Calculate the "Kretschmann scalar" $R_{abcd}R^{abcd}$.

Synopsis

- On a pseudo-Riemannian manifold with metric g, the "Kretschmann scalar" is a quadratic scalar invariant of the Riemann R tensor of g, defined as $R_{abcd}R^{abcd}$, where indices are contracted using the metric.

- In this worksheet we show how to calculate the Kretschmann scalar from a metric.

Example 1

Load the DifferentialGeometry package and its Tensor sub-package.

```
> with(DifferentialGeometry): with(Tensor):
```

Some optional aesthetic preferences:

```
> interface(typesetting = extended):
> Preferences("TensorDisplay", 1):
```

Define a coordinate chart on a 2-dimensional manifold.

```
> DGsetup([theta, phi], S);
```

Manifold: S (1

Define the metric. This is the standard "round" metric on the 2-sphere.

```
S > g := evalDG(dtheta &t dtheta + sin(theta)^2*dphi &t dphi);
```

$$g := d\theta \otimes d\theta + \sin(\theta)^2\, d\phi \otimes d\phi \tag{2}$$

Calculate the Riemann curvature tensor of the metric g.

```
S > R := CurvatureTensor(g);
```

$$R := \sin(\theta)^2\, \partial_\theta \otimes d\phi \otimes d\theta \otimes d\phi - \sin(\theta)^2\, \partial_\theta \otimes d\phi \otimes d\phi \otimes d\theta - \partial_\phi \otimes d\theta \otimes d\theta \otimes d\phi + \partial_\phi \otimes d\theta \otimes d\phi \otimes d\theta \tag{3}$$

Calculate the Kretschmann scalar, which is the metric inner product of the curvature with itself, $R_{abcd}R^{abcd}$. The TensorInnerProduct command takes the tensor product of any two tensors of the same type and then forms the scalar by contracting all indices using the metric.

```
S > TensorInnerProduct(g, R, R);
```

$$4 \tag{4}$$

Of course, one can also do all the tensor products and contractions "by hand":

```
S > h := InverseMetric(g);
```

$$h := \partial_\theta \otimes \partial_\theta + \frac{1}{\sin(\theta)^2}\, \partial_\phi \otimes \partial_\phi \tag{5}$$

```
S > ContractIndices(g &t h &t h &t h, R &t R, [[1,1], [2,5], [3,2], [4,6], [5,3], [6,7], [7, 4], [8, 8]]);
```

$$4 \tag{6}$$

Example 2

This same procedure will compute the scalar for any metric on any manifold. One simple chooses the appropriate coordinate chart and defines the appropriate metric. A famous example is the Kretschmann scalar of the Schwarzschild metric for a black hole spacetime.

```
> DGsetup([t, r, theta, phi], M);
```

Manifold: M (7)

```
M > f := 1 - 2*m/r;
```

$$f := 1 - \frac{2\,m}{r} \tag{8}$$

```
M > g := evalDG(- f * dt &t dt + 1/f * dr &t dr + r^2*(dtheta &t dtheta + sin(theta)^2*dphi &t dphi));
```

$$g := \frac{-r + 2\,m}{r}\, dt \otimes dt - \frac{r}{-r + 2\,m}\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \tag{9}$$

```
M > R := CurvatureTensor(g):
```

```
M > TensorInnerProduct(g, R, R);
```

$$\frac{48\,m^2}{r^6} \tag{10}$$

A complete set of scalar curvature invariants can be computed using the command RiemannInvariants.

Commands Illustrated

- DGsetup, evalDG, CurvatureTensor, TensorInnerProduct

Related Commands

- RiemannInvariants

References

Release notes

- Worksheet was executed using DG build "USU001-1460 9:25:48.66 Fri 01/03/2020"

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