathcal{D}} = \boldsymbol{\mathfrak{d}}$, since all lowercase Fraktur tensors are totally trace-free. So the totally skew piece is then defined by $\mathfrak{d}_{abc} = B_{[abc]}$.

Degrees of freedom for $\mathfrak{d}$

This sector's degrees of freedom are already counted by the Young tableaux calculation. We have the result given in equation (3.28).

$$\deg(\mathfrak{d}_{abc}) = \frac{1}{6}N(N-1)(N-2) \tag{3.28}$$

3.3 Summary of the $SO(p,q)$ Decomposition and Maple Procedures

Recall that the $GL(N)$ decomposition using the Young tableaux had 4 sectors, and it was given by equation (3.7), which we repeat here for convenience.

$$B_{abc} = \mathcal{A}_{abc} + \mathcal{B}_{abc} + \mathcal{C}_{abc} + \mathcal{D}_{abc}$$

Now the $SO(p,q)$ decomposition, with arbitrary signature, has 7 sectors and is given by:

$$B_{abc} = \mathfrak{A}_{abc} + \mathfrak{B}_{abc} + \mathfrak{C}_{abc} + \mathfrak{a}_{abc} + \mathfrak{b}_{abc} + \mathfrak{c}_{abc} + \mathfrak{d}_{abc} \tag{3.29}$$

Each of the definitions of these tensors is repeated below. For the $Y1$ sector, we have:

$$\mathcal{A}_{abc} = \mathfrak{a}_{abc} + \mathfrak{A}_{abc}$$

$$\mathfrak{A}_{abc} = \frac{3}{(N+2)} P_{(a}g_{bc)}$$

$$P_c = g^{ab}\mathcal{A}_{abc}$$

For the $Y2$ sectors, we can look at $Y2a$ and $Y2b$. For the $Y2a$ sector, we have:

$$\mathcal{B}_{abc} = \mathfrak{b}_{abc} + \mathfrak{B}_{abc}$$

$$\mathfrak{B}_{abc} = -\frac{2}{(N-1)} P_{[a}g_{|b|c]}$$

$$P_c = g^{ab}\mathcal{B}_{abc}$$

For the $Y2b$ sector, we have:

$$\mathcal{C}_{abc} = \mathfrak{c}_{abc} + \mathfrak{C}_{abc}$$

$$\mathfrak{C}_{abc} = -\frac{2}{(N-1)} R_{[a}g_{b]c}$$

$$R_b = g^{ac}\mathcal{C}_{abc}$$

For the $Y3$ sector, we have:

$$\mathfrak{d}_{[abc]} = \mathcal{D}_{[abc]} = B_{[abc]}$$

Finally, as a sanity check, we check that the degrees of freedom of each of these seven sectors add up to N^3 which they do.

We have created Maple procedures that calculate each of these seven tensors in N-dimensions. The procedures are explained below. Additional procedural details can be found in the modules and the files included with this work.

First, the code for the $Y1$ sector is given by: "Rank3FrankA", and "Rank3TraceA." These commands both take a general tensor (in our case this is B_{abc}) as the first argument and a metric as the second argument. Then they return the $\mathfrak{a}_{abc}$ and likewise $\mathfrak{A}_{abc}$ tensors. Furthermore, the following commands all take the same input. For the $Y2$ sectors, we have "Rank3FrankB", "Rank3TraceB", "Rank3FrankC", and "Rank3TraceC". Finally, for the $Y3$ sector, we have the command "Rank3FrankD" to be comprehensive, although this is really just the SymmetrizeIndices indices command with totally-skew indices.

3.4 Rank 6 Hermitian Spinor Decomposition

Next we move onto the relevant spinor decompositions for rank 3 tensors. Since tensors correspond to Hermitian spinors through the Infeld-Van der Waerden symbols ${\sigma^a}_{AA'}$, it is important to look at them with regard to our rank 3 tensor case. A general rank 3 tensor B_{abc} would turn into a general rank 6 spinor $\chi_{ABCA'B'C'}$. Applying our ideas from before, we can find the irreducible pieces of $\chi_{ABCA'B'C'}$ and thus decompose it. At first, we might think that just taking epsilon traces would be the most expeditious way to analyze $\chi_{ABCA'B'C'}$, but it is not. Instead, the efficient way to go about it is to use Young tableaux on our tensor B_{abc}. Then we build the $SO(p,q)$ irreducible representations of B_{abc}, the tensors $\mathfrak{a}, \mathfrak{A}, \ldots, \mathfrak{d}$. Last, we convert these pieces into their corresponding Hermitian spinors through the Infeld-Van der Waerden symbols. Then we build the $SL(2,\mathbb{C})$ irreducible spinors by taking traces of these spinors with epsilon spinors, ϵ_{AB}. Recall from chapter 2 that $SO(p,q)$ irreducible tensor components correspond to $SL(2,\mathbb{C})$ irreducible spinors.

On the other hand, there are some fruitful ideas which bloom from just starting with $\chi_{ABCA'B'C'}$ and decomposing it using traces. One of which is exposing the valency possibilities of the irreducible spinors. Another is enumerating the number of irreducible spinors. These are the only two fruits which can be gleaned easily from this approach, in the opinion of the author. This method is elaborated in the appendix.

After enumerating the types of spinors we can create (taking epsilons off), we find that we should have a set of: 1 $S_{(ABC)(A'B'C')}$ type spinor, 2 $\eta_{(ABC)A'}$ type spinors, and 4 $\omega_{AA'}$ type spinors. Note, the symmetry is due to a theorem of Stewart: irreducible spinors are symmetric in the same type of indices.

Each of these spinors matches a particular $SO(3,1)$ irreducible tensor. For instance, the $S_{(ABC)(A'B'C')}$ tensor matches the $\mathfrak{a}_{abc}$ tensor in equation (3.17) exactly. When we say match, we mean that the degrees of freedom match, and that there may be additional epsilon terms in a linear combination with the spinor needed for equality. In this instance, for $\mathfrak{a}_{abc}$, we have a direct equality from the solder form (Infeld-Van der Waerden symbols) given by equation (3.30).

$$S_{(ABC)(A'B'C')} = \mathfrak{a}_{abc}\ \sigma^a{}_{AA'}\sigma^b{}_{BB'}\sigma^c{}_{CC'} \tag{3.30}$$

We will also find equations such as:

$$\mathfrak{A}_{abc} \cong \omega_{AA'}\epsilon_{BC}\epsilon_{B'C'} + \omega_{BB'}\epsilon_{AC}\epsilon_{A'C'} + \omega_{CC'}\epsilon_{AB}\epsilon_{A'B'}$$

$$\omega_{CC'} \cong \frac{1}{6}\epsilon^{AB}\epsilon^{A'B'}\mathfrak{A}_{abc}$$

which illustrate how the epsilon trace decomposition works. To elaborate on the degrees of freedom matching, recall that the $SO\left(p,q\right)$ irreducible tensor $\mathfrak{a}$ has $\frac{1}{6}N\left(N+4\right)\left(N-1\right)$ degrees of freedom. To further elaborate on equation (3.30) and the correspondence of $SO\left(p,q\right)$ irreducible tensors to $SL\left(2,\mathbb{C}\right)$ irreducible spinors, we should count the degrees of freedom of $S_{(ABC)(A'B'C')}$. This is made simpler with Young tableaux, although here we have restricted ourselves to $N = 4$ dimensions when working with spinors; recall in $N = 4$ that $\deg\left(\mathfrak{a}_{abc}\right)|_{N=4} = 16$. So to count these, we count the dimension of the unprimed symmetric indices, and then multiply that by the dimension of the primed symmetric indices. The dimension of the tangent space at a point p, T_pM is 4, but each the primed and unprimed space lives only on a complex 2-dimensional space, recall $\mathcal{S}$ and $\overline{\mathcal{S}}$; thus the real $\dim\left(\mathcal{S}\otimes\overline{\mathcal{S}}\right) = 4$. For clarity, let n be the complex dimension of $\mathcal{S}$ and $\overline{\mathcal{S}}$ which is 2; this is halved for hermitian spinors by the reality condition of Penrose and Rindler (Penrose & Rindler, 1987a). This would tell us that the degrees of freedom of $S_{(ABC)(A'B'C')}$ has to be 16.

$$\begin{aligned}
\deg\left(S_{(ABC)(A'B'C')}\right)\Big|_{N=4} &= \left(\frac{1}{6}n\left(n+1\right)\left(n+2\right)\right)\Big|_{n=2}\left(\frac{1}{6}n\left(n+1\right)\left(n+2\right)\right)\Big|_{n=2} \\
&= \left(\left(\frac{1}{6}n\left(n+1\right)\left(n+2\right)\right)\Big|_{n=2}\right)^2 \\
&= 16
\end{aligned}$$

To elaborate more on the “additional epsilon terms, and a linear combination of that spinor needed for equality” statement from earlier in this section, we turn to the tensor $\mathfrak{A}$, and its corresponding spinor $\omega_{AA'}$ as an example. In this case, there were 4 degrees of freedom for $\boldsymbol{A}$. Likewise, there are 4 degrees of freedom for $\omega_{AA'}$; there are not 8 because although $\omega_{AA'}$ is a spinor, it is Hermitian, and its real dimension as just half of what it would have been had it had all coefficients in $\mathbb{C}$. Now we move onto how to find a formula for $\boldsymbol{A}$ in terms of $\omega_{AA'}$.

It may not be intuitively obvious, but it turns out that the easiest way to begin is to start with a spinor $\omega_{AA'}\epsilon_{BC}\epsilon_{B'C'}$ and then force the index symmetry $\mathfrak{A}_{(abc)}$ to hold on $\omega_{AA'}\epsilon_{BC}\epsilon_{B'C'}$. This "turn the crank method" results in the following decomposition for $\boldsymbol{A}$ given in equation 3.31. Additionally, $\omega_{AA'}$ is a hermitian spinor. Recall that we use the congruence/isomorphism symbols to mean equal once the Infeld-Van der Waerden symbols have been applied. See Penrose and Rindler (Penrose & Rindler, 1987a) for more details on their spinor notation, which we follow here.

$$\mathfrak{A}_{abc} \cong \omega_{AA'}\epsilon_{BC}\epsilon_{B'C'} + \omega_{BB'}\epsilon_{AC}\epsilon_{A'C'} + \omega_{CC'}\epsilon_{AB}\epsilon_{A'B'} \tag{3.31}$$

$$\omega_{CC'} \cong \frac{1}{6}\epsilon^{AB}\epsilon^{A'B'}\mathfrak{A}_{abc}$$

And again we can count the degrees of freedom of $\omega_{AA'}$ and find there to be 4.

$$\deg\left(\omega_{AA'}\right) = \left.(n)\right|_{n=2} \left.(n)\right|_{n=2} = 4$$

At this point, we still need to write the spinor form of each of the other $SO\left(p,q\right)$ irreducible tensors, and write the corresponding $SL\left(2,\mathbb{C}\right)$ irreducible spinors in terms of traces of the tensors. From before, we know that we have seven tensors which we need to write the spinor decomposition for: $\boldsymbol{\mathfrak{a}}, \boldsymbol{\mathfrak{A}}, \boldsymbol{\mathfrak{b}}, \boldsymbol{\mathfrak{B}}, \boldsymbol{\mathfrak{c}}, \boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{d}}$. We already have the trace piece $Y1$ sector spinor decomposition. For the tensor $\mathfrak{A}_{abc}$ this is given by equation (3.31) which is determined by the omega spinor $\omega_{AA'}$. The trace-free part of $Y1$ is $\mathfrak{a}_{abc}$ and is completely determined by the spinor $\boldsymbol{S}$. This spinor is already irreducible under $SL\left(2,\mathbb{C}\right)$ without taking any epsilon traces as in equation (3.32).

$$\mathfrak{a}_{(abc)} \cong S_{(ABC)(A'B'C')} \tag{3.32}$$

The $Y2$ and $Y3$ irreducible spinor decompositions are similar in structure. The spinor decompositions of $\mathfrak{b}_{abc}$, and $\mathfrak{c}_{abc}$ are given by equations (3.33), and (3.34). Note that we can get $\bar{\eta}_{(A'B'C')B}$ ($\bar{\sigma}_{(A'B'C')C}$) by taking a trace with ϵ^{AC} (ϵ^{AB}) instead of with $\epsilon^{A'C'}$ ($\epsilon^{A'B'}$).

$$\mathfrak{b}_{abc} \cong \epsilon_{A'C'}\ \eta_{(ABC)B'} + \epsilon_{AC}\ \bar{\eta}_{(A'B'C')B}, \quad \eta_{(ABC)B'} \cong \frac{1}{2}\mathfrak{b}_{abc}\epsilon^{A'C'} \tag{3.33}$$

$$\mathfrak{c}_{abc} \cong \epsilon_{A'B'}\ \sigma_{(ABC)C'} + \epsilon_{AB}\ \bar{\sigma}_{(A'B'C')C}, \quad \sigma_{(ABC)C'} \cong \frac{1}{2}\mathfrak{c}_{abc}\epsilon^{A'B'} \tag{3.34}$$

Furthermore, the spinor decompositions for $\mathfrak{B}_{[a|b|c]}$, and $\mathfrak{C}_{[ab]c}$ are given by equations (3.35), and (3.36). Additionally, $\tau_{CC'}$, and $\psi_{BB'}$ are hermitian spinors; this happens naturally when taking the epsilon traces and using the Infeld-Van der Waerden symbols.

$$\mathfrak{B}_{abc} \cong \tau_{AA'}\ \epsilon_{BC}\ \epsilon_{B'C'} - \tau_{CC'}\ \epsilon_{AB}\ \epsilon_{A'B'}, \quad \tau_{CC'} \cong -\frac{1}{3}\epsilon^{AB}\epsilon^{A'B'}\mathfrak{B}_{abc} \tag{3.35}$$

$$\mathfrak{C}_{abc} \cong \psi_{AA'}\ \epsilon_{BC}\ \epsilon_{B'C'} - \psi_{BB'}\ \epsilon_{AC}\ \epsilon_{A'C'}, \quad \psi_{BB'} \cong -\frac{1}{3}\epsilon^{AC}\epsilon^{A'C'}\mathfrak{C}_{abc} \tag{3.36}$$

To finish, the spinor decomposition for the totally skew piece is more complicated, requiring six different terms to produce the correct symmetry of the sector. Additionally, the irreducible spinor $\Xi_{DD'}$ which appears in equation (3.37) naturally occurs as an antihermitian spinor. It is curious that this occurs in this fashion. Nevertheless, we can turn this antihermitian spinor into a Hermitian spinor by removing an i from equation (3.37) and defining the new hermitian spinor. However, we will not do that here, as we believe this is indicative of some more general trend. Redefining things in this fashion and may obfuscate our classification aims later.

$$\begin{aligned} \mathfrak{d}_{abc} &\cong \epsilon_{BC}\epsilon_{A'C'}\Xi_{AB'} - \epsilon_{AC}\epsilon_{B'C'}\Xi_{BA'} \\ &+ \epsilon_{BC}\epsilon_{A'B'}\Xi_{AC'} - \epsilon_{AB}\epsilon_{B'C'}\Xi_{CA'} \\ &+ \epsilon_{AC}\epsilon_{A'B'}\Xi_{BC'} - \epsilon_{AB}\epsilon_{A'C'}\Xi_{CB'} \end{aligned} \tag{3.37}$$

$$\Xi_{DD'} = -\frac{1}{9}\epsilon^{AB}\epsilon^{A'C'}\mathfrak{d}_{ABDA'D'C'}, \quad \Xi_{DD'} = -\bar{\Xi}_{DD'}$$

We can further classify these spinors using the tools of Penrose and Rindler (Penrose & Rindler, 1987b), first by looking at the structural reducibility (algebraic irreducibility) of these $SL(2,\mathbb{C})$ irreducible spinors. What we mean by structural reducibility is that a given spinor can be written as a product of lower valence spinors. The spinor $\Xi_{DD'}$ for example, can be written as $\kappa_D \bar{\kappa}_{D'}$ for some components. This only occurs when the condition $\Xi_{DD'}\Xi^{DD'} = 0$ holds. Upon examining the structural reducibility of the spinors, we can find $SL(2,\mathbb{C})$ or $SO(3,1)$ invariant quantities. It is these quantities which we will use to classify possible geometries in ECSK theory. To further build on these ideas, Penrose (Penrose, 1972) used the ideas of algebraic geometry to examine the topology and singularity structure of the locus of these spinors ($\Xi_{DD'}\xi^D\eta^{D'} = 0$) as viewed on the null cone ($\epsilon_{AB}\epsilon_{A'B'}\xi^A\xi^B\eta^{A'}\eta^{B'} = 0$). We will not do this here since there is no algorithm given in the general case, but it is an exciting possibility for future work.

There are twelve new procedures coded up in the ECSK module to handle and compute these spinors. The first six which calculate the corresponding spinors given a tensor and a solder form are: *Rank3SSpinor*, *Rank3OmegaSpinor*, *Rank3TauSpinor*, *Rank3PsiSpinor*, *Rank3XiSpinor*, *Rank3EtaSpinor*, and *Rank3SigmaSpinor*. Out of these, the commands *Rank3EtaSpinor*, and *Rank3SigmaSpinor* can take an optional "barspinor" which returns the conjugate spinor instead of the original spinor. Next we have the other six procedures which generate a $SO(3,1)$ irreducible 3rd rank tensor from a given spinor of that type and a solder form: *Rank3GenerateSTensor*, *Rank3GenerateOmegaTensor*, *Rank3GenerateTauTensor*, *Rank3GeneratePsiTensor*, *Rank3GenerateXiTensor*, *Rank3GenerateEtaTensor*, and *Rank3GenerateSigmaTensor*. Where again the Eta and Sigma options can take an optional argument, their conjugate spinor as the third argument. We recommend using this optional argument.

In the next section, we establish and develop a classification for the torsion tensor up to structural reducibility.

3.5 $GL(N)$ and $SO(p,q)$ Reducibility of the Torsion Tensor

In this section, we establish and develop a classification for the torsion tensor up to structural reducibility. We will address the classification of the torsion tensor according to $GL\left(N\right)$, $SO\left(p,q\right)$, spinorally under $SL\left(2,\mathbb{C}\right)$, and then examine the structural reducibility (algebraic irreducibility) of the spinors in that order.

3.5.1 Decomposing the Torsion Tensor Under $GL\left(N\right)$

To begin, recall that the torsion tensor, which we denote by $T^{a}{}_{bc}$, is skew-symmetric on its last two indices: $T^{a}{}_{bc} = T^{a}{}_{[bc]}$. Because we have a metric in ECSK theory, we can lower the first index of the torsion tensor to make it totally covariant, $T_{abc} = T_{a[bc]}$. Now, we apply the ideas we developed for irreducible rank 3 tensors/spinors above to the torsion tensor. The torsion tensor has three irreducible $GL\left(N\right)$ sectors. These are given by the Young tableaux in equation (3.38).

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ ,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ ,\quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \tag{3.38}$$

However, we will find that the first tableaux do not share the same symmetries of the torsion tensor, and therefore they cannot be irreducible representations. Itin and Reches (Itin & Reches, 2021) give an excellent explanation of this and how to account for it. Since these tableaux do not have the same symmetry as the last two indices of the torsion tensor, we need to compensate because there still is an irreducible representation in the elbow Ferrers diagram:

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ .$$

We will call this irreducible representation $\mathcal{Q}_{abc}$, where $\mathcal{Q}_{abc}$ is defined as follows:

$$\mathcal{Q}_{abc} = \frac{1}{3}\left(2T_{abc} + T_{bac} + T_{cba}\right), \quad \mathcal{Q}_{abc} = \mathcal{Q}_{a[bc]} \tag{3.39}$$

This tensor has the same symmetry as the torsion tensor, and is in fact irreducible. We can see that it is irreducible by noting it is the piece leftover once the totally skew part ($Y3$ sector) of the torsion is subtracted off. A proof of this method is given by Itin and Reches

(Itin & Reches, 2021); furthermore, Fulton and Harris also guarantee that this methodology will work (Fulton & Harris, 1991). The $Y3$ Young sector is given by $\mathcal{D}_{abc}$ as before, but the formula simplifies to three terms because of the skew indices the torsion tensor has. This is given by equation (3.40).

$$\mathcal{D}_{abc} = \frac{1}{3}\left(T_{abc} + T_{bca} + T_{cab}\right) \tag{3.40}$$

The $GL\left(N\right)$ decomposition of the torsion tensor is given by:

$$T_{abc} = \mathcal{Q}_{abc} + \mathcal{D}_{abc} \tag{3.41}$$

As an aside, if we add the tensors $\mathcal{B}_{abc}$, and $\mathcal{C}_{abc}$ from equation (2.8) together, we find that they equal $\mathcal{Q}_{abc}$.

$$\mathcal{Q}_{abc} = \mathcal{B}_{abc} + \mathcal{C}_{abc} \tag{3.42}$$

Moving forward, we count the number of independent components of the irreducible sectors $\boldsymbol{\mathcal{Q}}$, and $\boldsymbol{\mathcal{D}}$. Formulas are given below.

$$\dim\left(\boldsymbol{\mathcal{Q}}\right) = \frac{1}{3}N\left(N+1\right)\left(N-1\right)$$

$$\dim\left(\boldsymbol{\mathcal{D}}\right) = \frac{1}{6}N\left(N-1\right)\left(N-2\right)$$

Let the space of the torsion tensor be $\boldsymbol{T}$, the space representing the elbow tableau be $\boldsymbol{\mathcal{Q}}$, and the space representing the vertical tableau space be $\boldsymbol{\mathcal{D}}$, then the final irreducible subspace decomposition is given by:

$$\boldsymbol{T} = \boldsymbol{\mathcal{Q}} \oplus \boldsymbol{\mathcal{D}}$$

3.5.2 Decomposing the Torsion Tensor Under $SO\left(p,q\right)$

Next, we derive the $SO\left(p,q\right)$ decomposition for the torsion tensor. The only element which is further reducible is $\mathcal{Q}_{abc}$ from equation (3.41) which splits into two tensors $\mathfrak{Q}_{abc}$, and $\mathfrak{q}_{abc}$. Keeping with the earlier notation, $\mathfrak{Q}_{abc}$ is the trace-full piece, and $\mathfrak{q}_{abc}$ is the trace-free not-totally-skew piece (or as the author likes to call it, the leftovers piece); note that both of these tensors are skew on the last two indices as well. The $SO\,(p,q)$ decomposition is then given by equation (3.43) which is shown below.

$$T_{abc} = \mathfrak{Q}_{abc} + \mathfrak{q}_{abc} + \mathfrak{d}_{abc} \tag{3.43}$$

$$\mathfrak{Q}_{abc} = \frac{1}{(N-1)}\left(P_c g_{ab} - P_b g_{ac}\right), \quad P_c = g^{ef}\mathcal{Q}_{efc} \tag{3.44}$$

$$\mathfrak{q}_{abc} = \mathcal{Q}_{abc} - \mathfrak{Q}_{abc} \tag{3.45}$$

As another aside, the new tensor $\boldsymbol{\mathfrak{Q}}$ can be calculated by adding the tensors $\boldsymbol{\mathfrak{B}}$, and $\boldsymbol{\mathfrak{C}}$ together from equations (3.21) and (3.24). Likewise, the new tensor $\boldsymbol{\mathfrak{q}}$ can be calculated by adding the tensors $\boldsymbol{\mathfrak{b}}$, and $\boldsymbol{\mathfrak{c}}$ together from equations (3.22) and (3.25). These both mirror equation (3.42), and are shown below. Recall that $\boldsymbol{\mathfrak{d}} = \boldsymbol{\mathcal{D}}$ from equation (3.40), we just rewrote it to keep with our notation.

$$\mathfrak{d}_{abc} = \mathcal{D}_{abc} \tag{3.46}$$

$$\mathfrak{Q}_{abc} = \mathfrak{B}_{abc} + \mathfrak{C}_{abc} \tag{3.47}$$

$$\mathfrak{q}_{abc} = \mathfrak{b}_{abc} + \mathfrak{c}_{abc} \tag{3.48}$$

Each of these $SO\,(p,q)$ irreducible tensors can be calculated with the following Maple procedures: *TorsionTraceQ*, *TorsionFrankQ*, and *TorsionFrankD*. They each take a torsion tensor and a metric as the inputs.

Before we move to the spinor decomposition of the torsion tensor, we remark that there are other versions of this decomposition presented in the literature. One of which is that of Shapiro (Shapiro, 2001). We explore the comparison in appendix A.

3.6 Spinor Decomposition of the Torsion Tensor

With the torsion tensor decomposed under $SO\,(p,q)$, we would like to find the spinor representation of the torsion tensor. Because we know the form for the spinors related to the tensors $\mathfrak{B}, (\mathfrak{C})$, and $\mathfrak{b}, (\mathfrak{c})$ we can write out the tensors $\mathfrak{Q}$, and $\mathfrak{q}$ in terms of their related spinors. We add these spinors together as we did in equations (3.42), (3.47), and (3.48), then use the Jacobi identity (see Stewart (Stewart, 1993)):

$$\epsilon_{A[B}\epsilon_{CD]} = 0$$

$$\epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC} = 0$$

and relabel terms to determine these spinors. Recall that we are using Penrose and Rindler's (Penrose & Rindler, 1987a) notation to suppress some indices. The formula for the tensor $\mathfrak{Q}$, in terms of its spinor equivalent, $\boldsymbol{\Theta}$ is:

$$\mathfrak{Q}_{abc} \cong \Theta_{BB'}\ \epsilon_{AC}\epsilon_{A'C'} - \Theta_{CC'}\ \epsilon_{AB}\epsilon_{A'B'}, \quad \Theta_{CC'} \cong -\frac{1}{3}\epsilon^{AB}\epsilon^{A'B'}\mathfrak{Q}_{abc} \tag{3.49}$$

Likewise, the formula for the tensor $\mathfrak{q}$ and its spinor equivalent is:

$$\mathfrak{q}_{abc} \ \cong \ \epsilon_{B'C'}\ \Omega_{(ABC)A'} + \epsilon_{BC}\ \bar{\Omega}_{(A'B'C')A}, \quad \Omega_{(ABC)A'} \cong \frac{1}{2}\mathfrak{q}_{abc}\epsilon^{B'C'} \tag{3.50}$$

Finally, for completeness we repeat the equation for the spinor equivalent to the tensor $\mathfrak{d}$ which was given by equation (3.37) here.

$$\begin{aligned} \mathfrak{d}_{abc} \ &\cong \ \epsilon_{BC}\epsilon_{A'C'}\Xi_{AB'} - \epsilon_{AC}\epsilon_{B'C'}\Xi_{BA'} \\ &+ \ \epsilon_{BC}\epsilon_{A'B'}\Xi_{AC'} - \epsilon_{AB}\epsilon_{B'C'}\Xi_{CA'} \\ &+ \ \epsilon_{AC}\epsilon_{A'B'}\Xi_{BC'} - \epsilon_{AB}\epsilon_{A'C'}\Xi_{CB'} \end{aligned} \tag{3.51}$$

$$\Xi_{DD'} = -\frac{1}{9}\epsilon^{AB}\epsilon^{A'C'}\mathfrak{d}_{ABDA'D'C'}, \quad \Xi_{DD'} = -\bar{\Xi}_{DD'}$$

In our spinor decomposition of the torsion tensor (in 4-D with signature $[+,-,-,-]$), we find that T_{abc} is entirely determined by 3 spinors: Θ, Ω, and Ξ (where we leave ϵ out because it is the object left invariant by $SL(2,\mathbb{C})$).

There are 6 new Maple procedures which allow us to work with these three spinors.

The procedures are: *TorsionThetaSpinor*, *TorsionOmegaSpinor*, *TorsionXiSpinor*, *TorsionGenerateOmegaTensor*, *TorsionGenerateThetaTensor*, and *TorsionGenerateXiTensor*. The first three take a torsion tensor and a solder form as their arguments and output the corresponding spinor. The last three take a spinor and a solder form as their arguments and form a corresponding $SO(3,1)$ irreducible part of the torsion tensor. Furthermore, both of the Omega spinor commands take optional third arguments. The first Omega command takes "barspinor" to return the conjugate spinor instead. The second Omega command takes the conjugate spinor as its third optional argument. We recommend using the optional argument when constructing the tensor from the Omega spinor.

Now we will look at the structural reducibility of each of the three spinors: Θ, Ω, and Ξ. The classifications of the spinors Θ, and Ξ are equivalent and relatively simple due to there being only one invariant which distinguishes the cases which aren't zero. The classification of the spinor Ω however is far more interesting and yields several new and unexplored possibilities for structural classification purposes.

Next we examine the structural classification for each of these spinors and enumerate all the different possibilities while neglecting any interplay between the spinors such as being considered in the excellent references (Zakhary & Carminati, 2001), (Carminati et al., 2002), (Carminati & Zakhary, 2002), (Carminati & McLenaghan, 1991). Applying the mindset of the references above would be yet another direction for future work. For clarity this means we neglect any co(in)variants resulting from spinors like $\zeta_{(BC)}$ (defined soon), (or Z) which would arise from calculations like: $\Omega_{(ABC)A'}\Xi^{AA'} = \zeta_{(BC)}$, (or $Z = \zeta_{(BC)}\zeta^{(BC)}$). Penrose and Rindler (Penrose & Rindler, 1987b) make mention of these types of co(in)variants on Pgs. 262-264, which again would be an excellent direction for further research.

There are three spinors which we need to structurally reduce: Θ, Ξ, and Ω, but because Θ, and Ξ have the same valence their structural reducibility is the same; this type of analysis holds for any spinor of said valence. First we will examine the structural reducibility of Ω, and then look at Θ, and Ξ. Then we will enumerate the number of distinct cases which can occur with respect to structural reducibility.

3.6.1 Structural Reducibility of: $\Omega_{(ABC)A'}$

For the torsion $\mathbf{\Omega}$ spinor there are eight different structural reducibility types. Recall that by structural reducibility we mean that the spinor cannot be written as a product; this coincides with the definition of algebraic irreducibility. The eight possible cases are enumerated below. Following the ideas of Penrose and Rindler (Penrose & Rindler, 1987b) Pg. 268 we develop a notation which describes the structural decomposition of the spinor $\Omega_{(ABC)A'}$; this is given in table (3.1) shown below. Table (3.1) enumerates the eight different

Cases for the Torsion Omega Spinor			
Cases	Spinor	Our Notation	Penrose Notation
Case 1	$\Omega_{(ABC)A'}$	[31]	$(3,1)$
Case 2	$\delta_{(A}\beta_{BC)A'}$	$[A,21]$	$(1,0)(2,1)$
Case 3	$\delta_{(A}\lambda_B\theta_{C)A'}$	$[A,B,11]$	$(1,0)(1,0)(1,1)$
Case 4	$\delta_{(A}\delta_B\theta_{C)A'}$	$[A^2,11]$	$(1,0)^2(1,1)$
Case 5	$\delta_{(A}\lambda_B\omega_{C)}\xi_{A'}$	$[A,B,C,D']$	$(1,0)(1,0)(1,0)(0,1)$
Case 6	$\delta_{(A}\delta_B\omega_{C)}\xi_{A'}$	$[A^2,B,C']$	$(1,0)^2(1,0)(0,1)$
Case 7	$\delta_{(A}\delta_B\delta_{C)}\xi_{A'}$	$[A^3,B']$	$(1,0)^3(0,1)$
Case 8	0	$[-]$	$(-)$

Table 3.1: Structural Reducibility Table of the Spinor $\Omega_{(ABC)A'}$

cases, the spinors corresponding to those cases, and two different notations to describe those cases. In each of the cases, we have a p,q spinor where there are p unprimed indices, and q primed indices; this is important for either notation.

In our notation, we use commas to separate different spinors. Unless the spinor is valence 1, we have a number which is meant to be read as a pair. The left number in this pair is the number of unprimed indices the spinor has, and the right number is the number of primed indices the spinor has. When we have a valence 1 spinor, we represent this with capital Latin letters or primed letters; the prime distinguishes a valence one spinor living on $\overline{\mathcal{S}}$ from one living on $\mathcal{S}$. For example, we would write A for a valence one unprimed spinor. Likewise, we would write A' for a valence one primed spinor. Any time a spinor of the same valence is repeated, we write a number representing how many times it has been repeated in the exponent. For example, if the spinor represented by A appears twice, we would write A^2. Our notation also represents complex conjugation in the sense that if we write AA' then the spinor represented by A' is the complex conjugate of A. If the spinor is not the complex conjugate, we would write AB'. The authors feel it useful to include our new notation here as a different approach which may be more intuitive with some readers than others.

Penrose and Rindler's notation accomplishes the same thing, where the numbers in parentheses represent the valence p, q of the spinor. Similarly to us, they also use an exponent to represent a repeated spinor.

As an example, in Case 1 in table (3.1), the spinor $\Omega_{(ABC)A'}$ is irreducible, and we would write [31] in our notation or $(3,)$ in Penrose and Rindler's notation. To clarify further, take Case 2 in table (3.1) for which we have $(1, 0)(2, 1)$. The parenthesis tell us that there are two spinors which are irreducible one of which is a single unprimed spinor $(1, 0)$, and the other is an irreducible spinor with two unprimed indices, and one primed index. Finally, to describe the last piece of possible notation, take case 4, where we have $(1, 0)^2(1, 1)$. The parenthesis indicate that there are three irreducible spinors, two of which are repeated, with one unprimed index, and the last spinor has one primed and one unprimed index $(1, 1)$. The very last case where we have $[-]$, and $(-)$ indicate that the entire spinor is zero, hence the lack of any spinors in the notation.

3.6.2 Structural Reducibility of: $\Theta_{CC'}$ and $\Xi_{DD'}$

Next we consider the spinors $\Theta_{CC'}$ and $\Xi_{DD'}$. We only need a classification for a spinor of type $\Theta_{CC'}$ and then we know the structural reducibility for both $\boldsymbol{\Theta}$, and $\boldsymbol{\Xi}$. Here the special names or notation are the same as before, but with one addition. We now have $|(1, 0)|^2$; the absolute value bars with the squared symbol suggests a "square modulus." This is different from $(1, 0)(0, 1)$ which would represent a spinor like $\xi_A \eta_{A'}$, because $|(1, 0)|^2$ means $\xi_A \bar{\xi}_{A'}$. The second spinor is the complex conjugate of the first, hence the "squared modulus."

In our notation, this is reflected by A, A' where we can tell by the repeated A that the spinor represented by A' is the complex conjugate of the spinor represented by A.

There are only three cases since $\mathbf{\Theta}$ is hermitian and $\mathbf{\Xi}$ is anti-hermitian. The decomposition for $\mathbf{\Xi}$ is slightly different in Case 2 because of the i included in front; this is to ensure the decomposition is also antihermitian, just like $\mathbf{\Xi}$ is.

We present table (3.2) next to illustrate each of these three cases, as we did for the spinor $\Omega_{(ABC)A'}$ in table (3.1). Now that we have the structural reducibility for $\mathbf{\Omega}$, $\mathbf{\Theta}$,

Cases for the Torsion Theta and Xi Spinors			
Cases	Spinor	Our Notation	Penrose Notation
Case 1	$\Theta_{CC'}, \Xi_{DD'}$	[11]	$(1,1)$
Case 2	$\kappa_C \bar{\kappa}_{C'}, i\kappa_C \bar{\kappa}_{C'}$	$[A, A']$	$\lvert(1,0)\rvert^2$
Case 3	0	[−]	(−)

Table 3.2: Structural Reducibility Table of the Spinors $\Theta_{CC'}$ and $\Xi_{DD'}$

and $\mathbf{\Xi}$, we find that the total number of cases would be $(8)\,(3)\,(3) = 72$. This is already a large number and could be larger still by refining the classification with the algebraic geometry methods of Penrose (Penrose, 1972). Future work would be to examine this "Penrose Refinement" and see how it correlates to physical frame independent observables in ECSK theory. Additionally, it would be excellent to have an algorithm built to determine these cases.

3.7 Algorithm to Classify the Torsion Tensor

In the last section, we enumerated all the different structural reducibility cases which were possible for the torsion tensor. In this section, we present an algorithm to determine each of the possibilities using covariants and invariants built from the spinors. First, we will begin with the $\mathbf{\Omega}$ spinor, and discuss the algorithm to determine its structural reducibility. We also discuss the co(in)variants that we use to determine the cases. Then we classify the spinors $\Theta_{CC'}$, and $\Xi_{DD'}$ with the same method, albeit different co(in)variants. With these three together, we will have classified the torsion tensor up to structural reducibility and provided an algorithm which determines the total structural reducibility type as well.

3.8 The Invariants and Covariants for $\Omega_{(ABC)A'}$

To begin with, there are seven different co(in)variants which determine the structural reducibility type of the $\mathbf{\Omega}$ spinor. The insight behind these follows very much from Zakhary and Carminati (Zakhary & Carminati, 2004), and the spirit with which the algorithm is constructed also follows them as well; furthermore the following references: (Acvevedo et al., 2006) (Zakhary et al., 2003) (Zakhary & Carminati, 2004) (Zakhary & Carminati, 2001) (Carminati et al., 2002) (Carminati & Zakhary, 2002), and (Carminati & McLenaghan, 1991) all expand upon this work as they discuss cross co(in)variants which we do not touch on here.

The spinor co(in)variants which determine the structural reducibility for $\mathbf{\Omega}$ will be denoted by: $\mathbf{\Omega}$, $\mathbf{N}$, $\mathbf{\Upsilon}$, $\mathbf{\Theta}$, $\boldsymbol{I_6}$, $\boldsymbol{R}$, and $\mathbf{\Delta}$. The $\mathbf{\Theta}$ here is not to be confused with the torsion theta spinor, but has an inconvenient name due to the nature of its development. We provide both a tensor form of the co(in)variants, and a Newman-Penrose (NP) inspired form which is useful for computer algebra calculations; this makes the calculations more computationally efficient than applying several tensor contractions. See Penrose and Rindler (Penrose & Rindler, 1987a), and Stewart (Stewart, 1993) for more details on the Newman-Penrose formalism. First, $\Omega_{(ABC)A'}$ is itself a covariant (in the words of Olver (Olver, 2003)); we can use $\mathbf{\Omega}$ to determine if we are in Case $8/(-)$ or not.

We define the $\mathbf{\Omega}$ components by contracting a spin basis o^A, ι^A for the two-dimensional, complex, symplectic vector space which we have been calling $\mathcal{S}$ onto $\Omega_{(ABC)A'}$.

$$\Omega_{(ABC)A'} \tag{3.52}$$

$$\begin{aligned}
\Omega_0 &= \Omega_{(ABC)A'}o^A o^B o^C \bar{o}^{A'} \\
\Omega_1 &= \Omega_{(ABC)A'}o^A o^B o^C \bar{\iota}^{A'} \\
\Omega_2 &= \Omega_{(ABC)A'}o^A o^B \iota^C \bar{o}^{A'} \\
\Omega_3 &= \Omega_{(ABC)A'}o^A o^B \iota^C \bar{\iota}^{A'} \\
\Omega_4 &= \Omega_{(ABC)A'}o^A \iota^B \iota^C \bar{o}^{A'} \\
\Omega_5 &= \Omega_{(ABC)A'}o^A \iota^B \iota^C \bar{\iota}^{A'} \\
\Omega_6 &= \Omega_{(ABC)A'}\iota^A \iota^B \iota^C \bar{o}^{A'} \\
\Omega_7 &= \Omega_{(ABC)A'}\iota^A \iota^B \iota^C \bar{\iota}^{A'}
\end{aligned}$$

Next, the spinor N is defined as, $N_{(AB)(A'B')}$ which is determined from the following equation:

$$N_{(AB)}{}^{(A'B')} = \Omega^{EF}{}_{(A}{}^{(A'}\Omega_{B)EF}{}^{B')} \tag{3.53}$$

where to get the covariant $N_{(AB)(A'B')}$ we just lower the indices in equation (3.53) with epsilon spinors. To get each of these covariants, we contract on the spin basis to $N_{(AB)(A'B')}$ as follows.

$$\begin{aligned}
N_0 &= N_{(AB)(A'B')}\ o^A o^B \bar{o}^{A'} \bar{o}^{B'} \\
N_1 &= N_{(AB)(A'B')}\ o^A o^B \bar{o}^{A'} \bar{\iota}^{B'} \\
N_2 &= N_{(AB)(A'B')}\ o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'} \\
N_3 &= N_{(AB)(A'B')}\ o^A \iota^B \bar{o}^{A'} \bar{o}^{B'} \\
N_4 &= N_{(AB)(A'B')}\ o^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'} \\
N_5 &= N_{(AB)(A'B')}\ o^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'} \\
N_6 &= N_{(AB)(A'B')}\ \iota^A \iota^B \bar{o}^{A'} \bar{o}^{B'} \\
N_7 &= N_{(AB)(A'B')}\ \iota^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'} \\
N_8 &= N_{(AB)(A'B')}\ \iota^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'}
\end{aligned}$$

The NP equations of the N covariant are given below in terms of the $\boldsymbol{\Omega}$ covariant.

$$\begin{aligned}
N_0 &= 2\Omega_0\Omega_4 - 2\left(\Omega_2\right)^2 \\
N_1 &= 2\Omega_0\Omega_5 + 2\Omega_1\Omega_4 - 4\Omega_2\Omega_3 \\
N_2 &= 2\Omega_1\Omega_5 - 2\left(\Omega_3\right)^2 \\
N_3 &= 2\Omega_0\Omega_6 - 2\Omega_2\Omega_4 \\
N_4 &= 2\Omega_0\Omega_7 + 2\Omega_1\Omega_6 - 2\Omega_2\Omega_5 - 2\Omega_3\Omega_4 \\
N_5 &= 2\Omega_1\Omega_7 - 2\Omega_3\Omega_5 \\
N_6 &= 2\Omega_2\Omega_6 - 2\left(\Omega_4\right)^2 \\
N_7 &= 2\Omega_2\Omega_7 + 2\Omega_3\Omega_6 - 4\Omega_4\Omega_5 \\
N_8 &= 2\Omega_3\Omega_7 - 2\left(\Omega_5\right)^2
\end{aligned}$$

We have shown that the N covariant is zero in cases 7 or 8 by a direct calculation proof. The spinor $N_{(AB)(A'B')}$ is zero if and only if we are in case 7 or 8.

Next, we define the covariant $\Upsilon_{(A'B'C'D')}$ by the following equations.

$$\Upsilon_{(A'B'C'D')} = N_{EF(A'B'}N^{EF}{}_{C'D')} \tag{3.54}$$

$$\begin{aligned}
\Upsilon_0 &= 2N_0N_6 - \frac{1}{2}\left(N_3\right)^2 \\
\Upsilon_1 &= 2N_0N_7 + 2N_1N_6 - N_3N_4 \\
\Upsilon_2 &= 2N_0N_8 - N_3N_5 + 2N_2N_6 + 2N_1N_7 - \frac{1}{2}\left(N_4\right)^2 \\
\Upsilon_3 &= 2N_1N_8 + 2N_2N_7 - N_4N_5 \\
\Upsilon_4 &= 2N_2N_8 - \frac{1}{2}\left(N_5\right)^2
\end{aligned}$$

The covariant $\Upsilon_{(A'B'C'D')}$ is zero in Cases 4,6,7,8 by direct calculation.

Next we have the $\Theta_{(ABCD)}$ covariant which is defined as follows.

$$\Theta_{(ABCD)} = \Omega_{(AB}{}^{FF'}\Omega_{CD)FF'} \tag{3.55}$$

$$\begin{aligned}
\Theta_0 &= 2\Omega_0\Omega_3 - 2\Omega_1\Omega_2 \\
\Theta_1 &= 4\Omega_0\Omega_5 - 4\Omega_1\Omega_4 \\
\Theta_2 &= 2\Omega_0\Omega_7 - 2\Omega_1\Omega_6 + 6\Omega_2\Omega_5 - 6\Omega_3\Omega_4 \\
\Theta_3 &= 4\Omega_2\Omega_7 - 4\Omega_3\Omega_6 \\
\Theta_4 &= 2\Omega_4\Omega_7 - 2\Omega_5\Omega_6
\end{aligned}$$

This covariant is given by a different contraction of the $\mathbf{\Omega}$ covariants. The covariant $\mathbf{\Theta}$ is zero in Cases 5,6,7,8 by direct calculation. This covariant combined with the covariant $\mathbf{\Upsilon}$ from equation (3.54) can be used to determine if we are in Cases 4,5, or 6.

Next we define the I_6 invariant, which is really Ω^2. It is given by equation (3.56) shown below.

$$I_6 = \Omega_{(ABC)A'}\Omega^{(ABC)A'} \tag{3.56}$$

$$I_6 = 2\Omega_0\Omega_7 - 2\Omega_1\Omega_6 - 6\Omega_2\Omega_5 + 6\Omega_3\Omega_4$$

The invariant I_6 is zero in cases 4,5,6,7,8 by direct calculation.

Our penultimate covariant is $R_{(ABCDEF)}$ which is defined by the following equations.

$$R_{(ABCDEF)} = 32\Theta_{(ABC}{}^{K}\Theta_{DE}{}^{LM}\Theta_{F)KLM} \tag{3.57}$$

$$\begin{aligned}
R_0 &= -8(\Theta_0)^2\Theta_3 + 4\Theta_0\Theta_1\Theta_2 - (\Theta_1)^3 \\
R_1 &= -32(\Theta_0)^2\Theta_4 - 4\Theta_0\Theta_1\Theta_3 + 8\Theta_0(\Theta_2)^2 - 2(\Theta_1)^2\Theta_2 \\
R_2 &= -40\Theta_0\Theta_1\Theta_4 + 20\Theta_0\Theta_2\Theta_3 - 5(\Theta_1)^2\Theta_3 \\
R_3 &= 20\Theta_0(\Theta_3)^2 - 20(\Theta_1)^2\Theta_4 \\
R_4 &= 40\Theta_0\Theta_3\Theta_4 - 20\Theta_1\Theta_2\Theta_4 + 5(\Theta_1)^2\Theta_3 \\
R_5 &= 32\Theta_0(\Theta_4)^2 + 4\Theta_1\Theta_3\Theta_4 - 8(\Theta_2)^2\Theta_4 + 2\Theta_2(\Theta_3)^2 \\
R_6 &= 8\Theta_1(\Theta_4)^2 - 4\Theta_2\Theta_3\Theta_4 + (\Theta_3)^3
\end{aligned}$$

The R covariant is zero in cases 3,4,5,6,7,8 by direct calculation. The factor of 32 is in the definition to simplify the denominator in the resulting NP formulas.

Our final invariant is Δ. To define Δ we also need to define the invariants I_p, and J_p which come from (Zakhary & Carminati, 2004). The invariants I_p, and J_p are defined by:

$$I_p = \Theta_{ABCD}\Theta^{ABCD} \tag{3.58}$$

$$I_p = 2\Theta_0\Theta_4 - \frac{1}{2}\Theta_1\Theta_3 + \frac{1}{6}(\Theta_2)^2 \tag{3.59}$$

$$J_p = \Theta_{ABCD}\Theta^{CDEF}\Theta_{EF}^{\ \ AB} \tag{3.60}$$

$$J_p = \Theta_0\Theta_2\Theta_4 - \frac{1}{36}(\Theta_2)^3 - \frac{3}{8}\Theta_0(\Theta_3)^2 - \frac{3}{8}(\Theta_1)^2\Theta_4 + \frac{1}{8}\Theta_1\Theta_2\Theta_3 \tag{3.61}$$

which allow us to define Δ as follows.

$$\Delta = (I_p)^3 - 6(J_p)^2 \tag{3.62}$$

The invariant Δ acts like a discriminant; it tells us when the structure is reducible, and thus is zero in cases 2,3,4,5,6,7,8. We determined that this is the case by a direct calculation.

Each of these co(in)variants can be calculated in Maple with the following new commands: *TorsionOmegaCovariant*, *TorsionNuCovariant*, *TorsionUpsilonCovariant*, *TorsionThetaCovariant*, *TorsionI6Invariant*, *TorsionRCovariant*, *TorsionIpInvariant*, *TorsionJpInvariant*, and *TorsionDeltaInvariant*. Each of these only take the Omega spinor as the input, and calculate the co(in)variant implied by the name as the output.

Having all of our invariants defined, it is now time to discuss how the algorithm works.

3.8.1 The Algorithm Classifying $\Omega_{(ABC)A'}$

We explain the algorithm by a "flow chart" in the same spirit as Zakhary and Carminati ((Zakhary & Carminati, 2004)). We also include a step chart to follow as well. Here is the step chart for the algorithm.

1.) Calculate Ω.

If $\Omega = 0$ then Case 8. $[-] - (-)$

If $\Omega \neq 0$ then:

2.) Calculate N.

If $N = 0$ then Case 7. $[A^3, B'] - (1,0)^3(0,1)$

If $N \neq 0$ then:

3.) Calculate Υ and Θ,

If $\Upsilon = 0$ then Case 6 or Case 4.

If $\Theta = 0$ then Case 6. $[A^2, B, C'] - (1,0)^2(1,0)(0,1)$

If $\Theta \neq 0$ then Case 4. $[A^2, 11] - (1,0)^2(1,1)$

If $\Upsilon \neq 0$ then:

If $\Theta = 0$ then Case 5. $[A, B, C, D'] - (1,0)(1,0)(1,0)(0,1)$

If $\Theta \neq 0$ then:

4.) Calculate R.

If $R = 0$ then Case 3. $[A, B, 11] - (1,0)(1,0)(1,1)$

If $R \neq 0$ then:

5.) Calculate Δ.

If $\Delta = 0$ then Case 2. $[A, 21] - (1,0)(2,1)$

If $\Delta \neq 0$ then Case 1. $[31] - (3,1)$

6.) The Algorithm is finished

We present the flowchart for the above step chart next in figure (3.1).

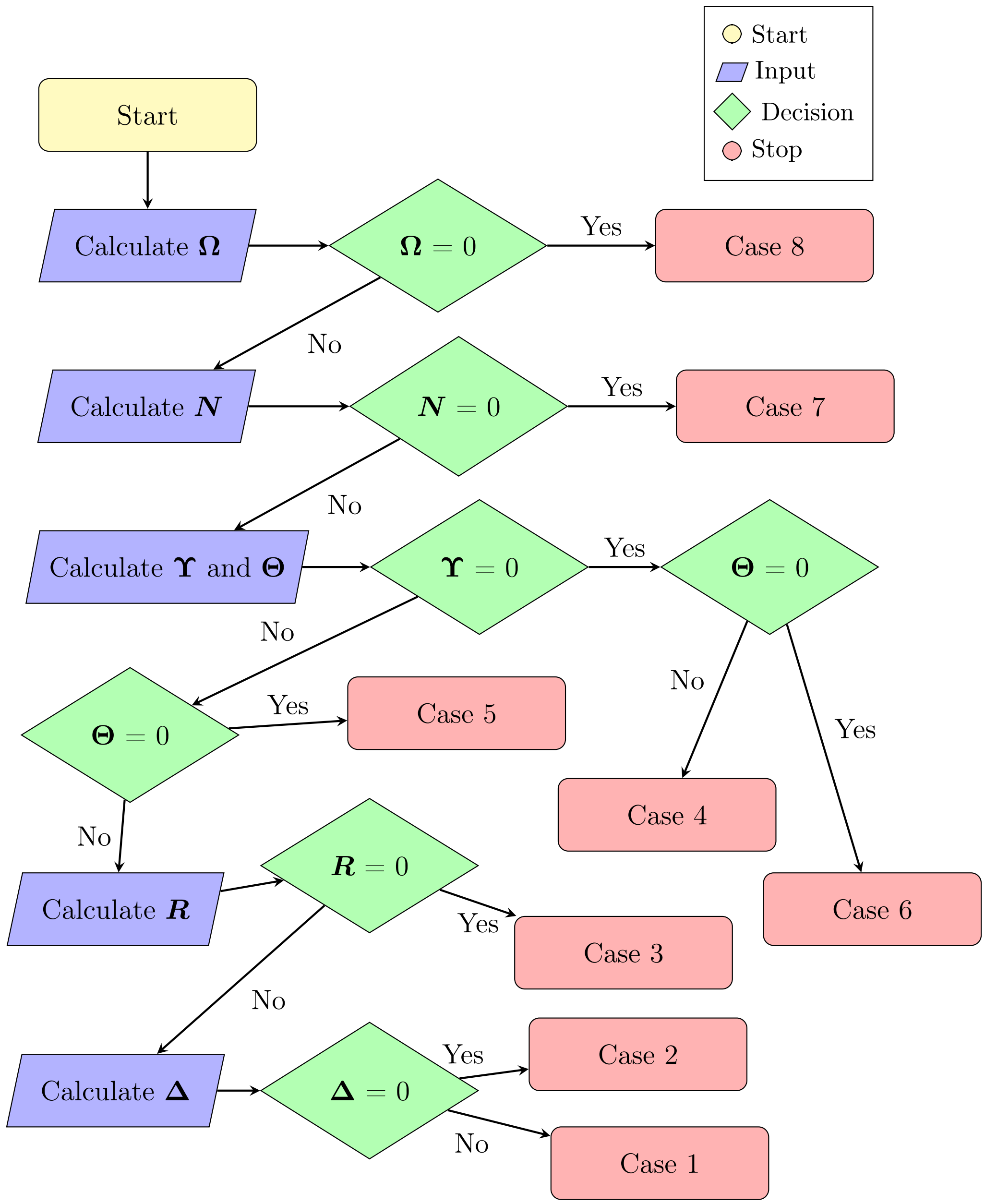


Figure 3.1: Flowchart for the $\Omega_{(ABC)A'}$ Spinor Classification Algorithm

Figure (3.1) is read starting at the yellow start box. Then each of the blue slanted rectangular boxes are calculations. Each of the green diamonds are checks that are performed. After each green diamonds, there is an arrow with either a "Yes" or "No" near it. If the condition in the box is true, we follow the "Yes" path. Likewise, if the condition in the box is false, we follow the "No" path. Each of the red boxes determines a case for which we stop. The cases are listed both in the step chart above the flow chart, or in the previous section. This algorithm can be called in Maple with the procedure: *TorsionOmegaSpinorReducibilityAlgorithm*. As inputs, it takes a torsion tensor or Omega spinor as its first argument, and solder form as its second argument. It will output the classification for the Omega spinor.

3.9 The Invariants and Covariants for $\Theta_{CC'}$ and $\Xi_{DD'}$

Now we move to examining the co(in)variants of spinors like $\Theta_{CC'}$ and $\Xi_{DD'}$. As we had before, we can formulate an algorithm using one invariant and one covariant. Given an arbitrary spinor $\alpha_{AA'}$, (not necessarily hermitian), we can determine its structural reducibility with the co(in)variants: $\boldsymbol{\alpha}$, and $\boldsymbol{O}$.

The covariant $\boldsymbol{\alpha}$ represents the components of its corresponding spinor. We define it as follows, where we take the components of $\boldsymbol{\alpha}$ NP style by using a spin basis o^A, ι^A like before.

$$\begin{aligned}
\alpha_{00} &= \alpha_{AA'} o^A \bar{o}^{A'} \\
\alpha_{01} &= \alpha_{AA'} o^A \bar{i}^{A'} \\
\alpha_{10} &= \alpha_{AA'} i^A \bar{o}^{A'} \\
\alpha_{11} &= \alpha_{AA'} i^A \bar{i}^{A'}
\end{aligned}$$

If the covariant $\boldsymbol{\alpha}$ is zero, then we have case $(-)$. The Omicron invariant $\boldsymbol{O}$ is zero in cases 2 and 3 is given by equations (3.63), and (3.64).

$$O = \alpha_{AA'} \alpha^{AA'} \tag{3.63}$$

$$O = 2\alpha_{00}\alpha_{11} - 2\alpha_{01}\alpha_{10} \tag{3.64}$$

There aren't as many invariants or covariants (only 2) that contribute to the structural decomposition of $\alpha_{AA'}$, and thus the algorithm to determine its structure is simpler.

These co(in)variants can be calculated newly in Maple with the following commands: *TorsionRank2AlphaCovariant*, and *TorsionRank2OmicronInvariant*. They both take as inputs either the $\boldsymbol{\Theta}$ or $\boldsymbol{\Xi}$ spinor.

3.9.1 The Algorithm Classifying $\Theta_{CC'}$ and $\Xi_{DD'}$

We explain the algorithm by a "flow chart" in the same spirit as Zakhary and Carminati ((Zakhary & Carminati, 2004)) as we did before. We also include a step chart to follow as well and use both the notation of Penrose and Rindler (Penrose & Rindler, 1987a), and our notation to describe the cases. Here is the step chart for the algorithm.

1.) Calculate α.

If $\alpha = 0$ then Case 3. $[-] - (-)$

If $\alpha \neq 0$ then:

2.) Calculate O.

If $O = 0$ then Case 2. $[A, A'] - |(1,0)|^2$

If $O \neq 0$ then Case 1. $[11] - (1,1)$

3.) The Algorithm is finished.

We have a flowchart included below.

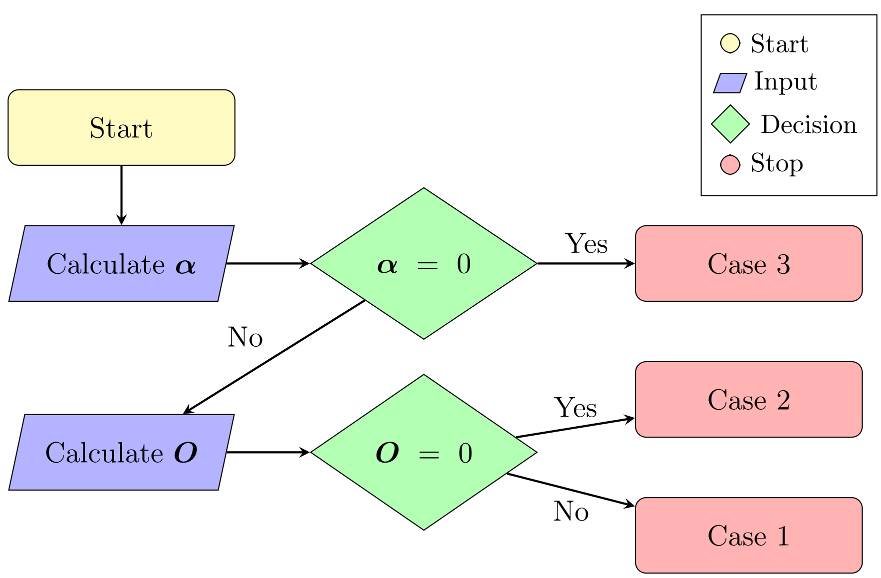


Figure 3.2: Flowchart for Classification Algorithm of the $\Theta_{AA'}$ and $\Xi_{AA'}$ Spinors

The flowchart presented in figure (3.2) is read in the same way that it was for the $\Omega_{(ABC)A'}$ spinor classification flowchart in figure (3.1).

This algorithm can be called newly in Maple with the following command: *TorsionRank2ReducibilityAlgorithm*. It takes as arguments either a torsion tensor, a solder form, and an argument either "Xi" or "Theta" depending on which spinor the user would like to look at. Or it can take a valence 1,1 spinor as the input. The algorithm will output the classification of the Theta and Xi spinors.

3.10 Summarizing the Classification of the Torsion Tensor

We have established and developed a classification for the torsion tensor. We have constructed an algorithm which classifies the torsion tensor into 72 possible types. This classification, although we apply it to the torsion tensor, technically applies to all tensors having the same rank and symmetry as the torsion tensor. The algorithm in the end takes a torsion tensor and a solder form (the Infeld-Van Der Waerden symbols) and outputs the structural reducibility type of the torsion tensor given to it. This command can be called in Maple by *TorsionTensorReducibilityAlgorithm* when using the "TorsionInvariantsModule" for calculations. In this algorithm, we construct the metric tensor g_{ab} from the inner product of solder forms. Then we calculate the spinors $\mathbf{\Theta}$, $\mathbf{\Xi}$, and $\mathbf{\Omega}$. We then take these spinors and determine their structural reducibility class as per the two algorithms given above.

Further work would include: 1.) how to implement cross co(in)variants as in Zakhary and Carminati (Zakhary & Carminati, 2001) but for torsion, 2.) how to implement Penrose's method from algebraic geometry to look at the singularity structure and topology of the curves defined by these spinors on the null cone of spacetime, see reference (Penrose, 1972), and 3.) developing physical characteristics, similar to the classification of principal null directions of the Weyl tensor (or something similar) which are analogous to the different reducibility types.

Examples of this classification will be found applied to examples in the NMC scalar field and Dirac chapters. See chapters 6 and 7 respectively.

CHAPTER 4

RANK 4 TENSOR DECOMPOSITIONS AND THE CURVATURE TENSOR

Now we move to decomposing rank 4 tensors in terms of $GL(N)$, $SO(p,q)$, and spinors. This will generalize the Ricci decomposition seen in GR to an arbitrary rank 4 tensor. We perform a similar analysis of a general 4th rank tensor and apply an analysis to the curvature tensor in the same spirit with which we analyzed the torsion tensor in chapter 3. This section outlines the process for decomposing rank 4 tensors using Young tableaux. We also provide the $SO(p,q)$ decomposition of a rank 4 tensor in any dimension.

We keep the notation we used before in the rank 2 and rank 3 tensor chapters (2), and (3) where trace-full tensors are denoted in capital Fraktur script i.e., $\mathfrak{A}$, and totally trace free tensors are denoted in lowercase Fraktur script i.e., $\mathfrak{a}$. There are new additions however to our notation for 4th rank tensors. We use curly script to denote tensors generated by a trace free second rank tensor i.e., $\mathscr{A}$. Furthermore, some of these "curly" tensors decompose further because they are not irreducible; their trace free second rank tensors are decomposable under $GL(N)$. In this case, as we will explain for the $Y2$ Young tensors, we write the symmetric part of these trace free rank 2 tensors with Hebrew script i.e., $\beth$, and we write the skew-symmetric part with Cyrillic script i.e., **Б**. We then find the $SL(2,\mathbb{C})$ irreducible spinors corresponding to the $SO(3,1)$ irreducible subspaces of an arbitrary rank 4 tensor in 4 dimensions. Additionally, we provide new Maple procedures for each step. Building upon the techniques described in the rank 2 and rank 3 tensor decomposition chapters, the same Young tableaux tools are applied to rank 4 tensors.

Rank 4 tensors decompose into 10 irreducible subspaces under $GL(N)$. These result from the 5 different types of Ferrers diagrams which occur. These are labeled by the capital script letters $\mathcal{A},\ldots,\mathcal{K}$ to follow the same notation used in the prior chapters. (We skip $\mathcal{I}$ because it is often associated with other things). Furthermore, under $SO(p,q)$ these 10 subspaces decompose further into 25 irreducible subspaces.

We list all the new notational additions in the rank 4 tensor case that do not occur in the rank 2 or rank 3 cases. These are the Hebrew ב, ג, ד, Cyrillic Б, Ц, Д, and curly $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$, $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$, and $\mathscr{J}$ type tensors. The Hebrew and Cyrillic type tensors specifically occur in the $Y2$ Ferrers diagram during the $SO(p,q)$ decomposition. These are given a special name because originally they would occur from a $\mathscr{B}$ type tensor.

In analogy to the Ricci decomposition of the Riemann tensor in GR, these script type tensors $\mathscr{B}$ etc. are analogous to the tensors generated by the trace-free Ricci tensor. They have one trace, but the tensor generating them is trace free. These would correspond to the trace free Ricci tensor in GR.

We give a few short examples of code in Maple which will do these same calculations in the "Rank4TensorModule" module. This module feeds into our ECSK module to apply the same calculations to the torsion tensor for classification purposes.

Next we move to spinor calculations for 4th rank tensors. For the total decomposition of an arbitrary rank 4 tensor in 4-dimensions, we find that a 4th rank tensor is completely characterized by 25 spinors (the same number which we got in the $SO(p,q)$ decomposition).

For the $Y1$ sector, these spinors are: $S_{(ABCD)(A'B'C'D')}$ for the totally trace free part which corresponds to the tensor $\mathfrak{a}_{abcd}$, υ for the trace-full part which corresponds to the tensor $\mathfrak{A}_{abcd}$, and $\iota_{(AB)(A'B')}$ for the curly tensor $\mathscr{A}_{abcd}$.

For the $Y2$ sectors, we find the spinors for the totally trace free parts, and the Hebrew and Cyrillic parts. The totally trace free spinors for the sections $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d})$ are: $\alpha_{(ABCD)(B'C')}$, $\beta_{(ABCD)(B'C')}$, and $\gamma_{(ABCD)(B'C')}$. For the Hebrew spinors corresponding to the tensors (ב, ג, ד) we have $\Omega_{(AB)(A'B')}$, $\kappa_{(AB)(A'B')}$, and $\theta_{(AB)(A'B')}$. For the last of the $Y2$ sectors, we have the Cyrillic tensors (Б, Ц, Д) which correspond to the spinors: $\mu_{(AD)}$, $\rho_{(AC)}$, and $o_{(AB)}$.

Similarly to the $Y1$ sector in structure, the $Y3$ sectors have totally trace free, trace-full, and curly spinors. For the totally trace free tensors, $(\mathfrak{e}, \mathfrak{f})$ we have the spinors: $\delta_{(ABCD)}$, and $\Psi_{(ABCD)}$. For the curly tensors. $(\mathscr{E}, \mathscr{F})$ we have the spinors: $\pi_{(AB)(A'B')}$, and $\Phi_{(AB)(A'B')}$. For the trace full tensors, $(\mathfrak{E}, \mathfrak{F})$ we have the spinors χ, and Λ.

