An Introduction to DifferentialGeometry with Maple

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The Octonions and the Exceptional Lie Algebra $\mathfrak{g}_2$

Synopsis

An algebra $\mathcal{A}$ over the real numbers is an algebraic structure with operations of addition, multiplication and scalar multiplication.

Algebras such as the quaternions, octonions, Jordan algebras, Clifford algebras play an important role in many constructions in differential geometry and Lie theory (see, for example, "Spinors and Calibrations" by R. Harvey).

In this worksheet we prove a famous theorem of Cartan (1908) regarding the derivations of the octonions (see the beautiful article "Octonions" by John Baez) .

The Quaternions

Before looking at the octonions, it will helpful to review the algebra of quaternions.

```
> restart:
> with(DifferentialGeometry): with(LieAlgebras):
```

The multiplication rules for working with the quaternions are stored in a DG library. Retrieve and initialize.

```
> LD := AlgebraLibraryData("Quaternions", Ham):
> DGsetup(LD, [e, i, j, k], [o]);
```

algebra name: Ham **(1.1)**

Here is the familiar multiplication table for the quaternions.

```
> MultiplicationTable(Ham);
```

$$\begin{bmatrix} & | & e & i & j & k \\ \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} \\ e & | & e & i & j & k \\ i & | & i & -e & k & -j \\ j & | & j & -k & -e & i \\ k & | & k & j & -i & -e \end{bmatrix} \tag{1.2}$$

We can calculate the inverse of a quaternion.

```
Ham > x := evalDG(2*e - i +3*j +k);
```

$$x := 2\,e - i + 3\,j + k \tag{1.3}$$

```
Ham > y := AlgebraInverse(x);
```

$$y := \frac{2}{15}\,e + \frac{1}{15}\,i - \frac{1}{5}\,j - \frac{1}{15}\,k \tag{1.4}$$

```
Ham > evalDG(x.y);
```

$$e \tag{1.5}$$

Let's us see what happens to the quaternion multiplication rules if we rotate our basis.

```
Ham > X := evalDG(cos(theta)*i - sin(theta)*j);
```

$$X := \cos(\theta)\,i - \sin(\theta)\,j \tag{1.6}$$

```
Ham > Y := evalDG(sin(theta)*i + cos(theta)*j);
```

$$Y := \sin(\theta)\,i + \cos(\theta)\,j \tag{1.7}$$

```
Ham > Z := k;
```

$$Z := k \tag{1.8}$$

```
Ham > evalDG(X.X +e), evalDG(Y.Y +e), evalDG(X.Y - Z);
```

$$0\,e, 0\,e, 0\,e \tag{1.9}$$

We say that Euclidean rotations of the basis i, j, k are automorphisms of the quaternions.

The derivations of an algebra are the "infinitesimal" automorphisms. These are easily calculated.

```
Ham > Derivations();
```

(1.10)

$$\left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\right] \quad \textbf{(1.10)}$$

Here is the formal definition. A linear transformation $\phi : \mathcal{A} \to \mathcal{A}$ is called a derivation if

$$\phi(xy) = \phi(x)y + x\,\phi(y) \text{ for all } x, y \in \mathcal{A}\,.$$

The Octonions

We use the AlgebraLibraryData command to obtain the multiplication rules for the octonions. We shall refer to this algebra as "oct".

```
> LD := AlgebraLibraryData("Octonions", oct):
```

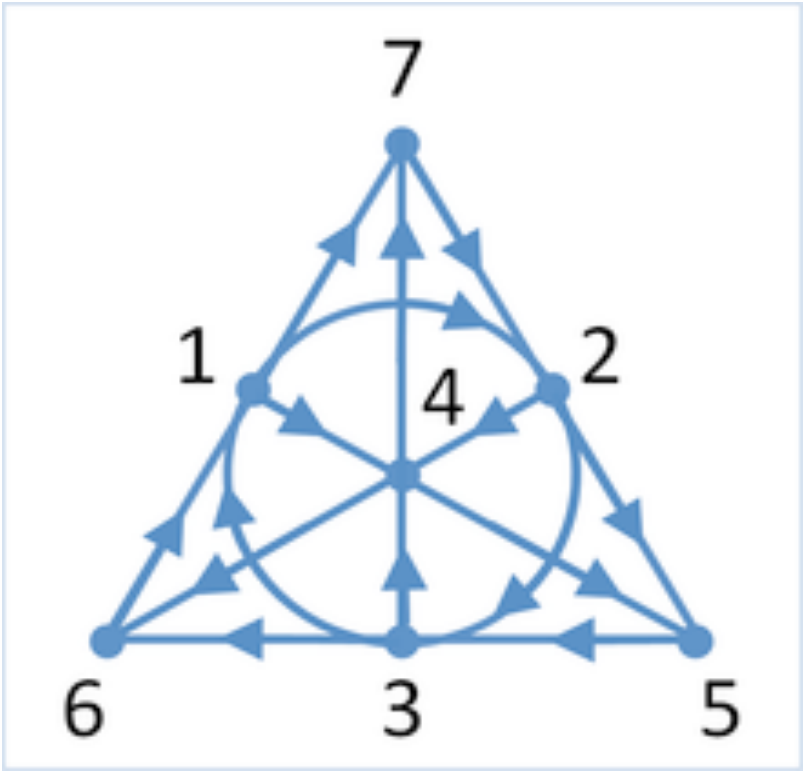
We use the DGsetup command to initialize this algebra and we display the multiplication table.

```
> DGsetup(LD,'[e, e1, e2, e3, e4, e5, e6, e7]', '[omega]'):
oct > MultiplicationTable();
```

