An Introduction to DifferentialGeometry with Maple
PCMI - July 5, 2013

Exact Solutions In General Relativity

Synopsis

In this worksheet we use Maple to find some exact solutions of the Einstein equations and to compute their properties.

Example 1

Perhaps the most important solution to the Einstein equations in general relativity is the Schwarzschild solution. This solution describes the gravitation field outside of a star and was the first solution used to experimentally confirm the validity of Einstein's theory of gravity.

With Maple it is easy to derive this solution.

```
> with(DifferentialGeometry): with(Tensor):
> DGsetup([t, r, theta, phi], M);
```

frame name: M **(1.1)**

Define the metric ansatz based upon symmetry considerations. The functions A(r) and B(r) are unknown functions which are to be determined by the Einstein equations.

```
M > g := evalDG(A(r)* dt &t dt + B(r)* dr &t dr + r^2*(dtheta &t
    dtheta + sin(theta)^2* dphi &t dphi));
```

$$g := A(r)\, dt \otimes dt + B(r)\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \quad \textbf{(1.2)}$$

The Einstein equations for this example are given by the vanishing of the Einstein tensor so we compute this tensor and pass the result to DGsolve.

```
M > EinTen := EinsteinTensor(g);
```

$$EinTen := -\frac{\left(\frac{d}{dr}B(r)\right)r + B(r)^2 - B(r)}{A(r)\,r^2\,B(r)^2}\,D_t \otimes D_t + \frac{\left(\frac{d}{dr}A(r)\right)r - A(r)\,B(r) + A(r)}{A(r)\,B(r)^2\,r^2}\,D_r \otimes D_r + \frac{1}{4\,r^3\,B(r)^2\,A(r)^2}\left(2\left(\frac{d^2}{dr^2}A(r)\right)A(r)\,B(r)\,r - \left(\frac{d}{dr}B(r)\right)\left(\frac{d}{dr}A(r)\right)A(r)\,r - \left(\frac{d}{dr}A(r)\right)^2 B(r)\,r - 2\left(\frac{d}{dr}B(r)\right)A(r)^2 + 2\left(\frac{d}{dr}A(r)\right)B(r)\,A(r)\right)D_\theta \otimes D_\theta - \frac{1}{4\,r^3\,B(r)^2\,A(r)^2\left(-1+\cos(\theta)^2\right)}\left(2\left(\frac{d^2}{dr^2}A(r)\right)A(r)\,B(r)\,r - \left(\frac{d}{dr}B(r)\right)\left(\frac{d}{dr}A(r)\right)A(r)\,r - \left(\frac{d}{dr}A(r)\right)^2 B(r)\,r - 2\left(\frac{d}{dr}B(r)\right)A(r)^2 + 2\left(\frac{d}{dr}A(r)\right)B(r)\,A(r)\right)D_\phi \otimes D_\phi \tag{1.3}$$

```
M > Soln := DGsolve(EinTen, g, [A(r), B(r)]);
```

$$Soln := \left\{\frac{(r+_C2)\,_C1}{r}\,dt \otimes dt + \frac{r}{r+_C2}\,dr \otimes dr + r^2\,d\theta \otimes d\theta + r^2\sin(\theta)^2\,d\phi \otimes d\phi\right\} \tag{1.4}$$

There are many commands available in DG to study the properties of this metric. For example, it is always useful to look at the scalar invariants of the curvature tensor (something one would almost never do without a computer).

```
M > RiemannInvariants(Soln[1], author = "CarminatiMcLenaghan");
```

$$\text{table}\left(\left[\text{"r1"} = 0, \text{"m5"} = 0, \text{"m4"} = 0, \text{"r2"} = 0, \text{"m3"} = 0, \text{"m2"} = 0, \text{"m1"} = 0, \text{"w2"} = -\frac{3\,_C2^3}{4\,r^9}, \text{"w1"} = \frac{3\,_C2^2}{2\,r^6}, \text{"r3"} = 0\right]\right) \tag{1.5}$$

We see that the constant of integration _C1 has no geometric significance, while the constant _C2 controls the amount of curvature. This constant turns out to be identified with the mass of the star.