The $Y4$ sectors are slightly different because there are no total trace pieces due to the skew symmetry of these sectors. Nevertheless, there are still curly pieces, and totally trace free pieces. In this case for the totally trace free tensors $(\mathfrak{g}, \mathfrak{h}, \mathfrak{j})$ we have the spinors: $\tau_{(AB)(C'D')}$, $\zeta_{(AB)(C'D')}$, and $\xi_{(AB)(C'D')}$. Furthermore, for the curly tensors $(\mathscr{G}, \mathscr{H}, \mathscr{J})$ we have the spinors: $\nu_{(AB)}$, $\eta_{(AB)}$, and $\sigma_{(AB)}$.

To finish, with the $Y5$ there is only one sector $\mathfrak{k}$, and its spinor equivalent is $\aleph$.

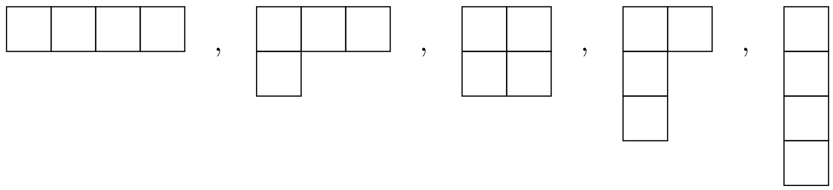
We can then use these spinors to classify the curvature tensor in 4-D. The curvature tensor decomposes into 6 irreducible tensors. The first three of these are analogous to the tensors which occur in the Ricci decomposition of the Riemann tensor $R\{\}_{abcd}$. But they are modified, and the interpretation is different due to the inclusion of torsion in the theory. The second three are pieces that occur because of the inclusion of torsion, and they do not appear in GR. From an $SO(p,q)$ decomposition perspective, we can write the tensors as $\mathfrak{f}_{abcd}$, $\mathscr{F}_{abcd}$, $\mathfrak{F}_{abcd}$, $\mathfrak{l}_{abcd}$, $\mathscr{L}_{abcd}$, and $\mathfrak{k}_{abcd}$. We have defined $\mathfrak{l}$, and $\mathscr{L}$ by $\mathfrak{l}_{abcd} = \mathfrak{h}_{abcd} + \mathfrak{j}_{abcd}$, and $\mathscr{L}_{abcd} = \mathscr{H}_{abcd} + \mathscr{J}_{abcd}$; this is in the same spirit in which the irreducible torsion tensor pieces $\mathfrak{Q}$ and $\mathfrak{q}$ from equations (3.44) and (3.45) were developed. Adding these sectors together results in an irreducible subspace for the curvature tensor retaining the correct and needed symmetries: $R_{abcd} = R_{[ab][cd]}$.

The irreducible tensor elements of the curvature tensor then decompose into the spinors: $\Psi_{(ABCD)}$, $\Phi_{(AB)(A'B')}$, Λ, Ж$_{(AB)(A'B')}$, Ю$_{(AB)}$, and $\aleph$. Interestingly enough, the spinors Ж, and $\aleph$ naturally occur as antihermitian spinors as opposed to Φ and Λ which occur as hermitian spinors; although no proof is given this seems to be a trend for spinors generated from a Young tableaux decomposition which have more skew-symmetries than symmetries.

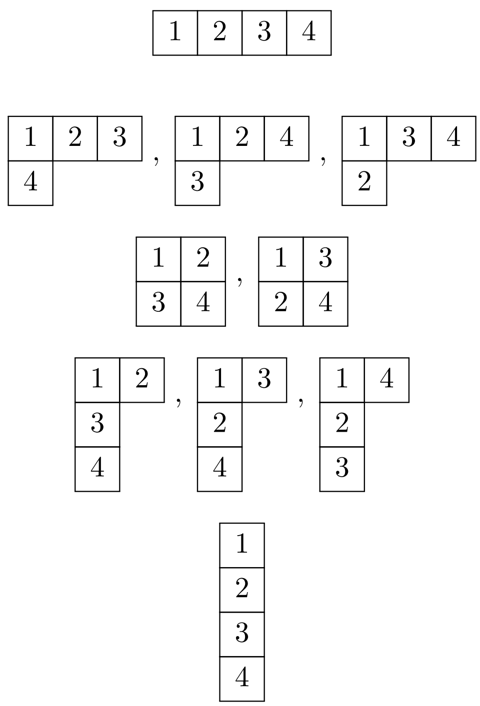
We would again like to point out that this classification can be refined using the tools of Penrose (Penrose, 1972) with algebraic geometry. We would examine the topology and multiple point structure of the curve (defined by the locus of the characteristic polynomial defined by the tensor) as viewed on a two-sphere when the real points are examined. This is again not done here, but seems to be an interesting and fruitful direction for future work.

4.1 Rank 4 $GL\,(N)$ Young Decomposition

We define a rank 4 tensor Q_{abcd}. We decompose it using Young tableaux. The decomposition gives us the 10 irreducible tensor components of an arbitrary rank 4 tensor. We begin by presenting the Ferrers diagrams for a 4th rank tensor.



There are more Young tableaux than Ferrers diagrams because of the way we can arrange the numbers in each sector. This provides us with the Young decomposition:



Using the notation we developed in the sections on Rank 2 and Rank 3 tensors, we can break a tensor Q_{abcd} into its $GL\,(N)$ irreducible sectors from the above Young tableaux as follows:

$$\begin{aligned} Q_{abcd} &= Y1\,(Q)_{abcd} \\ &= Y2a\,(Q)_{abcd} + Y2b\,(Q)_{abcd} + Y2c\,(Q)_{abcd} \\ &= Y3a\,(Q)_{abcd} + Y3b\,(Q)_{abcd} \\ &= Y4a\,(Q)_{abcd} + Y4b\,(Q)_{abcd} + Y4c\,(Q)_{abcd} \\ &= Y5\,(Q)_{abcd} \end{aligned}$$

Since this notation is cumbersome, we define the following tensors $\mathcal{A}, \ldots, \mathcal{K}$ (minus $\mathcal{I}$) to help us condense our notation.

$$\mathcal{A}_{abcd} = Y1\,(Q)_{abcd}\,,$$

$$\mathcal{B}_{abcd} = Y2a\,(Q)_{abcd}\,, \quad \mathcal{C}_{abcd} = Y2b\,(Q)_{abcd}\,, \quad \mathcal{D}_{abcd} = Y2c\,(Q)_{abcd}$$

$$\mathcal{E}_{abcd} = Y3a\,(Q)_{abcd}\,, \quad \mathcal{F}_{abcd} = Y3b\,(Q)_{abcd}$$

$$\mathcal{G}_{abcd} = Y4a\,(Q)_{abcd}\,, \quad \mathcal{H}_{abcd} = Y4b\,(Q)_{abcd}\,, \quad \mathcal{J}_{abcd} = Y4c\,(Q)_{abcd}$$

$$\mathcal{K}_{abcd} = Y5\,(Q)_{abcd}$$

Below, we have the decomposition of an arbitrary rank 4 tensor in terms of our notation above. This breaks the tensor $\boldsymbol{Q}$ into tensors living in each of its 10 $GL\,(N)$ irreducible subspaces:

$$\begin{aligned} Q_{abcd} &= \mathcal{A}_{abcd} \\ &+ \mathcal{B}_{abcd} + \mathcal{C}_{abcd} + \mathcal{D}_{abcd} \\ &+ \mathcal{E}_{abcd} + \mathcal{F}_{abcd} \\ &+ \mathcal{G}_{abcd} + \mathcal{H}_{abcd} + \mathcal{J}_{abcd} \\ &+ \mathcal{K}_{abcd} \end{aligned} \tag{4.1}$$

There are ten different subspaces in the rank 4 tensor case; this is a lot more than for the

3rd rank tensors, where we had four. Explicit formulas for the Young projectors can be found in appendix B.

If the tensor has additional symmetries, this may cause some of the components to vanish or combine together. For more information, see Itin and Reches (Itin & Reches, 2021). This happens in the case of the curvature tensor just as for the torsion tensor in chapter 3.

4.2 Rank 4 $SO\,(p,q)$ Trace Decomposition

Now we can begin taking traces to construct the irreducible $SO\,(p,q)$ decomposition of Q_{abcd}. It turns out that there are 25 irreducible subspaces for an arbitrary rank 4 tensor under $SO\,(p,q)$; we will show this below. To begin, following Hammermesh (Hammermesh, 1962/1989) we decompose an arbitrary rank 4 tensor with traces as follows:

$$\begin{aligned} Q_{abcd} &= \mathring{A}_{ab}g_{cd} + \mathring{B}_{ac}g_{bd} + \mathring{C}_{ad}g_{bc} + \mathring{D}_{bc}g_{ad} + \mathring{E}_{bd}g_{ac} + \mathring{F}_{cd}g_{ab} \\ &+ H g_{ab}g_{cd} + J g_{ac}g_{bd} + K g_{ad}g_{bc} + W_{abcd} \end{aligned} \tag{4.2}$$

where the tensor W_{abcd} is the totally trace free piece and $\mathring{A}_{ab} \dots \mathring{F}_{ab}$ are trace free second rank tensors with no presumed symmetries. We can take a second trace that we couldn't in the case of the rank 3 tensors; this is new and interesting. This decomposition is presented in Hammermesh (Hammermesh, 1962/1989). We want to be able to decompose this into a totally trace free part, trace free tensors, and scalars. We note that the superscript $\mathring{A}$ above the A indicates a trace free tensor.

We will use equation (4.2) to decompose each of the 10 Young sectors under $SO\,(p,q)$. As in the 3rd Rank tensor case, we define tensors that relate our above traces to just traces on a general fourth rank tensor Q_{abcd}. Later we will think of Q_{abcd} as $\mathcal{A}_{abcd} \dots \mathcal{K}_{abcd}$ as given by equation (4.1). It will be convenient to have the following definitions:

$$g^{ab}Q_{abcd} = P_{cd}, \quad g^{ac}Q_{abcd} = R_{bd}, \quad g^{ad}Q_{abcd} = U_{bc} \tag{4.3}$$

$$g^{bc}Q_{abcd} = X_{ad}, \quad g^{bd}Q_{abcd} = Y_{ac}, \quad g^{cd}Q_{abcd} = Z_{ab}$$

$$g^{ab}g^{cd}Q_{abcd} = P, \quad g^{ac}g^{bd}Q_{abcd} = R, \quad g^{ad}g^{bc}Q_{abcd} = U$$

With all of this established, we can determine all 25 $SO\,(p,q)$ irreducible tensors. Because there are an exorbitant amount of tensors to calculate, we will give two example calculations. We leave the proofs to appendix D for which we also repeat the methodology we show below.

Our examples will be from the $Y1$ and $Y2a$ cases because they provide nearly all the new information in the rank 4 tensor case

4.2.1 The $SO\,(p,q)$ Decomposition of the $Y1$ Sector of a Rank 4 Tensor

Here, we decompose the tensor $\boldsymbol{\mathcal{A}}$ given by the following Young tableaux:

1	2	3	4

into its irreducible components under the action of $SO\,(p,q)$ (taking traces). This example is here to illustrate how the new type of tensor (curly tensor) $\mathscr{A}$ comes about, and how it is calculated along with the more familiar totally trace free and trace-full tensors $\boldsymbol{\mathfrak{a}}$, and $\boldsymbol{\mathfrak{A}}$.

The tensor $\mathcal{A}_{abcd}$ is totally symmetric $\mathcal{A}_{abcd} = \mathcal{A}_{(abcd)}$. If we apply equation (4.2) to the tensor $\boldsymbol{\mathcal{A}}$ then we find the following formulas:

$$\mathcal{A}_{abcd} = \mathfrak{a}_{abcd} + \mathscr{A}_{abcd} + \mathfrak{A}_{abcd}$$

$$\mathscr{A}_{abcd} = \mathring{G}_{(ab)}g_{cd} + \mathring{G}_{(ac)}g_{bd} + \mathring{G}_{(ad)}g_{bc} + \mathring{G}_{(bc)}g_{ad} + \mathring{G}_{(bd)}g_{ac} + \mathring{G}_{(cd)}g_{ab} \tag{4.4}$$

$$\mathfrak{A}_{abcd} = L\left(g_{ab}g_{cd} + g_{ac}g_{bd} + g_{ad}g_{bc}\right)$$

We define $\mathring{G}_{ab}$ as a symmetric trace free rank 2 tensor, and L as a scalar, both of which need to be determined. To determine $\mathring{G}_{ab}$ and L, we apply the $Y1$ symmetry property (being totally symmetric) to the list of equations labeled by (4.3) to $\mathcal{A}$ where $\mathcal{A}_{abcd} = Q_{abcd}$. After applying the property, we find:

$$g^{ab}\mathcal{A}_{(abcd)} = P_{(cd)} = R_{(cd)} = U_{(cd)} = X_{(cd)} = Y_{(cd)} = Z_{(cd)} \tag{4.5}$$

$$g^{ab}g^{cd}\mathcal{A}_{(abcd)} = P = R = U$$

Which greatly simplifies the number of independent tensors down to two: P_{ab}, and P. Furthermore, since we can take traces, the two independent tensors are really $\mathring{P}_{ab}$, and P.

Once we solve these equations for $\mathring{G}_{ab}, L$ we find that they are given by the following list where N is a constant which represents the dimension of the space being worked on.

$$\mathring{G}_{(cd)} = \frac{1}{(N+4)}\mathring{P}_{(cd)}, \quad \mathring{P}_{(cd)} = P_{(cd)} - \frac{1}{N}Pg_{cd}$$

$$L = \frac{1}{N\,(N+2)}P$$

Thus, we can now write our $Y1$ decomposition in the following list of equations:

$$\mathcal{A}_{abcd} = \mathfrak{a}_{abcd} + \mathscr{A}_{abcd} + \mathfrak{A}_{abcd} \tag{4.6}$$

$$\mathscr{A}_{abcd} = \frac{6}{(N+4)}\mathring{P}_{(ab}g_{cd)} \tag{4.7}$$

$$\mathfrak{A}_{abcd} = \frac{3}{N\,(N+2)}Pg_{(ab}g_{cd)} \tag{4.8}$$

$$P_{(cd)} = g^{ab}\mathcal{A}_{(abcd)}, \quad P = g^{ab}g^{cd}\mathcal{A}_{(abcd)} \tag{4.9}$$

$$\mathring{P}_{(ab)} = P_{(ab)} - \frac{1}{N}Pg_{ab} \tag{4.10}$$

Equations (4.6)-(4.10) represent the $SO\,(p,q)$ irreducible decomposition of a totally symmetric rank 4 tensor. The methodology here is used throughout all the $SO\,(p,q)$ decompositions, and is useful to distinguish all the possibilities when the number of tensors increases so drastically with each rank of the tensor. We recommend this methodology for all types of $SO\,(p,q)$ irreducible decompositions.

How to use Maple to calculate the tensors above

We have written Maple procedures to calculate the tensors $\mathcal{A}$, $\mathfrak{A}$, and $\mathfrak{a}$. To calculate the tensors the following commands can be used: “Rank4FrankA” is used to calculate the left-hand side of equation (4.6), “Rank4CurlyA” is used to calculate the left-hand side of equation (4.7), and “Rank4TraceA” is used to calculate the left-hand side of equation (4.8). Each of these commands take two inputs: for slot 1: a totally covariant rank 4 tensor (Q_{abcd} for example), and for slot 2: a metric tensor g_{ab}.

4.2.2 The $SO\left(p,q\right)$ Decomposition of the $Y2a$ Sector of a Rank 4 Tensor

To illustrate two more new possibilities for irreducible tensors in the rank 4 case, we examine the $Y2a$ sector as our second example. We begin in the Young sector described by the following tableaux:

1	2	3
4		

for which we will make the tensor $Y2a\left(Q\right)_{abcd} = \mathcal{B}_{abcd}$. We would have the symmetry $\mathcal{B}_{abcd} = \mathcal{B}_{a(bc)d} = \mathcal{B}_{[a|bc|d]}$. If we use a degrees of freedom argument, we can find that the trace part: $\mathscr{B}_{abcd}$ is generated by a second rank general trace free tensor which we call $\mathring{b}_{ad}$. This then tells us that we can decompose $\mathcal{B}_{abcd}$ as:

$$\mathcal{B}_{abcd} = \mathscr{B}_{abcd} + \mathfrak{b}_{abcd} \tag{4.11}$$

$$\mathscr{B}_{abcd} = Y2a\left(\mathring{b}_{ad}g_{bc}\right)$$

with no total trace piece due to the skew symmetrization. Expanding out the Young symmetrizer, allows us to write $\mathscr{B}_{abcd}$ as:

$$\mathscr{B}_{abcd} = \left(\mathring{\Upsilon}_{ad} - \mathring{\Upsilon}_{da}\right)g_{bc} - \mathring{\Upsilon}_{ca}g_{bd} - \mathring{\Upsilon}_{ba}g_{cd} + \mathring{\Upsilon}_{bd}g_{ac} + \mathring{\Upsilon}_{cd}g_{ab} \tag{4.12}$$

The tensor $\mathring{b}_{ad}$ is related to the tensor $\mathring{\Upsilon}_{ad}$ by a factor of a half. Similarly to what we did for the $Y1$ piece, we make the trace equations:

$$g^{ab}\mathcal{B}_{abcd} = P_{cd} = R_{cd} = -Y_{cd} = -Z_{cd} = 2\mathring{\Upsilon}_{[cd]} + \mathring{\Upsilon}_{cd}N$$

$$g^{bc}\mathcal{B}_{abcd} = X_{ad} = 2P_{[ab]} = 2\mathring{\Upsilon}_{[ad]}(N+2)$$

$$g^{ad}\mathcal{B}_{abcd} = U_{bc} = 0, \quad g^{ab}g^{cd}\mathcal{B}_{abcd} = P = R = U = 0$$

in which we have used the symmetries $\mathcal{B}_{abcd} = \mathcal{B}_{a(bc)d} = \mathcal{B}_{[a|bc|d]}$ to simplify some of the equalities. From these, we find that we only have 1 independent equation. We solve it to find $\mathring{\boldsymbol{\Upsilon}}$ in terms of $\boldsymbol{P}$:

$$\mathring{\Upsilon}_{ab} = \frac{1}{N}P_{ab} - \frac{2}{N(N+2)}P_{[ab]}. \tag{4.13}$$

However, because $\mathring{\Upsilon}_{ab}$ is traceless, we can again use the structure contained in $GL(N)$ to break off a symmetric and skew piece. We will call the skew part of $\mathring{\Upsilon}_{ab}$ the Cyrillic Zhe: $Ж_{ab} = \mathring{\Upsilon}_{[ab]}$, and the resulting tensor generated by $Ж_{ab}$ through equation (4.12) will be called the Cyrillic Be: $Б_{abcd}$. Next, we will call the symmetric part of $\mathring{\Upsilon}_{ab}$ the Cyrillic Yu: $Ю_{ab} = \mathring{\Upsilon}_{(ab)}$, and the resulting tensor generated by $Ю_{ab}$ through equation (4.12) will be called the Hebrew Beth: $ב_{abcd}$.

To minimize space, we apply this methodology for splitting $\mathring{\boldsymbol{\Upsilon}}$ into Ж and Ю to equation (4.13) to get the next few equations instead of going through each step. This then gives us the irreducible decomposition for the tensor $\mathcal{B}_{abcd}$ through the following equations:

$$\mathcal{B}_{abcd} = Б_{abcd} + ב_{abcd} + \mathfrak{b}_{abcd} \tag{4.14}$$

$$Б_{abcd} = \frac{1}{N+2}\left(2P_{[ad]}g_{bc} - P_{[ca]}g_{bd} - P_{[ba]}g_{cd} + P_{[bd]}g_{ac} + P_{[cd]}g_{ab}\right) \tag{4.15}$$

$$ב_{abcd} = \frac{1}{N}\left(-P_{(ca)}g_{bd} - P_{(ba)}g_{cd} + P_{(bd)}g_{ac} + P_{(cd)}g_{ab}\right) \tag{4.16}$$

$$P_{cd} = g^{ab}\mathcal{B}_{abcd}$$

Equations (4.11,4.15,4.16) together result in the $SO\left(p,q\right)$ irreducible decomposition for the tensor $\mathcal{B}_{abcd}$. We have repeated how the tensor P_{cd} is defined for clarity.

How to use Maple to calculate the tensors above

To calculate the tensors above, the following commands can be used. "**Rank4FrankB**" is used to calculate the totally trace free tensor $\mathfrak{b}$ in equation (4.11), "**Rank4CyrillicB**" is used to calculate the left-hand side of equation (4.15), and "**Rank4HebrewB**" is used to calculate the left-hand side of equation (4.16). Each of these commands take two inputs, which are the same as in the $Y1$ case. Further code has been written which will calculate the rest of the 25 $SO\left(p,q\right)$ irreducible tensors and is explained in appendix C.

With both the $Y1$ example explaining how the tensor $\mathscr{A}$ comes to exist, and the $Y2a$ section above explaining how the tensors Б and $\beth$ come to exist, we now have all the information conceptually to determine the rest of 25 $SO\left(p,q\right)$ irreducible tensors.

4.3 The Explicit $SO\left(p,q\right)$ Decomposition for an Arbitrary 4th Rank Tensor

The $SO\left(p,q\right)$ decomposition of our tensor Q_{abcd} is given as follows:

$$
\begin{aligned}
Q_{abcd} &= \mathfrak{a}_{abcd} + \mathscr{A}_{abcd} + \mathfrak{A}_{abcd} \\
&+ \mathfrak{b}_{abcd} + \text{Б}_{abcd} + \beth_{abcd} \\
&+ \mathfrak{c}_{abcd} + \text{Ц}_{abcd} + \gimel_{abcd} \\
&+ \mathfrak{d}_{abcd} + \text{Д}_{abcd} + \daleth_{abcd} \\
&+ \mathfrak{e}_{abcd} + \mathscr{E}_{abcd} + \mathfrak{E}_{abcd} \\
&+ \mathfrak{f}_{abcd} + \mathscr{F}_{abcd} + \mathfrak{F}_{abcd} \\
&+ \mathfrak{g}_{abcd} + \mathscr{G}_{abcd} \\
&+ \mathfrak{h}_{abcd} + \mathscr{H}_{abcd} \\
&+ \mathfrak{j}_{abcd} + \mathscr{J}_{abcd} \\
&+ \mathfrak{k}_{abcd}
\end{aligned} \tag{4.17}
$$

where the rest of the tensors have been defined explicitly in the appendix. We have sorted equation (4.17) by each type of tenor in a column, except for the Hebrew and Cyrillic tensors. Since we have the $SO\,(p,q)$ decomposition of an arbitrary rank 4 tensor, we now move onto the spinor decomposition of each of these tensors in 4 dimensions, specifically beginning by counting degrees of freedom.

4.4 The Tensor-Spinor Correspondence for Rank 4 Tensors in 4 Dimensions

In this section, we enumerate the $SL\,(2,\mathbb{C})$ irreducible spinors corresponding to an arbitrary 4th rank tensor in $N = 4$ dimensions with signature $[+,-,-,-]$. There are 25 spinors which fully describe a tensor of this form, and there is a one-to-one correspondence between $SO\,(3,1)$ irreducible tensors and $SL\,(2,\mathbb{C})$ irreducible spinors. We list those spinors now, along with which irreducible tensor they correspond to. We follow Stewart (Stewart, 1993) in terms of our notation for spinors.

There is one rank 8 hermitian spinor: $S_{(ABCD)(A'B'C'D')}$ which corresponds to the totally trace free tensor $\mathfrak{a}_{abcd}$ residing in the $Y1$ Ferrers sector with tableau: $Y1$.

There are three rank 6 spinors: $\alpha_{(ABCD)(A'B')}$, $\beta_{(ABCD)(A'B')}$, and $\gamma_{(ABCD)(A'B')}$ which correspond to the totally trace free tensors $\mathfrak{b}_{abcd}$, $\mathfrak{c}_{abcd}$, and $\mathfrak{d}_{abcd}$ residing in the $Y2$ Ferrers sector with tableaux: $Y2a$, $Y2b$, and $Y2c$ respectively.

There are eleven rank 4 spinors which break into two categories: hermitian/antihermitian and non-hermitian.

There are two neither hermitian nor antihermitian spinors: $\delta_{(ABCD)}$, and $\Psi_{(ABCD)}$ which correspond to the totally trace free tensors $\mathfrak{e}_{abcd}$, and $\mathfrak{f}_{abcd}$ living in the $Y3$ Ferrers sector with tableaux: $Y3a$, and $Y3b$ respectively.

The other nine spinors are either hermitian or antihermitian.

The first six out of these nine are hermitian spinors: $\iota_{(AB)(A'B')}$, $\Omega_{(AB)(A'B')}$, $\kappa_{(AB)(A'B')}$, $\theta_{(AB)(A'B')}$, $\pi_{(AB)(A'B')}$, and $\Phi_{(AB)(A'B')}$ which correspond to the curly and Hebrew tensors: $\mathscr{A}_{abcd}$, $\beth_{abcd}$, $\gimel_{abcd}$, $\daleth_{abcd}$, $\mathscr{E}_{abcd}$, and $\mathscr{F}_{abcd}$. They live in the Ferrers sectors: $Y1$ ($\mathscr{A}$), $Y2$ ($\beth$, $\gimel$, $\daleth$), and $Y3$ ($\mathscr{E}$, $\mathscr{F}$). They have tableaux: $Y2a$, $Y2b$, and $Y2c$ for $\beth, \gimel$, and $\daleth$ respectively; we also have tableaux $Y3a$, and $Y3b$ for $\mathscr{E}$, and $\mathscr{F}$ respectively.

There are three antihermitian spinors $\tau_{(AB)(A'B')}$, $\zeta_{(AB)(A'B')}$, and $\xi_{(AB)(A'B')}$ which correspond to the totally trace free tensors $\mathfrak{g}_{abcd}$, $\mathfrak{h}_{abcd}$, and $\mathfrak{j}_{abcd}$ residing in the $Y4$ Ferrers sector with tableaux $Y4a$, $Y4b$, and $Y4c$ respectively.

There are six rank 2 spinors: $\mu_{(AB)}$, $\rho_{(AB)}$, $o_{(AB)}$, $\nu_{(AB)}$, $\eta_{(AB)}$, and $\sigma_{(AB)}$ which correspond to the curly and Cyrillic tensors: $Б_{abcd}$, $Ц_{abcd}$, $Д_{abcd}$, $\mathscr{G}_{abcd}$, $\mathscr{H}_{abcd}$, and $\mathscr{J}_{abcd}$. They live in the Ferrers sectors $Y2$ (Б, Ц, Д) and $Y4$ ($\mathscr{G}$, $\mathscr{H}$, $\mathscr{J}$). They have tableaux $Y2a$, $Y2b$, $Y2c$, $Y4a$, $Y4b$, and $Y4c$.

Finally, we have 3 rank 0 spinors: υ, χ, Λ, and $\aleph$ which correspond to the total trace tensors: $\mathfrak{A}_{abcd}$, $\mathfrak{E}_{abcd}$, $\mathfrak{F}_{abcd}$, and the totally trace free tensor $\mathfrak{k}_{abcd}$. These live in the Ferrers sectors $Y1$ (υ), $Y3$ (χ, Λ), and $Y5$ ($\aleph$). They have corresponding tableaux $Y1$, $Y3a$, $Y3b$, and $Y5$. Additionally, $\aleph$ is purely imaginary.

These are all the possible irreducible spinors.

We examine the degrees of freedom of each $SO(p,q)$ irreducible tensor, and compare the formula to that of the spinors in 4D. This is done in appendix E. There we label the sector with its Ferrers diagram, and then display the formulas for the degrees of freedom of both the tensors and the spinors. Additionally, we include the degrees of freedom of the $GL(N)$ irreducible tensors and separate it from the others with a semi-colon. We use the positive whole number n to represent the dimension of the spinor space $\mathcal{S}$, with a small sketch on how we calculated the degrees of freedom using Young tableaux ideas. Recall that for our purposes $n = 2$, and the tangent space at a point p: T_pM (where M is the manifold being worked on) is isomorphic to the tensor product $\mathcal{S} \otimes \overline{\mathcal{S}}$, where $\overline{\mathcal{S}}$ is the bar spinor space referenced by Stewart (Stewart, 1993).

4.5 Spinor-Tensor Correspondence Under $SO(3,1)$ and $SL(2,\mathbb{C})$

In this section, we establish an explicit description of each of the spinors; listed above, we build them out of the $SO(3,1)$ irreducible tensors and give epsilon trace equations which determine the $SL(2,\mathbb{C})$ irreducible spinors.

Many of the formulas here were determined by using Maple; these are found in the "Rank 3 Spinor Proofs" Maple file. It was useful to have a computer handle the laborious amounts of algebra. Example worksheets of this method are available on the Utah State University digital commons. The proofs go by having Maple manipulate symbolic variables in a way to have them match the Young symmetrizers. Examples for all 25 irreducible spinors are available in the "Rank 4 Spinor Proofs" Maple file.For example, we would give Maple a spinor $\mu_{(AB)}\epsilon_{CD}\epsilon_{A'B'}\epsilon_{C'D'}$, and it would return the correct permutation of indices to properly represent the corresponding rank 4 $SO\left(3,1\right)$ irreducible tensor.

Following Penrose and Rindler (Penrose & Rindler, 1987a) we will almost always use the isomorphism symbol $\cong$ to indicate equality up to the Infeld Van der Waerden symbols just like before in chapters 2 and 3. For example, we would write the equality below:

$$S_{(ABCD)(A'B'C'D')} = \mathfrak{a}_{abcd}\sigma^{a}{}_{AA'}\sigma^{b}{}_{BB'}\sigma^{c}{}_{CC'}\sigma^{d}{}_{DD'} \tag{4.18}$$

as follows:

$$S_{(ABCD)(A'B'C'D')} \cong \mathfrak{a}_{abcd}$$

where the $\cong$ symbol is suppressing the Infeld-Van der Waerden symbols.

We move to the examples now, the first being how we write the tensor $\mathscr{A}$ in terms of the spinor $\boldsymbol{\iota}$; the next being how we write the tensor $\mathfrak{b}$ in terms of the spinor $\boldsymbol{\alpha}$.

4.5.1 The $SL\left(2,\mathbb{C}\right)$ Decomposition of the Tensor $\mathscr{A}_{abcd}$

Given the tensor $\mathscr{A}$ from equation (4.7), we define $\boldsymbol{\iota}$ by:

$$\iota_{(AB)(A'B')} \cong \frac{1}{8}g^{ef}\mathscr{A}_{efab} \tag{4.19}$$

and then relate it back to $\mathscr{A}$ through:

$$\begin{aligned} \mathscr{A}_{abcd} \cong\ & \epsilon_{AB}\epsilon_{A'B'}\iota_{(CD)(C'D')} + \epsilon_{AC}\epsilon_{A'C'}\iota_{(BD)(B'D')} \\ +\ & \epsilon_{AD}\epsilon_{A'D'}\iota_{(BC)(B'C')} + \epsilon_{BC}\epsilon_{B'C'}\iota_{(AD)(A'D')} \\ +\ & \epsilon_{BD}\epsilon_{B'D'}\iota_{(AC)(A'C')} + \epsilon_{CD}\epsilon_{C'D'}\iota_{(AB)(A'B')} \end{aligned} \tag{4.20}$$

The Proof is given in the "Rank 4 Spinor Proofs" Maple worksheet. Iota and all the other spinors presented here and in appendix F are irreducible because they are completely symmetric in their unprimed and primed indices respectively.

Equations (4.20), and (4.19) show how the tensor $\mathscr{A}$ is given in terms of the spinor $\boldsymbol{\iota}$ and epsilon spinors. This is different from equation (4.18) because we can take epsilon traces off, and thus we find that the spinor given by: $\mathscr{A}_{abcd}\sigma^{a}{}_{AA'}\sigma^{b}{}_{BB'}\sigma^{c}{}_{CC'}\sigma^{d}{}_{DD'}$ is reducible or at least equivalent to the symmetric trace free rank 2 tensor we get from taking metric traces; this tensor is $\mathring{G}_{ab}$ from equation (4.4).

It may happen that we can take just one epsilon trace, or three epsilon traces, which would also be reducible under $SL(2,\mathbb{C})$. Recall from Stewart (Stewart, 1993) that the metric is given in terms of epsilon by $g_{ab} \cong \epsilon_{AB}\epsilon_{A'B'}$. With this being said, we move onto the case for $\mathfrak{b}_{abcd}$ for which this is the case.

4.5.2 The $SL(2,\mathbb{C})$ Decomposition of the Tensor $\mathfrak{b}_{abcd}$

Recall that the tensor $\mathfrak{b}$ is given in equation (3.21). We define its corresponding spinor to be $\alpha_{(ABCD)(B'C')}$ which is given by:

$$\alpha_{(ABCD)(B'C')} \cong \frac{1}{2}\mathfrak{b}_{abcd}\epsilon^{A'D'} \tag{4.21}$$

We can also write $\mathfrak{b}$ in terms of $\alpha_{(ABCD)(B'C')}$ as follows:

$$\mathfrak{b}_{abcd} \;\cong\; \alpha_{(ABCD)(B'C')}\epsilon_{A'D'} + \overline{\alpha}_{(A'B'C'D')(BC)}\epsilon_{AD} \tag{4.22}$$

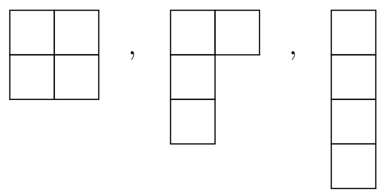
where, again, the proofs are given in the "Rank 4 Spinor Proofs" Maple worksheet. Similar calculations happen in all the other 25 cases. These decompositions along with the remaining 23 formulas can be found in appendix F.

4.6 Classification of the Curvature Tensor in ECSK Theory

This section provides a classification of the curvature tensor in ECSK theory. We will see that the curvature tensor classification is far more complex than that of the torsion tensor as there are six constituent spinors which determine it entirely.

4.6.1 Classification of the Curvature Tensor Under $GL(N)$

The completely covariant ECSK curvature tensor, which we call R_{abcd} decomposes into the following Ferrers sectors:



(4.23)

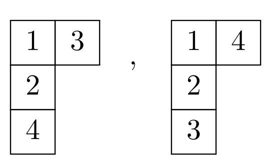
while the other sectors are forced to be zero by the symmetry:

$$R_{abcd} = R_{[ab][cd]} \tag{4.24}$$

Note, that more information on the curvature tensor, when torsion is present and in ECSK theory, can be found in Jensen (Jensen, 2005), and in our chapter 5. For the first Ferrers diagram in equation (4.23), only the $Y3b$ tableaux given by:

1	3
2	4

is non-zero. The $Y3a$ tableaux vanishes again because of symmetry of the curvature tensor as in equation (4.24). As shown in appendix C, we write an element in the $Y3b$ sector by the tensor $\boldsymbol{\mathcal{F}}$. The tensor $\boldsymbol{\mathcal{F}}$ is defined in appendix C. There is no $Y3a$ sector because taking the symmetric part of the first two indices results in zero. The second Ferrers diagram in equation (4.23) naively decomposes further into the $Y4b$ and $Y4c$ tableaux sectors:



However, neither of these sectors have the same symmetries as the curvature tensor. The set of tensors given by the two tableaux above are not invariant subspaces because they do not share the same symmetries as the curvature tensor. Nevertheless, there is a single invariant subspace defined by the elbow Ferrers diagram Y4 sector. We define the tensor $\mathcal{L}$ which represents this sector by:

$$\mathcal{L}_{abcd} = \frac{1}{2}\left(Q_{[ab][cd]} - Q_{[cd][ab]}\right) \tag{4.25}$$

It turns out that when we add the tensors $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{\mathcal{J}}$, defined in appendix B we produce the tensor $\boldsymbol{\mathcal{L}}$. The tensor $\boldsymbol{\mathcal{L}}$ has the same symmetries as the curvature tensor that are needed to form a proper representative element for the irreducible subspace represented by the elbow Ferrers diagram.

$$\mathcal{L}_{abcd} = \mathcal{H}_{abcd} + \mathcal{J}_{abcd} \tag{4.26}$$

The tensor $\mathcal{L}_{abcd}$ in equation (4.26) is analogous to the tensor $\mathcal{Q}_{abc}$ in chapter 3 in equation (3.39).

Finally, for the $Y5$ sector given by the tableau:

1
2
3
4

we have the corresponding tensor which we call $\boldsymbol{\mathcal{K}}$ as defined in appendix B; this is just the totally skew-symmetric part of our tensor Q_{abcd} from earlier. Viewing $\boldsymbol{\mathcal{L}}$, $\boldsymbol{\mathcal{F}}$, and $\boldsymbol{\mathcal{K}}$ as subspaces under $GL\left(N\right)$, the curvature tensor decomposes into the following subspaces:

$$\boldsymbol{R} = \boldsymbol{\mathcal{F}} \oplus \boldsymbol{\mathcal{L}} \oplus \boldsymbol{\mathcal{K}} \tag{4.27}$$

under $GL\left(N\right)$. Upon adding the degrees of freedom of the subspaces in equation (4.27) together, we find that there are 36 independent components, which completely determine the curvature tensor.

4.6.2 Classification of the Curvature Tensor Under $SO(p,q)$

Now we consider decomposing R_{abcd} under $SO(p,q)$. This decomposition was first given by Hehl (F. W. Hehl et al., 1995) in terms of differential forms. The $GL(N)$ decomposition further reduces into six subspaces which we label $\mathfrak{f}$, $\mathscr{F}$, $\mathfrak{F}$, $\mathfrak{l}$, $\mathscr{L}$, and $\mathfrak{k}$. The subspace $\mathcal{F}$ decomposes into $\mathfrak{f}$, $\mathscr{F}$, and $\mathfrak{F}$. The subspace $\mathcal{L}$ decomposes into $\mathfrak{l}$, and $\mathscr{L}$. Finally, the subspace $\mathcal{K}$ we relabel to $\mathfrak{k}$ to match our notation. Recall that lowercase Fraktur tensors are required to be totally trace free.

We have repeated the equations for the subspaces $\mathfrak{f}$, $\mathscr{F}$, and $\mathfrak{F}$ from appendix C here for convenience.

$$\mathcal{F}_{abcd} = \mathfrak{f}_{abcd} + \mathscr{F}_{abcd} + \mathfrak{F}_{abcd} \tag{4.28}$$

$$\mathscr{F}_{abcd} = \frac{1}{(N-2)}\left(\mathring{R}_{ac}g_{bd} - \mathring{R}_{bc}g_{ad} - \mathring{R}_{ad}g_{bc} + \mathring{R}_{bd}g_{ac}\right) \tag{4.29}$$

$$\mathfrak{F}_{abcd} = \frac{1}{N(N-1)}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{4.30}$$

$$R_{(bd)} = g^{ac}\mathcal{F}_{abcd}, \quad R = g^{ac}g^{bd}\mathcal{F}_{abcd}, \quad \mathring{R}_{(ab)} = R_{(ab)} - \frac{1}{N}Rg_{ab} \tag{4.31}$$

The subspace $\boldsymbol{\mathcal{L}}$ decomposes into two subspaces: $\mathfrak{l}$, and $\mathscr{L}$ in much the same way as all the other $Y4$ sectors decompose. Additionally, it decomposes similarly to how the tensor $\mathcal{Q}_{abc}$ from equation (3.39) decomposed into $\mathfrak{q}_{abc}$ and $\mathfrak{Q}_{abc}$ in equation (3.43); this is seen in chapter 3. We can calculate the tensors corresponding to those subspaces as follows:

$$R_{[bd]} = g^{ac}\mathcal{L}_{abcd}$$

$$\mathcal{L}_{abcd} = \mathscr{L}_{abcd} + \mathfrak{l}_{abcd}$$

$$\mathscr{L}_{abcd} = \frac{1}{(N-2)}\left(R_{[bd]}g_{ac} - R_{[bc]}g_{ad} - R_{[ad]}g_{bc} + R_{[ac]}g_{bd}\right)$$

$$\mathfrak{l}_{abcd} = \mathfrak{h}_{abcd} + \mathfrak{j}_{abcd} \tag{4.32}$$

$$\mathscr{L}_{abcd} = \mathscr{H}_{abcd} + \mathscr{J}_{abcd} \tag{4.33}$$

similarly to equation (4.26). Equation (4.32) shows us how to calculate $\mathfrak{l}$ in terms of $\mathfrak{h}$, and $\mathfrak{j}$; the tensor $\mathfrak{l}$ can be constructed with the Maple command "CurvatureFrankL". Next, equation (4.33) shows us how to calculate $\mathscr{L}$ in terms of $\mathscr{H}$, and $\mathscr{J}$; the tensor $\mathscr{L}$ can be constructed with the Maple command "CurvatureCurlyL". The tensor $\boldsymbol{\mathcal{K}}$ is the totally skew part of R_{abcd}, and does not reduce further under $SO\left(p,q\right)$. However, we change its notation to lowercase Fraktur script to match its totally trace free character.

$$\boldsymbol{\mathcal{K}} = \mathfrak{k}, \quad \mathfrak{k}_{abcd} = R_{[abcd]} \tag{4.34}$$

Equation (4.34) is the last of our tensors which describe how R_{abcd} decomposes under $SO\left(p,q\right)$. We let $\boldsymbol{R}$ be the vector space describing the curvature tensor. After that work, we now have how the curvature tensor decomposes under $SO\left(p,q\right)$:

$$\boldsymbol{R} = \mathfrak{f} \oplus \mathscr{F} \oplus \mathfrak{F} \oplus \mathfrak{l} \oplus \mathscr{L} \oplus \mathfrak{k} \quad . \tag{4.35}$$

4.6.3 Turning the Curvature Tensor Into $SL\left(2,\mathbb{C}\right)$ Irreducible Spinors

When we view the curvature tensor as a spinor it is always easiest to break it into its corresponding $SO\left(p,q\right)$ irreducible tensors first. Once we have those, there is a one to one correspondence between those tensors and irreducible spinors. We present the spinors here which are related to the decomposition of the curvature tensor starting first with the $Y3b$ spinors given by the following tableaux:

1	3
2	4

The spinors Ψ, Φ, and Λ are related to the tensors $\mathfrak{f}$, $\mathscr{F}$, and $\mathfrak{F}$ respectively. The following equations relate them:

$$\Psi_{(ABCD)} \cong \frac{1}{4}\mathfrak{f}_{abcd}\epsilon^{A'B'}\epsilon^{C'D'} \tag{4.36}$$

$$\mathfrak{f}_{abcd} \quad \cong \quad \Psi_{(ABCD)}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{(A'B'C'D')}\epsilon_{AB}\epsilon_{CD} \tag{4.37}$$

$$\Phi_{(AC)(A'C')} \cong \frac{1}{2} g^{bd} \mathscr{F}_{abcd} \tag{4.38}$$

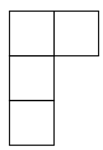
$$\begin{aligned} \mathscr{F}_{abcd} \;\; &\cong \;\; \Phi_{(AC)(A'C')}\epsilon_{BD}\epsilon_{B'D'} - \Phi_{(BC)(B'C')}\epsilon_{AD}\epsilon_{A'D'} \\ &+ \;\; \Phi_{(BD)(B'D')}\epsilon_{AC}\epsilon_{A'C'} - \Phi_{(AD)(A'D')}\epsilon_{BC}\epsilon_{B'C'} \end{aligned} \tag{4.39}$$

$$\Lambda \cong \frac{1}{12} g^{ac} g^{bd} \mathfrak{F}_{abcd} \tag{4.40}$$

$$\mathfrak{F}_{abcd} \cong \Lambda \left(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'} \right) \tag{4.41}$$

Equations (4.36-4.41) are all given in appendix F, and give the $SL(2,\mathbb{C})$ irreducible spinors for the $\boldsymbol{\mathcal{F}}$ sector. These equations also relate the spinors to their corresponding $SO(3,1)$, irreducible tensors. Note that $p = 2$, $q = 2$ is not allowed nor is $p = 4$, $q = 0$.

We have four new equations relating the new $\boldsymbol{\mathcal{L}}$ type spinors from the:



Ferrers diagram back to their corresponding tensors. We labeled these spinors by: Ж, and Ю. They relate to the tensors $\mathfrak{l}$, and $\mathscr{L}$ as follows.

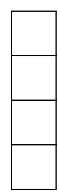
$$Ж_{(AB)(C'D')} \cong \frac{1}{2} \epsilon^{CD} \epsilon^{A'B'} \mathfrak{l}_{abcd} \tag{4.42}$$

$$\mathfrak{l}_{abcd} \;\; \cong \;\; \epsilon_{A'B'}\epsilon_{CD} \left(Ж_{(AB)(C'D')} \right) - \epsilon_{C'D'}\epsilon_{AB} \left(Ж_{(CD)(A'B')} \right) \tag{4.43}$$

$$Ю_{(AB)} \cong \frac{1}{4} g^{ef} \epsilon^{A'B'} \mathscr{L}_{eafb} \tag{4.44}$$

$$\begin{aligned}\mathscr{L}_{abcd} \;\cong\; & \epsilon_{BC}\epsilon_{B'C'}\left(\epsilon_{A'D'}\text{Ю}_{(AD)} + \epsilon_{AD}\overline{\text{Ю}}_{(A'D')}\right) \\ - \; & \epsilon_{AC}\epsilon_{A'C'}\left(\epsilon_{B'D'}\text{Ю}_{(BD)} + \epsilon_{BD}\overline{\text{Ю}}_{(B'D')}\right) \\ + \; & \epsilon_{AD}\epsilon_{A'D'}\left(\epsilon_{B'C'}\text{Ю}_{(BC)} + \epsilon_{BC}\overline{\text{Ю}}_{(B'C')}\right) \\ - \; & \epsilon_{BD}\epsilon_{B'D'}\left(\epsilon_{A'C'}\text{Ю}_{(AC)} + \epsilon_{AC}\overline{\text{Ю}}_{(A'C')}\right)\end{aligned} \tag{4.45}$$

Equations (4.42)-(4.44) tell us how to convert the tensors $\mathfrak{l}$, and $\mathscr{L}$ into the $\text{Ж}_{(AB)(C'D')}$, and $\text{Ю}_{(AB)}$ spinors; these spinor objects can be called with the following Maple commands: “CurvatureZheSpinor”, and “CurvatureYuSpinor”. Furthermore, equations (4.43)-(4.45) tell us how to convert these spinors with a mixture of epsilon spinors back into corresponding $SO(3,1)$ irreducible tensors; these objects can be constructed with the Maple commands: “CurvatureGenerateZheTensor”, and “CurvatureGenerateYuTensor”. For our last sector $\mathfrak{k}$, we again repeat the equations from appendix F. Recall that the tensor $\mathfrak{k}$ corresponds to the



Young tableau. Its two spinor equations are given by:

$$\aleph \cong \frac{1}{36}\epsilon^{AB}\epsilon^{CD}\epsilon^{A'C'}\epsilon^{B'D'}\mathfrak{k}_{abcd} \tag{4.46}$$

$$\begin{aligned}\mathfrak{k}_{abcd} \;=\; & \aleph\left(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'}\right) \\ + \; & \aleph\left(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'B'}\epsilon_{C'D'}\right) \\ + \; & \aleph\left(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}\right)\end{aligned} \tag{4.47}$$

Equations (4.46)-(4.47) tell us how to construct the aleph spinor, and how to convert the aleph spinor back into the tensor $\mathfrak{k}$. We only include these spinors because they are the ones which are important for the decomposition of the curvature tensor in ECSK theory. Moving forward, we then classify each of these 6 spinors using several tools, some already known, and some newly developed.

4.6.4 Classification of the Curvature Spinors

To classify the curvature tensor, we classify the 6 spinors:

$$\Psi_{(ABCD)}, \Phi_{(AB)(A'B')}, \Lambda, \text{Ж}_{(AB)(A'B')}, \text{Ю}_{(AB)}, \aleph \tag{4.48}$$

using several tools: the Petrov classification, the Plebanski classification, the Segre classification, a positive/negative/zero scalar classification, and a structural reducibility classification (algebraic irreducibility) developed from the ideas of Penrose and Rindler (Penrose & Rindler, 1987b). The structural reducibility classification follows in the same spirit as the classification that we did for the torsion tensor in chapter 3.

The first three spinors are similar to the spinors that are usually classified in GR: the Weyl spinor $\mathbf{\Psi}$, the Ricci spinor $\mathbf{\Phi}$, and the curvature/Ricci scalar $\mathbf{\Lambda}$; however, do not mistake these here for those because the spinors in equation (4.48) will include contributions from the torsion tensor; therefore they are not quite the same objects. Nevertheless, the classification of these objects is the same. The three new spinors: Ж, Ю, and ℵ are pieces of the curvature that arise due to the torsion tensor being non-zero.

We classify the spinor $\mathbf{\Psi}$ with the Petrov classification; see Acvevedo (Acvevedo et al., 2006) for an excellent computational reference, and see the following references: (Zakhary et al., 2003), (Letniowski & McLenaghan, 1988), (DÍnverno & Russell-Clark, 1971) for more details. We give a short description here for clarity.

4.6.5 The Petrov Classification of $\Psi_{(ABCD)}$

The Petrov classification can be found in many references in the literature, a good summary is provided in Stephani (Stephani et al., 2003). We can either examine the eigenbivectors of the Weyl tensor, or look at repeated roots in the symmetric spinor decomposition of the Weyl spinor $\Psi_{(ABCD)}$. We take the second approach and find that there are six different cases, the first of which is irreducible in the sense of Stewart's definition (Stewart, 1993). The Petrov classification classifies the spinor $\Psi_{(ABCD)}$ according to which of these

Cases for the Petrov Classification			
Cases	Form of $\Psi_{(ABCD)}$	Petrov Type	Degrees of Freedom
Case 1	$\alpha_A\beta_B\gamma_C\delta_D$	I	10
Case 2	$\alpha_A\alpha_B\beta_C\gamma_D$	II	8
Case 3	$\alpha_A\alpha_B\beta_C\beta_D$	D	6
Case 4	$\alpha_A\alpha_B\alpha_C\beta_D$	III	6
Case 5	$\alpha_A\alpha_B\alpha_C\alpha_D$	N	4
Case 6	0	O	0

Table 4.1: Petrov Classification of the Spinor $\Psi_{(ABCD)}$

six cases it falls into through calculating several invariants to distinguish the cases. Penrose and Rindler (Penrose & Rindler, 1987b) reference Grace and Young (Grace & Young, 2011), as do we in the construction of the invariants for the Petrov classification. We relabel some of the co(in)variants however. A computationally efficient Petrov type algorithm is given by Zakhary et al. (Zakhary et al., 2003). We begin with viewing the spinor $\boldsymbol{\Psi}$ as a covariant (like Olver (Olver, 2003)); this covariant is calculated the same way as in the Newman-Penrose formalism, see Stewart (Stewart, 1993) for more details on this. Then we choose a spin basis o^A, ι^A for the spinor space $\boldsymbol{\mathcal{S}}$. Then Ψ as a covariant is given by the following equations:

$$
\begin{aligned}
\Psi_0 &= \Psi_{(ABCD)}o^A o^B o^C o^D \qquad (4.49)\\
\Psi_1 &= \Psi_{(ABCD)}o^A o^B o^C \iota^D \\
\Psi_2 &= \Psi_{(ABCD)}o^A o^B \iota^C \iota^D \\
\Psi_3 &= \Psi_{(ABCD)}o^A \iota^B \iota^C \iota^D \\
\Psi_4 &= \Psi_{(ABCD)}\iota^A \iota^B \iota^C \iota^D
\end{aligned}
$$

If the Petrov type is O then all the $\boldsymbol{\Psi}$ covariants will be zero. We can then build other covariants out of the $\boldsymbol{\Psi}$ spinors.

The next of these will be the $\boldsymbol{Q}$ covariants, which if they are all zero and $\boldsymbol{\Psi}$ is non-zero then we are in a Petrov Type N space. We label the $\boldsymbol{Q}$ covariants the same way we labeled the $\boldsymbol{\Psi}$ covariants above in equation (4.49)

$$Q_{(ABCD)} = \Psi_{(AB}{}^{EF}\Psi_{CD)EF}$$

$$\begin{aligned} Q_0 &= 2\Psi_0\Psi_2 - 2\left(\Psi_1\right)^2 \\ Q_1 &= \Psi_0\Psi_3 - \Psi_1\Psi_2 \\ Q_2 &= \frac{1}{3}\Psi_0\Psi_4 + \frac{2}{3}\Psi_1\Psi_3 - \left(\Psi_2\right)^2 \\ Q_3 &= \Psi_1\Psi_4 - \Psi_2\Psi_3 \\ Q_4 &= 2\Psi_2\Psi_4 - 2\left(\Psi_3\right)^2 \end{aligned} \tag{4.50}$$

Next, to determine if we are in a type D space we calculate the R covariants. We calculate these if the other two prior covariants are non-zero. We can calculate the R covariants with the following equations:

$$R_{(ABCDEF)} = \Psi_{(ABC}{}^{K}\Psi_{DE}{}^{LM}\Psi_{F)KLM}$$

$$\begin{aligned} R_0 &= Q_0\Psi_1 - Q_1\Psi_0 \\ R_1 &= \frac{1}{2}\left(Q_0\Psi_2 - Q_2\Psi_0\right) \\ R_2 &= Q_1\Psi_2 - Q_2\Psi_1 \\ R_3 &= \frac{1}{2}\left(Q_1\Psi_3 - Q_3\Psi_1\right) \\ R_4 &= Q_2\Psi_3 - Q_3\Psi_2 \\ R_5 &= \frac{1}{2}\left(Q_2\Psi_4 - Q_4\Psi_2\right) \\ R_6 &= Q_3\Psi_4 - Q_4\Psi_3 \end{aligned} \tag{4.51}$$

which are zero in a type D space.

To determine the last few three cases, we need to calculate the I_p and J_p invariants. These are given below:

$$I_p = \frac{1}{3}\Psi_0\Psi_4 - \frac{4}{3}\Psi_1\Psi_3 + (\Psi_2)^2 \tag{4.52}$$

$$J_p = \Psi_0\Psi_2\Psi_4 + 2\Psi_1\Psi_2\Psi_3 - \Psi_0(\Psi_3)^2 - (\Psi_1)^2\Psi_4 - (\Psi_2)^3 \tag{4.53}$$

and if both $I_p = J_p = 0$ then we are in Petrov Type III space.

The last invariant Δ acts as a discriminant which tells us when there are repeated roots of the polynomial defined by $\mathbf{\Psi}$. Grace and Young explain more on the quartic polynomial which is generates by the spinor Ψ_{ABCD} (Grace & Young, 2011). The invariant Δ is given by:

$$\Delta = (I_p)^3 - (J_p)^2 \tag{4.54}$$

and differentiates the Petrov Type II and Type I cases. If $\Delta = 0$ we are in Petrov Type II, however if $\Delta \neq 0$ then we are in a Petrov Type I space. There are ways we could refine this classification further using ideas in invariant theory, seeing if the discriminant is positive, or negative, but this is made more difficult by the complex coefficients, and we will not examine it here. Excellent flow charts for this algorithm are presented in both Zakhary and D'Inverno (Zakhary et al., 2003)(DÍnverno & Russell-Clark, 1971). Furthermore, a direction for future work would be to incorporate the ideas of Penrose (Penrose, 1972), we could refine the classification and hopefully have a type for each degree of freedom 0 through 10 based on the multiplicities and classification of the singularities of the curves defined by the polynomial of Ψ.

4.6.6 The Plebanski Classification of $\Phi_{(AB)(A'B')}$ and Ж$_{(AB)(A'B')}$

The Plebanski classification is almost identical to the Petrov classification section. The only difference is that we begin with a spinor like $\mathbf{\Phi}$ instead of one like $\mathbf{\Psi}$. All the Plebanski classification does is turn $\mathbf{\Phi}$ into a $\mathbf{\Psi}$-like spinor, call it $\boldsymbol{\chi}$, through squaring it:

$$\chi_{(ABCD)} = \Phi_{(AB}{}^{E'F'}\Phi_{CD)E'F'}$$

and then applying the Petrov classification to $\boldsymbol{\chi}$. In NP fashion, the spinors $\mathbf{\Phi}$, and $\boldsymbol{\chi}$ when viewed as covariants, are given by the following equations:

$$\begin{aligned}
\Phi_{00} &= \Phi_{ABA'B'}o^A o^B \overline{o}^{A'}\overline{o}^{B'} \\
\Phi_{01} &= \Phi_{ABA'B'}o^A o^B \overline{o}^{A'}\overline{\iota}^{B'} \\
\Phi_{02} &= \Phi_{ABA'B'}o^A o^B \overline{\iota}^{A'}\overline{\iota}^{B'} \\
\Phi_{10} &= \Phi_{ABA'B'}o^A \iota^B \overline{o}^{A'}\overline{o}^{B'} \\
\Phi_{11} &= \Phi_{ABA'B'}o^A \iota^B \overline{o}^{A'}\overline{\iota}^{B'} \\
\Phi_{12} &= \Phi_{ABA'B'}o^A \iota^B \overline{\iota}^{A'}\overline{\iota}^{B'} \\
\Phi_{20} &= \Phi_{ABA'B'}\iota^A \iota^B \overline{o}^{A'}\overline{o}^{B'} \\
\Phi_{21} &= \Phi_{ABA'B'}\iota^A \iota^B \overline{o}^{A'}\overline{\iota}^{B'} \\
\Phi_{22} &= \Phi_{ABA'B'}\iota^A \iota^B \overline{\iota}^{A'}\overline{\iota}^{B'}
\end{aligned} \tag{4.55}$$

$$\begin{aligned}
\chi_0 &= 2\Phi_{00}\Phi_{02} - 2\left(\Phi_{01}\right)^2 \\
\chi_1 &= \Phi_{00}\Phi_{12} - 2\Phi_{01}\Phi_{11} + \Phi_{02}\Phi_{10} \\
\chi_2 &= \frac{1}{3}\left(\Phi_{00}\Phi_{22} - 2\Phi_{01}\Phi_{21} + \Phi_{02}\Phi_{20} + 4\Phi_{12}\Phi_{10} - 4\left(\Phi_{11}\right)^2\right) \\
\chi_3 &= \Phi_{10}\Phi_{22} - 2\Phi_{11}\Phi_{21} + \Phi_{12}\Phi_{20} \\
\chi_4 &= 2\Phi_{20}\Phi_{22} - 2\left(\Phi_{21}\right)^2
\end{aligned} \tag{4.56}$$

where $\overline{o}^{A'}$, and $\overline{\iota}^{B'}$ are a spin basis on $\overline{\boldsymbol{\mathcal{S}}}$. Upon calculating the χ covariants, we treat them as if they were the Ψ covariant and put them into the Petrov algorithm to determine the Plebanski type; just like how there are six Petrov types, there are six Plebanski types by the same reasoning.

The interesting thing about this classification in ECSK theory is that we now have the additional antihermitian spinor $Ж_{(AB)(A'B')}$ which we can classify with the same tools as we could with the $\mathbf{\Phi}$ spinor. This becomes particularly interesting when two nonidentical spacetimes have the same classification for $\mathbf{\Phi}$ but not for $Ж_{(AB)(A'B')}$.

4.6.7 The Segre Classification of $\Phi_{(AB)(A'B')}$ and $Ж_{(AB)(A'B')}$

The Segre classification further refines the Plebanski classification. We follow Hall's analysis (Hall, 1976) of the canonical forms of the Ricci tensor; this really applies to any second order symmetric tensor in a space with Lorentzian signature. These tools are then applied in Zakhary and Carminati (Zakhary & Carminati, 2004) to spinors which yield a complete algorithm for determining the Segre type of a given spinor in the form $\Phi_{(AB)(A'B')}$. Historically Churchill (Churchill, 1932) was the first to present the canonical forms for second order symmetric tensors. There are 15 distinct Segre types that can occur.

The Segre classification hinges on determining the Jordan normal form of either a second order symmetric tensor or equivalently a spinor of the form $\Phi_{(AB)(A'B')}$; see Horn (Horn & Johnson, 1985) for details. Reference (Stephani et al., 2003) presents how both of these viewpoints yield the same answer, and we will be examining how the tensor case relates to the spinor case. We begin by constructing the mixed null tetrad l^a, m^a, a^a, b^a of Hall where l^a and m^a are null vectors and a^a and b^a are spacelike vectors. The matrix representation of the metric tensor in this basis with mostly minus signature is given by:

$$g_{ab} \doteq \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We then write a second order symmetric tensor in this basis, call it A_{ab}:

$$\begin{aligned} A_{ab} &= -2A_1 l_{(a}m_{b)} - A_2 l_a l_b - A_3 m_a m_b - 2A_4 l_{(a}a_{b)} - 2A_5 l_{(a}b_{b)} \\ &- 2A_6 m_{(a}a_{b)} - 2A_7 m_{(a}b_{b)} - 2A_8 a_{(a}b_{b)} - A_9 a_a a_b - A_{10} b_a b_b \end{aligned} \tag{4.57}$$

where the coefficients $A_i \in \mathbb{R}$. Hall then proceeds to present two cases, one in which the tensor A_{ab} has a null eigenvector and the other being when the tensor A_{ab} does not have a null eigenvector. As an aside, Hall does his analysis with signature $[-,+,+,+]$ which is different that what we are using (mostly minus $[+,-,-,-]$) thus we decided to change equation (4.57) to have negatives in front for ease. A mostly minus signature can be used just as well, but the analysis done works regardless of the signature convention we pick. At this point we find it easiest to give canonical forms for every possible Segre type that Hall presents. We will then use these canonical forms to relate to the conditions that Zakhary and Carminati present in their Segre classification algorithm.

The Segre classification gives us information on the algebraic and geometric multiplicities of the eigenvalues and eigenvectors. We will provide how many eigenvectors of each type (timelike, spacelike, lightlike) there are later. Additionally, we can tell just from the Segre notation how many timelike, lightlike, and spacelike eigenvectors there are. For there to exist a timelike eigenvector the first number must be a one and must not appear in any of the parenthesis. If it does appear in parentheses then it must be a lightlike eigenvector. There are two cases for which there are complex eigenvectors and those also appear first, the other eigenvectors being spacelike. For example $[(1,111)]$ would have two null eigenvectors and two spacelike eigenvectors. There are always two null eigenvectors (except of course when the first number is a one and appears without parenthesis) unless the first number is greater than one, then there is just a single null eigenvector.

The Segre notation is meant to mirror the information given by the Jordan normal (canonical) form of the matrix. Plebanski and Stephani (Stephani et al., 2003), (Plebanski, 1964), additionally both give the Segre types in Plebanski notation, which we will not do here, but are good references.

The dimension of the eigenspace is represented in the Segre notation by the amount of numbers inside the square brackets; this is related to the geometric multiplicity. Recall that the geometric multiplicity for a given eigenvalue is the number of Jordan blocks corresponding to that value; we can also think of the geometric multiplicity as the dimension of the eigenspace for a given eigenvalue. For example in $[2, 11]$ we only have three eigenvectors, each with geometric multiplicity one. The dimension of the eigenspace of a matrix of type $[2, 11]$ is only 3. The presence of the 2 is telling us that the eigenspace is degenerate.

The algebraic multiplicity is represented in Segre notation by either normal parenthesis or by a number higher than one. Recall that the algebraic multiplicity for a given eigenvalue is the sum of the sizes of all the Jordan blocks corresponding to that value; we can also think of the algebraic multiplicity of an eigenvalue as the number of times it appears as a root of the characteristic polynomial of our matrix of interest. For clarity, when we see $[(2, 11)]$ we would have a single eigenvalue of algebraic multiplicity 4; the parenthesis are telling us about repeated eigenvalues.

Before we provide a list of all the Segre types and information about them, it is useful to describe the invariants we use to differentiate the different cases. The first invariant will be the Plebanski type discussed above. Zakhary and Carminati (Zakhary & Carminati, 2004), whom we will follow, first calculate the Plebanski type to sort the Segre types, and then calculate further invariants to determine the Segre type. Accordingly, we group the Segre types according to their Plebanski type first, then specialize to the Segre type.

Invariants and Covariants for the Segre Classification

There are several co(in)variants we need to determine the Segre classification after applying the Plebanski classification: Φ_{ab}, E_{ab}, I_6, I_7 χ_a, $\widetilde{\chi}_a$, H, k_a, and Δ. The co(in)variants: Φ_{ab}, χ_a, and Δ are given above in equations (4.55), (4.56), and (4.54). The other co(in)variants: E_{ab}, I_6, I_7, $\widetilde{\chi}_a$, H, and k_a we will define below.