		e	e1	e2	e3	e4	e5	e6	e7
	----	----	----	----	----	----	----	----	----
e	|	e	e1	e2	e3	e4	e5	e6	e7
e1	|	e1	-e	e3	-e2	e5	-e4	-e7	e6
e2	|	e2	-e3	-e	e1	e6	e7	-e4	-e5
e3	|	e3	e2	-e1	-e	e7	-e6	e5	-e4
e4	|	e4	-e5	-e6	-e7	-e	e1	e2	e3
e5	|	e5	e4	-e7	e6	-e1	-e	-e3	e2
e6	|	e6	e7	e4	-e5	-e2	e3	-e	-e1
e7	|	e7	-e6	e5	e4	-e3	-e2	e1	-e

(2.1)

Here is a picture which summarizes these rules.



The multiplication is non-associative.

```
oct >  evalDG(e1.evalDG(e3.e4));
```

$$e6 \tag{2.2}$$

```
oct > evalDG(evalDG(e1 .e3).e4);
```

$$-e6 \tag{2.3}$$

The Derivation Algebra for the Octonions

Let $\mathcal{A}$ be an algebra. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a derivation if it satisfies the "Leibniz" rule

$$\phi(xy) = \phi(x)y + x\phi(y) \quad x, y \in \mathcal{A}.$$

The algebra of derivations for the octonions is a matrix algebra of 8×8 matrices. We do not display all the matrices.

```
oct > aut := Derivations(oct):
oct > aut[3];
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(3.1)}$$

The derivation algebra is 14 dimensional.

```
oct > nops(aut);
```

$$14 \quad \textbf{(3.2)}$$

We use LieAlgebraData to prove that these 14 matrices form a Lie algebra and to calculate the structure equations.

```
oct > LD := LieAlgebraData(aut, alg):
oct > DGsetup(LD);
```

Lie algebra: alg **(3.3)**

```
alg > interface(rtablesize = 30);
```

$$14 \quad \textbf{(3.4)}$$

```
alg > MultiplicationTable("LieTable");
```

$$\begin{aligned}
&[[, |, e1, e2, e3, e4, e5, e6, e7, e8, e9, e10, e11, e12, e13, e14], \\
&\quad [, ----, ----, ----, ----, ----, ----, ----, ----, ----, ----, ----, ----, ----, ----], \\
&\quad [e1, |, 0, -e7, -e8, -e5 - e9, e4 - e10, -e11, e2, e3, e4 - e10, e5 + e9, e6, -e13, \\
&\quad e12, 0], \\
&\quad [e2, |, e7, 0, -e5, -2\,e6, e3, 2\,e4, -e1, e4, -e3 - e11, -e6, e5 + e9, -e14, 0, e12], \\
&\quad [e3, |, e8, e5, 0, -e12, -2\,e2 - 2\,e13, -e14, e10, -e1, e2 + e13, -e7, 0, e4, e5, e6], \\
&\quad [e4, |, e5 + e9, 2\,e6, e12, 0, -e14, -2\,e2, e11, -e2, -e1 + e14, 0, -e7, -e3, -e6, e5], \\
&\quad [e5, |, -e4 + e10, -e3, 2\,e2 + 2\,e13, e14, 0, -e12, e6 - e8, e7 - e12, 0, -e1 + e14, \\
&\quad -e2 - e13, e6, -e3, -e4], \\
&\quad [e6, |, e11, -2\,e4, e14, 2\,e2, e12, 0, -e5 - e9, 0, e7 - e12, e2, -e1, -e5, e4, -e3], \\
&\quad [e7, |, -e2, e1, -e10, -e11, -e6 + e8, e5 + e9, 0, e9, -e8, -2\,e11, 2\,e10, 0, -e14, e13 \\
&\quad], \\
&\quad [e8, |, -e3, -e4, e1, e2, -e7 + e12, 0, -e9, 0, 2\,e7 - 2\,e12, -e13, -e14, e9, e10, e11], \\
&\quad [e9, |, -e4 + e10, e3 + e11, -e2 - e13, e1 - e14, 0, -e7 + e12, e8, -2\,e7 + 2\,e12, 0, \\
&\quad -e14, e13, -e8, -e11, e10],
\end{aligned} \quad \textbf{(3.5)}$$

$$[e10, |, -e5 - e9, e6, e7, 0, e1 - e14, -e2, 2\,e11, e13, e14, 0, -2\,e7, e11, -e8, -e9],$$
$$[e11, |, -e6, -e5 - e9, 0, e7, e2 + e13, e1, -2\,e10, e14, -e13, 2\,e7, 0, -e10, e9, -e8],$$
$$[e12, |, e13, e14, -e4, e3, -e6, e5, 0, -e9, e8, -e11, e10, 0, -2\,e14, 2\,e13],$$
$$[e13, |, -e12, 0, -e5, e6, e3, -e4, e14, -e10, e11, e8, -e9, 2\,e14, 0, -2\,e12],$$
$$[e14, |, 0, -e12, -e6, -e5, e4, e3, -e13, -e11, -e10, e9, e8, -2\,e13, 2\,e12, 0]]$$

We use a sequence of Maple commands to make sense of this multiplication table, that is, to identify this Lie algebra. First we note that it is semi-simple.