Example 2 (A Parable - Part 1)

The other day a graduate student came to my office, exclaiming that an

interesting solution to the Einstein equations had been found and asking if we could please schedule the thesis defense.

The student's solution is:

```
M > DGsetup([x, y, z, w], P);
```

frame name: P **(2.1)**

```
P > g := evalDG((c2*dx &t dx)/z^2+c3*dx &s dz/z^2-(1/3)*c1*dx &s
    dw/z^4+(1/4)*(c1*dy &t dy)/z^4+c3*dx &s dz/z^2+(c4*dz &t dz)
    /z^2-(1/3)*c1*dx &s dw/z^4);
```

$$g := \frac{c2}{z^2}\, dx \otimes dx + \frac{c3}{z^2}\, dx \otimes dz - \frac{c1}{3\,z^4}\, dx \otimes dw + \frac{c1}{4\,z^4}\, dy \otimes dy + \frac{c3}{z^2}\, dz \otimes dx + \frac{c4}{z^2}\, dz \otimes dz - \frac{c1}{3\,z^4}\, dw \otimes dx \quad \textbf{(2.2)}$$

(How this metric was found is an (important) story for another day.)

The student said that this is a pure radiation solution with non-zero cosmological constant.

$$G^{ij} - \Lambda g^{ij} = kN^iN^j\,, \qquad N_iN^i = 0$$

Let's check this claim.

```
P > EinTens := EinsteinTensor(g):
P > CosmologicalTerm := Lambda*InverseMetric(g):
P > EinEqWithCosm := evalDG(EinTens - CosmologicalTerm);
```

$$EinEqWithCosm := \frac{3\,z^4\,(\Lambda\,c4-12)}{c1\,c4}\, D_x \otimes D_w - \frac{4\,z^4\,(\Lambda\,c4-12)}{c1\,c4}\, D_y \otimes D_y - \frac{z^2\,(\Lambda\,c4-12)}{c4^2}\, D_z \otimes D_z - \frac{3\,c3\,z^4\,(\Lambda\,c4-12)}{c1\,c4^2}\, D_z \otimes D_w + \frac{3\,z^4\,(\Lambda\,c4-12)}{c1\,c4}\, D_w \otimes D_x - \frac{3\,c3\,z^4\,(\Lambda\,c4-12)}{c1\,c4^2}\, D_w \otimes D_z + \frac{9\,z^6\,(\Lambda\,c2\,c4^2 - \Lambda\,c3^2\,c4 - 8\,c2\,c4 + 12\,c3^2)}{c4^2\,c1^2}\, D_w \otimes D_w \quad \textbf{(2.3)}$$

```
> T := DGsimplify(subs(Lambda = 12/c4, EinEqWithCosm));
```

$$T := \frac{36\,z^6\,c2}{c4\,c1^2}\, D_w \otimes D_w \quad \textbf{(2.4)}$$

```
P > TensorInnerProduct(g, D_w, D_w);
```

$$0 \quad \textbf{(2.5)}$$

So far so good.

Example 2 (A Parable - Part 2)

But is this solution new - or is it already known? Even with Google/Bing/Ask, this is almost impossible to answer.

But with DG it may be possible to make some progress. Let us find some additional properties of the metric.

```
P > KV := KillingVectors(g);
```

$$KV := \left[-\frac{3x}{c1} D_x - \frac{6y}{c1} D_y - \frac{3z}{c1} D_z - \frac{9w}{c1} D_w, -\frac{3}{c1} D_x, \frac{4x}{c1} D_y + \frac{3y}{c1} D_w, \frac{4}{c1} D_y, -\frac{3}{c1} D_w \right] \quad (3.1)$$

```
P > nops(KV);
```

$$5 \quad (3.2)$$

```
P > PetrovType(g);
```

$$\text{"O"} \quad (3.3)$$

```
P > MetricSearch();
```

Summary

- DG contains a wide range of tools for finding analytic solutions to geometric partial differential equations such as the Einstein equations. (In Demo5 we use Maple to classify the homogeneous metrics on the 7 sphere.)
- We are building extensive libraries of results from the mathematics and mathematical physics literature and easy to use tools for searching these libraries.
- Maple is more than just a computational engine.