To move forward we make a few observations. There is an erroneous minus sign in front of the invariant I_7 (their equation (25)) in the Zakhary and Carminati paper (Zakhary & Carminati, 2004); removing the minus sign there makes their equation (25) consistent with their equation (28). The covariant $\widetilde{\chi}_a$ is labeled the same way that the covariant $\boldsymbol{\Psi}$ is labeled in the Petrov classification. To calculate the invariants I_p and J_p in equations (4.52), and (4.53) we replace the covariant $\boldsymbol{\Psi}$ used in the Petrov classification with the covariant $\boldsymbol{\chi}$ used in the Plebanski classification, see equation (4.56).

$$E^{(AB)}{}_{(C'D')} = 2\Phi^{(A}{}_{EF'(C'}\Phi^{B)EF'}{}_{D')}$$

$$\begin{aligned}
E_{00} &= 4\left(\Phi_{00}\Phi_{11} - \Phi_{01}\Phi_{10}\right) \\
E_{01} &= 2\left(\Phi_{00}\Phi_{12} - \Phi_{02}\Phi_{10}\right) \\
E_{02} &= 4\left(\Phi_{01}\Phi_{12} - \Phi_{02}\Phi_{11}\right) \\
E_{10} &= 2\left(\Phi_{00}\Phi_{21} - \Phi_{01}\Phi_{20}\right) \\
E_{11} &= \Phi_{00}\Phi_{22} - \Phi_{02}\Phi_{20} \\
E_{12} &= 2\left(\Phi_{01}\Phi_{22} - \Phi_{02}\Phi_{21}\right) \\
E_{20} &= 4\left(\Phi_{10}\Phi_{21} - \Phi_{11}\Phi_{20}\right) \\
E_{21} &= 2\left(\Phi_{10}\Phi_{22} - \Phi_{12}\Phi_{20}\right) \\
E_{22} &= 4\left(\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}\right)
\end{aligned} \tag{4.58}$$

$$I_6 = \frac{1}{3}\Phi_{ABA'B'}\Phi^{ABA'B'}$$

$$I_6 = \frac{1}{3}\left(2\Phi_{00}\Phi_{22} - 4\Phi_{01}\Phi_{21} + 2\Phi_{02}\Phi_{20} - 4\Phi_{12}\Phi_{10} + 4\left(\Phi_{11}\right)^2\right) \tag{4.59}$$

$$I_7 = \frac{1}{3}\Phi_{ABC'D'}\Phi^{A\ C'}_{\ E\ F'}\Phi^{BED'F'} \tag{4.60}$$

$$\begin{aligned}
I_7 &= 2\left(\Phi_{00}\Phi_{11}\Phi_{22} + \Phi_{01}\Phi_{12}\Phi_{20} + \Phi_{02}\Phi_{10}\Phi_{21}\right) \\
&+ 2\left(-\Phi_{00}\Phi_{12}\Phi_{21} - \Phi_{01}\Phi_{10}\Phi_{22} - \Phi_{02}\Phi_{11}\Phi_{20}\right)
\end{aligned}$$

$$\widetilde{\chi}_{(ABCD)} = E_{(AB}{}^{E'F'}E_{CD)E'F'}$$

$$\begin{aligned}
\widetilde{\chi}_0 &= 2E_{00}E_{02} - 2\left(E_{01}\right)^2 \\
\widetilde{\chi}_1 &= E_{00}E_{12} - 2E_{01}E_{11} + E_{02}E_{10} \\
\widetilde{\chi}_2 &= \frac{1}{3}E_{00}E_{22} - \frac{2}{3}E_{01}E_{21} + \frac{1}{3}E_{02}E_{20} + \frac{4}{3}E_{12}E_{10} - \frac{4}{3}\left(E_{11}\right)^2 \\
\widetilde{\chi}_3 &= E_{10}E_{22} - 2E_{11}E_{21} + E_{12}E_{20} \\
\widetilde{\chi}_4 &= 2E_{20}E_{22} - 2\left(E_{21}\right)^2
\end{aligned} \tag{4.61}$$

$$H = I_6 I_p - J_p \tag{4.62}$$

$$k_a = J_p\left(E_{aa} - 2\frac{1}{H}I_7 I_p \Phi_{aa}\right), \quad (a = 0, 1, 2;\ \text{no summing}) \tag{4.63}$$

Below we have a list of the possible Segre types, their corresponding canonical form as a tensor, and conditions upon which an arbitrary tensor will be in that form. It is these forms for which the tensor A_{ab} in equation (4.57) can be put into if it is the corresponding Segre type. Note that the parameters $\lambda, \sigma, \rho_1, \rho_2, \rho_3 \in \mathbb{R}$. Plebanski (Plebanski, 1964) gives an excellent diagram on his pgs. 990 and 1001, describing how all the Segre types degenerate into each other upon changes in the invariants provided above. To determine the Segre type first we follow Zakhary and Carminati (Zakhary & Carminati, 2004), who first determine the Plebanski type, and then calculate extra invariants for the Segre classification.

Plebanski Type O

For Plebanski type O there are four different possible Segre types: $[(1, 111)]$, $[(2, 11)]$, $[1, (111)]$, and $[(1, 11)\,1]$. They are given below with their corresponding completely covariant canonical forms along with the conditions on the covariants which force them to be that particular Segre type. First we have Segre type $[(1, 111)]$ which has two null eigenvectors, and two spacelike eigenvectors with only one eigenvalue of algebraic multiplicity 4.

$[(1, 111)]$

$$A_{ab} = -2\rho_1 l_{(a}m_{b)} - \rho_1 a_a a_b - \rho_1 b_a b_b$$

$$\Phi_{ab} = 0$$

Next we have Segre type $[(2,11)]$, which has one null eigenvector, and two spacelike eigenvectors with only one eigenvalue of algebraic multiplicity 4.

$$[(2,11)]$$

$$A_{ab} = -\lambda l_a l_b - 2\rho_1 l_{(a} m_{b)} - \rho_1 a_a a_b - \rho_1 b_a b_b$$

$$\Phi_{ab} \neq 0, \text{ and } (\Phi_{11})^2 - \Phi_{01}\Phi_{21} = 0$$

Next we have Segre type $[1,(111)]$, which has one timelike eigenvector, and three spacelike eigenvectors with two distinct eigenvalues, one of algebraic multiplicity 3 and the other of algebraic multiplicity 1.

$$[1,(111)]$$

$$A_{ab} = -\rho_2 l_a l_b - 2\rho_1 l_{(a} m_{b)} - \rho_2 m_a m_b - (\rho_1 + \rho_2)\, a_a a_b - (\rho_1 + \rho_2)\, b_a b_b$$

$$E_{00} > 0$$

The last Segre type in Plebanski type O is $[(1,11)\,1]$, which has two null eigenvectors, and two spacelike eigenvectors with two distinct eigenvalues, one of algebraic multiplicity 3 and the other of algebraic multiplicity 1.

$$[(1,11)\,1]$$

$$A_{ab} = -2\rho_1 l_{(a} m_{b)} - \rho_1 a_a a_b - \rho_3 b_a b_b$$

$$E_{ab} \neq 0, \quad \text{for some } a, b, \text{ and } E_{00} \leq 0$$

Plebanski Type N

For Plebanski type N there are two different possible Segre types: $[(3,1)]$, and $[(2,1)\,1]$. They are given below with their corresponding completely covariant canonical forms along with the conditions on the covariants which force them to be that particular Segre type. First we have Segre type $[(3,1)]$ which has one null eigenvector, and one spacelike eigenvector with only one eigenvalue of algebraic multiplicity 4.

$$[(3,1)]$$

$$A_{ab} = -2\rho_1 l_{(a}m_{b)} - 2\sigma l_{(a}a_{b)} - \rho_1 a_a a_b - \rho_1 b_a b_b$$

$$I_6 = 0$$

The other Segre type in Plebanski type N is $[(2,1)\,1]$, which has one null eigenvector, and two spacelike eigenvectors with two distinct eigenvalues, one of algebraic multiplicity 3 and the other of algebraic multiplicity 1.

$$[(2,1)\,1]$$

$$A_{ab} = -\lambda l_a l_b - 2\rho_1 l_{(a}m_{b)} - \rho_1 a_a a_b - \rho_3 b_a b_b$$

$$I_6 \neq 0$$

Plebanski Type D

For Plebanski type D there are five different possible Segre types: $[(1,1)\,(11)]$, $[2,(11)]$, $\left[Z\overline{Z},(11)\right]$ $[1,1\,(11)]$, and $[(1,1)\,11]$. They are given below with their corresponding completely covariant canonical forms along with the conditions on the covariants which force them to be that particular Segre type. First we have Segre type $[(1,1)\,(11)]$ which has two null eigenvectors, and two spacelike eigenvectors with two distinct eigenvalues both of algebraic multiplicity 2.

$$[(1,1)\,(11)]$$

$$A_{ab} = -2\rho_1 l_{(a}m_{b)} - \rho_2 a_a a_b - \rho_2 b_a b_b$$

$$I_6 \neq 0, \text{ and } E_{ab} = 0$$

Next we have Segre type $[2,(11)]$, which has one null eigenvector, and two spacelike eigenvectors with two distinct eigenvalues, both of algebraic multiplicity 2.

$$[2,(11)]$$

$$A_{ab} = -\lambda l_a l_b - 2\rho_1 l_{(a} m_{b)} - \rho_2 a_a a_b - \rho_2 b_a b_b$$

$$I_6 \neq 0, \text{ and } (\widetilde{\chi}_0 = 0, \text{ if } \chi_0 \neq 0, \text{ or } \widetilde{\chi}_2 = 0 \text{ otherwise}) \text{ or just } \widetilde{\chi}_a = 0, \quad \forall a$$

Next we have Segre type $\left[Z\overline{Z},(11)\right]$, which has a pair of complex conjugate eigenvectors, and two spacelike eigenvectors with three distinct eigenvalues, one of algebraic multiplicity 2, and the other two being complex conjugate eigenvalues with algebraic multiplicities 1 each.

$$\left[Z\overline{Z},(11)\right]$$

$$A_{ab} = -\rho_2 l_a l_b - 2\rho_1 l_{(a} m_{b)} + \rho_2 m_a m_b - \rho_3 a_a a_b - \rho_3 b_a b_b$$

$$I_6 = 0, \text{ or } (I_6 \neq 0, \text{ and } H < 0)$$

Next we have Segre type $[1,1\,(11)]$, which has one timelike eigenvector, and three spacelike eigenvectors with three distinct eigenvalues, two of algebraic multiplicity 1, and the third of algebraic multiplicity 2.

$$[1,1\,(11)]$$

$$A_{ab} = -\rho_2 l_a l_b - 2\rho_1 l_{(a} m_{b)} - \rho_2 m_a m_b - \rho_3 a_a a_b - \rho_3 b_a b_b$$

$$I_6 \neq 0, \text{ and } H > 0, \text{ and } k_a < 0 \text{ for some } a$$

The last Segre type for Plebanski type D is $[(1,1)\,11]$, which has two null eigenvectors, and two spacelike eigenvectors with three distinct eigenvalues, two of algebraic multiplicity 1, and the third of algebraic multiplicity 2.

$$[(1,1)\,11]$$

$$A_{ab} = -2\rho_1 l_{(a}m_{b)} - \rho_2 a_a a_b - \rho_3 b_a b_b$$

$$I_6 \neq 0, \text{ and } H > 0, \text{ and } k_a > 0 \text{ for some } a$$

Plebanski Type III

This case of Plebanski type III requires no further calculations because the only possible Segre type is $[3,1]$. This type has one null eigenvector and one spacelike eigenvector. There are two distinct eigenvalues one with algebraic multiplicity 3 and the other of algebraic multiplicity 1.

$$[3,1]$$

$$A_{ab} = -2\rho_1 l_{(a}m_{b)} - 2\sigma l_{(a}a_{b)} - \rho_1 a_a a_b - \rho_2 b_a b_b$$

Plebanski Type II

This case of Plebanski type II requires no further calculations because the only possible Segre type is $[2,11]$. This type has one null eigenvector and two spacelike eigenvectors. There are three distinct eigenvalues one with algebraic multiplicity 2 and the other two of algebraic multiplicity 2.

$$[2,11]$$

$$A_{ab} = -\lambda l_a l_b - 2\rho_1 l_{(a}m_{b)} - \rho_2 a_a a_b - \rho_3 b_a b_b$$

Plebanski Type I

For Plebanski type I there are two different possible Segre types: $[1,111]$, and $\left[Z\overline{Z},11\right]$. They are given below with their corresponding completely covariant canonical forms along with the conditions on the covariants which force them to be that particular Segre type. We begin with Segre type $[1,111]$ which has one timelike eigenvector, and three spacelike eigenvectors with four distinct eigenvalues all of algebraic multiplicity 1.

$$[1,111]$$

$$A_{ab} = -\rho_2 l_a l_b - 2\rho_1 l_{(a} m_{b)} - \rho_2 m_a m_b - \rho_3 a_a a_b - \rho_4 b_a b_b$$

$$\Delta < 0$$

Next we have Segre type $\left[Z\overline{Z}, 11\right]$, which has a pair of complex conjugate eigenvectors, and two spacelike eigenvectors with four distinct eigenvalues, two of algebraic multiplicity 1, and the other two being complex conjugate eigenvalues with algebraic multiplicities 1 each.

$$\left[Z\overline{Z}, 11\right]$$

$$A_{ab} = -\rho_2 l_a l_b - 2\rho_1 l_{(a} m_{b)} + \rho_2 m_a m_b - \rho_3 a_a a_b - \rho_4 b_a b_b$$

$$\Delta > 0$$

Altogether there are 15 distinct Segre types that can occur. When we perform our classification of the curvature tensor in ECSK theory we often convert the curvature tensor into its 6 constituent spinors. We can then apply the Segre classification to the spinor $\mathbf{\Phi}$ because it corresponds to a symmetric rank 2 tensor. Recall that $\mathbf{\Phi}$ is a hermitian spinor. In our classification we also apply the Segre classification to the spinor Ӂ, which is antihermitian, in the same exact way. To turn this into a hermitian spinor, we could just multiply it by i but we believe it is better for classification purposes to keep the spinor in its naturally occurring form. Additionally, the Segre classification is sensitive to signs for some of the co(in)variants, so we do not take the route of multiplying by i. For the most part the classification does not change in either approach.

The classification still works on Ӂ because instead of all the co(in)variants working off all the real parts of the spinor, it instead works with the imaginary parts; e.g., we could just multiply by i. However, if the classification is applied to an arbitrary spinor of the form $A_{(AB)(A'B')}$, the resulting classification will acquire complex coefficients in several of the invariants and a new classification will be needed to account for complex values; this is not needed here since such a spinor would correspond to a complex valued world tensor in the words of Penrose and Rindler (Plebanski, 1964).

The Segre types can be calculated in Maple through the command "SegreType" which can take as its input the covariant $\mathbf{\Phi}$, then returning its Segre type.

There are ways to further refine the Segre classification such as through the ideas presented by Ludwig and Scanlan (Ludwig & Scanlan, 1971), and Penrose (Penrose, 1972). We do not examine these ideas here, but note that Crade and Hall (Crade & Hall, 1982) have an excellent summary article which compares all the different classifications including the Segre classification. This would be an excellent direction for future work.

4.6.8 The Structural Reducibility of $\Phi_{(AB)(A'B')}$ and Ж$_{(AB)(A'B')}$

Now we turn to the structural reducibility for $\Phi_{ABA'B'}$ and Ж$_{ABA'B'}$. Penrose and Rindler (Penrose & Rindler, 1987b) originally constructed all the cases we provide here, but did not present co(in)variants to differentiate the different types. We make use of the invariants in Zakhary and Carminati (Zakhary & Carminati, 2004), which differentiate all of Penrose and Rindler's cases except cases 2 and 3. There are eight different cases in total. We use the same definition for structural reducibility which we used with the torsion tensor

Cases for the Curvature Phi Spinor				
Cases	Spinor	Our Notation	Penrose Notation	Deg. Freedom
Case 1	$\Phi_{(AB)(A'B')}$	[22]	$(2,2)$	9
Case 2	$\Lambda_{AA'}\Upsilon_{BB'}$	[11, 11]	$(1,1)(1,1)$	7
Case 3	$\Gamma_{AA'}\bar{\Gamma}_{BB'}$	$[(11,\overline{11})]$	$\|(1,1)\|^2$	7
Case 4	$\Lambda_{AA'}\Lambda_{BB'}$	$[11^2]$	$(1,1)^2$	4
Case 5	$\Lambda_{AA'}\rho_B\bar{\rho}_{B'}$	$[(A,A'),11]$	$(1,1)\|(1,0)\|^2$	6
Case 6	$\rho_A\bar{\rho}_{A'}\sigma_B\bar{\sigma}_{B'}$	$[(A,A'),(B,B')]$	$\|(1,0)\|^2\|(1,0)\|^2$	5
Case 7	$\rho_A\bar{\rho}_{A'}\rho_B\bar{\rho}_{B'}$	$[A^3,B']$	$\|(1,0)^2\|^2$	3
Case 8	0	$[-]$	$(-)$	0

Table 4.2: Structural Reducibility Table of the Spinor $\Phi_{(AB)(A'B')}$

in chapter 3. Recall that a spinor is structurally reducible if it can be written as a product. Recall that for a valence 2 spinor, it is structurally reducible if it can be written as products of lower valence spinors. The eight different cases are given below with their classification type next and their degrees of freedom following; this can be found in Penrose and Rindler

(Penrose & Rindler, 1987b). These cases apply to a hermitian spinor $\Phi_{(AB)(A'B')}$. However, they can be made to work for an antihermitian spinor like $Ж_{(AB)(A'B')}$ if there is an i placed in front of each of the cases. For example, case 2 would become $Ж_{(AB)(A'B')} = i\Lambda_{AA'}\Upsilon_{BB'}$, and we would add an i similarly for each of the other cases as well. We present these cases in the following table, where all primed and unprimed indices, though not explicit for typesetting constraints, are totally symmetric. The spinors $\Lambda_{AA'}$, and $\Upsilon_{AA'}$, are hermitian, while $\Gamma_{AA'}$ is complex.

Since we have all the cases laid out, we must now define the invariants to distinguish each of these cases. The only cases we were unable to distinguish were cases 2 and 3, all the others have covariants determining their case. We present the algorithm through both a step chart and a flow chart:

1.) Calculate Φ_{ab}.

if $\Phi_{ab} = 0$ then Case 8. $[-] - (-)$

2.) Calculate I_6

if $I_6 = 0$ then Case 7. $[A^3, B'] - |(1,0)^2|^2$

3.) Calculate E_{ab}

if $E_{ab} = 0$ then Case 6. $[(A, A'), (B, B')] - |(1,0)|^2|(1,0)|^2$

4.) Calculate I_7

if $I_7 = 0$ then Case 5. $[(A, A'), 11] - (1,1)|(1,0)|^2$

5.) Calculate $\widetilde{\chi}_a$

if $\widetilde{\chi}_a = 0$ then Case 4. $[11^2] - (1,1)^2$

6.) Calculate R_a

if $R_a = 0$ then Case 2 or 3 $[11, 11] - (1,1)(1,1)$ or $[(11, \overline{11})] - |(1,1)|^2$,

else Case 1. $[22] - (2,2)$

7.) The Algorithm is finished

We present the flowchart for the above step chart next in figure (4.1). All the co(in)variants presented above can be found in the Segre type section. This classification in addition to the Segre classification can sometimes differentiate unique solutions that the Segre type alone

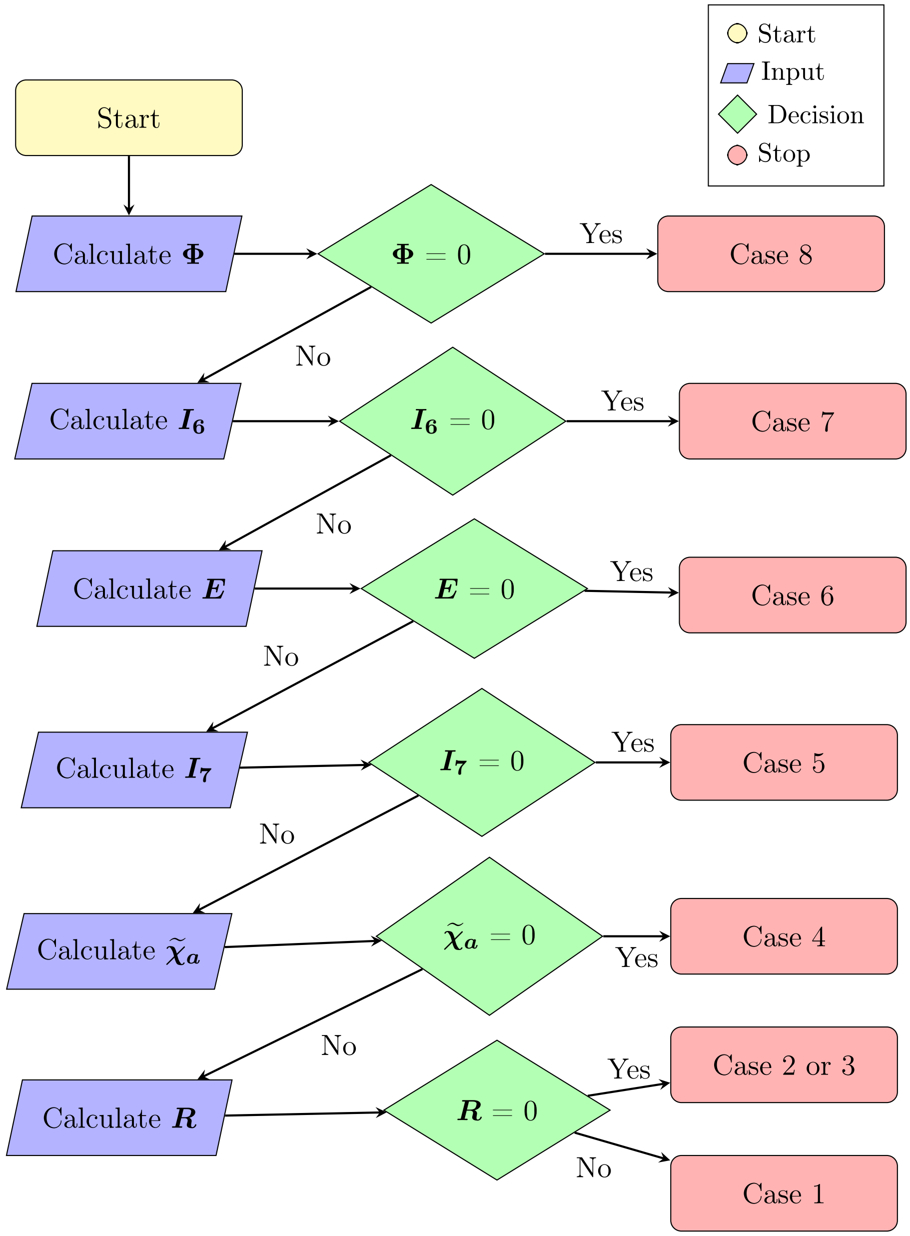


Figure 4.1: Flowchart for the $\Phi_{(AB)(A'B')}$ Spinor Classification Algorithm

cannot differentiate. Our labeling scheme seen here is the same notation that Penrose and Rindler (Penrose & Rindler, 1987b) use. Furthermore, figure (4.1) is read the same way as figure (3.1) is from chapter 3. The problem in differentiating cases 2 and 3 is that they have a very similar tensor structure.

4.6.9 The Classification of Λ

The classification of Λ is straightforward because it is a scalar. All that is needed is to see if Λ is positive, negative, or zero. Because lambda can vary from point to point, we classify it into these three sectors. The positive and negative cases are generic unless Λ is 0 everywhere. We use the notation: "$+, -, 0$" to differentiate these cases. In more complicated spacetimes this can be sufficiently interesting as the classification of Λ can vary from region to region in spacetime just as all the other classifications can. We use the terminology exceptional point for a point at which the classification of Λ changes, but for an open set around that point the type does not vary. For the nonzero case, an open set around an exceptional point is either positive or negative.

4.6.10 The Structural Reducibility Classification of $\text{Ю}_{(AB)}$

Here we classify the $\text{Ю}_{(AB)}$ spinor which arises as a new irreducible sector due to the inclusion of torsion. This classification is a primitive Petrov classification because it examines multiplicities of a lower degree polynomial. Unlike the Petrov classification which examines a quartic polynomial and its repeated roots, the classification of the spinor $\text{Ю}_{(AB)}$ examines a simpler quadratic polynomial which is significantly simpler. There is only one invariant O which determines when the roots are repeated, and it takes the same role that the invariant Δ in equation (4.54); the invariant O is the discriminant for the quadratic equation which $\text{Ю}_{(AB)}$ defines. This is different from what we had for the torsion spinor in table (3.2) in that we now only have unprimed indices. Having only unprimed indices automatically forces our spinor to decompose into two valence 1 spinors per Stewart's theorem (Stewart, 1993). Hence, the notation $[A, B]$. The case $\left[A^2\right]$ indicates that the spinors are repeated. The three different cases come as follows in table (4.3):

The notation with the square brackets we developed for all spinor structural reducibility classifications in $p = 3$, $q = 1$ signature spacetimes. Recall that it is read in the same way it was for the torsion spinor $\Omega_{(ABC)A'}$ from the torsion tensor in chapter 3. As a reminder, in the first case we use $[A, B]$ to represent that the spinor is irreducible.

The two co(in)variants: Ю, and O are used to differentiate the 3 cases from table

Cases for the Yu Spinor				
Cases	$Ю_{(AB)}$	Our Notation	Penrose Notation	Deg. Freedom
Case 1	$\alpha_A\beta_B$	$[A,B]$	$(1,0)(1,0)$	6
Case 2	$\alpha_A\alpha_B$	$[A^2]$	$(1,0)^2$	4
Case 3	0	$[-]$	$(-)$	0

Table 4.3: Structural Reducibility Table of the $Ю_{CC'}$ Spinor

(4.3). The co(in)variants are shown below in which we give a NP style form, and a spinor contraction form for them. We again pick a spin basis o^A, ι^A to form the NP style covariants.

$$\begin{aligned} Ю_{00} &= Ю_{(AB)}o^A o^B \\ Ю_{01} &= Ю_{(AB)}o^A \iota^A \\ Ю_{11} &= Ю_{(AB)}\iota^A \iota^A \end{aligned} \tag{4.64}$$

$$O = Ю_{(AB)}Ю^{(AB)}$$

$$O = 2\left(Ю_{00}Ю_{11} - (Ю_{01})^2\right) \tag{4.65}$$

To differentiate each of the three cases begin by calculating the Ю covariant. If Ю is zero then we are in case $[-]$. Then we calculate the invariant O, which if it is zero and $Ю \neq 0$, then we are in case $[(11)]$. Lastly, if neither Ю or O are zero then we are in the general case $[11]$. The algorithm goes as follows which we present in both a step chart and a flow chart.

1.) Calculate $Ю_{ab}$.

if $Ю_{ab} = 0$ then type $[-]$,

2.) Calculate O.

if $O = 0$ then type $\left[A^2\right]$

if $O \neq 0$ then type $[A, B]$

2.) The Algorithm is Complete.

We have a flowchart included below which illustrates the Ю spinor algorithm. This classification can be called in Maple by the command "CurvatureYuReducibilityAlgorithm". Likewise, the covariant Ю can be called with "CurvatureYuCovariant", and the invariant

O can be called with “CurvatureOmicronInvariant”. All of these three commands take the spinor Ю as an input

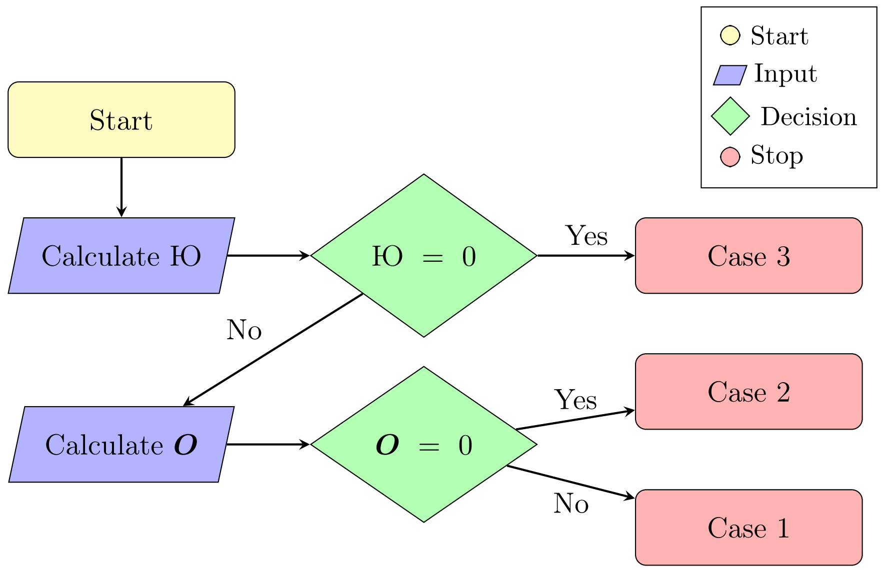


Figure 4.2: Flowchart for Classification Algorithm of the Ю Spinor

This is the classification for the Ю spinor.

4.6.11 The Classification of ℵ

The classification of the spinor ℵ is the same as it is for Λ other than the fact that it is purely imaginary. To classify ℵ we check if it is positive imaginary, negative imaginary, or zero. We use the notation “$+i$, $-i$, 0” to distinguish these cases. Again, like in the case for Λ, the spinor ℵ may also change its type in different in spacetime, or at exceptional points (just like for the Λ spinor). The type will remain consistent for some open set around a chosen generic point; if an open set is examined around an exceptional point there is a chance that the exceptional point is the boundary between two differing regions.

4.7 Refinements for the Classification of the Curvature Tensor in ECSK Theory

There are a few other viewpoints that can be taken in regard to the curvature tensor classification that do not arise in the classification of the torsion tensor. The first of these is that we are able to break the curvature into different pieces by using the Levi-Civita connection. This allows us to write $R_{abcd} = R\{\}_{abcd} + A_{abcd}$, where $R\{\}_{abcd}$ is the Levi-Civita generated curvature tensor, and A_{abcd} is the Alphonse tensor piece generated by torsion; we define this Alphonse tensor to be the part of the curvature purely generated by torsion and derivatives of torsion. This piece is explained further in the ECSK field equation chapter; see chapter 5.

With our classification algorithm already established, we could also classify the tensors $R\{\}_{abcd}$, and A_{abcd} in the same way we calculated the ECSK curvature tensor. The classification of $R\{\}_{abcd}$ is already well understood through the Petrov, Segre, and Λ types of classification we presented earlier, nevertheless there are ways to refine those classifications even further through either the tools of Ludwig and Scanlan (Ludwig & Scanlan, 1971), or Penrose (Penrose, 1972). The ideas in both of these are significantly developed but have yet to be implemented in computer algebra. In particular, there is no algorithm to determine the way to calculate out the Penrose type yet.

In the case of the Alphonse tensor it would be interesting to see how its classification differs from that of the ECSK Riemann tensor because the Alphonse tensor does not only contribute to the $Y4$, and $Y5$ irreducible sectors, it also contributes to the $Y3$ sector. This could lead to potentially different classifications and may yield new physical information such as a refinement of principal null directions, or a nuanced observable in terms of geodesic deviation. This also provides, in addition, a simple way to compare back to GR when the torsion tensor is zero. Furthermore, the tools of Ludwig and Scanlan and Penrose could be applied to this viewpoint as well, yielding a fruitful new direction for further work.

In the next section we examine the gravitational side of the field equations in ECSK theory with the aim of eventually applying the tools developed here for curvature, and before in chapter 3 for torsion, to physical problems.

CHAPTER 5
THE ECSK FIELD EQUATIONS

In this section we begin with the action of ECSK theory which depends on the fields $g^{ab}, T^{a}{}_{bc}$, and any arbitrary matter fields, for example scalar ϕ, or a spin $\frac{1}{2}$ fermionic Dirac field. We will explain the differences between this action and that of GR, which is due to the coupling of torsion to spin. Additionally, we will define several new objects such as the Cartan tensor $\mathscr{C}^{a}{}_{bc}$ which is an analogue to the Einstein tensor but for torsion, and the notation $R\{\}_{abcd}$, $R\{\}$ for the Levi-Civita generated curvature tensors. We will use ∇ for the metric affine covariant derivative and $\widetilde{\nabla}$ for the Levi-Civita covariant derivative.

We then proceed to discuss the symmetry properties of the curvature tensor in ECSK and how show how some differ from those in GR. Many of the definitions here follow Torre (Torre, 2020), Carroll (Carroll, 1997), and Poplawski (Poplawski, 2013) of which Poplawski gives wonderful clarity when it comes to densities; that clarity is especially useful in making an analogue in ECSK theory to what we already know in GR. After establishing some facts we vary the action with respect to the fields g^{ab}, and $T^{a}{}_{bc}$ while creating useful definitions along the way. From Hehl, (F. Hehl et al., 1976) we find that it is actually the contorsion tensor $C^{a}{}_{bc}$ which couples to spin, and we decide to vary with respect to it instead of the torsion tensor in the derivation; this decision also makes several of the computations easier as well.

We would also like to refer the reader to Wheeler (J. T. Wheeler, 2023) who clarifies the differences between the choice of field variables clearly. Additionally, the methodology he presents is applicable to our interests, and we would like to state that the Poincaré gauge theory approach to ECSK theory seems to be the most fruitful approach to take.

After the variation the first and second ECSK field equations are produced. Boundary terms are kept throughout, and we discuss the Gibbons-Hawking-York term as it appears in ECSK theory; this term changes slightly because of a modified/generalized definition of the extrinsic curvature of the boundary to include torsion. Finally, we discuss how to invert the spin tensor in terms of the contorsion tensor, along with making some final comments in a summary.

5.1 The Einstein-Hilbert Action for ECSK

We begin with the Einstein-Hilbert action S_{EH} (plus cosmological constant Λ), but without the torsion-free assumption, and its Lagrangian density.

$$S_{EH} = \int\limits_{\Omega} \mathcal{L}_{EH} d\Omega \tag{5.1}$$

$$\mathcal{L}_{EH} = \frac{1}{2\kappa} (R - 2\Lambda) \sqrt{-g} \tag{5.2}$$

This will later get modified to include the Gibbons-Hawking-York boundary term. R is the torsion-full metric affine Ricci scalar, Λ is the cosmological constant, g is the metric density, $\kappa = 8\pi G c^{-3}$ is the Einstein gravitational constant (this is different from the c^{-4} usually seen because we are using $dx^0 = cdt$ (one coordinate represents time) instead of without the c. An excellent explanation is given in Misner, Thorne, and Wheeler (Misner et al., 1973). The constant c is the speed of light in a vacuum, G is Newton's Gravitational constant $6.674 \times 10^{-11} m^3 kg^{-1} s^{-2}$, and Ω is the 4D region in spacetime of interest. Recall that the action has units of energy times time. Note the Ricci scalar has units of m^{-2}. Thus, when the units of κ, R, and Λ above are simplified, we find the units of the action to be energy times time just like we would expect.

Several of the usual properties of the curvature tensor in general change in ECSK due to the presence of torsion; Jensen (Jensen, 2005) provides an excellent explanation of these differences, although he uses a different notation than we do (he uses Wald's (Wald, 1984) convention for the curvature tensor). Specifically some of the algebraic tensor symmetries we would have had in GR are different. We will go over these properties now, and note when they are different from those in GR.

5.1.1 Symmetry Properties of the Curvature Tensor in ECSK

We will write these in a list, and make a sub list when the property differs from that in GR. For the boldfaced letters, we follow Wheeler's notation for differential forms (J. Wheeler, 2021). For instance $\boldsymbol{D}$ is the covariant exterior derivative, $\boldsymbol{R}^{\mu}{}_{\nu}$ is the curvature two-form, $\boldsymbol{e}^{\nu}$ is the solder form/orthonormal frame field, and $\wedge$ is the wedge product. Recall that there are five main algebraic tensor symmetries of the curvature tensor in GR. We have:

1) the skew-symmetry of the last two indices $R\{\}_{abcd} = R\{\}_{ab[cd]}$.

2) when lowered the skew-symmetry of the first two indices $R\{\}_{abcd} = R\{\}_{[ab]cd}$.

3) the interchange symmetry $R\{\}_{abcd} = R\{\}_{cdab}$

4) the first (algebraic) Bianchi identity $R\{\}_{a[bcd]} = 0$ or $\boldsymbol{e}^{\nu} \wedge \boldsymbol{R}^{\mu}{}_{\nu} = 0$, and

5) the second (differential) Bianchi identity $R\{\}_{ab[cd:e]} = 0$, or $\boldsymbol{D}\boldsymbol{R}^{\mu}{}_{\nu} = 0$.

These all define the GR curvature tensor which happens to lie irreducible in the $Y3b$ Young tableaux sector.

$$R\{\}_{abcd} \;\in\; \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \tag{5.3}$$

We proceed to go over these same five identities and see how they change in ECSK theory. We would not expect all of them to hold because now the curvature tensor breaks not only into the $Y3b$ sector, but a $Y4b, Y4c$ combination sector (they turn out to be linearly dependent) and a $Y5$ sector. We use the elbow Ferrers diagram to represent the combination sector.

$$R_{abcd} \;\in\; \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \;\oplus\; \begin{array}{|c|c|} \hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \;\oplus\; \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \tag{5.4}$$

Now we discuss how the symmetry properties are different than they were before:

1.) The first property in GR was $R\{\}_{abcd} = R\{\}_{ab[cd]}$. This skew symmetry of the last two indices is unchanged in ECSK.

$$R_{abcd} = R_{ab[cd]}$$

2.) The second property was $R\{\}_{abcd} = R\{\}_{[ab]cd}$. This skew symmetry of the first two indices is unchanged in ECSK.

$$R_{abcd} = R_{[ab]cd}$$

3.) The third property was $R\{\}_{abcd} = R\{\}_{cdab}$.

The interchange symmetry is no longer valid $R_{abcd} \neq R_{cdab}$. The obstruction to $R_{abcd} = R_{cdab}$ here is the addition of torsion in the connection; this can be inferred from the new Young sectors which appear in the irreducibility of R_{abcd}. We can recover the identity $R_{abcd} = R_{cdab}$ in the

1	3
2	4

piece of the Riemann tensor using Young tableaux, however. The new interchange formula is given in the equation which follows. We have checked and modified this equation from Jensen's formula (Jensen, 2005).

$$\begin{aligned} 2R_{abcd} &= 2R_{cdab} + 3\left(\nabla_{[b|}T_{a|cd]} + \nabla_{[a|}T_{b|cd]} + \nabla_{[d|}T_{c|ab]} + \nabla_{[c|}T_{d|ab]}\right) \\ &+ 3\left(T_{ae[b}T^{e}{}_{cd]} + T_{be[a}T^{e}{}_{cd]} + T_{ce[d}T^{e}{}_{ab]} + T_{de[c}T^{e}{}_{ab]}\right) \end{aligned} \tag{5.5}$$

4.) The fourth property was $R\{\}_{a[bcd]} = 0$.

The first Bianchi identity changes in ECSK to include a covariant derivative of torsion and a torsion squared term which is given as follows in both differential form language and abstract indices. Note that $\boldsymbol{d}$ is the exterior derivative.

$$R^{d}{}_{[cab]} = \nabla_{[b}T^{d}{}_{ca]} + T^{d}{}_{e[c}T^{e}{}_{ab]} \tag{5.6}$$

$$\boldsymbol{D}\boldsymbol{T}^{\mu} = \boldsymbol{e}^{\nu} \wedge \boldsymbol{R}^{\mu}{}_{\nu}$$

$$\boldsymbol{d}\boldsymbol{T}^{\mu} + \boldsymbol{T}^{\nu} \wedge \boldsymbol{\omega}^{\mu}{}_{\nu} = \boldsymbol{e}^{\nu} \wedge \boldsymbol{R}^{\mu}{}_{\nu}$$

5.) The fifth property was $R\{\}_{ab[cd:e]} = 0$.

The differential Bianchi identity changes in ECSK and now includes a torsion contracted with curvature term as follows in both differential form language and abstract indices.

$$\nabla_{[e|}R^{d}{}_{c|ab]} = -R^{d}{}_{cf[a}T^{f}{}_{be]} \tag{5.7}$$

$$\boldsymbol{DR}^{\mu}{}_{\nu} = 0$$

$$\boldsymbol{dR}^{\mu}{}_{\nu} + \boldsymbol{R}^{\alpha}{}_{\nu} \wedge \boldsymbol{\omega}^{\mu}{}_{\alpha} + \boldsymbol{R}^{\mu}{}_{\alpha} \wedge \boldsymbol{\omega}^{\alpha}{}_{\nu} = 0$$

These are how the ECSK curvature identities change in the presence of torsion. Next, we move onto the matter action and the entire gravity action.

5.2 The Matter Action and Whole Gravitational ECSK Action

We begin with defining the matter action S_M, which incorporates all the different variational matter sources for gravity. $\mathcal{L}_M$ is an arbitrary matter Lagrangian density; it can be broken into a Lagrangian times a metric/tetrad density: $\mathcal{L}_M = L_M\sqrt{-g} = L_M e$.

$$S_M = \int_{\Omega} \mathcal{L}_M d\Omega \tag{5.8}$$

Together we can write the total action $S = S_{EH} + S_M$ in equation (5.9) shown below.

$$S = \int_{\Omega} \left(\left[\frac{1}{2\kappa} (R - 2\Lambda) \right] \sqrt{-g} + \mathcal{L}_M \right) d\Omega \tag{5.9}$$

In ECSK, the action is a bit more complicated than in GR. Although the action looks the same as in GR it is different is because of the presence of torsion.

Now we have five equivalent choices on how to vary the action:

1) with respect to the metric and the torsion tensor g^{ab} and $T^{a}{}_{bc}$,

2) with respect to the metric and the contorsion tensor g^{ab} and $C^{a}{}_{bc}$,

3) with respect to the tetrad and the spin connection $e^{a}{}_{\mu}$ and $\omega^{\mu}{}_{\nu c}$,

4) with respect to the tetrad and the torsion tensor $e^{a}{}_{\mu}$ and $T^{a}{}_{bc}$, and

5) with respect to the tetrad and the contorsion tensor $e^a{}_\mu$ and $C^a{}_{bc}$.

In our case we will focus on 2), however we will examine parts of 1); furthermore, we will elaborate on 3) and 5) at the very end.

The contorsion tensor is defined by:

$$C_{abc} = \frac{1}{2}\left(-T_{abc} + T_{bac} + T_{cab}\right), \quad T_{abc} = -2C_{a[bc]} \tag{5.10}$$

$$C^f{}_{bf} = -T^f{}_{bf} = T^f{}_{fb} = T_b \tag{5.11}$$

Recall also that the contorsion tensor is skew in the first two indices.

$$C_{abc} = C_{[ab]c}$$

Now we vary equation (5.9) with respect to the fields g^{ab} and T^a_{bc} (or C^a_{bc}, we show both). We find

$$\delta S \;=\; \int_\Omega \left(\delta\sqrt{-g}\left[\frac{1}{2\kappa}\left(R - 2\Lambda\right)\right] + \sqrt{-g}\frac{1}{2\kappa}\delta R + \delta\mathcal{L}_M\right) d\Omega \tag{5.12}$$

Now since we have two independent fields: g^{ab} and T^a_{bc} we will break off those variations as separate parts; this will produce a variation looking like $\delta S = \delta S_g + \delta S_T$. Likewise, we can break off similar pieces in any of the five cases from earlier, which we will do for $e^a{}_\mu$ and $\omega^\mu{}_{\nu a}$ for example.

5.2.1 First ECSK Field Equation from Inverse Metric Tensor g^{ab} Variation

Following Hehl (F. Hehl et al., 1976), we find that the variation of the field equations with respect to the metric yields:

$$G_{(ab)} + \Lambda g_{ab} = \kappa T_{ab} \tag{5.13}$$

The first of the objects above is the Einstein Tensor G_{ab}, given as follows:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \quad . \tag{5.14}$$

where R_{ab} is the Ricci tensor, and R is the Ricci scalar; recall that the Ricci tensor is defined as the 1,3 contraction on the curvature tensor, and R is the metric contraction on the two indices of the Ricci tensor. Next we have the stress energy tensor T_{ab}; this is given as follows:

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{ab}} \tag{5.15}$$

Recall that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{ab} \delta g^{ab}$ from Carroll (Carroll, 1997).We note that the Einstein Field Equation has a spin contribution due to torsion inside the Einstein tensor as does Hehl in his equation (3.23) (F. Hehl et al., 1976).

For calculation purposes, it is also useful to define the stress energy density $\mathcal{T}_{ab}$ as in equation (5.16).

$$\mathcal{T}_{ab} = -2 \frac{\delta \mathcal{L}_M}{\delta g^{ab}} \tag{5.16}$$

Although we could look at the vielbein variation next, the author feels that it is better for clarity if the vielbein and spin connection variation is discussed later. Next we discuss some independent variable considerations which are important in ECSK theory.

5.2.2 Preliminaries Before the Second Field Equation, and Independent Variable Clarification for $g^{ab}, e^{a}{}_{\mu}, T^{a}{}_{bc}, C^{a}{}_{bc}$,and $\omega^{\mu}{}_{\nu a}$

Although we got the equation for the tetrad variation approximately above, there was an equation we used which related the metric variation and tetrad variation to do so. This subtlety addresses the importance of picking our independent variables in the variation. Above, we talked about five different cases in ECSK which we can pick to vary. Each of these will lead to different field equations and have different interpretations. Additionally, there is the need to be particular about the covariance/contravariance of the tensors we are varying. Although we say we were varying with respect to the metric, we really varied with respect to the inverse metric once the contravariance of the variation is considered. Likewise, in the tetrad section above, we chose the variance of the tetrad $e^a{}_\mu$ to be one abstract index up, and one orthonormal index down.

Tensor variance becomes important for the torsion tensor, contorsion tensor, and spin connection. The torsion tensor naturally comes in the form $T^a{}_{bc}$ so it is preferable to vary it with this variance; the interpretation becomes easier since in this form it is also completely decoupled from the metric. Because the contorsion tensor can be viewed as part of a connection 1-form, on one hand it makes sense to vary it as $C^a{}_{bc}$; on the other hand however, because the first two indices are skew it makes it easier computationally to vary C_{abc}; we will choose $C^a{}_{bc}$ as our object to vary because we can write $\left(A_a{}^{bc}\right)\delta C^a{}_{bc} = \left(g^{bd}A_{[ad]}{}^c\right)\delta C^a{}_{bc}$ (for some tensor $A_a{}^{bc}$ appearing in a variational principle) to still properly account for the skewness of the indices a, b once one is lowered. This is also what seems to work best for many formulas, and the author would recommend that $C^a{}_{bc}$ be varied instead of $C_{abc}$for simplicity. Note that if we choose g^{ab} and $C^a{}_{bc}$ as our independent variables, we can still relate back to g^{ab} and C_{abc} with equation (5.17).

$$\delta C^a{}_{bc} \quad = \quad \delta g^{ad} C_{dbc} + g^{ad}\delta C_{dbc} \tag{5.17}$$

These type equations relating different valences in variations are not terribly difficult to derive as all we did was lower an index with the metric and carry the variation through. The difficulty comes when the independent variables are not chosen exactly and explicitly from the start.

Moving forward, the spin connection naturally comes as $\omega^\mu{}_{\nu a}$ with the first two indices skew symmetric as well. Similarly to the contorsion case we could choose to vary $\omega_{\mu\nu a}$ instead. Note however, that $\delta\omega^\mu{}_{\nu a} = \eta^{\mu\alpha}\delta\omega_{\alpha\nu a}$, so there is really no difference here because $\eta_{\alpha\beta}$ does not vary. Nevertheless, to relate a theory with independent variables $e^a{}_\mu$ and $\omega_{\mu\nu a}$ back to a theory with independent variables $e^a{}_\mu$ and C_{abc} requires what is known as the Belinfante-Rosenfeld relation; see (Gotay & Marsden, 1986) for reference, and (Belinfante, 1940), (Rosenfeld, 1940) for the papers of Belinfante and Rosenfeld. Furthermore, we refer the reader to appendix G, and to Wheeler's paper (J. T. Wheeler, 2023) whom we believe gives the best explanation of the Belinfante-Rosenfeld relation. This relation is important because it produces a way to turn the tetrad energy momentum tensor back into the Hilbert stress energy tensor; this makes it incredibly useful for physical interpretations.

There are several equations which relate all these variations together however, the first of which is:

$$\delta g^{ab} = 2\eta^{\mu\nu}\delta^{(a}{}_{c}e^{b)}{}_{\nu}\delta e^{c}{}_{\mu} \tag{5.18}$$

This equation is lovely in the fact that there is no mixing of fields/variations as in equation (5.17). However, there is no way to turn a vielbein variation back into a metric variation in general because of the lack of guaranteed symmetry of the abstract indices. Past that, there is the equation relating the contorsion variation to the metric and torsion variation; we will choose the contorsion tensor $C^{a}{}_{bc}$ in natural form as our independent variable. This relation is given in equation (5.19); likewise, this relation can be inverted to give equation (5.20).

$$\delta C^{a}{}_{bc} = \frac{1}{2}\left(-\delta^{a}_{e}\delta^{d}_{b}\delta^{g}_{c} + g^{ad}g_{be}\delta^{g}_{c} + g^{ad}g_{ce}\delta^{g}_{b}\right)\delta T^{e}{}_{dg} \ + \tag{5.19}$$

$$\frac{1}{2}\left(\delta^{a}_{f}\left(T_{bgc} + T_{cgb}\right) - \left(T_{f}{}^{a}{}_{c}g_{bg} + T_{f}{}^{a}{}_{b}g_{cg}\right)\right)\delta g^{fg}$$

$$\delta T^{a}{}_{bc} = -2\delta^{e}_{[b}\delta^{f}_{c]}\delta C^{a}{}_{ef} \tag{5.20}$$

For clarity, we have also included these equations when we choose C_{abc} as the independent variable. It is an algebraic exercise to check that these equations are inversions of each other, in the sense that if we substitute $\delta T^{a}{}_{bc}$ into the δC_{abc} equation, that we get $\delta C_{abc} = \delta C_{abc}$.

$$\delta C_{abc} = -\frac{1}{2}\left(-g_{ag}T_{hbc} + g_{bg}T_{hac} + g_{cg}T_{hab}\right)\delta g^{gh} + \frac{1}{2}\left(-g_{ad}\delta^{e}_{b}\delta^{f}_{c} + g_{bd}\delta^{e}_{a}\delta^{f}_{c} + g_{cd}\delta^{e}_{[a}\delta^{f}_{b]}\right)\delta T^{d}{}_{ef}$$

$$\delta T^{a}{}_{bc} = \left(-2C_{d[bc]}\right)\delta g^{ad} + \left(-2g^{ad}\delta^{e}_{[b}\delta^{f}_{c]}\right)\delta C_{def}$$

The next equation which relates variations relates the torsion variation to the solder form $e^{a}{}_{\mu}$ and spin connection $\omega^{\mu}{}_{\nu a}$. When we look at the definition of torsion given by $T\left(X,Y\right) =$

$\nabla_X Y - \nabla_Y X - [X,Y]$, we can find an equivalent definition in terms of differential forms given by equation (5.21):

$$\boldsymbol{T}^{\alpha} = \boldsymbol{d}\boldsymbol{e}^{\alpha} + \boldsymbol{\omega}^{\alpha}{}_{\beta} \wedge \boldsymbol{e}^{\beta} \quad . \tag{5.21}$$

Recall that the bold letters denote differential forms. This is the form most often seen when working with Cartan geometries; see Sharpe (Sharpe, 1997), Choquet-Bruhat (Choquet-Bruhat et al., 1996), and Stone (Stone, 2008) for excellent references on these. Upon expanding equation (5.21), varying and simplifying it, we find how the torsion variation relates to the spin connection and solder form variations. One thing that is interesting about this relation however, is the piece $\partial_{[b}\left(e_{c]}{}^{\sigma}\delta^{a}_{f}\delta e^{f}{}_{\sigma}\right)$ which will turn into a boundary term in the action.

$$\delta T^{a}_{bc} = \left(2e^{a}{}_{\mu}e_{[c}{}^{\nu}\delta^{f}_{b]}\right)\delta\omega^{\mu}{}_{\nu f} \;+$$

$$\left(2e^{a}{}_{\mu}e_{f}{}^{\nu}\omega^{\mu}{}_{\nu[c}e_{b]}{}^{\sigma} - 2\delta^{a}_{f}\omega^{\sigma}{}_{\nu[c}e_{b]}{}^{\nu} + 2\delta^{a}_{f}\partial_{[b}e_{c]}{}^{\sigma} + 2e_{[c}{}^{\sigma}\partial_{b]}e^{a}{}_{\mu}e_{f}{}^{\mu}\right)\delta e^{f}{}_{\sigma} \;-$$

$$2\partial_{[b}\left(e_{c]}{}^{\sigma}\delta^{a}_{f}\delta e^{f}{}_{\sigma}\right)$$

Furthermore, we can use both of the prior variational relations to see how the spin connection variation can be written in terms of a contorsion and vielbein variation; the usefulness of this comes from the fact that once we do this, the coefficient of the vielbein variation becomes exactly the Hilbert stress energy momentum tensor, and can be interpreted the same way as in GR. This relation is what becomes the Belinfante-Rosenfeld relation once applied to the action principle. We will explore more on this later and in appendix G, and we also refer the reader to Wheeler (J. T. Wheeler, 2023) who provides an excellent explanation of this relation.

Before we move onto the variation of the action with respect to the torsion tensor, contorsion tensor (by the algebraic relation relating, this relation is given in equation (5.10), or the spin connection, it is important to define the quantities which appear once the matter Lagrangian is varied with respect to these objects. At least for the torsion and contorsion the algebraic relation relating them as we saw above is incredibly useful in relating the variations.

We define the following spin energy potential density $\mathfrak{M}_a{}^{bc}$, the Palatini potential density $\mathfrak{Y}^{a\mu\nu}$, and spin angular momentum density $\mathfrak{S}^{a\mu\nu}$ in equation (5.22); see Hehl (F. Hehl et al., 1976) with slight modification. The Palatini spin potential density $\mathfrak{Y}^{a\mu\nu}$ is a new object not defined in Hehl's paper. The index arrangement of both $\mathfrak{S}^{abc}$, and $\mathfrak{Y}^{a\mu\nu}$ are to follow the spirit of Hehl (F. Hehl et al., 1976) as in equation (3.6) there.

$$\mathfrak{M}_a{}^{bc} = -2\frac{\delta \mathcal{L}_M}{\delta T^a{}_{bc}}, \quad \mathfrak{Y}^{a\mu\nu} = -2\frac{\delta \mathcal{L}_M}{\delta \omega_{\mu\nu a}}, \quad \mathfrak{S}^a{}_b{}^c = -2\frac{\delta \mathcal{L}_M}{\delta C^b{}_{ca}} \tag{5.22}$$

In equation (5.22) $\omega^\mu{}_{\nu c}$ is the spin connection. The relation between the δC_{bca} and $\delta\omega_{\mu\nu a}$ is given by the Belinfante-Rosenfeld relation in appendix G. Recall that the spin connection can be broken into the Levi-Civita spin connection $\varpi^\nu{}_{\mu a}$ and contorsion tensor.

$$\omega^\nu{}_{\mu a} e^b_\nu = \varpi^\nu{}_{\mu a} e^b_\nu + C^b{}_{ca} e^c{}_\mu \tag{5.23}$$

Because the torsion and contorsion are related through equation (5.10) we can write $\mathfrak{M}_a{}^{bc}$ in terms of $\mathfrak{S}_a{}^{bc}$; lowering the indices makes the derivation easier. We then find the relations:

$$\mathfrak{M}_{abc} = \frac{1}{2}\left(\mathfrak{S}_{abc} + \mathfrak{S}_{cba} - \mathfrak{S}_{bca}\right), \quad \mathfrak{S}_{cab} = -2\mathfrak{M}_{[ab]c} \tag{5.24}$$

Likewise, we can make the densities $\mathfrak{M}_a{}^{bc}$, $\mathfrak{Y}^{a\mu\nu}$, and $\mathfrak{S}^{abc}$ into tensors with the following definitions for the spin energy potential tensor $\mu_a{}^{bc}$, Palatini potential tensor $y^{a\mu\nu}$, and spin angular momentum tensor s^{abc} (5.25).

$$\mu_a{}^{bc} = \frac{1}{\sqrt{-g}}\mathfrak{M}_a{}^{bc}, \quad y^{a\mu\nu} = \frac{1}{\sqrt{-g}}\mathfrak{Y}^{a\mu\nu}, \quad s^{abc} = \frac{1}{\sqrt{-g}}\mathfrak{S}^{abc} \tag{5.25}$$

If we apply the Belinfante-Rosenfeld relation from appendix G, which is just that the action variations for the tetrad & spin connection, and metric and torsion be equivalent, then we find that the Palatini spin potential density and spin angular momentum density are equal as in equation (5.26).

$$e_b{}^{[\mu} e_c{}^{\nu]} \mathfrak{S}^{abc} = \mathfrak{Y}^{a\mu\nu} \tag{5.26}$$

Before we go about the variation with respect to torsion and/or contorsion it is useful to know the formula for the Ricci scalar R in terms of the Levi-Civita connection generated Ricci scalar $R\{\}$. The difference is precisely given in terms of torsion and or contorsion terms. We can find this from the fact that the difference of covariant derivatives is a tensor; in this case the difference between the metric affine connection and the Levi-Civita connection is given by the contorsion tensor, see equation (5.27); X^a is an arbitrary vector field. This is the same idea that we used in our equation (5.23) which relates spin connections.

$$\left(\nabla_a - \widetilde{\nabla}_a\right) X^b = C^b{}_{ca} X^c \tag{5.27}$$

Applying this idea allows us to write the difference of curvature tensors in terms of torsion and or contorsion pieces; we need this in order to write the ECSK Ricci scalar in terms of the Levi-Civita generated Ricci scalar and torsion/contorsion pieces. We will now define the Alphonse Tensor $A^d{}_{cab}$, which we referenced in the summary section of chapter 4. It is a new object which is the difference between the curvature tensor generated by the metric affine connection $R^d{}_{cab}$ and the Riemann tensor of the Levi-Civita connection $R\{\}^a{}_{bcd}$ in equation (5.28).

$$A^d{}_{cab} = R^d{}_{cab} - R\{\}^d{}_{cab}\,, \quad A^d{}_{cab} = 2\widetilde{\nabla}_{[a} C^d{}_{|c|b]} + 2C^d{}_{f[a} C^f{}_{|c|b]} \tag{5.28}$$

Taking a d, a trace of the Alphonse Tensor gives us what we define to be and call the Edward Tensor E_{cb}. This is the same procedure as forming the Ricci tensor in GR. Note also that the Edward Tensor is the difference between the metric affine Ricci tensor and the Levi-Civita Ricci tensor. Equation (5.29) contains the definitions of what we call the Edward tensor.

$$E_{cb} = A^d{}_{cdb} = R_{cb} - R\{\}_{cb}\,, \quad E_{cb} = 2\widetilde{\nabla}_{[d} C^d{}_{|c|b]} + 2C^d{}_{f[d} C^f{}_{|c|b]} \tag{5.29}$$

This also lets us write a few more formulas regarding E_{cb}. E_{cb} contains the skew part of the Ricci tensor in ECSK, see equation (5.30). The symmetric part of E_{cb} is more than just the Levi-Civita generated piece of the Ricci tensor however; this is due to the presence of new

Young tableaux sectors for the curvature tensor in ECSK.

$$R_{[cb]} = E_{[cb]}, \quad R_{(cb)} = R\{\}_{cb} + E_{(cb)} \tag{5.30}$$

Lastly we have the Edward Scalar E which is the trace of the Edward Tensor, and again this corresponds to how we would form the Ricci Scalar in GR; this is given in equation (5.31). We have used $C_{abc} = C_{[ab]c}$ to combine the covariant derivative terms. Note also that the Edward scalar is the difference between the metric affine Ricci scalar and the Levi-Civita Ricci scalar; there is a parallel for each of these to the ones constructed naturally in GR. Additionally, we can finally write the Edward Scalar/Ricci scalar in terms of torsion; previously this was cumbersome because of the number of torsion terms. Finally, it is useful to define one more piece $\mathcal{U}$ which becomes useful in the variational principle for ECSK. We decided on $\mathcal{U}$ because Hehl describes ECSK as a U_4 theory (F. Hehl et al., 1976); this $\mathcal{U}$ will be called the Hehl scalar, see equation 5.33. The $\mathcal{U}$ scalar is the part of the Edward scalar which does not have a covariant derivative in it.

$$E = g^{cb}E_{cb} = R - R\{\}, \quad E = -2\widetilde{\nabla}_b C^{db}{}_d - C^{db}{}_d C^c{}_{bc} + C^{db}{}_c C^c{}_{bd} \tag{5.31}$$

$$E = \frac{1}{4}T_{abc}T^{abc} + \frac{1}{2}T_{abc}T^{bac} - T^a{}_{ab}T^{c}{}_{c}{}^{b} - 2\widetilde{\nabla}_b T^{a}{}_{a}{}^{b} \tag{5.32}$$

$$\mathcal{U} = -g^{ab}\left(C^d{}_{ad}C^c{}_{bc} - C^d{}_{ac}C^c{}_{bd}\right), \quad \mathcal{U} = E + 2\widetilde{\nabla}_b C^{db}{}_d \tag{5.33}$$

Now that we have defined several new objects, the variations with respect to the torsion and contorsion become manageable and easier to follow as well. We will move onto the differences in the next section.

5.2.3 Second ECSK Field Equations from Torsion $T^a{}_{bc}$, and Contorsion $C^a{}_{bc}$

We have the freedom to vary the action with respect to the torsion or the contorsion tensor. At first, it may make more sense to vary with respect to the torsion tensor because it is completely independent of the metric; the contorsion tensor depends on the metric through the raising and lowering of the torsion tensor. This does change a few things, but never too much because the torsion and contorsion are always related algebraically. Hehl et al. (F. Hehl et al., 1976) give a good explanation of why it is the contorsion which should couple to the spin of matter on page 398 the last paragraph in section F:

"We have seen that the presence of contortion in a U_4 supplies space-time with new rotational degrees of freedom. We know that matter possesses spin angular momentum in general, and, in the spirit of general relativity, we would like space-time to reflect the properties of matter. In the next section we will achieve this result by coupling the contortion of space-time to the spin of matter."

This comes about because now the metric-compatible affine connection is modified by the contorsion, not the torsion tensor. Nevertheless, for completeness the torsion variation may one day be valuable as a mathematical check elsewhere and for pedagogical reasons also. From this standpoint, we move onto the torsion variation.

5.2.4 The Torsion Variation

From equation (5.32) we can write the Edward scalar in terms of the difference of Ricci tensors and torsion. When E is varied with respect to torsion we find that the Levi-Civita generated Ricci scalar $R\{\}$ vanishes because it does not contain torsion; this yields equation (5.34).

$$\delta_T R = \left(\mathscr{A}_a{}^{bc}\right)\delta T^a{}_{bc} + \widetilde{\nabla}_f\left(-2\delta^b_a g^{fc}\delta T^a{}_{bc}\right) \tag{5.34}$$

Furthermore, when we take the variation in equation (5.12) to be with respect to torsion, we find the torsion variation δS_T is given by equation (5.35).

$$\delta S_T = \int_\Omega \left(\frac{1}{2\kappa}\frac{\delta R}{\delta T^a{}_{bc}} + \frac{\delta \mathcal{L}_M}{\delta T^a{}_{bc}}\right)\delta T^a{}_{bc}\sqrt{-g}d^4x \tag{5.35}$$

Applying equations (5.25) and (5.34) to equation (5.35), we then find the Second ECSK field equations, which we will call the Alpha Cartan equation, to be given by equation (5.36) with definition of the Alpha Cartan tensor in equation (5.37). It is precisely the Alpha Cartan tensor with which we would like to draw an analogy to the Einstein Tensor G_{ab}. Notice how both G_{ab} and $\mathscr{A}_{abc}$ appear on the left-hand side of their respective field equations, and how κT_{ab} and $\kappa\mu_{abc}$ appear on the right-hand side of their respective field equations. This is the good mathematical analogy to make in ECSK theory.

$$\mathscr{A}_a{}^{bc} = \kappa\mu_a{}^{bc} \tag{5.36}$$

$$\mathscr{A}_a{}^{bc} = \frac{1}{2}\left(T_a{}^{bc} - 2g_{ad}T^{[bc]d}\right) - 2\delta_a{}^{[b}T^{c]} \tag{5.37}$$

Although the $T_{abc} + T_{bac} - T_{cab}$ term looks like the definition of the contorsion tensor, it is not as it is off by just a few indices. The second ECSK field equation generated from the torsion variation is equivalent to the one generated from the contorsion variation as we will show. We now move onto the contorsion variation and then compare the field equations.

5.2.5 The Contorsion Variation

The contorsion variation is a bit different from the torsion tensor relation, however the difference is subtle. The contorsion tensor is skew on its first two indices when they have the same valence. However, the contorsion tensor naturally comes with the first two indices being different valences. We deal with this through the following relation: $\left(A_a{}^{bc}\right)\delta C^a{}_{bc} = \left(g^{bd}A_{[ad]}{}^c\right)\delta C^a{}_{bc}$. Without proper care taken in this relation we do not get out the proper field equations. Furthermore, it is precisely this relation which allows us to compare the torsion and contorsion field equations.

In the case of the contorsion variation, the variation of the Ricci tensor, much like and equivalent to equation (5.34), is given by equation (5.38). We can find this from the contorsion form of the Edward scalar in equation (5.31).

$$\delta_C R = \left(g^{bd}\mathscr{C}^c{}_{ad}\right)\delta C^a{}_{bc} - 2g^{db}\delta^c_a\widetilde{\nabla}_d\left(\delta C^a{}_{bc}\right) \tag{5.38}$$

Likewise, the contorsion variation δS_C is given by equation (5.39).

$$\delta S_C \quad = \quad \int\limits_{\Omega} \left(\frac{1}{2\kappa} \frac{\delta R}{\delta C^a{}_{bc}} + \frac{\delta \mathcal{L}_M}{\delta C^a{}_{bc}} \right) \delta C^a{}_{bc} \sqrt{-g} d\Omega \tag{5.39}$$

When we apply equations (5.25) and (5.38) to equation (5.39), we then find the Second ECSK field equation, what we will call the Cartan Equation, when we vary with respect to the contorsion in equation (5.40), with the definition of the Cartan tensor $\mathscr{C}^c{}_{ab}$ given in equation (5.41). Interestingly enough, the Cartan tensor, unlike the Alpha Cartan tensor $\mathscr{A}^a_{bc}$, can be defined purely in terms of the torsion, as in the second half of equation (5.41).

$$\mathscr{C}_{abc} = \kappa s_{abc} \tag{5.40}$$

$$\mathscr{C}^c{}_{ab} = -2 \left(C^c{}_{[ab]} + \delta^c{}_{[a} C^f{}_{b]f} \right), \quad \mathscr{C}^c{}_{ab} = T^c{}_{ab} + 2\delta^c_{[a} T^f{}_{b]f} \tag{5.41}$$

We have said that both of the Second field equations are equivalent, but it is important to show this as we have not found it anywhere else. A derivation of equivalence is shown in the "Equivalence between the Second field equations" section utilizing equations (5.24).