```
alg > Query("Semisimple");
```

$$true \tag{3.6}$$

We find a Cartan subalgebra. It has dimension 2 so that our derivation algebra has rank 2.

```
alg > CSA := CartanSubalgebra();
```

$$CSA := [e1, e4 + e10] \tag{3.7}$$

We see that all the roots in the root space decomposition are pure imaginary numbers. This means that the Lie algebra is of compact type.

```
alg > RSD := RootSpaceDecomposition(CSA);
```

$$\begin{aligned} RSD := \text{table}([&[I, 3\,I] = e2 - I\,e6 + I\,e7 + e11, [-I, -3\,I] = e2 + I\,e6 - I\,e7 + e11, \\ &[-I, I] = e2 - 2\,e3 - I\,e6 - I\,e7 + 2\,I\,e8 - e11 + 2\,I\,e12 + 2\,e13, [-I, -I] = e2 \\ &+ 2\,e3 + I\,e6 - I\,e7 - 2\,I\,e8 + e11 + 2\,I\,e12 + 2\,e13, [I, I] = e2 + 2\,e3 - I\,e6 \\ &+ I\,e7 + 2\,I\,e8 + e11 - 2\,I\,e12 + 2\,e13, [I, -I] = e2 - 2\,e3 + I\,e6 + I\,e7 - 2\,I\,e8 \\ &- e11 - 2\,I\,e12 + 2\,e13, [0, -2\,I] = e1 - I\,e5 + I\,e9 - 2\,e14, [-2\,I, 0] = e4 - I\,e5 \\ &- I\,e9 - e10, [0, 2\,I] = e1 + I\,e5 - I\,e9 - 2\,e14, [2\,I, 0] = e4 + I\,e5 + I\,e9 - e10, \\ &[-I, 3\,I] = e2 - I\,e6 - I\,e7 - e11, [I, -3\,I] = e2 + I\,e6 + I\,e7 - e11]) \end{aligned} \tag{3.8}$$

Finally, we find the simple roots and the Cartan matrix to conclude that the derivation algebra of the octonions is the compact real form of the exceptional Lie algebra $\mathfrak{g}_2$.

```
alg > SimRts := SimpleRoots(RSD, [<I, 0>, <0, I>]);
```

$$SimRts := \left[\begin{bmatrix} 0 \\ 2\,I \end{bmatrix}, \begin{bmatrix} I \\ -3\,I \end{bmatrix}\right] \tag{3.9}$$

```
alg > CM := CartanMatrix(SimRts, RSD);
```

$$CM := \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \tag{3.10}$$

```
alg > CartanMatrixToStandardForm(CM);
```

$$\left[\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{"G"}\right] \tag{3.11}$$

Summary

- Different types of algebras (Jordan algebras, Clifford algebras ...) play an important role in Lie theory and differential geometry. DG supports computations with these algebraic structures.
- There is a complete classification of simple Lie algebras. The tools to perform this classification (over the real or complex numbers) are available in Maple
- The representation theory of simple Lie algebras is forthcoming.