5.2.6 Equivalence Between the Second Field Equations

We begin with writing the spin angular momentum tensor in terms of the spin energy potential tensor by the skew symmetry relation and then plug in the definition of the Alpha Cartan tensor, and then simplify the resultant right-hand side to reproduce the Cartan equation. Recall that this is entirely due to the algebraic relation between the torsion tensor and the contorsion tensor.

$$s_{cab} = -2\mu_{[ab]c}$$

$$
\begin{aligned}
\kappa s_{cab} &= -2\left\{\frac{1}{2}\left[\left(\frac{1}{2}\left(T_{abc}+T_{bac}-T_{cab}\right)+g_{ab}T^{f}{}_{cf}-g_{ac}T^{f}{}_{bf}\right)\right]\right\} \\
&\quad +2\left\{\frac{1}{2}\left[\left(\frac{1}{2}\left(T_{bac}+T_{abc}-T_{cba}\right)+g_{ba}T^{f}{}_{cf}-g_{bc}T^{f}{}_{af}\right)\right]\right\} \\
&= -\left\{\frac{1}{2}\left(T_{abc}+T_{bac}-T_{cab}-T_{bac}-T_{abc}+T_{cba}\right)\right\} \\
&\quad +\left\{\cancelto{0}{-g_{ab}T^{f}{}_{cf}+g_{ba}T^{f}{}_{cf}}-g_{ac}T^{f}{}_{bf}+g_{bc}T^{f}{}_{af}\right\} \\
&= -\left\{\frac{1}{2}\left(-2T_{cab}\right)-2g_{c[a}T^{f}{}_{b]f}\right\} \\
&= \left\{T_{cab}+2g_{c[a}T^{f}{}_{b]f}\right\} \\
&= \mathscr{C}_{cab}
\end{aligned}
$$

This proves equivalence going one way, we now check everything the opposite way. For the second equivalence, we write the spin energy potential tensor in terms of the spin angular momentum tensor and then simplify the right-hand side.

$$
\mu_{abc} = \frac{1}{2}\left(-s_{abc}+s_{cba}+s_{bca}\right)
$$

$$
\begin{aligned}
\kappa\mu_{abc} &= \frac{1}{2}\left(-\left(-2\left(C_{c[ab]}+g_{c[a}C^{f}{}_{b]f}\right)\right)-\left(-2\left(C_{b[ca]}+g_{b[c}C^{f}{}_{a]f}\right)\right)\right) \\
&+ \frac{1}{2}\left(-2\left(C_{a[bc]}+g_{a[b}C^{f}{}_{c]f}\right)\right) \\
&= \frac{1}{2}\left(\left(2C_{c[ab]}+2C_{b[ca]}-2C_{a[bc]}\right)\right) \\
&+ \frac{1}{2}\left(2g_{c[a}C^{f}{}_{b]f}+2g_{b[c}C^{f}{}_{a]f}-2g_{a[b}C^{f}{}_{c]f}\right) \\
&= \frac{1}{2}\left(\left(-T_{cab}-T_{bca}+T_{abc}\right)+2g_{ac}C^{f}{}_{bf}-2g_{ab}C^{f}{}_{cf}\right) \\
&= \frac{1}{2}\left(\left(-T_{cab}-T_{bca}+T_{abc}\right)-4g_{a[b}C^{f}{}_{c]f}\right) \\
&= \frac{1}{2}\left(T_{abc}+T_{bac}-T_{cab}\right)+2g_{a[b}T^{f}{}_{c]f} \\
&= \mathscr{A}_{abc}
\end{aligned}
$$

And again we have equivalence, showing that either equation we choose to solve is equivalent to the other.

5.2.7 First and Second ECSK Field Equation from the Vielbein $e^a{}_\mu$, and Spin Connection $\omega^\mu{}_{\nu a}$ Variations

There are plenty of subtleties in comparing an ECSK theory built on the variables $e^a{}_\mu$, and $\omega^\mu{}_{\nu a}$ instead of g^{ab}, and $T^a_{\ bc}$ or $C^a_{\ bc}$. We can compare all of them and interpret them the same every time, but getting to that point requires a deep understanding of boundary terms, and a lot of clarity. Naively we could just use equation (5.18) and convert what we already discovered in the metric theory, however, this does not work because the spin connection has some dependence on the vielbein; this is unlike the torsion tensor and metric which do not depend on each other ab initio; the contorsion tensor has some similarities, but is not as complex as the spin connection and vielbein relation. Before we start, there are several reasons why we might be interested in an ECSK theory on the variables $e^a{}_\mu$, and $\omega^\mu{}_{\nu a}$.

Sometimes the matter action does not explicitly depend on the metric as in Dirac theory, but we would still like to have an ECSK theory of gravity in this case; more generally this becomes important when we discuss fermions. When this occurs, we can vary with respect to the vielbein $e^a{}_\mu$ instead of the metric. When we build on this idea it becomes important to define a few other terms which have not shown up yet. These new objects are related to the tetrad $e^a{}_\mu$. The first of these is the tetrad stress energy density $\mathfrak{T}_a{}^\mu$. We can turn $\mathfrak{T}_a{}^\mu$ into a density with abstract indices by contracting on a $e_{\mu b}$, which gives us $\mathfrak{T}_{ab}$; note that this is not necessarily symmetric unlike $\mathcal{T}_{ab}$. Furthermore, the next definition defines t_{ab} the tetrad stress energy tensor. Finally, $e = \det\left(e^a{}_\mu\right)$ is the determinant of the vielbein. These are all seen in equations (5.42).

$$\mathfrak{T}_a{}^\mu = -\frac{\delta \mathcal{L}_M}{\delta e^a{}_\mu}, \quad t_{ab} = \frac{1}{e}\mathfrak{T}_{ab} \tag{5.42}$$

This is shown by the Belinfante-Rosenfeld relation (in appendix G) to convert this theory into an ECSK theory built on the variables $e^a{}_\mu$, and $C^a_{\ bc}$. Note that the vielbein starts as a form in this theory instead of a vector; the conversion back is given by $\delta e_a{}^\mu = -e_f{}^\mu e_a{}^\sigma \delta e^f{}_\sigma$.

5.2.8 The Vielbein and Spin Connection Variation

We begin with the ECSK action given in terms of differential forms (and a matter Lagrangian density) as in equation (5.43); it can be shown that this is equivalent to the action we have in equation (5.9).

$$S = -\frac{1}{4\kappa}\int\limits_{\Omega}\left(\boldsymbol{R}^{\mu\nu}\wedge\boldsymbol{e}^{\alpha}\wedge\boldsymbol{e}^{\beta}+\frac{1}{6}\Lambda\boldsymbol{e}^{\mu}\wedge\boldsymbol{e}^{\nu}\wedge\boldsymbol{e}^{\alpha}\wedge\boldsymbol{e}^{\beta}\right)e_{\mu\nu\alpha\beta}+\int\limits_{\Omega}\mathcal{L}_M d\Omega \tag{5.43}$$

When we vary this with respect to $\boldsymbol{e}^{\mu}$ and $\boldsymbol{\omega}^{\mu}{}_{\nu}$ we find the following field equations in equations (5.44) and (5.45). See Wheeler (J. T. Wheeler, 2023) for more details.

$$R_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}R = t_{\mu\nu} \tag{5.44}$$

$$\mathscr{C}^{c}{}_{ab} = y^{c}{}_{ab} \tag{5.45}$$

5.3 The ECSK Action

The ECSK action we will use is the same action as the one used in GR, but with the addition of torsion. Furthermore, we also choose to include the Gibbons-Hawking-York (GHY) term so that we only need to specify that δg^{ab} vanish on the boundary for all the boundary terms to vanish; see York's paper for more specifics on why this works and is important (York, 1986).

The Gibbons-Hawking-York term is also modified as well such that the extrinsic curvature scalar (trace of the second fundamental form) K now contains torsion as well. The K_0 is the extrinsic curvature scalar of the boundary embedded flat spacetime, and Σ is some hypersurface subset of Ω. The action is given in equation (5.46).

$$S = \int\limits_{\Omega}\left[\frac{1}{2\kappa}\left(R-2\Lambda\right)\right]\sqrt{-g}d^4x + S_M + S_{GHY} \tag{5.46}$$

Through the rest of the sections we refer to equation (5.46) as the gravitational action.

CHAPTER 6

ECSK-NMC SCALAR FIELD THEORY

In this chapter we will examine a non-minimally coupled (NMC) scalar field and scalar field action and vary the action with respect to the metric, contorsion tensor, and the fields. We use the action provided by Bronnikov and Galiakhmetov (Bronnikov & Galiakhmetov, 2015). The purpose of this chapter is to derive and classify Bronnikov and Galiakhmetov's solution with the tools we developed in chapters 3 and 4. The Lagrangian density depends on the following fields: $\left\{\phi, \phi_{,a}, \psi, \psi_{,a}, g^{ab}, C^{a}{}_{bc}\right\}$. ϕ is the non-minimally coupled scalar field. ψ is a minimally coupled scalar field. $\phi_{,a}$ and $\psi_{,a}$ are derivatives of the respective fields. g^{ab} is the inverse metric tensor, and $C^{a}{}_{bc}$ is the contorsion tensor.

The action is given by Bronnikov and Galiakhmetov (Bronnikov & Galiakhmetov, 2015) and is special because the solutions to the field equations generated by the variational principle form a wormhole solution. Additionally, when no other spin-density is present, the field ϕ generates the $SO\left(p, q\right)$ irreducible trace $\mathfrak{Q}_{abc}$ element of the torsion. This is the only part of the torsion which exists in this solution. From earlier we know that the trace $\mathfrak{Q}_{abc}$ piece of the torsion corresponds through the Infeld–Van der Waerden symbols $\sigma^{a}{}_{AA'}$ directly to the torsion $\Theta_{AA'}$ spinor, which we have developed the tools to classify. The non-minimally coupled scalar field is a great example to see how this piece of the torsion not only fills out its own irreducible sector, but also to see the resulting effects on the total curvature as well.

While we are rederiving the equations Bronnikov and Galiakhmetov have, we will additionally include every boundary term arising in the variation. Usually we specify that the metric variation δg^{ab} vanishes on the boundary which is what we choose here. We do not actually have to specify that the contorsion tensor vanishes on the boundary. Gibbons-Hawking-York terms arise again but with a slight addition due to the non-minimal coupling. Furthermore, we will derive these NMC field equations in $N-$dimensions. As an aside, notice that the functions $V(\phi)$ and $W(\psi)$ are arbitrary and can contain the mass terms usually seen in a Klein-Gordon action. The sign in front of the Einstein-Hilbert piece is also a minus as opposed to the positive sign we had before; this will alter our explicit definition of the stress energy tensor. The relative sign is important here however to get the correct field equations. Note how it is different from our other derivation in the "Derivation of the ECSK Field Equations" section where we had $R + \mathcal{L}_M$.

For an action S of the following form in $N-$dimensions we define the following variables: Ω is the region of integration, $\epsilon = \pm 1 = \hat{n}^a \hat{n}_a$ is the signature of the unit vector normal to some three dimensional hypersurface $\partial\Omega$, R is the Ricci scalar, κ is the Einstein gravitational constant, g is the determinant of the metric, g^{ab} is the inverse metric, ϕ is a NMC scalar field, ξ is the NMC coupling constant, ψ is a minimally coupled scalar field. $\eta_1 = \pm 1$, $\eta_2 = \pm 1$, these two are for canonical (1) or phantom (-1) scalar fields. γ is the first fundamental form (for the boundary of the region Ω ($\partial\Omega$) being examined), K is the extrinsic curvature trace (mean curvature), K_0, K_1 are variables not depending on the fields which make the action finite, $\widetilde{\Box}$ is the Levi-Civita generated Dalembertian, $c^c{}_{ab}$ is the Cartan tensor, $s^c{}_{ab}$ is the spin density tensor, T_{ab} is the stress energy tensor, U_{ab} is the spin stress energy tensor, and $C^a{}_{bc}$ is the contorsion tensor. Additionally, S_{GHY} is the Gibbons-Hawking-York (GHY) term, and $S_{\partial NMC}$ is the GHY analogue term for NMC scalar fields coupled in this way. The new $S_{\partial NMC}$ term (labeled as such to signify "boundary of NMC") is included so that we have the freedom to prescribe that δg^{ab} vanishes on the boundary and not some combination of its derivatives. Notice the addition of the K_1 term; for minimally coupled fields we do not need to add a GHY type term to compensate for additional boundary effects. However, we need the K_1 here to prevent any divergent (running off to infinity) behavior due to properties of the scalar field ϕ; K_1 also does not contribute to the field equations. Further explanation of K_0 and K_1 can be found in Hawking and Horowitz (Hawking & Horowitz, 1996). The action is given as follows:

$$S = \int_{\Omega} \left[-\frac{R}{2\kappa}\right] \sqrt{-g} d^4x + S_\phi + S_\psi + S_{GHY} + S_{\partial NMC} \qquad (6.1)$$

$$S_{\phi} = \int\limits_{\Omega} \left[\frac{\eta_1}{2} \left(g^{ab} \phi_{,a} \phi_{,b} + \xi R \phi^2 - 2V(\phi) \right) \right] \sqrt{-g} d^4 x \tag{6.2}$$

$$S_{\psi} = \int\limits_{\Omega} \left[\frac{\eta_2}{2} \left(g^{ab} \psi_{,a} \psi_{,b} - 2W(\psi) \right) \right] \sqrt{-g} d^4 x \tag{6.3}$$

$$S_{GHY} = \oint\limits_{\partial\Omega} \left(\frac{\epsilon}{\kappa} (K - K_0) \right) \sqrt{\gamma} d^3 y \tag{6.4}$$

$$S_{\partial NMC} = \oint\limits_{\partial\Omega} \left(-\eta_1 \xi \phi^2 \epsilon (K - K_1) \right) \sqrt{\gamma} d^3 y \tag{6.5}$$

by the sum of the three terms S, S_{GHY}, and $S_{\partial NMC}$ above. Equation (6.1) is the same action as Bronnikov and Galiakhmetov (Bronnikov & Galiakhmetov, 2015) except for the S_{GHY} and $S_{\partial NMC}$ terms which we have included in the style of York (York, 1986), and the redefinition of $V(\phi)$, and $W(\psi)$ which equal $\frac{1}{\eta_1}, \frac{1}{\eta_2}$ times the same variables they defined. We redefined these terms to make the field equations more compact later. Erdmenger (Erdmenger et al., 2022) gives further discussion of boundary terms in ECSK theory, and there are many boundary terms which appear in NMC scalar field ECSK theory. Equations (6.2), and (6.3) are the actions for a NMC scalar field (ϕ), and a minimally-coupled (MC) scalar field (ψ). We vary the action constructed from equation (6.1) with respect to the fields: ϕ, ψ, g^{ab}, and $C^{a}{}_{bc}$. This produces the following field equations:

$$\widetilde{\Box}\phi - \xi R \phi + \frac{dV(\phi)}{d\phi} = 0 \tag{6.6}$$

$$\widetilde{\Box}\psi + \frac{dW(\psi)}{d\psi} = 0 \tag{6.7}$$

$$G\{\}_{ab} + U_{ab} = \kappa \left(T_{ab}[\phi] + T_{ab}[\psi] \right) \tag{6.8}$$

$$\mathscr{C}_{c}{}^{ab} = \kappa s_{c}{}^{ab} \tag{6.9}$$

Equation (6.6) is the field equation for ϕ; equation (6.7) is the field equation for ψ; equation (6.8) is the field equation for g^{ab}; lastly, equation (6.9) is the field equation for $C^{a}{}_{bc}$. In equation (6.8) the stress energy tensors are given by:

$$T_{ab}\left[\psi\right] = \frac{\eta_2}{2}\left[\psi_{,a}\psi_{,b} - \frac{1}{2}g_{ab}\psi^{,c}\psi_{,c} + g_{ab}W\left(\psi\right)\right] \tag{6.10}$$

$$\begin{aligned}
T_{ab}\left[\phi\right] &= \left[\eta_1\left(\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c} + g_{ab}V\left(\phi\right)\right)\right] && (6.11)\\
&+ \left[\eta_1\xi\left(2T_{(a}\widetilde{\nabla}_{b)}\phi^2 - g_{ab}T^c\widetilde{\nabla}_c\phi^2\right)\right] && (6.12)\\
&+ \left[\eta_1\xi\phi^2\left(G\left\{\right\}_{ab} + U_{ab}\right)\right]\\
&+ \left[\eta_1\xi\left(g_{ab}\widetilde{\Box}\phi^2 - \widetilde{\nabla}_{(a}\widetilde{\nabla}_{b)}\phi^2\right)\right]
\end{aligned}$$

$$U_{ab} = \mathcal{U}_{ab} - \frac{1}{2}\mathcal{U}g_{ab} \tag{6.13}$$

$$\mathcal{U}_{ab} = -\left(C^d{}_{ad}C^c{}_{bc} - C^d{}_{ac}C^c{}_{bd}\right)$$

$$\mathcal{U} = -g^{ab}\left(C^d{}_{ad}C^c{}_{bc} - C^d{}_{ac}C^c{}_{bd}\right)$$

where define what we call the Hehl-Gravity (HG) tensor by equation (6.13), with what we call the Hehl scalar defined below. Additionally we have used $C^f{}_{bf} = -T^f{}_{bf} = T^f{}_{fb} = T_b$ to write the trace of the contorsion tensor in terms of the trace of the torsion tensor T_a in equation (6.11). Finally we have removed the U_{ab} contribution to the ϕ stress energy tensor because we place it on the other side of the field equations to contribute to the Einstein tensor term. Equations (6.10), and (6.11) represent the contributions to the total stress energy by the fields ψ, and ϕ respectively.

In the second ECSK field equation (6.9) the spin density tensor generated by the matter content of the field ϕ is given by the following equation:

$$s^c{}_{ab} = 2\eta_1\xi\left(\mathscr{C}^c{}_{ab}\phi^2 + \delta^c{}_{[a}\widetilde{\nabla}_{b]}\phi^2\right) \tag{6.14}$$

$$\mathscr{C}^c{}_{ab} = \frac{1}{2}T^c{}_{ab} + \delta^c_{[a}T^f{}_{b]f} \tag{6.15}$$

where we have also repeated the definition of the Cartan tensor $\mathscr{C}^c{}_{ab}$ in equation (6.15).

We note that the spin density the would be generated by the field ψ is zero, as it does not contribute.

This action given by the by the sum of equations (6.1), (6.4), and (6.5) produes boundary terms, of which the metric variation is zero on the boundary (Dirischlet condition), and the contorion variaion is unconstrained. We use subscripts on the deltas to signify which field produced them; as an example δ_g signifies a variation with respect to the metric. The non-canceling boundary terms are presented below:

$$\delta_\phi S_{\partial\phi} = \oint_{\partial\Omega} \left(\eta_1 \epsilon \left(\hat{n}^b \phi_{,b} - 2\xi\phi\left(K - K_1\right)\right)\right) \delta\phi\sqrt{\gamma} d^3y \tag{6.16}$$

$$\delta_\psi S_{\partial\psi} = \oint_{\partial\Omega} \left(\epsilon\eta_2 \hat{n}^b \psi_{,b}\right) \delta\psi\sqrt{\gamma} d^3y \tag{6.17}$$

$$\delta_g S_{EHCB} = \oint_{\partial\Omega} \left(\frac{\epsilon}{\kappa}\hat{n}_a T_b\right) \delta g^{ab}\sqrt{\gamma} d^3y$$

$$\delta_g S_{NMCCB} = \oint_{\partial\Omega} \left(-\eta_1 \xi\epsilon\phi^2 \hat{n}_a T_b\right) \delta g^{ab}\sqrt{\gamma} d^3y$$

$$\delta_g S_{\Box\phi 1\partial} = \oint_{\partial\Omega} \left(\frac{\eta_1}{2}\xi\epsilon\hat{n}_a \widetilde{\nabla}_b \left(\phi^2\right)\right) \delta g^{ab}\sqrt{\gamma} d^3y$$

$$\delta_g S_{\Box\phi 2\partial} = \oint_{\partial\Omega} \left(-\frac{\eta_1}{2}\xi\epsilon\hat{n}^d \widetilde{\nabla}_d \left(\phi^2\right) g_{ab}\right) \delta g^{ab}\sqrt{\gamma} d^3y$$

Once we prescribe that the fields ϕ,ψ, and g^{ab} are fixed on the boundary, eg. $\delta\phi|_{\partial\Omega} = 0, \delta\psi|_{\partial\Omega} = 0, \delta g^{ab}\big|_{\partial\Omega} = 0$ then all the boundary terms given by equations (6.16) vanish.

We now move onto the variation of the action with respect to the fields: g^{ab}, $C^a{}_{bc}$, ϕ, and ψ. We begin with varying the action with respect to the field ϕ, then move onto the variation with respect to the field ψ. Before we vary the action with respect to g^{ab} or $C^a{}_{bc}$ we put the action in a form more useful to computation.

6.1 Variation with Respect to ϕ

In this section we varying the action with respect to ϕ. We first arrive at a piece with covariant derivatives which will be come our field equations and a new boundary piece due to $S_{\partial NMC}$. We give a quick sketch of our steps for the reader. First we perform integration by parts. Then break off the Levi-Civita part of the covariant derivative to apply Stokes' theorem. Then we simplify the result, winding up with the following field equation labeled by E_ϕ.

$$E_\phi = \eta_1 \left(- \Box \, \phi + T^b \phi_{,b} + \xi R \phi - V' \left(\phi \right) \right) \tag{6.18}$$

This equation can be simplified to what we have in equation (6.6) with an equation which relates the d'alembertian to the Levi-Civita connection generated d'alembertian.

$$\Box \, \phi = \widetilde{\Box} \phi + T^a \phi_{,a} \tag{6.19}$$

The d'alembertian equation (6.19) can be derived using the fact that the difference of covariant derivatives is a tensor (Wald, 1984). Upon applying this idea to our action, we find that the contorsion pieces cancel and that we now have the torsion free d'alembertian instead of the ECSK one in our field equation. Additionally we have the boundary pieces given too. We can see that when we apply the vanishing of the variation of ϕ on the boundary, the boundary term $\delta S_{\partial \phi}$ vanishes. We repeat the boundary term here for clarity and note it is given by equation (6.16) in the introduction also.

$$\delta_\phi S_{\partial \phi} = \oint\limits_{\partial \Omega} \left(\eta_1 \epsilon \left(\hat{n}^a \phi_{,a} - 2 \xi \phi \left(K - K_1 \right) \right) \right) \delta \phi \sqrt{\gamma} d^3 y$$

Once we extermize the action we find that the field equation for ϕ is given by:

$$\widetilde{\Box} \phi - \xi R \phi + \frac{dV \left(\phi \right)}{d \phi} = 0$$

which is the same as equation (6.6). We are now finished with the variation of the action with respect to ϕ.

6.2 Variation with Respect to ψ

Next we vary the action with respect to ψ. Similarly to the previous section, and after applying an integration by parts, we find the following to be the variation of the action; we have also included the boundary term. We can see that when we apply the vanishing of the variation of ψ on the boundary, that the boundary piece vanishes.

$$E_\psi \quad = \quad \eta_2 \left(- \Box \, \psi + T^a \psi_{,a} - W' \left(\psi \right) \right) \tag{6.20}$$

$$\delta S_{\partial \psi} = \oint\limits_{\partial \Omega} \left(\eta_2 \epsilon \hat{n}^a \psi_{,a} \right) \delta \psi \sqrt{\gamma} d^3 y$$

Upon applying the extremal action principal we get the following field equation:

$$\widetilde{\Box} \psi + \frac{dW\left(\psi\right)}{d\psi} = 0$$

which we repeat here for clarity. This is the same as equation (6.7) in the introduction.

6.3 Preparation Before the Variation with Respect to g^{ab} and $C^a{}_{bc}$

Before we move to varying equation (6.1) with respect to the metric g^{ab} and the contorsion tensor $C^a{}_{bc}$, it is useful to split off the contorsion contribution in the action from the ECSK Ricci scalar in equation (6.2). We need to recall how to expand R in terms of its Christoffel and contorsion pieces from chapter 5.

$$R = R \left\{\right\} - 2 g^{ab} \widetilde{\nabla}_a T_b + \mathcal{U}$$

We apply this decomposition to our action. The term S_{NMCCB} arises due to an integration by parts on the derivative of the torsion form T_a. The term S_{EHCB} arises from splitting off the contorsion in the gravitational part of the action. Due to the non-minimal coupling, we have to apply Stoke's theorem once to the $\xi\phi^2 R$ term. This yields the following action:

$$\begin{aligned} S &= \int\limits_{\Omega} \left[-\frac{1}{2\kappa} \left(R\{\} + \mathcal{U} \right) \right] \sqrt{-g} d^4x \\ &+ \int\limits_{\Omega} \mathcal{L}_{M\phi} \sqrt{-g} d^4x + \int\limits_{\Omega} \mathcal{L}_{M\psi} \sqrt{-g} d^4x \\ &+ S_{GHY} + S_{\partial NMC} + S_{EHCB} + S_{NMCCB} \end{aligned} \tag{6.21}$$

$$\begin{aligned} \mathcal{L}_{M\phi} &= \frac{\eta_1}{2} \left(g^{ab} \phi_{,a} \phi_{,b} - 2V(\phi) \right) \\ &+ \frac{\eta_1}{2} \xi \phi^2 \left(R\{\} + \mathcal{U} \right) + \eta_1 \xi g^{ab} T_a \widetilde{\nabla}_b \left(\phi^2 \right) \end{aligned} \tag{6.22}$$

$$\mathcal{L}_{M\psi} = \left[\frac{\eta_2}{2} \left(g^{ab} \psi_{,a} \psi_{,b} - 2W(\psi) \right) \right] \tag{6.23}$$

$$S_{EHCB} = \oint\limits_{\partial\Omega} \left(\frac{\epsilon}{\kappa} g^{ab} \hat{n}_a T_b \right) \sqrt{\gamma} d^3y \tag{6.24}$$

$$S_{NMCCB} = \oint\limits_{\partial\Omega} \left(-\epsilon \eta_1 \xi \phi^2 g^{ab} \hat{n}_a T_b \right) \sqrt{\gamma} d^3y \tag{6.25}$$

Following Bronnikov, and Galiakhmetov (Bronnikov & Galiakhmetov, 2015), we seperate the ϕ, and ψ matter fields in the action; this allows to in tern define a stress energy tensor related to each individual field as in equations (6.10), and (6.11).

6.4 Variation with Respect to g^{ab}

Now we vary the action with respect to the metric g^{ab}. This gives us the following form for the action variation:

$$\begin{aligned} \delta_g S &= \int\limits_{\Omega} \left[-\frac{1}{2\kappa} \left(G\{\}_{ab} + U_{ab} \right) \right] \delta g^{ab} \sqrt{-g} d^4x \\ &+ \delta_g S[\phi] + \delta_g S[\psi] \\ &+ \delta_g S_{\partial NMC1} + \delta_g S_{\partial NMC2} + \delta_g S_{NMCCB} + \delta_g S_G + \delta_g S_T \end{aligned} \tag{6.26}$$

$$\delta_g S[\psi] = \int_\Omega \left(\delta\mathcal{L}_{M\psi} - \frac{1}{2} g_{ab} \mathcal{L}_{M\psi} \delta g^{ab} \right) \sqrt{-g} d^4 x \tag{6.27}$$

$$\delta_g S[\phi] = \int_\Omega \left(\delta\mathcal{L}_{M\phi} - \frac{1}{2} g_{ab} \mathcal{L}_{M\phi} \delta g^{ab} \right) \sqrt{-g} d^4 x \tag{6.28}$$

$$\begin{aligned} \delta_g S_G &= \frac{\epsilon}{\kappa} \oint_{\partial\Omega} \left(\frac{1}{2} \hat{n}^c \overline{\nabla}_c g_{ab} + \frac{1}{2} K_0 \gamma_{ab} \right) \delta g^{ab} \sqrt{\gamma} d^3 y \\ &+ \frac{\epsilon}{\kappa} \oint_{\partial\Omega} \left(-\overline{\nabla}_{(a} \hat{n}_{b)} + \hat{n}_{(a} T_{b)} - \frac{1}{2} \hat{n}_c T^c g_{ab} \right) \delta g^{ab} \sqrt{\gamma} d^3 y \end{aligned} \tag{6.29}$$

$$\delta_g S_T = \oint_{\partial\Omega} \left(-\frac{\epsilon}{2\kappa} \hat{n}_{(a} \gamma_{b)}{}^d \right) \delta g^{ab}{}_{:d} \sqrt{\gamma} d^3 y \tag{6.30}$$

$$\delta_g S_{NMCCB} = \oint_{\partial\Omega} \left(-\eta_1 \xi \phi^2 \epsilon \left(\hat{n}_{(a} T_{b)} - \frac{1}{2} g^{ef} \hat{n}_e T_f g_{ab} \right) \right) \delta g^{ab} \sqrt{\gamma} d^3 y \tag{6.31}$$

$$\begin{aligned} \delta_g S_{\partial NMC1} &= \oint_{\partial\Omega} \left(-\eta_1 \xi \phi^2 \epsilon \left(-\overline{\nabla}_{(a} \hat{n}_{b)} \right) \right) \delta g^{ab} \sqrt{\gamma} d^3 y \\ &+ \oint_{\partial\Omega} \left(-\eta_1 \xi \phi^2 \epsilon \left(\frac{1}{2} \epsilon g^{cd} \overline{\nabla}_c \hat{n}_d \hat{n}_a \hat{n}_b \right) \right) \delta g^{ab} \sqrt{\gamma} d^3 y \\ &+ \oint_{\partial\Omega} \left(-\eta_1 \xi \phi^2 \epsilon \left(-\frac{1}{2} (K - K_1) \gamma_{ab} \right) \right) \delta g^{ab} \sqrt{\gamma} d^3 y \end{aligned} \tag{6.32}$$

$$\delta_g S_{\partial NMC2} = \oint_{\partial\Omega} \left(-\eta_1 \xi \phi^2 \epsilon \left(-\hat{n}_{(a} \gamma_{b)}{}^d + \frac{1}{2} \gamma_{ab} \hat{n}^d \right) \right) \delta g^{ab}{}_{:d} \sqrt{\gamma} d^3 y \tag{6.33}$$

The first line, which is the Einstein-Hilbert term, turns into a combination of the Einstein tensor and the spin stress energy term from before. The first also produces a boundary term δS_1 which results from the variation of the Ricci tensor with respect to the metric $\delta R\{\}_{ab}$. The next two lines correspond to the stress energy tensors for the ϕ and ψ fields respectively. Last, the final line is what happens to the boundary terms after the variation. The boundary term which resulted from the variation of the Ricci tensor with respect to the metric cancels out the one which results from the variation of the GHY term with respect to the metric. The δS_{GHY1} term is the variation of S_{GHY} with respect to the metric; the $\delta S_{\partial NMC1}$ is the variation of $S_{\partial NMC}$ with respect to the metric also; this follows for the others as well where a 1 at the end denotes variation with respect to the metric. The term $\delta_g S_G$ is a combination of the of the reamining parts of the GHY term and the EHCB term.

The boundary terms arising from the variation of the Ricci tensor and part of the GHY variation cancel each other out. Additionally, once we apply $\delta g^{ab}\big|_{\partial\Omega} = 0$ the $\delta_g S_{\partial NMC1}$, and $\delta_g S_{NMCCB}$ pieces vanish. We will see later that the normal derivative terms in $\delta_g S_{\partial NMC2}$ term will cancel out with the variation of the Ricci tensor piece arising from the NMC term. $G\left\{\right\}_{ab}$ is the Einstein tensor generated by the Levi-Civita connection, and U_{ab} is the Hehl gravity tensor. Both of these tensors are repeated below for convienence.

$$G\left\{\right\}_{ab} = R\left\{\right\}_{ab} - \frac{1}{2}R\left\{\right\}g_{ab}$$

$$U_{ab} = \mathcal{U}_{ab} - \frac{1}{2}\mathcal{U}g_{ab}$$

We have also used use the variation of the determinant of the metric which is given by $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}$ to simiplify equation (6.26). Now we vary the matter Lagrangians (ϕ, ψ) to make their corresponding stress energy tensors.

6.4.1 The Energy Momentum Tensor ψ

First we will vary the ψ piece to find its stress energy tensor, as in equation (6.27). After variation, we find that the piece of the action which corresponds to the gravity field generated by the ψ stress enert=gy tensor is given as follows:

$$\delta_g S\left[\psi\right] = \int\limits_{\Omega} \left(\frac{\eta_2}{2}\left[\psi_{,a}\psi_{,b} - \frac{1}{2}g_{ab}\psi^{,c}\psi_{,c} + g_{ab}W\left(\psi\right)\right]\right)\delta g^{ab}\sqrt{-g}d^4x \qquad (6.34)$$

which we will soon write as the ψ stress energy tensor. The stress energy tensor is the object which prescribes $G_{ab} - T_{ab} = 0$ in GR. When we apply that idea in ECSK theory there are a few modification, but the spirit is very much the same. Recall the definition of the stress energy tensor is:

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{ab}}$$

However, this definition only holds when there is not a minus sign outside the Ricci scalar; we need $G_{ab} - T_{ab} = 0$. In our case we have a minus sign. To get what we need the definition just changes by a sign to: $T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{ab}}$. Applying this the same exact way we did in our ECSK field equation derivation then gives us our ψ action piece and ψ stress energy tensor $T_{ab}[\psi]$ as:

$$\delta_g S[\psi] = \int_{\Omega} \left[\frac{1}{2} T_{ab}[\psi]\right] \delta g^{ab} \sqrt{-g} d^4x \tag{6.35}$$

$$T_{ab}[\psi] = \frac{\eta_2}{2} \left[\psi_{,a}\psi_{,b} - \frac{1}{2} g_{ab} \psi^{,c}\psi_{,c} + g_{ab} W(\psi)\right] \tag{6.36}$$

The metric action variation is now given by:

$$\begin{aligned} \delta_g S &= \int_{\Omega} \left[-\frac{1}{2\kappa}\left(G\{\}_{ab} + U_{ab}\right)\right] \delta g^{ab} \sqrt{-g} d^4x \\ &+ \delta_g S[\phi] + \int_{\Omega} \left[\frac{1}{2} T_{ab}[\psi]\right] \delta g^{ab} \sqrt{-g} d^4x \\ &+ \delta_g S_{\partial NMC1} + \delta_g S_{\partial NMC2} + \delta_g S_{NMCCB} + \delta_g S_G + \delta_g S_T \end{aligned} \tag{6.37}$$

with equation (6.36) substituted into equation (6.26).

6.4.2 The Energy Momentum Tensor for ϕ

Now we derive the stress energy tensor for the field ϕ as in equation (6.28). To vary $\mathcal{L}_{M\phi}$ we go through many of the prior steps as before: we vary $R\{\}, \mathcal{U}$, and get the boundary terms. Upon varying $\mathcal{L}_{M\phi}$ we find the following form for $\delta\mathcal{L}_{M\phi}$:

$$\begin{aligned} \delta\mathcal{L}_{M\phi} &= \left(\frac{\eta_1}{2}\phi_{,a}\phi_{,b} + \eta_1 \xi T_{(a}\widetilde{\nabla}_{b)}\phi^2\right) \delta g^{ab} + \frac{\eta_1}{2}\xi\phi^2 \widetilde{\nabla}_d B^d \\ &+ \left(\frac{\eta_1}{2}\xi\phi^2\left(R\{\}_{ab} + \mathcal{U}_{ab}\right)\right) \delta g^{ab} \end{aligned} \tag{6.38}$$

Now we substitute back into the action variation for ϕ given by equation (6.37); this allows us to examine the $\widetilde{\nabla}_d B^d$ term. We write out $\delta_g S\left[\phi\right] = \int_\Omega \left(\delta\mathcal{L}_{M\phi} - \frac{1}{2} g_{ab}\mathcal{L}_{M\phi}\delta g^{ab}\right)\sqrt{-g}d^4x$ with each of the terms explicitly shown.

$$\begin{aligned}
\delta_g S\left[\phi\right] &= \int\limits_\Omega \frac{\eta_1}{2}\left[\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c} + g_{ab}V\left(\phi\right)\right]\delta g^{ab}\sqrt{-g}d^4x \qquad (6.39)\\
&+ \int\limits_\Omega \eta_1\xi\left[T_{(a}\widetilde{\nabla}_{b)}\phi^2 - \frac{1}{2}g_{ab}T^c\widetilde{\nabla}_c\phi^2\right]\delta g^{ab}\sqrt{-g}d^4x\\
&+ \int\limits_\Omega \left[\frac{\eta_1}{2}\xi\phi^2\left[G\left\{\right\}_{ab} + U_{ab}\right]\right]\delta g^{ab}\sqrt{-g}d^4x\\
&+ \int\limits_\Omega \left[\frac{\eta_1}{2}\xi\phi^2\widetilde{\nabla}_d B^d\right]\sqrt{-g}d^4x
\end{aligned}$$

The top line in equation (6.39) mirrors that of the stress energy tensor for the MC scalar field ψ. The second and third lines arise from the NMC and each have the coupling constant ξ in front of them. The fourth line has an interesting term $\int_\Omega \left[\frac{\eta_1}{2}\xi\phi^2\widetilde{\nabla}_d B^d\right]\sqrt{-g}d^4x$ which arises again from the variation of the Ricci tensor, this time on the NMC part. We can further decompose equation (6.39) by expanding the last term. In the last line we use $B_d\hat{n}^d = \left(\gamma_{ab}\hat{n}^d - \hat{n}_{(a}\gamma_{b)}{}^d\right)\delta g^{ab}{}_{:d}$ which is shown by Carroll (Carroll, 1997).

$$\begin{aligned}
\int\limits_\Omega \left[\frac{\eta_1}{2}\xi\phi^2\widetilde{\nabla}_d B^d\right]\sqrt{-g}d^4x &= \int\limits_\Omega \left[\widetilde{\nabla}_d\left(\frac{\eta_1}{2}\xi\phi^2 B^d\right) - \frac{\eta_1}{2}\xi\widetilde{\nabla}_d\left(\phi^2\right)B^d\right]\sqrt{-g}d^4x\\
&= \oint\limits_{\partial\Omega} \epsilon\frac{\eta_1}{2}\xi\phi^2 B_d n^d\sqrt{\gamma}d^3y\\
&- \int\limits_\Omega \frac{\eta_1}{2}\xi\widetilde{\nabla}_d\left(\phi^2\right)B^d\sqrt{-g}d^4x\\
&= \oint\limits_{\partial\Omega} \left(\epsilon\frac{\eta_1}{2}\xi\phi^2\left(\gamma_{ab}\hat{n}^d - \hat{n}_{(a}\gamma_{b)}{}^d\right)\right)\delta g^{ab}{}_{:d}\sqrt{\gamma}d^3y\\
&- \int\limits_\Omega \frac{\eta_1}{2}\xi\widetilde{\nabla}_d\left(\phi^2\right)B^d\sqrt{-g}d^4x\\
&= \delta_g S_B + \delta_g S_{\Box\phi}
\end{aligned}$$

$$\delta_g S_B = \oint\limits_{\partial\Omega}\left(\eta_1\xi\phi^2\epsilon\left(-\frac{1}{2}\hat{n}_{(a}\gamma_{b)}{}^d + \frac{1}{2}\gamma_{ab}\hat{n}^d\right)\right)\delta g^{ab}{}_{:d}\sqrt{\gamma}d^3y$$

$$\delta_g S_{\Box\phi} = -\int\limits_{\Omega} \frac{\eta_1}{2}\xi\widetilde{\nabla}_d\left(\phi^2\right)B^d\sqrt{-g}d^4x$$

At this point we rewrite equation (6.37) in terms of these new definitions, which cancel some of the boundary terms. Most notably all the tangential derivative pieces are now given by $\delta_g S_{T'}$ which is a combination of $\delta_g S_B$, $\delta_g S_{\partial NMC2}$, and $\delta_g S_T$.

$$\begin{aligned} \delta_g S &= \int\limits_{\Omega}\left[-\frac{1}{2\kappa}\left(G\left\{\right\}_{ab}+U_{ab}\right)\right]\delta g^{ab}\sqrt{-g}d^4x \\ &+ \left(\delta_g S\left[\phi\right]-\delta_g S_B-\delta_g S_{\Box\phi}\right)+\int\limits_{\Omega}\left[\frac{1}{2}T_{ab}\left[\psi\right]\right]\delta g^{ab}\sqrt{-g}d^4x \\ &+ \delta_g S_{\Box\phi}+\delta_g S_{\partial NMC1}+\delta_g S_{NMCCB}+\delta_g S_G+\delta_g S_{T'} \end{aligned} \tag{6.40}$$

$$\begin{aligned} \delta_g S_{T'} &= \delta_g S_B+\delta_g S_{\partial NMC2}+\delta_g S_T \\ &= \oint\limits_{\partial\Omega}\left(\frac{\epsilon}{2}\left(\eta_1\xi\phi^2-\frac{1}{\kappa}\right)\hat{n}_{(a}\gamma_{b)}{}^{d}\right)\delta g^{ab}{}_{:d}\sqrt{\gamma}d^3y \end{aligned} \tag{6.41}$$

Now we will look at the $\delta S_{\Box\phi}$ term which we break into two terms: $\delta S_{\Box\phi1}$, and $\delta S_{\Box\phi2}$. This is done through the defnition of B^b which can be derived by modifying a few terms from what we already accomplished in the ECSK field equation chapter, and raising an index.

$$B^d = -\left(\delta^c_{(a}\delta^d_{b)}-g_{ab}g^{dc}\right)\delta g^{ab}{}_{:c}$$

$$\delta_g S_{\Box\phi} = \delta_g S_{\Box\phi1}+\delta_g S_{\Box\phi2} \tag{6.42}$$

$$\delta_g S_{\Box\phi1} = \int\limits_{\Omega}\frac{\eta_1}{2}\xi\widetilde{\nabla}_d\left(\phi^2\right)\delta^c_{(a}\delta^d_{b)}\delta g^{ab}{}_{:c}\sqrt{-g}d^4x \tag{6.43}$$

$$\delta_g S_{\Box\phi2} = -\int\limits_{\Omega}\frac{\eta_1}{2}\xi\widetilde{\nabla}_d\left(\phi^2\right)g_{ab}g^{dc}\delta g^{ab}{}_{:c}\sqrt{-g}d^4x \tag{6.44}$$

We will look at $\delta S_{\Box\phi1}$ in equation (6.43) first. This term ends up producing a boundary term and a symmetrized double covariant derivative term. We use Stokes' theorem to get the boundary term. It is interesting to note that this seemingly tangential derivative of the variation of the metric term turns into something that contributes to the field equations, along with a boundary term.

$$\delta_g S_{\Box\phi 1} \quad = \quad \delta S_{\Box\phi 1\partial} - \int\limits_{\Omega} \left(\frac{\eta_1}{2}\xi\widetilde{\nabla}_{(a}\widetilde{\nabla}_{b)}\phi^2\right)\delta g^{ab}\sqrt{-g}d^4x \tag{6.45}$$

$$\delta_g S_{\Box\phi 1\partial} = \oint\limits_{\partial\Omega} \left(\frac{\eta_1}{2}\xi\hat{n}_{(a}\widetilde{\nabla}_{b)}\phi^2\right)\delta g^{ab}\sqrt{\gamma}d^3y \tag{6.46}$$

We now look at $\delta S_{\Box\phi 2}$ given by equation (6.44) now. We find yet another boundary term, and a new piece which contributes to the field equations, the Dalembertian of ϕ^2.

$$\delta_g S_{\Box\phi 2} \quad = \quad \delta S_{\Box\phi 2\partial} + \int\limits_{\Omega} \left(\frac{\eta_1}{2}\xi g_{ab}\widetilde{\Box}\phi^2\right)\delta g^{ab}\sqrt{-g}d^4x \tag{6.47}$$

$$\delta_g S_{\Box\phi 2\partial} = \oint\limits_{\partial\Omega} \left(-\frac{\eta_1}{2}\xi\epsilon\hat{n}^d\widetilde{\nabla}_d\left(\phi^2\right)g_{ab}\right)\delta g^{ab}\sqrt{\gamma}d^3y \tag{6.48}$$

Now if we put all these terms back together, we find something suitable to be turned into a stress energy tensor for ϕ. By using $T_{ab} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{ab}}$ we find the compact form for $\delta_g S[\phi]$ along with the ϕ stress energy tensor $T_{ab}[\phi]$:

$$\begin{aligned} T_{ab}[\phi] \quad = \quad & \left[\eta_1\left(\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c} + g_{ab}V(\phi)\right)\right] \\ + \quad & \left[\eta_1\xi\left(2T_{(a}\widetilde{\nabla}_{b)}\phi^2 - g_{ab}T^c\widetilde{\nabla}_c\phi^2\right)\right] \\ + \quad & \left[\eta_1\xi\phi^2\left(G\{\}_{ab} + U_{ab}\right)\right] \\ + \quad & \left[\eta_1\xi\left(g_{ab}\widetilde{\Box}\phi^2 - \widetilde{\nabla}_{(a}\widetilde{\nabla}_{b)}\phi^2\right)\right] \end{aligned} \tag{6.49}$$

$$\begin{aligned}\delta_g S &= \int\limits_{\Omega} \left[-\frac{1}{2\kappa}\left(G\left\{\right\}_{ab} + U_{ab}\right)\right]\delta g^{ab}\sqrt{-g}d^4x \\ &+ \int\limits_{\Omega} \left[\frac{1}{2}T_{ab}\left[\phi\right]\right]\delta g^{ab}\sqrt{-g}d^4x \\ &+ \int\limits_{\Omega} \left[\frac{1}{2}T_{ab}\left[\psi\right]\right]\delta g^{ab}\sqrt{-g}d^4x \\ &+ \delta S_{\Box\phi 1\partial} + \delta S_{\Box\phi 2\partial} + \delta_g S_{\partial NMC1} + \delta_g S_{NMCCB} + \delta_g S_G + \delta_g S_{T'}\end{aligned} \tag{6.50}$$

All of the boundary terms in equation (6.50) vanish once we apply $\delta g^{ab}\big|_{\partial\Omega} = 0$. Furthermore, the stress energy tensor given by equation (6.49) is the same as the stress energy tensor in GR once torsion is set to zero; specifically the second line in equation (6.49) vanishes. Next, upon extremizing the action given by equation (6.49), we find our ECSK-NMC field equation for g^{ab} to be:

$$G\left\{\right\}_{ab} + U_{ab} = \kappa\left(T_{ab}\left[\phi\right] + T_{ab}\left[\psi\right]\right) \tag{6.51}$$

This is very similar to the Einstein-NMC Field equation in GR, but with the addition of the Hehl gravity tensor too.

6.4.3 Comparison of the ϕ Stress Energy Tensor to Bronnikov

In this section we will compare the ϕ stress energy tensor we derived to that of Bronnikov and Galiakhmetov (Bronnikov & Galiakhmetov, 2015). Bronnikov has a different stress energy tensor than we do, however they are equal when a condition on the torsion tensor holds. The condition, which we will call $\mathbb{S}$, on the torsion tensor is given by $T_{abc} \equiv \frac{2}{(N-1)} g_{a[b}T_{c]}$; in other words the torsion tensor is only determined by its irreducible trace $\mathfrak{Q}_{abc}$ piece.

To try and get this closer to Bronnikov we will use the definition of the contorsion tensor to relate back to the torsion tensor. Bronnikov's calculation only holds in 4 dimensions, whereas ours holds in N-dimensions. This is the reason why we have U_{ab} instead of Bronnikov's Λ_{ab} tensor. The condition above will help us simplify $\mathcal{U}$ and turn it into Bronnikov's Λ_{ab}. It is important to note that although it is not explicitly stated we can see that Bronnikov uses the following definition for the contorsion tensor C_B: $(C_B)_{abc} = -T_{abc} - T_{bca} + T_{cab}$; the B subscript is to let us distinguish the Bronnikov contorsion tensor from ours. The lack of the factor of a half in Bronnikov's definition means that each term with the Torsion vector T^a will have an extra 2 in it in his paper as opposed to ours; additionally we can see this from his definition of the torsion as $(T_B)^c{}_{ab} = \Gamma^c{}_{[ab]}$ instead of $T^c{}_{ab} = -2\Gamma^c_{[ab]}$ as we have defined it.

To be especially clear the convention which we are using has the torsion tensor defined as follows: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. This is the definition, along with the convention $\nabla_\mu e^c_\nu = \Gamma^c{}_{b\mu} e^b_\nu$, which produces the spirit of what we are doing through almost all of these calculations. The nabla covariant derivative above follows Misner, Thorne, and Wheeler. Greek indices are components/lists, and latin ones are Penrose abstract indices.

The first thing we will take a look at is the contorsion vector and how it relates to the torsion vector. Bronnikov defines the torsion covector T_a as $T_a = T^d{}_{da}$. By taking a trace of the contorsion tensor on the first and third index, we find a a relation between the contorsion trace and the torsion vector.

$$C^d{}_{ad} = T_a$$

The symmetry of the metric and $T^a{}_{bc} = -T^a{}_{cb}$ are used to produce the above relation. Next, recall that the torsion tensor can be decomposed into its $SO(p,q)$ irreducible elements as follows.

$$\begin{aligned} T_{abc} &= \Theta(N-3)\left(\frac{1}{6(N-3)!} S^{d_1 \dots d_N} \epsilon_{d_1 \dots d_N abc}\right) \\ &+ \Theta(N-2)\left(\frac{2}{(N-1)} g_{a[b} T_{c]} + q_{abc}\right) \end{aligned}$$

$\Theta(\dots)$ is the Heaviside step function; we have included it above because, similarly to the irreducible elements of the curvature tensor certain pieces of the torsion tensor do not exist in lower dimensions. Our other reference [2] states that for NMC scalar fields in ECSK only the torsion trace exists; we will prove this later. Because of that, since the action only has torsion vector pieces, that means we can write:

$$T_{abc}|_{\mathbb{S}} \equiv \frac{2}{(N-1)} g_{a[b}T_{c]}$$

We evaluate the torsion tensor on the condition $\mathbb{S}$. This then quickly shows our definition is consistent with the above decomposition, yielding $T^{a}{}_{ba} = -T_b$. Next we would like to apply this idea to $\mathcal{U}$; recall $\mathcal{U} = -g^{ab}\left(C^{d}{}_{ad}C^{c}{}_{bc} - C^{d}{}_{ac}C^{c}{}_{bd}\right)$. There are two pieces in $\mathcal{U}$: $C^{d}{}_{ad}C^{c}{}_{bc}$, and $C^{d}{}_{ac}C^{c}{}_{bd}$. The first term we readily find, and the second term appears after some algebra. Together we find that these terms and $\mathcal{U}$ are:

$$C^{d}{}_{ad}C^{c}{}_{bc}\Big|_{\mathbb{S}} = T_aT_b$$

$$C^{d}{}_{ac}C^{c}{}_{bd}\Big|_{\mathbb{S}} = \frac{1}{N-1}T_aT_b$$

$$\mathcal{U}|_{\mathbb{S}} = -\left(\frac{N-2}{N-1}\right)T_cT^c$$

Using these relations and $\mathbb{S}$ yield a new form for $T_{ab}[\phi]$. This is the form for $T_{ab}[\phi]$ that Bronnikov has. Additionally, Bronnikov defines a tensor Λ_{ab} which we also reproduce; this tensor is just the spin density tensor evaluated on $\mathbb{S}$: $U_{ab}|_{\mathbb{S}} = \Lambda_{ab}$.

$$\begin{aligned}
T_{ab}[\phi] &= \left[\eta_1\left(\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c} + g_{ab}V(\phi)\right)\right] \\
&+ \left[\eta_1\xi\left(2T_{(a}\widetilde{\nabla}_{b)}\phi^2 - g_{ab}T^c\widetilde{\nabla}_c\phi^2\right)\right] \\
&+ \left[\eta_1\xi\phi^2\left(G\{\}_{ab} + U_{ab}\right)\right] \\
&+ \left[\eta_1\xi\left(g_{ab}\widetilde{\Box}\phi^2 - \widetilde{\nabla}_{(a}\widetilde{\nabla}_{b)}\phi^2\right)\right]
\end{aligned}$$

$$\Lambda_{ab} = \left(\frac{N-2}{N-1}\right)T_aT_b - \frac{1}{2}g_{ab}\left(\frac{N-2}{N-1}\right)T_cT^c \tag{6.52}$$

Upon choosing our dimension to be 4, $N = 4$, we reproduce Bronnikov's (7) & (9). The factors of 2 are different because of the definiton of the torison tensor. Recall that $C^{a}{}_{bc}$ is equal to Bronnikov's contorsion but with an extra factor of 2.

$$\Lambda_{ab}\big|_{N=4} = \frac{2}{3}T_aT_b - g_{ab}\frac{1}{3}T_cT^c$$

Our torsion terms differ by a factor of a half because of Bronnokov did not include that in his definition of the connection; see the bottom of his page 1. Now we move on to the variation with respect to the contorsion.

6.5 Variation with Respect to $C^a{}_{bc}$

To vary with respect to the contorsion $C^a{}_{bc}$ we start again by looking at our prepared action given by equation (6.26). Using what we know from before, we find that two terms vanish outright: $R\{\}$, and $\mathcal{L}_{M\psi}$ because they do not depend at all on the contorsion tensor. The variation with respect to the contorsion tensor of S_{GHY} and the other action pieces will have a C in them at the end to denote variation of the contorsion tensor just like the 1 at the end denoted variation with respect to the metric. The contorsion variation of S is then:

$$\begin{aligned}
\delta_C S &= \int\limits_{\Omega} \left[-\frac{1}{2\kappa}\left(\delta_C\mathcal{U}\right)\right]\sqrt{-g}d^4x \\
&+ \int\limits_{\Omega} \delta_C\mathcal{L}_{M\phi}\sqrt{-g}d^4x \\
&+ \delta_C S_{GHY} + \delta_C S_{\partial NMC} + \delta_C S_{EHCB} + \delta_C S_{NMCCB}
\end{aligned}$$

$$\delta_C K = \left(\hat{n}_a\gamma^{bc}\right)\delta C^a{}_{bc}$$

$$\delta_C\mathcal{U} = \left(g^{bd}\mathscr{C}^c{}_{ad}\right)\delta C^a{}_{bc}$$

$$\mathscr{C}^c{}_{ab} = -2\left(C^c{}_{[ab]} + \delta^c{}_{[a}C^f{}_{b]f}\right), \quad \mathscr{C}^c{}_{ab} = T^c{}_{ab} + 2\delta^c_{[a}T^f{}_{b]f}$$

Where the boundary terms are again given by:

$$\delta_C S_{EHCB} = \oint\limits_{\partial\Omega}\left(-\frac{\epsilon}{\kappa}g_{ad}\hat{n}^{[d}\gamma^{b]c}\right)\delta C^a{}_{bc}\sqrt{\gamma}d^3y$$

$$\delta_C S_{GHY} = \oint_{\partial\Omega} \left(\frac{\epsilon}{\kappa} g_{ad}\hat{n}^{[d}\gamma^{b]c}\right) \delta C^a{}_{bc}\sqrt{\gamma} d^3y$$

$$\delta_C S_{\partial NMC} \;\; = \;\; \oint_{\partial\Omega} \left(-\epsilon\eta_1\xi\phi^2 g^{bd}\hat{n}_{[a}\gamma_{d]}^{\;\;c}\right) \delta C^a{}_{bc}\sqrt{\gamma} d^3y$$

$$\delta_C S_{NMCCB} = \oint_{\partial\Omega} \left(\epsilon\eta_1\xi\phi^2 g^{bd}\hat{n}_{[a}\gamma_{d]}^{\;\;c}\right) \delta C^a{}_{bc}\sqrt{\gamma} d^3y$$

All the boundary terms vanish; the $\delta_C S_{GHY}$ term and the $\delta_C S_{EHCB}$ term cancel; the term $\delta_C S_{\partial NMC}$ cancels with the term $\delta_C S_{NMCCB}$. This is interesting in its own right because we didn't ever need to impose $\delta C^a{}_{bc}|_{\partial\Omega} = 0$. Now we just need what the contorsion variation of $\mathcal{U}$ and $\mathcal{L}_{M\phi}$ are. We already know that the variation of $\mathcal{U}$ is given by a combination with the Cartan tensor: $\delta_C \mathcal{U} = \left(g^{bd}\mathscr{C}^c{}_{ad}\right)\delta C^a{}_{bc}$. The variation of $\mathcal{L}_{M\phi}$ is slightly longer. For now the action simplifies to the following form.

$$\begin{aligned} \delta_C S \;\; &= \;\; \int_{\Omega} \left[-\frac{1}{2\kappa}\left(g^{bd}\mathscr{C}^c{}_{ad}\right)\delta C^a{}_{bc}\right]\sqrt{-g} d^4x \\ &+ \;\; \int_{\Omega} \delta_C \mathcal{L}_{M\phi}\sqrt{-g} d^4x \end{aligned}$$

$$\mathscr{C}^c{}_{ab} = -2\left(C^c{}_{[ab]} + \delta^c{}_{[a}C^f{}_{b]f}\right) = T^c{}_{ab} - 2\delta^c_{[a}T_{b]}$$

Our next course of action is to vary $\mathcal{L}_{M\phi}$. Since the contorsion tensor is skew in the first 2 indices, i.e. $C_{cad} = C_{acd}$ we obtain some skew brackets in our equation above. Thus we arrive an our $\delta\mathcal{L}_{M\phi}$ equation.

$$\delta_C \mathcal{L}_{M\phi} \;\; = \;\; \frac{\eta_1}{2}\xi g^{bd}\left(\phi^2\mathscr{C}^c{}_{ad} + 2\delta^c{}_{[a}\widetilde{\nabla}_{d]}\left(\phi^2\right)\right)\delta C^a{}_{bc}$$

Thus, all together, this gives us:

$$\begin{aligned}\delta_C S &= \int\limits_\Omega \left[-\frac{1}{2\kappa}\left(g^{bd}\mathscr{C}^c{}_{ad}\right)\right]\delta C^a{}_{bc}\sqrt{-g}d^4x \\ &+ \int\limits_\Omega \left(\frac{\eta_1}{2}\xi g^{bd}\left(\phi^2\mathscr{C}^c{}_{ad} + 2\delta^c{}_{[a}\widetilde{\nabla}_{d]}\left(\phi^2\right)\right)\right)\delta C^a{}_{bc}\sqrt{-g}d^4x\end{aligned}$$

The bottom piece in the equation here is the part which corresponds to the spin density since it was generated by $\mathcal{L}_{M\phi}$. Recall that the spin density is defined by equation $\mathscr{C}_a{}^{bc} - s_a{}^{bc} = 0$. Originally we had this relation given by $\mathfrak{S}^a{}_b{}^c = -2\frac{\delta\mathcal{L}_M}{\delta C^b{}_{ca}}$, but it is modified here to $\mathfrak{S}^a{}_b{}^c = 2\frac{\delta\mathcal{L}_M}{\delta C^b{}_{ca}}$ to make the above equation hold. This arises from the negative in front of the Ricci scalar. This gives us our spin density and action variation as:

$$\begin{aligned}\delta_C S &= \int\limits_\Omega \left[-\frac{1}{2\kappa}\left(g^{bd}\mathscr{C}^c{}_{ad}\right)\right]\delta C^a{}_{bc}\sqrt{-g}d^4x \\ &+ \int\limits_\Omega \left[\frac{1}{2}g^{bd}s^c{}_{ad}\right]\delta C^a{}_{bc}\sqrt{-g}d^4x\end{aligned} \tag{6.53}$$

$$s^c{}_{ad} = \eta_1\xi\left(\phi^2\mathscr{C}^c{}_{ad} + 2\delta^c{}_{[a}\widetilde{\nabla}_{d]}\left(\phi^2\right)\right) \tag{6.54}$$

To be specific we say $s^c{}_{ad} = s^c{}_{ad}[\phi]$ and $s^c{}_{ad}[\psi] = 0$; this mirrors our notation from earlier for example, in equation (6.35). Now we can extremize the action to find our field equation.

$$\mathscr{C}^c{}_{ad} = \kappa s^c{}_{ad} \tag{6.55}$$

Next we can examine how this corresponds to Bronnikov's T_a equation in the limit when we apply the condition $\mathbb{S}$.

6.5.1 Comparison of the Spin Stress Energy Tensor to Bronnikov

Our goal in this section is to reproduce what Bronnikov has in his paper. It is also here we will show that only the trace $\mathfrak{Q}_{abc}$ piece of the torsion tensor is non-vanishing for NMC scalar fields in ECSK. We will prove that the $\mathfrak{d}$ and $\mathfrak{q}$ pieces of the torsion tensor vanish for NMC fields. To do this we will use the field equations and the definition of the Cartan tensor. We now find the $SO\left(p,q\right)$ irreducible elements of these two tensors because the field equation for them will give us insight into what is really going on.

Theorem: For a NMC scalar field subject to the second ECSK field equation (6.55), the only non-zero irreducible element of the torsion tensor is the trace piece of the torsion tensor, the $\mathfrak{Q}$ piece. This piece is in turn given by the formula in $N-$dimensions:

$$T_a = -2\left(\frac{N-1}{N-2}\right)\xi\Psi\phi\phi_{,a}$$

$$\Psi = \frac{\kappa}{\left(\frac{1}{\eta_1} - \kappa\xi\phi^2\right)}$$

Proof: We begin by getting the irreducible elements of the Cartan and Spin density tensors respectively from the following definitions. The Cartan tensor definition (in terms of torsion), and the spin density tensor are repeated here with all indices lowered.

$$\mathscr{C}^c{}_{ab} = T^c{}_{ab} + 2\delta^c_{[a}T^f{}_{b]f}$$

$$s_{cab} = \eta_1\xi\left(\mathscr{C}_{cab}\phi^2 + 2g_{c[a}\widetilde{\nabla}_{b]}\phi^2\right)$$

First we have the totally skew pieces given by the $\mathfrak{d}$ element.

$$\mathscr{C}_{[cab]} = \mathfrak{d}_{[cab]}$$

$$s_{[cab]} = \eta_1\xi\phi^2\mathfrak{d}_{[cab]}$$

Next we have the trace pieces given by the $\mathfrak{Q}$ element.

$$\mathscr{C}_{cab}|_{\mathbb{S}} = -2\left(\frac{N-2}{N-1}\right)g_{c[a}T_{b]}$$

$$s_{cab}|_{\mathbb{S}} = 2\eta_1\xi\left(-\left(\frac{N-2}{N-1}\right)g_{c[a}T_{b]}\phi^2 + g_{c[a}\widetilde{\nabla}_{b]}\phi^2\right)$$

We finally have the leftover pieces $\mathfrak{q}$, which are marked by an evaluation on $\mathfrak{q}$ symbol.

$$\mathscr{C}_{cab}\big|_{\mathfrak{q}} = \mathfrak{q}_{c[ab]}$$

$$s_{cab}\big|_{\mathfrak{q}} = \eta_1 \xi \phi^2 \mathfrak{q}_{c[ab]}$$

Now we can apply the field equation $\mathscr{C}_{abc} = \kappa s_{abc}$ to each different part and prove that only the $\mathfrak{Q}$ piece is non-zero. When we apply the field equation to the $\mathfrak{d}$ piece, we find that $T_{[abc]} \equiv 0$ from $\frac{1}{2}\left(1 - \kappa\eta_1\xi\phi^2\right) T_{[abc]} = 0$; note that $\left(1 - \kappa\eta_1\xi\phi^2\right)$ cannot vanish because oherwise Ψ would diverge. Similarly for the $\mathfrak{q}$ piece, once we apply the field equation we find that $\mathfrak{q}_{a[bc]} = 0$ from $\frac{1}{2}\left(1 - \kappa\eta_1\xi\phi^2\right) \mathfrak{q}_{a[bc]} = 0$. The only piece that remains is the T_a piece which corresponds to the $\mathfrak{Q}$ sector. Now we move onto getting the simplified field equation for T_a. Like before once we apply the field equations the prior relations to the left hand side of our field equation gives us:

$$g_{c[a}T_{b]} = -\left(\frac{N-1}{N-2}\right)\Psi\xi g_{c[a}\widetilde{\nabla}_{b]}\phi^2$$

$$\Psi = \frac{\kappa}{\left(\frac{1}{\eta_1} - \kappa\xi\phi^2\right)}$$

This is non zero, and upon taking a trace, we find that the torsion vector is related to the scalar field ϕ by the following equation.

$$T_a = -2\left(\frac{N-1}{N-2}\right)\xi\Psi\phi\phi_{,a}$$

and thus we are done with our proof $\blacksquare$.

To compare with Bronnokov's form, we choose $\eta_1 = \pm 1$ and $N = 4$. Doing so reduced our form of the torsion equation in our proof to a simplier form. $T_b\big|_{N=4,\eta_1=\pm 1} = -3\xi\Psi\phi\phi_{,b}$, and $\Psi\big|_{\eta_1=\pm 1} = \frac{\kappa}{(\eta_1 - \kappa\xi\phi^2)}$. Note that the factor of 2 is here again. To convert back to Bronnikov, take $T_a \to -2T_a$ and we reproduce his results. This is again because of the difference in the definition of the torsion; we had minus 2 of what he has. Finally , we can reproduce full torsion tensor using just the trace with our trace only formula of the torsion: $T_{abc} \equiv \frac{1}{(N-1)} (g_{ab}T_c - g_{ac}T_b)$.

$$T^a_{\ bc} = \frac{-4}{(N-2)} \xi \Psi \phi \delta^a_{\ [b} \phi_{,c]}$$

We have now reproduced all of Bronnikov's field equations, but in N-dimensions with the extermal action principal g^{ab}, ψ, ϕ, and $C^a_{\ bc}$. Additionally we only needed to prescribe the following three things vanish on the boundary: $\delta\phi|_{\partial\Omega} = 0, \delta\psi|_{\partial\Omega} = 0, \delta g^{ab}\big|_{\partial\Omega} = 0$.

6.6 Classifying the Bronnikov Solution

Using the classification tools we developed in the Rank 3 Tensors and Rank 4 Tensors sections, we classify Bronnikov's and Galiakhmetov's solution (Bronnikov & Galiakhmetov, 2015). We examine their ECSK-NMC scalar field solution where the fields of interest are: ϕ, ψ, g^{ab}, and $C^a_{\ bc}$. The field equations for the fields g^{ab}, $C^a_{\ bc}$, ϕ, and ψ are:

$$G\left\{\right\}_{ab} + U_{ab} = \kappa \left(T_{ab}\left[\phi\right] + T_{ab}\left[\psi\right]\right)$$

$$\mathscr{C}_{abc} = \kappa s_{abc}$$

$$\widetilde{\Box}\phi - \xi R \phi + \frac{dV\left(\phi\right)}{d\phi} = 0$$

$$\widetilde{\Box}\psi + \frac{dW\left(\psi\right)}{d\psi} = 0$$

The solution presented has coordinates: (t, u, θ, ϕ). The metric g_{ab} is presented with signature $[+, -, -, -]$ in the following general static, spherically symmetric form:

$$g_{ab} = A\left(u\right) dt \otimes dt - \frac{1}{A\left(u\right)} du \otimes du - r^2\left(u\right) \left(d\theta \otimes d\theta + \sin^2\left(\theta\right) d\phi \otimes d\phi\right)$$

They make the coordinate transformation $x = \frac{u}{b}$, with $b \in \mathbb{R} - \{0\}$. This puts the metric in the following form:

$$g_{ab} = A(x)\, dt \otimes dt - \frac{b^2}{A(x)} dx \otimes dx - r^2(x) \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\phi \otimes d\phi \right)$$

We then have the exact solution given by:

$$A(x) = \left(x^2 + 1\right) \left[B_0 - 2\frac{\arctan(x)}{x} - \arctan^2(x) \right]$$

$$r(x) = b^2 \sqrt{x^2 + 1}$$

$$\phi(x) = \left(\kappa\xi\left(x^2 + 1\right)\right)^{-\frac{1}{2}}$$

$$\psi(x) = \psi_0 \pm 2x \left(\kappa\left(x^2 + 1\right)\right)^{-\frac{1}{2}}, \quad \psi_0 \in \mathbb{R}$$

$$\begin{aligned} W(x) &= W_0 + \left(8\kappa b^2 \left(x^2 + 1\right)^2\right)^{-1} \left[8B_0 x^2 \left(2 + x^2\right)\right. \\ &+ 25x^2 \left(5 + 4x^2\right) - 2\left(75x^4 + 125x^2 + 32\right) \frac{\arctan(x)}{x} \\ &- \left.\left(75x^4 + 150x^2 + 67\right) \arctan^2 x\right] \end{aligned}$$

$$\begin{aligned} V(x) &= \left(8\kappa b^2 \left(x^2 + 1\right)^2\right)^{-1} \left[-8\kappa b^2 W_0\right. \\ &+ -8B_0 \left(4x^4 + 6x^2 + 1\right) - 100x^4 - 101x^2 + 16 \\ &+ 2\left(99x^4 + 149x^2 + 32\right) \frac{\arctan(x)}{x} \\ &+ \left.\left(99x^4 + 182x^2 + 75\right) \arctan^2 x\right] \end{aligned}$$

Where $T_{ab}[\phi]$, and $T_{ab}[\psi]$ are given respectively by equations (6.11), and (6.10) evaluated on the solution above. We take $C_1 = 0$ following Bronnikov's reasoning that $A(x)$ should be regular everywhere. Additionally we take $\eta_1 = \eta_2 = 1$ to specify that both scalar fields are canonical and non-phantom. Evaluating the torsion tensor trace on the solution gives:

$$T_a = -\frac{3}{2\left(x^2+1\right)x}dx \tag{6.56}$$

We can now classify the solution using our tools from before.

6.6.1 Torsion Tensor Classification

We take equation (6.56), and build the torsion tensor out of it using the trace piece of the torsion tensor from equation (3.44) in chapter 3.

$$\mathfrak{Q}_{abc} = \frac{2}{(N-1)} g_{a[b}T_{c]}$$

Then upon applying our algorithm we recieve:

$$\left[\Theta = (1,1)\,, \Xi = (-)\,, \Omega = (-)\right] \tag{6.57}$$

as our classification. We expect that the spinors $\boldsymbol{\Xi}$, and $\boldsymbol{\Omega}$ would be zero because the axial part ($\boldsymbol{\Xi}$) and leftovers piece ($\boldsymbol{\Omega}$) vanish in our derivation. Additionally we find that the spinor $\boldsymbol{\Theta}$ which corresponds to the trace part of the torsion is of general type. We look to see if there are any exceptional points on which $\boldsymbol{\Theta}$ decomposes further. To do so we examine the $\boldsymbol{O}$ (Omicron) invariant. In this case it is given by:

$$O = \frac{1}{4}\frac{\arctan^2(x)\,x - B_0 x + 2\arctan(x)}{x^3 b^2\left(x^2+1\right)}$$

The only way O can be zero is if x is the root O. Because of the parameters B_0, b and the transcendental nature of the equation it is difficult to find x for which $O(x) = 0$. However, if $b = 1$, and $2.35 \gtrsim B_0 > 2$ there exist two roots, and thus there 2 exceptional points where the character of $\boldsymbol{\Theta}$ degenerates to be of type $|(1,0)|^2$; we note here that the number 2.35 is not an upper bound on the existence of roots, there exist higher numbers, but the bound is close to this number. Additionally, this may change for other values of b, but should not change the character of the roots much since it appears in the denominator.

6.6.2 Curvature Tensor Classification

Now we present the classification of the ECSK curvature tensor. We see that $\Psi = D$ implies Petrov type D. That $\Phi = \{[11,11] \quad OR \quad [(11,\overline{11})], D, [1,1(11)]\}$ implies the structural reducibility type for the Phi spinor, the Plebanski type, and the Segre type. That $\Lambda =$ "*Indeterminant*" means that the algorithm was unable to tell if *Lambda* is positive negative or zero. That $\text{Ж} = \{(-), O, [(1,111)]\}$ means that the structural reducibility type is empty, the Plebanski type is zero, and the Segre type is trivial. That $\text{Ю} = [-]$ means that the Yu spinor is zero. Finally, that $\aleph = 0$ means that the aleph spinor is zero everywhere.

$$\left[\Psi = D, \Phi = \{[11,11] \quad OR \quad [(11,\overline{11})], D, [1,1(11)]\}, \Lambda = \text{"}Indeterminant\text{"}\right] \tag{6.58}$$

$$\left[\text{Ж} = \{(-)\,, O, [(1,111)]\}\,, \text{Ю} = [-], \aleph = 0\right] \tag{6.59}$$

Equation (6.59) is interesting because it implies that $\mathscr{L}_{abcd}$, $\mathfrak{l}_{abcd}$, and $\mathfrak{k}_{abcd}$ are all zero. Thus, the torsion here only contributes to the $\boldsymbol{\mathcal{F}}$sectors as defined in the Rank 4 Tensors section. Recall that the $\boldsymbol{\mathcal{F}}$ type tensors have tableaux:

1	3
2	4

which decomposes into the three other $\boldsymbol{\mathcal{F}}$ sectors: $\mathfrak{F}$, $\mathscr{F}$, and $\mathfrak{f}$ under $SO\,(p,q)$.

We will first examine if there is any way for the Petrov type to degenerate further at exceptional points. Petrov type D is determined by the the covariant $\boldsymbol{\Psi}$ being non-zero and the covariant R being equal to zero. According to Penrose and Rindler (Penrose & Rindler, 1987b), the only ways a general type D spacetime can degenerate further is either into type N, or type O. Recall that for a type N spacetime that the covariant $\boldsymbol{\Psi}$ is non-zero and the covariant $\boldsymbol{Q}$ is zero; similarly for a type O spacetime the covariant $\boldsymbol{\Psi}$ must be zero. Upon calculating these co(in)variants we find that

$$\Psi_0 = -\frac{1}{3}\Psi_2 = \Psi_4 = \frac{1}{2}\frac{\arctan(x)\,x^2 + \arctan(x) - x}{x\,(x^2+1)}$$

$$\Psi_1 = \Psi_3 = 0$$

$$Q_0 = -\frac{1}{3}Q_2 = Q_4 = -\frac{1}{6}\frac{\left(\arctan(x)\,x^2 + \arctan(x) - x\right)^2}{x^6\,(x^2+1)^2}$$

$$Q_1 = Q_3 = 0$$

Thus, $\boldsymbol{Q} = 0$ if and only if $\boldsymbol{\Psi} = 0$, where $\boldsymbol{Q}$, $\boldsymbol{\Psi}$ signify Q_a, Ψ_a for all a. This tells us that the only way that this solution can degenerate in Petrov type is straight from type D to type O. There is no way for the solution to degenerate to a type N spacetime.

The structural reduciblity algorithm for Φ tells us that it is either type $(1,1)\,(1,1)$, or $|(1,1)|^2$. The co(in)variants arising from this classification are rather large and complicated so we do not examine the exceptional points, if any, that could arise.

The Segre classifiation when applied in Maple was unable to determine what Segre type we are dealing with originally; in equation (6.58) the Segre algorithm originally fails. However we will show that this solution is type $[1,1\,(11)]$. We begin by noting that we are in a Plebanski type D spacetime, and find that the algorithm was unable to determine the sign of the H invariant defined in equation 4.62. The H invariant once simplified shows us that each term is squared and can therefore only be positive; it has several terms so we omit writing it here.

The next thing we calculate in the case of Plebanski type D and $H > 0$ is the covariant k_a defined in equation 4.63. In a similar fashion to H, we find every term except the leading coefficient is squared. The leading coefficient is negative and thus we determine the Segre type to be $[1,1\,(11)]$ per the algorithm of Zakhary and Carminati (Zakhary & Carminati, 2004).

Finally we examine the classification of Λ. Here the rank 0 spinor Λ is given by:

$$\Lambda = \frac{1}{24}\frac{\left(-24x^5-6x^3-9x\right)\arctan^2(x)}{x^3\left(x^2+1\right)} + \frac{1}{24}\frac{\left(-48x^4+4x^2-38\right)\arctan(x)}{x^3\left(x^2+1\right)} + \frac{1}{24}\frac{24B_0x^5+(6B_0-24)\,x^3+(9B_0+20)\,x}{x^3\left(x^2+1\right)} \tag{6.60}$$

Upon analyzing Λ we again find that for $2.35 \gtrsim B_0 > 2$ that there are 2 roots. We can classify this further however. First for $B_0 \leq 2$, we have that $\Lambda(x) < 0 \quad \forall x$. For $B_0 \geq 2.5$ we can see that $\Lambda(x) > 0 \quad \forall x$. In the region $2.5 \geq B_0 \gtrsim 2.35$ it is difficult to tell if Λ is always positive or if it changes based on the value of x. To clarify on the region we also determined that for $2.35 \gtrsim B_0 > 2$ there are 3 regions distinguished by the roots of equation (6.60). Let the first root be called x_{-1}, and the second root be called x_1, where $x_{-1} < x_1$. Then the regions can be defined as follows: for $x < x_{-1}$ we know $\Lambda(x) < 0$; for $x_{-1} < x < x_1$ we know $\Lambda(x) > 0$; finally for $x > x_1$ we know $\Lambda(x) < 0$. This again only occurs in the region $2.35 \gtrsim B_0 > 2$.

One more interesting thing about the classification of this solution is that the spinors: $Ж(AB)(A'B')$, $Ю_{(AB)}$, and $\aleph$ are all zero even with non-vanishing torsion. With this our classification is complete.

CHAPTER 7
ECSK-DIRAC THEORY

Much of the background here was expressed well in reference (Poplawski, 2013); it is precisely those ideas that helped cement so much of this chapter. The clarity of the ideas in reference (Poplawski, 2013) led to the development of code which upon completion of this dissertation will hopefully be added to USU's "DifferentialGeometry" package for Dirac spinors. Reference (Poplawski, 2013) has been invaluable in many aspects and ideas beyond just this Dirac spinor section as well. Wheeler (J. Wheeler, 2021) has also been incredibly useful in solidifying technical points relating to Dirac spinors. We begin the chapter with the section on Dirac spinors, gamma matrices, and how Lorentz transformations act on Dirac spinors. Next we examine derivatives of spinors and the Dirac spinor connection, or the Fock-Ivaneko coefficients. Following that we have a short section on the curvature spinor.

The last three sections focus on:

1) the Dirac Lagrangian and field equations with torsion,

2) the irreducible sectors in ECSK theory generated by a Dirac spinor, and

3) the classification of a solution presented by Platania.

7.1 Dirac Spinors, Gamma Matrices, and Lorentz Transformations on Spinors

In this section we will examine Dirac spinors and the gamma matrices. First, we will define what a Dirac spinor ψ (or if we include the Dirac indices $\psi^{\mathring{A}}$) is in terms of a complex mapping. We will express this mapping as a column vector and then proceed to define what the derivative of a Dirac spinor, $\partial_\mu \psi$, is. Next we will present the gamma matrices γ_μ as a set of complex linear transformations which satisfy a Clifford algebra relation. The Clifford algebra relation relates the gamma matrices back to the spacetime Minkowski metric in special relativity. This relation can be upgraded in a sense to include gravity by including orthonormal frame fields ${e_a}^\mu$ which can be combined with the gamma matrices, to make the spacetime dependent gamma matrices $\gamma_a = e_a{}^\mu \gamma_\mu$. To end this section we examine how Lorentz transformations on spinors are carried out in this new projective unitary representation of the Lorentz group. A Lorentz transformation is then applied to the gamma matrices so we can see how they transform.

7.1.1 The Dirac spinor ψ, $\overline{\psi}$ and their derivatives

We begin by looking at the Dirac spinor ψ. Notationally we will flip between ψ and $\psi^{\mathring{A}}$ which are equivalent; here the superscript A indicates a column vector. Usually this index is suppressed in the literature, but we believe that including it can help elucidate concepts in a few areas like transformations of spinors. Similarly, we will also represent $\overline{\psi}$ by $\overline{\psi}_{\mathring{A}}$, which will indicate a row vector. We think of ψ as the mapping:

$$\psi : M \to \mathbb{C}^4$$

The spinor ψ viewed in this way can be written as a column vector. Note that $\overline{\psi}$ is given in terms of the hermitian conjugate of ψ multiplied by γ^0 i.e., $\overline{\psi} = \psi^\dagger \gamma^0$ which will be elaborated on later in more detail; this ensures that it transforms the correct way under Lorentz transformations. $\overline{\psi}$ viewed in this way can be written as a row vector. The 2×2 identity matrix is denoted I_2.

$$\overline{\psi} = \psi^\dagger \gamma^0$$

$$\gamma_0 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}$$

Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, and x^α be coordinates on M, then ψ is given by the following column vector. Similarly, we can also write $\overline{\psi}$ as a row vector.

$$\psi = \begin{bmatrix} \alpha(x^\alpha) \\ \beta(x^\alpha) \\ \gamma(x^\alpha) \\ \delta(x^\alpha) \end{bmatrix}, \quad \overline{\psi} = \begin{bmatrix} \gamma^*(x^\alpha) & \delta^*(x^\alpha) & \alpha^*(x^\alpha) & \beta^*(x^\alpha) \end{bmatrix}$$

If we explicitly include indices: ψ becomes $\psi^{\mathring{A}}$, and $\overline{\psi}$ becomes $\overline{\psi}_{\mathring{A}}$. The $\mathring{A}$ index is an abstract Dirac spinor index. We also write the derivative of ψ: $\partial_\mu \psi$, and the derivative of $\overline{\psi}$: $\partial_\mu \overline{\psi}$ as partials onto the column and row vectors.

$$\partial_\mu \psi = \begin{bmatrix} \partial_\mu \alpha \\ \partial_\mu \beta \\ \partial_\mu \gamma \\ \partial_\mu \delta \end{bmatrix}, \quad \partial_\mu \overline{\psi} = \begin{bmatrix} \partial_\mu \overline{\gamma} & \partial_\mu \overline{\delta} & \partial_\mu \overline{\alpha} & \partial_\mu \overline{\beta} \end{bmatrix}$$

This will be the basic notation we use when working with Dirac spinors. We will also use $\eta_{\mu\nu}$ to raise and lower Greek indices.

7.1.2 The gamma matrices γ_μ

To work further with spinors we need to define the gamma matrices. These allow us to relate special relativity to our spinors, and perform Lorentz transformations. The gamma matrices in our case are four (indexed by μ) linear transformations on $\mathbb{C}^4$. The following mapping gives them.

$$\gamma_\mu : \mathbb{C}^4 \to \mathbb{C}^4$$

We could write these as a list of linear transformations on the spinor space represented by $\gamma_{\mu}{}^{\mathring{A}}{}_{\mathring{B}}$. Here we can view the $\mathring{A}, \mathring{B}$ indices as explicitly creating this linear transformation. Furthermore, the gamma matrices satisfy the following relation:

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I_4 \tag{7.1}$$

$$\gamma_{\mu}{}^{\mathring{A}}{}_{\mathring{C}} \gamma_{\nu}{}^{\mathring{C}}{}_{\mathring{B}} + \gamma_{\nu}{}^{\mathring{A}}{}_{\mathring{C}} \gamma_{\mu}{}^{\mathring{C}}{}_{\mathring{B}} = 2\eta_{\mu\nu} \delta^{\mathring{A}}_{\mathring{B}}$$

Below in equation (7.1) we have also included explicitly the $\mathring{A}, \mathring{B}$ indices for further clarity.

The set of gammas with $\{,\}$ as the multiplication (anti-commutator i.e., $\{a, b\} = ab + ba$) generate a **Clifford Algebra** which is used to define the spinor representation of the Lorentz group. The Dirac matrices are not uniquely determined by their anti-commutation relations.

It is possible to find other, equivalent representations. However, Pauli showed that if two sets of matrices γ_μ and γ'_μ satisfy the anti-commutation relations and the Hermiticity relations of the gamma matrices as shown above, then there is a unitary transformation U on $\mathbb{C}^4$ such that the following relation holds.

$$\gamma'_\mu = U^{-1}\gamma_\mu U$$

As we can see in equation (7.1) above, $\eta_{\mu\nu}$ is the Minkowski metric, relates spacetime to the gamma matrices. The gamma matrices are constructed from the Pauli matrices as follows:

$$\gamma_0 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \quad \gamma_j = \begin{bmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{bmatrix}, j \in \{1, 2, 3\} \tag{7.2}$$

This is often called the Weyl or Chiral representation of the gamma matrices. The σ_i's above represent the Pauli matrices in quantum mechanics. The Pauli matrices are repeated below for clarity.

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7.3}$$

Additionally, there are some valuable relations of the gamma matrices to each other. These relations are derived from the structure of the Pauli matrices. The first few relations are given by the hermitian adjoints, †, of the gamma matrices. The next several relations give us how contractions of the gamma matrices act, and simplify to much simpler objects. The second to last relation gives us how three gamma matrices act on each other to give back $\eta_{\mu\nu}$'s. The last relations show what the anti commutator & commutator of the skew symmetrized product of the gamma matrices becomes.

$$\gamma_0^\dagger = \gamma_0, \gamma_j^\dagger = -\gamma_j, \gamma_0^\dagger = \gamma_0^{-1}, \gamma_j^\dagger = \gamma_j^{-1}$$

$$\gamma^\mu \gamma_\mu = 4I_4, \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma_\mu = -2\gamma^\rho \gamma^\lambda \gamma^\nu \tag{7.4}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4\eta^{\nu\lambda}I_4 \tag{7.5}$$

$$\gamma^\mu\gamma^\nu\gamma^\lambda = \gamma^{[\mu}\gamma^\nu\gamma^{\lambda]} + \eta^{\mu\nu}\gamma^\lambda + \eta^{\nu\lambda}\gamma^\mu - \eta^{\mu\lambda}\gamma^\nu \tag{7.6}$$

$$\left\{\gamma^\mu, \gamma^{[\nu}\gamma^{\rho]}\right\} = 2\gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \tag{7.7}$$

$$\left[\gamma^\mu, \gamma^{[\nu}\gamma^{\rho]}\right] = 4\eta^{\mu[\nu}\gamma^{\rho]} \tag{7.8}$$

Moving forward, when we do relativistic quantum mechanics, we need an additional matrix to correctly reflect the addition of time t into the geometry; think Lorentzian boosts. This additional matrix, which we will call σ_0, is the 2×2 identity matrix. Recall that this same idea leads to the Infeld-Van der Waerden symbols used in Stewart (Stewart, 1993). We get something that makes more intuitive sense if we write γ_0 in terms of σ_0 as well.

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma_0 = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$$

7.1.3 The Infeld-Van Der Waerden symbols, and the two-component spinor/Dirac spinor correspondence

Now we have each of the γ's written in terms of the σ's. Each of the sigmas: $\sigma_0, \ldots, \sigma_3$ correspond to $\sigma_t, \ldots, \sigma_z$ etc. On a bit of a side note which will be explored later, the set of these four σ's can be used to create a useful correspondence to two-component spinors. These objects are called the Infeld-Van Der Waerden symbols ${\sigma_a}^{AA'}$ (the capital indices label the entries of the matrix like below) where the line element is given as the determinant of a combination of the gamma matrices.

$$dx^\alpha\sigma_\alpha = dt\sigma_t + dx\sigma_x + dy\sigma_y + dz\sigma_z = \begin{bmatrix} dt + dz & dx - idy \\ dx + idy & dt - dz \end{bmatrix}$$

$$\begin{aligned} \det\left(dx^{\alpha}\sigma_{\alpha}\right) &= (dt+dz)\,(dt-dz)-(dx-idy)\,(dx+idy) \\ ds^2 &= dt^2-dx^2-dy^2-dz^2 \end{aligned}$$

More generally on a manifold we use an orthonormal frame $\left\{e^{a}{}_{\mu},\ldots\right\}$ in 4D. This is different than usual, but let $e_a{}^{\mu}$ be given as a list indexed by a of 1-forms indexed by μ; here we have swapped our usual notation to make a point that we can do this in any orthonormal frame; the Θ's form an orthonormal dual basis. Then we have the same relation as above, but in an orthonormal frame.

$$e^{a}{}_{\mu} = \begin{bmatrix} \Theta_0 & \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix}$$

$$e^{a}{}_{\mu}\sigma_a{}^{AA'} = \begin{bmatrix} \Theta_0+\Theta_3 & \Theta_1-i\Theta_2 \\ \Theta_1+i\Theta_2 & \Theta_0-\Theta_3 \end{bmatrix}$$

$$\begin{aligned} \det\left(e_a{}^{\mu}\sigma_{\mu}{}^{AA'}\right) &= (\Theta_0+\Theta_3)\otimes(\Theta_0-\Theta_3)-(\Theta_1-i\Theta_2)\otimes(\Theta_1+i\Theta_2) \\ g_{ab} &= \Theta_0\otimes\Theta_0-\Theta_1\otimes\Theta_1-\Theta_2\otimes\Theta_2-\Theta_3\otimes\Theta_3 \end{aligned}$$

Next we introduce a useful quantity, γ^5. γ^5 allows us to make a projection from our Dirac spinors to 2-component spinors; we will show how in a moment. First γ^5 is defined in terms of the other gamma matrices, see equation (7.9); the matrix representation of γ^5 is given below as well.

$$\gamma^5 = \frac{i}{24}e_{\mu\nu\lambda\rho}\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\rho} = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{7.9}$$

$$\gamma^5 \doteq \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$$

γ^5 has some unique properties in terms of the anti-commutator, and the derivative D_a. The first is that the anti-commutator of γ^5 with any of the other gamma matrices is zero. The next is that the square of γ^5 is the identity matrix. The penultimate one can be derived from the anti-commutator relation and lets us move γ^5 through the other γ's. The last is that the derivative of γ^5 is zero.

$$\left\{\gamma^\mu, \gamma^5\right\} = 0, \quad \left(\gamma^5\right)^2 = I_4, \quad \gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu, \quad D_a \gamma^5 = 0$$

We can now define projections to make 2 component spinors by using γ^5. These projections ψ_L, and ψ_R project us to 2-component spinors of left and likewise right chirality (handedness).

$$\psi_L = \frac{I_4 - \gamma^5}{2}\psi, \quad \psi_R = \frac{I_4 + \gamma^5}{2} \tag{7.10}$$

Now we move on to the spacetime gamma matrices and finding a representation of the Lorentz group on spinors.

7.1.4 Spacetime gamma matrices and a representation of the Lorentz group on Dirac spinors

So far, we have not included gravity in our formalism for Dirac spinors. We can begin to include gravity with a simple addition to the gamma matrices: $e^a{}_\mu$. By using the vielbein to convert our orthonormal Greek indices to abstract Latin indices, we can couple gravity in easily.

$$\gamma_a = e_a{}^\mu \gamma_\mu \tag{7.11}$$

As a reminder, we know the vielbein satisfies the orthonormality relation $\eta_{\mu\nu} e_a{}^\mu e_b{}^\nu = g_{ab}$, where g_{ab} is the metric on spacetime. γ_a are the spacetime dependent gamma matrices, and now satisfy the new Clifford algebra relation, where the metric g replaces η. This also changes the matrix representation for γ_a:

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2 g_{ab} I_4 \tag{7.12}$$

We get the spacetime-Clifford algebra relation from writing γ_μ in terms of γ_a in equation (7.1). Next we need to look at how spinors transform under Lorentz transformations. We really want to know how the $\mathring{A}$ index of the spinor transforms. It would be something like: $\psi^{\mathring{A}} \to L^{\mathring{A}}{}_{\mathring{B}}\psi^{\mathring{B}}$. But how do we determine $L^{\mathring{A}}{}_{\mathring{B}}$? This question is asking about the spinor representation of the Lorentz group, which we need to determine. We know that infinitesimal Lorentz transformations are given by $\omega^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon\lambda^{\mu}{}_{\nu}$. ϵ is some small parameter, which we would say for good measure is three orders of magnitude smaller than the smallest value in the relevant equation being transformed. Here we use $\omega^{\mu}{}_{\nu}$ to symbolize only infinitesimal Lorentz transformations. These $\omega^{\mu}{}_{\nu}$ objects form a one parameter subgroup of the Lorentz group when Taylor expanded to first order in the parameter ϵ.

A side note on Lorentz generators:

The λ's are the generators of the Lorentz group, and satisfy $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$; note that the standard matrix representation of the generators is written from $\lambda^{\mu}{}_{\nu}$ not $\lambda_{\mu\nu}$. As a clarifying example we write a t-x boost generator $\lambda^{\mu}{}_{\nu}$ as the 4×4 matrix below.

$$\lambda^{\mu}{}_{\nu} \doteq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We write the λ generators in this form because it forms a basis. This basis is a basis for the Lorentz Lie algebra using antisymmetric matrices for the vector space, and the matrix commutator as the algebra structure.

To provide clarity, we could technically give the λ's an abstract Lie algebra index $\mathfrak{a}$ like $\lambda^{\mathfrak{a}\mu}{}_{\nu}$, this is the most transparent notation for what these objects actually are. However, if we write this $\mathfrak{a}$ index as $[\mathfrak{bc}]$, an anti-symmetric pair, we are able to write some formulas which also give intuitive understanding to our group; pedagogically, we can think of the pair as labeling rotations and boosts in some orthonormal frame, i.e., "$[tx]$" labels one element etc. The first formula is given by:

$$\lambda^{[\mathfrak{ab}]\mu\nu} = \eta^{\mathfrak{a}\mu}\eta^{\mathfrak{b}\nu} - \eta^{\mathfrak{b}\mu}\eta^{\mathfrak{a}\nu}$$

If we lower in index, we find that the above formula reproduces the 4×4 matrix representation of the generators.

$$\lambda^{[\mathfrak{12}]\mu}{}_{\nu} \doteq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Lie bracket of the generators (with proper μ, ν indices on the λ's of the right-hand side) is given by:

$$\left[\lambda^{[\mathfrak{ab}]}, \lambda^{[\mathfrak{cd}]}\right] = \left(\lambda^{\mathfrak{db}}\eta^{\mathfrak{ac}} + \lambda^{\mathfrak{ad}}\eta^{\mathfrak{bc}} + \lambda^{\mathfrak{bc}}\eta^{\mathfrak{ad}} + \lambda^{\mathfrak{ca}}\eta^{\mathfrak{bd}}\right) \tag{7.13}$$

Any representation of the Lorentz algebra must satisfy (7.13). We can use this idea to determine a representation of the Lorentz group for our Dirac spinors. If we consider the commutator of the gamma matrices, call it $S_{\mu\nu}$, it turns out we can find a representation of the Lorentz group on our Dirac spinors. This $S_{\mu\nu}$ object is a set of 6 independent linear transformations on $\mathbb{C}^4$. We have also included what this looks like with Dirac indices.

$$S_{\mu\nu} = \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right] \tag{7.14}$$

$$S_{\mu\nu}{}^{\mathring{A}}{}_{\mathring{B}} = \frac{1}{4}\left(\gamma_{\mu}{}^{\mathring{A}}{}_{\mathring{C}}\gamma_{\nu}{}^{\mathring{C}}{}_{\mathring{B}} - \gamma_{\nu}{}^{\mathring{A}}{}_{\mathring{C}}\gamma_{\mu}{}^{\mathring{C}}{}_{\mathring{B}}\right)$$

Using the anti-commutator, we can simplify this:

$$S_{\mu\nu} = \frac{1}{2}\left(\gamma_\mu\gamma_\nu - \eta_{\mu\nu}I_4\right)$$

To show that $S_{\mu\nu}$ defines a representation of the Lorentz group, we need to show that $\left[S^{\mu\nu}, S^{\alpha\beta}\right]$ satisfies the same bracket relation as the λ's do. Before we show this, it is advantageous to examine $[S^{\mu\nu}, \gamma^\alpha]$.

$$\left[S^{\mu\nu}, \gamma^\alpha\right] = \frac{1}{2}\left(\gamma^\mu\eta^{\nu\alpha} - \eta^{\alpha\mu}\gamma^\nu\right)$$

With this identity, we can now show that $S^{\mu\nu}$ satisfies the Lorentz generator relations, and is a spinor representation of the Lorentz group. It is shown below for convenience and can be derived from the definition of $S_{\mu\nu}$ and the identity above.

$$\left[S^{\mu\nu}, S^{\alpha\beta}\right] = S^{\mu\beta}\eta^{\nu\alpha} + S^{\beta\nu}\eta^{\alpha\mu} + S^{\alpha\mu}\eta^{\nu\beta} + S^{\nu\alpha}\eta^{\beta\mu}$$

Then given any Lorentz generator (any linear combination of the $\lambda^{\mu}{}_{\nu}$'s) we can define the representative $S(\lambda)$ or $L^{\mathring{A}}{}_{\mathring{B}}$ on $\mathbb{C}^4$ by equation (7.15).

$$l = S(\lambda) = \frac{1}{2}\lambda^{\mu\nu}S_{\mu\nu}$$

$$l^{\mathring{A}}{}_{\mathring{B}} = \frac{1}{2}\lambda^{\mu\nu}S_{\mu\nu}{}^{\mathring{A}}{}_{\mathring{B}} \tag{7.15}$$

Note that equation (7.15) cannot be inverted to solve for $S_{\mu\nu}$ because $S_{\mu\nu}$ is defined by a commutator and the trace of a commutator is zero. Now that we know what l is, we can get the corresponding Lorentz group transformation which acts on spinors from our infinitesimal one. To do so, we take the matrix exponential e^{lt}, with t the parameter, of the generator l; we will write this Lorentz group transformation on spinors as $L^{\mathring{A}}{}_{\mathring{B}} = e^{\left(l^{\mathring{A}}{}_{\mathring{B}}t\right)}$. We have a projective spinor representation of both the Lorentz generators and the Lorentz group. These l constitute the generators of the Lorentz group in terms of the group's spinor representation on $\mathbb{C}^4$.

We can use these l's to define infinitesimal Lorentz transformations on spinors, call them $\mathfrak{L}$, by: $\mathfrak{L}^{\mathring{A}}{}_{\mathring{B}} = \delta^{\mathring{A}}{}_{\mathring{B}} + \epsilon l^{\mathring{A}}{}_{\mathring{B}}$.

To make an analogy with what we said before, we know that $\mathfrak{L}^{\mathring{A}}{}_{\mathring{B}}$ is a one parameter subgroup of the Lorentz group up to a first order Taylor expansion in ϵ. From all this information, we can determine how our spinors transform in general under a Lorentz transformation. The Lorentz group acts on ψ from the left, and $\overline{\psi}$ as an inverse from the right.

$$\psi' \to L\psi, \quad \overline{\psi}' = \overline{\psi}L^{-1} \tag{7.16}$$

$$\left(\psi^{\mathring{A}}\right)' \to L^{\mathring{A}}{}_{\mathring{B}}\psi^{\mathring{B}}, \quad \left(\overline{\psi}_{\mathring{A}}\right)' = \overline{\psi}_{\mathring{B}}\left(L^{-1}\right)^{\mathring{B}}{}_{\mathring{A}}$$

There is some ambuiguity however as to why $\overline{\psi}_{\mathring{A}}$ transforms as $\overline{\psi}L^{-1}$ however. We came up with $\overline{\psi} = \psi^{\dagger}\gamma^{0}$ because we needed an object which transforms as $\overline{\psi}L^{-1}$. Initially, we may have started by guessing that $\psi^{\dagger}$ transforms in this way. It however does not, and transforms as follows:

$$\left(\psi'\right)^{\dagger} = (L\psi)^{\dagger} = \psi^{\dagger}L^{\dagger}$$

We know that $L^{\dagger}$ is not necessarily equal to L^{-1}. There are some identities however, which facilitate how we determine that $\overline{\psi} = \psi^{\dagger}\gamma^{0}$. The first of which is the relation which expresses hermitian adjoints in terms of the original gamma matrix and γ^{0}'s.

$$(\gamma^{\mu})^{\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0}$$

This can be proven easily with some component manipulation due to the properties of the Pauli matrices; see also Poplawski pg. 111 for comments (Poplawski, 2013). With that identity we can now prove our next relation, which shows how γ^{0} can turn $L^{\dagger}$ into L^{-1}, see equation (7.17) below.

$$L^{\dagger}\gamma^{0} = \gamma^{0}L^{-1} \tag{7.17}$$

The relation above can be realized by applying the γ^{0} formula and noting that $\mathfrak{L}^{-1} = I_4 - \epsilon\frac{1}{8}\lambda^{\mu\nu}\left(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}\right)$; we work out the transformation using an infinitesimal Lorentz transformation. With all this, we can prove that $\overline{\psi}$ transforms with L^{-1}. When we let

$\psi' \to L\psi$, we find:

$$\begin{aligned} \overline{\psi}' &= \left(\psi^{\dagger}\gamma^{0}\right)' \\ &= \left(\psi'\right)^{\dagger}\left(\gamma^{0}\right)' \\ &= \left(L\psi\right)^{\dagger}\gamma^{0} \\ &= \psi^{\dagger}L^{\dagger}\gamma^{0} \\ &= \psi^{\dagger}\gamma^{0}L^{-1} \\ &= \overline{\psi}L^{-1} \end{aligned}$$

Now we can examine how the gamma matrices transform under a Lorentz transformation. There are two interpretations or viewpoints we can take regarding how the gamma matrices transform; one is they do not transform, and the other is that they do. The transformation properties of the gamma matrices are tricky to dissect because they have two spinor indices and one spacetime/orthonormal index. We will examine how to interpret this by going through an example in quantum mechanics.

Recall states $|\psi\rangle$ in quantum mechanics and the Pauli matrices σ_i. We transform the states and not the matrix operators themselves; there is no $\hat{x}'$, only $\hat{x}$. Here is the example, we take a state $|\psi\rangle$ and say we want to calculate the expected angular momentum in the x-direction for this state; this is given by $\langle\psi|\,\hat{J}_x\,|\psi\rangle$; $\hat{J}_x$ is the x-direction angular momentum operator. If we rotate the state around the z-axis first we find that although we rotated the states $|\psi\rangle$, that we could interpret this transformation as the operator transforming and the states not transforming, as in line two below.

$$\begin{aligned} \left\langle\psi'\right|J_x\left|\psi'\right\rangle &= \langle\psi|\,e^{i\sigma_z\frac{\theta}{2}}J_x e^{-i\sigma_z\frac{\theta}{2}}\,|\psi\rangle \\ &= \langle\psi|\left(\cos\left(\theta\right)J_x - \sin\left(\theta\right)J_y\right)|\psi\rangle \\ &= \cos\left(\theta\right)\langle J_x\rangle - \sin\left(\theta\right)\langle J_y\rangle \end{aligned}$$

Note that $\langle\psi|\,J_x\,|\psi\rangle = \langle J_x\rangle$. "It's as if we left our particle alone and rotated the Pauli matrices. But note that if we apply the rotation to $|\psi\rangle$, then we don't touch the matrices. Also, we never say that we transformed the matrices. We just transformed the state, and then found out that we could leave it alone and rotate the matrices. The situation for a Dirac spinor is similar" (Javier, 2015). This example can also be found in Sakurai (Sakurai & Napolitano, 2021) on pg. In analogue to the classical quantum mechanics approach, we similarly take the approach that the gamma matrices "don't transform" we only transform the "states" (spinors) like before.

When we write $\overline{\psi}\gamma^{\mu}\psi$, we would like for it to be a vector. Then it must transform as a vector as in equation (7.18) below.

$$\left(\overline{\psi}\gamma^{\mu}\psi\right)' = \Lambda^{\mu}{}_{\nu}\overline{\psi}\gamma^{\nu}\psi \tag{7.18}$$

Likewise, if we perform a Lorentz transformation in analogue to classical quantum mechanics, we only transform the states (the gamma matrices can be interpreted as an operator in this sense." This leads to the relation in equation (7.19).

$$\left(\widetilde{\overline{\psi}\gamma^{\mu}\psi}\right) = \widetilde{\overline{\psi}}\gamma^{\mu}\widetilde{\psi} = \overline{\psi}L^{-1}\gamma^{\mu}L\psi \tag{7.19}$$

To have consistency in how we transform, we require that $\left(\overline{\psi}\gamma^{\mu}\psi\right)' = \left(\widetilde{\overline{\psi}\gamma^{\mu}\psi}\right)$, which results in relation (7.20).

$$\Lambda^{\mu}{}_{\nu}\gamma^{\nu} = L^{-1}\gamma^{\mu}L \tag{7.20}$$

Like before, we can find similarities to the "States do not transform, but operators do" interpretation by moving the L's to the other side of the equation, resulting in equation (7.21). Although this looks like a transformation, it is not because of the lack of primes; this is important.

$$\Lambda^{\mu}{}_{\nu}L\gamma^{\nu}L^{-1} = \gamma^{\mu} \tag{7.21}$$

We now have all the tools we need to calculate Lorentz transformations of spinors.

7.2 Derivatives of Dirac Spinors & the Spin Connection

Since we have finished with Dirac spinor algebra, we now move onto Dirac spinor analysis and examine how derivatives of spinors transform. We will begin with the coordinate partial derivative, and show how this derivative acting on a spinor fails to transform as a spinor. Because of the failure of this derivative to transform the way a spinor does, we introduce the Dirac spinor connection Γ_a, or $\Gamma^{\mathring{A}}{}_{\mathring{B}a}$. With this object we can define a spinor covariant derivative which indeed transforms the way a normal spinor does. Furthermore, we develop a formula for this spin connection in terms of the gamma matrices γ_μ, the spin connection $\omega^\mu{}_{\nu a}$, and an arbitrary form A_a. What is called the Dirac spinor connection here is the connection form of the Fock-Ivaneko coefficients. We finish this section with some notes on how Dirac spinor connections differ if there is torsion present in the spacetime, and how to break off the torsion-full part of this connection as well.

7.2.1 The Covariant Derivative of a spinor ψ, and a spinor $\overline{\psi}$

To begin we examine the derivative of a Dirac spinor. We quickly see that this derivative does not transform as a spinor.

$$\partial_\mu \psi' = \partial_\mu \left(L\psi\right) = \partial_\mu L\psi + L\partial_\mu \psi$$

This is a problem, but we can rectify it with the Dirac spinor covariant derivative. To build a spinor covariant object, we need a connection on the Dirac spinor bundle. We introduce this connection as Γ_a, or $\Gamma^{\mathring{A}}{}_{\mathring{B}a}$. Γ_a transforms according to the usual connection law given in equation (7.22).

$$\left(\Gamma'\right)_a = L\Gamma_a L^{-1} + \partial_a L L^{-1} \tag{7.22}$$

$$\left(\Gamma'\right)^A{}_{Ba} = L^A{}_C \Gamma^C{}_{Da} \left(L^{-1}\right)^D{}_B + \partial_a L^A{}_C \left(L^{-1}\right)^C{}_B$$

This is exactly the way any connection transforms under a group transformation (in our case we use the Lorentz group). Recall that this is exactly the same way a connection on a principal fiber bundle transforms. Following this idea, we can deduce that the covariant derivative of a spinor transforms the same way a spinor would. The ∇ we use here is the ∇ of Wald. (Wald, 1984)

$$\begin{aligned}\nabla_a \psi' &= \partial_a \psi' - \left(\Gamma'\right)_a \psi' \\ &= \left(\partial_a L \psi + L \partial_a \psi\right) - \left(L \Gamma_a L^{-1} + \partial_a L L^{-1}\right) (L\psi) \\ &= \partial_a L \psi + L \partial_a \psi - L \Gamma_a \psi - \partial_a L \psi \\ &= L \left(\partial_a \psi - \Gamma_a \psi\right) \\ &= L \nabla_a \psi\end{aligned}$$

$$\nabla_a \psi = \partial_a \psi - \Gamma_a \psi \tag{7.23}$$

We can also examine $\overline{\psi}'$ which we can derive the transformation from by the fact that $\overline{\psi}\psi$ must be a scalar. We begin by assuming the following form for how $\overline{\psi}$ transforms.

$$\nabla_a \overline{\psi}' = \partial_a \overline{\psi}' - \overline{\psi}' \widetilde{\Gamma}'_a$$

$\widetilde{\Gamma}_a$ is a new connection to be determined in terms of Γ_a. After shoving this into $\nabla_a \left(\overline{\psi}' \psi'\right)$ and solving for $\widetilde{\Gamma}'_a$, we find that:

$$\widetilde{\Gamma}'_a = -\left(\Gamma'\right)_a$$

Which tells us about the covariant derivative of the bar spinor in terms of the connection on the non-bar spinor.:

$$\nabla_a \overline{\psi} = \partial_a \overline{\psi} + \overline{\psi} \Gamma_a \tag{7.24}$$

7.3 The Gamma Matrices and the Fock-Ivaneko Coefficients

Next we examine the covariant derivative of the gamma matrices. This becomes more interesting than before because of the addition of the orthonormal index μ. Taking the covariant derivative of the gamma matrices gives the relation in equation (7.25).

$$\nabla_a \gamma_\mu = \cancelto{0}{\partial_a \gamma_\mu} - \Gamma_a \gamma_\mu + \gamma_\mu \Gamma_a$$

$$\nabla_a \gamma_\mu = -2 \left[\Gamma_a, \gamma_\mu\right] \tag{7.25}$$

The nabla derivative of Wald does not see the orthonormal indices. Equation (7.25) by itself gives us no additional information on how to determine what Γ_a is, nevertheless it is useful conceptually. To get any more information we need to use the universal covariant derivative which we will call D; this is the same covariant derivative that Carroll (Carroll, 1997) uses, and it is different from Wald's (Wald, 1984); D sees both abstract and orthonormal indices. Recall that D is defined by leaving the tetrad covariantly constant (Tetrad Postulate).

$$D_a {e^b}_\mu = 0 \tag{7.26}$$

In the case of ONLY the Dirac spinors ψ and $\overline{\psi}$ and scalars we know that ∇ and D are the same because they do not contain any orthonormal indices.

$$D_a \psi = \nabla_a \psi, \quad D_a \overline{\psi} = \nabla_a \overline{\psi}$$

One important result from this is that we can use the spin connection to see how orthonormal indices vary. We can see the need for this in multiple ways. The first is when we take the D covariant derivative of the gamma matrices. With this derivative we get the spin connection contribution ${\omega^\nu}_{\mu a}$ which in turn causes this whole formula to be zero.

To clarify we have three different connections floating around. These are sections of a bundle which we label with three different indices, lowercase Latin, Greek, and uppercase Latin with a circle above: a, μ, $\mathring{A}$. The first represents the connection for vector fields determined by $\nabla g_{ab} = 0$. The next is the connection determined by the orthonormal frame field $\nabla {e^a}_\mu = 0$. The last is the connection for spinors determined from $\nabla \gamma = 0$. All of these connections tell us the same information, but in different parts of the whole fiber bundle.

$$D_a \gamma_\mu = -{\omega^\nu}_{\mu a} \gamma_\nu - 2 \left[\Gamma_a, \gamma_\mu\right] = 0$$

Since γ_μ is written purely in terms of $\eta_{\mu\nu}$ and Dirac pieces, we can say $D_a \gamma_\mu = 0$. This small idea originally was incredibly confusing, but what makes it coherent is the fact that the spinor indices on $[\gamma_\mu]^{\mathring{A}}{}_{\mathring{B}}$ get hit along with the orthonormal μ index. The combination of these both together with their respective connection pieces becomes zero; this can be shown in maple after making some connections; additionally we can see this from the fact that the gamma matrices need to be covariantly constant, otherwise the spinors would not be coupled to spacetime correctly. One other way we can think of this, is it had better be zero or else we are no longer generally covariant. Moving on, once we multiply this relation by a γ^μ from the right we find a relation that gives us Γ_a in terms of the spin connection and the gamma matrices.

$$D_a \gamma_\mu \gamma^\mu = 0 = -\omega_{\mu\nu a}\gamma^\mu\gamma^\nu - 4\Gamma_a + \gamma_\mu \Gamma_a \gamma^\mu \tag{7.27}$$

To solve this for Γ_a we try a solution which is the spin connection term plus some other piece A_a, or $A_a{}^{\mathring{A}}{}_{\mathring{B}}$; this is equation (7.28). Applying this assumption to equation (7.27), we find that A_a must be some form V_a times the identity matrix.

$$\Gamma_a = -\frac{1}{4}\omega_{\mu\nu a}\gamma^\mu\gamma^\nu - A_a \tag{7.28}$$

$$A_a = V_a I_4, \quad V_a \in T^*(M)$$

Therefore, up to an arbitrary form V_a we have a calculatable form for the Dirac connection given in equation (7.29). This object is usually called the **Fock-Ivaneko coefficients**, and acts as a connection on the Dirac spinor bundle.

$$\Gamma_a = -\frac{1}{4}\omega_{\mu\nu a}\gamma^\mu\gamma^\nu \tag{7.29}$$

$$\nabla_a \psi = \partial_a \psi + \frac{1}{4}\omega_{\mu\nu a}\gamma^\mu\gamma^\nu\psi, \quad \nabla_a\overline{\psi} = \partial_a\overline{\psi} - \frac{1}{4}\omega_{\mu\nu a}\overline{\psi}\gamma^\mu\gamma^\nu$$

7.3.1 Dirac Connection with Torsion

The formula we had for the Fock-Ivaneko coefficients in general includes torsion. Since the difference of any 2 connections is a tensor, we can do our usual trick and find the Levi-Civita generated Fock-Ivaneko coefficients and break them off with a contorsion piece. To do this, since we already know the spin connection, we can just break off the Levi-Civita piece from Γ_a and we will already have our solution. Since we know $\omega_{\mu\nu a} = \varpi_{\mu\nu a} + e^b{}_\mu e^c{}_\nu C_{bca}$, where $\varpi_{\mu\nu a}$ is the Levi-Civita generated Fock-Ivaneko coefficients, and C_{bca} is the contorsion tensor, we can get the following formula for the affine Dirac connection in terms of the Levi-Civita Dirac connection minus the contorsion tensor.

$$\Gamma_a = \Gamma_a \{\} - \frac{1}{4} C_{bca} \gamma^b \gamma^c \tag{7.30}$$

$$\Gamma_a \{\} = -\frac{1}{4} \varpi_{\mu\nu a} \gamma^\mu \gamma^\nu$$

Finally, we can apply this principle to the derivatives of our Dirac spinors, and likewise break off the contorsion from them. This results in the following two equations where $\widetilde{\nabla}$ is the Levi-Civita covariant derivative.

$$\nabla_a \psi = \widetilde{\nabla}_a \psi + \frac{1}{4} C_{bca} \gamma^{[b} \gamma^{c]} \psi, \quad \nabla_a \overline{\psi} = \widetilde{\nabla}_a \overline{\psi} - \frac{1}{4} C_{bca} \overline{\psi} \gamma^{[b} \gamma^{c]}$$

With our new connection, we are ready to move onto what the curvature and torsion spinors look like in the Dirac formalism.

7.4 The Curvature Spinor

In this section, we will delve into the curvature and torsion of our connection of interest, represented by Γ_a. The torsion tensor, as described by Cartan, is only present on the tangent bundle, whereas the curvature can exist with indices in the fibers. To understand the concept of a connection on the spin bundle better, we calculate its curvature. We will apply the curvature operator to a spinor ψ^A and observe the outcome. This generates the curvature spinor which we label K_{ab} or $K^{\mathring{A}}{}_{\mathring{B}cd}$. A detailed explanation is provided below.

Since we have a connection, we can ask about its curvature and torsion, which is what we will do in this section. The connection of interest will be Γ_a. The torsion tensor only exists as an object on the tangent bundle from a Cartan viewpoint (Sharpe, 1997), but the curvature can exist with indices in the fibers; this is what will happen in the case of Γ_a. We will apply the curvature operator to a spinor ψ^A and see what happens as a result. This will generate the curvature spinor K_{ab}, or $K^A{}_{Bcd}$. A derivation is given below.

$$\begin{aligned}
\left(\nabla_a\nabla_b - \nabla_b\nabla_a + T^d{}_{ab}\nabla_d\right)\psi &= \nabla_a\nabla_b\psi - \nabla_b\nabla_a\psi + T^d{}_{ab}\nabla_d\psi \\
&= \partial_a\left(\nabla_b\psi\right) - \Gamma^c{}_{ba}\nabla_c\psi - \Gamma_a\nabla_b\psi \\
&+ -\partial_b\left(\nabla_a\psi\right) + \Gamma^c{}_{ab}\nabla_c\psi + \Gamma_b\nabla_a\psi \\
&+ T^d{}_{ab}\nabla_d\psi \\
&= \partial_a\left(\partial_b\psi - \Gamma_b\psi\right) - \Gamma_a\left(\partial_b\psi - \Gamma_b\psi\right) \\
&+ -\partial_b\left(\partial_a\psi - \Gamma_a\psi\right) + \Gamma_b\left(\partial_a\psi - \Gamma_a\psi\right) \\
&= -\partial_a\Gamma_b\psi + \partial_b\Gamma_a\psi + \Gamma_a\Gamma_b\psi - \Gamma_b\Gamma_a\psi \\
&= \Gamma_{a,b}\psi - \Gamma_{b,a}\psi + \Gamma_a\Gamma_b\psi - \Gamma_b\Gamma_a\psi \\
&= K_{ab}\psi
\end{aligned}$$

We now have an expression for the curvature spinor in terms of the Dirac connection. This is given in equation (7.31).

$$K_{ab} = \Gamma_{a,b} - \Gamma_{b,a} + \left[\Gamma_a, \Gamma_b\right] \tag{7.31}$$

$$K^{\mathring{C}}{}_{\mathring{D}ab} = \Gamma^{\mathring{C}}{}_{\mathring{D}a,b} - \Gamma^{\mathring{C}}{}_{\mathring{D}b,a} + \Gamma^{\mathring{C}}{}_{\mathring{E}a}\Gamma^{\mathring{E}}{}_{\mathring{D}b} - \Gamma^{\mathring{C}}{}_{\mathring{E}b}\Gamma^{\mathring{E}}{}_{\mathring{D}a}$$

There is a way to get the curvature spinor straight from the curvature tensor $R^a{}_{bcd}$ by using the gamma matrices. Recall that $D_a\gamma_b = 0$. By using this idea and looking at $D_{[a}D_{b]}\gamma_c = 0$, we can find a similar formula relating K_{ab} and $R^a{}_{bcd}$. Recall the Ricci Identity:

$$\left(\nabla_a\nabla_b - \nabla_b\nabla_a\right)v^c = R^c{}_{dab}v^d - T^f{}_{ab}\nabla_f v^c \tag{7.32}$$

If we apply this to γ^c we find an equation which relates the curvature tensor to the curvature spinor.

$$R_{cdab}\gamma^c\gamma^d + \gamma^c K_{ab}\gamma_c - 4K_{ab} = 0 \tag{7.33}$$

We can solve this equation for K_{ab} by making an assumption that Kibble (Kibble, 1961) does. Following Poplawski (Poplawski, 2013), who makes this assumption, we try a solution which has curvature pieces along with an arbitrary spinor-tensor B_{ab} where $B_{ab} \equiv B^{\mathring{A}}{}_{\mathring{B}ab}$.

$$K_{ab} = \frac{1}{4}R_{cdab}\gamma^c\gamma^d + B_{ab} \tag{7.34}$$

Upon substituting this assumption into our equation relating $\boldsymbol{R}$ and $\boldsymbol{K}$, see equation (7.33), we then find that B_{ab} has to be a skew-symmetric tensor $\mathtt{B}_{[ab]}$ censored to the Dirac indexed Kronecker delta tensor. Finding this relation requires the triple gamma matrix relation in (7.5).

$$B^{\mathring{A}}{}_{\mathring{B}cd} = \mathtt{B}_{[cd]}\delta^{\mathring{A}}_{\mathring{B}}$$

We can relate B_{ab} back to our $A_a = V_a I_4$ which we used in the Fock-Ivaneko coefficients by looking at our formula for K_{ab}. If we put in the A_a here, we can find a formula for B_{ab} as well. Let $\Gamma'_a = \Gamma_a - A_a$, then $K'_{ab} = K_{ab} + B_{ab}$. We use the minus here because that is how A_a originally appeared; this in turn makes the B_{ab} term appear with a plus sign in front.

$$K'_{ab} = \Gamma'_{a,b} - \Gamma'_{b,a} + 2\left[\Gamma'_a, \Gamma'_b\right]$$

$$K_{ab} + B_{ab} = \Gamma_{a,b} - A_{a,b} - \Gamma_{b,a} + A_{b,a} + 2\left[\Gamma_a - A_a, \Gamma_b - A_b\right]$$

Since A_a is a vector times the identity, we can simplify the bracket term:

$$\begin{aligned}
[\Gamma_a - A_a, \Gamma_b - A_b] &= [\Gamma_a, \Gamma_b - A_b] - [A_a, \Gamma_b - A_b] \\
&= [\Gamma_a, \Gamma_b] - [\Gamma_a, A_b] - [A_a, \Gamma_b] + [A_a, A_b] \\
&= [\Gamma_a, \Gamma_b]
\end{aligned}$$

And now we have the formula relating our B_{ab} tensor to A_a. Additionally, we can relate $\mathtt{B}_{ab}$ to V_a as well with the relations from before.

$$B_{ab} = -A_{a,b} + A_{b,a}, \quad \mathtt{B}_{ab} = -2V_{[a,b]}$$

If we set $A_a = 0$ as in the Fock-Ivaneko case, then likewise $B_{ab} = 0$ for our curvature spinor. Thus, we find a formula for the curvature spinor in terms of the curvature tensor.

$$K_{ab} = \frac{1}{4} R_{cdab} \gamma^c \gamma^d \tag{7.35}$$

Now since we have all the connection and multilinear algebra pieces well established for Dirac spinors, we can move onto the generally covariant Dirac action for ECSK and consider its variation with respect to various fields.

7.5 Dirac Action and Field Equations

The Dirac action with a mass term is given in the form of equation (7.36).

$$S = \int\limits_{\Omega} \left[\frac{i\hbar c}{2} \left(\overline{\psi} \gamma^a D_a \psi - D_a \overline{\psi} \gamma^a \psi \right) - mc^2 \overline{\psi} \psi \right] e d^4 x \tag{7.36}$$

Recall that $\hbar$ has units of Js. Then $\hbar c$ has units of Jm, and since D_a adds units of $\frac{1}{m}$, we then have the units of Js which we need for our action. Now it is time to vary the action. We will vary it with respect to the tetrad $e^a{}_\mu$, the spin connection $\omega^\mu{}_{\nu a}$, and the Dirac fields Ψ, and $\overline{\Psi}$. From Poplawski (Poplawski, 2013) there is an equivalence between varying the spin connection and varying the contorsion tensor; this can also been seen in Sciama, and Kibble's papers (Kibble, 1961), (Sciama, 1962).

First, we need to write the spacetime gamma matrices in terms of orthonormal gamma matrices i.e., $\gamma^a = e^a{}_\mu \gamma^\mu$. Then we must also decompose the Fock-Ivaneko coefficients in the connection into a Levi-Civita generated part and a torsion part. This step is important because the torsion terms are concentrated in the spin connection. We can use the normal partial derivative to break the coefficients off.

$$S = \int_{\Omega} \left[\frac{i\hbar c}{2} e^a{}_\mu \left(\overline{\psi}\gamma^\mu \psi_{,a} - \overline{\psi}_{,a}\gamma^\mu \psi - \overline{\psi} \left\{ \gamma^\mu, \Gamma_a \right\} \psi \right) - mc^2 \overline{\psi}\psi \right] ed^4x$$

Next recall from equation (7.27), that $\Gamma_a = -\frac{1}{4}\omega_{\mu\nu a}\gamma^\mu\gamma^\nu$, and note we have also used equation (7.7) $\left\{\gamma^\mu, \gamma^{[\nu}\gamma^{\rho]}\right\} = 2\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$. This renders us the following action which will be much easier to vary in terms of what we are interested in.

$$S = \int_{\Omega} \left[\frac{i\hbar c}{2} \left(\overline{\psi}\gamma^a \partial_a \psi - \partial_a \overline{\psi}\gamma^a \psi + \frac{1}{2}\omega_{\mu\nu a} \overline{\psi}\gamma^{[a}\gamma^\mu\gamma^{\nu]}\psi \right) - mc^2 \overline{\psi}\psi \right] ed^4x \quad (7.37)$$

Equivalently, we can write the action in terms of the contorsion tensor in equation (7.38). This comes from breaking apart the connection ∇_a in terms of its Levi-Civita and contorsion pieces.

$$S = \int_{\Omega} \left[\frac{i\hbar c}{2} \left(\overline{\psi}\gamma^a \widetilde{\nabla}_a \psi - \widetilde{\nabla}_a \overline{\psi}\gamma^a \psi + \frac{1}{2} C_{bca} \overline{\psi}\gamma^{[a}\gamma^b\gamma^{c]}\psi \right) - mc^2 \overline{\psi}\psi \right] ed^4x \quad (7.38)$$

Now we vary the Dirac-ECSK action with respect to the fields: $\psi, \overline{\psi}, e^a{}_\mu, \omega_{\alpha\beta a}$ which we choose as our independent variables. First we will vary the ψ field.

7.5.1 Variation with respect to ψ

For this one it is easier to start with the Dirac action before it is broken apart as in (7.38). This results in:

$$\delta S = \int_{\Omega} \left(E_\psi \right) \delta\psi ed^4x + \oint_{\partial\Omega} \epsilon n_a \Theta^a \left\{ \delta\psi \right\} d^3y$$

The variational piece E_ψ (given in two equivalent forms) and boundary piece $\Theta^a \left\{\delta\psi\right\}$ are given by:

$$E_\psi = i\hbar c \left(-\widetilde{\nabla}_a \overline{\psi}\gamma^a + \frac{1}{4} C_{abc} \overline{\psi}\gamma^{[a}\gamma^b\gamma^{c]} \right) - mc^2 \overline{\psi}$$

$$E_\psi = i\hbar c\left(-\nabla_a\overline{\psi}\gamma^a + \frac{1}{2}C^a{}_{ba}\overline{\psi}\gamma^b\right) - mc^2\overline{\psi} \tag{7.39}$$

$$\Theta^a\left\{\delta\psi\right\} = \frac{i\hbar c}{2}\overline{\psi}\gamma^a\delta\psi e$$

The critical points of the action are $E_\psi = 0$. We note that in the literature people often set $C^a{}_{ba} = 0$ because of the spin connection field equation; the spin connection field equation, which we called the Cartan equation in chapter 5, specifies that the contorsion is totally skew, thus the trace we presented $C^a{}_{ba}$ is identically zero. We choose not to set $C^a{}_{ba} = 0$ because it obfuscates any further couplings that can occur with the addition of other matter.

Notice, that once the second field equation is applied this looks very similar to the Dirac equation for ψ in a flat spacetime. We will mainly use equation (7.39) instead of the one directly above it because, as we will see it closely follows exactly what happens in GR. Later, we will show that the trace piece of the torsion/contorsion drops out for a pure ECSK-Dirac theory, and thus the contorsion trace above vanishes. However, if other matter is present this piece will not necessarily vanish. Thus, we are done with the ψ variation.

7.5.2 Variation with respect to $\overline{\psi}$

In a similar manner, we can derive the field equations for $\overline{\psi}$.

$$\delta S = \int_\Omega \delta\overline{\psi}\left(E_{\overline{\psi}}\right) e d^4x + \oint_{\partial\Omega} \epsilon n_a \Theta^a\left\{\delta\overline{\psi}\right\} d^3y$$

The variational piece $E_{\overline{\psi}}$ and boundary piece $\Theta^a\left\{\delta\overline{\psi}\right\}$ are given by:

$$E_{\overline{\psi}} = i\hbar c\left(\gamma^a\widetilde{\nabla}_a\psi + \frac{1}{4}C_{abc}\gamma^{[a}\gamma^b\gamma^{c]}\psi\right) - mc^2\psi = 0$$

$$E_{\overline{\psi}} = i\hbar c\left(\gamma^a\nabla_a\psi - \frac{1}{2}C^a{}_{ba}\gamma^b\psi\right) - mc^2\psi = 0 \tag{7.40}$$

$$\Theta^a\left\{\delta\overline{\psi}\right\} = -\frac{i\hbar c}{2}\delta\overline{\psi}\gamma^a\psi e$$

Notice that once again this looks very similar to the Dirac equation for $\overline{\psi}$ in a flat spacetime.

Again, the same logic applies here, we will find later that the trace piece of the torsion/contorsion vanishes in a pure ECSK-Dirac theory. Thus, we are done with the $\overline{\psi}$ variation.

7.5.3 Variation with respect to $e^a{}_\mu$ and $\omega^\mu{}_{\nu a}$

Originally, we would not be so wrong to vary the tetrad and spin connection separately and get the field equations relatively quickly. However, there is some nuance here that needs to be accounted for. Namely, that $\omega^\mu{}_{\nu a}$ partly depends on the vielbein; it also depends on the torsion of course. When we assume that these are independent in our variation we get different field equations. We need to account for this to get the Hilbert stress-energy tensor. Nevertheless, the field equations are still correct without converting to the Hilbert stress energy tensor with the Belinfante-Rosenfeld relation. See Wheeler (J. T. Wheeler, 2023), and our appendix G for more details. There is nothing wrong with varying the tetrad and spin connection separately. This is the mentality which we will follow here. For interpretation however, it is useful to convert back to the Hilbert stress energy tensor.

The variation with respect to $e^a{}_\mu$ closely follows Wheeler (J. Wheeler, 2021). We begin with the action in which the spin connection is separated out as in equation (7.37). First we need to break off the $e^a{}_\mu$ dependence on the spacetime gamma matrices, this puts the action in the form of equation (7.41)

$$S = \int\limits_{\Omega} \left[\frac{i\hbar c}{2} \left(\overline{\psi}\gamma^\lambda \partial_a \psi - \partial_a \overline{\psi}\gamma^\lambda \psi + \frac{1}{2}\omega_{\mu\nu a}\overline{\psi}\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}\psi \right) e^a{}_\lambda - mc^2\overline{\psi}\psi \right] e d^4x \tag{7.41}$$

And now we begin with the variation. We get two pieces, one from the variation of $e^a{}_\mu$ and the other from the variation of $\omega_{\mu\nu a}$. This gives us the action variation in the form $\delta S = \delta_e S + \delta_\omega S$. We first will look at $\delta_e S$, and then look at $\delta_\omega S$.

7.5.4 Variation with respect to $e^a{}_\mu$

When we vary S with respect to $e^a{}_\mu$ we get two pieces, one from the variation of e and the other from the variation of the $e^a{}_\mu$ which comes from the gamma matrices. To calculate out these pieces it is useful to break $\delta_e S$ into two pieces: I_1 for the $e^a{}_\mu$ piece and I_2 from the e part; this lets us write $\delta_e S = I_1 + I_2$. The I_1 term, after simplification is given by equation (7.42). Likewise, the I_2 after simplification is given by equation (7.43); to simplify this term we use the equation for the variation of the determinant of the vielbein in terms of the variation of the vielbein: $\delta e = e e_a{}^{\mu} \delta e^a{}_{\mu}$.

$$I_1 = \int\limits_{\Omega} \left[\frac{i\hbar c}{2} \left(\overline{\psi}\gamma^{\lambda}\nabla_a\psi - \nabla_a\overline{\psi}\gamma^{\lambda}\psi \right) - mc^2\overline{\psi}\psi \right] \delta e^a{}_{\lambda} e d^4x \tag{7.42}$$

$$I_2 = \int\limits_{\Omega} \left(\frac{1}{2} \left[\overline{\psi}E_{\overline{\psi}} + E_{\psi}\psi \right] e_a{}^{\lambda} \right) \delta e^a{}_{\lambda} e d^4x \tag{7.43}$$

Putting I_1 and I_2 together we find the tetrad energy momentum tensor $t_a{}^{\lambda}$ for an ECSK-Dirac theory, which is given by equation (7.44). From this definition, we can use the ψ, and $\overline{\psi}$ equations of motion: $E_{\psi} = 0$ and $E_{\overline{\psi}} = 0$ respectively to simplify the EMT. All of this together yields the vielbein variation in the form of equation (7.45).

$$t_a{}^{\lambda} = -\frac{i\hbar c}{2} \left(\overline{\psi}\gamma^{\lambda}\nabla_a\psi - \nabla_a\overline{\psi}\gamma^{\lambda}\psi \right) - \frac{1}{2} \left\{ \overline{\psi}E_{\overline{\psi}} + E_{\psi}\psi \right\} e_a{}^{\lambda} \tag{7.44}$$

$$\delta_e S = \int\limits_{\Omega} \left[-t_a{}^{\lambda} \right] \delta e^a{}_{\lambda} e d^4x \tag{7.45}$$

Next we move onto the variation with respect to the spin connection.

7.5.5 Variation with respect to $\omega_{\mu\nu a}$

The variation of the action with respect to the spin connection $\delta_{\omega} S$, once simplified, is given by equation (7.46). We can write this piece as the Palatini spin potential $\mathfrak{Y}^{a\mu\nu}$, but a more useful object is the spin angular momentum density $\mathfrak{S}^{a\mu\nu}$; The Belinfante-Rosenfeld relation tells us that these objects are equivalent, and thus we can write the spin angular momentum density in the form of equation (7.47) (once we make the density into a tensor). This results in the action being written by $\delta_{\omega} S = \int_{\Omega} \left(-\frac{1}{2}\mathfrak{S}^{abc} e_b{}^{\mu} e_c{}^{\nu} \right) \delta\omega_{\mu\nu a} e d^4x$.

$$\delta_{\omega} S = \int\limits_{\Omega} \left(\frac{i\hbar c}{4} \overline{\psi}\gamma^{[a}\gamma^{\mu}\gamma^{\nu]}\psi \right) \delta\omega_{\mu\nu a} e d^4x \tag{7.46}$$

$$s^{abc} = -\frac{i\hbar c}{2}\overline{\psi}\gamma^{[a}\gamma^{b}\gamma^{c]}\psi \tag{7.47}$$

With all these together, we have all the field equation information we need to describe an ECSK-Dirac theory. The field equations are given in the form: $R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = t_{\mu\nu}$, and $\mathscr{C}^{c}{}_{ab} = s^{c}{}_{ab}$. Since we have this, we now move onto classifying an ECSK-Dirac example from Platania and Rosania (Platania & Rosania, 1997).

7.6 Classifying an ECSK-Dirac Example: "A Universe with Torsion"

Platania and Rosania (Platania & Rosania, 1997) found a solution to the ECSK-Dirac field equations. Their solution models a universe with torsion. In this section we will use the tools from chapters 3 and 4 to classify the curvature and torsion tensor of their solution.

To classify this example using the tools we've built, fundamentally, we need the metric tensor g_{ab} and the torsion tensor $T^{a}{}_{bc}$. These in turn allow us to make the metric affine connection $\Gamma^{a}{}_{bc}$, and then the curvature tensor $R^{a}{}_{bcd}$. We are given from the beginning a line element, and thus a metric tensor in the form of equation (7.48).

$$ds^2 = dt^2 - R^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{7.48}$$

We are then given an axial torsion vector in the form of equation (7.49); this is equation (17) in (Platania & Rosania, 1997); we have a factor of two difference because of the way our contorsion tensor is defined as opposed to theirs, additionally we have not set $\hbar = c = 1$ but included it; finally we have the "5" on γ^5 up instead of down like theirs for consistency with our notation. This θ^d is the same object as our $\mathfrak{d}_{abc}$ from equation (3.46) from chapter 3 upon applying a Levi-Civita symbol to it ϵ_{dabc}. We further note that without loss of generality the vector θ^a is spacelike.

$$\theta^d = \mathfrak{d}_{abc}\epsilon^{dabc}$$

$$\theta^d = \frac{\hbar c}{2}\left(\overline{\psi}\gamma^5\gamma^d\psi\right) \tag{7.49}$$

To show that $\theta^d \epsilon_{dabc} = s_{abc}$ (in their equation 47) and compare our results to that of (Platania & Rosania, 1997) it is useful to have the following equations: $\gamma^5\gamma^a = -\gamma^a\gamma^5$ from earlier, and equation (7.50) which is shown below. Noting this, we can now see that we have a consistent notation and can move forward towards writing the torsion tensor. We can take the Levi-Civita tensor dual of θ^a to get what we need through the following identity:

$$i\gamma^{[a}\gamma^b\gamma^{c]} = e^{abcd}\gamma_d\gamma^5 \tag{7.50}$$

Recall that, in the ECSK field, the Cartan equation (5.40) is given by $\mathscr{C}_{abc} = \kappa s_{abc}$. This equation reduces to $T_{[abc]} = \kappa s_{[abc]}$ in an ECSK-Dirac theory because $s_{abc} = s_{[abc]}$ from equation (7.47). Furthermore, because the torsion tensor is completely determined by the irreducible torsion tensor subspace $\mathfrak{d}$, which is totally skew-symmetric, we know $T_{abc} = \mathfrak{d}_{abc}$, from equation (7.51).

$$T_{abc} = -\frac{i\hbar c\kappa}{2}\left(\overline{\psi}\gamma_{[a}\gamma_b\gamma_{c]}\psi\right) \tag{7.51}$$

Platania and Rosania (Platania & Rosania, 1997) take a slightly different approach than just substituting equation (7.51) into the Cartan equation however, and they write θ^d as τ^d with components given by equation (7.52), and the relation $\tau^\mu = \Lambda^\mu{}_\nu\left(x^i\right)\theta^\nu$. The object $\Lambda^\mu{}_\nu\left(x^i\right)$ is a local Lorentz transformation; through the application of one boost, and two rotations, any spacelike vector can be put into the form of equation (7.52) shown below. Note that $\boldsymbol{e}_1$ is an orthonormal basis vector.

$$\tau^\mu = \tau^1\boldsymbol{e}_1 \tag{7.52}$$

We then use the same relation $\tau^d\epsilon_{dabc} = s_{abc}$ to write s_{abc}, and then convert it to the torsion tensor with the second field equation. We then can write the torsion tensor as in equation (7.53).

$$T_{abc} = \kappa\tau^d\epsilon_{dabc} \tag{7.53}$$

We now have everything we need tensor wise to calculate our solution and classify it.

Platania and Rosania present the solution shown below in equation (7.54), where $\mu, R_0, t_1, t_2 \in \mathbb{R}$. The symbols R_0 and b are real integration constants, B, C, D, and F are complex integration constants. We note that $t = t_1$ and $t = t_2$ are singularities of the metric tensor and spin vector. They are physical because the curvature scalar diverges at $t = t_1, t_2$.

$$\left[R\left(t\right) = \sqrt[3]{R_0 \left(t - t_1\right)\left(t - t_2\right)}, \tau^1 = \frac{\mu}{R_0 \left(t - t_1\right)\left(t - t_2\right)} \right] \tag{7.54}$$

$$\begin{aligned}
\psi_1 &= \frac{1}{2R^{3/2}} e^{-imt} \left(B e^{-iG(t)} + C e^{iG(t)} \right) \\
\psi_2 &= \frac{1}{2R^{3/2}} e^{-imt} \left(B e^{-iG(t)} - C e^{iG(t)} \right) \\
\psi_3 &= \frac{1}{2R^{3/2}} e^{imt} \left(D e^{-iG(t)} + F e^{iG(t)} \right) \\
\psi_4 &= \frac{1}{2R^{3/2}} e^{imt} \left(D e^{-iG(t)} - F e^{iG(t)} \right)
\end{aligned}$$

$$G(t) = \frac{1}{2} \ln \frac{t - t_1}{t - t_2}$$

$$\begin{aligned}
t_1 &= -b + \frac{1}{2|R_0|}\sqrt{3\gamma} \\
t_2 &= -b - \frac{1}{2|R_0|}\sqrt{3\gamma}
\end{aligned}$$

$$\gamma = 3\left(8\pi G\mu\right)^2$$

$$\begin{aligned}
\nu &= \frac{1}{2}\left(|B|^2 + |C|^2 - |D|^2 - |F|^2\right) \\
\mu &= \frac{1}{8}\left(|C|^2 + |F|^2 - |B|^2 - |D|^2\right)
\end{aligned}$$

$$R_0 = -6\pi G m \nu$$

Platania and Rosania point out that there are two different types of solution. One for $\nu < 0$, and one for $\nu > 0$. For $\nu < 0$, $t \in [t_1, \infty) \cup (-\infty, t_2]$. For $\nu > 0$, $t \in [t_2, t_1]$. In the most general case for the constants arbitrary, we cannot determine the Segre type, but we are able to determine all the other types; additionally we can determine the Plebanski Type when the constants are arbitrary as well. We first examine the classification of the torsion tensor, then the curvature tensor classification.

7.6.1 Classification of the Torsion tensor in "A Universe with Torsion"

These results are presented below for the torsion tensor classification, see equation (7.55).

$$[\Theta = (-)\,, \Xi = (1,1)\,, \Omega = (-)] \tag{7.55}$$

We find that the solution in the general case is irreducible. However, we can examine when Ξ becomes reducible by examining the omicron invariant from equation (3.63) in chapter 3. In this case the omicron invariant is given by (7.56). The omicron invariant is only zero when μ is zero or when t approaches infinity ($t \to \infty$) and $t > t_1$ and $t > t_2$ hold. However, it is trivially reducible because the alpha invariant is also zero when $\mu = 0$ and in the $t \to \infty$ case. Therefore, it is not possible to have the $|(1,0)|^2$ case for Ξ in this solution; we could only possibly reduce to the case where $\Xi = (-)$, and thus the entire torsion tensor would be zero.

$$O = \frac{1}{9}\frac{\kappa^2\mu^2}{(R_0)^2\,(t-t_1)^2\,(t-t_2)^2} \tag{7.56}$$

Next we move to the classification of the curvature tensor.

7.6.2 Classification of the Curvature tensor in "A Universe with Torsion"

The results in the arbitrary constant case are given in (7.57), and (7.58). Notice that the Λ classification is indeterminant. This is because of its dependence on t, and the values of the constants. All the structural reducibility pieces were completely determined generally without dependence on any parameters. The same holds for the Plebanski classifications of the spinors Φ and Ӿ from equations (4.38) and (4.42). The Plebanski type is D for both the Φ spinor and the Ӿ spinor. We now present the information in the same way that we did for the NMC classification of the Bronnikov and Galiakhmetov solution of chapter 6.

$$\left[\Psi = O, \Phi = \left\{(1,1)\,(1,1) \quad OR \quad |(1,1)|^2, D, "FAIL"]\right\}, \Lambda = "Indeterminant",\right] \quad (7.57)$$

$$\left[Ж = \left\{(1,1)\,|(1,0)|^2, D, "FAIL"\right\}, Ю = [A, B], \aleph = 0\right] \quad (7.58)$$

To help the Segre classification algorithm along we picked some parameters to determine the generic type. We evaluate the curvature tensor at a point in the parameter space of μ, R_0, t_1, t_2, additionally we evaluate t at some value not equal to t_1 or t_2. We pick the values: $t_1 = 1$, $t_2 = 2$, $\mu = 1$, $R_0 = 1$, and $\kappa = 1$. We set $\kappa = 1$ here not because it is actually that value, but to help the computer along in the calculation. Maple can encounter some difficulty in determining if certain values are zero in this calculation due to the cubic roots. Furthermore, we choose $t = 0$ because it is a generic point and not exceptional. This results in the following classification of the ECSK-Dirac solution at a point given by (7.59), and (7.60). The classification around an open set of this point will also be the same type.

$$\left[\Psi = O, \Phi = \left\{(1,1)\,(1,1) \quad OR \quad |(1,1)|^2, D, [1,1\,(11)]\right\}, \Lambda = -\right] \quad (7.59)$$

$$\left[Ж = \left\{(1,1)\,|(1,0)|^2, D, [1,1\,(11)]\right\}, Ю = [A, B], \aleph = 0\right] \quad (7.60)$$

Now we can see more information than before. Λ is now negative, and Φ and Ж have the same Segre type; this is interesting because they are still distinct in their structural reducibilities. This illustrates the usefulness of the structural reducibility as a useful tool on top of the Segre classification. We can explore the entire reducibility structure in depth by looking at how the parameters change the co(in)variants and how this in turn changes the type this solution is with regard to its classification. However, we will not do this fully, but we will explore both the structure making Λ originally indeterminant, and if there is a possibility for Ю to reduce further.

First we look at Λ in general. We can determine that there are values of the parameters for which Λ is positive, negative, and zero from equation (7.61). This clarifies why we got the indeterminant result earlier.

$$\Lambda = \frac{\left(-32\left(R_0\right)^2 t^2 + 32\left(R_0\right)^2\left(t_1+t_2\right)t + 16\left(\left(t_1\right)^2 - 4t_1t_2 + \left(t_2\right)^2\right)\left(R_0\right)^2 - 9\kappa^2\mu^2\right)}{288\left(R_0\right)^2\left(t-t_2\right)^2\left(t-t_1\right)^2} \tag{7.61}$$

Upon evaluating equation (7.61) on the values we picked earlier except for t, we find that its structure simplifies considerably as seen in equation (7.62).

$$\Lambda = \frac{1}{288}\frac{-32t^2+96t-57}{\left(t-2\right)^2\left(t-1\right)^2} \tag{7.62}$$

Upon examining the roots of the numerator for t, we find that there are three different regions of classification for Λ. Λ is negative in the region: $t \in \left(-\infty, \frac{3}{2} - \frac{1}{8}\sqrt{30}\right) \cup \left(\frac{3}{2} + \frac{1}{8}\sqrt{30}, \infty\right)$. Lambda is zero in the region $t \in \left\{\frac{3}{2} - \frac{1}{8}\sqrt{30}, \frac{3}{2} + \frac{1}{8}\sqrt{30}\right\}$. Lastly Λ is positive in the region $t \in \left(\frac{3}{2} - \frac{1}{8}\sqrt{30}, 1\right) \cup (1,2) \cup \left(2, \frac{3}{2} + \frac{1}{8}\sqrt{30}\right)$. This should be able to be refined for arbitrary parameters μ, R_0, t_1, t_2. Finally, a figure of the different regions of Lambda is presented in figure (7.1) below: Here the solid blue line is the plot of Λ on the vertical vs. t on the horizontal. The dashed blue lines represent the values we chose for t_1 and t_2. Additionally, the green dots represent the roots of the graph, which are also exceptional points.

Next we move onto looking at the IO spinor's reducibility. Similarly to what happens to the torsion tensor, there is no way for the IO spinor to be of type [2]. For the same parameters we chose above in the case for Λ, the curvature omicron invariant is given by (7.63).

$$O = \frac{1}{288}\frac{\left(2t-3\right)^2}{\left(t-1\right)^4\left(t-2\right)^4} \tag{7.63}$$

This can be zero when $t = \frac{3}{2}$, but this also forces the entire IO spinor to be zero as well, thus precluding the [2] case as only [11] and [(11)] are possible. We now end with a short summary.

7.7 ECSK-Dirac Classification Summary

For an ECSK-Dirac theory, the only non-zero piece of the torsion tensor is generated

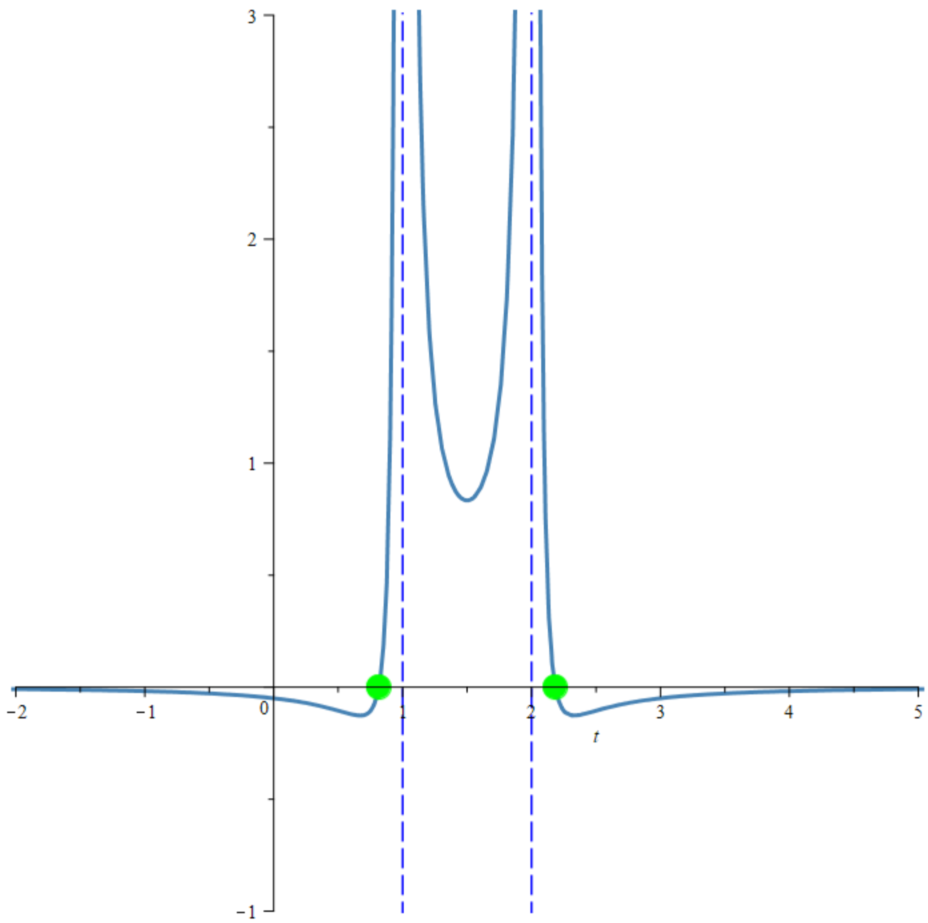


Figure 7.1: Plot of Lambda Spinor for Platania and Rosania Solution

by $\mathfrak{d}_{abc}$. We described the action and the field equations, along with the boundary terms for ψ, and $\overline{\psi}$. Interestingly enough, the Dirac equation is modified in this format due to spin coupling to torsion; this is what augments the Dirac equation to the Hehl-Datta equation as we found in equations (7.39), and (7.40). Furthermore, the contorsion piece in both of these equations reduces to a Dirac-like equation because of the second field equation's constraint on the torsion; having only axial torsion $\left(T_{[abc]}\right)$ forces the contorsion tensor to also be purely skew-symmetric. This causes Dirac particles to behave differently just by loosening the constraint that torsion be zero from GR. The classification was also interesting because it shows us just how complicated things can get. We are successfully able to classify the solution and see that there is no Weyl curvature, which is particularly interesting.

CHAPTER 8

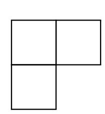
RELATIVISTIC SPINNING FLUIDS AND EXTENDED ECSK

We include this section although somewhat short and incomplete on its background to cover the last irreducible piece of torsion $\mathfrak{q}_{abc}$ as in the rank 3 tensors chapter 3 in equation (3.45); we will do this with a short note on spinning fluids, including references for the reader. We will present an example from Chen (Chen et al., 2018) in an extended ECSK theory which also acts as a source for $\mathfrak{q}_{abc}$. We begin with the spinning fluid. Further references on spinning fluids include Poplawski (Poplawski, 2013) in his section (2.5.11) on “Relativistic spin fluids”, and Westenholz (Westenholz, 1978) in his section on “Relativistic Fluid Mechanics” see pg 456 and onward.

8.1 A Short Note on Spinning Fluids

Spinning fluids are fluids which include some macroscopic spin density in them; the microscopic spin density does not average out to zero. Weyssenhoff justifies this by calling it the density of angular momentum per unit rest volume. The author has been unable to find a kinetic, moment, or variational approach which leads to the spin density presented by Weyssenhoff. Nevertheless, this variational/kinetic problem seems to be an interesting heading for future work. Tsoubelis (Tsoubelis, 1981) presents a spinning fluid solution, although we will not examine it here. We will not examine any more background on spinning fluids here, but refer the reader to the prior citations.

A source for the leftovers sector of torsion is a spinning fluid. Spinning fluids date back to Weyssenhoff and Raabe (Weyssenhoff & Raabe, 1947) in the late 1940s. Spinning fluids also source an axial torsion piece which we called $\mathfrak{q}_{abc}$ in equation (3.46). Recall that we represented the leftovers sector by $\mathfrak{q}_{abc}$ in equation (3.45) additionally $\mathfrak{q}_{abc}$ resides in the



Ferrers diagram as the trace free part of this diagram.

8.2 An Example From Extended ECKS Theory

In this section our goal will be to apply the classification tools we have developed to three solutions presented by Chen (Chen et al., 2018). We also take note that although Chen uses a modified ECSK theory which includes a quadratic torsion term, our classification tools are still applicable because they focus purely on the curvature and torsion without regard to the field equations. Furthermore, in this theory the torsion satisfies a differential, not algebraic equation for torsion. This allows the torsion tensor to be non-zero despite lacking any matter source in the action. This arises from the $R\mathcal{T}$ term in the action.

Chen's action is given as follows:

$$S = \int\limits_{\Omega} \left[\frac{1}{2\kappa} \left(R + R\mathcal{T} \right) \right] \sqrt{-g} d^4 x \tag{8.1}$$

in which R is the ECSK Ricci scalar and $\mathcal{T}$ is given by:

$$\mathcal{T} = a_1 T_{abc} T^{abc} + a_2 T_{abc} T^{bac} + a_3 T^{a}{}_{a}{}^{b} T^{c}{}_{bc} \tag{8.2}$$

with a_1, a_2, $a_3 \in \mathbb{R}$ being some arbitrary torsion coupling constants. The field equations for equation (8.1) are complicated, and as such we refer the reader to reference (Chen et al., 2018). Nevertheless, we produce the field equations here albeit without a compensating GHY term. The GHY term will not cancel the boundary terms in the given action and additional terms will be needed. This would be interesting to explore.

As our field variables, we choose the inverse metric (which we will again just call the metric), and the torsion tensor: g^{ab}, and $T^{a}{}_{bc}$. To make varying equation (8.1) easier, we include the following equations:

$$\delta_g \mathcal{T} = \mathcal{T}_{ab} \delta g^{ab} \tag{8.3}$$

$$\mathcal{T}_{ab} = \left(a_1 \left(-T_{(a}{}^{ef} T_{b)ef} + 2 T_{ef(a} T^{ef}{}_{b)} \right) + \left(a_2 T_{ef(a} T^{fe}{}_{b)} - a_3 T_{(a} T_{b)} \right) \right) \tag{8.4}$$

$$\delta_T \mathcal{T} = \mathscr{T}_a{}^{bc} \delta T^a{}_{bc} \tag{8.5}$$

$$\mathscr{T}_a{}^{bc} = 2\left(a_1 T_a{}^{bc} + a_2 T^b{}_a{}^c - a_3 \delta_a{}^{[b} T^{c]}\right) \tag{8.6}$$

$$\delta_g R = \left(R\left\{\right\}_{ab} + \mathcal{U}_{ab}\right)\delta g^{ab} + \widetilde{\nabla}_b\left(-2T_a \delta g^{ab} + \left(g^{qb} g_{ef} - \delta^b_f \delta^q_e\right)\delta g^{ef}{}_{:q}\right) \tag{8.7}$$

Recall that $\mathcal{U}_{ab}$ is the Hehl Tensor defined in equation (6.13).

$$\delta_T R = \mathscr{A}_a{}^{bc} \delta T^a{}_{bc} + \widetilde{\nabla}_q\left(-2g^{cq} \delta^b_a \delta T^a{}_{bc}\right) \tag{8.8}$$

Recall that $\mathscr{A}_a{}^{bc}$ is the Alpha Cartan Tensor defined in equation (5.37).

The equations above represent the variation of R, and $\mathcal{T}$ with respect to the fields g_{ab} and $T^a{}_{bc}$. The metric variation of equation (8.1) produces the following field equations:

$$\left(R\left\{\right\}_{ab} + \mathcal{U}_{ab}\right)(1 + \mathcal{T}) + R\mathcal{T}_{ab} - \frac{1}{2} g_{ab} R (1 + \mathcal{T}) = 0 \tag{8.9}$$

$$\mathscr{A}_a{}^{bc}(1 + \mathcal{T}) + R\mathscr{T}_a{}^{bc} = 0 \tag{8.10}$$

where we have used $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{ab} \delta g^{ab}$ and equations (8.3)-(8.8) above.

Moving forward to find solutions to equations (8.9), and (8.10) we begin with coordinates (t, r, θ, ϕ) and a metric g_{ab} with signature $[+, -, -, -]$ in the following form where the functions $H(r)$ and $F(r)$ are part of what we will solve for in the field equations.

$$g_{ab} = H(r)\, dt \otimes dt - \frac{1}{F(r)} dr \otimes dr - r^2\left(d\theta \otimes d\theta + \sin^2(\theta)\, d\phi \otimes d\phi\right) \tag{8.11}$$

This is Chen's metric up to the signature change. We have changed the signature to mostly minus, which is different from Chen's mostly plus signature, so that we can apply our Maple algorithm which requires the mostly minus signature. Furthermore, the torsion tensor is given by:

$$T^r{}_{tr} = -T^r{}_{rt} = A(r) \tag{8.12}$$

$$T^\theta{}_{t\theta} = -T^\theta{}_{\theta t} = T^\phi{}_{t\phi} = -T^\phi{}_{\phi t} = B(r) \tag{8.13}$$

with $A(r)$, and $B(r)$ being functions which only depending on r. Chen provides 3 different solutions which we have checked, and adapted to our mostly minus signature. The three solutions are black hole solutions in braneworlds which he names after the branes in string theory. The third solution is similar to a Reissner-Nordström black hole with charge q. To classify these solutions, we begin by presenting the first solution.

8.3 Braneworld I Solution

Chen's first solution adapted to our signature is given by the following three equations:

$$F(r) = \frac{(2r + C_1)(2m - r)}{(3m - 2r)\, r}$$

$$H(r) = -\varphi \left(\frac{2m - r}{r} \right)$$

$$B(r) = \sqrt{1 - \frac{2m}{r}}, \quad A(r) = -\frac{1}{2} B(r)$$

where φ, C_1, C_2, and γ are constants and φ is given by $\varphi = -\frac{9}{4}(2a_1 - 2a_2 + a_3)$ in terms of the a_i constants in equation (8.2). Next we classify the curvature and torsion of this solution using our tools from the previous sections.

8.3.1 Torsion Classification of Solution 1

Upon running our algorithm, we find that the classification of the torsion tensor is given by:

$$[\Theta = (1,1)\,, \Xi = (-)\,, \Omega = (1,0)\,(1,0)\,(1,1)]$$

where the spinors $\mathbf{\Theta}$, $\mathbf{\Xi}$, and $\mathbf{\Omega}$ are defined as before in the Rank 3 tensor section. Upon examining the co(in)variants, we find that there are no exceptional points in this solution for either $\mathbf{\Theta}$ or $\mathbf{\Omega}$.

8.3.2 Curvature Classification of Solution 1

We find that the subspaces $\mathfrak{F}$, $\mathfrak{l}$, and $\mathfrak{k}$ for this solution are empty, meaning that their corresponding tensors are zero. This in turn implies that the spinors Λ, ӜК, and ℵ are zero, implying that they are types 0, (−), and 0 respectively.

Upon running the classification algorithm we find the following classification for the curvature given as Maple output.

$$\left[\Psi = D, \Phi = \left\{(1,1)\,(1,1) \quad OR \quad |(1,1)|^2, D, "Indeterminant"]\right\}, \Lambda = 0\right]$$

$$[\text{ӜК} = \{(-), O, [(1,111)]\}, \text{Ю} = [A, B], \aleph = 0]$$

There is some information we can determine about the Segre type. For instance, the sign of the H invariant depends on the choice of parameters. However, the k_a invariant is always negative, except at some possible exceptional points. We can make the following remarks: when $H > 0$ we have Segre Type $[1, 1\,(11)]$, when $H = 0$ then the Segre type is $[2, (11)]$, finally when $H < 0$, then the Segre type $\left[Z\overline{Z}, (11)\right]$.

8.4 Braneworld II Solution

Chen's second solution adapted to our signature is given by the following three equations:

$$F(r) = 1 - \frac{2\gamma m}{r}$$

$$H(r) = \varphi\left(C_1 + C_2\sqrt{\frac{2\gamma m - r}{r}}\right)^2$$

$$B(r) = C_1 + C_2\sqrt{\frac{2\gamma m - r}{r}}, \quad A(r) = -\frac{1}{2}B(r)$$

where φ, C_1, C_2, and γ are constants and φ is given by $\varphi = -\frac{9}{4}(2a_1 - 2a_2 + a_3)$ in terms of the a_i constants in equation (8.2). Next, we classify the curvature and torsion of this solution using our tools from the previous sections.

8.4.1 Torsion Classification of Solution 2

Upon running our algorithm, we find that the classification of the torsion tensor is given by:

$$[\Theta = (1,1)\,, \Xi = (-)\,, \Omega = (1,0)\,(1,0)\,(1,1)]$$

where the spinors $\mathbf{\Theta}$, $\mathbf{\Xi}$, and $\mathbf{\Omega}$ are defined as before in the Rank 3 tensor section. Upon examining the co(in)variants, we find that there are no exceptional points in this solution for either $\mathbf{\Theta}$, or $\mathbf{\Omega}$.

8.4.2 Curvature Classification of Solution 2

We find that this classification is generally the same as for solution 1. However, the Plebanski and Segre types may be different depending on the parameters than in solution.

8.5 Extended ECSK Reissner-Nordström Black Hole Solution

Chen's third solution adapted to our signature is given by the following three equations:

$$F\,(r) = 1 - \frac{2\gamma m}{r} + \frac{q^2}{r^2}$$

$$H\,(r) = \varphi \frac{(C_1)^2 \left(\gamma m r + C_2 q \sqrt{2\gamma m r - q^2 - r^2} - q^2\right)^2}{r^2\left(\gamma^2 m^2 - q^2\right)}$$

$$B\,(r) = \frac{C_1 \left(\gamma m r + C_2 q \sqrt{2\gamma m r - q^2 - r^2} - q^2\right)}{r\sqrt{\gamma^2 m^2 - q^2}}, \quad A\,(r) = -\frac{1}{2} B\,(r)$$

where φ, C_1, C_2, and γ are constants and φ is given by $\varphi = -\frac{9}{4}\left(2a_1 - 2a_2 + a_3\right)$ in terms of the a_i constants in equation (8.2). Next we classify the curvature and torsion of this solution using our tools from the previous sections.

8.5.1 Torsion Classification of Solution 3

Upon running our algorithm, we find that the classification of the torsion tensor is given by:

$$[\Theta = (1,1)\,, \Xi = (-)\,, \Omega = (1,0)\,(1,0)\,(1,1)]$$

where the spinors $\mathbf{\Theta}$, $\mathbf{\Xi}$, and $\mathbf{\Omega}$ are defined as before in the Rank 3 tensor section. Upon examining the co(in)variants, we find that there are exceptional points in this solution.

The spinors $\mathbf{\Theta}$, and $\mathbf{\Omega}$ share the same exceptional points. We find that there are exceptional points for these spinors if $C_1 = 0$, or at roots of either of the expressions: $\left(\gamma mr + C_2 q\sqrt{2\gamma mr - q^2 - r^2} - q^2\right)$, $\left(2\gamma mr - q^2 - r^2\right)$. In each of these cases $\mathbf{\Theta}$ can only reduce to type $(-)$; type $|(1,0)|^2$ is not possible. Similarly, $\mathbf{\Omega}$ can only reduce to type $(-)$ as well.

8.5.2 Curvature Classification of Solution 3

We find that this classification is generally the same as for solution 1. However, the Plebanski and Segre types may vary differently depending on the parameters than in solutions 1 and 2. Additionally, there are some exceptional points depending on the parameters, which we do not get in the other two cases.

8.6 Summary

Altogether, we presented information and references (Westenholz, 1978), (Poplawski, 2013), (Weyssenhoff & Raabe, 1947), and (Tsoubelis, 1981) on spinning fluids, commented on how our classification still works in a generalized ECSK theory, and classified the three solutions of Chen (Chen et al., 2018). Although we classified these solutions, we were unable to tell any general differences other than at some exceptional points. This presents the question on how to develop a better classification, and to what extent these classification tools can tell us about the spacetime we are in. For example, if we were to try to make a Cartan-Karlhede (Karlhede, 1980) algorithm for ECSK, these invariants would not be enough information to determine when solutions are equivalent. We would need more information. To be more specific, take for example that all three solutions are of Petrov type D. This invariant alone is not enough to distinguish the 3 different solutions that Chen et al. (Chen et al., 2018) present.

CHAPTER 9

CONCLUSION

The dissertation presents a classification scheme, Petrov/Plebanski/Segre-like (PPS), for the curvature and torsion tensors in Einstein-Cartan-Sciama-Kibble theory (ECSK). ECSK is an extension of General Relativity that naturally couples spin to the geometry through torsion. The PPS classification method was developed for ECSK to classify solutions and provide an equivalence method, which is not currently available due to the inclusion of the torsion tensor. This classification scheme is important because it allows for the invariant distinction of solutions in ECSK theory, which includes the addition of torsion and the coupling of spin to geometry. The new software package in Maple includes computational and PPS classification tools, which were used to classify six different solutions given in references. In addition to the PPS classification, the dissertation includes several new works, such as the decomposition of an arbitrary 4th rank tensor under $SO(p,q)$, equivalent $SL(2,\mathbb{C})$ irreducible spinor decompositions to arbitrary 3rd and 4th rank tensors, an algorithm to determine the algebraic decomposition of the torsion spinors, an algorithm for the algebraic decomposition of the ECSK $\mathcal{F}$ spinor (a part of the curvature tensor), and Gibbons-Hawking-York complete ECSK-non-minimally-coupled boundary terms. Overall, the work presented in this dissertation is significant in furthering our understanding of ECSK theory and its applications.

We provide a table, see table (9.1) below which shows whwhich types of matter are sources for torsion. This is not an exhaustive list, but it does touch on many types of matter. We present increasing spin as the list descends, ending with the spinning fluid example presented by Weyssenhoff and Raabe (Weyssenhoff & Raabe, 1947), and Tsoubelis (Tsoubelis, 1981). We include a spin $\frac{3}{2}$, Rarita-Schwinger field the same way Wheeler presents it (J. T. Wheeler, 2023); we note he provides a proof that Rarita-Schwinger is a source for all three irreducible representations of the torsion tensor. Additionally, Wheeler provides the form for higher spin $\frac{n}{2}$ with $n > 3$ and odd. Future work would be to examine the irreducible representations for the torsion tensor for these higher spin cases.

The table is read as follows: if there is a circle that means that this matter type does not generate the corresponding torsion of that type, if there is a checkmark it means that this matter source does generate torsion of this type, if there is a question mark that means that it is unknown if this matter type generates the corresponding torsion type or not. We use the symbols $\mathfrak{Q}$, $\mathfrak{D}$, and $\mathfrak{q}$ to represent the irreducible representations of the torsion tensor as in chapter 3. These represent the trace, axial, and leftovers parts as in equations (3.44), (3.46), and (3.45) respectively.

We do not include a similar table for curvature; although we would like to, it would be a significant undertaking. Furthermore, the nonlinearity of the field equations makes determining a general Petrov type for the Weyl tensor an undertaking that will hopefully one day be examined in future work.

Matter Sources for Torsion			
Matter	$\mathfrak{Q}$	$\mathfrak{d}$	$\mathfrak{q}$
MC Scalar	$\circ$	$\circ$	$\circ$
NMC Scalar	$\checkmark$	$\circ$	$\circ$
Dirac	$\circ$	$\checkmark$	$\circ$
Maxwell	$\circ$	$\circ$	$\circ$
Rarita-Schwinger	$\checkmark$	$\checkmark$	$\checkmark$
Spin $\frac{n}{2}$, n odd, n>3	$\checkmark$	$\checkmark$	$\checkmark$
Spinning Fluid	$\circ$	$\checkmark$	$\checkmark$

Table 9.1: Table for Matter Sources of Torsion

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APPENDICES

APPENDIX A
THE TORSION TENSOR DECOMPOSITION IN THE LITERATURE

There are other forms of the decomposition we presented of the torsion tensor in the literature. The most common of these is that given by Shapiro (Shapiro, 2001). To compare our notation to that of (Shapiro, 2001) we first define two objects: the trace of the torsion tensor given by T_a in equation (A.1). Recall, this is also defined in the ECSK field equation section, see chapter 5. The axial trace of the torsion tensor given by S^a in equation (A.2).

$$T_a = T^e{}_{ea} \tag{A.1}$$

$$S^a = T_{efg}\epsilon^{efga} \tag{A.2}$$

The $SO\,(p,q)$ decomposition of the torsion tensor is often given in the form of equation (A.3); this decomposition is only valid in $N = 4$ dimensions however, and must be modified if the dimension is changed.

$$T_{abc} = \frac{1}{6}S^e\epsilon_{eabc} + \frac{2}{3}g_{a[b}T_{c]} + q_{abc} \tag{A.3}$$

In N-dimensions the decomposition in equation (A.3) can be written as (A.4). Where we have defined $\Theta\,(N-3)$ as the Heaviside step function and where the number of dots is equal to $N-3$. Note that $S^{\cdots}$ doesn't exist in dimensions lower than 3, and the torsion tensor doesn't exist in dimensions lower than 2 because it would be instantly zero due to its skew-symmetry. The dots are in the axial "vector" due to the number changing in each dimension; for every additional dimension S becomes a bivector of higher rank, for clarity in $N = 5$ we have an axial bivector S^{ab}. Lastly q_{abc} is what is defined as the "leftovers" piece (the trace free, and axial trace free piece).

$$T_{abc} = \frac{\Theta\left(N-3\right)}{6\left(N-3\right)!}S^{\cdots}\epsilon_{\ldots abc} + \frac{2\Theta\left(N-2\right)}{\left(N-1\right)}g_{a[b}T_{c]} + q_{abc} \tag{A.4}$$

The decomposition (A.4) relates to our $SO\left(p,q\right)$ decomposition as by the use of equations: (A.5), (A.6), and (A.7).

$$\mathfrak{Q}_{abc} = \frac{2\Theta\left(N-2\right)}{\left(N-1\right)}g_{a[b}T_{c]} \tag{A.5}$$

$$\mathfrak{q}_{abc} = q_{abc} \tag{A.6}$$

$$\mathfrak{d}_{abc} = \frac{\Theta\left(N-3\right)}{6\left(N-3\right)!}S^{\cdots}\epsilon_{\ldots abc} \tag{A.7}$$

This concludes the comparison of our $SO\left(p,q\right)$ decomposition to the literature, and we find that everything matches exactly.

APPENDIX B

FORMULAS FOR THE $GL\left(N\right)$ DECOMPOSITION OF AN ARBITRARY RANK 4 TENSOR

The Young decomposition is a powerful mathematical tool that provides a way to break down, in this case, an arbitrary rank 4 tensor into its irreducible subspaces. Since the Young tableaux are a graphical representation of the decomposition, they act as a shortcut to understanding the complexities of irreducible $GL\left(N\right)$ subspaces quickly. This results in a comprehensive understanding of the internal structure of the tensor and its relationships with the corresponding subspaces. Below we present formulas which produce the proper Young decomposition of an arbitrary rank 4 tensor.

One issue that may arise however, is that these subspaces may not be linearly independent, and this must be checked for before deciding that the subspaces are irreducible. We follow the definition of Itin and Reches on independence (Itin & Reches, 2021) (See page 7): **Independence**: The sub-tensors $T_{(p)a\ldots b}$ must be linearly independent, i.e., any equation of the form $\sum_{p=1}^{n} \alpha_p \left(T_{(p)a\ldots b}\right) = 0$ yields $\alpha_p = 0$ for all p. For more on this phenomenon see (Itin & Reches, 2021) (pages 25-26) for an excellent explanation of what occurs for the piezoelectric tensor; we take that methodology and expand on it here.

First we present the $Y1$ decomposition of our an arbitrary rank 4 tensor Q_{abcd}. We symmetrize on the indices a, b, c, d and then we have $\boldsymbol{\mathcal{A}}$.

1	2	3	4

$$\begin{aligned}
\mathcal{A}_{abcd} &= \frac{1}{24}\left(Q_{abcd} + Q_{abdc} + Q_{acbd} + Q_{acdb} + Q_{adbc} + Q_{adcb}\right) \\
&+ \frac{1}{24}\left(Q_{bacd} + Q_{badc} + Q_{bcad} + Q_{bcda} + Q_{bdac} + Q_{bdca}\right) \\
&+ \frac{1}{24}\left(Q_{cabd} + Q_{cadb} + Q_{cbad} + Q_{cbda} + Q_{cdab} + Q_{cdba}\right) \\
&+ \frac{1}{24}\left(Q_{dabc} + Q_{dacb} + Q_{dbac} + Q_{dbca} + Q_{dcab} + Q_{dcba}\right)
\end{aligned}$$

Next we have the $Y2$ decomposition which results in three tensors $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, and $\boldsymbol{\mathcal{D}}$. For $\boldsymbol{\mathcal{B}}$ we symmetrize on a, b, c first and then skew symmetrize on a, d. Next, for $\boldsymbol{\mathcal{C}}$ we symmetrize on a, b, d first and then skew symmetrize on a, c. Finally, for $\boldsymbol{\mathcal{D}}$ we symmetrize on a, c, d first and then skew symmetrize on a, b.

Here is the $Y2a$ piece:

1	2	3
4		

$$\begin{aligned}
\mathcal{B}_{abcd} &= \frac{1}{8}\left(+Q_{abcd} + Q_{bcad} + Q_{cabd} + Q_{acbd} + Q_{cbad} + Q_{bacd}\right) \\
&+ \frac{1}{8}\left(-Q_{dbca} - Q_{bcda} - Q_{cdba} - Q_{dcba} - Q_{cbda} - Q_{bdca}\right)
\end{aligned}$$

Here is the $Y2b$ piece:

1	2	4
3		

$$\begin{aligned}
\mathcal{C}_{abcd} &= \frac{1}{8}\left(+Q_{abcd} + Q_{adcb} + Q_{bacd} + Q_{bdca} + Q_{dacb} + Q_{dbca}\right) \\
&+ \frac{1}{8}\left(-Q_{cbad} - Q_{cdab} - Q_{bcad} - Q_{bdac} - Q_{dcab} - Q_{dbac}\right)
\end{aligned}$$

Here is the $Y2c$ piece:

1	3	4
2		

$$\begin{aligned}
\mathcal{D}_{abcd} &= \frac{1}{8}\left(+Q_{abcd} + Q_{abdc} + Q_{cbad} + Q_{cbda} + Q_{dbac} + Q_{dbca}\right) \\
&+ \frac{1}{8}\left(-Q_{bacd} - Q_{badc} - Q_{cabd} - Q_{cadb} - Q_{dabc} - Q_{dacb}\right)
\end{aligned}$$

There are two irreducible $GL(N)$ subspaces for the $Y3$ tableaux. They are given by $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{\mathcal{F}}$. For $\boldsymbol{\mathcal{E}}$ we first symmetrize on the pairs a, b and c, d, (not totally symmetrized though as per the tableaux) then we skew symmetrize on the pairs a, c and b, d. Then for $\boldsymbol{\mathcal{F}}$ we first symmetrize on the pairs a, c and b, d, then we skew symmetrize on the pairs a, b and c, d. As an aside the tensor $\boldsymbol{\mathcal{F}}$ has the same symmetry as the Riemann tensor of GR: i.e. $\mathcal{F}_{abcd} = \mathcal{F}_{[ab][cd]} = \mathcal{F}_{cdab}$. These sectors also have the additional simultaneous symmetric index swap; the last equality in above for $\boldsymbol{\mathcal{F}}$, and for $\boldsymbol{\mathcal{E}}$ we would have $\mathcal{E}_{abcd} = \mathcal{E}_{badc}$.

Here is the $Y3a$ piece:

1	2
3	4

$$
\begin{aligned}
\mathcal{E}_{abcd} &= \frac{1}{12}\left(+Q_{abcd} + Q_{abdc} + Q_{bacd} + Q_{badc}\right)\\
&+ \frac{1}{12}\left(-Q_{cbad} - Q_{cbda} - Q_{bcad} - Q_{bcda}\right)\\
&+ \frac{1}{12}\left(-Q_{adcb} - Q_{adbc} - Q_{dacb} - Q_{dabc}\right)\\
&+ \frac{1}{12}\left(+Q_{cdab} + Q_{cdba} + Q_{dcab} + Q_{dcba}\right)
\end{aligned}
$$

Here is the $Y3b$ piece:

1	3
2	4

$$
\begin{aligned}
\mathcal{F}_{abcd} &= \frac{1}{12}\left(+Q_{abcd} + Q_{adcb} + Q_{cbad} + Q_{cdab}\right)\\
&+ \frac{1}{12}\left(-Q_{bacd} - Q_{bdca} - Q_{cabd} - Q_{cdba}\right)\\
&+ \frac{1}{12}\left(-Q_{abdc} - Q_{acdb} - Q_{dbac} - Q_{dcab}\right)\\
&+ \frac{1}{12}\left(+Q_{badc} + Q_{bcda} + Q_{dabc} + Q_{dcba}\right)
\end{aligned}
$$

The Y4 subspaces are the more skew symmetric ones and are presented by the tensors $\boldsymbol{\mathcal{G}}$, $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{\mathcal{J}}$; we skipped $\boldsymbol{\mathcal{I}}$ which, although is alphabetically before $\boldsymbol{\mathcal{J}}$, reminds us too much of the identity. Moving forward, to get $\boldsymbol{\mathcal{G}}$ we first symmetrize on a, b, then we skew symmetrize on a, c, d. To get $\boldsymbol{\mathcal{H}}$ we first symmetrize on a, c, then we skew symmetrize on a, b, d. Finally, to get $\boldsymbol{\mathcal{J}}$ we first symmetrize on a, d, then we skew symmetrize on a, b, c.

Here is the $Y4a$ piece:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} \mathcal{G}_{abcd} &= \frac{1}{8}\left(Q_{abcd} + Q_{cbda} + Q_{dbac} - Q_{abdc} - Q_{dbca} - Q_{cbad}\right) \\ &+ \frac{1}{8}\left(Q_{bacd} + Q_{bcda} + Q_{bdac} - Q_{badc} - Q_{bdca} - Q_{bcad}\right) \end{aligned}$$

Here is the $Y4b$ piece:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} \mathcal{H}_{abcd} &= \frac{1}{8}\left(Q_{abcd} + Q_{bdca} + Q_{dacb} - Q_{adcb} - Q_{dbca} - Q_{bacd}\right) \\ &+ \frac{1}{8}\left(Q_{cbad} + Q_{cadb} + Q_{cdba} - Q_{cbda} - Q_{cdab} - Q_{cabd}\right) \end{aligned}$$

Here is the $Y4c$ piece:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} \mathcal{J}_{abcd} &= \frac{1}{8}\left(Q_{abcd} + Q_{bcad} + Q_{cabd} - Q_{acbd} - Q_{cbad} - Q_{bacd}\right) \\ &+ \frac{1}{8}\left(Q_{dbca} + Q_{dcab} + Q_{dabc} - Q_{dbac} - Q_{dacb} - Q_{dcba}\right) \end{aligned}$$

Lastly we have the $Y5$ piece for which we totally skew symmetrize on the a, b, c, d indices.

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

$$\begin{aligned} \mathcal{K}_{abcd} &= \frac{1}{24}\left(Q_{abcd} - Q_{abdc} - Q_{acbd} + Q_{acdb} + Q_{adbc} - Q_{adcb}\right) \\ &+ \frac{1}{24}\left(-Q_{bacd} + Q_{badc} + Q_{bcad} - Q_{bcda} - Q_{bdac} + Q_{bdca}\right) \\ &+ \frac{1}{24}\left(Q_{cabd} - Q_{cadb} - Q_{cbad} + Q_{cbda} - Q_{cdba} + Q_{cdab}\right) \\ &+ \frac{1}{24}\left(-Q_{dabc} + Q_{dacb} + Q_{dbac} - Q_{dbca} + Q_{dcba} - Q_{dcab}\right) \end{aligned}$$

If we were to follow the wonderful notation of Itin and Reches (Itin & Reches, 2021) on page 9, then we would write a first reducible decomposition as:

$$\boldsymbol{Q} = \boldsymbol{\mathcal{A}} \oplus \boldsymbol{B} \oplus \boldsymbol{E} \oplus \boldsymbol{G} \oplus \boldsymbol{\mathcal{K}} \tag{B.1}$$

Here in equation (B.1) $\boldsymbol{\mathcal{A}}$, and $\boldsymbol{\mathcal{K}}$ are irreducible but $\boldsymbol{B}$,$\boldsymbol{E}$, and $\boldsymbol{G}$ are not irreducible. However $\boldsymbol{\mathcal{A}}, \boldsymbol{B}, \boldsymbol{E}, \boldsymbol{G}$, and $\boldsymbol{\mathcal{K}}$ are orthogonal to each other, i.e.

$$\mathcal{A}_{abcd}\mathcal{K}^{abcd} = B_{abcd}E^{abcd} = \cdots = 0 \tag{B.2}$$

In equations (B.1), and (B.2) we have that $\boldsymbol{B}$ is given by the irreducible decomposition $\boldsymbol{B} = \boldsymbol{\mathcal{B}} \oplus \boldsymbol{\mathcal{C}} \oplus \boldsymbol{\mathcal{D}}$. Likewise $\boldsymbol{E}$ and $\boldsymbol{G}$ are given irreducibly by: $\boldsymbol{E} = \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{F}}$, and $\boldsymbol{G} = \boldsymbol{\mathcal{G}} \oplus \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{J}}$. Similarly to equation (B.2) there are some orthogonality conditions on these tensors. The first of which is that the sub decompositions are all orthogonal if the decomposition is linearly independent. We would write this as:

$$\mathcal{G}_{abcd}\mathcal{H}^{abcd} = \cdots = 0$$

Where the sub decompositions are given by the $\boldsymbol{B}$, $\boldsymbol{E}$, and $\boldsymbol{G}$ equals lines above. Furthermore, these subspaces may not necessarily be orthogonal to the other subspaces:

$$\mathcal{G}_{abcd}\mathcal{E}^{abcd} \neq \mathcal{B}_{abcd}\mathcal{H}^{abcd} \neq \mathcal{C}_{abcd}\mathcal{F}^{abcd} \neq 0$$

All in all this results in the $GL\left(N\right)$ irreducible, non-orthogonal decomposition $\boldsymbol{Q}$ being given by:

$$\boldsymbol{Q} = \boldsymbol{\mathcal{A}} \oplus \boldsymbol{\mathcal{B}} \oplus \boldsymbol{\mathcal{C}} \oplus \boldsymbol{\mathcal{D}} \oplus \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{F}} \oplus \boldsymbol{\mathcal{G}} \oplus \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{J}} \oplus \boldsymbol{\mathcal{K}} \tag{B.3}$$

The decomposition in equation (B.3) can be shown to decompose further under $SO\left(p, q\right)$.

APPENDIX C

FORMULAS FOR THE $SO\left(p,q\right)$ DECOMPOSITION OF AN ARBITRARY RANK 4 TENSOR

In this appendix we present all 25 $SO\left(p,q\right)$ rank 4 irreducible tensors; given an arbitrary tensor these can all be calculated out. However, just as in the Rank 3 tensor case, it is also important for one to ensure linear independence of these tensors, otherwise the decomposition will be dependent if the given tensor has any additional $GL\left(N\right)$ type index symmetries. This section will delve into the concept of $SO\left(p,q\right)$ irreducible tensors by examining the interplay between the structure of the tensors and the structure of the group, this section provides a comprehensive list of rank 4 $SO\left(p,q\right)$ irreducible tensors.

As a refresher, the concept of irreducibility is fundamental to the study of Lie groups and Lie algebras. It refers to the property of a tensor that cannot be decomposed into a direct sum of simpler tensors under a given transformation. In the context of $SO\left(p,q\right)$ irreducible tensors, this means that the tensor cannot be broken down into simpler tensors that transform differently under the action of the special orthogonal group. This property makes $SO\left(p,q\right)$ irreducible tensors particularly useful for describing physical phenomena, as it ensures that all the degrees of freedom of the tensor are captured by its irreducible representation. The decomposition of a tensor into irreducible components provides a way to understand the symmetry structure of the tensor, as well as its relationships with other mathematical objects. Additionally, the analysis of irreducible tensors can reveal new mathematical relationships and structures that are hidden in the tensor.

We will define each of these 25 irreducible tensors below, beginning with the $Y1$ sector, and ending with the $Y5$ sector.

For the $Y1$ sector, we begin with the tableaux:

1	2	3	4

which determines the $GL\left(N\right)$ irreducible tensor $\boldsymbol{\mathcal{A}}$. We call the subspace $\boldsymbol{\mathcal{A}}$ the subspace determined by the tensor $\boldsymbol{\mathcal{A}}$ through an abuse of notation; context should make it apparent which of the two we are using at any given moment. The subspace $\boldsymbol{\mathcal{A}}$ decomposes under $SO\left(p,q\right)$ into the following three subspaces:

$$\boldsymbol{\mathcal{A}} = \mathfrak{a} \oplus \mathscr{A} \oplus \mathfrak{A}$$

The three pieces: $\mathfrak{a}$, $\mathscr{A}$, and $\mathfrak{A}$ determined by the following equations:

$$\mathfrak{a}_{abcd} = \mathcal{A}_{abcd} - \mathscr{A}_{abcd} - \mathfrak{A}_{abcd}$$

$$\mathscr{A}_{abcd} = \frac{6}{\left(N+4\right)} \mathring{P}_{(ab}g_{cd)}, \quad \mathfrak{A}_{abcd} = \frac{3}{N\left(N+2\right)} P g_{(ab}g_{cd)}$$

$$P_{(ab)} = g^{ef}\mathcal{A}_{(efab)}, \quad P = g^{ab}g^{cd}\mathcal{A}_{(abcd)}, \quad \mathring{P}_{(ab)} = P_{(ab)} - \frac{1}{N} P g_{ab}$$

For the $Y2$ sector, we begin with the Ferrers diagram:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

for which there are three Young tableaux:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

which correspond to 3 $GL\left(N\right)$ irreducible tensors $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, and $\boldsymbol{\mathcal{D}}$. The subspaces $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, and $\boldsymbol{\mathcal{D}}$ decompose under $SO\left(p,q\right)$ in the following way:

$$\begin{aligned} \boldsymbol{\mathcal{B}} &= \mathfrak{b} \oplus \text{Б} \oplus \beth \\ \boldsymbol{\mathcal{C}} &= \mathfrak{c} \oplus \text{Ц} \oplus \gimel \\ \boldsymbol{\mathcal{D}} &= \mathfrak{d} \oplus \text{Д} \oplus \daleth \end{aligned}$$

The nine pieces: $\mathfrak{b}$,$\mathfrak{c}$,$\mathfrak{d}$, Б, Ц, Д, $\beth$, $\gimel$, and $\daleth$ are explicitly given below grouped by their corresponding tableaux to make finding them easier while doing research.

For the $Y2a$ sector we have:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}$$

$$\mathfrak{b}_{abcd} = \mathcal{B}_{abcd} - \text{Б}_{abcd} - \beth_{abcd}$$

$$\text{Б}_{abcd} = \frac{1}{N+2}\left(2P_{[ad]}g_{bc} - P_{[ca]}g_{bd} - P_{[ba]}g_{cd} + P_{[bd]}g_{ac} + P_{[cd]}g_{ab}\right)$$

$$\beth_{abcd} = \frac{1}{N}\left(-P_{(ca)}g_{bd} - P_{(ba)}g_{cd} + P_{(bd)}g_{ac} + P_{(cd)}g_{ab}\right)$$

$$P_{cd} = g^{ab}\mathcal{B}_{abcd}$$

For the $Y2b$ sector we have:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$$\mathfrak{c}_{abcd} = \mathcal{C}_{abcd} - \text{Ц}_{abcd} - \gimel_{abcd}$$

$$\text{Ц}_{abcd} = \frac{1}{N+2}\left(-2P_{[ac]}g_{bd} + P_{[ba]}g_{cd} - P_{[bc]}g_{ad} + P_{[da]}g_{bc} - P_{[dc]}g_{ab}\right)$$

$$\gimel_{abcd} = \frac{1}{N}\left(-P_{(ba)}g_{cd} + P_{(bc)}g_{ad} - P_{(da)}g_{bc} + P_{(dc)}g_{ab}\right)$$

$$P_{cd} = g^{ab}\mathcal{C}_{abcd}$$

For the $Y2c$ sector we have:

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$\mathfrak{d}_{abcd} = \mathcal{D}_{abcd} - \text{Д}_{abcd} - \daleth_{abcd}.$$

$$\text{Д}_{abcd} = \frac{1}{N+2}\left(-2R_{[ab]}g_{bd} + R_{[ca]}g_{bd} - R_{[cb]}g_{ad} + R_{[da]}g_{bc} - R_{[db]}g_{ac}\right)$$

$$\daleth_{abcd} = \frac{1}{N}\left(-R_{(ca)}g_{bd} + R_{(cb)}g_{ad} - R_{(da)}g_{bc} + R_{(db)}g_{ac}\right)$$

$$R_{bd} = g^{ac}\mathcal{D}_{abcd}$$

For the $Y3$ sector, we begin with the Ferrers diagram:

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$$

for which there are two Young tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

which correspond to 2 $GL(N)$ irreducible tensors $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{\mathcal{F}}$. The subspaces $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{\mathcal{F}}$ decompose under $SO(p,q)$ in the following way:

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \boldsymbol{\mathfrak{e}} \oplus \mathscr{E} \oplus \boldsymbol{\mathfrak{E}} \\ \boldsymbol{\mathcal{F}} &= \boldsymbol{\mathfrak{f}} \oplus \mathscr{F} \oplus \boldsymbol{\mathfrak{F}} \end{aligned}$$

The six pieces: $\boldsymbol{\mathfrak{e}}$,$\boldsymbol{\mathfrak{f}}$, $\mathscr{E}$, $\mathscr{F}$, $\boldsymbol{\mathfrak{E}}$ and $\boldsymbol{\mathfrak{F}}$ are explicitly given below grouped by their corresponding tableaux to make finding them easier while doing research.

For the $Y3a$ sector we have:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$\mathfrak{e}_{abcd} = \mathcal{E}_{abcd} - \mathscr{E}_{abcd} - \mathfrak{E}_{abcd}$$

$$\mathscr{E}_{abcd} = \frac{1}{(N-2)} \left(\mathring{P}_{ab} g_{cd} - \mathring{P}_{cb} g_{ad} - \mathring{P}_{ad} g_{cb} + \mathring{P}_{cd} g_{ab} \right)$$

$$\mathfrak{E}_{abcd} = \frac{1}{N(N-1)} P \left(g_{ab} g_{cd} - g_{ad} g_{cb} \right)$$

$$P_{cd} = g^{ab} \mathcal{E}_{abcd}, \quad P = g^{ab} g^{cd} \mathcal{E}_{abcd}, \quad \mathring{P}_{(ab)} = P_{(ab)} - \frac{1}{N} P g_{ab}$$

For the $Y3b$ sector we have:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$$\mathfrak{f}_{abcd} = \mathcal{F}_{abcd} - \mathscr{F}_{abcd} - \mathfrak{F}_{abcd}$$

$$\mathscr{F}_{abcd} = \frac{1}{(N-2)} \left(\mathring{R}_{ac} g_{bd} - \mathring{R}_{bc} g_{ad} - \mathring{R}_{ad} g_{bc} + \mathring{R}_{bd} g_{ac} \right)$$

$$\mathfrak{F}_{abcd} = \frac{1}{N(N-1)} R \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right)$$

$$R_{(bd)} = g^{ac} \mathcal{F}_{abcd}, \quad R = g^{ac} g^{bd} \mathcal{F}_{abcd}, \quad \mathring{R}_{(ab)} = R_{(ab)} - \frac{1}{N} R g_{ab}$$

For the $Y4$ sector, we begin with the Ferrers diagram:

for which there are three Young tableaux:

1	2
3	
4	

,

1	3
2	
4	

,

1	4
2	
3	

which correspond to 3 $GL(N)$ irreducible tensors $\boldsymbol{\mathcal{G}}$, $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{\mathcal{J}}$. The subspaces $\boldsymbol{\mathcal{G}}$, $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{\mathcal{J}}$ decompose under $SO(p,q)$ in the following way:

$$\begin{aligned} \boldsymbol{\mathcal{G}} &= \mathfrak{g} \oplus \mathscr{G} \\ \boldsymbol{\mathcal{H}} &= \mathfrak{h} \oplus \mathscr{H} \\ \boldsymbol{\mathcal{J}} &= \mathfrak{j} \oplus \mathscr{J} \end{aligned}$$

The nine pieces: $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{j}$, $\mathscr{G}$, $\mathscr{H}$, and $\mathscr{J}$ are explicitly given below grouped by their corresponding tableaux to make finding them easier while doing research.

1	2
3	
4	

$$\mathfrak{g}_{abcd} = \mathcal{G}_{abcd} - \mathscr{G}_{abcd}$$

$$\mathscr{G}_{abcd} = \frac{1}{(N-2)}\left(P_{[cd]}g_{ab} - P_{[ad]}g_{bc} + P_{[ac]}g_{bd}\right)$$

$$P_{cd} = g^{ab}\mathcal{G}_{abcd}$$

For the $Y4b$ sector we have:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\mathfrak{h}_{abcd} = \mathcal{H}_{abcd} - \mathscr{H}_{abcd}$$

$$\mathscr{H}_{abcd} = \frac{1}{(N-2)}\left(R_{[bd]}g_{ac} - R_{[ad]}g_{bc} + R_{[ab]}g_{cd}\right)$$

$$R_{cd} = g^{ac}\mathcal{H}_{abcd}$$

For the $Y4c$ sector we have:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\mathfrak{j}_{abcd} = \mathcal{J}_{abcd} - \mathscr{J}_{abcd}$$

$$\mathscr{G}_{abcd} = \frac{1}{(N-2)}\left(U_{[bc]}g_{ad} - U_{[ac]}g_{bd} + U_{[ab]}g_{cd}\right)$$

$$U_{cd} = g^{ad}\mathcal{J}_{abcd}$$

The $Y5$ sector does not compose further under $SO(p,q)$. To keep with the notation however we write:

$$\boldsymbol{\mathcal{K}} = \mathfrak{k}$$

$$\mathfrak{k}_{abcd} = Q_{[abcd]}$$

Like we did in the rank 4 $GL\left(N\right)$ appendix, see appendix (B), we follow the notation of Itin and Reches (Itin & Reches, 2021) on page 9. This results in the $SO\left(p,q\right)$ decomposition:

$$
\begin{aligned}
\boldsymbol{Q} \;=\; & \boldsymbol{\mathfrak{a}} \oplus \mathscr{A} \oplus \boldsymbol{\mathfrak{A}} \oplus \\
& \boldsymbol{\mathfrak{b}} \oplus \text{Б} \oplus \beth \oplus \boldsymbol{\mathfrak{c}} \oplus \text{Ц} \oplus \gimel \oplus \boldsymbol{\mathfrak{d}} \oplus \text{Д} \oplus \daleth \oplus \\
& \boldsymbol{\mathfrak{e}} \oplus \mathscr{E} \oplus \boldsymbol{\mathfrak{E}} \oplus \boldsymbol{\mathfrak{f}} \oplus \mathscr{F} \oplus \boldsymbol{\mathfrak{F}} \oplus \\
& \boldsymbol{\mathfrak{g}} \oplus \mathscr{G} \oplus \boldsymbol{\mathfrak{h}} \oplus \mathscr{H} \oplus \boldsymbol{\mathfrak{j}} \oplus \mathscr{J} \oplus \\
& \boldsymbol{\mathfrak{k}}
\end{aligned} \tag{C.1}
$$

which we have grouped in lines according to their Ferrers diagram. This is the irreducible $SO\left(p,q\right)$ decomposition for an arbitrary rank 4 tensor; again if the tensor $\boldsymbol{Q}$ has any additional symmetries one must check that these spaces are independent in a similar manner to how we checked in the $GL\left(N\right)$ case.

APPENDIX D

RANK 4 $SO(p,q)$ DECOMPOSITION PROOFS

In this appendix we provide proofs for the $SO(p,q)$ decomposition of a 4th rank tensor. Recall that the $GL(N)$ irreducible tensors are given by the following decomposition:

$$\begin{aligned} Q_{abcd} &= \mathcal{A}_{abcd} \\ &+ \mathcal{B}_{abcd} + \mathcal{C}_{abcd} + \mathcal{D}_{abcd} \\ &+ \mathcal{E}_{abcd} + \mathcal{F}_{abcd} \\ &+ \mathcal{G}_{abcd} + \mathcal{H}_{abcd} + \mathcal{J}_{abcd} \\ &+ \mathcal{K}_{abcd} \end{aligned} \tag{D.1}$$

where each of the calligraphic letters represent the Young projectors. We follow Hammermesh (Hammermesh, 1962/1989) and apply the decomposition:

$$\begin{aligned} Q_{abcd} &= \mathring{A}_{ab}g_{cd} + \mathring{B}_{ac}g_{bd} + \mathring{C}_{ad}g_{bc} + \mathring{D}_{bc}g_{ad} + \mathring{E}_{bd}g_{ac} + \mathring{F}_{cd}g_{ab} \\ &+ H g_{ab}g_{cd} + J g_{ac}g_{bd} + K g_{ad}g_{bc} + W_{abcd} \end{aligned} \tag{D.2}$$

to each of the $GL(N)$ irreducible tensors to decompose them under $SO(p,q)$. The tensor W_{abcd} is the totally trace free piece and $\mathring{A}_{ab} \ldots \mathring{F}_{ab}$ are trace free second rank tensors with no presumed symmetries. We will use equation (D.2) to decompose each of the 10 Young sectors under $SO(p,q)$. As in the 3rd Rank tensor case, we define tensors that relate our above traces to just traces on a general fourth rank tensor Q_{abcd}. Later we will think of Q_{abcd} as $\mathcal{A}_{abcd} \ldots \mathcal{K}_{abcd}$ as given by equation (D.1). It will be convenient to have the following definitions:

$$\begin{gathered} g^{ab}Q_{abcd} = P_{cd}, \quad g^{ac}Q_{abcd} = R_{bd}, \quad g^{ad}Q_{abcd} = U_{bc} \\ g^{bc}Q_{abcd} = X_{ad}, \quad g^{bd}Q_{abcd} = Y_{ac}, \quad g^{cd}Q_{abcd} = Z_{ab} \end{gathered} \tag{D.3}$$

$$g^{ab}g^{cd}Q_{abcd} = P, \quad g^{ac}g^{bd}Q_{abcd} = R, \quad g^{ad}g^{bc}Q_{abcd} = U$$

With all of this established, we can determine all 25 $SO(p,q)$ irreducible tensors, and we give proofs to show how we decomposed them. Hammermesh (Hammermesh, 1962/1989) guarantees that taking traces in this fashion and applying Young tableaux methods will produce the $SO(p,q)$ irreducible elements.

D.1 Y1 - $\mathfrak{a}, \mathscr{A}, \mathfrak{A}$

Below we give a proof for the $Y1$ sector $SO(p,q)$ decomposition.

1	2	3	4

Let us first look at the totally symmetric piece: $Y1(Q)_{abcd} = \mathcal{A}_{abcd}$. We would have the symmetry $\mathcal{A}_{abcd} = \mathcal{A}_{(abcd)}$. Let us also define $Y1(W)_{abcd} = \mathfrak{a}_{abcd}$. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned}
\mathcal{A}_{abcd} &= \frac{1}{24}\left(Q_{abcd} + Q_{abdc} + Q_{acbd} + Q_{acdb} + Q_{adbc} + Q_{adcb}\right) \\
&+ \frac{1}{24}\left(Q_{bacd} + Q_{badc} + Q_{bcad} + Q_{bcda} + Q_{bdac} + Q_{bdca}\right) \\
&+ \frac{1}{24}\left(Q_{cabd} + Q_{cadb} + Q_{cbad} + Q_{cbda} + Q_{cdab} + Q_{cdba}\right) \\
&+ \frac{1}{24}\left(Q_{dabc} + Q_{dacb} + Q_{dbac} + Q_{dbca} + Q_{dcab} + Q_{dcba}\right)
\end{aligned}$$

This then shows us that we can write $\mathcal{A}_{abcd}$ as:

$$\mathcal{A}_{abcd} = \mathring{G}_{(ab)}g_{cd} + \mathring{G}_{(ac)}g_{bd} + \mathring{G}_{(ad)}g_{bc} + \mathring{G}_{(bc)}g_{ad} + \mathring{G}_{(bd)}g_{ac} + \mathring{G}_{(cd)}g_{ab} + L\left(g_{ab}g_{cd} + g_{ac}g_{bd} + g_{ad}g_{bc}\right) + \mathfrak{a}_{abcd}$$

Next we will define 2 more tensors:

$$\mathscr{A}_{abcd} = \mathring{G}_{(ab)}g_{cd} + \mathring{G}_{(ac)}g_{bd} + \mathring{G}_{(ad)}g_{bc} + \mathring{G}_{(bc)}g_{ad} + \mathring{G}_{(bd)}g_{ac} + \mathring{G}_{(cd)}g_{ab}$$

$$\mathfrak{A}_{abcd} = L\left(g_{ab}g_{cd} + g_{ac}g_{bd} + g_{ad}g_{bc}\right)$$

Let us apply total symmetry to the equations we got from maple before. Doing this by hand would have been a pain! We find: We can write:

$$g^{ab}\mathcal{A}_{(abcd)} = P_{(cd)}$$

$$g^{ac}\mathcal{A}_{(abcd)} = R_{(bd)} = g^{ac}\mathcal{A}_{(acbd)} = P_{(bd)}$$

$$g^{ad}\mathcal{A}_{(abcd)} = U_{(bc)} = g^{ad}\mathcal{A}_{(adbc)} = P_{(bc)}$$

$$g^{bc}\mathcal{A}_{(abcd)} = X_{(ad)} = g^{bc}\mathcal{A}_{(bcad)} = P_{(ad)}$$

$$g^{bd}\mathcal{A}_{(abcd)} = Y_{(ac)} = g^{bd}\mathcal{A}_{(bdac)} = P_{(ac)}$$

$$g^{cd}\mathcal{A}_{(abcd)} = Z_{(ab)} = g^{cd}\mathcal{A}_{(cdab)} = P_{(ab)}$$

$$g^{ab}g^{cd}\mathcal{A}_{(abcd)} = P$$

$$g^{ac}g^{bd}\mathcal{A}_{(abcd)} = R = g^{ac}g^{bd}\mathcal{A}_{(acbd)} = P$$

$$g^{ad}g^{bc}\mathcal{A}_{(abcd)} = U = g^{ad}g^{bc}\mathcal{A}_{(adbc)} = P$$

Solving these equations for $\mathring{G}_{ab}, L$ gives us:

$$\mathring{G}_{(cd)} = \frac{1}{(N+4)}P_{(cd)} - \frac{(N+2)}{(N+4)}Lg_{cd}$$

$$L = \frac{1}{N\,(N+2)}P$$

Simplifying, we find:

$$\mathring{G}_{(cd)} = \frac{1}{(N+4)}\mathring{P}_{(cd)}$$

$$\mathring{P}_{(cd)} = P_{(cd)} - \frac{1}{N}Pg_{cd}$$

$$L = \frac{1}{N\,(N+2)}P$$

This in turn lets us write $\mathscr{A}_{abcd}$, and $\mathfrak{A}_{abcd}$ as:

$$\mathscr{A}_{abcd} = \frac{1}{(N+4)}\left(\mathring{P}_{(ab)}g_{cd} + \mathring{P}_{(ac)}g_{bd} + \mathring{P}_{(ad)}g_{bc} + \mathring{P}_{(bc)}g_{ad} + \mathring{P}_{(bd)}g_{ac} + \mathring{P}_{(cd)}g_{ab}\right)$$

$$\mathfrak{A}_{abcd} = \frac{1}{N\left(N+2\right)}P\left(g_{ab}g_{cd} + g_{ac}g_{bd} + g_{ad}g_{bc}\right)$$

Thus, we can now write our $Y1$ decomposition as:

$$\mathcal{A}_{abcd} = \mathfrak{a}_{abcd} + \mathscr{A}_{abcd} + \mathfrak{A}_{abcd}$$

$$\mathscr{A}_{abcd} = \frac{6}{(N+4)}\mathring{P}_{(ab}g_{cd)}$$

$$\mathfrak{A}_{abcd} = \frac{3}{N\left(N+2\right)}Pg_{(ab}g_{cd)}$$

$$P_{(cd)} = g^{ab}\mathcal{A}_{(abcd)}$$

$$P = g^{ab}g^{cd}\mathcal{A}_{(abcd)}$$

$$\mathring{P}_{(ab)} = P_{(ab)} - \frac{1}{N}Pg_{ab}$$

D.2 Y2 - $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \beth, \gimel, \daleth$, Б, Ц, Д

Below we give proofs for the $Y2$ sector $SO\left(p,q\right)$ decomposition.

D.3 Y2a - $\mathfrak{b}, \beth$, Б

Below we give a proof for the $Y2a$ sector $SO\left(p,q\right)$ decomposition.

1	2	3
4		

Let us first look at the $Y2a$ piece: $Y2a\left(Q\right)_{abcd} = \mathcal{B}_{abcd}$. We would have the symmetry$\mathcal{B}_{abcd} = \mathcal{B}_{a(bc)d} = \mathcal{B}_{[a|bc|d]}$. Let us also define $Y2a\left(W\right)_{abcd} = \mathfrak{b}_{abcd}$. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned}\mathcal{B}_{abcd} &= \frac{1}{8}\left(+Q_{abcd}+Q_{bcad}+Q_{cabd}+Q_{acbd}+Q_{cbad}+Q_{bacd}\right)\\ &+ \frac{1}{8}\left(-Q_{dbca}-Q_{bcda}-Q_{cdba}-Q_{dcba}-Q_{cbda}-Q_{bdca}\right)\end{aligned}$$

If we use a degrees of freedom argument, we can find that the trace part: $\mathscr{B}_{abcd}$ is generated by a second rank general trace free tensor $\mathring{a}_{ad}$. This then tells us that we can write $\mathcal{B}_{abcd}$ as:

$$\mathcal{B}_{abcd} = \mathscr{B}_{abcd} + \mathfrak{b}_{abcd}$$

$$\mathscr{B}_{abcd} = Y2a\left(\mathring{a}_{ad}g_{bc}\right)$$

After applying the Young symmetrizer, we have:

$$\mathscr{B}_{abcd} = 2\left(\mathring{a}_{ad}-\mathring{a}_{da}\right)g_{bc} - 2\mathring{a}_{ca}g_{bd} - 2\mathring{a}_{ba}g_{cd} + 2\mathring{a}_{bd}g_{ac} + 2\mathring{a}_{cd}g_{ab}$$

Let us absorb the factor of 2 into $\mathring{a}_{ab}$ and call it Υ:

$$\mathscr{B}_{abcd} = \left(\mathring{\Upsilon}_{ad}-\mathring{\Upsilon}_{da}\right)g_{bc} - \mathring{\Upsilon}_{ca}g_{bd} - \mathring{\Upsilon}_{ba}g_{cd} + \mathring{\Upsilon}_{bd}g_{ac} + \mathring{\Upsilon}_{cd}g_{ab}$$

We will now go on to show that this is not the irreducible form of $\mathcal{B}$. We now decompose $\mathcal{B}$ further. Recall from earlier that we defined the P, R, U, X, Y, Z tensors and the P, R, U scalars. Their definitions on any 4th rank tensor $\mathcal{B}$ are below:

$$g^{ab}\mathcal{B}_{abcd} = P_{cd}$$

$$g^{ac}\mathcal{B}_{abcd} = R_{bd}$$

$$g^{ad}\mathcal{B}_{abcd} = U_{bc}$$

$$g^{bc}\mathcal{B}_{abcd} = X_{ad}$$

$$g^{bd}\mathcal{B}_{abcd} = Y_{ac}$$

$$g^{cd}\mathcal{B}_{abcd} = Z_{ab}$$

$$g^{ab}g^{cd}\mathcal{B}_{abcd} = P$$

$$g^{ac}g^{bd}\mathcal{B}_{abcd} = R$$

$$g^{ad}g^{bc}\mathcal{B}_{abcd} = U$$

However, we know that in our case $\mathcal{B}$ has symmetries. This will in turn make some of these definitions overlap. Let us begin by shoving our definition of $\mathcal{B}$, $\mathcal{B}_{abcd} = \left(\mathring{A}_{ab} - \mathring{F}_{ba}\right) g_{cd} + \mathring{C}_{ad}g_{bc} + H g_{ab}g_{cd} + W_{abcd}$ into the above equations. Then we find:

$$\mathcal{B}_{abcd} = \left(\mathring{\Upsilon}_{ad} - \mathring{\Upsilon}_{da}\right) g_{bc} - \mathring{\Upsilon}_{ca}g_{bd} - \mathring{\Upsilon}_{ba}g_{cd} + \mathring{\Upsilon}_{bd}g_{ac} + \mathring{\Upsilon}_{cd}g_{ab} + \mathfrak{b}_{abcd}$$

$$g^{ab}\mathcal{B}_{abcd} = P_{cd} = 2\mathring{\Upsilon}_{[cd]} + \mathring{\Upsilon}_{cd}N$$

$$g^{ac}\mathcal{B}_{abcd} = R_{bd} = 2\mathring{\Upsilon}_{[bd]} + \mathring{\Upsilon}_{bd}N$$

$$g^{ad}\mathcal{B}_{abcd} = 0 = -\mathring{\Upsilon}_{cb} - \mathring{\Upsilon}_{bc} + \mathring{\Upsilon}_{bc} + \mathring{\Upsilon}_{cb} = 0$$

$$g^{bc}\mathcal{B}_{abcd} = X_{ad} = 2\mathring{\Upsilon}_{[ad]}\left(N+2\right)$$

$$g^{bd}\mathcal{B}_{abcd} = Y_{ac} = 2\mathring{\Upsilon}_{[ac]} - \mathring{\Upsilon}_{ca}N$$

$$g^{cd}\mathcal{B}_{abcd} = Z_{ab} = 2\mathring{\Upsilon}_{[ab]} - \mathring{\Upsilon}_{ba}N$$

$$g^{ab}g^{cd}\mathcal{B}_{abcd} = 0$$

$$g^{ac}g^{bd}\mathcal{B}_{abcd} = 0$$

$$g^{ad}g^{bc}\mathcal{B}_{abcd} = 0$$

From here we will use the symmetries $\mathcal{B}_{abcd} = \mathcal{B}_{a(bc)d} = \mathcal{B}_{[a|bc|d]}$ to cut down some definitions. For example, we will find that $R_{bd} = P_{bd}$, etc. This very much simplifies the computation down to something manageable.

$$g^{ab}\mathcal{B}_{abcd} = P_{cd}$$

$$g^{ac}\mathcal{B}_{abcd} = R_{bd} = g^{ac}\mathcal{B}_{acbd} = P_{bd}$$

$$g^{ad}\mathcal{B}_{abcd} = 0$$

$$g^{bc}\mathcal{B}_{abcd} = X_{[ad]}$$

$$g^{bd}\mathcal{B}_{abcd} = Y_{ac} = -g^{bd}\mathcal{B}_{dbca} = -P_{ca}$$

$$g^{cd}\mathcal{B}_{abcd} = Z_{ab} = -g^{cd}\mathcal{B}_{dcba} = -P_{ba}$$

$$g^{ab}g^{cd}\mathcal{B}_{abcd} = P$$

$$g^{ac}g^{bd}\mathcal{B}_{abcd} = R = g^{ac}g^{bd}\mathcal{B}_{acbd} = P$$

$$g^{ad}g^{bc}\mathcal{B}_{abcd} = 0$$

Now since we have simplified some equations, we shove them back into our starting 6.

$$\mathcal{B}_{abcd} = \left(\mathring{\Upsilon}_{ad} - \mathring{\Upsilon}_{da}\right) g_{bc} - \mathring{\Upsilon}_{ca} g_{bd} - \mathring{\Upsilon}_{ba} g_{cd} + \mathring{\Upsilon}_{bd} g_{ac} + \mathring{\Upsilon}_{cd} g_{ab} + \mathfrak{b}_{abcd}$$

$$P_{cd} = P_{cd} = 2\mathring{\Upsilon}_{[cd]} + \mathring{\Upsilon}_{cd} N$$

$$P_{bd} = R_{bd} = 2\mathring{\Upsilon}_{[bd]} + \mathring{\Upsilon}_{bd} N$$

$$0 = 0$$

$$X_{[ad]} = 2\mathring{\Upsilon}_{[ad]} \left(N + 2\right)$$

$$-P_{ca} = 2\mathring{\Upsilon}_{[ac]} - \mathring{\Upsilon}_{ca} N$$

$$-P_{ba} = 2\mathring{\Upsilon}_{[ab]} - \mathring{\Upsilon}_{ba} N$$

$$P = 0$$

There are many indices floating around now. To relate these equations, let us relabel these only using the indices ab, and then get rid of repeated equations:

$$\mathcal{B}_{abcd} = \left(\mathring{\Upsilon}_{ad} - \mathring{\Upsilon}_{da}\right) g_{bc} - \mathring{\Upsilon}_{ca} g_{bd} - \mathring{\Upsilon}_{ba} g_{cd} + \mathring{\Upsilon}_{bd} g_{ac} + \mathring{\Upsilon}_{cd} g_{ab} + \mathfrak{b}_{abcd}$$

$$P_{ab} = 2\mathring{\Upsilon}_{[ab]} + \mathring{\Upsilon}_{ab} N$$

$$X_{[ab]} = 2\mathring{\Upsilon}_{[ab]} \left(N + 2\right)$$

$$P = 0$$

Before that, we can find something out using the skew part of P_{ab}:

$$P_{[ab]} = \frac{1}{2}\left(P_{ab} - P_{ba}\right) = \mathring{\Upsilon}_{[ab]} \left(N + 2\right)$$

We find that:

$$P_{[ab]} = \frac{1}{2} X_{[ab]}$$

Therefore, we really only have 1 independent equation:

$$\mathcal{B}_{abcd} = \left(\mathring{\Upsilon}_{ad} - \mathring{\Upsilon}_{da}\right) g_{bc} - \mathring{\Upsilon}_{ca} g_{bd} - \mathring{\Upsilon}_{ba} g_{cd} + \mathring{\Upsilon}_{bd} g_{ac} + \mathring{\Upsilon}_{cd} g_{ab} + \mathfrak{b}_{abcd}$$

$$P_{ab} = 2\mathring{\Upsilon}_{[ab]} + \mathring{\Upsilon}_{ab} N$$

$$P_{[ab]} = \mathring{\Upsilon}_{[ab]} \left(N + 2\right)$$

$$P = 0$$

Now we solve. The Second equation gives us:

$$\mathring{\Upsilon}_{[ab]} = \frac{1}{(N+2)} P_{[ab]}$$

Shoving the second into the first gives us:

$$\mathring{\Upsilon}_{ab} = \frac{1}{N}P_{ab} - \frac{2}{N\left(N+2\right)}P_{[ab]}$$

We are done. Now we have a way to calculate our equation.

$$\mathscr{B}_{abcd} = 2\mathring{\Upsilon}_{[ad]}g_{bc} - \mathring{\Upsilon}_{ca}g_{bd} - \mathring{\Upsilon}_{ba}g_{cd} + \mathring{\Upsilon}_{bd}g_{ac} + \mathring{\Upsilon}_{cd}g_{ab}$$

However, because $\mathring{\Upsilon}_{ab}$ is only traceless we can again use the structure contained in $GL\left(N\right)$ to break off a symmetric and skew piece. We will call the Skew part of $\mathring{\Upsilon}_{ab}$ the Cyrillic Zhe: $Ж_{ab} = \mathring{\Upsilon}_{[ab]}$, and the resulting tensor generated by $Ж_{ab}$ will be called the Cyrillic Be: $Б_{abcd}$. Next, we will call the symmetric part of $\mathring{\Upsilon}_{ab}$ the Cyrillic Yu: $Ю_{ab} = \mathring{\Upsilon}_{(ab)}$, and the resulting tensor generated by $Ю_{ab}$ will be called the Hebrew Beth: $\beth_{abcd}$.This then gives us:

$$\mathscr{B}_{abcd} = Б_{abcd} + \beth_{abcd}$$

$$Б_{abcd} = 2Ж_{[ad]}g_{bc} - Ж_{ca}g_{bd} - Ж_{ba}g_{cd} + Ж_{bd}g_{ac} + Ж_{cd}g_{ab}$$

$$\beth_{abcd} = -Ю_{ca}g_{bd} - Ю_{ba}g_{cd} + Ю_{bd}g_{ac} + Ю_{cd}g_{ab}$$

And we are left with what we should actually have as our trace decomposition of $Y2a$:

$$\mathcal{B}_{abcd} = Б_{abcd} + \beth_{abcd} + \mathfrak{b}_{abcd}$$

$$\mathfrak{b}_{abcd} = \mathcal{B}_{abcd} - Б_{abcd} - \beth_{abcd}$$

$$\mathring{\Upsilon}_{ab} = \frac{1}{N}P_{ab} - \frac{2}{N\left(N+2\right)}P_{[ab]}$$

$$Ж_{ab} = \mathring{\Upsilon}_{[ab]}$$

$$Ю_{ab} = \mathring{\Upsilon}_{(ab)}$$

Then we find:

$$\text{Ж}_{ab} = \frac{1}{N+2} P_{[ab]}$$

$$\text{Ю}_{ab} = \frac{1}{N} P_{(ab)}$$

Which gives us:

$$\text{Б}_{abcd} = \frac{1}{N+2}\left(2P_{[ad]}g_{bc} - P_{[ca]}g_{bd} - P_{[ba]}g_{cd} + P_{[bd]}g_{ac} + P_{[cd]}g_{ab}\right)$$

$$\beth_{abcd} = \frac{1}{N}\left(-P_{(ca)}g_{bd} - P_{(ba)}g_{cd} + P_{(bd)}g_{ac} + P_{(cd)}g_{ab}\right)$$

The irreducible form of $\mathcal{B}_{abcd}$ is given by:

$$\mathcal{B}_{abcd} = \text{Б}_{abcd} + \beth_{abcd} + \mathfrak{b}_{abcd}$$

$$\mathfrak{b}_{abcd} = \mathcal{B}_{abcd} - \text{Б}_{abcd} - \beth_{abcd}$$

$$\text{Б}_{abcd} = \frac{1}{N+2}\left(2P_{[ad]}g_{bc} - P_{[ca]}g_{bd} - P_{[ba]}g_{cd} + P_{[bd]}g_{ac} + P_{[cd]}g_{ab}\right)$$

$$\beth_{abcd} = \frac{1}{N}\left(-P_{(ca)}g_{bd} - P_{(ba)}g_{cd} + P_{(bd)}g_{ac} + P_{(cd)}g_{ab}\right)$$

$$P_{cd} = g^{ab}\mathcal{B}_{abcd}$$

D.4 Y2b - $\mathfrak{c}, \gimel,$ Ц

Below we give a proof for the $Y2b$ sector $SO(p,q)$ decomposition.

1	2	4
3		

Let us first look at the $Y2b$ piece: $Y2b\,(Q)_{abcd} = \mathcal{C}_{abcd}$. We would have the symmetry$\mathcal{C}_{abcd} = \mathcal{C}_{a(b|c|d)} = \mathcal{C}_{[a|b|c]d}$. Let us also define $Y2b\,(W)_{abcd} = \mathfrak{c}_{abcd}$. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned}\mathcal{C}_{abcd} &= \frac{1}{8}\left(+Q_{abcd}+Q_{adcb}+Q_{bacd}+Q_{bdca}+Q_{dacb}+Q_{dbca}\right)\\ &+ \frac{1}{8}\left(-Q_{cbad}-Q_{cdab}-Q_{bcad}-Q_{bdac}-Q_{dcab}-Q_{dbac}\right)\end{aligned}$$

If we use a degrees of freedom argument, we can find that the trace part: $\mathscr{C}_{abcd}$ is generated by a second rank general trace free tensor $\mathring{b}_{ac}$. This then tells us that we can write $\mathcal{C}_{abcd}$ as:

$$\mathcal{C}_{abcd} = \mathscr{C}_{abcd} + \mathfrak{c}_{abcd}$$

$$\mathscr{C}_{abcd} = Y2b\left(\mathring{b}_{ac}g_{bd}\right)$$

After applying the Young symmetrizer, we have:

$$\mathscr{C}_{abcd} = 2\left(\mathring{b}_{ac}-\mathring{b}_{ca}\right)g_{bd} - 2\mathring{b}_{ba}g_{cd} + 2\mathring{b}_{bc}g_{ad} - 2\mathring{b}_{da}g_{bc} + 2\mathring{b}_{dc}g_{ab}$$

Let us absorb the factor of 2 into $\mathring{b}_{ab}$ and call it Υ:

$$\mathscr{C}_{abcd} = \left(\mathring{\Upsilon}_{ac}-\mathring{\Upsilon}_{ca}\right)g_{bd} - \mathring{\Upsilon}_{ba}g_{cd} + \mathring{\Upsilon}_{bc}g_{ad} - \mathring{\Upsilon}_{da}g_{bc} + \mathring{\Upsilon}_{dc}g_{ab}$$

We will now go on to show that this is NOT the simplest form of $\mathcal{C}$ as we previously thought. The derivation comes next. Recall from earlier that we defined the P, R, U, X, Y, Z tensors and the P, R, U scalars. Their definitions on any 4th rank tensor $\mathcal{B}$ are below:

$$g^{ab}\mathcal{C}_{abcd} = P_{cd}$$

$$g^{ac}\mathcal{C}_{abcd} = R_{bd}$$

$$g^{ad}\mathcal{C}_{abcd} = U_{bc}$$

$$g^{bc}\mathcal{C}_{abcd} = X_{ad}$$

$$g^{bd}\mathcal{C}_{abcd} = Y_{ac}$$

$$g^{cd}\mathcal{C}_{abcd} = Z_{ab}$$

$$g^{ab}g^{cd}\mathcal{C}_{abcd} = P$$

$$g^{ac}g^{bd}\mathcal{C}_{abcd} = R$$

$$g^{ad}g^{bc}\mathcal{C}_{abcd} = U$$

From here we will use the symmetries $\mathcal{C}_{abcd} = \mathcal{C}_{a(b|c|d)} = \mathcal{C}_{[a|b|c]d}$ to cut down some definitions. For example, we will find that $R_{bd} = P_{bd}$, etc. This very much simplifies the computation down to something manageable.

$$g^{ab}\mathcal{C}_{abcd} = P_{cd}$$

$$g^{ac}\mathcal{C}_{abcd} = 0$$

$$g^{ad}\mathcal{C}_{abcd} = U_{bc} = g^{ad}\mathcal{C}_{adcb} = P_{bc}$$

$$g^{bc}\mathcal{C}_{abcd} = X_{ad} = -g^{bc}\mathcal{C}_{cbad} = -P_{ad}$$

$$g^{bd}\mathcal{C}_{abcd} = Y_{[ac]}$$

$$g^{cd}\mathcal{C}_{abcd} = Z_{ab} = -g^{cd}\mathcal{C}_{cdab} = -P_{ab}$$

$$g^{ab}g^{cd}\mathcal{C}_{abcd} = P$$

$$g^{ac}g^{bd}\mathcal{C}_{abcd} = 0$$

$$g^{ad}g^{bc}\mathcal{C}_{abcd} = U = g^{ad}g^{bc}\mathcal{C}_{adcb} = P$$

However, we know that in our case $\mathcal{C}$ has symmetries. This will in turn make some of these definitions overlap. Let us begin by shoving our definition of $\mathcal{C}$ into the above equations. Then we find:

$$\mathscr{C}_{abcd} = \left(\mathring{\Upsilon}_{ac} - \mathring{\Upsilon}_{ca}\right) g_{bd} - \mathring{\Upsilon}_{ba}g_{cd} + \mathring{\Upsilon}_{bc}g_{ad} - \mathring{\Upsilon}_{da}g_{bc} + \mathring{\Upsilon}_{dc}g_{ab}$$

$$P_{cd} = 2\mathring{\Upsilon}_{[dc]} + \mathring{\Upsilon}_{dc} N$$

$$Y_{[ac]} = 2\left(N+2\right)\mathring{\Upsilon}_{[ac]}$$

$$P = 0$$

There are many indices floating around now. To relate these equations, let us relabel these only using the indices ab, and then get rid of repeated equations:

$$\mathscr{C}_{abcd} = \left(\mathring{\Upsilon}_{ac} - \mathring{\Upsilon}_{ca}\right) g_{bd} - \mathring{\Upsilon}_{ba} g_{cd} + \mathring{\Upsilon}_{bc} g_{ad} - \mathring{\Upsilon}_{da} g_{bc} + \mathring{\Upsilon}_{dc} g_{ab}$$

$$P_{cd} = 2\mathring{\Upsilon}_{[dc]} + \mathring{\Upsilon}_{dc} N$$

$$Y_{[ab]} = 2\left(N+2\right)\mathring{\Upsilon}_{[ab]}$$

$$P = 0$$

Before that, we can find something out using the skew part of P_{ab}:

$$P_{ab} = 2\mathring{\Upsilon}_{[ba]} + \mathring{\Upsilon}_{ba} N$$

$$P_{[ab]} = \frac{1}{2}\left(P_{ab} - P_{ba}\right) = -\mathring{\Upsilon}_{[ab]}\left(N+2\right)$$

$$\mathring{\Upsilon}_{[ab]} = -\frac{1}{(N+2)} P_{[ab]}$$

We find that:

$$Y_{[ab]} = -2P_{[ab]}$$

Therefore, we really only have 1 independent equation:

$$P_{ab} = 2\mathring{\Upsilon}_{[ba]} + \mathring{\Upsilon}_{ba} N$$

Now solve for $\mathring{\Upsilon}_{ab}$. Let us use the skew equation.

$$P_{[ba]} = -\mathring{\Upsilon}_{[ba]} \left(N + 2\right)$$

Then we find that:

$$P_{ab} = \frac{2}{(N+2)} P_{[ab]} + \mathring{\Upsilon}_{ba} N$$

$$\mathring{\Upsilon}_{ba} = \frac{1}{N} P_{ab} - \frac{2}{N\,(N+2)} P_{[ab]}$$

$$\mathring{\Upsilon}_{[ab]} = -\frac{1}{(N+2)} P_{[ab]}$$

And, we are done! And now we have a way to calculate our equation!

$$\mathscr{C}_{abcd} = 2\mathring{\Upsilon}_{[ac]} g_{bd} - \mathring{\Upsilon}_{ba} g_{cd} + \mathring{\Upsilon}_{bc} g_{ad} - \mathring{\Upsilon}_{da} g_{bc} + \mathring{\Upsilon}_{dc} g_{ab}$$

However, because $\mathring{\Upsilon}_{ab}$ is only traceless we can again use the structure contained in $GL\,(N)$ to break off a symmetric and skew piece. We will call the Skew part of $\mathring{\Upsilon}_{ab}$ the Cyrillic Zhe: $\text{Ж}_{ab} = \mathring{\Upsilon}_{[ab]}$, and the resulting tensor generated by Ж_{ab} will be called the Cyrillic Ce: Ц_{abcd}. Next, we will call the symmetric part of $\mathring{\Upsilon}_{ab}$ the Cyrillic Yu: $\text{Ю}_{ab} = \mathring{\Upsilon}_{(ab)}$, and the resulting tensor generated by Ю_{ab} will be called the Hebrew Gimel: $\gimel_{abcd}$. This then gives us:

$$\mathscr{C}_{abcd} = \text{Ц}_{abcd} + \gimel_{abcd}$$

$$\text{Ц}_{abcd} = 2\text{Ж}_{[ac]} g_{bd} - \text{Ж}_{ba} g_{cd} + \text{Ж}_{bc} g_{ad} - \text{Ж}_{da} g_{bc} + \text{Ж}_{dc} g_{ab}$$

$$\gimel_{abcd} = -\text{Ю}_{ba} g_{cd} + \text{Ю}_{bc} g_{ad} - \text{Ю}_{da} g_{bc} + \text{Ю}_{dc} g_{ab}$$

And we are left with what we should actually have as our trace decomposition of $Y2a$:

$$\mathcal{C}_{abcd} = \text{Ц}_{abcd} + \gimel_{abcd} + \mathfrak{c}_{abcd}$$

$$\mathfrak{c}_{abcd} = \mathcal{C}_{abcd} - \text{Ц}_{abcd} - \gimel_{abcd}$$

$$\mathring{\Upsilon}_{ba} = \frac{1}{N} P_{ab} - \frac{2}{N\left(N+2\right)} P_{[ab]}$$

$$\text{Ж}_{ab} = -\mathring{\Upsilon}_{[ab]}$$

$$\text{Ю}_{ab} = \mathring{\Upsilon}_{(ab)}$$

We then find that:

$$\text{Ж}_{ab} = -\frac{1}{N+2} P_{[ab]}$$

$$\text{Ю}_{ab} = \frac{1}{N} P_{(ab)}$$

$$\text{Ц}_{abcd} = \frac{1}{N+2}\left(-2P_{[ac]}g_{bd} + P_{[ba]}g_{cd} - P_{[bc]}g_{ad} + P_{[da]}g_{bc} - P_{[dc]}g_{ab}\right)$$

$$\gimel_{abcd} = \frac{1}{N}\left(-P_{(ba)}g_{cd} + P_{(bc)}g_{ad} - P_{(da)}g_{bc} + P_{(dc)}g_{ab}\right)$$

The irreducible form of $\mathcal{C}_{abcd}$ is given by:

$$\mathcal{C}_{abcd} = \text{Ц}_{abcd} + \gimel_{abcd} + \mathfrak{c}_{abcd}$$

$$\mathfrak{c}_{abcd} = \mathcal{C}_{abcd} - \text{Ц}_{abcd} - \gimel_{abcd}$$

$$\text{Ц}_{abcd} = \frac{1}{N+2}\left(-2P_{[ac]}g_{bd} + P_{[ba]}g_{cd} - P_{[bc]}g_{ad} + P_{[da]}g_{bc} - P_{[dc]}g_{ab}\right)$$

$$\gimel_{abcd} = \frac{1}{N}\left(-P_{(ba)}g_{cd} + P_{(bc)}g_{ad} - P_{(da)}g_{bc} + P_{(dc)}g_{ab}\right)$$

$$P_{cd} = g^{ab}\mathcal{C}_{abcd}$$

Notice that the indices on $\mathring{\Upsilon}$ are reversed from the definition in $Y2a$

$$\mathring{\Upsilon}_{ba} = \frac{1}{N} P_{ab} - \frac{2}{N\left(N+2\right)} P_{[ab]}$$

D.5 Y2c - $\mathfrak{d}$, ℸ, Д

Below we give a proof for the $Y2c$ sector $SO\left(p,q\right)$ decomposition.

1	3	4
2		

Let us look at the $Y2c$ piece: $Y2c\left(Q\right)_{abcd} = \mathcal{D}_{abcd}$. We would have the symmetry $\mathcal{D}_{abcd} = D_{[ab](cd)}$. Let us also define $Y2c\left(W\right)_{abcd} = \mathfrak{d}_{abcd}$. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned}\mathcal{D}_{abcd} &= \frac{1}{8}\left(+Q_{abcd} + Q_{abdc} + Q_{cbad} + Q_{cbda} + Q_{dbac} + Q_{dbca}\right) \\ &+ \frac{1}{8}\left(-Q_{bacd} - Q_{badc} - Q_{cabd} - Q_{cadb} - Q_{dabc} - Q_{dacb}\right)\end{aligned}$$

If we use a degrees of freedom argument, we can find that the trace part: $\mathscr{C}_{abcd}$ is generated by a second rank general trace free tensor $\mathring{c}_{ac}$. This then tells us that we can write $\mathcal{D}_{abcd}$ as:

$$\mathcal{D}_{abcd} = \mathscr{D}_{abcd} + \mathfrak{d}_{abcd}$$

$$\mathscr{D}_{abcd} = Y2c\left(\mathring{c}_{ab}g_{cd}\right)$$

After applying the Young symmetrizer, we have:

$$\mathscr{D}_{abcd} = 4\mathring{c}_{[ab]}g_{bd} - 2\mathring{c}_{ca}g_{bd} + 2\mathring{c}_{cb}g_{ad} - 2\mathring{c}_{da}g_{bc} + 2\mathring{c}_{db}g_{ac}$$

Let us absorb the factor of 2 into $\mathring{c}_{ab}$ and call it Υ:

$$\mathscr{D}_{abcd} = 2\mathring{\Upsilon}_{[ab]}g_{bd} - \mathring{\Upsilon}_{ca}g_{bd} + \mathring{\Upsilon}_{cb}g_{ad} - \mathring{\Upsilon}_{da}g_{bc} + \mathring{\Upsilon}_{db}g_{ac}$$

However, because $\mathring{\Upsilon}_{ab}$ is only traceless we can again use the structure contained in $GL\left(N\right)$ to break off a symmetric and skew piece. We will call the Skew part of $\mathring{\Upsilon}_{ab}$ the Cyrillic Zhe: $Ж_{ab} = \mathring{\Upsilon}_{[ab]}$, and the resulting tensor generated by $Ж_{ab}$ will be called the Cyrillic De: $Д_{abcd}$. Next, we will call the symmetric part of $\mathring{\Upsilon}_{ab}$the Cyrillic Yu: $Ю_{ab} = \mathring{\Upsilon}_{(ab)}$, and the resulting tensor generated by $Ю_{ab}$ will be called the Hebrew Daleth: $ד_{abcd}$. The same methodology we applied for the $Y2a$, and $Y2b$ sectors works here, and we leave out the intermediate steps; after establishing the proofs for $Y2a$, and $Y2b$, we can see by examination the way the $Y2c$ sector decomposes. This then gives us:

$$\mathscr{D}_{abcd} = Д_{abcd} + ד_{abcd}.$$

$$Ж_{ab} = -\mathring{\Upsilon}_{[ab]}$$

$$Ю_{ab} = \mathring{\Upsilon}_{(ab)}$$

$$\mathring{\Upsilon}_{ba} = \frac{1}{N} R_{ab} - \frac{2}{N\,(N+2)} R_{[ab]}$$

$$Ж_{ab} = -\frac{1}{N+2} R_{[ab]}$$

$$Ю_{ab} = \frac{1}{N} R_{(ab)}$$

$$Д_{abcd} = 2Ж_{[ab]} g_{bd} - Ж_{ca} g_{bd} + Ж_{cb} g_{ad} - Ж_{da} g_{bc} + Ж_{db} g_{ac}$$

$$ד_{abcd} = -Ю_{ca} g_{bd} + Ю_{cb} g_{ad} - Ю_{da} g_{bc} + Ю_{db} g_{ac}$$

And we are left with what we should actually have as our trace decomposition of $Y2c$:

$$\mathcal{D}_{abcd} = Д_{abcd} + ד_{abcd}. + \mathfrak{d}_{abcd}$$

We will now go on to show that this is NOT the simplest form of $\mathcal{C}$ as we previously thought. The derivation comes next. The simplest form of $\mathcal{D}_{abcd}$ is given by:

$$\mathcal{D}_{abcd} = Д_{abcd} + ד_{abcd}. + \mathfrak{d}_{abcd}$$

$$\mathfrak{d}_{abcd} = \mathcal{D}_{abcd} - Д_{abcd} - ד_{abcd}.$$

$$Д_{abcd} = \frac{1}{N+2}\left(-2R_{[ab]} g_{bd} + R_{[ca]} g_{bd} - R_{[cb]} g_{ad} + R_{[da]} g_{bc} - R_{[db]} g_{ac}\right)$$

$$\daleth_{abcd} = \frac{1}{N}\left(-R_{(ca)}g_{bd} + R_{(cb)}g_{ad} - R_{(da)}g_{bc} + R_{(db)}g_{ac}\right)$$

$$R_{bd} = g^{ac}\mathcal{D}_{abcd}$$

D.6 Y3 - $\mathfrak{e}, \mathfrak{f}, \mathscr{E}, \mathscr{F}, \mathfrak{E}, \mathfrak{F}$

Below we give proofs for the $Y3$ sector $SO\left(p,q\right)$ decomposition.

D.7 Y3a - $\mathfrak{e}, \mathscr{E}, \mathfrak{E}$

Below we give a proof for the $Y3a$ sector $SO\left(p,q\right)$ decomposition.

1	2
3	4

Let us look at the $Y3a$ piece: $Y3a\left(Q\right)_{abcd} = \mathcal{E}_{abcd}$. We would have the symmetry$\mathcal{E}_{abcd} = \mathcal{E}_{[a|b|c]d} = \mathcal{E}_{a[b|c|d]} = \mathcal{E}_{bacd}$. Let us also define $Y3a\left(W\right)_{abcd} = \mathfrak{e}_{abcd}$. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned}
\mathcal{E}_{abcd} &= \frac{1}{12}\left(+Q_{abcd} + Q_{abdc} + Q_{bacd} + Q_{badc}\right) \\
&+ \frac{1}{12}\left(-Q_{cbad} - Q_{cbda} - Q_{bcad} - Q_{bcda}\right) \\
&+ \frac{1}{12}\left(-Q_{adcb} - Q_{adbc} - Q_{dacb} - Q_{dabc}\right) \\
&+ \frac{1}{12}\left(+Q_{cdab} + Q_{cdba} + Q_{dcab} + Q_{dcba}\right)
\end{aligned}$$

We can define the following orthogonal sectors:

$$\mathscr{E}_{abcd} = \frac{4}{(N-2)}Y3a\left(\mathring{P}_{ab}g_{cd}\right)$$

$$\mathfrak{E}_{abcd} = \frac{2P}{N\left(N-1\right)}Y3a\left(g_{ab}g_{cd}\right)$$

All we have to do now is apply symmetrizations:

$$\begin{aligned}
\mathscr{E}_{abcd} &\rightarrow \frac{4}{(N-2)}\mathring{P}_{ab}g_{cd} \\
&\rightarrow \frac{2}{(N-2)}\left(\mathring{P}_{ab}g_{cd}-\mathring{P}_{cb}g_{ad}\right) \\
&= \frac{1}{(N-2)}\left(\mathring{P}_{ab}g_{cd}-\mathring{P}_{cb}g_{ad}-\mathring{P}_{ad}g_{cb}+\mathring{P}_{cd}g_{ab}\right)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{E}_{abcd} &\rightarrow \frac{2}{N(N-1)}Pg_{ab}g_{cd} \\
&\rightarrow \frac{1}{N(N-1)}P\left(g_{ab}g_{cd}-g_{cb}g_{ad}\right) \\
&= \frac{1}{2N(N-1)}P\left(g_{ab}g_{cd}-g_{cb}g_{ad}-g_{ad}g_{cb}+g_{cd}g_{ab}\right) \\
&= \frac{1}{N(N-1)}P\left(g_{ab}g_{cd}-g_{ad}g_{cb}\right)
\end{aligned}$$

Recall that we had:

$$g^{ab}\mathcal{A}_{abcd}=P_{cd}$$

$$g^{ac}\mathcal{A}_{abcd}=R_{bd}$$

$$g^{ad}\mathcal{A}_{abcd}=U_{bc}$$

$$g^{bc}\mathcal{A}_{abcd}=X_{ad}$$

$$g^{bd}\mathcal{A}_{abcd}=Y_{ac}$$

$$g^{cd}\mathcal{A}_{abcd}=Z_{ab}$$

$$g^{ab}g^{cd}\mathcal{A}_{abcd}=P$$

$$g^{ac}g^{bd}\mathcal{A}_{abcd}=R$$

$$g^{ad}g^{bc}\mathcal{A}_{abcd}=U$$

Now before we sub in the $\mathcal{E}$ equation, we can simplify the above relations due to the symmetries: $\mathcal{E}_{abcd} = \mathcal{E}_{[a|b|c]d} = \mathcal{E}_{a[b|c|d]} = \mathcal{E}_{bacd}$. We find:

$$X_{ab} = U_{ab}$$

$$P_{ab} = Z_{ab}$$

$$R_{bd} = 0$$

$$Y_{ac} = 0$$

$$P_{ab} = -X_{ab}$$

Where the last relation comes from $P_{cd} = g^{ab}\mathcal{E}_{abcd} = -g^{ab}\mathcal{E}_{cbad} = -g^{ab}\mathcal{E}_{cabd} = -X_{cd}$. These relations also imply that:

$$P = -U$$

$$R = 0$$

Last there are some symmetry relations on these as well. We have:

$$P_{ab} = P_{(ab)} = Z_{(ab)} = -X_{(ab)} = -U_{(ab)}$$

Applying these relations to our very first messy equations gives us:

$$\begin{aligned} \nu \mathring{A}_{ab} &= P_{ab}N^3 + \left(-Pg_{ab} + 6P_{ab}\right)N^2 \\ &+ \left(-6Pg_{ab} + 8P_{ab}\right)N \\ &+ \left(-8Pg_{ab}\right) \end{aligned}$$

$$\mathring{B}_{ab} = 0$$

$$\begin{aligned} \nu \mathring{C}_{ab} &= -P_{ab}N^3 + \left(Pg_{ab} - P_{ab} - 4P_{ab} - P_{ab}\right)N^2 \\ &+ \left(6Pg_{ab} - 3P_{ab} + P_{ba} - 3P_{ab} + P_{ab} - 4P_{ab}\right)N \\ &+ \left(8Pg_{ab}\right) \end{aligned}$$

$$\begin{aligned} \nu \mathring{D}_{ab} &= -P_{ab}N^3 + \left(Pg_{ab} - P_{ab} - 4P_{ab} - P_{ba}\right)N^2 \\ &+ \left(6Pg_{ab} + P_{ab} - 3P_{ba} + P_{ab} - 3P_{ab} - 4P_{ab}\right)N \\ &+ \left(8Pg_{ab}\right) \end{aligned}$$

$$\mathring{E}_{ab} = 0$$

$$\begin{aligned} \nu \mathring{F}_{ab} &= P_{ab}N^3 + \left(-Pg_{ab} + 2P_{ab} + 4P_{ab}\right)N^2 \\ &+ \left(-6Pg_{ab} + 4P_{ab} + 2P_{ab} + 2P_{ab}\right)N \\ &+ \left(-8Pg_{ab} - 4P_{ab} - 4P_{ba} + 4P_{ab} + 4P_{ab}\right) \end{aligned}$$

$$H = \frac{1}{\mu}P\left(N+2\right)$$

$$J = 0$$

$$K = -\frac{1}{\mu}P\left(N+2\right)$$

$$g^{ab}\mathcal{E}_{abcd} = P_{cd} = g_{cd} + \left(H-K\right)Ng_{cd}$$

$$g^{ac}\mathcal{E}_{abcd} = 0 = \mathring{A}_{db} - \mathring{C}_{db} - \mathring{D}_{bd} + \mathring{F}_{db} + \left(H-K\right)g_{db}$$

And two traces from those equations:

$$g^{ab}g^{cd}\mathcal{E}_{abcd} = P = N + \left(H-K\right)N^2$$

$$g^{ac}g^{bd}\mathcal{E}_{abcd} = 0 = \left(H-K\right)N$$

$$g^{ab}\mathcal{E}_{abcd} = P_{cd} = g_{cd} + (H - K)\, N g_{cd}$$

$$g^{ac}\mathcal{E}_{abcd} = R_{bd} = 0 = \mathring{A}_{db} - \mathring{C}_{db} - \mathring{D}_{bd} + \mathring{F}_{db} + (H - K)\, g_{db}$$

$$g^{ad}\mathcal{E}_{abcd} = U_{bc} = \mathring{A}_{cb} - \mathring{C}_{cb} - \mathring{D}_{bc} + \mathring{F}_{cb} + (H - K)\, g_{cb}$$

$$g^{bc}\mathcal{E}_{abcd} = X_{ad} = \left(\mathring{A}_{ab} - \mathring{C}_{ab} - \mathring{D}_{ba} + \mathring{F}_{ab}\right) g_{cd} + (H - K)\, g_{ab} g_{cd}$$

$$g^{bd}\mathcal{E}_{abcd} = Y_{ac} = \left(\mathring{A}_{ab} - \mathring{C}_{ab} - \mathring{D}_{ba} + \mathring{F}_{ab}\right) g_{cd} + (H - K)\, g_{ab} g_{cd}$$

$$g^{cd}\mathcal{E}_{abcd} = Z_{ab} = \left(\mathring{A}_{ab} - \mathring{C}_{ab} - \mathring{D}_{ba} + \mathring{F}_{ab}\right) g_{cd} + (H - K)\, g_{ab} g_{cd}$$

$$g^{ab} g^{cd}\mathcal{E}_{abcd} = P = \left(\mathring{A}_{ab} - \mathring{C}_{ab} - \mathring{D}_{ba} + \mathring{F}_{ab}\right) g_{cd} + (H - K)\, g_{ab} g_{cd}$$

$$g^{ac} g^{bd}\mathcal{E}_{abcd} = R = \left(\mathring{A}_{ab} - \mathring{C}_{ab} - \mathring{D}_{ba} + \mathring{F}_{ab}\right) g_{cd} + (H - K)\, g_{ab} g_{cd}$$

$$g^{ad} g^{bc}\mathcal{E}_{abcd} = U = \left(\mathring{A}_{ab} - \mathring{C}_{ab} - \mathring{D}_{ba} + \mathring{F}_{ab}\right) g_{cd} + (H - K)\, g_{ab} g_{cd}$$

Or we can skip to the end with Maple:

$$\mathscr{E}_{abcd} = \frac{4}{(N-2)} \mathring{P}_{ab} g_{cd}$$

$$\mathfrak{E}_{abcd} = \frac{2}{N\,(N-1)} P g_{ab} g_{cd}$$

$$g^{ab}\mathcal{E}_{abcd} = P_{cd}$$

$$\mathring{P}_{(ab)} = P_{(ab)} - \frac{1}{N} P g_{ab}$$

Therefore, when we make orthogonal sectors we define the following:

$$\mathcal{E}_{abcd} = \mathfrak{e}_{abcd} + \mathscr{E}_{abcd} + \mathfrak{E}_{abcd}$$

$$\mathfrak{e}_{abcd} = \mathcal{E}_{abcd} - \mathscr{E}_{abcd} - \mathfrak{E}_{abcd}$$

$$\mathscr{E}_{abcd} = \frac{1}{(N-2)} \left(\mathring{P}_{ab} g_{cd} - \mathring{P}_{cb} g_{ad} - \mathring{P}_{ad} g_{cb} + \mathring{P}_{cd} g_{ab}\right)$$

$$\mathfrak{E}_{abcd} = \frac{1}{N(N-1)} P\left(g_{ab}g_{cd} - g_{ad}g_{cb}\right)$$

$$P_{cd} = g^{ab}\mathcal{E}_{abcd}$$

$$P = g^{ab}g^{cd}\mathcal{E}_{abcd}$$

$$\mathring{P}_{(ab)} = P_{(ab)} - \frac{1}{N}Pg_{ab}$$

D.8 Y3b - $\mathfrak{f}, \mathscr{F}, \mathfrak{F}$

Below we give a proof for the $Y3b$ sector $SO(p,q)$ decomposition.

1	3
2	4

Let us look at the $Y3b$ piece: $Y3b\,(Q)_{abcd} = \mathcal{F}_{abcd}$. We would have the symmetry$\mathcal{F}_{abcd} = \mathcal{F}_{[ab][cd]} = \mathcal{F}_{cdab}$. Let us also define $Y3b\,(W)_{abcd} = \mathfrak{f}_{abcd}$. Similarly to what we did in the $Y2c$ section, we apply the same rationale here to make the proof shorter. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned}
\mathcal{F}_{abcd} &= \frac{1}{12}\left(+Q_{abcd} + Q_{adcb} + Q_{cbad} + Q_{cdab}\right) \\
&+ \frac{1}{12}\left(-Q_{bacd} - Q_{bdca} - Q_{cabd} - Q_{cdba}\right) \\
&+ \frac{1}{12}\left(-Q_{abdc} - Q_{acdb} - Q_{dbac} - Q_{dcab}\right) \\
&+ \frac{1}{12}\left(+Q_{badc} + Q_{bcda} + Q_{dabc} + Q_{dcba}\right)
\end{aligned}$$

We can define the following orthogonal sectors:

$$\mathscr{F}_{abcd} = \frac{4}{(N-2)} Y3b\left(\mathring{R}_{ac}g_{bd}\right)$$

$$\mathfrak{F}_{abcd} = \frac{2R}{N(N-1)} Y3b\left(g_{ac}g_{bd}\right)$$

All we have to do now is apply symmetrizations:

$$\begin{aligned}\mathscr{F}_{abcd} &\rightarrow \frac{4}{(N-2)}\mathring{R}_{ac}g_{bd} \\ &\rightarrow \frac{2}{(N-2)}\left(\mathring{R}_{ac}g_{bd} - \mathring{R}_{bc}g_{ad}\right) \\ &= \frac{1}{(N-2)}\left(\mathring{R}_{ac}g_{bd} - \mathring{R}_{bc}g_{ad} - \mathring{R}_{ad}g_{bc} + \mathring{R}_{bd}g_{ac}\right)\end{aligned}$$

$$\begin{aligned}\mathfrak{F}_{abcd} &\rightarrow \frac{2}{N(N-1)}Rg_{ac}g_{bd} \\ &\rightarrow \frac{1}{N(N-1)}R\left(g_{ac}g_{bd} - g_{bc}g_{ad}\right) \\ &= \frac{1}{2N(N-1)}R\left(g_{ac}g_{bd} - g_{bc}g_{ad} - g_{ad}g_{bc} + g_{bd}g_{ac}\right) \\ &= \frac{1}{N(N-1)}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)\end{aligned}$$

Therefore, when we make orthogonal sectors we define the following:

$$\mathcal{F}_{abcd} = \mathfrak{f}_{abcd} + \mathscr{F}_{abcd} + \mathfrak{F}_{abcd}$$

$$\mathfrak{f}_{abcd} = \mathcal{F}_{abcd} - \mathscr{F}_{abcd} - \mathfrak{F}_{abcd}$$

$$\mathscr{F}_{abcd} = \frac{1}{(N-2)}\left(\mathring{R}_{ac}g_{bd} - \mathring{R}_{bc}g_{ad} - \mathring{R}_{ad}g_{bc} + \mathring{R}_{bd}g_{ac}\right)$$

$$\mathfrak{F}_{abcd} = \frac{1}{N(N-1)}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

$$R_{(bd)} = g^{ac}\mathcal{F}_{abcd}$$

$$R = g^{ac}g^{bd}\mathcal{F}_{abcd}$$

$$\mathring{R}_{(ab)} = R_{(ab)} - \frac{1}{N}Rg_{ab}$$

D.9 Y4 - $\mathfrak{g}, \mathfrak{h}, \mathfrak{j}, \mathscr{G}, \mathscr{H}, \mathscr{J}$

Below we give proofs for the $Y4$ sector $SO(p,q)$ decomposition.

D.10 Y4a - $\mathfrak{g}, \mathscr{G}$

Below we give a proof for the $Y4a$ sector $SO(p,q)$ decomposition.

1	2
3	
4	

Let us look at the $Y4a$ piece: $Y4a(Q)_{abcd} = \mathcal{G}_{abcd}$. We would have the symmetry$\mathcal{G}_{abcd} = \mathcal{G}_{[a|b|cd]}$. Let us also define $Y4a(W)_{abcd} = \mathfrak{g}_{abcd}$. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned} \mathcal{G}_{abcd} &= \frac{1}{8}\left(Q_{abcd} + Q_{cbda} + Q_{dbac} - Q_{abdc} - Q_{dbca} - Q_{cbad}\right) \\ &+ \frac{1}{8}\left(Q_{bacd} + Q_{bcda} + Q_{bdac} - Q_{badc} - Q_{bdca} - Q_{bcad}\right) \end{aligned}$$

Which implies that $\mathcal{G}$ is of the form:

$$\mathcal{G}_{abcd} = \left(\mathring{C}_{cd} - \mathring{C}_{dc}\right) g_{ab} - \left(\mathring{C}_{ad} - \mathring{C}_{da}\right) g_{bc} + \left(\mathring{C}_{ac} - \mathring{C}_{ca}\right) g_{bd} + \mathfrak{g}_{abcd}$$

Simplifying gives us:

$$\mathcal{G}_{abcd} = 2\left(\mathring{C}_{[cd]} g_{ab} - \mathring{C}_{[ad]} g_{bc} + \mathring{C}_{[ac]} g_{bd}\right) + \mathfrak{g}_{abcd}$$

$$\mathcal{G}_{abcd} = \mathring{\Upsilon}_{[cd]} g_{ab} - \mathring{\Upsilon}_{[ad]} g_{bc} + \mathring{\Upsilon}_{[ac]} g_{bd} + \mathfrak{g}_{abcd}$$

$$\mathscr{G}_{abcd} = \mathring{\Upsilon}_{[cd]} g_{ab} - \mathring{\Upsilon}_{[ad]} g_{bc} + \mathring{\Upsilon}_{[ac]} g_{bd}$$

Recall that we had:

$$g^{ab}\mathcal{G}_{abcd} = P_{cd}$$

$$g^{ac}\mathcal{G}_{abcd} = R_{bd}$$

$$g^{ad}\mathcal{G}_{abcd} = U_{bc}$$

$$g^{bc}\mathcal{G}_{abcd} = X_{ad}$$

$$g^{bd}\mathcal{G}_{abcd} = Y_{ac}$$

$$g^{cd}\mathcal{G}_{abcd} = Z_{ab}$$

$$g^{ab}g^{cd}\mathcal{G}_{abcd} = P$$

$$g^{ac}g^{bd}\mathcal{G}_{abcd} = R$$

$$g^{ad}g^{bc}\mathcal{G}_{abcd} = U$$

Now before we sub in the $\mathcal{G}$ equation, we can simplify the above relations due to the symmetries: $\mathcal{G}_{abcd} = \mathcal{G}_{[a|b|cd]}$. We find:

$$R_{bd} = 0$$

$$U_{bc} = 0$$

$$Z_{ab} = 0$$

$$P_{cd} = P_{[cd]}$$

$$R = 0$$

$$U = 0$$

$$P = 0$$

$$g^{bc}\mathcal{G}_{abcd} = X_{ad} = -P_{ad}$$

$$g^{bd}\mathcal{G}_{abcd} = Y_{ac} = P_{ac}$$

Shoving these all into our original equations we find that:

$$\mathscr{G}_{abcd} = \mathring{\Upsilon}_{[cd]}g_{ab} - \mathring{\Upsilon}_{[ad]}g_{bc} + \mathring{\Upsilon}_{[ac]}g_{bd}$$

$$P_{cd} = (N-2)\,\mathring{\Upsilon}_{[cd]}$$

$$\mathring{\Upsilon}_{[ab]} = \frac{1}{(N-2)} P_{[ab]}$$

This gives us:

$$\mathscr{G}_{abcd} = \frac{1}{(N-2)} \left(P_{[cd]} g_{ab} - P_{[ad]} g_{bc} + P_{[ac]} g_{bd} \right)$$

We have:

$$\mathcal{G}_{abcd} = \mathscr{G}_{abcd} + \mathfrak{g}_{abcd}$$

$$\mathfrak{g}_{abcd} = \mathcal{G}_{abcd} - \mathscr{G}_{abcd}$$

$$\mathscr{G}_{abcd} = \frac{1}{(N-2)} \left(P_{[cd]} g_{ab} - P_{[ad]} g_{bc} + P_{[ac]} g_{bd} \right)$$

$$P_{cd} = g^{ab} \mathcal{G}_{abcd}$$

D.11 Y4b - $\mathfrak{h}, \mathscr{H}$

Below we give a proof for the $Y4b$ sector $SO(p,q)$ decomposition.

1	3
2	
4	

Let us look at the $Y4b$ piece: $Y4b\,(Q)_{abcd} = \mathcal{H}_{abcd}$. We would have the symmetry$\mathcal{H}_{abcd} = \mathcal{H}_{[ab|c|d]}$. Let us also define $Y4b\,(W)_{abcd} = \mathfrak{h}_{abcd}$. Similarly to what we did in the $Y2c$ section, we apply the same rationale here to make the proof shorter. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned} \mathcal{H}_{abcd} &= \frac{1}{8} \left(Q_{abcd} + Q_{bdca} + Q_{dacb} - Q_{adcb} - Q_{dbca} - Q_{bacd} \right) \\ &+ \frac{1}{8} \left(Q_{cbad} + Q_{cadb} + Q_{cdba} - Q_{cbda} - Q_{cdab} - Q_{cabd} \right) \end{aligned}$$

Which implies that $\mathcal{H}$ is of the form:

$$\mathcal{H}_{abcd} = \mathring{\Upsilon}_{[bd]}g_{ac} - \mathring{\Upsilon}_{[ad]}g_{bc} + \mathring{\Upsilon}_{[ab]}g_{cd} + \mathfrak{h}_{abcd}$$

Simplifying gives us:

$$\mathscr{H}_{abcd} = \mathring{\Upsilon}_{[bd]}g_{ac} - \mathring{\Upsilon}_{[ad]}g_{bc} + \mathring{\Upsilon}_{[ab]}g_{cd}$$

We have:

$$\mathcal{H}_{abcd} = \mathscr{H}_{abcd} + \mathfrak{h}_{abcd}$$

$$\mathfrak{h}_{abcd} = \mathcal{H}_{abcd} - \mathscr{H}_{abcd}$$

$$\mathscr{H}_{abcd} = \frac{1}{(N-2)}\left(R_{[bd]}g_{ac} - R_{[ad]}g_{bc} + R_{[ab]}g_{cd}\right)$$

$$R_{cd} = g^{ac}\mathcal{H}_{abcd}$$

D.12 Y4c - $\mathfrak{j}$, $\mathscr{J}$

Below we give a proof for the $Y4c$ sector $SO(p,q)$ decomposition.

1	4
2	
3	

Let us look at the $Y4c$ piece: $Y4c\,(Q)_{abcd} = \mathcal{J}_{abcd}$. We would have the symmetry$\mathcal{J}_{abcd} = \mathcal{J}_{[abc]d}$. Let us also define $Y4c\,(W)_{abcd} = \mathfrak{j}_{abcd}$. Similarly to what we did in the $Y2c$ section, we apply the same rationale here to make the proof shorter. Then, breaking up using the idea of the general trace gives us the following formula:

$$\begin{aligned}\mathcal{J}_{abcd} &= \frac{1}{8}\left(Q_{abcd} + Q_{bcad} + Q_{cabd} - Q_{acbd} - Q_{cbad} - Q_{bacd}\right)\\ &+ \frac{1}{8}\left(Q_{dbca} + Q_{dcab} + Q_{dabc} - Q_{dbac} - Q_{dacb} - Q_{dcba}\right)\end{aligned}$$

Which implies that $\mathcal{J}$ is of the form:

$$\mathcal{J}_{abcd} = \mathring{\Upsilon}_{[bc]}g_{ad} - \mathring{\Upsilon}_{[ac]}g_{bd} + \mathring{\Upsilon}_{[ab]}g_{cd} + \mathfrak{j}_{abcd}$$

Simplifying gives us:

$$\mathscr{J}_{abcd} = \mathring{\Upsilon}_{[bc]}g_{ad} - \mathring{\Upsilon}_{[ac]}g_{bd} + \mathring{\Upsilon}_{[ab]}g_{cd}$$

We have:

$$\mathcal{J}_{abcd} = \mathscr{J}_{abcd} + \mathfrak{j}_{abcd}$$

$$\mathfrak{j}_{abcd} = \mathcal{J}_{abcd} - \mathscr{J}_{abcd}$$

$$\mathscr{G}_{abcd} = \frac{1}{(N-2)}\left(U_{[bc]}g_{ad} - U_{[ac]}g_{bd} + U_{[ab]}g_{cd}\right)$$

$$U_{cd} = g^{ad}\mathcal{J}_{abcd}$$

D.13 Y5 - $\mathfrak{k}$

The $Y5$ sector does not decompose further under $SO(p,q)$. We however repeat results here on it from before.

1
2
3
4

Let us look at the $Y5$ piece: $Y5(Q)_{abcd}$. We would have the symmetry $Y5(Q)_{abcd} = Y2b(Q)_{[abcd]}$. Breaking up:

$$\begin{aligned}
\mathfrak{k}_{abcd} &= \frac{1}{24}(Q_{abcd} - Q_{abdc} - Q_{acbd} + Q_{acdb} + Q_{adbc} - Q_{adcb}) \\
&+ \frac{1}{24}(-Q_{bacd} + Q_{badc} + Q_{bcad} - Q_{bcda} - Q_{bdac} + Q_{bdca}) \\
&+ \frac{1}{24}(Q_{cabd} - Q_{cadb} - Q_{cbad} + Q_{cbda} - Q_{cdba} + Q_{cdab}) \\
&+ \frac{1}{24}(-Q_{dabc} + Q_{dacb} + Q_{dbac} - Q_{dbca} + Q_{dcba} - Q_{dcab})
\end{aligned}$$

We only have one piece, therefore:

$$Y5\,(Q)_{abcd} = \mathfrak{t}_{abcd}$$

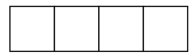
$$\mathfrak{t}_{abcd} = Q_{[abcd]}$$

APPENDIX E
DEGREES OF FREEDOM OF IRREDUCIBLE RANK 4 TENSORS AND THEIR RELATED SPINORS IN 4-DIMENSIONS

In the Rank 4 Tensors section, we began talking about degrees (notated "deg") of freedom and how they correspond regarding rank 4 tensors and spinors. Here we show how the degrees of freedom match exactly between all the rank 4 $SO(p,q)$ irreducible tensors, and $SL(2,\mathbb{C})$ irreducible spinors in 4D. We will go through each sector labeled by their Ferrers diagrams, starting with $Y1$, and ending with $Y5$. We also include the $GL(N)$ irreducible tensors like $\boldsymbol{\mathcal{A}}$ and count their components. This is important in helping us calculate the components of the $SO(p,q)$ irreducible tensors. What we do to find those, is we can calculate through Young tableaux methods the number of components of the $GL(N)$ irreducible tensors. Then we can calculate the curly tensors $\boldsymbol{A}\ldots$ through the same methods applied to trace free rank two tensors. The difference of these sectors gives the totally trace free degrees of freedom for tensors like $\boldsymbol{\mathfrak{a}}$.

Furthermore, to calculate the degrees of the freedom of the spinors, we can also use Young tableaux tools because we can count the degrees of freedom of totally symmetric tensors. Applying this method to both $\mathcal{S}$ and $\overline{\mathcal{S}}$ and then multiplying the results together because of the tensor product $\mathcal{S}\otimes\overline{\mathcal{S}}$ being the spinor space then gives us the degrees of freedom of any of the spinors that we are interested in. Notice that for each of these, the degrees of freedom of all the $SO(p,q)$ irreducible tensors add up to the degrees of freedom of the single $GL(N)$ irreducible tensor. Now we move onto the presentation of the results.

For the $Y1$ sector, we have the following tensors and spinors: $\mathcal{A}_{abcd}$; $\mathfrak{a}_{abcd}$, $\mathscr{A}_{abcd}$, $\mathfrak{A}_{abcd}$, $S_{(ABCD)(A'B'C'D')}$, $\iota_{(AB)(A'B')}$, and υ. The degrees of freedom of each of the tensors are given below. The degrees of freedom of $\boldsymbol{S}$, and $\boldsymbol{\mathfrak{a}}$ are equal in 4D; likewise the degrees of freedom of $\mathscr{A}$ and $\boldsymbol{\iota}$ are equal in 4D; last the degrees of freedom of $\mathfrak{A}$ and υ are equal in 4D.

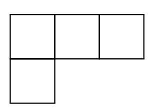


$$\deg\left(\boldsymbol{\mathcal{A}}\right) = \frac{1}{24}\left(N\right)\left(N+1\right)\left(N+2\right)\left(N+3\right)$$

$$\deg\left(\boldsymbol{\mathfrak{a}}\right) = \frac{1}{24}\left(N\right)\left(N+1\right)\left(N-1\right)\left(N+6\right)$$

$$\deg\left(\mathscr{A}\right) = \frac{1}{2}\left(N\right)\left(N+1\right) - 1, \quad \deg\left(\boldsymbol{\mathfrak{A}}\right) = 1$$

For the $Y2$ sector, we have the following tensors and spinors: $\mathcal{B}_{abcd}$, $\mathcal{C}_{abcd}$, $\mathcal{D}_{abcd}$; $\mathfrak{b}_{abcd}$, $\mathfrak{c}_{abcd}$, $\mathfrak{d}_{abcd}$, $\beth_{abcd}$, $\gimel_{abcd}$, $\daleth_{abcd}$, Б$_{abcd}$, Щ$_{abcd}$, Д$_{abcd}$, $\alpha_{(ABCD)(A'B')}$, $\beta_{(ABCD)(A'B')}$, $\gamma_{(ABCD)(A'B')}$, $\Omega_{(AB)(A'B')}$, $\kappa_{(AB)(A'B')}$, $\theta_{(AB)(A'B')}$, $\mu_{(AB)}$, $\rho_{(AB)}$, and $o_{(AB)}$. The degrees of freedom of all the totally trace free tensors are the same; likewise the same holds for all the Cyrillic tensors, and the Hebrew tensors. Note that we multiply both the spinors $\boldsymbol{\alpha}$ and $\boldsymbol{\mu}$ by 2 because they are arbitrary complex spinors with no reality condition of Penrose (Penrose & Rindler, 1987a); in a similar vein the spinor $\boldsymbol{\Omega}$ does not get multiplied by 2 because of hermiticity/the same reality condition. The degrees of freedom of each of the tensors and relevant spinors are given below. The degrees of freedom of the Hebrew tensors match those of the hermitian spinors in 4D; likewise the totally trace free tensors' degrees of freedom match that of the 6th rank spinors in 4D; finally the degrees of freedom of the Cyrillic tensors match that of the rank 2 spinors in 4D.



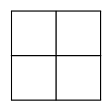
$$\deg\left(\boldsymbol{\mathcal{B}}\right) = \frac{1}{8}\left(N\right)\left(N+1\right)\left(N+2\right)\left(N-1\right)$$

$$\deg\left(\boldsymbol{\mathfrak{b}}\right) = \frac{1}{8}\left(N-1\right)\left(N-2\right)\left(N+4\right)\left(N+1\right)$$

$$\deg\left(\boldsymbol{\beth}\right) = \frac{1}{2}N\left(N+1\right) - 1, \quad \deg\left(\text{Б}\right) = \frac{1}{2}N\left(N-1\right)$$

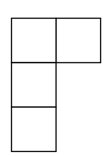
For the $Y3$ sector, we have the following tensors and spinors: $\mathcal{E}_{abcd}$, $\mathcal{F}_{abcd}$; $\mathfrak{e}_{abcd}$, $\mathfrak{f}_{abcd}$, $\mathscr{E}_{abcd}$,

$\mathscr{F}_{abcd}$, $\mathfrak{C}_{abcd}$, F_{abcd}, $\delta_{(ABCD)}$, $\Psi_{(ABCD)}$, $\pi_{(AB)(A'B')}$, $\Phi_{(AB)(A'B')}$, χ and Λ. The degrees of freedom of all the totally trace free tensors are the same; likewise the same holds for all the curly tensors, and the scalars. Like before, we multiply the degrees of freedom of $\boldsymbol{\delta}$ and $\boldsymbol{\Psi}$ by two because they are generally complex; with the same idea the spinors $\boldsymbol{\pi}$ and $\boldsymbol{\Phi}$ do not get multiplied by two just like before. The degrees of freedom of each of the tensors and relevant spinors are given below. The degrees of freedom of the curly tensors match those of the hermitian (4th rank) spinors in 4D; likewise the totally trace free tensors' degrees of freedom match that of the 4th rank spinors in 4D; finally the degrees of freedom of the trace-full tensors match that of the rank 0 spinors in 4D.



$$\deg\left(\boldsymbol{\mathcal{E}}\right) = \frac{1}{12}\left(N^2\right)\left(N+1\right)\left(N-1\right)$$
$$\deg\left(\mathfrak{e}\right) = \frac{1}{12}\left(N\right)\left(N+1\right)\left(N+2\right)\left(N-3\right)$$
$$\deg\left(\boldsymbol{E}\right) = \frac{1}{2}\left(N\right)\left(N+1\right) - 1, \quad \deg\left(\mathfrak{E}\right) = 1$$

For the $Y4$ sector, we have the following tensors and spinors: the following tensors and spinors: $\mathcal{G}_{abcd}$, $\mathcal{H}_{abcd}$, $\mathcal{J}_{abcd}$; $\mathfrak{g}_{abcd}$, $\mathfrak{h}_{abcd}$, $\mathfrak{j}_{abcd}$, $\mathscr{G}_{abcd}$, $\mathscr{H}_{abcd}$, $\mathscr{J}_{abcd}$, $\tau_{(AB)(A'B')}$, $\zeta_{(AB)(A'B')}$, $\xi_{(AB)(A'B')}$, $\nu_{(AB)}$, $\eta_{(AB)}$, and $\sigma_{(AB)}$. Again, the degrees of freedom of all the totally trace free tensors are the same; likewise the same holds for all the curly tensors. The degrees of freedom of each of the tensors and relevant spinors are given below. The degrees of freedom of the curly tensors match those of the rank 2 spinors in 4D; finally the totally trace free tensors' degrees of freedom match that of the antihermitian (4th rank) spinors in 4D.

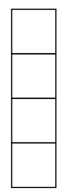


$$\deg\left(\mathcal{G}_{abcd}\right) = \frac{1}{8}\left(N\right)\left(N+1\right)\left(N-1\right)\left(N-2\right)$$

$$\deg\left(\mathfrak{g}_{abcd}\right) = \frac{1}{8}\left(N\right)\left(N-1\right)\left(N-3\right)\left(N+2\right),$$

$$\deg\left(\mathscr{G}_{abcd}\right) = \frac{1}{2}\left(N\right)\left(N-1\right)$$

For the $Y5$ sector we have the following tensor and spinor: $\mathcal{K}_{abcd}$; $\mathfrak{k}_{abcd}$, and $\aleph$. The degrees of freedom of the tensors and relevant spinors are given below. The degrees of freedom of the totally trace free, skew-symmetric tensor matches that of a rank 0 spinor in 4D.



$$\deg\left(\mathfrak{k}_{abcd}\right) = \frac{1}{24}\left(N\right)\left(N-1\right)\left(N-2\right)\left(N-3\right)$$

$$\deg\left(\mathfrak{k}_{abcd}\right)\big|_{N=4} = \deg\left(\boldsymbol{\aleph}\right) = 1$$

And with that we have all the relations on degrees of freedom for rank 4 spinors in relation to real 4th rank tensors (in N-dimensions for the tensors, while the spinor degrees of freedom we looked at matched in 4D which interests us because it is the dimension of spacetime in ECSK theory).

APPENDIX F

FORMULAS FOR ALL 25 $SL\,(2,\mathbb{C})$ IRREDUCIBLE SPINORS IN TERMS OF THEIR $SO\,(3,1)$ IRREDUCIBLE RANK 4 TENSORS

Two component spinors are mathematical objects used to describe the spin of particles in physics; they are made through the Pauli matrices and an identity matrix. They are represented by two complex numbers on a vector space $\mathcal{S}$. The degrees of freedom of a two component spinor describe the number of real independent variables needed to specify the state of the spinor. In the case of a two component spinor the two complex variables correlate directly to the four coordinates of spacetime through the Pauli matrices.

We can describe the state of the spinor in terms of its orientation in a complex plane, which is closely related to the properties of spacetime through the above correspondence. It is also worth noting that the number of degrees of freedom of a two component spinor is related to the number of independent components of the rank 2 spinor field that it represents. Furthermore, these spinors are related to spacetime through the Infeld-Van Der Waerden symbols ${\sigma_a}^{AA'}$; we can convert a vector v^a to a spinor $\omega^{AA'}$ with these symbols. Expressions such as $\epsilon^{A'D'}\epsilon^{BC}\epsilon^{B'C'}\text{Б}_{abcd}$ really represent:

$$\epsilon^{A'D'}\epsilon^{BC}\epsilon^{B'C'}\text{Б}_{abcd} \cong \epsilon^{A'D'}\epsilon^{BC}\epsilon^{B'C'}\text{Б}_{AA'BB'CC'DD'}$$

Where we have defined the Cyrillic spinor in terms of the Infeld-Van der Waerden symbols by:

$$\text{Б}_{AA'BB'CC'DD'} = \text{Б}_{abcd}{\sigma^a}_{AA'}{\sigma^b}_{BB'}{\sigma^c}_{CC'}{\sigma^d}_{DD'}$$

but we omit this to save space in our equations below. This is in the same fashion as Penrose and Rindler in (Penrose & Rindler, 1987a).

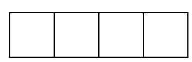
For the purposes of computer algebra, we also give the calling commands to use in the ECSK package below the subsequent equations in each section. Many of the commands just require a tensor and a soldering form. Any of the commands to generate a tensor from a given spinor require the spinor and a soldering form; we write this as the inputs: 1.) a 4th rank tensor, 2.) a solder form. The solder forms we use are the Infeld-Van Der Waerden symbols. Furthermore, we do not say this every time, but it will be implied.

For the cases in which the formulas have a spinor bar, an optional argument is able to be added in the third slot which is the bar spinor. For example in the $Y2a$ equations later on there is an $\boldsymbol{\alpha}$ spinor and an $\overline{\boldsymbol{\alpha}}$ spinor in the equation to generate the tensor $\mathfrak{b}$. The inputs are: 1.) a spinor, 2.) a solder form, 3.) {optional argument} a spinor conjugate to input 1. To calculate $\mathfrak{b}$ we would use the command "Rank4GenerateAlphaTensor" with the following inputs: ($\boldsymbol{\alpha}$, ${\sigma_a}^{AA'}$, $\overline{\boldsymbol{\alpha}}$). This also helps the computer to better handle complex conjugation.

The sections below begin with the $Y1$ sector and end with the $Y5$ sector.

F.1 The Y1 Spinors - S, ι, υ

The $Y1$ spinors are $\boldsymbol{S}$, $\boldsymbol{\iota}$, and $\boldsymbol{\upsilon}$. A list of equations relating them to their corresponding 4th rank $SO\left(p,q\right)$ irreducible tensors in 4 dimensions is given below.



$$S_{(ABCD)(A'B'C'D')} \cong \mathfrak{a}_{abcd} \tag{F.1}$$

$$\iota_{(AB)(A'B')} \cong \frac{1}{8} g^{ef} \mathscr{A}_{efab} \tag{F.2}$$

$$\begin{aligned} \mathscr{A}_{abcd} \;\cong\;& \epsilon_{AB}\epsilon_{A'B'}\iota_{(CD)(C'D')} + \epsilon_{AC}\epsilon_{A'C'}\iota_{(BD)(B'D')} \\ +\;& \epsilon_{AD}\epsilon_{A'D'}\iota_{(BC)(B'C')} + \epsilon_{BC}\epsilon_{B'C'}\iota_{(AD)(A'D')} \\ +\;& \epsilon_{BD}\epsilon_{B'D'}\iota_{(AC)(A'C')} + \epsilon_{CD}\epsilon_{C'D'}\iota_{(AB)(A'B')} \end{aligned} \tag{F.3}$$

$$\upsilon = \frac{1}{24} g^{ef} g^{gh} \mathfrak{A}_{efgh} \tag{F.4}$$

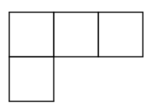
$$\begin{aligned} \mathfrak{A}_{abcd} &\cong \upsilon\left(\epsilon_{AB}\epsilon_{A'B'}\epsilon_{CD}\epsilon_{C'D'}\right) \\ &+ \upsilon\left(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'}\right) \\ &+ \upsilon\left(\epsilon_{AD}\epsilon_{A'D'}\epsilon_{BC}\epsilon_{B'C'}\right) \end{aligned} \tag{F.5}$$

Here we list the code used to calculate the left-hand side of each of the above equations; first though the right-hand side of equation (F.1) can be calculated from a given $\boldsymbol{S}$ spinor and soldering from by the command “Rank4GenerateSTensor”.

We have: “Rank4SSpinor” for equation (F.1), “Rank4IotaSpinor” for equation (F.2), “Rank4GenerateIotaTensor” for equation (F.3), “Rank4UpsilonSpinor” for equation (F.4), “Rank4GenerateUpsilonTensor” for equation (F.5)

F.2 The Y2 Spinors - $\alpha, \beta, \gamma, \Omega, \kappa, \theta, \mu, \rho, o$

The $Y2$ spinors are $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\Omega}$, $\boldsymbol{\kappa}$, $\boldsymbol{\theta}$, $\boldsymbol{\mu}$, $\boldsymbol{\rho}$, and $\boldsymbol{o}$. We split the spinors into the $Y2a$, $Y2b$, and $Y2c$ tableaux sectors to sort them further.



F.3 The Y2a Spinors - α, Ω, μ

The $Y2a$ spinors are $\boldsymbol{\alpha}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\mu}$. A list of equations relating them to their corresponding 4th rank $SO\left(p,q\right)$ irreducible tensors in 4 dimensions is given below.

$$\alpha_{(ABCD)(B'C')} \cong \frac{1}{2}\mathfrak{b}_{abcd}\epsilon^{A'D'} \tag{F.6}$$

$$\mathfrak{b}_{abcd} \cong \alpha_{(ABCD)(B'C')}\epsilon_{A'D'} + \overline{\alpha}_{(A'B'C'D')(BC)}\epsilon_{AD} \tag{F.7}$$

$$\mu_{(CD)} \cong \frac{1}{12}\epsilon^{C'D'}g^{ab}\text{Б}_{abcd} \tag{F.8}$$

$$
\begin{aligned}
\text{Б}_{abcd} \;\cong\;& 2g_{bc}\left(\epsilon_{A'D'}\mu_{(AD)} + \epsilon_{AD}\overline{\mu}_{(A'D')}\right) \\
+\;& g_{ab}\left(\epsilon_{C'D'}\mu_{(CD)} + \epsilon_{CD}\overline{\mu}_{(C'D')}\right) \\
+\;& g_{ac}\left(\epsilon_{B'D'}\mu_{(BD)} + \epsilon_{BD}\overline{\mu}_{(B'D')}\right) \\
+\;& g_{bd}\left(\epsilon_{A'C'}\mu_{(AC)} + \epsilon_{AC}\overline{\mu}_{(A'C')}\right) \\
+\;& g_{cd}\left(\epsilon_{A'B'}\mu_{(AB)} + \epsilon_{AB}\overline{\mu}_{(A'B')}\right)
\end{aligned} \tag{F.9}
$$

$$
\Omega_{(CD)(C'D')} \cong \frac{1}{4} g^{ab} \sqsupset_{abcd} \tag{F.10}
$$

$$
\begin{aligned}
\sqsupset_{abcd} \;\cong\;& g_{ab}\Omega_{(CD)(C'D')} + g_{ac}\Omega_{(BD)(B'D')} \\
-\;& g_{bd}\Omega_{(AC)(A'C')} - g_{cd}\Omega_{(AB)(A'B')}
\end{aligned} \tag{F.11}
$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: “Rank4AlphaSpinor” for equation (F.6), “Rank4OmegaSpinor” for equation (F.10), “Rank4MuSpinor” for equation (F.8), “Rank4GenerateAlphaTensor” for equation (F.7), “Rank4GenerateOmegaTensor” for equation (F.11), and “Rank4GenerateMuTensor” for equation (F.9).

F.4 The Y2b Spinors - β, κ, ρ

The $Y2b$ spinors are $\boldsymbol{\beta}$, $\boldsymbol{\kappa}$, and $\boldsymbol{\rho}$. A list of equations relating them to their corresponding 4th rank $SO\,(p,q)$ irreducible tensors in 4 dimensions is given below.

$$
\beta_{(ABCD)(B'D')} \cong \frac{1}{2} \mathfrak{c}_{abcd} \epsilon^{A'C'} \tag{F.12}
$$

$$
\mathfrak{c}_{abcd} \;\cong\; \beta_{(ABCD)(B'D')}\epsilon_{A'C'} + \overline{\beta}_{(A'B'C'D')(BD)}\epsilon_{AC} \tag{F.13}
$$

$$
\rho_{(CD)} \cong \frac{1}{12} \epsilon^{C'D'} g^{ab} \text{Ц}_{abcd} \tag{F.14}
$$

$$\begin{aligned} \text{Ц}_{abcd} &\cong -2g_{bd}\left(\epsilon_{A'C'}\rho_{(AC)} + \epsilon_{AC}\overline{\rho}_{(A'C')}\right) \\ &+ g_{ab}\left(\epsilon_{C'D'}\rho_{(CD)} + \epsilon_{CD}\overline{\rho}_{(C'D')}\right) \\ &- g_{ad}\left(\epsilon_{B'C'}\rho_{(BC)} + \epsilon_{BC}\overline{\rho}_{(B'C')}\right) \\ &- g_{bc}\left(\epsilon_{A'D'}\rho_{(AD)} + \epsilon_{AD}\overline{\rho}_{(A'D')}\right) \\ &- g_{cd}\left(\epsilon_{A'B'}\rho_{(AB)} + \epsilon_{AB}\overline{\rho}_{(A'B')}\right) \end{aligned} \quad (F.15)$$

$$\kappa_{(CD)(C'D')} \cong \frac{1}{4} g^{ab} \text{ꞁ}_{abcd} \quad (F.16)$$

$$\begin{aligned} \text{ꞁ}_{abcd} &\cong g_{ab}\kappa_{(CD)(C'D')} + g_{ad}\kappa_{(BC)(B'C')} \\ &- g_{bc}\kappa_{(AD)(A'D')} - g_{cd}\kappa_{(AB)(A'B')} \end{aligned} \quad (F.17)$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: "Rank4BetaSpinor" for equation (F.12), "Rank4KappaSpinor" for equation (F.16), "Rank4RhoSpinor" for equation (F.14), "Rank4GenerateBetaTensor" for equation (F.13), "Rank4GenerateKappaTensor" for equation (F.17), and "Rank4GenerateRhoTensor" for equation (F.15).

F.5 The Y2c Spinors - γ, θ, o

The $Y2c$ spinors are $\boldsymbol{\gamma}$, $\boldsymbol{\theta}$, and $\boldsymbol{o}$. A list of equations relating them to their corresponding 4th rank $SO(p,q)$ irreducible tensors in 4 dimensions is given below.

$$\gamma_{(ABCD)(C'D')} \cong \frac{1}{2} \mathfrak{d}_{abcd} \epsilon^{A'B'} \quad (F.18)$$

$$\mathfrak{d}_{abcd} \cong \gamma_{(ABCD)(C'D')}\epsilon_{A'B'} + \overline{\gamma}_{(A'B'C'D')(CD)}\epsilon_{AB} \quad (F.19)$$

$$o_{(BD)} \cong \frac{1}{12} \epsilon^{B'D'} g^{ac} \text{Д}_{abcd} \quad (F.20)$$

$$\begin{aligned}
\text{Д}_{abcd} &\cong -2g_{cd}\left(\epsilon_{A'B'}o_{(AB)} + \epsilon_{AB}\overline{o}_{(A'B')}\right) \\
&+ g_{ac}\left(\epsilon_{B'D'}o_{(BD)} + \epsilon_{BD}\overline{o}_{(B'D')}\right) \\
&+ g_{ad}\left(\epsilon_{B'C'}o_{(BC)} + \epsilon_{BC}\overline{o}_{(B'C')}\right) \\
&- g_{bc}\left(\epsilon_{A'D'}o_{(AD)} + \epsilon_{AD}\overline{o}_{(A'D')}\right) \\
&- g_{bd}\left(\epsilon_{A'C'}o_{(AC)} + \epsilon_{AC}\overline{o}_{(A'C')}\right)
\end{aligned} \tag{F.21}$$

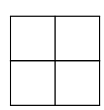
$$\theta_{(BD)(B'D')} \cong \frac{1}{4} g^{ac} \text{˥}_{abcd} \tag{F.22}$$

$$\begin{aligned}
\text{˥}_{abcd} &\cong g_{ac}\theta_{(BD)(B'D')} + g_{ad}\theta_{(BC)(B'C')} \\
&- g_{bc}\theta_{(AD)(A'D')} - g_{bd}\theta_{(AC)(A'C')}
\end{aligned} \tag{F.23}$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: "Rank4GammaSpinor" for equation (F.18), "Rank4ThetaSpinor" for equation (F.22), "Rank4OmicronSpinor" for equation (F.20), "Rank4GenerateGammaTensor" for equation (F.19), "Rank4GenerateThetaTensor" for equation (F.23), and "Rank4GenerateOmicronTensor" for equation (F.21).

F.6 The Y3 Spinors - $\delta, \Psi, \pi, \Phi, \chi, \Lambda$

The $Y3$ spinors are $\boldsymbol{\delta}$, $\boldsymbol{\Psi}$, $\boldsymbol{\pi}$, $\boldsymbol{\Phi}$, $\boldsymbol{\chi}$, and $\boldsymbol{\Lambda}$. We split the spinors into the $Y3a$, and $Y3b$ tableaux sectors to sort them further.



F.7 The Y3a Spinors - δ, π, χ

The $Y3a$ spinors are $\boldsymbol{\delta}$, $\boldsymbol{\pi}$, and $\boldsymbol{\chi}$. A list of equations relating them to their corresponding 4th rank $SO(p,q)$ irreducible tensors in 4 dimensions is given below.

$$\delta_{(ABCD)} \cong \frac{1}{4}\mathfrak{e}_{abcd}\epsilon^{A'C'}\epsilon^{B'D'} \tag{F.24}$$

$$\mathfrak{e}_{abcd} \cong \delta_{(ABCD)}\epsilon_{A'C'}\epsilon_{B'D'} + \overline{\delta}_{(A'B'C'D')}\epsilon_{AC}\epsilon_{BD} \tag{F.25}$$

$$\pi_{(AB)(A'B')} = \frac{1}{2}g^{cd}\mathscr{E}_{abcd} \tag{F.26}$$

$$\begin{aligned}\mathscr{E}_{abcd} &\cong \pi_{(AB)(A'B')}\epsilon_{CD}\epsilon_{C'D'} - \pi_{(BC)(B'C')}\epsilon_{AD}\epsilon_{A'D'} \\ &+ \pi_{(CD)(C'D')}\epsilon_{AB}\epsilon_{A'B'} - \pi_{(AD)(A'D')}\epsilon_{BC}\epsilon_{B'C'}\end{aligned} \tag{F.27}$$

$$\chi \cong \frac{1}{12}g^{ab}g^{cd}\mathfrak{E}_{abcd} \tag{F.28}$$

$$\mathfrak{E}_{abcd} \cong \chi\left(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'B'}\epsilon_{C'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}\right) \tag{F.29}$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: “Rank4DeltaSpinor” for equation (F.24), “Rank4PiSpinor” for equation (F.26), “Rank4ChiSpinor” for equation (F.28), “Rank4GenerateDeltaTensor” for equation (F.25), “Rank4GeneratePiTensor” for equation (F.27), and “Rank4GenerateChiTensor” for equation (F.29).

F.8 The Y3b Spinors - Ψ, Φ, Λ

The $Y3b$ spinors are $\mathbf{\Psi}$, $\mathbf{\Phi}$, and $\mathbf{\Lambda}$. A list of equations relating them to their corresponding 4th rank $SO\left(p,q\right)$ irreducible tensors in 4 dimensions is given below.

$$\Psi_{(ABCD)} \cong \frac{1}{4}\mathfrak{f}_{abcd}\epsilon^{A'B'}\epsilon^{C'D'} \tag{F.30}$$

$$\mathfrak{f}_{abcd} \cong \Psi_{(ABCD)}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{(A'B'C'D')}\epsilon_{AB}\epsilon_{CD} \tag{F.31}$$

$$\Phi_{(AC)(A'C')} \cong \frac{1}{2}g^{bd}\mathscr{F}_{abcd} \tag{F.32}$$

$$\begin{aligned}\mathscr{F}_{abcd} &= \Phi_{(AC)(A'C')}\epsilon_{BD}\epsilon_{B'D'} - \Phi_{(BC)(B'C')}\epsilon_{AD}\epsilon_{A'D'} \\ &+ \Phi_{(BD)(B'D')}\epsilon_{AC}\epsilon_{A'C'} - \Phi_{(AD)(A'D')}\epsilon_{BC}\epsilon_{B'C'}\end{aligned} \tag{F.33}$$

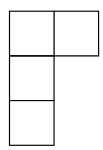
$$\Lambda \cong \frac{1}{12} g^{ac} g^{bd} \mathfrak{F}_{abcd} \tag{F.34}$$

$$\mathfrak{F}_{abcd} \cong \Lambda \left(\epsilon_{AC} \epsilon_{BD} \epsilon_{A'C'} \epsilon_{B'D'} - \epsilon_{AD} \epsilon_{BC} \epsilon_{A'D'} \epsilon_{B'C'} \right) \tag{F.35}$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: “Rank4PsiSpinor” for equation (F.30), “Rank4PhiSpinor” for equation (F.32), “Rank4LambdaSpinor” for equation (F.34), “Rank4GeneratePsiTensor” for equation (F.31), “Rank4GeneratePhiTensor” for equation (F.33), and “Rank4GenerateLambdaTensor” for equation (F.35).

F.9 The Y4 Spinors - $\tau, \zeta, \xi, \nu, \eta, \sigma$

The $Y4$ spinors are $\boldsymbol{\tau}$, $\boldsymbol{\zeta}$, $\boldsymbol{\xi}$, $\boldsymbol{\nu}$, $\boldsymbol{\eta}$, and $\boldsymbol{\sigma}$. We split the spinors into the $Y2a$, $Y2b$, and $Y2c$ tableaux sectors to sort them further.



F.10 The Y4a Spinors - τ, ν

The $Y4a$ spinors are $\boldsymbol{\tau}$, and $\boldsymbol{\nu}$. A list of equations relating them to their corresponding 4th rank $SO\,(p,q)$ irreducible tensors in 4 dimensions is given below.

$$\tau_{(AB)(C'D')} = \frac{1}{2} \mathfrak{g}_{abcd} \epsilon^{A'B'} \epsilon^{CD} \tag{F.36}$$

$$\begin{aligned} \mathfrak{g}_{abcd} \cong\; & \epsilon_{A'C'} \epsilon_{BD} \tau_{(AC)(B'D')} - \epsilon_{B'D'} \epsilon_{AC} \tau_{(BD)(A'C')} \\ +\; & \epsilon_{A'B'} \epsilon_{CD} \tau_{(AB)(C'D')} - \epsilon_{C'D'} \epsilon_{AB} \tau_{(CD)(A'B')} \\ -\; & \epsilon_{A'D'} \epsilon_{BC} \tau_{(AD)(B'C')} + \epsilon_{B'C'} \epsilon_{AD} \tau_{(BC)(A'D')} \end{aligned} \tag{F.37}$$

$$\nu_{(CD)} \cong \frac{1}{4} \epsilon^{C'D'} g^{ab} \mathscr{G}_{abcd} \tag{F.38}$$

$$\begin{aligned} \mathscr{G}_{abcd} &= \epsilon_{AB}\epsilon_{A'B'}\left(\epsilon_{C'D'}\nu_{(CD)} + \epsilon_{CD}\overline{\nu}_{(C'D')}\right) \\ &- \epsilon_{BC}\epsilon_{B'C'}\left(\epsilon_{A'D'}\nu_{(AD)} + \epsilon_{AD}\overline{\nu}_{(A'D')}\right) \\ &+ \epsilon_{BD}\epsilon_{B'D'}\left(\epsilon_{A'C'}\nu_{(AC)} + \epsilon_{AC}\overline{\nu}_{(A'C')}\right) \end{aligned} \tag{F.39}$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: "Rank4TauSpinor" for equation (F.36), "Rank4NuSpinor" for equation (F.38), "Rank4GenerateTauTensor" for equation (F.37), and "Rank4GenerateNuTensor" for equation (F.39).

F.11 The Y4b Spinors - ζ, η

The $Y4b$ spinors are $\boldsymbol{\zeta}$, and $\boldsymbol{\eta}$. A list of equations relating them to their corresponding 4th rank $SO\left(p,q\right)$ irreducible tensors in 4 dimensions is given below.

$$\zeta_{(AB)(C'D')} = \frac{1}{2}\mathfrak{h}_{abcd}\epsilon^{A'B'}\epsilon^{CD} \tag{F.40}$$

$$\begin{aligned} \mathfrak{h}_{abcd} &\cong \epsilon_{A'C'}\epsilon_{BD}\zeta_{(AC)(B'D')} - \epsilon_{B'D'}\epsilon_{AC}\zeta_{(BD)(A'C')} \\ &+ \epsilon_{A'B'}\epsilon_{CD}\zeta_{(AB)(C'D')} - \epsilon_{C'D'}\epsilon_{AB}\zeta_{(CD)(A'B')} \\ &+ \epsilon_{A'D'}\epsilon_{BC}\zeta_{(AD)(B'C')} - \epsilon_{B'C'}\epsilon_{AD}\zeta_{(BC)(A'D')} \end{aligned} \tag{F.41}$$

$$\eta_{(BD)} \cong \frac{1}{4}\epsilon^{B'D'}g^{ac}\mathscr{H}_{abcd} \tag{F.42}$$

$$\begin{aligned} \mathscr{H}_{abcd} &= \epsilon_{AC}\epsilon_{A'C'}\left(\epsilon_{B'D'}\eta_{(BD)} + \epsilon_{BD}\overline{\eta}_{(B'D')}\right) \\ &- \epsilon_{BC}\epsilon_{B'C'}\left(\epsilon_{A'D'}\eta_{(AD)} + \epsilon_{AD}\overline{\eta}_{(A'D')}\right) \\ &+ \epsilon_{CD}\epsilon_{C'D'}\left(\epsilon_{A'B'}\eta_{(AB)} + \epsilon_{AB}\overline{\eta}_{(A'B')}\right) \end{aligned} \tag{F.43}$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: "Rank4ZetaSpinor" for equation (F.40), "Rank4EtaSpinor" for equation (F.42), "Rank4GenerateZetaTensor" for equation (F.41), and "Rank4GenerateEtaTensor" for equation

(F.43).

F.12 The Y4c Spinors - ξ, σ

The $Y4c$ spinors are $\boldsymbol{\xi}$, and $\boldsymbol{\sigma}$. A list of equations relating them to their corresponding 4th rank $SO(p,q)$ irreducible tensors in 4 dimensions is given below.

$$\xi_{(AB)(C'D')} = \frac{1}{2}\mathfrak{j}_{abcd}\epsilon^{A'B'}\epsilon^{CD} \tag{F.44}$$

$$\begin{aligned} \mathfrak{j}_{abcd} &\cong \epsilon_{A'C'}\epsilon_{BD}\xi_{(AC)(B'D')} - \epsilon_{B'D'}\epsilon_{AC}\xi_{(BD)(A'C')} \\ &- \epsilon_{A'B'}\epsilon_{CD}\xi_{(AB)(C'D')} + \epsilon_{C'D'}\epsilon_{AB}\xi_{(CD)(A'B')} \\ &+ \epsilon_{A'D'}\epsilon_{BC}\xi_{(AD)(B'C')} - \epsilon_{B'C'}\epsilon_{AD}\xi_{(BC)(A'D')} \end{aligned} \tag{F.45}$$

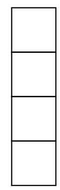
$$\sigma_{(BC)} \cong \frac{1}{4}\epsilon^{B'C'}g^{ad}\mathscr{J}_{abcd} \tag{F.46}$$

$$\begin{aligned} \mathscr{J}_{abcd} &= \epsilon_{AD}\epsilon_{A'D'}\left(\epsilon_{B'C'}\sigma_{(BC)} + \epsilon_{BC}\overline{\sigma}_{(B'C')}\right) \\ &- \epsilon_{BD}\epsilon_{B'D'}\left(\epsilon_{A'C'}\sigma_{(AC)} + \epsilon_{AC}\overline{\sigma}_{(A'C')}\right) \\ &+ \epsilon_{CD}\epsilon_{C'D'}\left(\epsilon_{A'B'}\sigma_{(AB)} + \epsilon_{AB}\overline{\sigma}_{(A'B')}\right) \end{aligned} \tag{F.47}$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: "Rank4XiSpinor" for equation (F.44), "Rank4SigmaSpinor" for equation (F.46), "Rank4GenerateXiTensor" for equation (F.45), and "Rank4GenerateSigmaTensor" for equation (F.47).

F.13 The Y5 Spinor - $\aleph$

The $Y5$ spinor is $\boldsymbol{\aleph}$. Two equations relating this spinor to its corresponding 4th rank $SO\left(p,q\right)$ irreducible tensor in 4 dimensions are given below.



$$\aleph \cong \frac{1}{36}\epsilon^{AB}\epsilon^{CD}\epsilon^{A'C'}\epsilon^{B'D'}\mathfrak{t}_{abcd} \tag{F.48}$$

$$\begin{aligned} \mathfrak{t}_{abcd} &= \aleph\left(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'}\right) \\ &+ \aleph\left(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'B'}\epsilon_{C'D'}\right) \\ &+ \aleph\left(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}\right) \end{aligned} \tag{F.49}$$

Here we list the code used to calculate the left-hand side of each of the above equations. We have: "Rank4AlephSpinor" for equation (F.48), and "Rank4GenerateAlephTensor" for equation (F.49).

APPENDIX G

THE BELINFANTE-ROSENFELD RELATION

We will follow Poplawski Pg. 68, but have modified it slightly to suit our definitions and needs. Also see Wheeler (J. T. Wheeler, 2023) for clarification. The total variation of the matter action with respect to geometrical variables is either equation (G.1) or equation (G.2). Equation (G.2) is in a way more fundamental because we can interpret g^{ab} as a function of $e^a{}_\mu$ from the orthonormality relation $g^{ab} = e^a{}_\mu e^b{}_\nu \eta^{\mu\nu}$.

$$\delta S_M = -\frac{1}{2}\int_\Omega \mathcal{T}_{ab}\delta g^{ab} d\Omega - \frac{1}{2}\int_\Omega \mathfrak{M}_a{}^{bc}\delta T^a_{bc} d\Omega \tag{G.1}$$

$$\delta S_M = -\int_\Omega \mathfrak{T}_a{}^\mu \delta e^a{}_\mu d\Omega - \frac{1}{2}\int_\Omega \mathfrak{Y}^{a\mu\nu}\delta\omega_{\mu\nu a} d\Omega \tag{G.2}$$

Equivalently, we could write the torsion piece in terms of the contorsion. The motivation for this comes from Hehl. Hehl (F. Hehl et al., 1976) examines parallelly transported tetrads in an ECSK (he uses the term U_4) spacetime. Going further he says that the connection 1-forms $\omega^\mu{}_{\nu a}$ describe "the rotation of the parallelly transported tetrad relative to the given tetrad system. This rotation consists of two pieces, the Ricci rotation $\varpi^\mu{}_{\nu a}$ due to the Riemannian metric [...], and an independent "added twist" $C^b{}_{ca}$ proportional to the contortion. The tetrad vectors in a U_4 thus have new degrees of freedom - independent rotations not specified by the metric structure" (F. Hehl et al., 1976) (See section II.F Pg 6). In this light Hehl couples the spin angular momentum to the contorsion tensor, not the torsion tensor. This equivalent variation is given by equation (G.3).

$$\delta S_M = -\frac{1}{2}\int_\Omega \mathcal{T}_{ab}\delta g^{ab} d\Omega - \frac{1}{2}\int_\Omega \left(g_{bd}\mathfrak{S}^{adc}\right)\delta C^b{}_{ca} d\Omega \tag{G.3}$$

In each of these equations (1,2,3) we have only specified our independent variables: for equation (G.1) the independent variables are g^{ab} and $T^{a}_{\ bc}$, for equation (G.2) the independent variables are $e^{a}_{\ \mu}$ and $\omega^{\mu}_{\ \nu a}$, and for equation (G.3) that the independent variables are g^{ab} and $C^{a}_{\ bc}$.

Our goal here is to relate these two actions together, the metric/contorsion one, and the vielbein/spin connection one. This relation is called the Belinfante-Rosenfeld relation which we will derive shortly. See Belinfante (Belinfante, 1940), and Rosenfeld (Rosenfeld, 1940) for the historical references. To begin with, this relation exists in the first place because of the orthonormality relation generated by $e^{a}_{\ \mu}$, and because the spin connection's Levi-Civita generated piece $\varpi^{\mu}_{\ \nu a}$ depends on the vielbein. These ideas give us a way to relate these two different action variations under the same principles. The relation is essential in understanding a gravity theory coupled to Dirac spinors for example because the action does not depend on the metric, only on the vielbein. A relation between equation (G.2) and equation (G.3) is what we ultimately want.

To get down to it, we start by writing the metric variation δg^{ab} in terms of the vielbein variation $\delta e^{a}_{\ \mu}$. If we use the orthonormality relation $g_{ab} = e_{a}^{\ \mu} e_{b}^{\ \nu} \eta_{\mu\nu}$, and then vary it, $\delta g^{ab} = 2\eta^{\mu\nu}\delta^{(a}_{c} e^{b)}_{\ \nu}\delta e^{c}_{\ \mu}$, then we find that the metric variation piece in equation (G.1) can be written as is in equation (G.4.)

$$\frac{1}{2}\int\limits_{\Omega} \mathcal{T}_{ab}\delta g^{ab} d\Omega = \int\limits_{\Omega} \mathcal{T}_{ab} e^{a\mu}\delta e^{b}_{\ \mu} d\Omega \tag{G.4}$$

Although we want to get a relation between equations (G.2) and (G.3), it is beneficial calculation wise to examine equation (G.1) because of a relation on the torsion tensor in terms of the spin connection.

Similarly to what we did above, we can examine the torsion part. By making use of the tetrad postulate, we can expand the torsion tensor in terms of the spin connection and tetrad as in equation (G.6); see Poplawski (Poplawski, 2013) pg 45 (1.5.31).

$$D_a e^{b}_{\ \mu} = \partial_a e^{b}_{\ \mu} + \Gamma^{b}_{\ ca} e^{c}_{\ \mu} - \omega^{\nu}_{\ \mu a} e^{b}_{\ \nu} = 0 \tag{G.5}$$

Recall that even in an anholonomic basis, we can use Lie's third theorem, turning the lie algebra basis into a coordinate one with coordinate coefficients, to write the torsion as $T^a_{\ bc} = -2\Gamma^a{}_{[bc]}$. This combined with the tetrad postulate results in a way to write the torsion tensor in terms of the spin connection and tetrad, see equation (G.6).

$$T^a_{\ bc} = 2\omega^a{}_{[cb]} + 2e^a{}_\mu \partial_{[b} e_{c]}{}^\mu \tag{G.6}$$

The variation of equation (G.6) becomes equation (G.7). It has been simplified considerably using an incredibly useful relation between the vielbein and inverse vielbein: $\delta e_c{}^\beta = -e_a{}^\beta e_c{}^\mu \delta e^a{}_\mu$, and by an integration by parts to isolate an eventual boundary term.

$$\delta T^a_{\ bc} = 2\delta\omega^a{}_{[cb]} + \left(2\delta^a_f \partial_{[b} e_{c]}{}^\sigma + 2e_{[c}{}^\sigma \partial_{b]} e^a{}_\mu e_f{}^\mu\right) \delta e^f{}_\sigma - 2\partial_{[b}\left(e_{c]}{}^\sigma \delta^a_f \delta e^f{}_\sigma\right) \tag{G.7}$$

Now substitute the torsion variation above into the torsion variation piece of equation (G.1). We can simplify off some pieces first by isolating the boundary term $\delta S\{e\}$ and the spin connection variation piece $\delta S\{\omega\}$. We will then examine the tetrad variation piece $\delta S\{e\}$ and simplify it. We begin by turning the abstract indices on $\delta\omega$ into orthonormal ones: i.e. $\delta\omega^a{}_{cb} = \delta\left(e^a{}_\mu e_c{}^\nu \omega^\mu{}_{\nu b}\right)$. This results in equation (G.7) simplifying down to equation (G.8).

$$\delta T^a_{\ bc} = \tag{G.8}$$

$$\left(2e^a{}_\mu e_{[c}{}^\nu \delta^f_{b]}\right)\delta\omega^\mu{}_{\nu f} - 2\partial_{[b}\left(e_{c]}{}^\sigma \delta^a_f \delta e^f{}_\sigma\right)$$
$$+\left(2e^a{}_\mu e_f{}^\nu \omega^\mu{}_{\nu[c} e_{b]}{}^\sigma - 2\delta^a_f \omega^\sigma{}_{\nu[c} e_{b]}{}^\nu + 2\delta^a_f \partial_{[b} e_{c]}{}^\sigma + 2e_{[c}{}^\sigma \partial_{b]} e^a{}_\mu e_f{}^\mu\right)\delta e^f{}_\sigma$$

Below we have substituted the torsion variation in terms of the vielbein and spin connection and simplified. Furthermore, we have broken off each of the pieces corresponding to the vielbein variation $\delta S\{e\}$, the spin connection variation $\delta S\{\omega\}$, and the vielbein boundary piece $\delta S\{\partial e\}$.

$$\frac{1}{2}\int_\Omega \mathfrak{M}_a{}^{bc} \delta T^a_{\ bc} d\Omega =$$

$$
\begin{aligned}
&= \int\limits_{\Omega} \mathfrak{M}_{a}{}^{bc} \left(\delta \left(e^{a}{}_{\mu} e_{c}{}^{\nu} \omega^{\mu}{}_{\nu b} \right) + \left(\delta^{a}_{f} \partial_{b} e_{c}{}^{\sigma} + e_{c}{}^{\sigma} \partial_{b} e^{a}{}_{\mu} e_{f}{}^{\mu} \right) \delta e^{f}{}_{\sigma} \right) d\Omega \\
&- \int\limits_{\Omega} \mathfrak{M}_{a}{}^{bc} \partial_{b} \left(e_{c}{}^{\sigma} \delta^{a}_{f} \delta e^{f}{}_{\sigma} \right) d\Omega \\
&= \delta S\left\{e\right\} + \delta S\left\{\omega\right\} + \delta S\left\{\partial e\right\}
\end{aligned}
$$

Here we have each of the pieces $\delta S\{e\}$, $\delta S\{\omega\}$, and $\delta S\{\partial e\}$ defined as follows in equations: (G.9), (G.10), and (G.11). We have used $\mathfrak{S}_{cab} = -2\mathfrak{M}_{[ab]c}$ in equation G.11, and we have used Stokes' theorem in equation (G.9) to convert the region Ω integral to a closed hypersurface Σ integral.

$$
\begin{aligned}
\delta S\left\{e\right\} &= \int\limits_{\Omega} \left(\mathfrak{M}_{a}{}^{bc} \delta \left(e^{a}{}_{\mu} e_{c}{}^{\nu} \right) \omega^{\mu}{}_{\nu b} \right) d\Omega \\
&+ \int\limits_{\Omega} \left(\left(\mathfrak{M}_{a}{}^{bc} \delta^{a}_{f} \partial_{b} e_{c}{}^{\sigma} + \mathfrak{M}_{a}{}^{bc} e_{c}{}^{\sigma} \partial_{b} e^{a}{}_{\mu} e_{f}{}^{\mu} \right) \delta e^{f}{}_{\sigma} \right) d\Omega \\
&+ \int\limits_{\Omega} \left(\left(\partial_{b} \mathfrak{M}_{a}{}^{bc} e_{c}{}^{\sigma} \delta^{a}_{f} \right) \delta e^{f}{}_{\sigma} \right) d\Omega
\end{aligned}
\tag{G.9}
$$

$$
\delta S\left\{\partial e\right\} = -\int\limits_{\Omega} \partial_{b} \left(\mathfrak{M}_{a}{}^{bc} e_{c}{}^{\sigma} \delta^{a}_{f} \delta e^{f}{}_{\sigma} \right) d\Omega = -\oint\limits_{\Sigma} n_{b} \mathfrak{M}_{f}{}^{bc} e_{c}{}^{\sigma} \delta e^{f}{}_{\sigma} d\Sigma \tag{G.10}
$$

$$
\delta S\left\{\omega\right\} = -\int\limits_{\Omega} \mathfrak{M}_{\mu}{}^{\nu b} \delta \omega^{\mu}{}_{\nu b} d\Omega = \int\limits_{\Omega} \left(\frac{1}{2} \mathfrak{S}^{b\mu\nu} \right) \delta \omega_{\mu\nu b} d\Omega \tag{G.11}
$$

We will now simplify the non-boundary tetrad piece and give a derivation.

$$
\delta S\left\{e\right\} =
$$

$$
\begin{aligned}
= & \int\limits_{\Omega} \left(\mathfrak{M}_a{}^{bc} \delta \left(e^a{}_\mu e_c{}^\nu \right) \omega^\mu{}_{\nu b} \right) d\Omega \\
+ & \int\limits_{\Omega} \left(\left(\mathfrak{M}_a{}^{bc} \delta^a_f \partial_b e_c{}^\sigma + \mathfrak{M}_a{}^{bc} e_c{}^\sigma \partial_b e^a{}_\mu e_f{}^\mu \right) \delta e^f{}_\sigma \right) d\Omega \\
+ & \int\limits_{\Omega} \left(\left(\partial_b \mathfrak{M}_a{}^{bc} e_c{}^\sigma \delta^a_f \right) \delta e^f{}_\sigma \right) d\Omega
\end{aligned}
$$

$$
\begin{aligned}
= & \int\limits_{\Omega} \left(\mathfrak{M}_f{}^{bc} \delta e^f{}_\sigma e_c{}^\nu \omega^\sigma{}_{\nu b} + \mathfrak{M}_a{}^{bc} e^a{}_\mu \left(-e_f{}^\nu e_c{}^\sigma \delta e^f{}_\sigma \right) \omega^\mu{}_{\nu b} \right) d\Omega \\
+ & \int\limits_{\Omega} \left(\left(\mathfrak{M}_a{}^{bc} \delta^a_f \partial_b e_c{}^\sigma + \mathfrak{M}_a{}^{bc} e_c{}^\sigma \partial_b e^a{}_\mu e_f{}^\mu \right) \delta e^f{}_\sigma \right) d\Omega \\
+ & \int\limits_{\Omega} \left(\left(\partial_b \mathfrak{M}_a{}^{bc} e_c{}^\sigma \delta^a_f \right) \delta e^f{}_\sigma \right) d\Omega
\end{aligned}
$$

$$
\begin{aligned}
= & \int\limits_{\Omega} \left(\mathfrak{M}_f{}^{bc} e_c{}^\nu \omega^\sigma{}_{\nu b} - \mathfrak{M}_a{}^{bc} e^a{}_\mu e_f{}^\nu e_c{}^\sigma \omega^\mu{}_{\nu b} \right) \delta e^f{}_\sigma d\Omega \\
+ & \int\limits_{\Omega} \left(\mathfrak{M}_a{}^{bc} \delta^a_f \partial_b e_c{}^\sigma + \mathfrak{M}_a{}^{bc} e_c{}^\sigma \partial_b e^a{}_\mu e_f{}^\mu \right) \delta e^f{}_\sigma d\Omega \\
+ & \int\limits_{\Omega} \left(\partial_b \mathfrak{M}_a{}^{bc} e_c{}^\sigma \delta^a_f \right) \delta e^f{}_\sigma d\Omega
\end{aligned}
$$

$$
\begin{aligned}
= & \int\limits_{\Omega} \left(\mathfrak{M}_f{}^{bc} e_c{}^\nu \omega^\sigma{}_{\nu b} - \mathfrak{M}_a{}^{bc} e^a{}_\mu e_f{}^\nu e_c{}^\sigma \omega^\mu{}_{\nu b} + \right) \delta e^f{}_\sigma d\Omega \\
+ & \int\limits_{\Omega} \left(\mathfrak{M}_a{}^{bc} \delta^a_f \left(\Gamma^f{}_{cb} e_f{}^\sigma - \omega^\sigma{}_{\lambda b} e_c{}^\lambda \right) + \mathfrak{M}_a{}^{bc} e_c{}^\sigma \partial_b e^a{}_\mu e_f{}^\mu \right) \delta e^f{}_\sigma d\Omega \\
+ & \int\limits_{\Omega} \left(\partial_b \mathfrak{M}_a{}^{bc} e_c{}^\sigma \delta^a_f \right) \delta e^f{}_\sigma d\Omega
\end{aligned}
$$

$$= \int\limits_\Omega \left(\mathfrak{M}_a{}^{bc} e_c{}^\sigma e_f{}^\mu \left(\partial_b e^a{}_\mu - \omega^\nu{}_{\mu b} e^a{}_\nu\right)\right) \delta e^f{}_\sigma d\Omega$$

$$+ \int\limits_\Omega \left(\mathfrak{M}_a{}^{bc} \delta^a_f \Gamma^d{}_{cb} e_d{}^\sigma + \partial_b \mathfrak{M}_a{}^{bc} e_c{}^\sigma \delta^a_f\right) \delta e^f{}_\sigma d\Omega$$

$$+ \int\limits_\Omega \left(\partial_b \mathfrak{M}_a{}^{bc} e_c{}^\sigma \delta^a_f\right) \delta e^f{}_\sigma d\Omega$$

$$= \int\limits_\Omega \left(\mathfrak{M}_a{}^{bc} e_c{}^\sigma e_f{}^\mu \left(-\Gamma^a{}_{db} e^d{}_\mu\right) + \mathfrak{M}_a{}^{bc} \delta^a_f \Gamma^d{}_{cb} e_d{}^\sigma\right) \delta e^f{}_\sigma d\Omega$$

$$= \int\limits_\Omega \left(\partial_b \mathfrak{M}_a{}^{bc} e_c{}^\sigma \delta^a_f\right) \delta e^f{}_\sigma d\Omega$$

$$= \int\limits_\Omega \left(-\mathfrak{M}_a{}^{bc} e_c{}^\sigma e_f{}^\mu \Gamma^a{}_{db} e^d{}_\mu + \mathfrak{M}_a{}^{bc} \delta^a_f \Gamma^d{}_{[cb]} e_d{}^\sigma\right) \delta e^f{}_\sigma d\Omega$$

$$= \int\limits_\Omega \left(\left(D_b \mathfrak{M}_a{}^{bc} + \Gamma^d{}_{ab} \mathfrak{M}_d{}^{bc} - 2\Gamma^g{}_{[bg]} \mathfrak{M}_a{}^{bc} - \frac{1}{2} T^c{}_{bd} \mathfrak{M}_a{}^{bd}\right) e_c{}^\sigma \delta^a_f\right) \delta e^f{}_\sigma d\Omega$$

$$= \int\limits_\Omega \left(D_b \mathfrak{M}_f{}^{bc} + T^g{}_{bg} \mathfrak{M}_f{}^{bc}\right) e_c{}^\sigma \delta e^f{}_\sigma d\Omega$$

In the derivation above, we needed to use the covariant derivative of a tensor density. This is given by equation (G.12) because it does not come up very often. Upon the many steps of simplification, equation (G.9) reduces to the compact form of equation (G.13).

$$D_f \mathfrak{M}_a{}^{bc} = \partial_f \mathfrak{M}_a{}^{bc} - \Gamma^d{}_{af} \mathfrak{M}_d{}^{bc} + \Gamma^b{}_{df} \mathfrak{M}_a{}^{dc} + \Gamma^c{}_{df} \mathfrak{M}_a{}^{bd} - \Gamma^g{}_{gf} \mathfrak{M}_a{}^{bc} \tag{G.12}$$

$$\delta S\left\{e\right\} = \int\limits_\Omega \left(\left(D_b \mathfrak{M}_f{}^{bc} + T^g{}_{bg} \mathfrak{M}_f{}^{bc}\right) e_c{}^\sigma\right) \delta e^f{}_\sigma d\Omega \tag{G.13}$$

We can furthermore simplify δS to write the torsion variation as the tetrad and contorsion variation. Recall: $T_{abc} = -2C_{a[bc]}$, $\mathfrak{M}_{abc} = \frac{1}{2}\left(\mathfrak{S}_{abc} + \mathfrak{S}_{cba} - \mathfrak{S}_{bca}\right)$. These result in an interesting formula which shows us how the contorsion tensor variation is related to the torsion tensor variation in equation (G.14). Recall also that $\mathfrak{S}_{cab} = -2\mathfrak{M}_{[ab]c}$.

$$\frac{1}{2}\int\limits_{\Omega} \mathfrak{M}_a{}^{bc}\delta T^a{}_{bc} d\Omega = \int\limits_{\Omega} \left(\frac{1}{2}g_{bd}\mathfrak{S}^{adc}\right)\delta C^b{}_{ca} d\Omega \tag{G.14}$$

To move forward, we make one assumption which will later be found out true in the same vein as the arguments from Belinfante, Rosenfeld, and Poplawski ((Belinfante, 1940), (Rosenfeld, 1940) (Poplawski, 2013) respectively). We assume that the equations of motion for a variational theory starting from the same action with independent variables $e^a{}_\mu$,and $C^a{}_{bc}$ must always be the same; Wheeler assumes less (J. T. Wheeler, 2023), and thus his argument shows this procedure is even more general; to clarify, we can always vary whatever we would like, this just means that if we vary with respect to a variable not $e^a{}_\mu$,and $C^a{}_{bc}$ that the resulting equations can always be written in terms of a variation of $e^a{}_\mu$, and $C^a{}_{bc}$. This assumption results in equation (G.15) where $\mathfrak{S}^{a\mu\nu} = e_a{}^{[\mu} e_b{}^{\nu]}\mathfrak{S}^{abc}$. It will also result in the Belinfante-Rosenfeld relation, equation (G.18).

$$\mathfrak{S}^{a\mu\nu} = \mathfrak{Y}^{a\mu\nu} \tag{G.15}$$

This turns out to be at the core of the argument that Belinfante and Rosenfeld used to prove that the Belinfante-Rosenfeld tensor was the same as the Hilbert energy momentum tensor. With this assumption, just by using equation (G.14) and that equations (G.10), (G.11), and (G.13) together equal the left-hand side of equation (G.14), we find how a spin connection variation decomposes into tetrad and contorsion pieces if we choose them as the independent variables in our theory. This is one of the most useful relations that we will show eventually relates our physics back to the Hilbert energy momentum tensor of general relativity. Substituting this all in we get a relation for the spin connection variation of matter in terms of the vielbein and contorsion variation plus a boundary term as in equation (G.16).

$$\frac{1}{2}\int\limits_{\Omega} \mathfrak{Y}^{a\mu\nu}\delta\omega_{\mu\nu a} d\Omega = \tag{G.16}$$

$$\int\limits_{\Omega} \left(\left(D_c \mathfrak{M}_f{}^{bc} - T^g{}_{gc} \mathfrak{M}_f{}^{bc} \right) e_b{}^{\sigma} \right) \delta e^f{}_{\sigma} d\Omega$$

$$+ \int\limits_{\Omega} \left(\frac{1}{2} g_{bd} \mathfrak{S}^{adc} \right) \delta C^b{}_{ca} d\Omega + \oint\limits_{\Sigma} n_b \mathfrak{M}_f{}^{bc} e_c{}^{\sigma} \delta e^f{}_{\sigma} d\Sigma$$

To use everything we have so far, and to make the Belinfante-Rosenfeld relation all we need to do it substitute equation (G.16) into our original variation, equation (G.2). This results in equation (G.17), where now our independent variables are the vielbein and contorsion tensor $C^a{}_{bc}$.

$$\begin{aligned} \delta S_M &= \int\limits_{\Omega} \left(-\mathfrak{T}_f{}^{\sigma} - \left(D_c \mathfrak{M}_f{}^{bc} - T^g{}_{gc} \mathfrak{M}_f{}^{bc} \right) e_b{}^{\sigma} \right) \delta e^f{}_{\sigma} d\Omega \\ &- \int\limits_{\Omega} \left(\frac{1}{2} g_{ad} \mathfrak{S}^{fde} \right) \delta C^a{}_{ef} d\Omega - \oint\limits_{\Sigma} n_b \mathfrak{M}_f{}^{bc} e_c{}^{\sigma} \delta e^f{}_{\sigma} d\Sigma \end{aligned} \tag{G.17}$$

In GR, Belinfante (Belinfante, 1940) and Rosenfeld (Rosenfeld, 1940) showed that the coefficient of $\delta e^f{}_{\sigma}$ in equation (G.17) is exactly the Hilbert energy momentum tensor; of course in GR the torsion tensor is zero, and the relation is simplified. By an extension of their logic we can say that the coefficient of $\delta e^f{}_{\sigma}$ above is exactly the Hilbert energy momentum tensor in ECSK theory, making this the natural extension of Belinfante and Rosenfeld's logic when we include torsion. Furthermore, this resolves our assumption that " that the equations of motion for a variational theory starting from the same action with independent variables $e^a{}_{\mu}$, and $C^a{}_{bc}$ must always be the same." The assumption turns out to be true always by Belinfante & Rosenfeld's arguments. Thus, we wind up with a way to interpret tetrad and spin connection energy momentum tensors in ECSK to metric and contorsion energy momentum tensors in ECSK as well how to relate Palatini spin potential tensors to spin angular momentum tensors. This Belinfante-Rosenfeld relation is shown below in equation (G.18).

$$\mathcal{T}_{af} e^{a\sigma} = \mathfrak{T}_f{}^{\sigma} + D_c \mathfrak{M}_f{}^{bc} e_b{}^{\sigma} - T^g{}_{gc} \mathfrak{M}_f{}^{bc} e_b{}^{\sigma} \tag{G.18}$$

Hehl (Belinfante, 1940) on pg. 75 has a form version of the Belinfante-Rosenfeld Relation also for an ECSK spacetime in terms of differential forms, however he drops the boundary term as implied by his approximate equal sign in equations (5.2.19) on said page. Applying the Belinfante-Rosenfeld relation to equation (G.2) is insightful into the structure we are used to in general relativity. Once we use the Belinfante-Rosenfeld relation on equation (G.17), we get equation (G.19).

$$\delta S_M = \int_\Omega \left(-\mathcal{T}_{af} e^{a\sigma}\right) \delta e^f{}_\sigma d\Omega - \int_\Omega \left(\frac{1}{2} g_{bd} \mathfrak{S}^{adc}\right) \delta C^b{}_{ca} d\Omega - \oint_\Sigma n_b \mathfrak{M}_f{}^{bc} e_c{}^\sigma \delta e^f{}_\sigma d\Sigma \qquad \text{(G.19)}$$

Upon comparing equation (G.19) to equation (G.3) with equation (G.4) applied, we find that we have the same exact theory up to a boundary term. This furthermore goes along with Belinfante and Rosenfeld's work and is incredibly clarifying when we look physically at what couples to gravity matter wise.

A slight nuance when examining the boundary term occurs, however. If one takes the viewpoint that the spin connection depends on the tetrad and contorsion tensor; this is an important philosophical and mathematical distinction. For instance if we are able to vary with respect to the tetrad and contorsion at the beginning, and we are able to vary with respect to the tetrad and spin connection, we must wind up with the same thing if we pick the tetrad and contorsion as our independent variables.

This comes up in Dirac theory where it is easier to vary with respect to the spin connection and tetrad, and use the Belinfante relation. The difficulty in varying with respect to contorsion and the tetrad at the get go comes from the fact that the Levi-Civita spin connection ($\varpi^\mu{}_{\nu a}$) appears when breaking off the contorsion tensor from the covariant derivative, resulting in the same boundary term as we found above.

All in all, this relation relates many of the different theories of gravity together.

CURRICULUM VITAE

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Interested in the curvature of spacetime because of wormholes. The math that goes into these objects is beautiful. Talented in: Multilinear Algebra, Maple, PowerPoint, Word, Lie Theory, General Relativity, Electrodynamics & E/M Radiation, ODE and PDE Theory, Extensive Differential Geometry, and 3D Printing. Other pertaining skills include: Python, Mathematica, Excel, Maple, and Fortran. Works well with others, and is incredibly passionate and described as "bubbly" by his students.

University of Mount Union

Graduated Cum Laude (GPA 3.699) - (May 2018)
Bachelor of Science in Physics (Honors) - 2018
Bachelor of Science in Mathematics (Honors) - 2018
Costume Shop - design & manufacturing under Deborah Lotsof
Mathematics Department: "W. & B. Stuckey Mathematics Prize (2015)"
Mathematics Department: Tutor
Physics Department: "Physics Faculty Award/Scholarship (2017)"
Founders Scholarship
Theatre Scholarship
Library Aid under Linda Scott

Scholar Day

Oral Presentations

1.) (2017) "Sound demonstration with constructed instruments"
2.) (2018) "Visualizing Spacetime"

Poster Presentations

3.) (2016) "Evolving Interval Solar Conditions, How the Sun's internal hydrogen content changes over time"
4.) (2018) "3D Printing with Mathematica and Differential Geometry"

STEMCoding with Ohio State and Mount Union

- Presented on STEMCoding project with OSU and UMU at Appalachian Section of the American Association of Physics Teachers – Fall 2017 Meeting
- Presented on STEMCoding at Mount Union during 2018 Ohio-PKAL conference titled "Promoting Effective Learning in a Diverse STEM Environment." The Conference was a part of the AACU. (May 19)

Honors Projects

Mathematics

Differential Equations: Laplace Transform project
Mathematical Statistics: Chi squared test project
Honors Thesis: 5 Mathematical Problems in ODE's & PDE's

Physics

Astrophysics: Pulsar, White & Brown Dwarfs, Neutron Star & Black Holes
Analytical Mechanics: Chaotic motion project

Electromagnetic Theory: Visualizing Electromagnetism

Honors Societies

- **Sigma Pi Sigma—2016/17**

Physics Honors Society—President

- **Pi Mu Epsilon—2016/17**

Mathematics Honor Society—President

- **Alpha Mu Gamma – 2016/17**

Foreign Language Honor Society—President

- **Alpha Lambda Delta—2015/16**

National Scholastic Honors Society—Treasurer

Brother of Sigma Alpha Epsilon National Fraternity

- Scholarship Chair
- Eminent Chaplain
- Fred Archibald Leadership Award — recipient (2015)
- National Leadership School — attendee (2015)

Michigan State University

NS3– Nuclear Science Summer School (2017)

Southwest University for Nationalities (Chengdu, China) (Study Abroad)

Advanced study of the Chinese Language (Summer 2016)

Utah State University (2018-Present)

- "Howard L. Blood Award" (2018) – Summer Research on: Scalar field Solutions to the Einstein field Equations. Compiled six solutions to the EFE's to be added to the USU Library.
- USU Physics Colloquium (September 2018) "Scalar Field Solutions to the Einstein Field Equations"
- 35th Jim Isenberg Pacific Coast Gravity Meeting (March 2019): "Codimensions, Curvature, and Coding"
- USU Explicit Methods for Abelian and K3 Varieties (July 2019): "Immersions and the Gauss, Codazzi, Mainardi Equations"
- Lab instructor PHYS 2215 (Physics 1 lab, Mechanics) (Fall 2018)
- Lab instructor PHYS 2225 (Physics 2 lab, Electromagnetism) (Spring 2019, Fall 2019)
- APS 4-Corners Meeting at Embry-Riddle Aeronautical University (October 2019): "(Hyper)Surface Software in Physics" & Award for Outstanding Student Presentation.
- APS 4-Corners Meeting (2020) "Symbolic Tools in ECSK Theory"
- USU TPT (Theoretical Physics Talks) Invited Speaker. "What is a Wormhole?" (3/9/2022)

Papers:

- "An Easy Algorithm for Students to Generate Implicit N-Holed Tori in Maple and Mathematica" unpublished.

NIWC San Diego:

- Summer internships with Stephen Hobbs Summer (2021-Present)

Others:

- ❖ PhD Candidate - Passed Qualifying Exam (Fall 2021)
- ❖ Women's Welsh Club Scholarship recipient (2015-2019)
- ❖ 200 Hr. Yoga Teacher
- ❖ Usui Shiki Ryoho Reiki Master
- ❖ DoD – SMART Scholar (Granted Fall 2020)

Skills:

- ❖ <u>Computer Programing</u>
    - Python
    - Mathematica
    - Maple
    - Matlab
    - Latex
    - Fortran
- ❖ <u>Microsoft</u>
    - Excel
    - Powerpoint
    - Word
- ❖ <u>Mathematics and Physics</u>
    - Extensive Multilinear Algebra
    - Lie Theory
    - General Relativity
    - Electrodynamics & Electromagnetic Radiation
    - ODE & PDE Theory
    - Extensive Differential Geometry
- ❖ <u>3D Printing</u>